

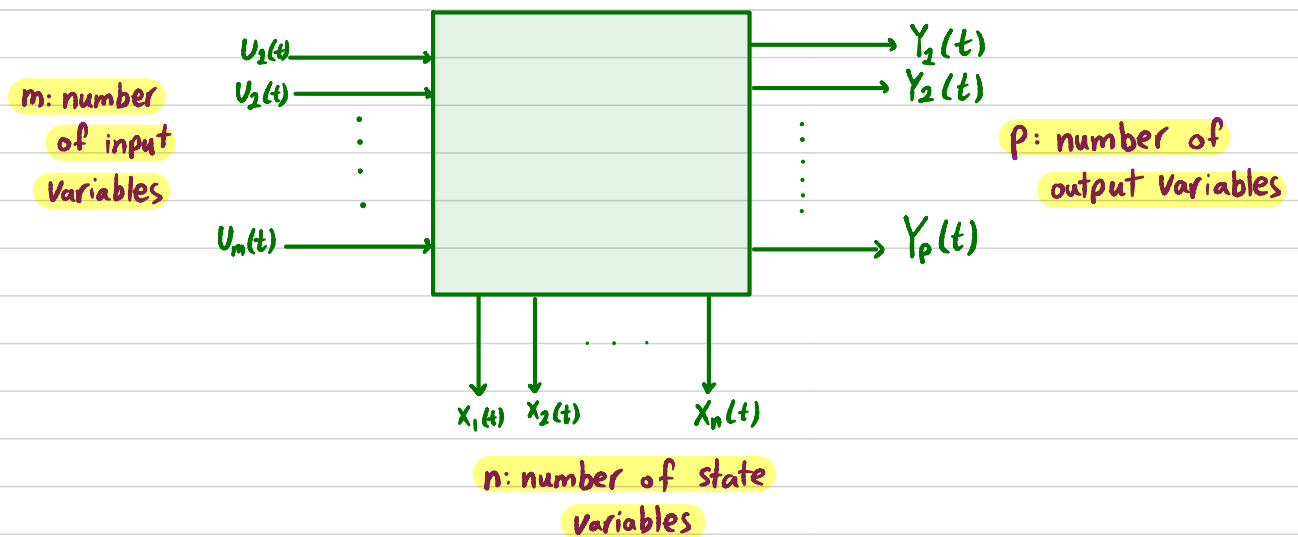
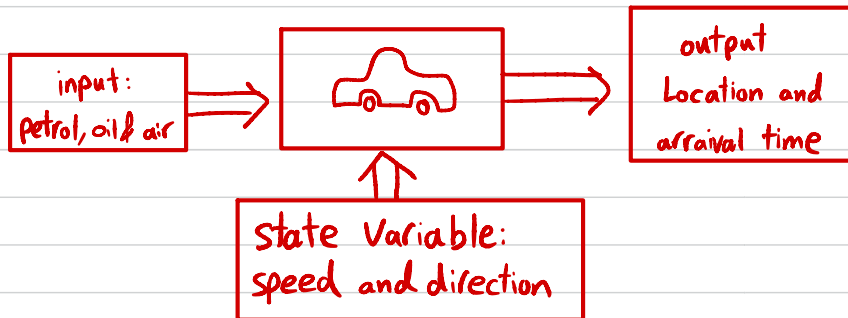
نظرية التحكم الحديث

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الطالب المبدع
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* State Variable:

A set of variables which describes the system at any time instants are called state variables.



* State Variable Equation

$$\dot{X}(t) = AX(t) + BU(t) \Rightarrow \text{state equation.}$$

$$Y(t) = CX(t) + DU(t) \Rightarrow \text{output equation.}$$

first derivative

- $X(t)$:- state Vector.
- $U(t)$:- Input vector.
- $Y(t)$:- output vector.
- A :- system matrix.
- B :- Input matrix.
- C :- output matrix.
- D :- Feedback forward matrix.

Ex: 3 state variables (n)
 2 inputs (m)
 2 outputs (p)

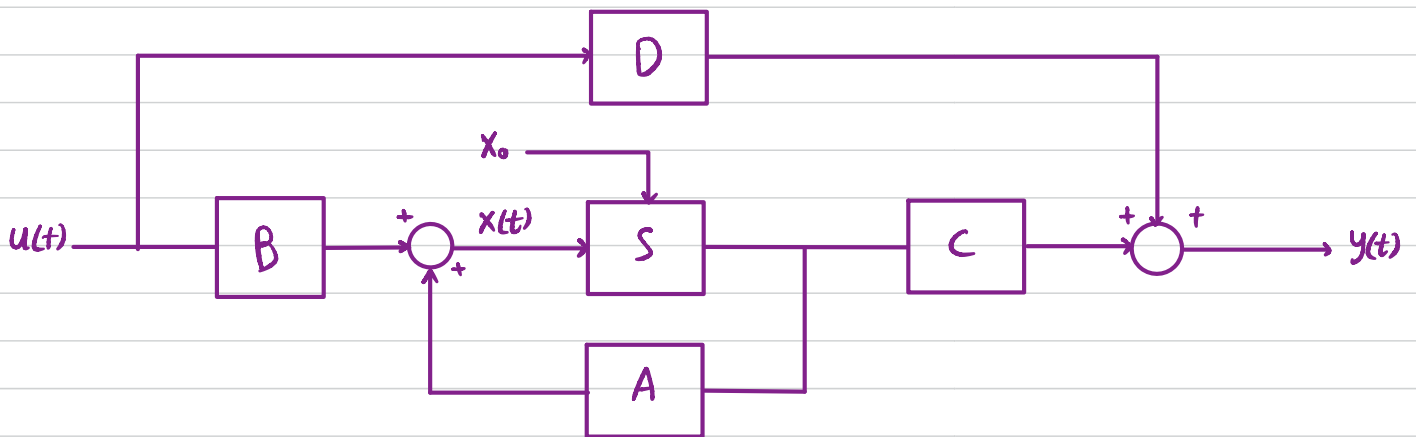
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

(n x n) (n x m)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} u(t)$$

(p x n) (p x m)

↪ state variable vector



- state variable equation block diagram -

* Example:- consider a system model given by the following D.E:

$$\frac{\partial^3 y}{\partial t^3} + 6 \frac{\partial^2 y}{\partial t^2} + 11 \frac{\partial y}{\partial t} + 6y = u(t)$$

where $y(t)$ is output & $u(t)$ is input, find state-space model.

$$\begin{aligned}x_1 &= y(t) \\x_2 &= \dot{x}_1 = \dot{y}(t) \\x_3 &= \dot{x}_2 = \ddot{y}(t)\end{aligned}$$

* number of state variables depend on the order of the diff equation.

we know that:

$$\begin{aligned}\dot{X} &= AX + BU \\Y &= CX + DU\end{aligned}$$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \ddot{y}(t) = u(t) - 6x_3 - 11x_2 - 6x_1\end{aligned}$$

from the given equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$[y] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 0$$

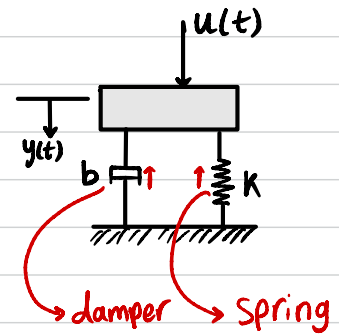
no information about the D matrix

* Example: mass-spring Damper System

friction & weight = 0

$k, b \rightarrow$ is a storage element

components that store something like force



$$\Sigma F = m\vec{a}$$

$$u(t) - ky - b\dot{y} = m\ddot{y}$$

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{x}_1 = \dot{y} \end{aligned}$$

second order equation
so two state variables.

$$\dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = \frac{u(t)}{m} - \frac{k}{m}x_1 - \frac{b}{m}x_2$$

$$\dot{x} = Ax(t) + Bu(t)$$

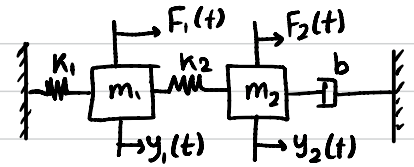
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$Y(t) = Cx(t) + Du(t)$$

$$D = 0$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

* Example: find S.S.M.



$$\Sigma F_1 = m_1 a_1$$

$$F_1(t) - k_1 y_1 - k_2 (y_1 - y_2) = m_1 \ddot{y}_1$$

$$\ddot{y}_1 = \frac{F_1(t)}{m_1} - \frac{k_1}{m_1} y_1 - \frac{k_2}{m_1} y_1 + \frac{k_2}{m_1} y_2$$

inputs : $F_1(t)$, $F_2(t)$
outputs : $y_1(t)$, $y_2(t)$

$$\Sigma F_2 = m_2 a_2$$

$$F_2(t) - k_2 (y_2 - y_1) - b \dot{y}_2 = m_2 \ddot{y}_2$$

$$\ddot{y}_2 = \frac{F_2(t)}{m_2} - \frac{k_2}{m_2} y_2 + \frac{k_2}{m_2} y_1 - \frac{b}{m_2} \dot{y}_2$$

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= \dot{y}_1 \\ x_3 &= y_2 \\ x_4 &= \dot{y}_2 \end{aligned}$$

$$\dot{x}_1 = \dot{y}_1 = x_2$$

$$\dot{x}_2 = \ddot{y}_1 = \frac{u_1(t)}{m_1} - \left(\frac{k_1 + k_2}{m_1} \right) x_1 + \frac{k_2}{m_1} x_3$$

$$\dot{x}_3 = \dot{y}_2 = x_4$$

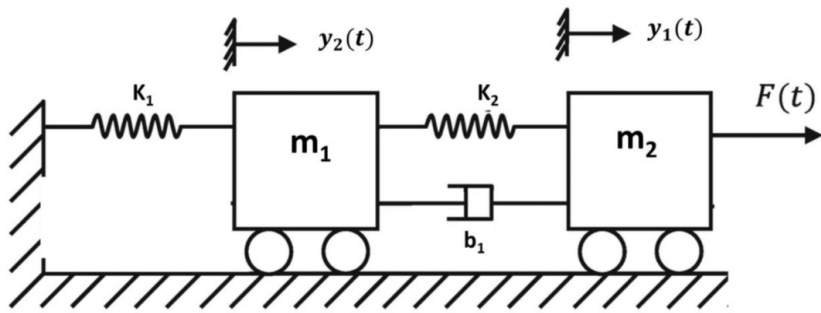
$$\dot{x}_4 = \ddot{y}_2 = \frac{u_2(t)}{m_2} - \frac{k_2}{m_2} x_3 + \frac{k_2}{m_2} x_1 - \frac{b}{m_2} x_4$$

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1 + k_2)}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & -\frac{b}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = Cx + Du$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$\begin{aligned} k_1 &= 4 \\ K_2 &= 3 \\ m_1 &= 12 \\ m_2 &= 24 \\ b_1 &= 2 \end{aligned}$$

$$\begin{aligned} \Sigma F_1 &= m_1 \ddot{y}_2 \\ -k_1 y_2 - K_2 (y_2 - y_1) - b_1 (\dot{y}_2 - \dot{y}_1) &= m_1 \ddot{y}_2 \\ -4 y_2 - 3 y_2 + 3 y_1 - 2 \dot{y}_2 + 2 \dot{y}_1 &= 12 \ddot{y}_2 \\ \ddot{y}_2 &= \frac{1}{4} \dot{y}_1 - \frac{7}{12} \dot{y}_2 + \frac{1}{6} \dot{y}_1 - \frac{1}{6} \dot{y}_2 \end{aligned}$$

$$\Sigma F_2 = m_2 \ddot{y}_1$$

$$F(t) - K_2 (y_1 - y_2) - b_1 (\dot{y}_1 - \dot{y}_2) = m_2 \ddot{y}_1$$

$$-\frac{(K_1 + K_2)}{m_1} y_2 + \frac{K_2}{m_1} y_1 - \frac{b_1}{m_1} \dot{y}_2 + \frac{b_1}{m_1} \dot{y}_1 = \ddot{y}_2$$

$$\ddot{y}_1 = \frac{F(t)}{m_2} - \frac{K_2}{m_2} y_1 + \frac{K_2}{m_2} y_2 - \frac{b_1}{m_2} \dot{y}_1 + \frac{b_1}{m_2} \dot{y}_2$$

$$\ddot{y}_1 = \frac{F(t)}{24} - \frac{1}{8} y_1 + \frac{1}{8} y_2 - \frac{1}{12} \dot{y}_1 + \frac{1}{12} \dot{y}_2$$

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= \dot{y}_1 \\ x_3 &= y_2 \\ x_4 &= \dot{y}_2 \end{aligned}$$

$$\dot{x}_1 = \dot{y}_1 = x_2$$

$$\ddot{x}_2 = \ddot{y}_1 = \frac{F(t)}{24} - \frac{1}{8} x_1 + \frac{1}{8} x_3 - \frac{1}{12} x_2 + \frac{1}{12} x_4$$

$$\dot{x}_3 = \dot{y}_2 = x_4$$

$$\dot{x}_4 = \ddot{y}_2 = \frac{1}{4} x_1 - \frac{7}{12} x_3 + \frac{1}{6} x_2 - \frac{1}{6} x_4$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/8 & -1/12 & 1/8 & 1/12 \\ 0 & 0 & 0 & 1 \\ 1/4 & 1/6 & -7/12 & -1/6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/24 \\ 0 \\ 0 \end{bmatrix} u$$

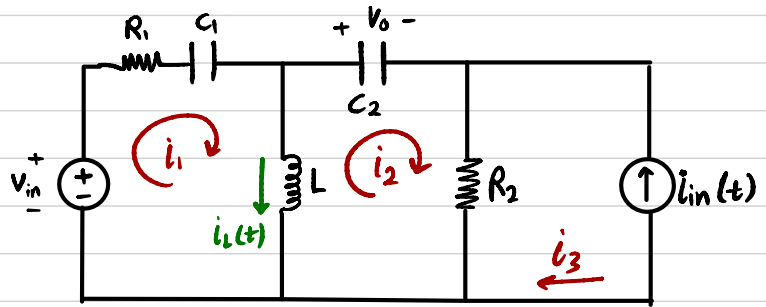
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0$$

* Example: find state space model for the circuit below:

$$\begin{aligned} u_1 &= V_{in}(t) \\ u_2 &= i_{in}(t) \end{aligned} \quad \left. \vphantom{\begin{aligned} u_1 &= V_{in}(t) \\ u_2 &= i_{in}(t) \end{aligned}} \right\} \text{inputs}$$

$$\begin{aligned} x_1 &= V_{C_1}(t) \\ x_2 &= V_{C_2}(t) \\ x_3 &= i_L(t) \end{aligned} \quad \left. \vphantom{\begin{aligned} x_1 &= V_{C_1}(t) \\ x_2 &= V_{C_2}(t) \\ x_3 &= i_L(t) \end{aligned}} \right\} \text{storage elements as state variables}$$

$$y = V_{C_2}(t) = x_2$$



$$* i_L(t) = i_1 - i_2$$

$$\text{mesh 1: } V_{in}(t) = R_1 i_1(t) + V_{C_1}(t) + L \frac{d}{dt} [i_1(t) - i_2(t)]$$

$$\text{mesh 2: } L \frac{d}{dt} [i_2(t) - i_1(t)] + V_{C_2}(t) + R_2 [i_2(t) - i_3(t)] = 0$$

$$\text{mesh 3: } i_3(t) = -i_{in}(t)$$

$$i_1(t) = C_1 \frac{dV_{C_1}}{dt} = C_1 \dot{x}_1(t)$$

$$i_2(t) = C_2 \frac{dV_{C_2}}{dt} = C_2 \dot{x}_2(t)$$

$$x_3(t) = i_L(t) = C_1 \dot{x}_1(t) - C_2 \dot{x}_2(t)$$

$$u_1(t) - x_1(t) = R_1 C_1 \dot{x}_1(t) + L \dot{x}_3(t)$$

$$-L \dot{x}_3(t) + x_2(t) + R_2 C_2 \dot{x}_2(t) + R_2 u_2(t)$$

$$C_1 \dot{x}_1(t) - C_2 \dot{x}_2(t) = x_3(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} R_1 C_1 & 0 & L \\ 0 & R_2 C_2 & -L \\ C_1 & -C_2 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -R_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \right)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

This method is harder and take time to solve because of the inverse.

OR:-

$$X_3 = C_1 \dot{X}_1 - C_2 \dot{X}_2$$

$$V_{in} = i_1 R + V_{C1} + L \frac{d(i_1 - i_2)}{dt}$$

$$L \frac{d(i_2 - i_1)}{dt} + V_{C2} + R_2 (i_1 + i_2) = 0$$

$$U_1 = R_1 C_1 \dot{X}_1 + X_1 + L \dot{X}_3 \quad \text{eq 1}$$

$$-L \dot{X}_3 + X_2 + R_2 C_2 \dot{X}_2 + R_2 U_2 = 0 \quad \text{eq 2}$$

$$* U_1 = X_1 + X_2 + R_1 C_1 \dot{X}_1 + R_2 C_2 \dot{X}_2 + R_2 U_2$$

$$C_1 \dot{X}_1 = X_3 + C_2 \dot{X}_2$$

$$U_1 = X_1 + X_2 + R_1 X_3 + R_1 C_2 \dot{X}_2 + R_2 C_2 \dot{X}_2 + R_2 U_2$$

$$\dot{X}_2 = \frac{U_1 - X_1 - X_2 - R_1 X_3 + R_2 U_2}{C_2 (R_1 + R_2)}$$

$$C_2 \dot{X}_2 = C_1 \dot{X}_1 - X_3$$

$$U_1 = X_1 + X_2 + R_1 C_1 \dot{X}_1 + R_2 C_1 \dot{X}_1 - R_2 X_3 + R_2 U_2$$

$$\dot{X}_1 = \frac{U_1 - X_1 - X_2 + R_2 X_3 - R_2 U_2}{C_1 (R_1 + R_2)}$$

to get $\dot{X}_3 \rightarrow$ Sub. either $\dot{X}_1$ in eq 1 or $\dot{X}_2$ in eq 2 and simplify.

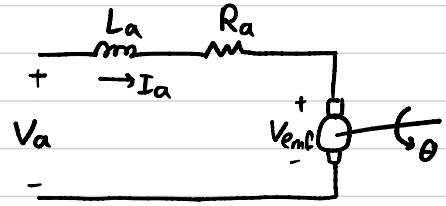
$$U_1 = \frac{R_1 U_1 - R_1 X_1 - R_1 X_2 + R_1 R_2 X_3 - R_1 R_2 U_2}{(R_1 + R_2)} + X_1 + L \dot{X}_3$$

$$\dot{X}_3 = \frac{U_1 R_2 + R_1 X_2 - R_1 R_2 X_3 + R_1 R_2 U_2 - R_2 X_1}{L (R_1 + R_2)}$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} \frac{-1}{C_1 (R_1 + R_2)} & \frac{-1}{C_1 (R_1 + R_2)} & \frac{R_2}{C_1 (R_1 + R_2)} \\ \frac{-1}{C_2 (R_1 + R_2)} & \frac{-1}{C_2 (R_1 + R_2)} & \frac{-R_1}{C_2 (R_1 + R_2)} \\ \frac{-R_2}{L (R_1 + R_2)} & \frac{R_1}{L (R_1 + R_2)} & \frac{-R_1 R_2}{L (R_1 + R_2)} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1 (R_1 + R_2)} & \frac{-R_2}{C_1 (R_1 + R_2)} \\ \frac{1}{C_2 (R_1 + R_2)} & \frac{R_2}{C_2 (R_1 + R_2)} \\ \frac{R_2}{L (R_1 + R_2)} & \frac{R_1 R_2}{L (R_1 + R_2)} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

* Ex: find S.S.M?

$$V_a = L_a \frac{dI_a}{dt} + I_a R_a + V_{emf} \dots (1)$$



$$\begin{aligned} \Sigma \tau &= J \ddot{\theta} \\ T_m - T_f &= J \ddot{\theta} \\ T_m - b \dot{\theta} &= J \ddot{\theta} \end{aligned} \quad \left\{ \begin{aligned} T_m &= k_m I_a \\ V_{emf} &= k_p \dot{\theta} \end{aligned} \right.$$

$$\begin{aligned} x_1 &= \theta = y \\ x_2 &= \dot{\theta} = \dot{x}_1 \\ x_3 &= I_a \end{aligned}$$

output $\rightarrow \theta$
input $\rightarrow V_a$

$$\ddot{\theta} = \frac{k_m}{J} I_a - \frac{b}{J} \dot{\theta} \dots (2)$$

$$\dot{x}_2 = \ddot{\theta} = \frac{k_m}{J} x_3 - \frac{b}{J} x_2$$

$$\dot{x}_3 = \frac{dI_a}{dt} = \frac{V_a}{L_a} - \frac{R_a}{L_a} x_3 - \frac{k_p}{L_a} x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -b/J & k_m/J \\ 0 & -k_p/L_a & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} V_a$$

* Transfer function T.F for state space.

$$T.F = \frac{y(s)}{u(s)} = C(SI - A)^{-1} B$$

$$* (SI - A)^{-1} = \frac{\text{Adj}(SI - A)}{\det(SI - A)}$$

A, C, B are matrices.

I: identity matrix
because s is scalar.

$$(SI - A) = \begin{bmatrix} s & 1 \\ -3s & 5 \end{bmatrix}$$

$$\det(SI - A) = 5s - 3s = 2s$$

$$\text{adj}(SI - A) = \begin{bmatrix} 5 & -1 \\ 3s & s \end{bmatrix}$$

$$(SI - A)^{-1} = \begin{bmatrix} \frac{5}{2s} & \frac{-1}{2s} \\ \frac{3s}{2s} & \frac{s}{2s} \end{bmatrix}$$

$$(SI - A) = \begin{bmatrix} 0 & s & 0 \\ -2s & 0 & s \\ 1 & 2 & 3s \end{bmatrix}$$

$$\det(SI - A) = 0 - (-6s^2 - s) + 0 = 6s^2 + s$$

$$\text{adj}(SI - A) = \begin{bmatrix} -2s & 6s^2 + s & -4s \\ -3s^2 & 0 & s \\ s^2 & 0 & 2s^2 \end{bmatrix}$$

takes a lot of time

Examples:

$$\dot{x}(t) = \begin{bmatrix} 0 & -1/c \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/c \\ 0 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -1/c \\ 1/L & -R/L \end{bmatrix} \quad B = \begin{bmatrix} 1/c \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & R \end{bmatrix}$$

$$T.F = C(SI - A)^{-1} \cdot B$$

$$(SI - A)^{-1} = \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -1/c \\ 1/L & -R/L \end{bmatrix} \right)^{-1} = \left(\begin{bmatrix} s & 1/c \\ -1/L & s + R/L \end{bmatrix} \right)^{-1}$$

$$\det(SI - A) = s^2 + \frac{sR}{L} - \frac{1}{LC} = \Delta$$

$$\text{adj}(SI-A) = \begin{bmatrix} S + \frac{R}{L} & -\frac{1}{L} \\ \frac{1}{L} & S \end{bmatrix}$$

$$(SI-A)^{-1} = \frac{\text{adj}(SI-A)}{\det(SI-A)} = \begin{bmatrix} \frac{S + \frac{R}{L}}{\Delta} & -\frac{1}{\Delta L} \\ \frac{1}{\Delta L} & \frac{S}{\Delta} \end{bmatrix}$$

$$\text{T.F} = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} \frac{S + \frac{R}{L}}{\Delta} & -\frac{1}{\Delta L} \\ \frac{1}{\Delta L} & \frac{S}{\Delta} \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{R}{L\Delta} & -\frac{RS}{\Delta} \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$= \frac{R}{CL\Delta} = \frac{R}{CLs^2 + RCs + 1} \quad \#$$

T.F $\rightarrow$ S.S.M

$$\boxed{\text{I}} \quad \text{T.F} = \frac{\overset{\text{constant}}{b_0}}{\underbrace{s^n + a_1 s^{n-1} + a_2 s^{n-2} \dots + a_n}_{\text{function}}}$$

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$$\Rightarrow \text{S.S.M} \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix}, \quad C = [1 \ 0 \ 0]$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & \dots & -a_1 \end{bmatrix}$$

Ex:- T.F = $\frac{5}{s^3 + 3s^2 + 2s + 1}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$

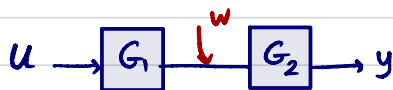
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \emptyset^0$$

II T.F = $\frac{b_1 s^{n-1} + b_2 s^{n-2} \dots b_{n-1} s^1 + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} \dots a_{n-1} s^1 + a_n}$

درجة البسط
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المقام

$$T.F = \underbrace{\frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}}_{G_1} \cdot \underbrace{\frac{b_1 s^2 + b_2 s + b_3}{1}}_{G_2}$$

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$$\frac{w}{u} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\begin{matrix} x_1 = w \\ x_2 = \dot{w} \\ x_3 = \ddot{w} \end{matrix} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$w = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$y = w G_2$$

$$y(s) = w(b_1 s^2 + b_2 s + b_3) \Rightarrow y(t) = f'$$

$$y(t) = b_1 \ddot{w} + b_2 \dot{w} + b_3 w = b_1 x_3 + b_2 x_2 + b_3 x_1$$

$$y = \begin{bmatrix} b_3 & b_2 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

final state space model to use.

$$y = \begin{bmatrix} b_3 & b_2 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \cancel{0}^0$$

must be written.

$$\text{III} \quad \text{T.F} = \frac{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} \dots + a_{n-1} s + a_n}$$

سⁿ با b₀
لازم يكون
II

$$\text{T.F} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\text{T.F} = b_0 + \frac{(b_1 - a_1 b_0) s^2 + (b_2 + a_2 b_0) s + (b_3 - a_3 b_0)}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

always

$$y = \begin{bmatrix} (b_3 - a_3 b_0) & (b_2 - a_2 b_0) & (b_1 - a_1 b_0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_0] u$$

$$\frac{y}{u} = b_0 + \Delta$$

$$y = \textcircled{b_0 u} + \textcircled{\Delta u}^*$$

final state model to use.

Ex:- T.F = $\frac{2s^3 + 4s^2 + 2s + 3}{s^3 + 5s^2 + 2s + 7}$, find S.S.M

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [(b_3 - a_3 b_0) \quad (b_2 - a_2 b_0) \quad (b_1 - a_1 b_0)] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_0] u$$

$$y = [(3 - (7)(2)) \quad (2 - (2)(2)) \quad (4 - (5)(2))] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [2] u$$

$$y = [-11 \quad -2 \quad -6] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [2] u$$

Ex:- T.F = $\frac{s^3 + 1}{s^3 + 3s^2 + 2s + 1}$, find S.S.M

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [(1 - (1)(1)) \quad (0 - (2)(1)) \quad (0 - (3)(1))] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_0] u$$

$$y = [0 \quad -2 \quad -3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [1] u$$

Ex: T.F = $\frac{4s^2 + 2s + 3}{s^3 + 5s^2 + 2s + 7}$, find S.S.M

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + D^0$$

Linearization:-

* in a linear system:- $x_1(t) + x_2(t) = y_1(t) + y_2(t)$
 $ax_1(t) + bx_2(t) = ay_1(t) + by_2(t)$ Linear system.

superposition ↗

★ if a system is non-linear, a linear one can be obtained using Taylor series expansion.

★ Linear models are easier to understand (than nonlinear models) and are necessary for most control system design methods.

★ Linearized system is always a local approximation about the operation point.

After linearization, we can write the state-space model as:

$$\begin{bmatrix} \frac{d(x_1 - x_{1s})}{dt} \\ \frac{d(x_2 - x_{2s})}{dt} \end{bmatrix} = \begin{bmatrix} \left. \frac{\partial f_1}{\partial x_1} \right|_{x_{1s}, x_{2s}, u_s} & \left. \frac{\partial f_1}{\partial x_2} \right|_{x_{1s}, x_{2s}, u_s} \\ \left. \frac{\partial f_2}{\partial x_1} \right|_{x_{1s}, x_{2s}, u_s} & \left. \frac{\partial f_2}{\partial x_2} \right|_{x_{1s}, x_{2s}, u_s} \end{bmatrix} \begin{bmatrix} x_1 - x_{1s} \\ x_2 - x_{2s} \end{bmatrix} + \begin{bmatrix} \left. \frac{\partial f_1}{\partial u} \right|_{x_{1s}, x_{2s}, u_s} \\ \left. \frac{\partial f_2}{\partial u} \right|_{x_{1s}, x_{2s}, u_s} \end{bmatrix} [u - u_s]$$

$$y - y_s = \begin{bmatrix} \left. \frac{\partial g}{\partial x_1} \right|_{x_{1s}, x_{2s}, u_s} & \left. \frac{\partial g}{\partial x_2} \right|_{x_{1s}, x_{2s}, u_s} \end{bmatrix} \begin{bmatrix} x_1 - x_{1s} \\ x_2 - x_{2s} \end{bmatrix} + \left[\left. \frac{\partial g}{\partial u} \right|_{x_{1s}, x_{2s}, u_s} \right] [u - u_s]$$

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}$$

$$\bar{y} = C\bar{x} + D\bar{u}$$

Generalization:-

• Elements of the linearization matrices:

$$A_{ij} = \left. \frac{\partial f_i}{\partial x_j} \right|_{x_s, u_s}$$

$$B_{ij} = \left. \frac{\partial f_i}{\partial u_j} \right|_{x_s, u_s}$$

$$C_{ij} = \left. \frac{\partial g_i}{\partial x_j} \right|_{x_s, u_s}$$

$$D_{ij} = \left. \frac{\partial g_i}{\partial u_j} \right|_{x_s, u_s}$$

• state-space form:-

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + B\bar{u} \\ \bar{y} &= C\bar{x} + D\bar{u} \end{aligned}$$

Ex:- $\dot{X}_1 = 2X_1^2 + 3X_2 + u$ ← non-linear systems
 $\dot{X}_2 = 2X_1^3 + 3X_2^2 + u^2$
 $\dot{X}_3 = 2X_1 + 3X_2 + 1X_3$
 $y = X_1^2$

$X_s = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_s = 1$, find linear S.S.M?

$$A = \begin{bmatrix} (2X_1^2)'|_{X_s} & (3X_2)'|_{X_s} & (0)'|_{X_s} \\ (2X_1^3)'|_{X_s} & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \Rightarrow A = \begin{bmatrix} 4 & 3 & 0 \\ 6 & 6 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}$$

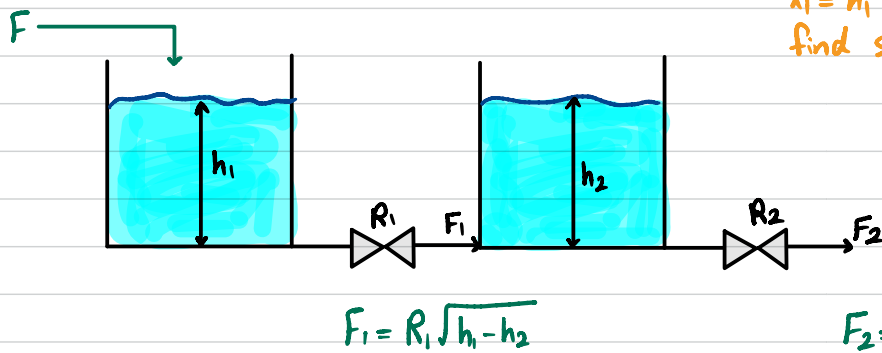
$$D = 0$$

* even though we found A, B, C, and D
 we're not done because the question
 wants the S.S.M.

$$\begin{bmatrix} \dot{\bar{X}}_1 \\ \dot{\bar{X}}_2 \\ \dot{\bar{X}}_3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ 6 & 6 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \bar{u}$$

$$\bar{y} = \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \end{bmatrix} + \mathcal{D}^0$$

Example:- Interacting Tanks



$y = h_2$, input = F
 $x_1 = h_1$, $x_2 = h_2$
 find state model.

• Modeling equations:-

$$\frac{dh_1}{dt} = \frac{F}{A_1} - \frac{R_1}{A_1} \sqrt{h_1 - h_2} = f_1(h_1, h_2, F)$$

$$\frac{dh_2}{dt} = \frac{R_1}{A_1} \sqrt{h_1 - h_2} - \frac{R_2}{A_2} \sqrt{h_2} = f_2(h_1, h_2, F)$$

$$\Delta h_1 = \frac{F}{A_1} - \frac{F_1}{A_1}$$

$$\Delta h_2 = \frac{F_1}{A_1} - \frac{F_2}{A_2}$$

state variable $\Rightarrow x_1 = h_1$, $x_2 = h_2$

Input Variable $\Rightarrow u = F$

- assume only the second tank height is measured.

$$h_s = \begin{bmatrix} h_{1s} \\ h_{2s} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h_1 - h_{1s} \\ h_2 - h_{2s} \end{bmatrix}$$

$$u = F - F_s$$

$$y = h_2 - h_{2s}$$

$$A_{11} = \left. \frac{df_1}{dh_1} \right|_{h_s, F_s} = \frac{-R_1}{2A_1 \sqrt{h_{1s} - h_{2s}}}, \quad A_{22} = \left. \frac{df_1}{dh_2} \right|_{h_s, F_s} = \frac{R_1}{2A_1 \sqrt{h_{1s} - h_{2s}}}$$

$$A_{21} = \left. \frac{df_2}{dh_1} \right|_{h_s, F_s} = \frac{R_1}{2A_1 \sqrt{h_{1s} - h_{2s}}}, \quad A_{22} = \left. \frac{df_2}{dh_2} \right|_{h_s, F_s} = \frac{-R_1}{2A_1 \sqrt{h_{1s} - h_{2s}}} - \frac{R_2}{2A_2 \sqrt{h_{2s}}}$$

$$B_{11} = \left. \frac{\partial f_1}{\partial F} \right|_{h_s, F_s} = \frac{1}{A_1}, \quad B_{21} = \left. \frac{\partial f_2}{\partial F} \right|_{h_s, F_s} = 0$$

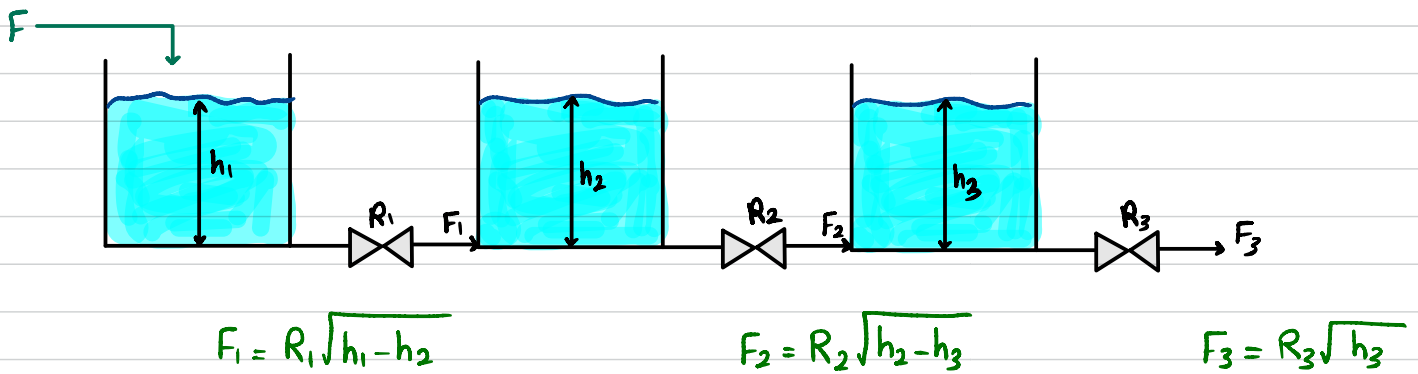
$$y = g(h_1, h_2, F) = h_2 - h_{2s}$$

$$C_{11} = \left. \frac{\partial g}{\partial h_1} \right|_{h_s, F_s} = 0, \quad C_{12} = \left. \frac{\partial g}{\partial h_2} \right|_{h_s, F_s} = 1$$

• state - space model :-

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-R_1}{2A_1\sqrt{h_{1s}-h_{2s}}} & \frac{R_1}{2A_1\sqrt{h_{1s}-h_{2s}}} \\ \frac{R_1}{2A_2\sqrt{h_{1s}-h_{2s}}} & \frac{-R_1}{2A_2\sqrt{h_{1s}-h_{2s}}} - \frac{R_2}{2A_2\sqrt{h_{2s}}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} [u]$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (y = x_2 = h_2 - h_{2s})$$



↪ must be able to solve

* State Flow Diagrams:-

$$y = ax$$

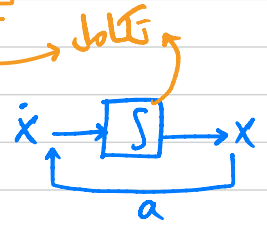
$$x \xrightarrow{a} y$$

$$y = ax_1 + bx_2$$

$$\begin{matrix} x_1 \xrightarrow{a} y \\ x_2 \xrightarrow{b} y \end{matrix}$$

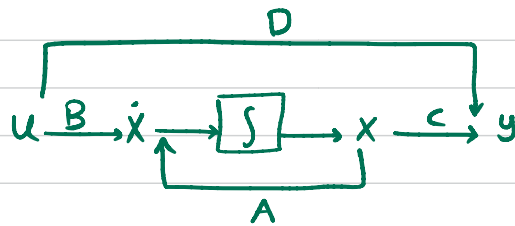
$$\boxed{s^{-1}} \text{ or } \boxed{\frac{1}{s}}$$

$$\dot{x} = ax$$



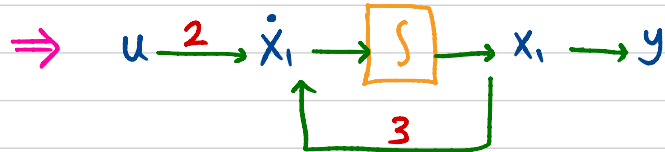
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

⇒



* Ex₁:-

$$\begin{aligned} \dot{x}_1 &= 3x_1 + 2u \\ y &= x_1 \end{aligned}$$



* Ex₂:-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

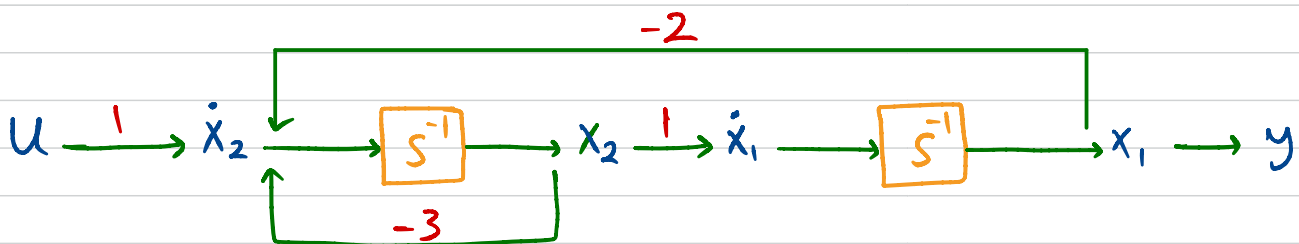
Draw signal flow diagram.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_1 - 3x_2 + u$$

$$y = x_1$$

← order of x's
... x₃ x₂ x₁



* Ex₃ :-

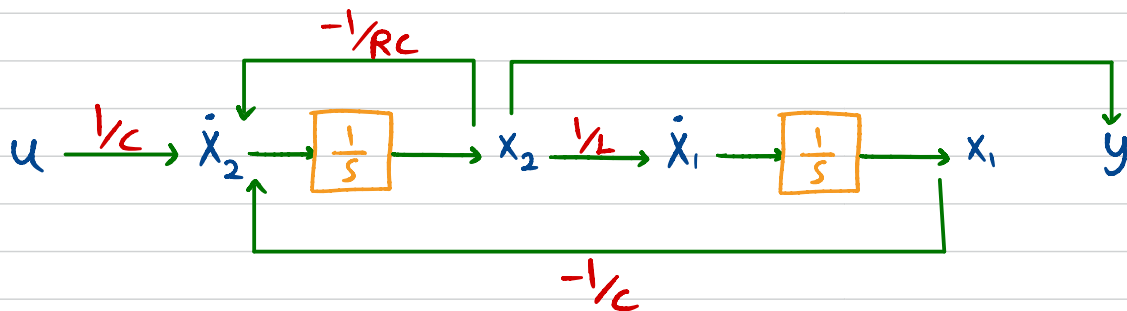
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/RC \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/C \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{Draw signal flow diagram?}$$

$$\dot{x}_1 = \frac{1}{L} x_2$$

$$\dot{x}_2 = -\frac{1}{C} x_1 - \frac{1}{RC} x_2 + \frac{1}{C} u$$

$$y = x_2$$



* Ex₄ :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$

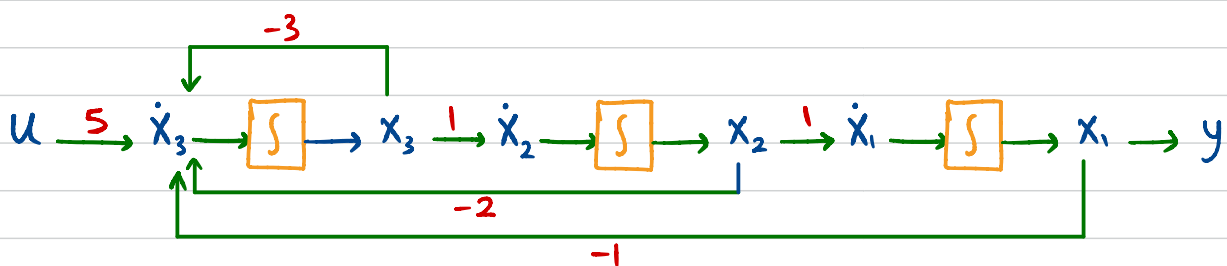
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\dot{x}_1 = x_2$$

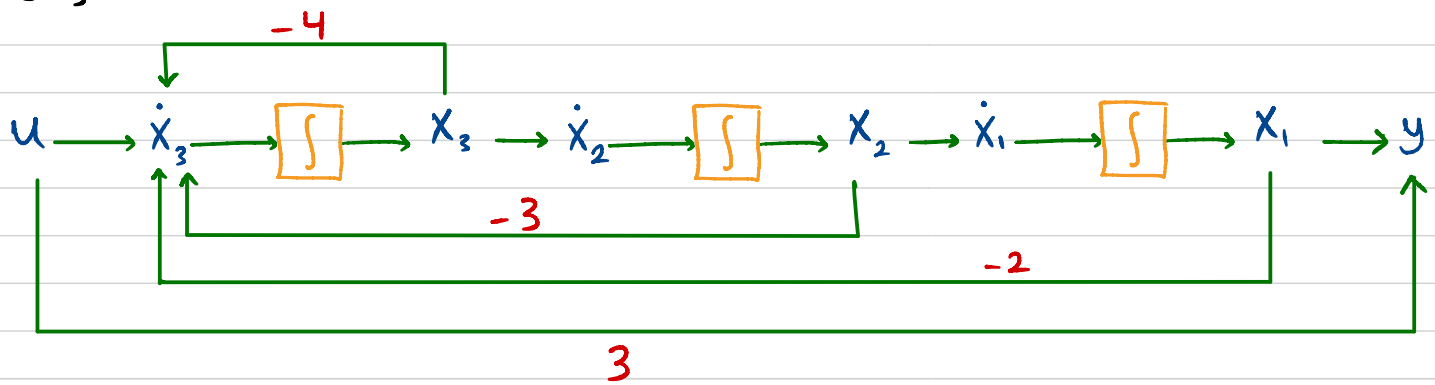
$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -1x_1 - 2x_2 - 3x_3 + 5u$$

$$y = x_1$$



* Ex 5 :-



Find the state space model?

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

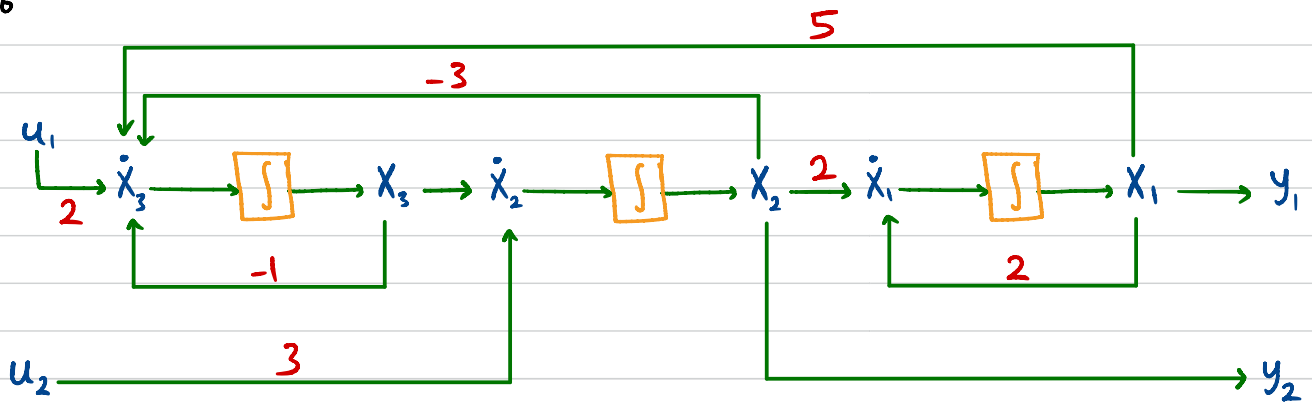
$$\dot{x}_3 = -2x_1 - 3x_2 - 4x_3 + u$$

$$y = x_1 + 3u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 \end{bmatrix} u$$

* Ex₆ :-



find state space model

$$\dot{x}_1 = 2x_1 + 2x_2$$

$$\dot{x}_2 = x_3 + 3u_2$$

$$\dot{x}_3 = 5x_1 - 3x_2 - x_3 + 2u_1$$

$$y_1 = x_1$$

$$y_2 = x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 1 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

* Canonical form of the system:-

[1] controllable canonical form (C.C.F).

$$T.F = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & \dots & \dots & -a_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [(b_n - a_n b_0) \quad (b_{n-1} - a_{n-1} b_0) \quad \dots \quad (b_1 - a_1 b_0)] \quad , \quad D = [b_0]$$

[2] observable Canonical form (O.C.F)

$$T.F = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ 0 & 1 & \dots & 0 & \vdots \\ \vdots & \vdots & \dots & 1 & a_1 \end{bmatrix}}_A \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} + \underbrace{\begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix}}_B u$$

$$y = \underbrace{[1 \quad 0 \quad \dots \quad 0]}_C \underbrace{\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}}_X + \underbrace{[b_0]}_D u$$

[3] Diagonal Canonical form (D.C.F)

$$T.F = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

poles of T.F in the diagonal ^{أول مقام}

after division

$$T.F = b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \dots + \frac{c_n}{s+p_n}$$

$$A = \begin{bmatrix} -p_1 & 0 & 0 & \dots & 0 \\ 0 & -p_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & -p_n \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

vector of ones

$$C = [c_1 \ c_2 \ c_3 \ \dots \ c_n], \quad D = [b_0]$$

$$T.F = b_0 + \frac{(b_1 - a_1 b_0) s^2 + (b_2 - a_2 b_0) s + (b_3 + a_3 b_0)}{s^3 + a_1 s^2 + a_2 s + a_3}$$

أول مقام : $s = p_1$
 $s = p_2$
 $s = p_3$

$$\frac{(b_1 - a_1 b_0) s^2 + (b_2 - a_2 b_0) s + (b_3 + a_3 b_0)}{(s - p_1)(s - p_2)(s - p_3)}$$

$$T.F = b_0 + \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \frac{c_3}{s-p_3}, \text{ iff } p_1 \neq p_2 \neq p_3$$

(Distinct poles)

$$A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C = [c_1 \ c_2 \ c_3], \quad D = [b_0]$$

[4] Jordan Canonical form (J.C.F)

$$T.F = \frac{b_0 s^n + b_1 s^{n-1} + \dots + b_n}{s^n + a_1 s^{n-1} + \dots + a_n}$$

$$T.F = b_0 + \frac{(b_1 - a_1 b_0) s^2 + (b_2 - a_2 b_0) s + (b_3 + a_3 b_0)}{s^3 + a_1 s^2 + a_2 s + a_3}$$

الخطوات : $s = p_1$
 $s = p_2$
 $s = p_3$

$$\frac{(b_1 - a_1 b_0) s^2 + (b_2 - a_2 b_0) s + (b_3 + a_3 b_0)}{(s - p_1)^2 (s - p_3)}$$

$$T.F = b_0 + \frac{c_1}{(s - p_1)^2} + \frac{c_2}{s - p_1} + \frac{c_3}{s - p_3}, \text{ iff } \underline{p_1 = p_2 \neq p_3}$$

separate square matrix

$$A = \begin{bmatrix} p_1 & 1 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$C = [c_1 \ c_2 \ c_3], \quad D = [b_0]$$

* for matrix A, we put repeated poles in a separate square matrix then put an upper diagonal of ones in that new square matrix.

* for matrix B, we put zeros for the repeated poles and the last repeated pole we put 1.

Ex: $p_1, p_2 = p_3 = p_4, p_5, p_6$

$$T.F = \frac{c_1}{s - p_1} + \frac{c_2}{(s - p_2)^3} + \frac{c_3}{(s - p_2)^2} + \frac{c_4}{s - p_2} + \frac{c_5}{s - p_5} + \frac{c_6}{s - p_6}$$

$$A = \begin{bmatrix} p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & p_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & p_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

* Eigen Values are the same as poles
 $\lambda = p$

$$C = [c_1 \ c_2 \ c_3 \ c_4 \ c_5 \ c_6], \quad D = 0$$

To find Eigen Values $\rightarrow |\lambda I - A| = 0$

C.C.F (original)

$$\dot{X} = AX + Bu$$
$$Y = CX + Du$$

$$Z = PX$$

new matrix $\rightarrow$ original matrix

Transformation matrix

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^n & \lambda_2^n & \lambda_3^n & \lambda_4^n \end{bmatrix}$$

Transformation Matrix only
for getting D.C.F from C.C.F
(C.C.F $\rightarrow$ D.C.F).

matrix of constants

D.C.F (new)

$$\dot{Z} = A_{\text{new}} Z + B_{\text{new}} U$$
$$Y = C_{\text{new}} Z + D U$$

$$P \dot{X} = A(PX) + Bu$$

$$\dot{X} = \boxed{P^{-1} A P} X + \boxed{P^{-1} B} U$$

A_{new} B_{new}

$$Y = \boxed{C P} X + D U$$

C_{new}

Steps:-

- ① find λ
- ② construct P
- ③ find $A_{\text{new}} = P^{-1} A_{\text{old}} P$
 $B_{\text{new}} = P^{-1} B_{\text{old}}$
 $C_{\text{new}} = C_{\text{old}} P$
 $D_{\text{new}} = D_{\text{old}}$
- ④ write S.S.M

Ex: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$ find D.C.f

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \cancel{0}$$

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix} = 0$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -3 & -1 & -2 \\ 4 & 1 & 4 \end{bmatrix}$$

$$\begin{aligned} \lambda(\lambda(\lambda+6)+11)+6 &= 0 \\ \lambda^3 + 6\lambda^2 + 11\lambda + 6 &= 0 \\ \lambda &= -3, -1, -2 \end{aligned}$$

← using calculator

$$A_{\text{new}} = P^{-1} A P = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$B_{\text{new}} = P^{-1} B = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}$$

$$C_{\text{new}} = C P = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Ex: T.F = $\frac{3}{s+2} + \frac{2}{(s+2)^2} + \frac{1}{s+3}$, find D.C.f / J.c.f

$$\frac{2}{(s+2)^2} + \frac{3}{(s+2)} + \frac{1}{s+3}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \cancel{0}$$

Example:

$$A = \begin{bmatrix} 8 & -5 & 10 \\ 0 & -1 & 1 \\ -8 & 5 & -9 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$$

$$D = 0$$

find the diagonal conical form

$$|\lambda I - A| = \begin{vmatrix} \lambda - 8 & 5 & -10 \\ 0 & \lambda + 1 & -1 \\ 8 & -5 & \lambda + 9 \end{vmatrix}$$

$$= (\lambda - 8)((\lambda + 1)(\lambda + 9) - 5) - 40 - 10(-8(\lambda + 1))$$

$$\begin{aligned} &= (\lambda - 8)(\lambda^2 + 10\lambda + 9 - 5) - 40 + 80\lambda + 80 \\ &= \lambda^3 + 10\lambda^2 + 4\lambda - 8\lambda^2 - 8\lambda - 32 - 40 + 80\lambda + 80 \\ &= \lambda^3 + 2\lambda^2 + 4\lambda + 8 = 0 \end{aligned}$$

$$\lambda_1 = -2$$

$$\lambda_2 = 2i$$

$$\lambda_3 = -2i$$

Ex: $T.F = \frac{s+3}{s^2+3s+2}$ find c.c.f o.c.f and D.c.f/J.c.f

c.c.f

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [3 \quad 1]$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [3 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathcal{D}^0$$

O.c.f

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u$$

$$y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathcal{D}^0$$

D.c.f / J.c.f

poles : -1, -2

$$T.F = \frac{s+3}{s^2+3s+2}$$

$$T.F = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2}$$

$$T.F = \frac{C_1}{s+1} + \frac{C_2}{s+2}$$

$$C_1(s+2) + C_2(s+1) = s+3$$

$$@ s = -1 \Rightarrow C_1 = 2$$

$$@ s = -2 \Rightarrow C_2 = -1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [2 \quad -1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathcal{D}^0$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

if $\det(A) = 0 \rightarrow$ singular matrix.

if A is singular $\rightarrow A^{-1}$ doesn't exist.

Matrix A is non-singular iff $\det(A) \neq 0$

* Matrix Rank $\Rightarrow R(A) = \#$ of independent columns/rows.

* if A is non-singular $\rightarrow A$ is a full Rank matrix

★ Controllability

if we can drive state space variables from $X(t_0)$ to $X(t_f)$ in finite time using (u) , system is controllable.

$$C_o = M = [B \mid AB \mid A^2B \mid \dots \mid A^{n-1}B]_{(n \times (r \cdot m))}$$

The system is controllable iff $R(M) \geq n$ $\rightarrow$ number of state variables.

Ex: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} u$ is the system controllable?

$$M = [B \mid AB]$$

$$= \begin{bmatrix} -2 & 8 \\ 2 & -8 \end{bmatrix} \Rightarrow \det(M) = 0 \therefore \text{not full rank}$$

$R(M) \neq n \Rightarrow$ system isn't controllable

Ex: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} u$ is the system controllable?

$$M = [B \mid AB \mid A^2B] = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 4 & 4 \\ 3 & -6 & 12 \end{bmatrix}$$

$|M| = -144 \neq 0 \therefore$ matrix is full

Rank $\rightarrow R(M) = 3 \geq n$

$\therefore$ system is controllable.

Ex:
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} u$$

$$M = [B \mid AB \mid A^2B \mid A^3B \mid A^4B]$$

$$= \begin{bmatrix} 0 & 1 & 0 & -2 & 3 & 4 & -6 & -8 & 24 & 16 \\ 0 & 0 & 3 & 0 & 0 & 6 & 12 & 0 & 0 & 0 \\ 3 & 0 & 6 & 0 & 12 & 0 & 24 & 0 & 48 & 0 \\ 0 & 0 & 2 & 1 & -20 & -10 & 150 & 75 & -1000 & -500 \\ 2 & 1 & -10 & -5 & 50 & 25 & -80 & -125 & 250 & 625 \end{bmatrix}$$

$$|M| = -189$$

since $\det \neq 0$ there is a sub-matrix with $R=5$
so system is controllable

if it was $= 0$ we shift
the matrix to the right one
by one until we find the $\det$
that $\neq 0$. (if not found then not controllable).

★ Observability

system is observable if $x(t)$ at time (t_0) can be approximated
by observing the output $y(t)$

$$O_b = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \left. \vphantom{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}} \right\} n \times m$$

system is observable
if $R(O_b) \geq n$

★ output controllability

ability to control y through u

$$O_c = [CB \mid CAB \mid CA^2B \mid \dots \mid CA^{n-1}B \mid D]$$

$$(n+1)r \times m$$

→ # of output
→ # of input

* The system is output controllable
iff $\text{Rank}(O_c) \geq \underline{m}$

Example: find if the system is controllable or observable or output controllable.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\textcircled{1} M = [B \ AB \ A^2B] = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 4 \\ 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 1 & 4 & 1 \end{bmatrix}$$

$$\det = 0 - 1(3 - 0) + 0 = -3 \neq 0$$

→ The matrix is full Rank = 3 ≥ n so the system is cont.

$$\textcircled{2} O_b = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

all zeros
hence there
is no sub
matrix with
 $R \geq n$
⇒ system is not obs.

$$\textcircled{3} O_c = [CB : CAB : CA^2B : D]$$

$$= \left[\begin{array}{cc|cc|cc|cc} 0 & 1 & 0 & 2 & 0 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 \end{array} \right]$$

$$\det = 0 - 1(1) = -1 \neq 0$$

$R = 2 \geq m$ so the system is output cont.

LTI solution (linear time invariant solution):-

$$\dot{X}(t) = Ax(t) + Bu(t)$$

LTI solution is to find $X(t)$

$$\mathcal{L}[\dot{X}(t) = Ax(t) + Bu(t)]$$

Laplace

$$sX(s) - X(0) = AX(s) + Bu(s)$$

$$sX(s) - AX(s) = X(0) + Bu(s)$$

$$X(s)(sI - A) = X(0) + Bu(s)$$

$$X(s) = (sI - A)^{-1}(X(0) + Bu(s))$$

$$X(s) = \underbrace{\Phi(s) X(0)}_{\text{Natural Response}} + \underbrace{\Phi(s) Bu(s)}_{\text{Forced Response}}$$

Natural
Response

Forced
Response

initial
state

$$X(t) = \mathcal{L}^{-1}(X(s))$$

* Revision :-

$$X(s) = \mathcal{L} X(t) = \int_0^{\infty} e^{-st} x(t) \cdot dt$$

$$\mathcal{L} \dot{X}(t) = sX(s) - X(0)$$

$$\mathcal{L} A = A/s$$

$$\mathcal{L} At = A/s^2$$

$$\mathcal{L} e^{at} = \frac{1}{s-a}$$

$$\mathcal{L} t e^{at} = \frac{1}{(s-a)^2}$$

$$\mathcal{L} \cos at = \frac{s}{s^2 + a^2}$$

$$\mathcal{L} \sin at = \frac{a}{s^2 + a^2}$$

$$\mathcal{L} e^{at} \cos bt = \frac{s-a}{(s-a)^2 + b^2}$$

$$\text{step response} \rightarrow u(s) = \frac{1}{s}$$

Example: $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $x(t)$?

$$X(s) = \Phi(s) X(0) + \cancel{\Phi(s)} B u(s)$$

0 $\rightarrow$ there is no B

$$\Phi(s) = (sI - A)^{-1}$$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix} \quad (sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s+6}{s^2+6s+8} & \frac{1}{s^2+6s+8} \\ \frac{-8}{s^2+6s+8} & \frac{s}{s^2+6s+8} \end{bmatrix}$$

$$X(s) = (sI - A)^{-1} X(0)$$

$$X(s) = \begin{bmatrix} \frac{s+6}{s^2+6s+8} & \frac{1}{s^2+6s+8} \\ \frac{-8}{s^2+6s+8} & \frac{s}{s^2+6s+8} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s+6}{s^2+6s+8} \\ \frac{-8}{s^2+6s+8} \end{bmatrix}$$

$$X(s) = \begin{bmatrix} \frac{s+6}{s^2+6s+8} \\ \frac{-8}{s^2+6s+8} \end{bmatrix} = \begin{bmatrix} \frac{s+6}{(s+4)(s+2)} \\ \frac{-8}{(s+4)(s+2)} \end{bmatrix}$$

$$\frac{s+6}{(s+4)(s+2)} = \frac{A}{s+4} + \frac{B}{s+2} = \frac{A(s+2) + B(s+4)}{(s+4)(s+2)}$$

$$@ s = -2 \rightarrow \boxed{B = 2} \quad @ s = -4 \rightarrow \boxed{A = -1}$$

$$\frac{-8}{(s+4)(s+2)} = \frac{A}{s+4} + \frac{B}{s+2} = \frac{A(s+2) + B(s+4)}{(s+4)(s+2)}$$

$$@ s = -2 \rightarrow \boxed{B = -4} \quad @ s = -4 \rightarrow \boxed{A = 4}$$

$$X(t) = \mathcal{L}^{-1} X(s) = \begin{bmatrix} 2e^{-2t} - e^{-4t} \\ -4e^{-2t} + 4e^{-4t} \end{bmatrix} \rightarrow \begin{matrix} x_1(t) \\ x_2(t) \end{matrix}$$

Example: if the input $u(t)$ is unit step function find $x(t)$:-

↖, the initial state is zero.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$X(s) = \cancel{\phi(s)} X(0) + \phi(s) B u(s)$$

$$\phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix}$$

↙ det = $s^2 + 3s + 2$

$$X(s) = \begin{bmatrix} \frac{s+3}{s^2+3s+2} & \frac{1}{s^2+3s+2} \\ \frac{-2}{s^2+3s+2} & \frac{s}{s^2+3s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s} \end{bmatrix}$$

$$X(s) = \begin{bmatrix} \frac{1}{s^2+3s+2} \\ \frac{s}{s^2+3s+2} \end{bmatrix} \begin{bmatrix} \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{1}{s(s^2+3s+2)} \\ \frac{1}{(s^2+3s+2)} \end{bmatrix}$$

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{s}$$

$$= A(s)(s+2) + B(s)(s+1) + C(s+1)(s+2) = 1$$

$$@ s=0 \rightarrow C = 0.5$$

$$@ s=-1 \rightarrow A = -1$$

$$@ s=-2 \rightarrow B = 0.5$$

$$x_1(t) = \mathcal{L}^{-1} \left(\frac{-1}{s+1} + \frac{0.5}{s+2} + \frac{0.5}{s} \right)$$

$$= -e^{-t} + 0.5e^{-2t} + 0.5$$

$$\frac{1}{(s^2+3s+2)} = \frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$= A(s+2) + B(s+1) = 1$$

$$@ s=-2 \rightarrow B = -1$$

$$@ s=-1 \rightarrow A = 1$$

$$x_2(t) = \mathcal{L}^{-1} \left(\frac{1}{s+1} + \frac{-1}{s+2} \right)$$

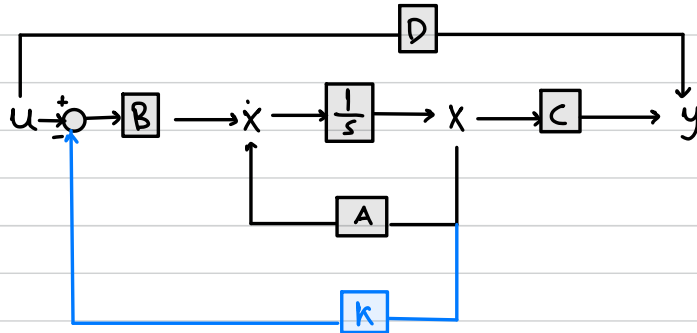
$$= e^{-t} - e^{-2t}$$

$$x(t) = \begin{bmatrix} -e^{-t} + 0.5e^{-2t} + 0.5 \\ e^{-t} - e^{-2t} \end{bmatrix}$$

* Pole Replacement .-

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y = cx(t) + Du(t)$$



$$\dot{x}(t) = Ax(t) + B(-kx(t) + u)$$

$$\dot{x}(t) = \underbrace{(A - BK)}_{A_{\text{new}}} x(t) + Bu(t)$$

$$A_{\text{new}} = A - BK$$

$$y = cx(t) + Du(t) \quad \text{Same}$$

* pole replacement iff system is controllable, so first step is to verify that the system is controllable. if it is, we can proceed.

$$\begin{aligned} * \Delta_{\text{old}} &= |sI - A| \\ &= s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n \end{aligned}$$

$$\begin{aligned} * \Delta_{\text{new}} &= |sI - A_{\text{new}}| \\ &= (s + p_{1\text{new}})(s + p_{2\text{new}}) \dots (s + p_{n\text{new}}) \\ &= s^n + \alpha_1 s^{n-1} + \dots + \alpha_n \end{aligned}$$

$$* k = [(\alpha_n - a_n) \quad (\alpha_{n-1} - a_{n-1}) \quad \dots \quad (\alpha_1 - a_1)]$$

$$* A_{\text{new}} = A_{\text{old}} - BK$$

* Pole Replacement steps:-

1. controllability test (find M).
(system must be controllable)

2. find Δ_{old} and Δ_{new}
to find (α) to find (α)

3. find K_i (initial) $K_i = [(\alpha_n - a_n) \quad (\alpha_{n-1} - a_{n-1}) \quad \dots \quad (\alpha_1 - a_1)]$

4. construct T to find (T^{-1})
 $T = MW$

where M :- controllability test Matrix

$$\Delta_{old} = |SI - A_{old}| = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$$

from Δ_{old} coefficient ↙

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & 1 & 0 \\ a_{n-2} & \dots & a_1 & 1 & 0 & 0 \\ a_{n-3} & \dots & a_1 & 1 & 0 & 0 \\ \vdots & \leftarrow \vdots & \vdots & \vdots & 0 & 0 \\ \vdots & \leftarrow \vdots & \vdots & \vdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

T^{-1} should be Identity if system is cont.

5. $K_{final} = K_i T^{-1}$

6. $A_{new} = A_{old} - BK_{final}$

Example: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$, $P_{\text{new}} = [-2+j4 \quad -2-j4 \quad -10]$
 $C = [1 \quad 0 \quad 0]$
 $D = [0]$

① controllability

$$M = [B \quad AB \quad A^2B]$$

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix} \quad \det(M) = -1 \neq 0$$

$$\text{Rank}(M) = n = 3$$

$\therefore$ system is controllable

② $\Delta_{\text{old}} = |sI - A|$

$$= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{vmatrix} = s^3 + 6s^2 + 5s + 1$$

$$\alpha_1 = 6$$

$$\alpha_2 = 5$$

$$\alpha_3 = 1$$

$$\Delta_{\text{new}} = (s - p_1)(s - p_2)(s - p_3)$$

$$= (s - (-2+j4))(s - (-2-j4))(s - (-10))$$

$$= s^3 + 14s^2 + 60s + 200$$

$$\alpha_1 = 14$$

$$\alpha_2 = 60$$

$$\alpha_3 = 200$$

③ $k_i = [\alpha_3 - \alpha_1 \quad \alpha_2 - \alpha_2 \quad \alpha_1 - \alpha_1]$

$$k_i = [(200 - 1) \quad (60 - 5) \quad (14 - 6)]$$

$$k_i = [199 \quad 55 \quad 8]$$

④ $T = MW$

$$W = \begin{bmatrix} \alpha_2 & \alpha_1 & 1 \\ \alpha_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix} \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = T^{-1}$$

⑤ $k_{\text{final}} = k_i * T^{-1} = [199 \quad 55 \quad 8] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [199 \quad 55 \quad 8]$

⑥ $A_{\text{new}} = A_{\text{old}} - B k_{\text{final}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -200 & -60 & -14 \end{bmatrix}$

Example: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $C = [1 \ 0]$, $p_{new} = [-3 \ -5]$

find new s s m

full rank

$$M = [B \ AB] = \begin{bmatrix} 0 & 2 \\ 2 & -6 \end{bmatrix} \quad \det(M) = -4 \neq 0 \quad \therefore \text{Rank} = 2$$

Rank ≥ 2 $\therefore$ system is cont.

$$\Delta_{old} = |sI - A| = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s(s+3) + 2 = s^2 + \underset{a_1}{3}s + \underset{a_2}{2}$$

$$\Delta_{new} = (s+3)(s+5) = s^2 + \underset{\alpha_1}{8}s + \underset{\alpha_2}{15}$$

$$k_i = [(15-2) \quad (8-3)] = [13 \quad 5]$$

$$T = MW \quad W = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} , \quad T^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$k_{final} = [13 \quad 5] \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = [6.5 \quad 2.5]$$

$$A_{new} = A_{old} - BK_i = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} [6.5 \quad 2.5] = \begin{bmatrix} 0 & 1 \\ -15 & -8 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -15 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = [1 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0$$

Example: given T.f = $\frac{10}{(s+1)(s+2)(s+3)}$, $p_{new} = [-2+j2\sqrt{3} \quad -2-j2\sqrt{3} \quad -12]$

find new s.s.m

$$\Delta_{old} = (s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

① controllability

$$M = [B \ AB \ A^2B] = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 10 & -60 \\ 10 & -60 & 250 \end{bmatrix} \rightarrow \det = -1000 \neq 0$$

$\therefore$ matrix is non-singular
 $\therefore$ // is full rank
 $\therefore \text{Rank}(M) \geq n$
 $\therefore$ system is controllable.

② $\Delta_{old} = |sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{vmatrix} = s(s(s+6)+11)+6$
 $= s^3 + 6s^2 + 11s + 6$ $a_3 = 6 \quad a_2 = 11 \quad a_1 = 6$

$$\Delta_{new} = (s+2-j2\sqrt{3})(s+2+j2\sqrt{3})(s+10)$$

$$s^2 + 2s + j2\sqrt{3}s + 2s + 4 + j4\sqrt{3} - j2\sqrt{3}s - j4\sqrt{3} + 12$$

$$= (s^2 + 4s + 16)(s+10)$$

$$= s^3 + 4s^2 + 16s + 10s^2 + 40s + 160$$

$$= s^3 + 14s^2 + 56s + 160$$

$\alpha_3 = 160 \quad \alpha_2 = 56 \quad \alpha_1 = 14$

$$T = MW = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 10 & -60 \\ 10 & -60 & 250 \end{bmatrix} \begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

③ $k_i = [154 \quad 45 \quad 8]$

$$k_{final} = k_i T^{-1} = [15.4 \quad 4.5 \quad 0.8]$$

$$T^{-1} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

$$A_{new} = A_{old} - B k_{final} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}$$

