ElCoM
ELECTRICAL COMPUTER MECHATRONICS

تقدم لجنة ElCoM الاكاديمية

دفتر لمادة:

تحليل عددي

من شرح:

احمد بني ياسين

جزيل الشكر للطالبة:

ميس الشريف

No: 1 Date:

Ch 3 :- Approximation & Round-off errors

* significant figures :- (sig. fig) number of figures or "digits" that can be read exactly

E.x $\pi \xrightarrow{3 \text{ sig.fig}} 3.14$

$\xrightarrow{5 \text{ sig.fig}} 3.1415$

E.x How many sig.fig are in each of the following number :-

a) 2371 → 4 sig.fig

b) 0.0035 → 2 sig fig **(leading zeros are not sig.fig)

c) 1.080 → 4 sig fig (zeros after decimal Point)
لا قيمة

d) 2.97×10^5 → 3 sig.fig   297000 → 3

إذا كانت توجد الفاصلة العشرية نعد إلى بعدها من اصفار (f)

e) 10000 → 1 sig.fig   إذا لم يوجد فاصلة عشرية (e)   (1) sig.fig

f) 1.000 → 4 sig.fig

* قبل الفاصلة صفر وبعد أصفار ثم رقم (b)
لا نعد الاصفار

* قبل الفاصلة رقم ومن ثم أصفار وبعد ذلك رقم نعدهم

* في حالة وجود الفاصلة
⇐ نعد الأصفار دائماً أينما كانت
ما عدا   0.005 — 1 sig.fig

* في حالة عدم وجود فاصلة
لا نعد الأصفار   7500
2 sig.fig.

** **Error** :- a measure of the estimated difference between the observed or calculated value of a quantity & it's true value

مقدر

True value (exact value) = Approximation + error (E_t)

⇒ **Type of errors** :-

1) Round-off errors :- تقريب
- without chopping (Rounding)
- with Chopping (with Rounding)

cut

2) Truncation error اقتطاع

E.x 1) 3.4156 → 3.416 with Rounding

2) 3.4156 → 3.415 with Chopping تقطيع

** The error $E_t = |exact - Approx|$

** Relative error $\varepsilon_t = \frac{|exact - Approx|}{exact} * 100\%$

true

No: 3 Date:

E.x suppose that exact value 10 000, approximation value 9999 find E_t, ε_t ??

$E_t = 10000 - 9999 = 1$

$$\varepsilon_t = \frac{|10000 - 9999|}{10000} * 100\% = 0.01\%$$

** **Disadvantge** :- if we don't know the exact value, like real application (**itration methods**)

** Approximation error $(\varepsilon_a) = \frac{\text{approximate error}}{\text{approximation}} * 100\%$

$$\varepsilon_a = \left| \frac{\text{Present value} - \text{Previous value}}{\text{Present Value}} \right| * 100\%$$

** How many iterations do we need ?? (stopping criteria)

$|\varepsilon_a| < \varepsilon_s$, $\boxed{\varepsilon_s = 0.5 * 10^{2-N}\ \%}$

stopping

N:- sig.fig

E.x if $N = 5$, $\varepsilon_s = ??$

$$\varepsilon_s = 0.5 \times 10^{2-5}\ \% = 0.5 \times 10^{-3} = 0.0005\ \%$$

E.x using series expansion (Maclarin series)

نعوض 0.5

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

How many terms need to find approx value of $e^{0.5}$ with $\varepsilon_s = 0.05\%$

من الآلة الحاسبة

exact value → $e^{0.5} = 1.648721271$

نأخذ الحد الأول من السلسلة ونحل

# terms	Approx Value	ε_a %	ε_t %
1	1	— لا يوجد نحسبها في أول مرة	39.3 %
2	1.5	$\varepsilon_a = \frac{1.5-1}{1.5} * 100\% = 33.3\%$	9.02 %
3	1.625	$\varepsilon_a = \frac{1.625-1.5}{1.625} * 100\% = 7.69\%$	1.44 %
4	1.645833	$\varepsilon_a = 1.27\%$	0.175 %
5	1.6424375	$\varepsilon_a = 0.158\%$	0.172 %
6	1.6489791	$= 0.0158\%$	0.00142 %

$\varepsilon_a < \varepsilon_s$ stop

C.h (4) Taylor series

Def :- Approximation (Numerical) method can be used to predict a function f(x) value at a point (x) & it's derivatives at another point.

$$f(x_{i+1}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_i)}{n!} * (x_{i+1} - x_i)^n$$

$\Downarrow$

مجموع h^n

where x_i : Base Point

$x_{i+1} - x_i$:- step size (h)

n : derivative order of

⟹ special cas

1) n=0 (Zero order)

$$\boxed{f(x_{i+1}) \triangleq f(x_i)}$$

2) n=1 (first order)

$$f(x_{i+1}) \simeq f(x_i) + \frac{f'(x_i)}{1!} h^1$$

$$= \boxed{f(x_{i+1}) = f(x_i) + f'(x_i) h}$$

3) $n = 2$ (second-order)

$$\boxed{f(x_{i+1}) = f(x_i) + f'(x_i)\,h + \frac{f''(x_i)\,h^2}{2}}$$

E.x : use zero to fourth order Taylor series expansion to approximate the function

$$\left(f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \right)$$

from $x_i = 0$ with $h = 1$, that is to predict the function value at $x_{i+1} = 1$

$f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$

$f''(x) = -1.2x^2 - 0.9x - 1$

$f'''(x) = -2.4x - 0.9$

$f^{(4)}(x) = -2.4$

→ $\boxed{n=0}$ (zero order)

$f(1) \cong f(0) \cong 1.2$

→ $\boxed{n=1}$ (first order)

$f(1) \cong f(0) + f'(0)\,h = 1.2 + (-0.25) = 0.95$

→ $\boxed{n=2}$ (second order)

$$f(1) \cong f(0) + f'(0)\,h + \frac{f''(0)\,h^2}{2}$$

$$0.95 - \frac{1}{2} = 0.45$$

* إذا طلب منا order كبير جداً "15 وأكثر"
بنعوض مباشرة في الاقتران
x_{i+1}
وذلك للحصول على exact

→ $\boxed{n=3}$ (third order)

$$f(1) \cong f(0) + f'(0)\,h + \frac{f''(0)\,h^2}{2} + \frac{f'''(0)\,h^3}{3!}$$

$= 0.3$

→ $\boxed{n=4}$ (fourth order)

$$f(1) \simeq f(0) + f'(0)h + \frac{f''(0)\,h^2}{2} + \frac{f'''(0)\,h^3}{6} + \frac{f^{(4)}(0)\,h^4}{24}$$

$= 0.2$

exact solution

اذا كانت (taylor series) ـه كثير حدود
تكون ال exact sol عند أعلى درجة لكثير حدود
في المثال ـه exact عند $n=4$

order	f(x) approximation	E_t → (Truncation error)
0	1.2	-1
1	0.95	0.75
2	0.45	0.25
3	0.3	0.1
4	0.2	0

** Remaindar (R_n)

$$R_n = f_{exact} - f_{approx}$$

$$R_n = \frac{f^{(n+1)}(\xi)\, h^{n+1}}{(n+1)!}$$ where ξ : value lies between x_{i+1} & x_i

Note :-

if $n\uparrow \longrightarrow R_n \downarrow \longrightarrow$ accuracy $\uparrow$ (دقة)

if $n\uparrow \longrightarrow$ Round off $\uparrow$

if ∞ number of terms were used → we get the exact solution

No: 6/11
3-4

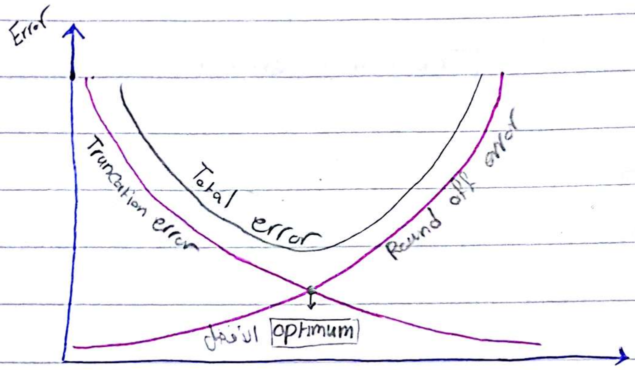
Date: 8

if h(step size) ↓ ⟶ accuracy ↑

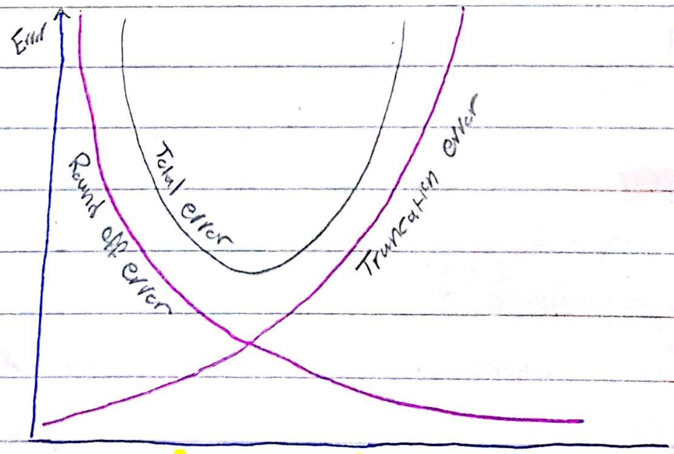
if h(step size) ↓ ⟶ Round-off ↑

Round off ↑ ⟶ Truncation ↓

* Total error = Truncation error + Round off error



# of calculation



h(step size)

E.x :- The Taylor 2^{nd} order approximation of $f(1.8)$ about $a = 0.5$ is 3.671 if $f''(a) = 1.72$, $f^{(4)}(a) = 0.93$

Base point x_i

What is the taylor 3^{rd} order approximation of $f(1.8)$ about $a = 0.5$??

Sol :-

$$h = \overset{1.8}{x_{i+1}} - \overset{a}{x_i}$$

$$= 1.8 - 0.5 \Rightarrow \boxed{h = 1.3}$$

$$f_3(1.8) = \underbrace{f(0.5) + f'(0.5)\,h + \frac{f''(0.5)\,h^2}{2}}_{\text{معطى second order}} + \frac{f^{(3)}(0.5)\,h^3}{6}$$

$$f_3(1.8) = 3.671 + 1.72 * \frac{1.3^3}{6} = 4.301$$

لنفس المثال

*إذا طلب منا second order وبدنا $f_3(1.8)$ ومعنا المشتقة (٣) نفرض $f_3(1.8)$ - المشتقة

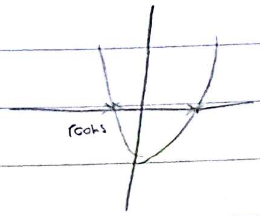
No: .... Date: ....

C.h (5) Roots of equations

$f(x) = x^2 - 4$

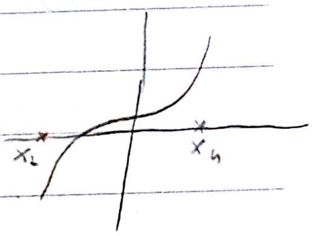
$x^2 - 4 = 0$

$x = \pm 2$



Roots of equation → Bracketing methods
→ open methods

⇒ Bracketing methods → التنصيف Bisection method
→ false position method



** if function $f(x)$ has a root X_L lies in the interval $[X_L, X_u]$ (Lower, uper)
and $f(X_L)\, f(X_u) < 0$ then Bisection method
Compute X_r as

① Bisection

$$\boxed{X_r = \frac{X_L + X_u}{2}}$$

Date: ....

** Solution Proceders :-

① select X_L & X_u (interval guesses), make sure $f(x_L) \cdot f(x_u) < 0$

② find $X_r = \frac{X_l + X_u}{2}$

③ Compute $f(x_r)$
- → if $f(x_r) = 0$ ⇒ x_r : exact root
- → if $f(x_r) \cdot f(x_L) > 0$, ⊕ sign change X_L by X_r ⇒ go to step ②
- → if $f(x_r) \cdot f(x_L) < 0$, ⊖ sign. Change X_u by X_r ⇒ go to step ②

العملية عكسية

XL ← يتغير ⊕

Xu يتغير ⊖

E.X use Bisection method to find the approx root of $\left(f(x) = e^x - 2\right)$, between $X_L = 0$, $X_u = 1$, if $\mathcal{E}_a < 5\%$

[illegible]

approx root

iter	X_L	X_u	X_r	$f(x_r)$	$f(X_L)$	sign	$\mathcal{E}_a$
1	0	1	0.5	-0.35	-1	⊕	-
2	0.5	1	0.75	0.12	-0.35	⊖	33.3% ⇝ $\frac{0.75 - 0.5}{0.75} \times 100\%$
3	0.5	0.75	0.625	-0.13	-0.35	⊕	20%
4	0.625	0.75	0.6875	-0.011	-0.13	⊕	9.1%
5	0.6875	0.75	0.71875	0.052			4.3%

① ③ ②

①

Solution $X_r = 0.71875$ after 5 iteration $\mathcal{E}_a = 4.3\%$

$e^x = 2$

$x = \ln 2$

$x = 0.69314718$ (exact solution

YASSIN

Scanned by CamScanner

** Required # of iterations

$$n \geq \left(\frac{\log(\Delta x / E_a)}{\log(2)} \right) \qquad \Delta x = x_u - x_L$$

from previous example

$E_a = 0.05 \qquad \Delta x = 1 - 0 = 1$

$$n \geq \left(\frac{\log(1/0.05)}{\log 2} \right) = 4.32$$

ناخد int اكبر منها

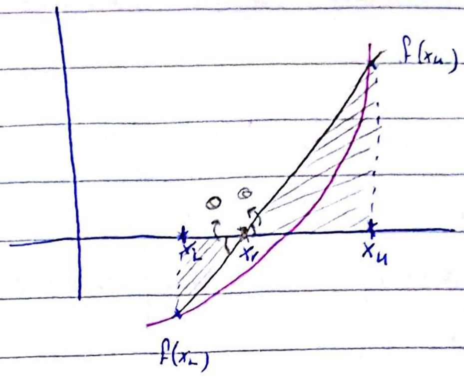
$n = 5 \qquad 4.005 \rightarrow n = 5$

Scanned by CamScanner

No: Date:

2 false position method

if a function $f(x)$ has a root (x_r) lies in the interval $[x_L, x_u]$.
and $f(x_L) f(x_u) < 0$ then FPM compute x_r as :-



$$\boxed{x_r = x_u - \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)}}$$

$\tan\theta = \frac{f(x_u)}{x_u - x_r}$

$\tan\theta = \frac{f(x_L)}{x_r - x_L}$

** Solution procedure (simular to bisection method)

① select x_L & x_u, make $f(x_L) \cdot f(x_u) < 0$

$$\frac{f(x_u)}{x_u - x_r} = \frac{f(x_L)}{x_r - x_L}$$

$$x_r = x_u - \frac{f(x_u)(x_L - x_u)}{f(x_L) - f(x_u)}$$

Ex using false position method to find the root of $f(x) = e^x - 2$ where $X_L = 0$, $X_u = 1$

iter	X_L	X_u	$f(X_L)$	$f(X_u)$	X_r	$f(X_r)$	Sign	ε_a %
1	0	1	-1	0.7183	0.5819	-0.2106	+	—
2	0.5819	1	-0.2106	0.7183	0.677	-0.032	+	14%
3	0.677	1	-0.032	0.7183	0.6879	-0.01	+	1.6%

sol $X_r = 0.6879$ after 3 iteration

$\varepsilon_a = 1.6\%$ more accurate than bisection

Ex $f(x) = \ln(\cos(x))$, $f\left(\frac{\pi}{2}\right) = ??$

$x_i = 0$

fourth order of Taylor series

$$f\left(\frac{\pi}{2}\right) = f(0) + \frac{f'(0)}{1}\left(\frac{\pi}{2}\right) + \frac{f''(0)}{2!}\left(\frac{\pi}{2}\right)^2 + \frac{f'''(0)}{3!}\left(\frac{\pi}{2}\right)^3 + \frac{f^{(4)}(0)}{4!}\left(\frac{\pi}{2}\right)^4$$

$$0 + 0 + \frac{-\pi^2}{8} + 0 + \frac{-2}{24}\,\frac{\pi^4}{16}$$

$$= -1.741$$

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$

$$f''(x) = -\sec^2 x$$

$$f'''(x) = -2\sec^2 x \tan x$$

$$f^{(4)}(x) = -4\sec^2 \tan^2 x - 2\sec^4 x$$

Scanned by CamScanner

(C.h 6) Rooting of equation (open method)

Rooting of equation
- → ① simple fixed Point method
- → ② Newton Raphson method (N-R)
- → ③ Modified of (N.R) (Multiple roots)
- → ④ secant method
- → ⑤ Modified of secant.

1) Simple fixed Point

use to find root of equation (intial guess X_0)

⟹ Method Procedures :-

(I) Convert $f(x)$ to a form $x = g(x)$ by setting $f(x) = 0$
- → simple manipulation
- → Adding X to both sides

E.x $f(x) = x^2 - 4x + 3$ → convert $x = g(x)$

1) $x = \underbrace{\sqrt{4x+3}}_{g(x)}$

2) $x(x-4)+3=0 \Rightarrow x = \dfrac{-3}{x-4}$

3) $x = \dfrac{x^2+3}{4}$

E.x $f(x) = e^{-x} - 2\sin x$

convert to $x = g(x)$

1) $e^{-x} - 2\sin x = 0 \Rightarrow x = -\ln(2\sin(x))$

2) $x = e^{-x} - 2\sin(x) + x$

II) Do iteration with initial guess x_0 for

$$x_{i+1} = g(x_i) \quad , \quad i = 0, 1, 2 \ldots \infty$$

E.x

use fixed point iteration to find the root of $(f(x) = e^{-x} - x)$, use $x_0 = 0$

$\varepsilon_a < 10\%$

Sol :- $x = e^{-x} \Rightarrow x_{i+1} = e^{-x_i}$

Calc $\boxed{0} = AC$

x_0

e^{-Ans} =

مع كل كبسة (=) بمشي iter

# iter	x_{i+1}	ε_a %
1	1	–
2	0.3679	171.8%
3	0.6922	46.8%
4	0.5005	38.3%
5	0.6062	17.43%
6	0.5454	11.2%
7	0.5796	5.9%

first → $i = 0$

$x_1 = e^{-x_0} = 1$

second → $i = 1$

$x_2 = e^{-x_1} = e^{-1} = 0.3679$

Third → $i = 2$

$x_3 = e^{-x_2} = e^{-0.3679}$

fourth → $i = 3$

$x_4 = e^{-x_3} = e^{-0.6922}$

$x_r = 0.5796$ after 7 iteration

$\varepsilon_a = 5.9\%$

* Condition of Convergence :-

1) Simple fixed point doesn't always Convergent (it may divergent)

2) Simple fixed Point Converge to unique solution if
$g(x) \in [a,b]$ for $\forall\ x \in [a,b]$ & $|g'(x)| \in [a,b]$

$\boxed{|g'(x_0)| \leq 1}$
- $=1$ Converge slowly
- <1 converg
- >1 diverge

ونعوض بدل x_0

⟹ from Previous example

معرفة X_0

$g(x) = x^2 - x$

$-1 < g'(x) \leq 1$ (نعمل المطابقة)

$-1 < 2x - 1 \leq 1$

$0 \leq 2x \leq 2$

$0 \leq x \leq 1 \rightarrow$ slowly Converge

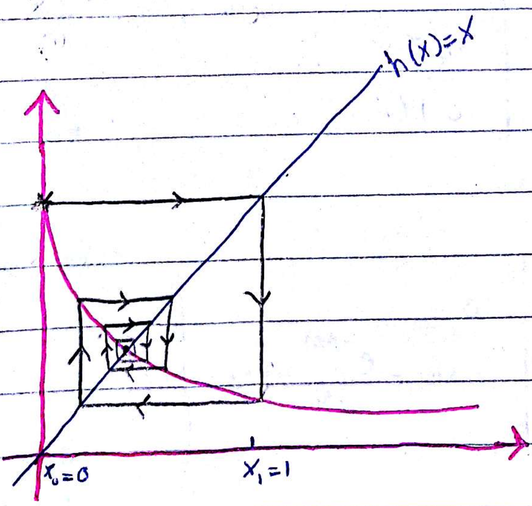
لمعرفة الفترة الي بنا نعوض منها (X_0) وتكون Converg لويسأل عن الفترة الي تكون Converge

$g(x) = e^{-x}$

$g'(x) = -e^{-x}$

$g'(0) = -1$

$|-1| \leq 1 \Rightarrow 1$ slowly converge



$f(x) = e^{-x} - x$, $x_0 = 0$

$e^{-x} - x = 0$

$x = e^{-x} \rightarrow g(x)$

$x_{i+1} = e^{-x_i}$ $i = 0, 1, 2$

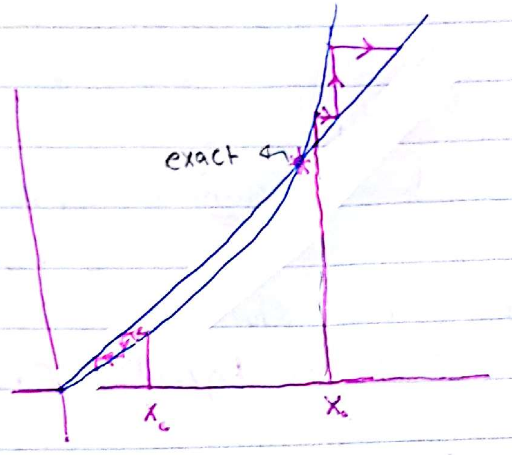
$h(x) = g(x)$

* طريقة حل الرسمة :- ١) نرسم محور التماثل

٢) نحدد X_0 على الاقتران ومن ثم نمد خطاً إلى محور التماثل

٣) نسقط النقطة إلى الاقتران.. وهكذا

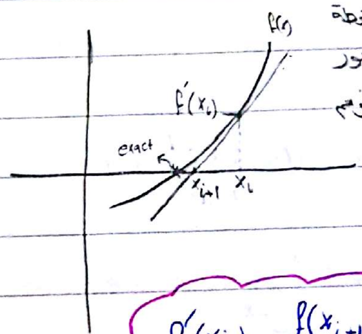
E.x



Simple fixed تفشل
diverge X_0 لأنو
تبعد عن exact

2) Newton Raphson method

* نحدد X_0 على f(x)
ثم نرسم مماس ونقطة
تقاطع المماس مع محور
السينات هي X_{i+1} ونرسم
مماس مرة أخرى



$$\boxed{X_{i+1} = X_i - \frac{f(x_i)}{f'(x_i)}}$$ where $i = 0, 1, 2$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{X_{i+1} - X_i}$$

slope

E.x use (N.R) to find the root of
$f(x) = e^{-x} - x$, $X_0 = 0$ after 3 iteration

Sol: $f'(x) = -e^{-x} - 1$

first iteration $X_1 = X_0 - \frac{f(x_0)}{f'(x_0)}$

$= 0 - \frac{1}{-2}$

iter	X_{i+1}	ε_a %
1	0.5	-
2	0.56681	11.7 %
3	0.56714	0.14 %

على Calc

$$\boxed{0 = Ans - \frac{e^{-Ans} - Ans}{-e^{-Ans} - 1}}$$

Scanned by CamScanner

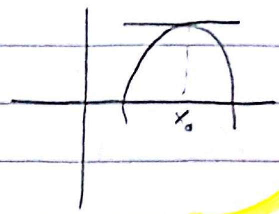
No: .... Date: ....

*** Note :**

The main problems that may arise when using N-R method

1) if $f' \approx 0$, at x_0, Chang x_0

2) if f & f' has the same root (Multiple root)



* يمكن أن يطالب منا بالسؤال متى يفشل

$f'(x) = 0$

ونكون قيم x التي طلعتها

E.x

Find the root $f(x) = \cos(x) - xe^{x}$, $x_0 = 2$

use N-R method if $\varepsilon_a < 1\%$

$x_5 = 0.5178$

$\varepsilon_a = 0.74\%$

No: Date:

3) Modified Newton Raphson method (Multiple roots)

$$X_{i+1} = X_i - \frac{f(x_i) \cdot f'(x_i)}{[f'(x_i)]^2 - f(x_i) \cdot f''(x_i)}$$

E.x

Compere between N-R & M.N.R for

$f(x) = x^3 - 5x^2 + 7x - 3$, at $X_0 = 0$

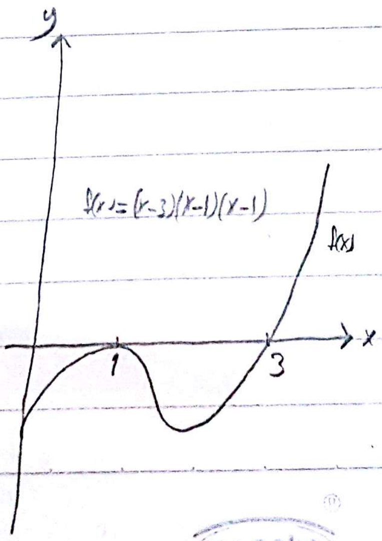
$f(x) = (x-3)(x-1)(x-1)$

* iter	x_i	ε_a %
1	0.4286	–
2	0.6857	31%
3	0.83286	17%
4	0.91332	8.7%
5	0.9557	4.4%

N-R

* iter	x_i	ε_a %
1	1.1052	–
2	1.00308	0.31%
3	1.000002	0.0024 %

M.N.R



$(x-1)(x-1)$

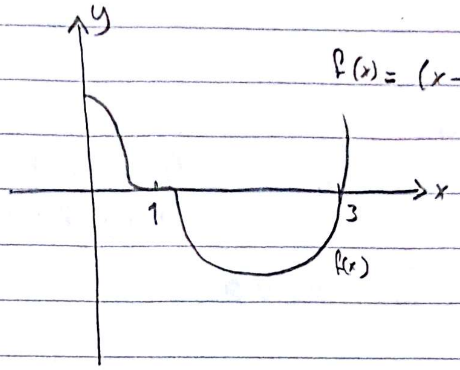
⇐ لما يكون root متكرر لا يقطع X-axis عند هدا root ← نفس الرسمة الماضية عند (1)

even → لا يقطع x-axis

odd → يقطع x-axis

(x=3) مثل عند

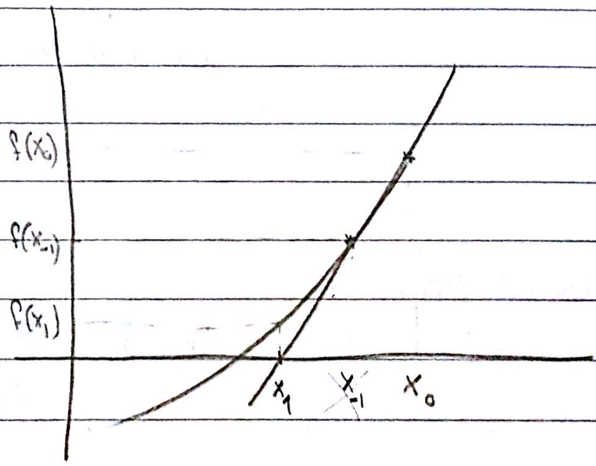
$f(x) = (x-3)(x-1)(x-1)(x-1)$



4) Secant method

use to find the root of $f(x)$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}}$$



$x_{-1} - x_0 \rightarrow x_1$

$x_0 - x_1 \rightarrow x_2$

$x_1 - x_2 \rightarrow x_3$

No: Date:

E.x use: secant to find the root of $f(x) = e^{-x} - x$

use $x_{-1} = 0$, $x_0 = 1$, $\varepsilon_a < 5\%$:

→ first iteration

$i = 0$ $$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$x_1 = 0.61271$$

→ second iteration

$i = 1$ $$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$x_2 = 0.56384$, $\varepsilon_a = 8.77$

→ thired iteration

$i = 2$ $$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$x_3 = 0.56717$, $\varepsilon_a = 0.59\%$

No: .... Date: ....

5) Modified secant method

use to find root of $f(x)$ as :-

Delta

$$\boxed{x_{i+1} = x_i - \frac{\delta x_i f(x_i)}{f(x_i + \delta x_i) - f(x_i)}}$$

$\delta x_i = (x_i - x_{i-1})$

with δ 8 x_0 معطي

E.x

use M. secant Method to find the root $f(x) = e^{-x} - x$, $x_0 = 1$

$\delta = 0.01$, $\varepsilon_a < 5\%$

* $i=0 \Rightarrow x_1 = 1 - \frac{0.01 * 1 \, f(1)}{f(1.01) - f(1)} = 0.5376$

* $i=1 \Rightarrow x_2 = 0.5376 - \frac{0.01 * 0.5376 \, f(0.53727)}{f(0.53727 + 0.01 * 0.53727) - f(0.53727)}$

$x_2 = 0.56701$ $\varepsilon_a = 5.2\%$

* $i=2 \Rightarrow x_3 = 0.56714$

$\varepsilon_a = 0.023\%$

** system of Non linear eq's :-

$2x + 3y = 10$
$3x - 4y = 5$ } $\rightarrow$ linear system

$x^2 + xy = 10$
$y + 3xy^2 = 57$ } $\rightarrow$ Non linear system

1) Fixed Point iteration

(I) Convert to $x = g_1(x, y)$
$y = g_2(x, y)$

(II) Do iteration

$x_{i+1} = g_1(x_i, y_i)$ $\quad$ $y_{i+1} = g_2(x_{i+1}, y_i)$

** Convergenc Condition

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| < 1$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| < 1$$

No: .... Date: ....

E.x Solve the following Non-linear system by using f.p iteration with $x_0 = 1.5$, $y = 3.5$ after 2 iteration

$x^2 + xy - 10 = 0$

$y + 3xy^2 - 57 = 0$

Exact solution	$x = 2$
	$y = 3$

Sol

$$x_{i+1} = \sqrt{10 - x_i y_i} \quad , \quad y_{i+1} = \sqrt{\frac{57 - y_i}{3x_i + 1}}$$

→ first iteration

$i = 0$

$$x_1 = \sqrt{10 - 1.5 * 3.5} = 2.17945$$

$$y_1 = \sqrt{\frac{57 - 3.5}{3 * 2.17945}} = 2.86051$$

→ second iteration

$i = 1$

$$x_2 = \sqrt{10 - 2.17945 * 2.86051} = 1.94053$$

$$y_2 = \sqrt{\frac{57 - 2.86051}{3 * 1.94053 + 1}} =$$

ε_a ??

No: Date:

2) Newton Raphson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{, if single variable } x$$

$$x_{i+1} = x_i - \frac{u_i \frac{dv_i}{dy} - v_i \frac{du_i}{dy}}{\frac{du_i}{dx} \cdot \frac{dv_i}{dy} - \frac{du_i}{dy} \cdot \frac{dv_i}{dx}}$$

$$y_{i+1} = y_i - \frac{v_i \frac{du_i}{dx} - u_i \frac{dv_i}{dx}}{\frac{du_i}{dx} \cdot \frac{dv_i}{dy} - \frac{du_i}{dy} \cdot \frac{dv_i}{dx}}$$

E.x

use N.R method to solve the non linear system with
$x_0 = 1.5$, $y_0 = 3.5$

$x^2 + xy = 10 \Rightarrow u(x,y) = x^2 + xy - 10$

$y + 3xy^2 = 57 \Rightarrow v(x,y) = y + 3xy^2 - 57$

$\frac{\partial u}{\partial x} = 2x + y \qquad \frac{\partial v}{\partial x} = 3y^2$

$\frac{\partial u}{\partial y} = x \qquad \frac{\partial v}{\partial y} = 1 + 6xy$

$i = 0$

$$X_1 = 1.5 - \frac{-2.5*32.5 - 1.625*1.5}{6.5*32.5 - 1.5*36.75} = 2.036$$

$$y_1 = 2.8438$$

Ch 9 | linear algebraic equations

1) The graphical method

2) Cramer's rule

↳

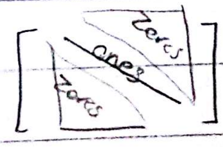
note :-

$$a_{11}X_1 + a_{12}X_2 = b_1$$
$$a_{21}X_1 + a_{22}X_2 = b_2$$

$$\underset{m \times n}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \underset{n \times k}{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}} = \underset{m \times k}{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$

$$[A][A]^{-1} = [I]$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = (a_{11}a_{22}) - (a_{12}a_{21})$$



No: .... Date: ....

Cramer's rule :-

تفشل Cramer's عندما يكون $D = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

E.x

آلة حاسبة simple Newton M.NF , M.seca

$2x_1 + 2x_2 = 18$

$-x_1 + 2x_2 = 2$

$$\begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 2 \end{bmatrix}$$

$$x_1 = \frac{\begin{bmatrix} 18 & 2 \\ 2 & 2 \end{bmatrix}}{4+2} = 5.3333$$

$$x_2 = \frac{\begin{bmatrix} 2 & 18 \\ -1 & 2 \end{bmatrix}}{6} = 3.666$$

Scanned by CamScanner

No: .... Date: ....

* solve linear system

⇒ Gauss Elimination

Like
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

pivot

$$\begin{array}{l} \text{row 1} \rightarrow \\ \text{row 2} \rightarrow \\ \text{row 3} \rightarrow \end{array} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & \overset{\text{pivot}}{a_{22}} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

factor

$$R_2' = R_2 - \frac{a_{21}}{a_{11}} R_1$$

$$R_3' = R_3 - \frac{a_{31}}{a_{11}} R_1$$

$$R_3'' = R_3' - \frac{a_{32}'}{a_{22}'} R_2'$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}' & a_{23}' & b_2' \\ 0 & 0 & a_{33}'' & b_3'' \end{array}\right]$$

1) $x_3 = \dfrac{b_3''}{a_{33}''}$ 2) $x_2 = \dfrac{b_2' - a_{23}' x_3}{a_{22}'}$ 3) $x_1 = \dfrac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}}$

E.x use gauss elimination to solve the following system :-

$x_1 + x_2 + x_3 = 6$

$2x_1 + 3x_2 + 4x_3 = 20$

$3x_1 + 4x_2 + 2x_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & 4 & 20 \\ 3 & 4 & 2 & 17 \end{array}\right]$$

$R_2' = R_2 - \frac{2}{1} R_1$

$R_3' = R_3 - \frac{3}{1} R_1$

$R_3'' = R_3' - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -3 & -9 \end{array}\right]$$

E.x

$x + y + z = 5$

$2x + 5z + 3y = 8$

$4x + 5z = 2$

$x = 3$

$y = 4$

$z = -2$

No: .... Date: ....

* Gauss Jordan elmination

like

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Cinvert to this form :-

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{array}\right]$$

* Pitfalse of Gauss elimination :-

1) division by zero

like
$$2x_2 + 3x_3 = 8$$
$$4x_1 + 6x_2 + 7x_3 = -3$$
$$2x_1 + x_2 + 6x_3 = 5$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 4 & 6 & 7 & -3 \\ 2 & 1 & 6 & 5 \end{array}\right]$$

Solu

=> change of Rows

No: Date:

2) division by almost Zero

like $0.0003x_1 + 3x_2 = 2.0001$

$x_1 + x_2 = 1$

$$\left[\begin{array}{cc|c} 0.0003 & 3 & 2.0001 \\ 1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 0.0003 & 3 & 2.0001 \\ 0 & -9999 & -6666 \end{array}\right]$$

$$R_2' = R_2 - \frac{1}{0.0003} R_1$$

$x_2 = 0.666667$

$x_1 = 0.33333$

# sig. fig	x_2	x_1
3	0.667	-3
4	0.6667	0
5	0.66667	0.3

* ممكن يطالب بالسؤال باستخدام 3 sig. fig من يتغير Rows

$\Rightarrow$ change

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0.0003 & 3 & 2.0001 \end{array}\right]$$

$$R_2' = R_2 - \frac{0.0003}{1} R_1$$

# sig. fig	x_2	x_1
3	0.667	0.333
4	0.6667	0.3333
5		
⋮		⋮

No: .... Date: ....

3) Ill-Conditioned system

I) Determinant of Coeff matrix ≈ 0, $\left(Det(A) \approx 0\right)$

Like
$$x_1 + 2x_2 = 10$$
$$1.05x_1 + 2x_2 = 1.4$$

$$\left[\begin{array}{cc|c} 1 & 2 & 10 \\ 1.05 & 2 & 1.4 \end{array}\right] \quad x_1 = 8, \ x_2 = 1 \quad \text{(rounding)} \quad \left[\begin{array}{cc|c} 1 & 2 & 10 \\ 1.1 & 2 & 1.4 \end{array}\right] \quad x_1 = 4, \ x_2 = 3$$

Det

$1 \times 2 - 1.05 \times 2 = -0.1$

Det

$= -0.2$

ولكون الخطين لهم تقريباً نفس الميل

II) Scaling

$$\begin{bmatrix} 100 & -101 \\ 200 & -200 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$$

$Det = 200$

$$\begin{bmatrix} 1 & -1.01 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$Det = 0.02$

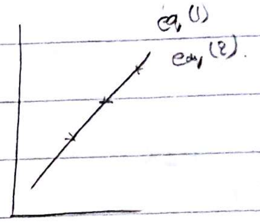
No: Date:

4) Singular system $Det(A) = 0$

(I) system with infinit solutions

Like

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



$Det(A) = 0$

الخطين منطبقين على بعضهم

=> elimination

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$0 x_2 = 0$

(II) System with no solution

Like

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

elimination

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & -1 \end{array}\right]$$

$0 x_2 = -1$ ✗

No solution

Ch 10 LU-Decomposition & inverse Matrix

set of linear eqs

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = C_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = C_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = C_3$

$$\underset{}{[A]}\ \underset{3\times1}{[X]} = \underset{3\times1}{[C]} \Rightarrow [L]\underbrace{[U][X]}_{[D]} = [C]$$

$\underset{3\times3}{[L]}$ $\underset{3\times3}{[U]}$

forward sub

$[L]\underset{3\times1}{[D]} = [C] \Rightarrow$ we get $\{D\}$ intermediater vector

$[U][x] = [D] \Rightarrow$ we get $\{x\}$

(1) Decomposition $[A] = [L][U]$

$[U] \Rightarrow$ get from gauss elimination

$[L] \Rightarrow$ assume Matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$$

No: .... Date: ....

$$[U] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

$$\begin{array}{ccccc} [L] & & [U] & & [A] \\ \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} & & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{12} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} & = & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

الضرب يكون صف بعمود

1) $L_{21} \times a_{11} = a_{21} \Rightarrow \boxed{L_{21} = \frac{a_{21}}{a_{11}}}$

2) $L_{31} \times a_{11} = a_{31} \Rightarrow \boxed{L_{31} = \frac{a_{31}}{a_{11}}}$

3) $L_{31} \times a_{12} + L_{32} a'_{22} = a_{32} \Rightarrow \boxed{L_{32} = \frac{a_{32} - L_{31} a_{12}}{a'_{22}}}$

2) $[L][D] = [C]$

$$\begin{array}{ccc} \text{known} & \text{unknown} & & \text{known} \\ \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} & \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} & = & \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \downarrow \end{array}$$

forward sub

$\rightarrow 1 \times d_1 = C_1 \Rightarrow \boxed{d_1 = C_1}$

$\rightarrow L_{21} d_1 + d_2 + 0 = C_2 \Rightarrow \boxed{d_2 = C_2 - L_{21} d_1}$

$\rightarrow L_{31} d_1 + L_{32} d_2 + d_3 = C_3 \Rightarrow \boxed{d_3 = C_3 - L_{31} d_1 - L_{32} d_2}$

3)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \uparrow$$

Backward sub

$\rightarrow a_{33}'' X_3 = d_3 \Rightarrow \boxed{X_3 = \dfrac{d_3}{a_{33}''}}$

$\rightarrow a_{11} X_1 + a_{12} X_2 + a_{13} X_3 = d_3$

$\rightarrow a_{22}' X_2 + a_{33}' X_3 = d_2$

$\Rightarrow \boxed{X_2 = \dfrac{d_2 - a_{23}' X_3}{a_{22}'}}$

Scanned by CamScanner

NO:

E.x

use LU-Decomposition to solve the following system

$x_1 + x_2 + x_3 = 6$

$2x_1 + 3x_2 + 4x_3 = 20$

$3x_1 + 4x_2 + 2x_3 = 17$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 17 \end{bmatrix}$$

$$[U] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$R_2' = R_2 - \boxed{2}R_1$ 2

$R_3' = R_3 - \boxed{3}R_1$ 3

$R_3'' = R_3' - \boxed{1}R_2$ 1

$$\underset{[L]}{\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}} \underset{[U]}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}} = \underset{[A]}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix}}$$

ما في داعي لهاي الخطوة

$\boxed{L_{21} = \boxed{2}}$ $\boxed{L_{31} = \boxed{3}}$ $L_{31} \times 1 + L_{32} \times 1 = 4 \Rightarrow \boxed{L_{32} = \boxed{1}}$

* ملاحظة هامة

coefficient في G.E هي نفسها $\boxed{L_{21}}$ $\boxed{L_{31}}$ $\boxed{L_{32}}$ بنفس الترتيب

YASSIN

No: Date: 11/12

$$[L][D]=[C]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 17 \end{bmatrix}$$

$\boxed{d_1=6}$

$d_2 = 20-12 = 8$

$\boxed{d_2=8}$

$3d_1+d_2+d_3=17$

$\boxed{d_3=-9}$

$$[U][X]=[D]$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ -9 \end{bmatrix}$$

$x_3 = 3$

$x_2 = 8-6 = 2$

$x_1 = 6-2-3 = 1$

No: .... Date: ....

* Inverse matrix

$[A] \Rightarrow [A]^{-1}$

$[A][I] = [A]$

$[A][A]^{-1} = [I]$

$[L][U]\underbrace{[A]^{-1}}_{[D]} = [I]$

$[L][D] = [I] \Rightarrow$ we get $[D]$

$[U][A]^{-1} = [D] \Rightarrow$ we get $[A]^{-1}$

E.x use LU-Decomposition to find the inverse of $[A]$:-

$$[A] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \end{bmatrix}$$

$\rightarrow$

Scanned by CamScanner

No:-------------- Date:--------------

from gauss => [U]

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

[L] =

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

=> first iteration

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_1' \\ d_2' \\ d_3' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

second $\begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}$ third $\begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$

$d_1' = 1$

$d_2' = -2$

$d_3' = -1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} a_{11}^* \\ a_{21}^* \\ a_{31}^* \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$a_{31}^* = 1/3$

$a_{21}^* = -2 - \frac{2}{3} = \frac{-8}{3}$

$a_{11} = 1 + \frac{8}{3} - \frac{1}{3} = \frac{10}{3}$

$$A^{-1} = \begin{vmatrix} 10/3 & -2/3 & -1/3 \\ -8/3 & 1/3 & 2/3 \\ 1/3 & 1/3 & -1/3 \end{vmatrix}$$

No: Date:

C.h (11) Gauss-seidel Method and Jordan Jacobi Method

like

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

⇒ Procedure of solution :-

جعل المعادلة بدلالة (X)

1) → $x_1 = \dfrac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$

→ $x_2 = \dfrac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$

→ $x_3 = \dfrac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$

اذا مو معطى بلش من Zero

2) use initial guesses for $x_2 = x_3 = 0$

No:............. Date:.............

3) find $x_1 = \frac{b_1}{a_{11}}$

from (x_1) find (x_2) $\quad x_2 = \frac{b_2 - a_{21} x_1}{a_{22}}$

from (x_1, x_2) find (x_3) $\quad x_3 = \frac{b_3 - a_{31} x_1 - a_{32} x_2}{a_{33}}$

4) repeat step I and II until $\varepsilon_a \leq \varepsilon_s$

$\Rightarrow$ update the value of variable from the iteration $\Leftarrow$

$\Rightarrow$ Condition :-

1) $a_{11}, a_{22}, a_{33} \neq$ Zero — اذا كانت Zero نعمل change between rows

2) $|a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| > |a_{21}| + |a_{23}|$ — شرط أن يتحققوا كلهم $\Leftarrow$

$|a_{33}| > |a_{31}| + |a_{32}|$

Dominant Matrix

$\rightarrow$ to converge the solution

solution:- try to change rows if fail then this method is diverge

Scanned by CamScanner

No: .... Date: ....

E.x

use gauss seidel Method to solve with $x_2 = x_3 = 0$ after 2 iter

$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$

$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$

$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$

Sol

check condition :- converge

cont :-

$$x_1 = \frac{b_1 - a_{12}x_2 + 0.2x_3}{3} \qquad x_1 =$$

$$x_2 = \frac{-19.3 - 0.1x_2 + 0.3x_3}{7} \qquad x_2 =$$

$$x_3 = \frac{71.4 - 0.3x_1 + 0.2x_2}{10}$$

⇒ first iteration

$$x_1 = \frac{7.85}{3} = 2.6167$$

$x_2 = -2.7945$ $\qquad$ $x_3 = 7.0056$

⇒ second iteration

$$x_1 = \frac{7.85 + 0.1 \times -2.7945 + 0.3 \times 7.0056}{3}$$

$$\boxed{x_1 = 2.9906}$$

No: Date:

$$x_2 = \frac{-19.3 - 0.1 \times 2.9906 + 0.3 \times 7.0056}{7}$$

$$\boxed{x_2 = -2.4996}$$

$$x_3 = \frac{71.4 - 0.3 \times 2.9906 + 0.2 \times -2.4996}{10}$$

$x_3 = 7.003$

$\Rightarrow$ to find $\varepsilon_a = \frac{\text{Present} - \text{Previous}}{\text{Present}} * 100\ \%$

$\varepsilon_{a_{x_1}} = 12.5\ \%$ $\varepsilon_{a_{x_2}} = 11.8\ \%$ $\varepsilon_{a_{x_3}} = 0.075\ \%$

$\Rightarrow$ in general :-

$$X_1^{(i)} = \frac{b_1 - a_{12} X_2^{(i-1)} - a_{13} X_3^{(i-1)}}{a_{11}}$$

where (i) start from (1)
↓
number of iteration

$$X_2^{(i)} = \frac{b_2 - a_{21} X_1^{(i)} - a_{23} X_3^{(i-1)}}{a_{22}}$$

$$X_3^{(i)} = \frac{b_3 - a_{31} X_1^{(i)} - a_{32} X_2^{(i)}}{a_{33}}$$

No: .... Date: ....

** Jacobi Method

very simelar to Gauss seidel Method

⇒ In Gerenal case :-

$$x_1^{(i)} = \frac{b_1 - a_{12}(x_2^{i-1}) - a_{13}x_3^{(i-1)}}{a_{11}}$$

$$x_2^{(i)} = \frac{b_2 - a_{21}x_1^{(i-1)} - a_{23}x_3^{(i-1)}}{a_{22}}$$

$$x_3^{(i)} = \frac{b_3 - a_{31}x_1^{(i-1)} - a_{32}x_2^{(i-1)}}{a_{33}}$$

Need intial guess

** Solve the Previous example by using Jacobi method after 2 iter with intial Guess $x_1 = x_2 = x_3 = 0$

$\boxed{i=1}$

$x_1 = 2.6167$

$x_2 = -2.7571$

$x_3 = 7.14$

$\boxed{i=2}$

$x_1 = 3.008$

$x_2 = -2.4885$

$x_3 = 7.0074$

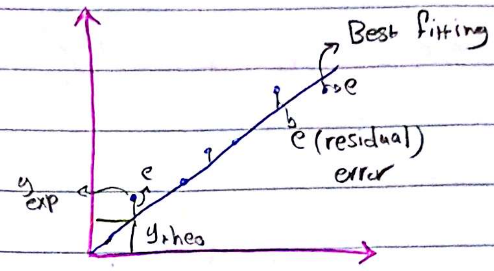
$\varepsilon_{a_{x_1}} = 12.8\%$ $\varepsilon_{a_{x_2}} = 16.8\%$ $\varepsilon_{a_{x_3}} = 1.9\%$

Scanned by CamScanner

No: ------------- Date: -------------

Ch 17 | Curve fitting

* least squar method or regression.



⇒ linear regression

$y = a_0 + a_1 X$

for one point $\quad y_{exp} = (a_0 + a_1 x)_{the} + e$

for all points $\quad \Sigma y_i = \Sigma (a_0 + a_1 x_i + e_i)$

$$\Sigma e_i = \Sigma (y_i - (a_0 + a_1 x_i))$$

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

squar

⇒ the goal of least squar method is to find a_0 & a_1 with min (S_r)

$$\frac{\partial S_r}{\partial a_0} = 0 \Rightarrow -2 \Sigma (y_i - a_0 - a_1 x_i) = 0 \text{ ---- eq (1)}$$

$$\frac{\partial S_r}{\partial a_1} = 0 \Rightarrow -2 \Sigma (y_i - a_0 - a_1 x_i) x_i = 0 \text{ ---- eq (2)}$$

Scanned by CamScanner

No: .... Date: ....

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} a_0 + \sum_{i=1}^{n} a_1 x_i \quad \text{--- eq (1)}$$

$a_0 n$

$$\sum_{i=1}^{n} y_i x_i = \sum_{i=1}^{n} a_0 x_i + \sum_{i=1}^{n} a_1 x_i^2 \quad \text{--- eq (٢)}$$

Matrix مهم

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1} x_i & \sum_{i=1} x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} y_i x_i \end{bmatrix}$$

E.x

Using linear regression fit to fit the following data :-

الحل

				Σ
x	2	3.25	5.1	10.35
y	3.5	5.6	7.8	16.9
x^2	4	10.5625	26.01	40.5725
xy	7	18.2	39.78	64.98

Calc

mode → statistics (6) →

(2) $y = a + bx$ → ندخل قيم y & x →

→ AC → option → a & b نجيب

$$\begin{bmatrix} 3 & 10.35 \\ 10.35 & 40.5725 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 16.9 \\ 10.35 \end{bmatrix}$$

$a_0 = 0.8999$

$a_1 = 1.372$

curve fit

$$y = 0.8994 + 1.372x$$

* Polynomial regression

Given (n) data points to be fitted to m^{th} order of polynomial

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m \quad , \text{where } n \geq (m+1)$$

↳ # Points

The regression form will be :-

$$\begin{bmatrix} n & \Sigma x & \Sigma x^2 & \Sigma x^3 \dots & \Sigma x^m \\ \Sigma x & \Sigma x^2 & \Sigma x^3 & \Sigma x^4 \dots & \Sigma x^{m+1} \\ \Sigma x^2 & \vdots & & \Sigma x^5 & \Sigma x^{m+2} \\ \vdots & \vdots & & & \vdots \\ \Sigma x^m & \Sigma x^{m+1} & \Sigma x^{m+2} & \Sigma x^{m+3} \dots & \Sigma x^{2m} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_m \end{bmatrix} \Rightarrow \begin{bmatrix} \Sigma y \\ \Sigma y \\ \\ \\ \Sigma x^m y \end{bmatrix}$$

1) Mean (averge)

$$\boxed{\bar{x} = \frac{\Sigma x}{n} \quad , \quad \bar{y} = \frac{\Sigma y}{n}}$$

الانحراف المعياري

2) Standard deviation (S_y)

$$\boxed{S_y = \sqrt{\frac{\Sigma (y_i - \bar{y})^2}{n-1}} = \sqrt{\frac{S_t}{n-1}}}$$

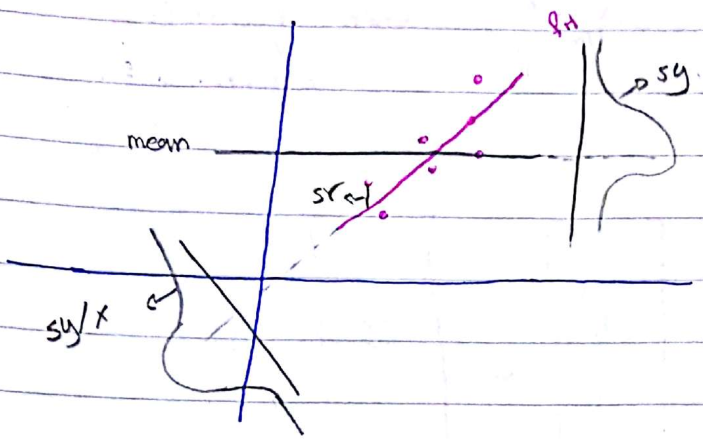
S_t :- is the total sum of the squars of the residuals the data point of the mean

3) Variance (S_y^2)

$$\boxed{S_y^2 = \frac{S_t}{n-1}}$$

4) Standard error of estimate ($s_{y/x}$)

$$\boxed{s_{y/x} = \sqrt{\frac{S_r}{n-2}}} \quad , \quad \boxed{S_r = \sum (y_i - a_0 - a_1 x_i)^2}$$



5) Coefficient of determination (r^2)

$$\boxed{r^2 = \frac{S_t - S_r}{S_t}}$$

6) Correlation coefficient (r) = $\sqrt{r^2}$

$$\boxed{r = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} y_i}{\sqrt{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \cdot \sqrt{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2}}}$$

Goodness of fit :

$0 < r < 1$

↓ Bad fit ↓ Best fit

No: -------------- Date: --------------

→ from previous example

$r = 0.9952$

Now find S_y, $S_{y/x}$

بنضيف عالجدول الثاني $(y_i)^2$ فقط

Calculation

1) $S_y \rightarrow S_t$

2) $r \rightarrow r^2$

3) coeffecient of det
↓ S_r

4) $S_{y/x} = \sqrt{\frac{S_r}{n-1}}$

بنجد القيم
بالمعادلة
A, B, C

Application for linear regression (lineratization)
To convert non linear relations to linear form

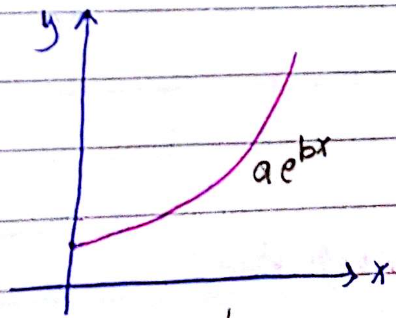
1) Exponantial function

$$y = a e^{bx}$$

take ln to both side

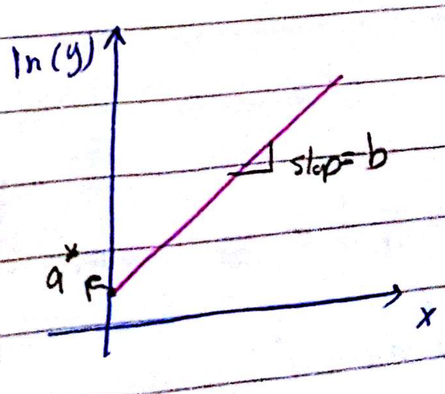
$$\ln y = \ln(a \cdot e^{bx})$$
$$\ln y = \ln a + bx$$

$$\ln y = \ln a + bx$$
$$y^* = a^* + bx$$



↓

linearization



No: .... Date: ....

2) Power function

$y = a x^b$

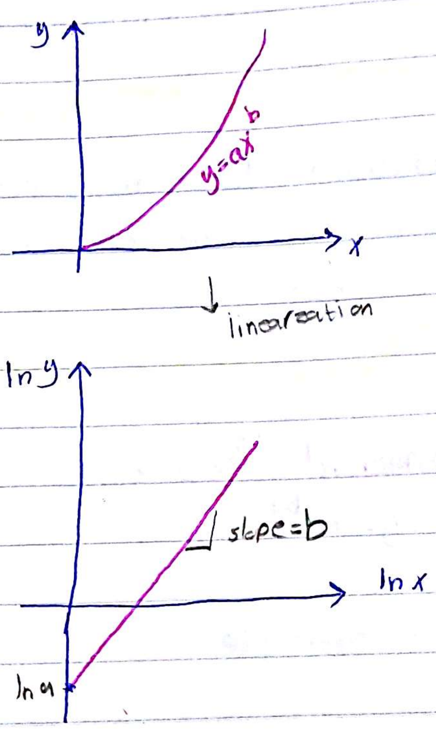
$\ln y = \ln a + b \ln x$

$y^* = \ln a + b x^*$

take ln to both sides :-

$\ln y = \ln(a . x^b)$

$\ln y = \ln a + b \ln x$



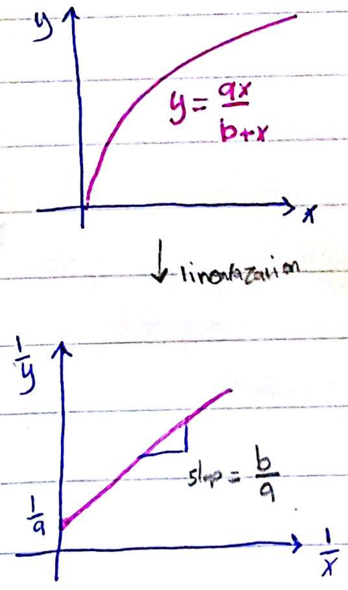
3) Saturation - growth rate

$y = \frac{ax}{b+x}$

take inverse

$\frac{1}{y} = \frac{b+x}{ax} \Rightarrow \frac{1}{y} = \frac{b}{a} \frac{1}{x} + \frac{1}{a}$

$y^* = b^* x^* + a^*$



No: .... Date: ....

E.x $y = axe^{bx}$

Sol $\frac{y}{x} = ae^{bx}$

$\ln\left(\frac{y}{x}\right) = \ln a + bx$

$\left(\ln(y) - \ln(x)\right) = \ln a + bx$

$y^* = a^* + bx$

* E.x

use the following data to fit the function $\boxed{y = ae^{bx}}$

x	0.5	1	1.2	2	2.4	$\Sigma x = 7.1$
y	0.9111	1.6601	2.1103	5.5116	8.9071	-
$y^* = \ln y$	-0.0931	0.5069	0.7469	1.7069	2.1869	$\Sigma \ln y = 5.0543$
x^2	0.25	1	1.44	4	5.76	$\Sigma x^2 = 12.45$
xy^*	-0.0466	0.5069	0.8962	3.4138	5.24856	$\Sigma xy^* = 10.0187$

$y = ae^{bx}$

$\ln y = \ln a + \ln e^{bx} \Rightarrow$ $\ln y = \ln a + bx$

$y^* = a^* + bx$

No: Date:

$$\begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

$$\begin{bmatrix} 5 & 7.1 \\ 7.1 & 12.45 \end{bmatrix} \begin{bmatrix} a^* \\ b \end{bmatrix} = \begin{bmatrix} 5.0543 \\ 10.0187 \end{bmatrix}$$

$a^* = -0.6931$ $\quad$ $b = 1.2$

$a = e^{a^*}$

$e^{-0.6931} = 0.5$

$\boxed{y = 0.5e^{1.2x}}$

$r = 0.9999$

Calc

1) Menu → Statistics

2) $y = a.e^\wedge(bx)$ نختار

3) x | y — ندخل قيم x و y

4) AC

5) option → Regression calc

أو shift → 1 → regretion

6) a =

b = → A

r =

1) $Sr = \Sigma (y_i - a^* - bx_i)^2$

2) $Sy/x = \sqrt{\frac{Sr}{n-2}}$

3) $r^2 = \frac{St - Sr}{St}$

4) $St = \Sigma (y_i - \bar{y})^2$

E.x $y = Axe^{Bx}$

take ln to Both side

$\ln y = \ln A + \ln x + Bx$

$\ln y - \ln x = \ln a + Bx$

$\ln(y/x) = \ln a + bx$

$y^* = a^* + bx$

على الآله الحاسبة

نختار $y = aX + b$

x	$\ln(y/x)$

ندخلها مكان y

E.x $y = \dfrac{1}{\sqrt{a + bx^2}}$

$\dfrac{1}{y} = \sqrt{a + bx^2}$

$\dfrac{1}{y^2} = a + bx^2 \Rightarrow y^* = a + bx^2 + cx$

نختار من الآلة

$y = a + bx + cx^2$

E.x $y = Ae^{2x} - Bx$

$\hookrightarrow$ linearization ما بتنضبط

نرجع إلى التعريف الأصلي

$Sr = 0$

$Sr = \sum(y_i - Ae^{2x} + Bx_i)^2$

$\dfrac{\partial Sr}{\partial A} = -2e^{2x}(y_i - Ae^{2x} + Bx_i) = 0$

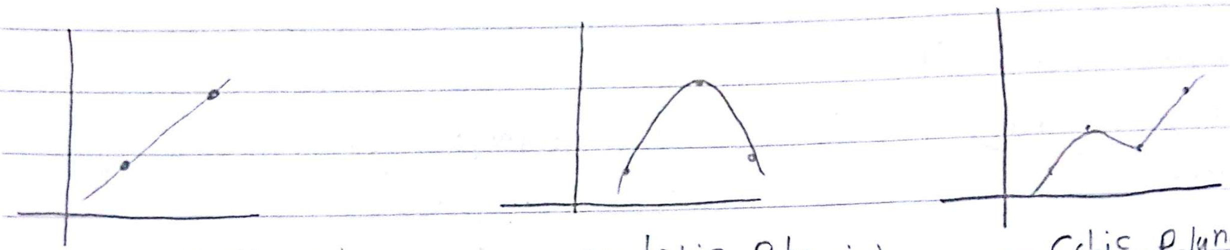
$= \sum y_i e^{2x_i} - \sum Ae^{4x_i} + Bx_i e^{2x_i} = 0 \longrightarrow (1)$

$\dfrac{\partial Sr}{\partial B} = 0 \longrightarrow (2)$

Matrix

No: .... Date: ....

* $\boxed{\text{Ch 18}}$:- Interpolation



linear Polynomial — quadratic Polynomial — Cubic Polynomial

* polynomial of interpolation

$$f_n(x) = a_0 + a_1 X + a_2 X^2 + \text{------} + a_n X^n$$

degree of Polynomial

⇒ $\boxed{\text{for } (n+1)}$ data points , there is one & only one Polynomial of $\boxed{\text{order } (n)}$ that passes through all the Points .

** How to find the polynomial interpolation ??

1) Newton's divided difference
2) lagrange Polynomial interPolation

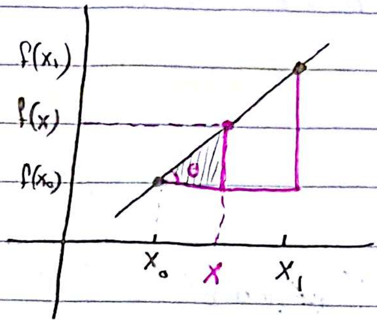
الطريقتين بنحصل على نفس الاقتران

No: ------------- Date: -------------

1) Newton divided diff → difference

→ Liner interpolation

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



$$f(x)_1 = f(x_0) + \frac{(f(x_1) - f(x_0))(x - x_0)}{x_1 - x_0}$$

$$\tan\theta = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f(x)_1 = b_0 + b_1(x - x_0)$$

⇒ first divided difference

$$\tan\theta = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

→ Quadratic Polynomial

$$f(x)_2 = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

⇒ second divided differnce

when $b_0 = f(x_0)$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

→ Cubic Polynomial

$$f(x)_3 = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$b_3 =$

الفرق

No: .... Date: ....

* General form of Newton interpolation Polynomial

$$f_n(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + \ldots + b_n(x-x_0)(x-x_1) * \ldots * (x-x_{n-1})$$

$b_0 = f(x_0)$ $\quad b_1 = f[x_1, x_0]$ $\quad b_2 = f[x_2, x_1, x_0]$

$$b_3 = f[x_3, x_2, x_1, x_0] = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0}$$

** Errors of Newton interpolation

$$R_n \cong f[x_{n+1}, x_n, x_{n-1}, \ldots, x_0](x-x_0)(x-x_1)\ldots(x-x_n)$$

E.x

use linear interpolation (N.D.D) for the following data to find $x = 2$ $f(2)$

$x_0 = 1 \rightarrow f(x_0) = 0$

$x_1 = 6 \rightarrow f(x_1) = 1.79176$

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$= 0 + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x-x_0) \Rightarrow 0 + \frac{1.79176 - 0}{6 - 1}(2-1)$$

$f(2) = 0.35835$

← اذا اعطانا نقط نعطيني لحالنا لازم نعرف انو linear

← اذا اعطانا ٣ نقاط يكون Quadratic

No: .... Date: ....

E.X

ايجاد الأقتران بشكل تام

use (N.D.D) to fit the following data with third order Polynomial

$x_0 = 1$, $f(x_0) = 0$

$x_1 = 4$, $f(x_1) = 1.38629$

$x_2 = 5$, $f(x_2) = 1.60944$

$x_3 = 6$, $f(x_3) = 1.7976$

$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

i	x_i	$f(x_i)$	first D.D	second D.D	third D.D
0	x_0	$f(x_0)$	$F[x_1, x_0]$	$F[x_2, x_1, x_0]$	$F[x_3, x_2, x_1, x_0]$
1	x_1	$f(x_1)$			
2	x_2	$f(x_2)$	$F[x_2, x_1]$	$F[x_3, x_2, x_1]$	
3	x_3	$f(x_3)$	$F[x_3, x_2]$		

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{F[x_2, x_1] - F[x_1, x_0]}{x_2 - x_0}$$

$$b_3 = \frac{F[x_3, x_2, x_1] - F[x_2, x_1, x_0]}{x_3 - x_0}$$

يكون الطرح في X الذي بيسم (2)
$(x_3 - x_1)$ أو $(x_2 - x_0)$

		b_0	b_1	b_2	b_3
x0	1.	0	0.46209	-0.05974	0.007864
x1	4	1.38629			
x2	5	1.50944	0.22315	-0.02042	
x3	6	1.7976	0.18232		

$$b_3 = \frac{-0.02042 - b_2}{5}$$

$$f_3(x) = 0.462098(x-1) - 0.05974(x-1)(x-4) + 0.007864(x-1)(x-4)(x-5)$$

Scanned by CamScanner

No: .... Date: ....

2) Lagrange Interpolation Polynomials

$$f_n(x) = \sum_{i=0}^{n} L_i(x) \cdot f(x_i)$$

where $$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

E.x

Find the lagrange of the first order $(n=1)$

$$f(x) = \sum_{i=0}^{1} L_i(x) \cdot f(x) = L_0(x) \cdot f(x_0) + L_1(x) \cdot f(x_1)$$

$$\rightarrow L_0(x) = \prod_{\substack{j=0 \\ j \neq i}}^{1} \frac{x - x_j}{x_i - x_j} = \frac{x - x_1}{x_0 - x_1}$$

$$\rightarrow L_1(x) = \prod_{\substack{j=0 \\ j \neq i}}^{1} \frac{x - x_j}{x_i - x_j} = \frac{x - x_0}{x_1 - x_0}$$

$$\boxed{f_1(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)}$$

Scanned by CamScanner

No: .... Date: ....

$\Rightarrow$ Second order $(n=2)$, x_0 , x_1 , x_2

$$f_2(x) = \sum_{i=0}^{2} L_i(x) \cdot f(x_i)$$

$$f_2(x) = L_0(x) f(x_0) + L_1(x) \cdot f(x_1) \cdot L_2(x) \cdot f(x_2)$$

$$L_0(x) = \prod_{\substack{j=0 \\ j\neq i}}^{2} \frac{x-x_j}{x_i-x_j} = \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)$$

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq i}}^{2} \frac{x-x_j}{x_i-x_j} = \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)$$

$$L_2(x) = \prod_{\substack{j=0 \\ j\neq i}}^{2} \frac{x-x_j}{x_i-x_j} = \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)$$

حدين

$$L_3(x) \prod_{\substack{i=0 \\ j\neq 0}}^{4} \frac{x-x_j}{x_i-x_j} = \left(\frac{x-x_0}{x_3-x_0}\right)\left(\frac{x-x_1}{x_3-x_1}\right)\left(\frac{x-x_2}{x_3-x_2}\right)\left(\frac{x-x_4}{x_3-x_4}\right)$$

No: .... Date: ....

E.X

Use lagrange interpolation of the first & second order to evaluate $f(2)$ on the basic of the following data

$x_0 = 1 \Rightarrow f(x_0) = 0$

$x_1 = 4 \Rightarrow f(x_1) = 1.386294$

$x_2 = 6 \Rightarrow f(x_2) = 1.791760$

* طلبا بالسؤال f(2) و 2 تقع بين 1 و 4 لذلك بأخذ هاي النقطتين

$$\Rightarrow f_1(x) = \cancel{L_0(x) f(x_0)} + L_1 f(x_1)$$

$f(x_0) = 0$

$$1)\ L_1(x) = \prod_{\substack{i \neq j \\ j=0}}^{2} \frac{x - x_j}{x_i - x_j} = \frac{x - x_0}{x_1 - x_0} = \frac{x-1}{4-1} = \frac{x-1}{3}$$

$$f_1(x) = \left(\frac{x-1}{3}\right) 1.386294 \Rightarrow f_1(2) = 0.4620981$$

$$\Rightarrow f_2(x) = \cancel{L_0(x) f(x_0)} + L_1 f(x_1) + L_2(x) f(x_2)$$

ثانيا:

first order

$$1)\ L_1(x) = \prod_{\substack{j=0 \\ i \neq j}}^{2} \frac{x - x_j}{x_i - x_j} = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) \Rightarrow L_1(x) = \left(\frac{x-1}{4-1}\right)\left(\frac{x-6}{4-6}\right)$$

$$2)\ L_2(x) = \prod_{\substack{j=0 \\ i \neq j}}^{2} \frac{x - x_j}{x_i - x_j} = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right) \Rightarrow L_2(x) = \left(\frac{x-1}{6-1}\right)\left(\frac{x-4}{6-4}\right)$$

$$f_2(2) = \left(\frac{2-1}{4-1}\right)\left(\frac{2-6}{4-6}\right) * 1.386294 + \left(\frac{2-1}{6-1}\right)\left(\frac{2-4}{6-4}\right) * 1.79176$$

$$f_2(2) = 0.565844$$

No: Date:

E.X $f_2(x) = 0.2(x-5)(x-7) - 3(x-2)(x-7) + 4(x-2)(x-5)$ lagrange

$b_1 = ??$

$f_2(x) = b_0 + b_1 (x-2) + b_2 (x-2)(x-5)$ (N.D.D)

$(x-x_0) \quad (x-x_0)(x-x_1)$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f_2(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) \cdot f(x_2)$$

$$f_2(x) = \left(\frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2}\right) f(x_0) + \left(\frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2}\right) f(x_1) + \left(\frac{x-x_0}{x_2-x_0} \cdot \frac{x-x_1}{x_2-x_1}\right) f(x_2)$$

$x_0 = 2$, $x_1 = 5$, $x_2 = 7$ من السؤال

$$\frac{f(x_0)}{(2-5)(2-7)} = 0.2 \qquad \frac{f(x_1)}{(5-2)(5-7)} = -3 \qquad \frac{f(x_2)}{(7-2)(7-5)} = 4$$

$\boxed{f(x_0) = 3}$ $\quad \boxed{f(x_1) = 18}$ $\quad f(x_2) = 40$

$$\Rightarrow \frac{18-3}{5-2} = \frac{15}{3} = 5$$

$$f(x_{i+1}) = \sum_{n=0}^{\infty} \frac{f^n(x_i)}{n!} h^n, \qquad R_n = \frac{f^{n+1}(x_i)}{(n+1)!} h^{n+1}$$

$$x_r = \frac{x_l + x_u}{2}, \qquad x_{i+1} = g(x_i)$$

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(u)}, \quad x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{\acute{f}(x_i)}, \qquad n = \frac{\log(\Delta x / E_a)}{log2}$$

$$x_{i+1} = x_i - \frac{f(x_i)\delta x_i}{f(x_i + \delta x_i) - f(x_i)}$$

$$x_{i+1} = x_i - \frac{f(x_i)\acute{f}(x_i)}{[\acute{f}(x_i)]^2 - f(x_i)f''(x_i)}$$

ElCoM ELCOM-HU.COM

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}, \quad O(h)$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h}, \quad O(h^2)$$

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}, \quad O(h)$$

$$f''(x_i) = \frac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2}, O(h^2)$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}, \quad O(h^2)$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}, O(h^4)$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}, \quad O(h^2)$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}, \quad O(h)$$

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h}, \quad O(h^2)$$

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}, \quad O(h)$$

$$f''(x_i) = \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2}, \quad O(h^2)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}, \quad f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$f_n(x) = b_o + b_1(x - x_o) + b_2(x - x_o)(x - x_1) + \cdots + b_n(x - x_o)\ldots(x - x_{n-1})$$

$$b_0 = f(x_0), b_1 = f[x_1, x_0], b_2 = f[x_2, x_1, x_0]$$

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}, f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

$$\begin{bmatrix} n & \left(\sum x\right) & \left(\sum x^2\right) \\ \left(\sum x\right) & \left(\sum x^2\right) & \left(\sum x^3\right) \\ \left(\sum x^2\right) & \left(\sum x^3\right) & \left(\sum x^4\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \left(\sum y\right) \\ \left(\sum xy\right) \\ \left(\sum x^2 y\right) \end{bmatrix}$$

$$S_{y/x} = \sqrt{\frac{S_r}{n-2}}, \; S_y = \sqrt{\frac{S_t}{n-1}}, \; S_t = \sum(y_i - \bar{y})^2$$

$$r = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\left(\sqrt{n\sum x_i^2 - (\sum x_i)^2}\right)\left(\sqrt{n\sum y_i^2 - (\sum y_i)^2}\right)}$$

Euler's Method $y_{i+1} = y_i + f(x_i, y_i)h$

Heun's Method $y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$

$k_1 = f(x_i, y_i), \qquad k_2 = f(x_i + h, y_i + k_1 h)$

Midpoint Method $y_{i+1} = y_i + (k_2)h$

$k_1 = f(x_i, y_i), \qquad k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$

Ralston's Method $y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$

$k_1 = f(x_i, y_i), k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$

$$y_{i+1} = y_i + {}^1\!/_6 (k_1 + 4k_2 + k_3)h$$

$$k_1 = f(x_i, y_i), \quad k_2 = f(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}hk_1)$$

$$k_3 = f(x_i + h, y_i - hk_1 + 2hk_2)$$

$$y_{i+1} = y_i + {}^1\!/_6 (k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i), \quad k_2 = f(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}hk_1)$$

$$k_3 = f(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}hk_2), k_4 = f(x_i + h, y_i + hk_3)$$

$$I \cong \frac{h}{2}(f(b) + f(a)), \qquad I \cong \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$$

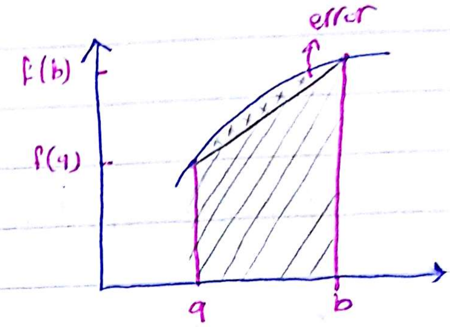
$$I \cong \frac{3h}{8}(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3))$$

$$I \cong \frac{h}{2}\left[f(x_o) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$$I \cong \frac{h}{3}\left[f(x_o) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{i=2,4,6}^{n-2} f(x_i) + f(x_n)\right]$$

CH 21 Newton Coles Integration Formula

1) Trapezoidal Rule



$$I = \int_a^b f(x)\,dx$$

$$I \cong \frac{1}{2}\left(f(a) + f(b)\right)(b-a)$$

$\Rightarrow E_t = I_{exact} - I_{approx}$ E_t :- relative error , $\boxed{\text{Truncation}}$ error

$\Rightarrow$ Approximation error = $\boxed{E_a = \frac{-1}{12} \bar{f}''(x)(b-a)^3}$ *

$$\bar{f}''(x) = \frac{\int_a^b f''(x)\,dx}{b-a} = \boxed{\frac{f'(b) - f'(a)}{b-a} = \bar{f}''(x)}$$

E.x

use trapezoidal rule to intergrate $f(x)$ from $a=0$, $b=0.8$

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

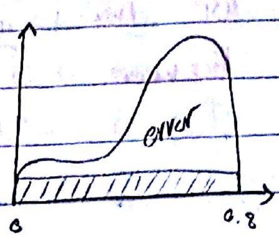
I exact $= 1.640533$ calc ...

$$I \cong \frac{1}{2}\left(f(0) + f(0.8)\right)(0.8 - 0)$$

No: .... Date: ....

$I \cong \frac{1}{2}(0.2 + 0.232)(0.8)$

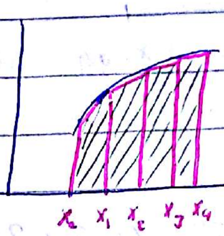
$I \cong 0.1728$



$\varepsilon_t = \frac{1.640533 - 0.1728}{1.640533} * 100\% = 89.4\%$

2) Multiple - Trapezoidal rule.

use n segments with constant width



$step = \frac{b-a}{n}$

$$I \cong \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx \ldots\ldots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$I \cong \frac{1}{2}\left(f(x_1) - f(x_0)\right)(x_1 - x_0) + \frac{1}{2}\left(f(x_2) - f(x_1)\right)(x_2 - x_1) + \ldots\ldots \frac{1}{2}\left(f(x_n) - f(x_{n-1})\right)(x_n - x_{n-1})$$

كل رقم تكرر مرتين

$$\boxed{I \cong \frac{1}{2}\,\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{n} \cdot (b-a)}$$

$n \rightarrow$ # of segments

* Approximation error :-

$$\boxed{E_a = \frac{-(b-a)^3}{12n^2}\,\bar{f}''(x)}$$

No: .... Date: ....

E.x use two segments of trap-rule to find integral of $f(x)$ between 0 & 0.8 ??

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

نفس المثال السابق

Sol

$$\text{step} = \frac{0.8 - 0}{2} = 0.4$$

$f(x_0) = 0.2$ $\quad$ $f(x_1) = ?$ $\quad$ $f(x_2) = 0.232$

$x_0 = 0$ $\quad$ $x_1 = 0.4$ $\quad$ $x_2 = 0.8$

ضروري جدا الرسم
لمعرفة قيم x

$(2-1) \to i = 1$

$\sum_{i=1} f(x_i)$

$$I \cong \frac{(0.8-0)}{2 \times 2} \left(0.2 + 2 \times f(0.4) + 0.232\right)$$

$I = 1.0688$ $\qquad$ $\varepsilon_t = 34.8\ \%$

E.x

$x_0 = 0$, $f(x) = 0.2$

$x_1 = 0.12$, $f(x_1) = 1.309$

$x_2 = 0.4$, $f(x_2) = 2.451$

$x_3 = 0.7$, $f(x_3) = 2.363$

$x_4 = 0.8$, $f(x_4) = 0.232$

find the Integration for the following using trap.rule ?

$$I \cong \frac{1}{2}\Big[(0.2 + 1.309)(0.12 - 0) + (2.451 - 1.309)(0.4 - 0.12) + (2.363 - 2.451)(0.7 - 0.4) + (0.232 - 2.363)(0.8 - 0.7)\Big]$$

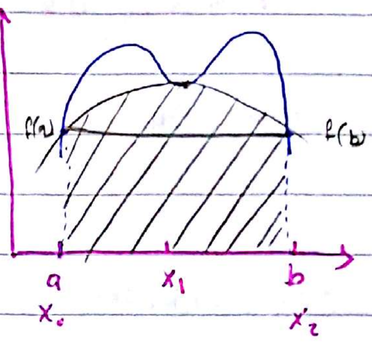
$I \cong 1.4711$

Scanned by CamScanner

NO: Date:

2) simpson's 1/3 rule

Quadratic relation



$$I = \int_a^b f(x)\,.dx$$

$$I \cong \int_a^b f_2(x)\,.dx$$

$$I \cong \int_a^b b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1)\,.dx$$

$$\boxed{h = \frac{b-a}{2}}$$

$$I \cong \frac{1}{3}\left(\frac{b-a}{2}\right)\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$\boxed{I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]}$$

⇒ Truncation error of simpsion $\frac{1}{3}$

$$E_t = I_{exact} - I_{approx}$$

$$\boxed{E_t = \frac{-1}{90} f^{(4)}(\xi) h^5}$$

$$\varepsilon_t = \frac{E_t}{exact} \times 100\%$$

⇒ Approximation error

$$\boxed{E_a = \frac{-1}{90} \bar{f}^{(4)}(x).\,h^5}$$

$$\varepsilon_a = \frac{E_a}{approximation} \times 100\%$$

$$\bar{f}^{(4)}(x) = \frac{\int_a^b f^{(4)}(x)\,dx}{b-a} = \frac{f^{(3)}(b) - f^{(3)}(a)}{b-a}$$

↓

averge forth derivative

E.x use simpsons 1/3 to integrate $f(x)=0.2+25x-200x^2+675x^3-900x^4+400x^5$ from 0 to 0.8

$I_{exact}=1.640533$

$h=\frac{0.8-0}{2}=0.4$

$f(0)=0.2$ $f(0.4)=2.459$ $f(0.8)=0.232$

$x_0=0$ $x_1=0.4$ $x_2=0.8$

$I \cong \frac{0.4}{3}[0.2+4\times 2.459+0.232]$

$=1.36747$ $\varepsilon_t=16.6$

* Trapezidal → exact (linear, Constant, infinit # of segment)

* simpson 1/3 → exact (linear, Quadratic)

* simpson 3/8 → exact (cubic)

Sigment → simpson 1/3 ⟹ even زوجي

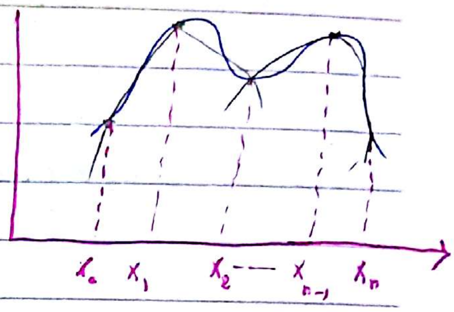
E.x $f(x)=4x+9$

find $\int_0^2 f(x)\,dx$ using trap-rule ?

No: Date:

⇒ Multiple segments of simpson's 1/3 rule

$$I \cong \int_{x_0}^{x_n} f_2(x)\,dx$$



$$I \cong \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx \ldots \int_{x_{n-2}}^{x_n} f(x)\,dx$$

segment even دائماً

$$I \cong \frac{h}{3}\left[f(x_0)+4f(x_1)+f(x_2)\right]+\frac{h}{3}\left[f(x_2)+4f(x_3)+f(x_4)\right]+\ldots\frac{h}{3}\left[f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

$$I = \frac{h}{3}\left[f(x_0)+2\sum_{\substack{i=2,4\ldots \\ \text{even}}}^{n-2} f(x_i)+4\sum_{\substack{j=1,3\ldots \\ \text{odd}}}^{n-1} f(x_j)+f(x_n)\right]$$

$$\boxed{h = \frac{b-a}{n}}$$

* Approximation error :-

$$\boxed{E_a = \frac{-(b-a)^5\,\bar{f}^{(4)}(x)}{180\,n^4}}$$

E.x use simpson's 1/3 to integrate $f(x) = 0.2+25x-200x^2+675x^3-900x^4+400x^5$ with $n=4$ from 0 to 0.8

$$h = \frac{0.8-0}{4} = 0.2$$

$f(0)=0.2$	$f(0.2)=1.265$	$f(0.4)=2.469$	$f(0.6)=3.464$	$f(0.8)=0.232$
0	0.2	0.4	0.6	0.8
x_0	x_1	x_2	x_3	x_4

$$I \cong \frac{0.2}{3}\left[0.2+2(2.469)+4(1.265+3.464)+0.232\right]$$

$I \cong 1.623466$ $\varepsilon_a = 1.051\%$

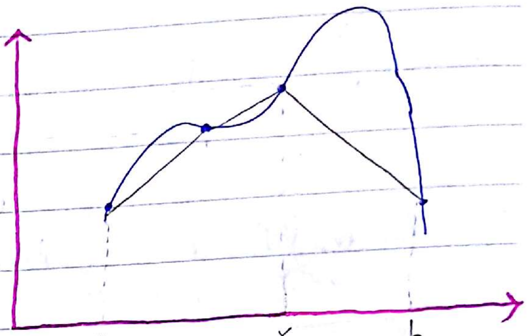
No: Date:

3) Simpson's $\left(\frac{3}{8}\right)$ Rule

(more acurte)

Cubic relation 4 points

$$I \simeq \int_a^b b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_2)(x-x_1)\,dx$$



$f_3(x) =$

$$I \simeq \frac{3}{8} h \left[f(x_0) + 3\left(f(x_1) + f(x_2)\right) + f(x_3) \right]$$

→ how to find h

$$h = \frac{(x_3 - x_0)}{3} = \frac{b-a}{3}$$

** Truncation Error :-

$$E_t = \frac{-3}{80} h^5 f^{(4)}(\xi)$$

** Approximation Error :-

$$E_a = \frac{-3}{80} \cdot h^5 \bar{f}^{(4)}(x)$$

$$\bar{f}^{(4)}(x) = \int_a^b \frac{f^{(4)}(x)}{b-a}\,dx = \frac{f^{(3)}(b) - f^{(3)}(a)}{b-a}$$

مثال

E.x

use Simpson's $\frac{3}{8}$ Rule to integrate $f(x)$ between $0, 0.8$:-

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

$h = \frac{0.8 - 0}{3} = 0.2667$

عدد الشرائح ← 3

$f(x)$:-	0.2	1.43272	3.48717	0.232
x :-	x_0	x_1	x_2	x_3
	0	0.2667	0.533	0.8

$$I \cong \frac{3}{8}(0.2667)\left[0.2 + 3(1.43272 + 3.48717) + 0.232\right]$$

$I \cong 1.51917$ / $I_{exact} = 1.640533$ / $\varepsilon_t = 7.4\%$

E.x

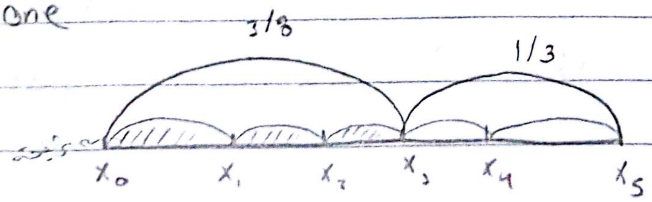
use simpson's Rule to integrate $f(x)$ (from Previous example) with five sigments from (0 to 0.8)

$h = \text{step} = \frac{0.8 - 0}{5} = 0.16$

0.2	1.296919	1.743393	3.186015	3.181929	0.232
x_0	x_1	x_2	x_3	x_4	x_5
0	0.16	0.32	0.48	0.64	0.8

→ Case one

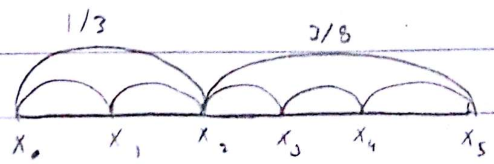
مرة أطبق الأول $(\frac{3}{8})$ ومرة $(\frac{1}{3})$



$(\frac{1}{3})$ ← شريحتين

$(\frac{3}{8})$ ← ٣ شرائح

→ Case two



$$\Rightarrow I_{1/3} = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right]$$

$$I_{1/3} = 0.3803237$$

$$\Rightarrow I_{3/8} = \frac{3}{8} \times 0.16 \left[1.743393 + 3(3.186015 + 3.18929) + 0.232 \right]$$

$$I_{3/8} = 1.264754$$

$$I_{total} = 1.645077$$

$$I_{exact} = 1.646533 \qquad \varepsilon_t = 0.28\%$$

Case 2 more acurte than 1

E.x

time	0	0.16	0.32	~~0.4~~	0.48	0.5	0.64
velocity	0	1	1.7	~~2.3~~	3.4	3.6	4.2

3/8

find distance at $t = 0.4$

* بما أن المسافات غير متساوية ما بقدر اطلع simpson's لذلك نحذف النقطة الغلط

$$\text{distance} = \int_0^{0.48} V(t)\,.\,dt$$

نحل على $\frac{3}{8}$ simpson's ⟸

Date: ....

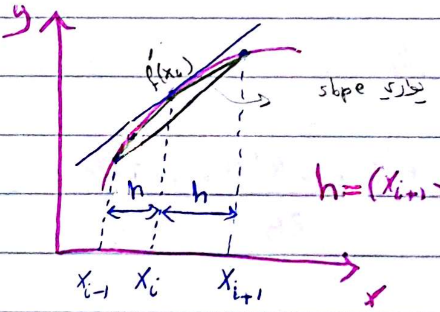
** Numerical Differensiation (C.H 23)

$$f'(x) = \lim_{h \to 0} \frac{f(x_i + h) - f(x_i)}{h}$$

$$f(x_{i+1}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_i)}{n!} h^n \qquad \text{Tylor series}$$

$\Rightarrow f(x_{i+1}) = f(x_i) + f'(x_i)h + \underbrace{\dots}_{O(h)}$ first derivative from Taylor series

$$\boxed{f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}}$$



$h = (x_{i+1} - x_i)$

(Forward finite differnce method) $(O(h))$

$$f'(x_i) \cong \frac{f(x_{i+1}) - f(x_i)}{h}$$

(Backword finite differnce method) $(O(h)) \rightarrow$ (first order of accuracy)

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$$

(Center finite differnce method) $(O(h^2)) \rightarrow$ (second order of accuracy)

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

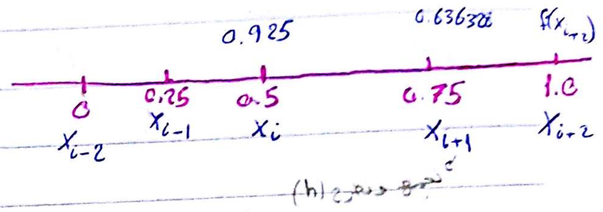
E.x

Use forward & Backward with ($O(h)$, $O(h^2)$) & centered with ($O(h^2)$ & $O(h^4)$) to find $f'(x)$ at $x = 0.5$ by using $h = 0.25$

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$f'_{exact}(0.5) = -0.9125$$

⇒ **forward** $O(h)$



$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} =$$

$$\frac{0.636308 - 0.925}{0.25} = -1.155 \qquad \varepsilon_t = 26.5\%$$

⇒ $O(h^2)$

$$f'(x_i) = \frac{f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} = \frac{-0.2 + 4 \times 0.636308 - 3 \times 0.925}{2 \times 0.25} = -0.859375, \quad \varepsilon_t = 5.8\%$$

⇒ **Backward**

$O(h)$
$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$$

$$f'(0.5) = \frac{f(0.5) - f(0.25)}{0.25} = -0.714, \quad \varepsilon_t = 21.7\%$$

$O(h^2)$
$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h}$$

$$f'(0.5) = -0.8781 \qquad \varepsilon_t = 3.7\ \%$$

No: .... Date: ....

⇒ $\boxed{\text{Center}}$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} \qquad O(h^2)$$

$f'(0.5) = -0.934 \qquad \varepsilon_t = 2.4\%$

$f'(x_i) = -0.9125$ exact $(O(h^4))$

$\varepsilon_t = 0\%$

t	0	1	2	2.5	3
P	0	0.5	0.75	1.25	2

شرطها طالما انه h نتكلم عنها

* افضل طريقة الحل هي Center method

$\frac{dP}{dt} = V(t)$

$V(1) \rightarrow$ Center

$V(3) \rightarrow$ Backward ← تعينا لانو اخر رقم ناخد 2 و 1 معه

$O(h^2)$ ← يجب استخدامه لانو more accuracy

No: Date:

Ch 25 :- Ordinary Differenial eqution (ODE)

$$\frac{dy}{dx} = f(x,y) \Rightarrow \int dy = \int f(x,y)\,dx \quad , \quad y(x) = ----$$

$y(x_0) = y_0$

intial condition

→ Numerical method to solve ODE :-

1) Euler's method
   - → Heun's method
   - → Mid point method
   - → Ralston's method

2) Runge - Kutta method (R.K 2^{nd} order to 4^{th} order)

1) Euler's method

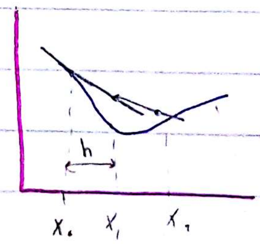
To solve the (ODE) as $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$

$$y_{i+1} = y_i + f(x_i, y_i)\, h$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$$

$$\frac{dy}{dx} = \frac{f\overset{y_{i+1}}{(x_{i+1})} - \overset{y_i}{f(x_i)}}{h}$$

$$y_{i+1} = y_i + \frac{dy}{dx}\, h$$



$y(x_0) = y_0$

$i = 0, 1, 2 ----$

$$h = x_{i+1} - x_i$$

* الحل دائماً يكون بوضع $\frac{dy}{dx}$ بطرف واحد

E.x

Use Euler's method to solve the following ODE

$$\frac{dy}{dx} + 2x^3 - 12x^2 + 20x - 8.5 = 0 \quad \text{from } x=0 \text{ to } x=1 \text{ with } h=0.5,\ y(0)=1$$

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

بعد الحل

$y_0(x_0)=1$ $\qquad$ $y_1(x_1)=5.25$ $\qquad$ $y_2(x_2)=5.875$

x_0 $\qquad$ $x_1=0.5$ $\qquad$ $x_2=1$

$$y_{i+1} = y_i + f(x_i, y_i)\, h$$

$i=0$ $\quad y_1 = y_0 + f(x_0, y_0) \times 0.5$

$y_1 = 1 + 8.5 \times 0.5$

$y_1 = 5.25$

$f(0,1) = 8.5$

$f(0.5, 5.25) = 1.25$

لا يوجد هنا بالاقتران (y)

لذلك بس بنعوض قيمة X

$i=1$ $\quad y_2 = y_1 + f(x_1, y_1) \times 0.5$

$y_2 = 5.25 + 1.25 \times 0.5$

$y_2 = 5.875$

** Approximation error :-

$$\boxed{E_a = \frac{f'(x_i, y_i)\, h^2}{2}}$$

No: .... Date: ....

E.x

Use Euler's method to solve (ODE)

$\frac{dy}{dx} - 5x^2y - 1 = 0$ where $y(0) = 1$, $h = 0.25$ Find $y(0.5)$??

& E_a at y_2

$\frac{dy}{dx} = 5x^2y + 1$ $\boxed{f(x_i, y_i) = 5x_i^2 y_i + 1}$

$y_0 = 1$ $y_1 = 1.25$ $y_2 (0.5)$

x_0 x_1 x_2

0 0.25 0.5

$i = 0$ $y_1 = y_0 + f(x_0, y_0) * 0.25$

$y_1 = 1 + f(0, 1) * 0.25 = 1.25$

$i = 1$ $y_2 = y_1 + f(x_1, y_1) * 0.25$

$y_2 = 1.25 + f(0.25, 1.25) * 0.25$

$y_2 = 1.5976$

2) $f(x_i, y_i)' = 10xy$ اشتقينا بالنسبة لـ (x)

$f(x_1, y_1) = 10 * 0.25 * 1.25$

$= 3.125$

$E_a = \frac{3.125 * (0.25)^2}{2} = 0.09765$

$f(1.25)$

$x_0 = 0$ $x_1 = 0.25$ $x_2 = 0.5$ $x_3 = 0.75$ $x_4 = 1$ $x_5 = 1.25$

Scanned by CamScanner

No: Date:

2) Runge-Kutta method

→ 2^{nd} order
→ 3^{rd} "
→ 4^{th} "

⇒ Second order R.K

$$f(x_{i+1}) = \frac{f(x_i)h^0}{0!} + \frac{f'(x_i)h^1}{1!} + \frac{f''(x_i)h^2}{2!}$$

$\frac{dy}{dx} = f(x,y)$, General formula R.K 2^{nd} order

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

where $k_1 = f(x_i, y_i)$

$k_2 = f(x_i + P_1 h, y_i + q_{11} k_1 h)$

→ 4 unknowns a_1, a_2, P_1, q_{11}

$a_1 + a_2 = 1$

$a_2 P_1 = \frac{1}{2}$

$a_2 q_{11} = \frac{1}{2}$

1) Midpoint method assume $a_2 = 1$ accuracy (2)

$a_1 = 0$, $P_1 = \frac{1}{2}$, $q_{11} = \frac{1}{2}$

$$\boxed{y_{i+1} = y_i + k_2 h}$$

where $k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{1}{2} k_1 h)$

No: .... Date: ....

2) Heun's method $a_2 = \frac{1}{2}$

accuracy اقل (3)

$a_1 = \frac{1}{2}$, $P_1 = 1$, $q_{11} = 1$

$$\boxed{y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h}$$

where $k_1 = f(x_i, y_i)$

$k_2 = f(x_i + h, y_i + k_1 h)$

3) Ralston's method $a_2 = \frac{2}{3}$

more accuracy (1)

$a_1 = \frac{1}{3}$ $P_1 = \frac{3}{4}$, $q_{11} = \frac{3}{4}$

$$\boxed{y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h}$$

where $k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h)$

⇒ third order R.K

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h$$

where $k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}h k_1)$

$k_3 = f(x_i + h, y_i - k_1 h + 2k_2 h)$

⇒ Forth order R.K

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)h$$

where

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1)$

$k_3 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_2)$

$k_4 = f(x_i + h, y_i + h k_3)$

No: .... Date: ....

Ex use Mid point & Ralston's method to evaluate $y(0.5)$ where $h = 0.5$, $y(0) = 1$,

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5 \quad / \quad f(x_i, y_i) = -2x^3 + 12x^2 - 20x + 8.5$$

$y_0 = 1$ $\quad$ $y_1 = ??$

$x_0 = 0$ $\quad$ $x_1 = 0.5$

Mid $\Rightarrow$ $y_{i+1} = y_i + k_2 h$, $\quad k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_1)$

$\Rightarrow i = 0$ $\quad k_1 = f(x_0, y_0) = f(0, 1) = 8.5$

$k_2 = f(\underline{\underline{0.25}}, 1 + \frac{1}{2} \times 0.5 \times 8.5) = 4.21875$

بس نعوض هاد الرقم لأنو لا يوجد Y في
fun

$i = 0$

$y_1 = y_0 + k_2 h$

$y_1 = 1 + 4.21875 \times 0.5$

$y_1 = 3.109375$

Ralston's $\Rightarrow$ $y_{i+1} = y_i + (\frac{1}{3}k_1 + \frac{2}{3}k_2)h$

$k_1 = f(x_i, y_i)$

$k_2 = f(x_i + \frac{3}{4}h, y_i + \frac{3}{4}hk_1)$

$k_1 = f(0, 1) = 8.5$

$k_2 = f(0 + \frac{3}{4} \times \frac{1}{2}, 1 + \frac{3 \times 1}{4 \times 2} \times 8.5) \Rightarrow k_2 = f(0.375, 4.1875) = 2.582$

$y_1 = y_0 + (\frac{1}{3} \times 8.5 + \frac{2}{3} \times 2.582) \times 0.5$

$y_1 = 3.2773$

No: .... Date: ....

Example use R.K 4^{th} order to evaluate $y(0.5)$, where $h = 0.5$)

$y(0) = 1$ $\qquad \frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$

$$y_{i+1} = y_i + \frac{1}{6}\left(k_1 + 2(k_2 + k_3) + k_4\right) h$$

$k_1 = f(x_i, y_i) \Rightarrow \boxed{k_1 = 8.5}$

$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} k_1\right) \Rightarrow \boxed{k_2 = 4.22}$

$k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2} k_2\right) \Rightarrow \boxed{k_3 = 4.22}$

$k_4 = f(x_i + h, y_i + h k_3) \Rightarrow \boxed{k_4 = 1.25}$

$i = 0$

$$y_1 = 1 + \frac{1}{6}\left(8.5 + 2(4.22 + 4.22) + 1.25\right) 0.5$$

$y_1 = 3.2192$

No: .... Date: ....

** System of eq's

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2 \ldots\ldots y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots\ldots y_n)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2 \ldots\ldots y_n)$$

\# n of initial Conditions

$y_1(x_0) = y_1^0$

$y_2(x_0) = y_2^0$

$y_n(x_0) = y_n^0$

⟹ **Euler's**

$$y_{i+1}^1 = y_i^1 + f_1(x_i, y_i^1, y_i^2, \ldots\ldots y_i^n)\, h$$

$$y_{i+1}^2 = y_i^2 + f_2(x_i, y_i^1, y_i^2 \ldots\ldots y_i^n)\, h$$

$$\boxed{y_{i+1}^n = y_i^n + f_n(x_i, y_i^1, y_i^2 \ldots\ldots y_i^n)\, h}$$

E.x solve the following set of eq's by using Euler's method with $y(0) = 4$, $z(0) = 6$ to evaluate $y(1)$, $z(1)$, $h = 0.5$

لازم نتنبه — intial x_0

$$\frac{dy}{dx} = -0.5y \quad , \quad \frac{dz}{dy} = 4 - 0.3z - 0.1y$$

$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$
$z_0 = 6$	$z_1 =$	$z_2 =$
$y_0 = 4$	$y_1 =$	$y_2 =$

$$f_1(x_i, y_i, z_i) = -0.5y$$

$$f_2(x_i, y_i, z_i) = 4 - 0.3z - 0.1y$$

No: Date:

$y_{i+1} = y_i + f_1 h$ / $z_{i+1} = y_i + f_2 h$

$i=0$ $y_1 = 4 + f_1(0, 4, 6) \times 0.5$ / $z_1 = 6 + f_2(0, 4, 6) \times 0.5$

$y_1 = 3$ $z_1 = 6.9$

$i=1$

$y_2 = y_1 + f_1(x_1, y_1, z_1) h \Rightarrow y_2 = 3 + f_1(0.5, 3, 6.9) \times 0.5$

$y_2 = 3 + (-1.5) \times 0.5 = 2.25$

$z_2 = z_1 + f_2(x_1, y_1, z_1) h \Rightarrow z_2 = 6.9 + f_2(0.5, 3, 6.9) \times 0.5$.

$z_2 = 6.9 + 1.63 \times 0.5 = 7.715$

* $y_{i+1} = y_i + \frac{1}{6}\left(k_1^y + 2(k_2^y + k_3^y) + k_4^y\right) h$

* $z_{i+1} = z_i + \frac{1}{6}\left(k_1^z + 2(k_2^z + k_3^z) + k_4^z\right) h$

$k_1^y = f_1(x_i, y_i, z_i)$ $K_2 = f(x_i + \frac{h}{2}, y_i + \frac{h}{2} k_1)$

$k_1^z = f_2(x_i, y_i, z_i)$ $k_2^y = f_1(x_i + \frac{h}{2}, y_i + \frac{h}{2} k_1^y, z_i + \frac{h}{2} k_1^z)$

No: .... Date: ....

** Higher order ODE

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots\cdots + a_1 \frac{dy}{dx} + a_0 y = f(x)$$

with $n-1$ initial Condition

$y = Z_1$

$\frac{dy}{dx} = \frac{dZ_1}{dx} = Z_2$ where $\frac{dZ_1}{dx} = Z_2 = f_1(x, Z_1, Z_2, \ldots\ldots Z_n)$

$\frac{d^2y}{dx^2} = \frac{d^2 Z_1}{dx^2} = \frac{dZ_2}{dx} = Z_3$ $\qquad \frac{dZ_2}{dx} = Z_3 = f_2(x, Z_1, Z_2, \ldots\ldots Z_n)$

$\vdots$

$\frac{dZ_n}{dx} = Z_n = f_n(x, Z_1, Z_2, \ldots\ldots Z_n)$

Ex Use R.K 3^{rd} order with $y(0) = 1$, $y'(0) = 2$, $h = 0.5$ to find $y(0.5)$?

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}$$

$y' = z = 2$ $\qquad$ $z =$

$y = 1$ $\qquad$ $y =$

$x_0 = 0$ $\qquad$ $x_1 = 0.5$

Sol

let $\frac{dy}{dx} = z$, $\frac{dz}{dx} + 2z + y = e^{-x}$

No: .... Date: ....

$$\frac{dy}{dx} = f_1(x_i, y_i, z_i) = z \qquad y_{i+1} = y_i + \frac{1}{6}\left(k_1^y + 4k_2^y + k_3^y\right)h$$

$$\frac{dz}{dx} = f_2(x_i, y_i, z_i) = e^{-x} - 2z - y \qquad z_{i+1} = z_i + \frac{1}{6}\left(k_1^z + 4k_2^z + k_3^z\right)h$$

$y_1 = 1.60657$

$z_1 = y_1' = 0.51368$

The Hashemite University

Faculty of Engineering - Department of Mechanical Engineering

Numerical Analysis - Fall Semester . 2017/2018

First Test: Closed book, closed sheet

20

Student Name : **Student ID# :** **Section :**	**Instructor:** ○ **Eng. A.Bani yaseen** ○ **Dr. Sameer Al-Dahidi** ○ **Dr. Mustafa Jaradat**

Consider the following function for Q1, Q2

$$f(x) = xe^{-x} - 0.2$$

Q1) Using bisection method on the interval [1, 5].What is the root after 3rd iteration?

A) 1.5 B) 2.5 C) 3.5 D) 3.875 E) 2.875 F) 1.875

Q2) Using Newton-Raphson method with initial guess $x_0 = 0$, What is the root of the equation after the 2nd iteration?

A) 0.1117 B) 0.4499 C) 2.4526 D) 0.2554 E) 1.7813 F) 3.5500

Q3) The Taylor 3rd order approximation of $f(1.8)$ about x_i=0.1 is <u>4.845</u> is $f'''(x_i) = 1.72$ & $f^{(4)}(x_i) = 0.93$ What is the Taylor second approximation of $f(1.8)$ about x_i ?

A) 3.4366 B) 3.5241 C) 3.8775 D) 3.7548 E) 3.6208 F) 4.2152

Q4) Iteration started to find a root of a function; the 4th iteration obtained value of $x = 1.2$ with $\varepsilon_t = 23\%$,What is the value of the root for the previous iteration that gives $\varepsilon_t = 57\%$

A) 0.72597 B) 0.67013 C) 0.52484 D) 0.83766 E) 0.94935 F) 0.87241

Q5) The following system of equations is given by $y = -x^2 + x + 0.75$, $y + 5xy = x^2$, starting with (x_o, y_o)=(1,0.2), Using fixed point iteration method to find the solution after one iteration.

Use: $g_1(x,y) = \sqrt{x - y + 0.75}$, $g_2(x,y) = x^2/(1+5x)$

A) x=1.2042 y=0.1667 B) x=1.2042 y=0.2065 C) x=1.1180 y=0.1897 D) x=1.3229 y=0.2298 E) x=1.2449 y=0.2145 F) x=1.2449 y=0.1667

Q6) As the step size increases in the Taylor series expansion:

A) Round-off error increases B) Truncation error increases C) Round off error decreases

D) Truncation error decreases E) A&B F) C&D

G) B&C H) A&D I) None of the above

Q7) Using Gauss elimination method to solve the system of equation,:-

$$2x_1 + 4x_2 + 3 = -6$$
$$2x_1 + 2x_2 - 2x_3 = 4$$
$$x_1 - x_2 + 4x_3 = 8$$

The third row of the coefficient matrix at the end of the elimination process is:-

A) (0,0,-8) B) (0,0,16) C) (0,0,10) D) (0,0,-2) E) (0,0,6) F) (0,0,2)

Q8) In root finding of $f(x)$ using bracketing methods over the interval $[x_l, x_u]$ when $f(x_l) \times f(x_u) < 0$ then $f(x)$:

A) Has one root B) Has an odd number of roots C) Has no root

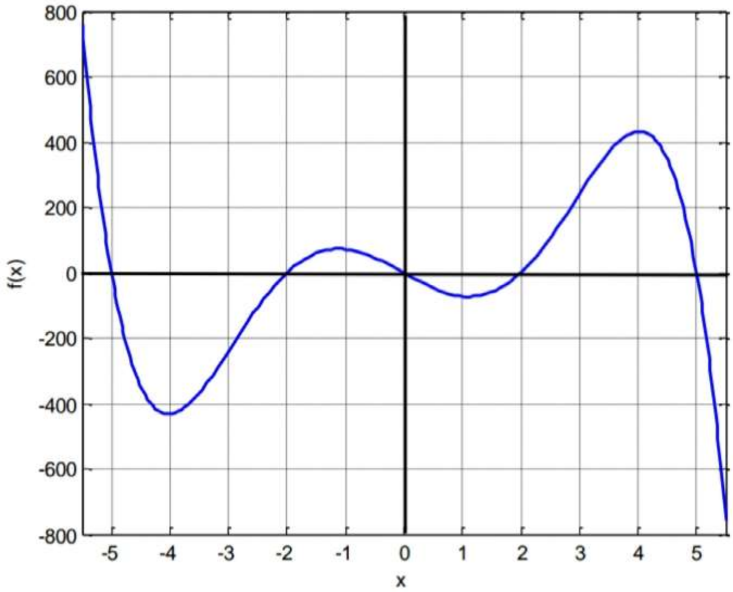
D) Has an Even number of roots E) A or D F) B or C

G) A or B H) B or D I) A or C

Q9) The secant method formula to find the square root of a real number of R^2 from $x^2 - R^2 = 0$ is :

A) $\frac{x_i x_{i-1}+R^2}{x_{i-1}+x_i}$ **B)** $\frac{-x_i^2-R^2}{x_{i-1}+x_i}$ **C)** $\frac{x_i^2+R^2}{x_{i-1}+x_i}$ **D)** $\frac{-x_i^2+R^2}{x_{i-1}+x_i}$ **E)** $\frac{x_i^2-R^2}{x_{i-1}+x_i}$ **F)** $\frac{x_i x_{i-1}-R^2}{x_{i-1}+x_i}$

Use the following figure to answer the question:-



Q10) If the false position is used with $x_l = -2.5$, $x_u = 4$ the method will converge to:

A) -3 **B) -2** **C) 0** **D) 2** **E) 3** **F) Diverge solution**

Useful Formulas

$$f(x_{i+1}) = \sum_{n=0}^{\infty} \frac{f^n(x_i)}{n!} h^n, \qquad R_n = \frac{f^{n+1}(x_i)}{(n+1)!} h^{n+1}, \quad x_r = \frac{x_l + x_u}{2}, \quad x_{i+1} = g(x_i)$$

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}, \quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

$$x_{i+1} = x_i - \frac{f(x_i)\delta x_i}{f(x_i + \delta x_i) - f(x_i)}, \qquad x_{i+1} = x_i - \frac{f(x_i) f'(x_i)}{[f'(x_i)]^2 - f(x_i) f''(x_i)}$$

$$n = \frac{\log(\Delta x / E_a)}{\log 2}, \quad x_{i+1} = x_i - \frac{u_i \frac{\partial v_i}{\partial y} - v_i \frac{\partial u_i}{\partial y}}{\frac{\partial u_i}{\partial x}\frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y}\frac{\partial v_i}{\partial x}}, \quad y_{i+1} = y_i - \frac{v_i \frac{\partial u_i}{\partial x} - u_i \frac{\partial v_i}{\partial x}}{\frac{\partial u_i}{\partial x}\frac{\partial v_i}{\partial y} - \frac{\partial u_i}{\partial y}\frac{\partial v_i}{\partial x}}$$

Good Luck

The Hashemite University

Faculty of Engineering - Department of Mechanical Engineering

Numerical Analysis - Fall Semester 2017/2018

Second Test: Closed book, closed sheet

20

Student Name :		**Instructor:**
Student ID# :		○ **Eng. A.Bani yaseen**
Section :		○ **Dr. Sameer Al-Dahidi**
		○ **Dr. Mustafa Jaradat**

Consider the following function for Q1, Q2

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$7x_2 + 3x_1 + 13x_3 = 76$$

Q1) Using Gauss Seidel iteration method with initial guess x_o=[1, 1 , 1]. The value of x_3 after first iteration will be:

A) 3.09 **B) 2.96** **C) 3.12** **D) 0.25** **E) 2.85** **F) 4.95**

Q2) Using Jacobi iteration method with initial guess x_o=[1, 1 , 1]. The value of x_3 after the first iteration will be :

A) 5.31 **B) 5.62** **C) 5.55** **D) 5.08** **E) 5.85** **F) 5.8**

The coefficient matrix A is decomposed into the following matrix $\begin{bmatrix} 4 & 1 & 4 \\ 0 & 2.5 & -2 \\ 0 & 0 & -0.3 \end{bmatrix}$ & $\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.9 & 1 \end{bmatrix}$

Q3) What is coefficient matrix element a_{33}?

A) 5.5 **B) 3.5** **C) 4.5** **D) 2.5** **E) 1.5** **F) 4.1**

Q4) What is the intermediate vector resulting from the calculation of the 1st column of inverse of matrix A?

A) $[1,-0.5,-0.95]^T$ **B) $[1,0.5,-0.7]^T$** **C) $[1,-0.5,-0.7]^T$** **D) $[1,-0.5,-1.2]^T$** **E) $[1,-0.5,0.95]^T$** **F) $[1,-0.5,-1.45]^T$**

Q5) Let $f(x) = \left(\frac{b\sqrt{x}}{a+\sqrt{x}}\right)^2$, The linearized form for the function $y = f(x)$ to used when using linear regression will be:

A) $\sqrt{\frac{1}{y}} = \frac{b}{a}\frac{1}{\sqrt{x}} + \frac{1}{b}$ B) $\frac{1}{y^2} = \frac{b}{a}\frac{1}{\sqrt{x}} + b$ C) $\sqrt{\frac{1}{y}} = \frac{a}{b}\frac{1}{\sqrt{x}} + \frac{1}{b}$

D) $\sqrt{y} = \frac{a}{b}\frac{1}{\sqrt{x}} + \frac{1}{b}$ E) $\sqrt{\frac{1}{y}} = \frac{b}{a}x^2 + b$ F) $y^2 = \frac{b}{a}\frac{1}{\sqrt{x}} + \frac{1}{b}$

Consider the following table for Q6, Q7

Given the Following data Points

x	1	1.5	2	2.5
y=f(x)	2.5	3.5	5.5	7.2

Q6) If a quadratic interpolation polynomial $f_2(x) = b_o + b_1(x - x_o) + b_2(x - x_o)(x - x_1)$ was used to determine *f(1.2)*. What is the value of b_2?

A) 2 **B) 1** **C) -3** **D) -1** **E) -2** **F) 3**

Q7) The lagrangian $L_3(1.2)$ for the third order polynomial equal to:

A) -0.0640 **B) 0.0640** **C) -0.0320** **D) 0.0480** **E) 0.0320** **F) -0.0480**

This paragraph applies to Q8-Q9.

Given the Following data Points

x	0.2	0.7	1.5	2.6	4
$y=f(x)$	0.7	0.48	0.23	0.065	0.012

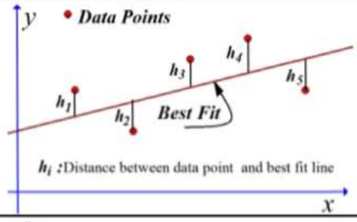
Q8) The following data is used to fit the function $y = \frac{1}{1+ae^{bx}}$, using the least squares method to find the values of coefficients a and b, respectively.

A) a= 0.4352 b= 1.3324
B) a= 0.4629 b= 1.3029
C) a= 0.3859 b= 1.3642
D) a= 1.3324 b= 0.4352
E) a= 0.5346 b= 1.2479
F) a= 1.3642 b= 0.3859

Q9) The correlation coefficient of the fit is:-

A) 0.9997 **B)** 0.9904 **C)** 0.9979 **D)** 0.9992 **E)** 0.9982 **F)** 0.9964

Q10) The figure aside shows the distance between data points and their best fit line, if h_1=1.5, h_2=2.5, h_3=2, h_4=1.5, h_5=4.



What is the standard error of estimate?

A) 2.1602 **B)** 3.2016 **C)** 2.9297
D) 1.7795 **E)** 3.7859 **F)** 3.0957

Q11) To ensure the following system of equation converge use Gauss-Seidel method.

$$x_1 + 3x_2 - 8x_3 = 9 \;\dots\dots\dots\; eq(1)$$
$$7x_1 + 3x_2 + 2x_3 = 16 \;\dots\dots.\; eq(2)$$
$$2x_1 + 4x_2 + x_3 = 5 \;\dots\dots.\; eq(3)$$

the equations has to be re-ordered as follows

A) *eq(1), eq(2),eq(3)* **B)** *eq(1), eq(3),eq(2)* **C)** *eq(2), eq(1),eq(3)*
D) *eq(2), eq(3),eq(1)* **E)** *eq(3), eq(2),eq(1)* **F)** *eq(3), eq(1),eq(2)*
G) System will never converge **H)** System will always converge

Useful Formulas

$$L_i(x) = \prod_{\substack{j=o \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \quad , f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$f_n(x) = b_o + b_1(x-x_o) + b_2(x-x_o)(x-x_1) + \cdots + b_n(x-x_o)\dots(x-x_{n-1})$$

$$b_0 = f(x_0) \;, b_1 = f[x_1, x_0] \;, b_2 = f[x_2, x_1, x_0]$$

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j} \;,\; f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

$$\begin{bmatrix} n & (\sum x) \\ (\sum x) & (\sum x^2) \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \end{bmatrix} = \begin{bmatrix} (\sum y) \\ (\sum xy) \end{bmatrix} \;,\; S_{y/x} = \sqrt{\frac{S_r}{n-2}} \;,\; S_y = \sqrt{\frac{S_t}{n-1}} \;, S_r = \sum (y_i - f(x_i))^2$$

$$r = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\left(\sqrt{n\sum x_i^2 - (\sum x_i)^2}\right)\left(\sqrt{n\sum y_i^2 - (\sum y_i)^2}\right)} \quad , \quad S_t = \sum (y_i - \bar{y})^2$$

Good Luck

Chapter 08.05
On Solving Higher Order Equations for Ordinary Differential Equations

After reading this chapter, you should be able to:

1. *solve higher order and coupled differential equations,*

We have learned Euler's and Runge-Kutta methods to solve first order ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y),\ y(0) = y_0 \tag{1}$$

What do we do to solve simultaneous (coupled) differential equations, or differential equations that are higher than first order? For example an n^{th} order differential equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1 \frac{dy}{dx} + a_o y = f(x) \tag{2}$$

with $n-1$ initial conditions can be solved by assuming

$$y = z_1 \tag{3.1}$$

$$\frac{dy}{dx} = \frac{dz_1}{dx} = z_2 \tag{3.2}$$

$$\frac{d^2 y}{dx^2} = \frac{dz_2}{dx} = z_3 \tag{3.3}$$

$$\vdots$$

$$\frac{d^{n-1}y}{dx^{n-1}} = \frac{dz_{n-1}}{dx} = z_n \tag{3.n}$$

$$\frac{d^n y}{dx^n} = \frac{dz_n}{dx}$$

$$= \frac{1}{a_n}\left(-a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} \ldots - a_1 \frac{dy}{dx} - a_0 y + f(x)\right)$$

$$= \frac{1}{a_n}\left(-a_{n-1}z_n \ldots - a_1 z_2 - a_0 z_1 + f(x)\right) \tag{3.n+1}$$

The above Equations from (3.1) to (3.n+1) represent n first order differential equations as follows

$$\frac{dz_1}{dx} = z_2 = f_1(z_1, z_2, \ldots, x) \tag{4.1}$$

$$\frac{dz_2}{dx} = z_3 = f_2(z_1, z_2, \ldots, x) \tag{4.2}$$

$$\vdots$$

$$\frac{dz_n}{dx} = \frac{1}{a_n}\left(-a_{n-1}z_n \ldots - a_1 z_2 - a_0 z_1 + f(x)\right) \tag{4.n}$$

Each of the n first order ordinary differential equations are accompanied by one initial condition. These first order ordinary differential equations are simultaneous in nature but can be solved by the methods used for solving first order ordinary differential equations that we have already learned.

Example 1

Rewrite the following differential equation as a set of first order differential equations.

$$3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x},\ y(0) = 5,\ y'(0) = 7$$

Solution

The ordinary differential equation would be rewritten as follows. Assume

$$\frac{dy}{dx} = z,$$

Then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

Substituting this in the given second order ordinary differential equation gives

$$3\frac{dz}{dx} + 2z + 5y = e^{-x}$$

$$\frac{dz}{dx} = \frac{1}{3}\left(e^{-x} - 2z - 5y\right)$$

The set of two simultaneous first order ordinary differential equations complete with the initial conditions then is

$$\frac{dy}{dx} = z,\ y(0) = 5$$

$$\frac{dz}{dx} = \frac{1}{3}\left(e^{-x} - 2z - 5y\right),\ z(0) = 7\,.$$

Now one can apply any of the numerical methods used for solving first order ordinary differential equations.

Example 2

Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t},\ y(0) = 1,\ \frac{dy}{dt}(0) = 2\text{, find by Euler's method}$$

a) $y(0.75)$

b) the absolute relative true error for part(a), if $y(0.75)\big|_{exact} = 1.668$

c) $\frac{dy}{dt}(0.75)$

Use a step size of $h = 0.25$.

Solution

First, the second order differential equation is written as two simultaneous first-order differential equations as follows. Assume

$$\frac{dy}{dt} = z$$

then

$$\frac{dz}{dt} + 2z + y = e^{-t}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y$$

So the two simultaneous first order differential equations are

$$\frac{dy}{dt} = z = f_1(t,y,z),\ y(0) = 1 \tag{E2.1}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y = f_2(t,y,z),\ \ z(0) = 2 \tag{E2.2}$$

Using Euler's method on Equations (E2.1) and (E2.2), we get

$$y_{i+1} = y_i + f_1(t_i, y_i, z_i)h \tag{E2.3}$$

$$z_{i+1} = z_i + f_2(t_i, y_i, z_i)h \tag{E2.4}$$

a) To find the value of $y(0.75)$ and since we are using a step size of 0.25 and starting at $t = 0$, we need to take three steps to find the value of $y(0.75)$.

For $i = 0, t_0 = 0,\ y_0 = 1,\ z_0 = 2$,

From Equation (E2.3)

$$\begin{aligned} y_1 &= y_0 + f_1(t_0, y_0, z_0)h \\ &= 1 + f_1(0,1,2)(0.25) \\ &= 1 + 2(0.25) \\ &= 1.5 \end{aligned}$$

y_1 is the approximate value of y at

$$t = t_1 = t_0 + h = 0 + 0.25 = 0.25$$

$$y_1 = y(0.25) \approx 1.5$$

From Equation (E2.4)

$$\begin{aligned} z_1 &= z_0 + f_2(t_0, y_0, z_0)h \\ &= 2 + f_2(0,1,2)(0.25) \\ &= 2 + \left(e^{-0} - 2(2) - 1\right)(0.25) \\ &= 1 \end{aligned}$$

z_1 is the approximate value of z (same as $\frac{dy}{dt}$) at $t = 0.25$

$$z_1 = z(0.25) \approx 1$$

For $i = 1,\ t_1 = 0.25,\ y_1 = 1.5,\ z_1 = 1$,

From Equation (E2.3)

$$\begin{aligned} y_2 &= y_1 + f_1(t_1, y_1, z_1)h \\ &= 1.5 + f_1(0.25, 1.5, 1)(0.25) \\ &= 1.5 + (1)(0.25) \\ &= 1.75 \end{aligned}$$

y_2 is the approximate value of y at

$$t = t_2 = t_1 + h = 0.25 + 0.25 = 0.50$$

$$y_2 = y(0.5) \approx 1.75$$

From Equation (E2.4)

$$\begin{aligned} z_2 &= z_1 + f_2(t_1, y_1, z_1)h \\ &= 1 + f_2(0.25, 1.5, 1)(0.25) \\ &= 1 + \left(e^{-0.25} - 2(1) - 1.5\right)(0.25) \\ &= 1 + (-2.7211)(0.25) \\ &= 0.31970 \end{aligned}$$

z_2 is the approximate value of z at

$$t = t_2 = 0.5$$

$$z_2 = z(0.5) \approx 0.31970$$

For $i = 2,\ t_2 = 0.5,\ y_2 = 1.75,\ z_2 = 0.31970$,

From Equation (E2.3)

$$\begin{aligned} y_3 &= y_2 + f_1(t_2, y_2, z_2)h \\ &= 1.75 + f_1(0.50, 1.75, 0.31970)(0.25) \\ &= 1.75 + (0.31970)(0.25) \\ &= 1.8299 \end{aligned}$$

y_3 is the approximate value of y at

$$t = t_3 = t_2 + h = 0.5 + 0.25 = 0.75$$

$$y_3 = y(0.75) \approx 1.8299$$

From Equation (E2.4)

$$\begin{aligned} z_3 &= z_2 + f_2(t_2, y_2, z_2)h \\ &= 0.31972 + f_2(0.50, 1.75, 0.31970)(0.25) \\ &= 0.31972 + \left(e^{-0.50} - 2(0.31970) - 1.75\right)(0.25) \\ &= 0.31972 + (-1.7829)(0.25) \\ &= -0.1260 \end{aligned}$$

z_3 is the approximate value of z at

$$t = t_3 = 0.75$$

$$z_3 = z(0.75) \approx -0.12601$$

$$y(0.75) \approx y_3 = 1.8299$$

b) The exact value of $y(0.75)$ is

$$y(0.75)\big|_{exact} = 1.668$$

The absolute relative true error in the result from part (a) is

$$|\in_t| = \left|\frac{1.668 - 1.8299}{1.668}\right| \times 100$$
$$= 9.7062\%$$

c) $\dfrac{dy}{dx}(0.75) = z_3 \approx -0.12601$

Example 3

Given

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{-t}, y(0) = 1, \frac{dy}{dt}(0) = 2,$$

find by Heun's method

a) $y(0.75)$

b) $\dfrac{dy}{dx}(0.75)$.

Use a step size of $h = 0.25$.

Solution

First, the second order differential equation is rewritten as two simultaneous first-order differential equations as follows. Assume

$$\frac{dy}{dt} = z$$

then

$$\frac{dz}{dt} + 2z + y = e^{-t}$$
$$\frac{dz}{dt} = e^{-t} - 2z - y$$

So the two simultaneous first order differential equations are

$$\frac{dy}{dt} = z = f_1(t,y,z), y(0) = 1 \tag{E3.1}$$

$$\frac{dz}{dt} = e^{-t} - 2z - y = f_2(t,y,z), z(0) = 2 \tag{E3.2}$$

Using Heun's method on Equations (1) and (2), we get

$$y_{i+1} = y_i + \frac{1}{2}\left(k_1^y + k_2^y\right)h \tag{E3.3}$$

$$k_1^y = f_1\left(t_i, y_i, z_i\right) \tag{E3.4a}$$

$$k_2^y = f_1\left(t_i + h, y_i + hk_1^y, z_i + hk_1^z\right) \tag{E 3.4b}$$

$$z_{i+1} = z_i + \frac{1}{2}\left(k_1^z + k_2^z\right)h \tag{E3.5}$$

$$k_1^z = f_2\left(t_i, y_i, z_i\right) \quad \text{(E3.6a)}$$
$$k_2^z = f_2\left(t_i + h, y_i + hk_1^y, z_i + hk_1^z\right) \quad \text{(E3.6b)}$$

For $i = 0,\ t_o = 0,\ y_o = 1,\ z_o = 2$

From Equation (E3.4a)

$$\begin{aligned}k_1^y &= f_1\left(t_o, y_o, z_o\right)\\ &= f_1\left(0,1,2\right)\\ &= 2\end{aligned}$$

From Equation (E3.6a)

$$\begin{aligned}k_1^z &= f_2\left(t_0, y_0, z_0\right)\\ &= f_2\left(0,1,2\right)\\ &= e^{-0} - 2\left(2\right) - 1\\ &= \text{-}4\end{aligned}$$

From Equation (E3.4b)

$$\begin{aligned}k_2^y &= f_1\left(t_0 + h, y_0 + hk_1^y, z_0 + hk_1^z\right)\\ &= f_1\left(0 + 0.25, 1 + \left(0.25\right)\left(2\right), 2 + \left(0.25\right)\left(-4\right)\right)\\ &= f_1\left(0.25, 1.5, 1\right)\\ &= 1\end{aligned}$$

From Equation (E3.6b)

$$\begin{aligned}k_2^z &= f_2\left(t_0 + h, y_0 + hk_1^y, z_0 + hk_1^z\right)\\ &= f_2\left(0 + 0.25, 1 + \left(0.25\right)\left(2\right), 2 + \left(0.25\right)\left(-4\right)\right)\\ &= f_2\left(0.25, 1.5, 1\right)\\ &= e^{-0.25} - 2\left(1\right) - 1.5\\ &= -2.7212\end{aligned}$$

From Equation (E3.3)

$$\begin{aligned}y_1 &= y_0 + \frac{1}{2}\left(k_1^y + k_2^y\right)h\\ &= 1 + \frac{1}{2}\left(2 + 1\right)\left(0.25\right)\\ &= 1.375\end{aligned}$$

y_1 is the approximate value of y at

$$t = t_1 = t_0 + h = 0 + 0.25 = 0.25$$
$$y_1 = y\left(0.25\right) \cong 1.375$$

From Equation (E3.5)

$$\begin{aligned}z_1 &= z_0 + \frac{1}{2}\left(k_1^z + k_2^z\right)h\\ &= 2 + \frac{1}{2}(-4 + (-2.7212))(0.25)\\ &= 1.1598\end{aligned}$$

z_1 is the approximate value of z at

$$t = t_1 = 0.25$$

$$z_1 = z(0.25) \approx 1.1598$$

For $i=1,\ t_1 = 0.25,\ y_1 = 1.375,\ z_1 = 1.1598$

From Equation (E3.4a)

$$\begin{aligned} k_1^y &= f_1(t_1, y_1, z_1) \\ &= f_1(0.25, 1.375, 1.1598) \\ &= 1.1598 \end{aligned}$$

From Equation (E3.6a)

$$\begin{aligned} k_1^z &= f_2(t_1, y_1, z_1) \\ &= f_2(0.25, 1.375, 1.1598) \\ &= e^{-0.25} - 2(1.1598) - 1.375 \\ &= -2.9158 \end{aligned}$$

From Equation (E3.4b)

$$\begin{aligned} k_2^y &= f_1(t_1 + h, y_1 + hk_1^y, z_1 + hk_1^z) \\ &= f_1(0.25 + 0.25, 1.375 + (0.25)(1.1598), 1.1598 + (0.25)(-2.9158)) \\ &= f_1(0.50, 1.6649, 0.43087) \\ &= 0.43087 \end{aligned}$$

From Equation (E3.6b)

$$\begin{aligned} k_2^z &= f_2(t_1 + h, y_1 + hk_1^y, z_1 + hk_1^z) \\ &= f_2(0.25 + 0.25, 1.375 + (0.25)(1.1598), 1.1598 + (0.25)(-2.9158)) \\ &= f_2(0.50, 1.6649, 0.43087) \\ &= e^{-0.50} - 2(0.43087) - 1.6649 \\ &= -1.9201 \end{aligned}$$

From Equation (E3.3)

$$\begin{aligned} y_2 &= y_1 + \frac{1}{2}(k_1^y + k_2^y)h \\ &= 1.375 + \frac{1}{2}(1.1598 + 0.43087)(0.25) \\ &= 1.5738 \end{aligned}$$

y_2 is the approximate value of y at

$$t = t_2 = t_1 + h = 0.25 + 0.25 = 0.50$$

$$y_2 = y(0.50) \approx 1.5738$$

From Equation (E3.5)

$$\begin{aligned} z_2 &= z_1 + \frac{1}{2}(k_1^z + k_2^z)h \\ &= 1.1598 + \frac{1}{2}(-2.9158 + (-1.9201))(0.25) \\ &= 0.55533 \end{aligned}$$

z_2 is the approximate value of z at

$$t = t_2 = 0.50$$

$$z_2 = z(0.50) \approx 0.55533$$

For $i = 2,\ t_2 = 0.50,\ y_2 = 1.57384,\ z_2 = 0.55533$

From Equation (E3.4a)

$$\begin{aligned} k_1^y &= f_1(t_2, y_2, z_2) \\ &= f_1(0.50, 1.5738, 0.55533) \\ &= 0.55533 \end{aligned}$$

From Equation (E3.6a)

$$\begin{aligned} k_1^z &= f_2(t_2, y_2, z_2) \\ &= f_2(0.50, 1.5738, 0.55533) \\ &= e^{-0.50} - 2(0.55533) - 1.5738 \\ &= -2.0779 \end{aligned}$$

From Equation (E3.4b)

$$\begin{aligned} k_2^y &= f_2\left(t_2 + h, y_2 + hk_1^y, z_2 + hk_1^z\right) \\ &= f_1(0.50 + 0.25, 1.5738 + (0.25)(0.55533), 0.55533 + (0.25)(-2.0779)) \\ &= f_1(0.75, 1.7126, 0.035836) \\ &= 0.035836 \end{aligned}$$

From Equation (E3.6b)

$$\begin{aligned} k_2^z &= f_2\left(t_2 + h, y_2 + hk_1^y, z_2 + hk_1^z\right) \\ &= f_2(0.50 + 0.25, 1.5738 + (0.25)(0.55533), 0.55533 + (0.25)(-2.0779)) \\ &= f_2(0.75, 1.7126, 0.035836) \\ &= e^{-0.75} - 2(0.035836) - 1.7126 \\ &= -1.3119 \end{aligned}$$

From Equation (E3.3)

$$\begin{aligned} y_3 &= y_2 + \frac{1}{2}\left(k_1^y + k_2^y\right)h \\ &= 1.5738 + \frac{1}{2}(0.55533 + 0.035836)(0.25) \\ &= 1.6477 \end{aligned}$$

y_3 is the approximate value of y at

$$t = t_3 = t_2 + h = 0.50 + 0.25 = 0.75$$

$$y_3 = y(0.75) \approx 1.6477$$

b) From Equation (E3.5)

$$\begin{aligned} z_3 &= z_2 + \frac{1}{2}\left(k_1^z + k_2^z\right)h \\ &= 0.55533 + \frac{1}{2}(-2.0779 + (-1.3119))(0.25) \\ &= 0.13158 \end{aligned}$$

z_3 is the approximate value of z at

$$t = t_3 = 0.75$$

$$z_3 = z(0.75) \cong 0.13158$$

The intermediate and the final results are shown in Table 1.

Table 1 Intermediate results of Heun's method.

i	0	1	2
t_i	0	0.25	0.50
y_i	1	1.3750	1.5738
z_i	2	1.1598	0.55533
k_1^y	2	1.1598	0.55533
k_1^z	-4	-2.9158	-2.0779
k_2^y	1	0.43087	0.035836
k_2^z	-2.7211	-1.9201	-1.3119
y_{i+1}	1.3750	1.5738	1.6477
z_{i+1}	1.1598	0.55533	0.13158