ElCoM
ELECTRICAL COMPUTER MECHATRONICS

تقدم لجنة ElCoM الاكاديمية

دفتر لمادة:

تفاضل و تكامل (3)

من شرح:

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جزيل الشكر للطالبة:

سهى حنيحن

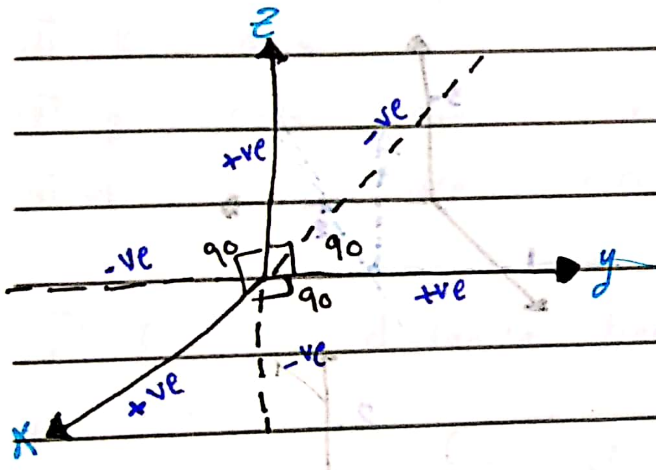
Subject CH 12 Day Date

Vector and Geomtry of space.

Sec 12.1

3- dimensional coordinate system

Def: The coordinate system in (x, y, z) "called the cartesian coordinate system" consistes of 3 perpendicular axes intersected in a point (pt) denoted by "O" called the origin.



1. 3- space.
2. 3- coordinate system
3. 3D

A pt in 3-space is written A(a, b, c).

a : x - axis (coordinate).

b : y - axis "

c : z - axis "

perpendicular متعامد.

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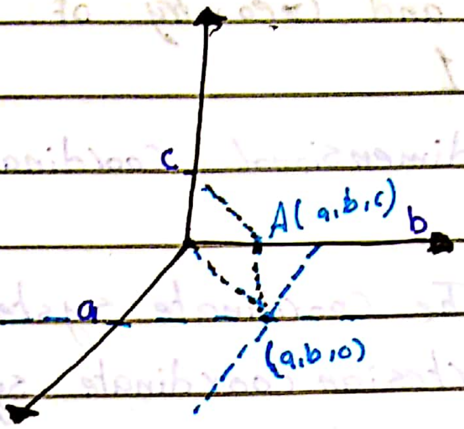
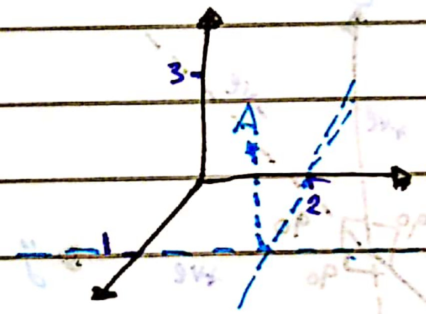
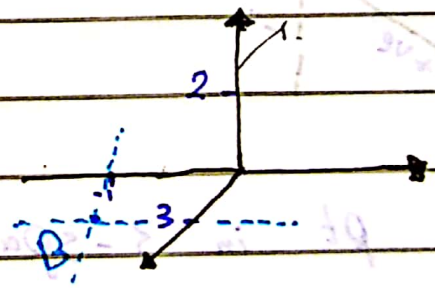


Ex:- In the space sketch the graph of the following pts.

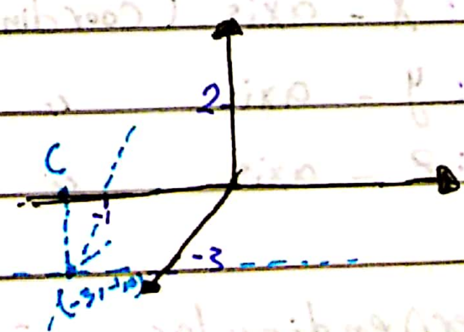
[1] A (1,2,3)



[2] B (3,-1,0)



[3] C (3,-1,2)



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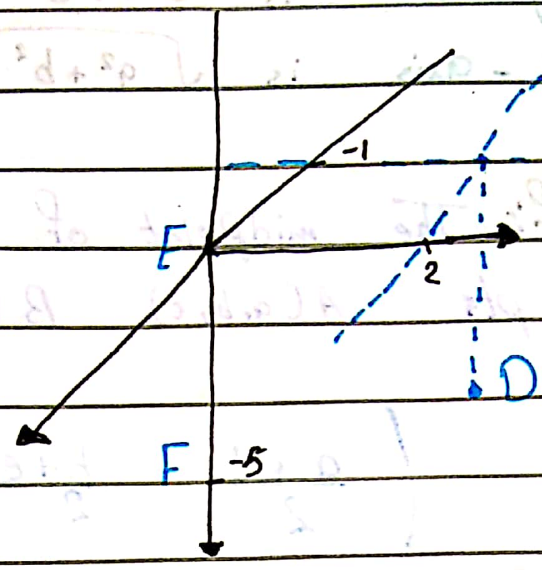
D $(-1, 2, -5)$
E $(0, 0, 0)$
F $(0, 0, -5)$

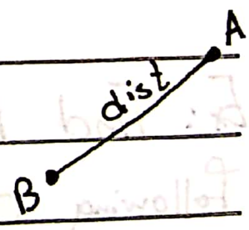


Remark: The points on the:

1. x - axis are $(x, 0, 0)$
2. y - axis are $(0, y, 0)$
3. z - axis are $(0, 0, z)$

Def: The distance between $A(a, b, c)$ and $B(d, e, f)$

$$\text{dist}(A, B) = \sqrt{(a-d)^2 + (b-e)^2 + (c-f)^2}$$



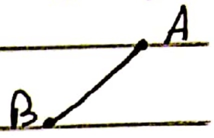
Rule: The distance between the pt $A(a, b, c)$ and the:

1. xy - plane is $|c|$
2. xz - " " $|b|$
3. yz - " " $|a|$
4. x - axis " $\sqrt{b^2 + c^2}$

5. y-axis is $\sqrt{a^2 + c^2}$

6. z-axis is $\sqrt{a^2 + b^2}$

Def: The midpoint of the Line segement joining the pts $A(a,b,c)$ $B(d,e,f)$ is



$$\left(\frac{a+d}{2}, \frac{b+e}{2}, \frac{c+f}{2}\right)$$

Def: The equation of the sphere Center at $A(a,b,c)$ of radius r is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

في حال r [1] موجب ← كرة   المعاملات متساوية

[2] صفر ← نقطة   في نفس الجهة

Ex: Find the eq of the sphere in each of the following Cases:-

1. centerd at $A(2,1,0)$ of radius 2

$$(x-2)^2 + (y-1)^2 + z^2 = 4$$

2. " " $B(-4,7,9)$ and tangent to yz-plane.

$$r = \text{dis}(B, yz\text{-plane}) = |a| = 4$$

$$(x+4)^2 + (y-7)^2 + (z-9)^2 = 16$$

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$\boxed{3}$ " " $B(-4,7,9)$ and tangent to z-axis

$r = \text{dist}(B, z\text{-axis}) = \sqrt{(-4)^2 + (7)^2} = \sqrt{65}$

eq: $(x+4)^2 + (y-7)^2 + (z-9)^2 = 65$.

$\boxed{4}$ " " $B(-4,7,9)$ and passing through the pt $C(-7,1,6)$ ~~D(-1,5,1)~~ ~~E(1,-1,4)~~

$r = \text{dist}(B,C) = \sqrt{(-4-7)^2 + (7-1)^2 + (9-6)^2} = \sqrt{54}$

eq $\Rightarrow$ $(x+4)^2 + (y-7)^2 + (z-9)^2 = 54$

$\boxed{5}$ one of its diameters has end pts

$D(-1,5,1)$ $E(1,-1,4)$.

Center = midpt (D,E)

$= \left(\frac{-1+1}{2}, \frac{5+-1}{2}, \frac{1+4}{2}\right) = \left(0, 2, \frac{5}{2}\right)$

$r = \frac{1}{2} \text{dist}(D,E) = \frac{1}{2}\sqrt{(-1-1)^2 + (5--1)^2 + (1-4)^2}$

$= \frac{7}{2}$ $\Rightarrow$ eq $\Rightarrow$ $x^2 + (y-2)^2 + (z-\frac{5}{2})^2 = \frac{49}{4}$

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Ex :- which of the following represent an eq of a sphere and find the center and radius of a sphere.

① $x^2 - 3x - 2y^2 + z^2 = 4$ No

معامل $y^2 \neq$ معامل x^2

② $2x^2 + 12x + 2y^2 = -70 - 2y + z$ No

② y معامل $\neq$ z^2 معامل

③ $2x^2 + 12x + 2y^2 = -100 - 2z^2 + 6y$

$2x^2 + 12x + 2y^2 + 2z^2 - 6y = -100$ $\boxed{\div 2}$

$x^2 + 6x + y^2 + z^2 - 3y = -50$ اكمال مربع

$x^2 + 6x + 9 + y^2 - 3y + \frac{9}{4} + z^2 = -50 + 9 + \frac{9}{4}$

$(x+3)^2 + (y - \frac{3}{2})^2 + z^2 = \frac{-155}{4}$ No ~~eq~~ sphere.

سالب

④ $2x^2 + 12x + 2y^2 = -2z^2 + 6y$

$2x^2 + 12x + 2y^2 + 2z^2 - 6y = 0$

$x^2 + 6x + y^2 + z^2 - 3y = 0$

$(x+3)^2 + (y - \frac{3}{2})^2 + z^2 = 9 + \frac{9}{4} = \frac{45}{4}$

Sphere $\Rightarrow$ Center $(-3, 3/2, 0)$

radius $\sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2}$

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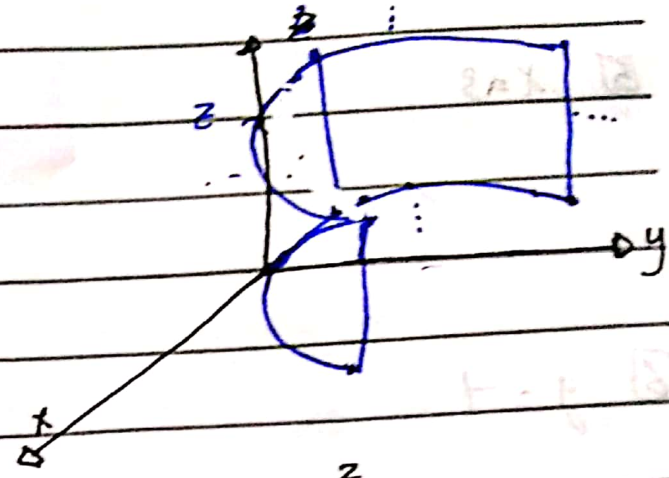
Subject

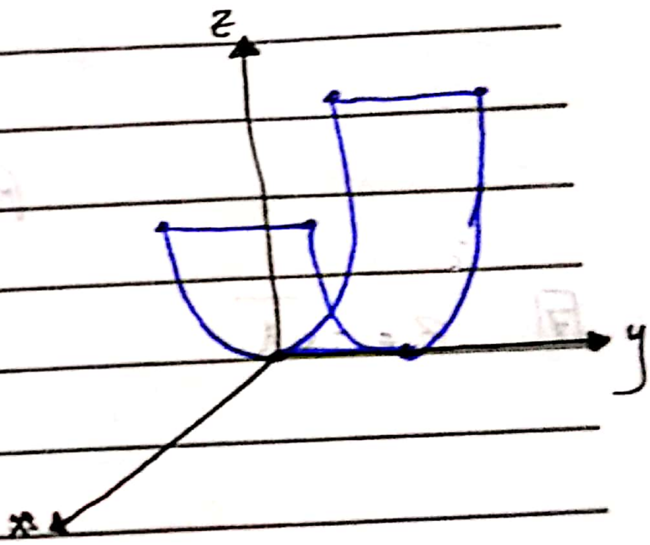
Def:- A cylindrical surface is a surface obtained by moving a curve along a fixed axis

sketch the surface of each eq:

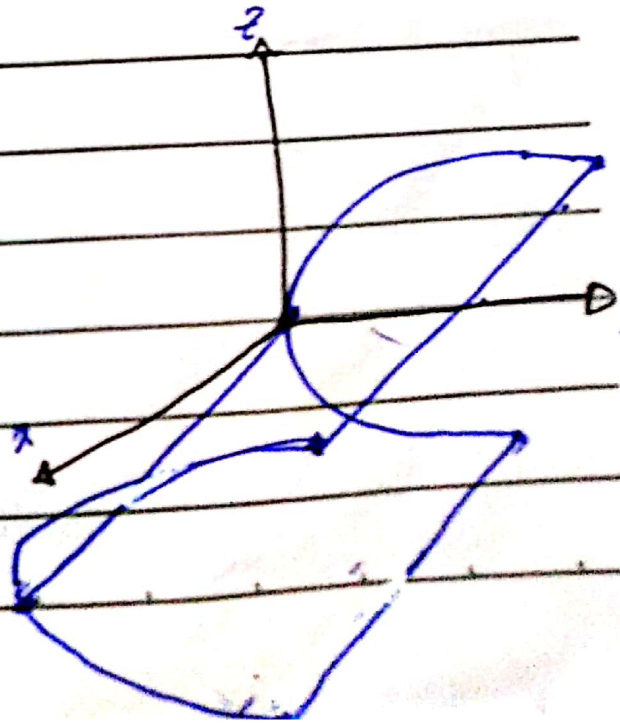
1. $y = x^2$



2. $z = x^2$



3. $y = z^2$



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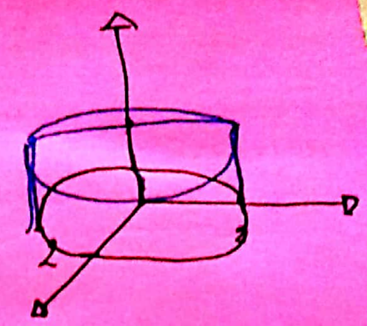
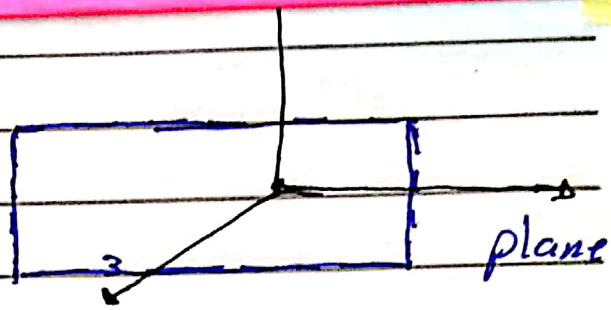
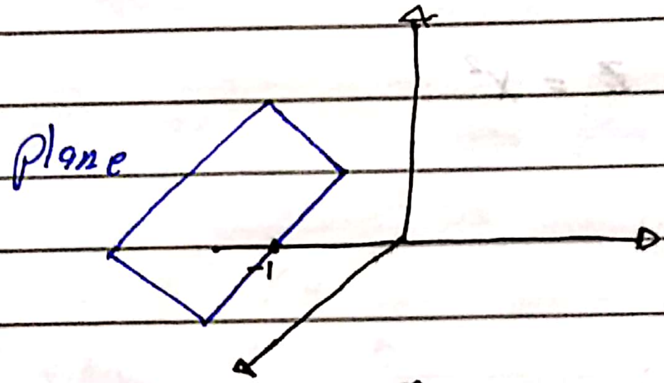
$\boxed{4}$ $x^2+y^2=4$



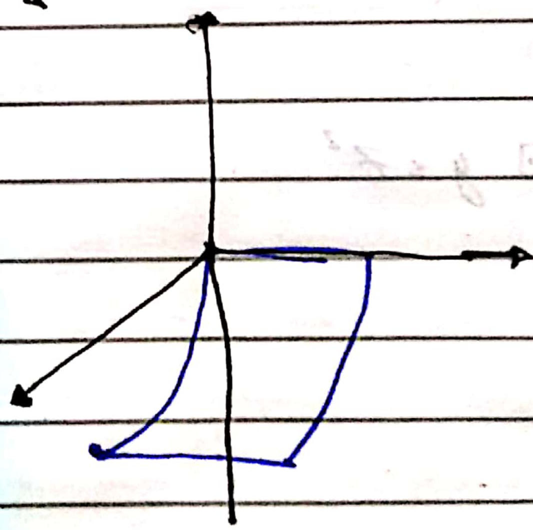
$\boxed{5}$ $x=3$



$\boxed{6}$ $y=-1$



$\boxed{7}$ $z=-\sqrt{x}$



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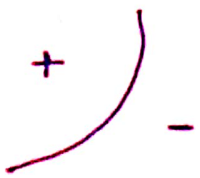

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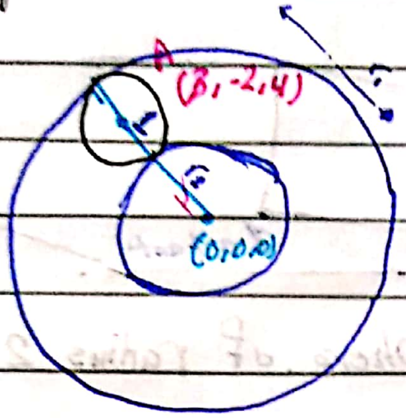
Exercise : Find the eq of the two spheres that are centered at the origin and are tangent to the sphere of radius 1 and centered at A(3, -2, 4)



$$\text{dist}(A, 0) = \sqrt{(3)^2 + (-2)^2 + (4)^2} = \sqrt{29}$$

$r_1 = \sqrt{29} + 1$ $\qquad$ $r_2 = \sqrt{29} - 1$

eq : $x^2 + y^2 + z^2 = (\sqrt{29} + 1)^2$

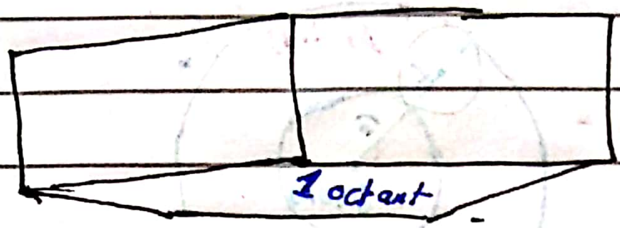
$x^2 + y^2 + z^2 = (\sqrt{29} - 1)^2$

Def:- The coordinate axes x-axis, y-axis, z-axis

Def:- " " planes are xy-plane, yz plane, xz-plane.

Remark:- The " planes divides (يقسم) the 3 space into 8 parts

First octant (ثُمن) Consists of the ~~positive~~ positive. Coordinate axes



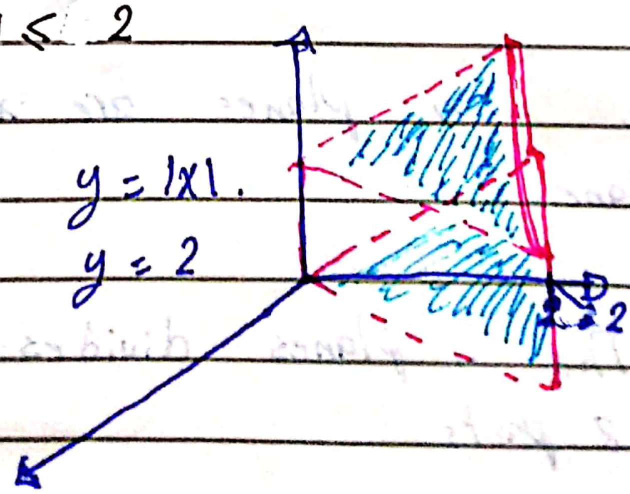
Ex:- Find the eq of the sphere of radius 2 and touches the coordinate planes.

Sol:- Center $(2,2,2)$, $(2,-2,2)$ $(-2,2,2)$, $(2,2,-2)$ $(-2,-2,2)$, $(2,-2,-2)$, $(-2,2,-2)$ $(-2,-2,-2)$

وتكتب المعادلات

Exercies:- Find the eq of the sphere in the first octant of radius 2 and touches the coordinate planes. $(x-2)^2+(y-2)^2+(z-2)^2=4$

Ex:- Sketch the region in the space satisfying $|x| < y \leq 2$



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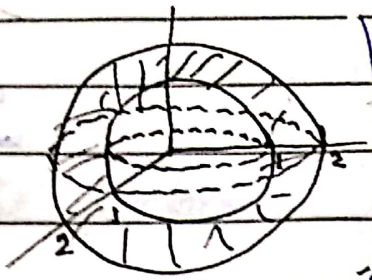
$\boxed{2}$ $1 \le x^2+y^2+z^2 \le 4$

$x^2+y^2+z^2=4$



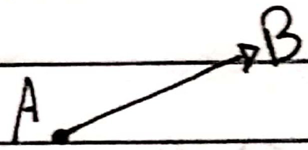
Sec 12.2 Vector

Def:- A vector written as $\vec{u}, \vec{v}, \vec{w}, \vec{x}, \vec{a}, \dots$ is a quantity that has

$\boxed{1}$ magnitued (Length) written $|\vec{a}|$

$\boxed{2}$ direction.

Def:- A vector $\vec{v}$ with initial pt A and terminal pt B written as $\vec{v} = \overrightarrow{AB}$



$$|\vec{v}| = \text{dist}(A, B).$$

Def:- Two vector $\vec{u}, \vec{v}$ are equal written $(\vec{u} = \vec{v})$ if they have the same dierction and Length.

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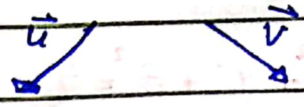
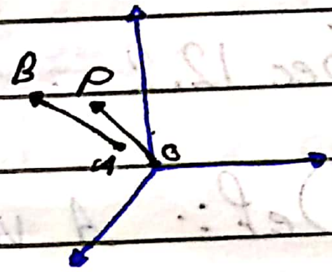
$\vec{u} = \vec{v}$

$\vec{u} \neq \vec{v}$

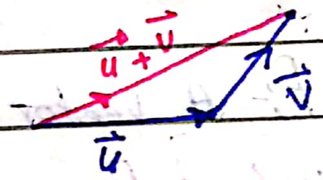


Def:- If $\vec{v} = \overrightarrow{AB}$
the position vector of $\vec{v}$ is $\vec{v} = \overrightarrow{OP}$



Def: The sum of $\vec{u}, \vec{v}$ is a vector $\vec{u}+\vec{v}$ defined by

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$



Def: Zero vector $\vec{O}$ vector of Length O in any dierction

Zero vector $\vec{O}$ is a vector with the same initial and terminal بدايته = نهايته

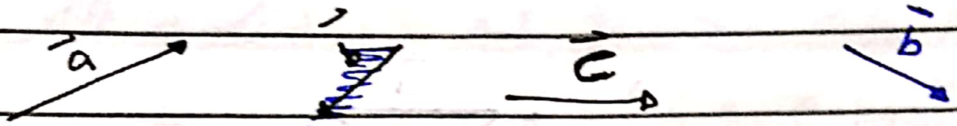
$$|\vec{O}| = \overrightarrow{AA} = \overrightarrow{BB}$$

Ex: write the vector in a simple form

$\vec{AB} + \vec{BA} = \vec{AA} = \vec{O}$

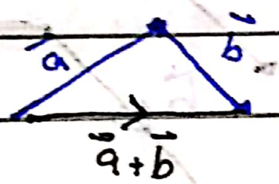
Ex:- sketch the vectors

$\vec{a} + \vec{b}$, $\vec{a} + \vec{b} + \vec{c}$

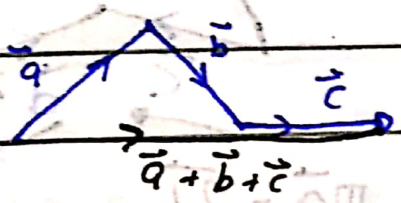


Sol:-

$\boxed{1}$



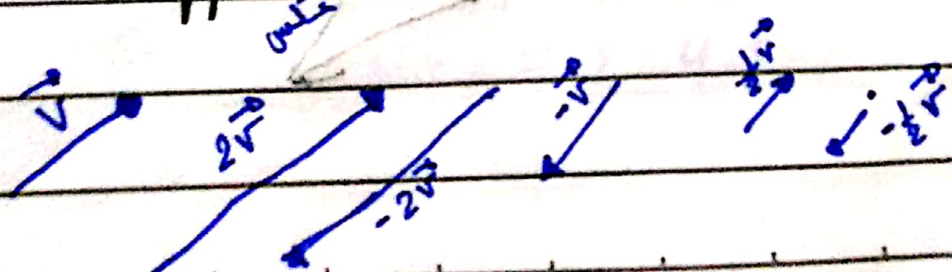
$\boxed{2}$



Def:- Let c be a scaler , $\vec{v}$ $c\vec{v}$ is the vector of Length $|c|\,|\vec{v}|$ and with

$\boxed{1}$ the same dierction of $\vec{v}$ if $c > 0$

$\boxed{2}$ " opposite " " " " $c < 0$



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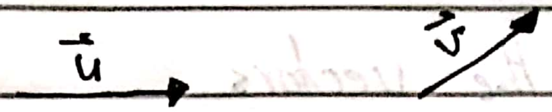
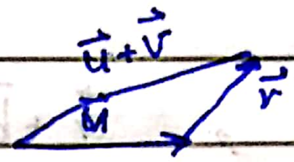
Remark: ① $\overrightarrow{AB} \Rightarrow -\overrightarrow{AB} = \overrightarrow{BA}$

② $\vec{u} - \vec{v} = \vec{u} + -\vec{v}$

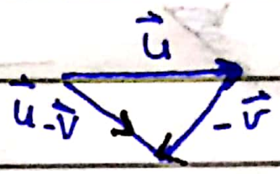
Remark



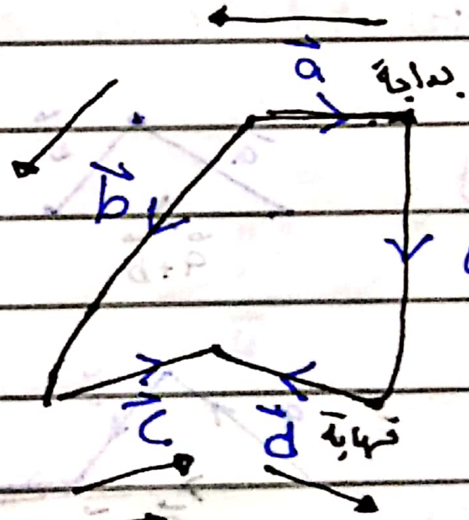
$\vec{u} + \vec{v}$



$\vec{u} - \vec{v}$

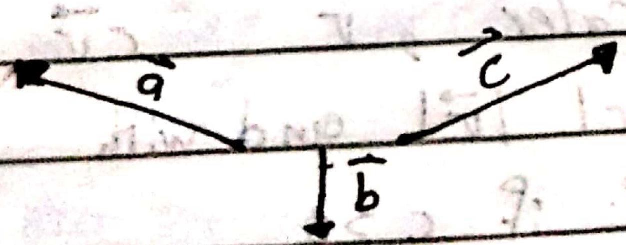


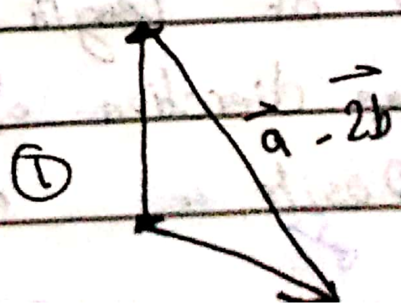
$\vec{e} = -\vec{a} + \vec{b} + \vec{c} - \vec{d}$



Ex:- draw the vectors ① $\vec{a} - 2\vec{b}$

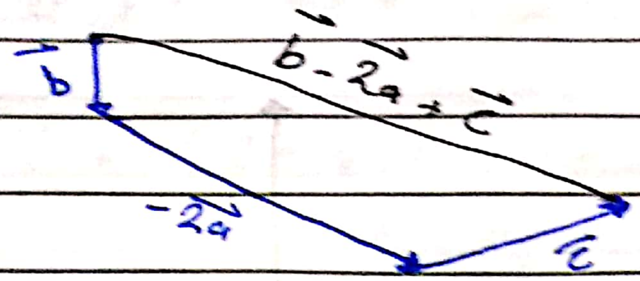
② $\vec{b} - 2\vec{a} + \vec{c}$





①

$\boxed{2}$

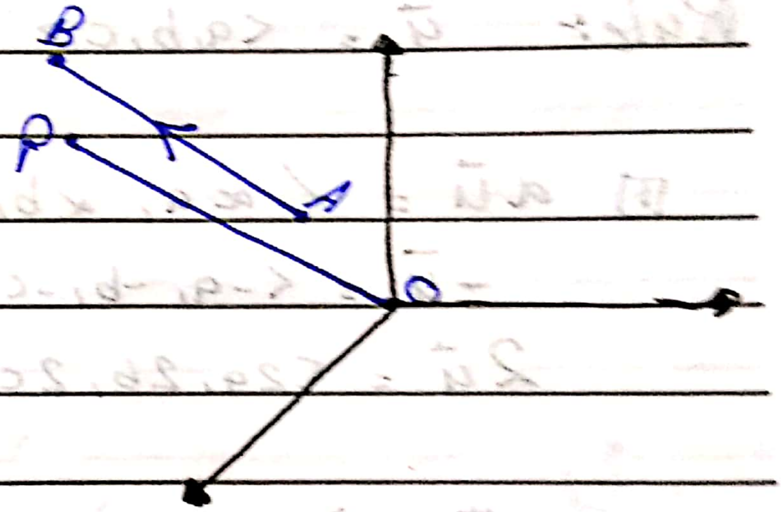


Def:- Let $A(a,b,c)$, $B(d,e,f)$. A Component (مركبي) form of the vector $\vec{v}=\overrightarrow{AB}$ is $\overrightarrow{AB} = \langle d-a, e-b, f-c \rangle$ $\langle B-A \rangle$.

$= OP$, $P=(d-a, e-b, f-c)$.

positive vector.

$$|\vec{v}| = \sqrt{(d-a)^2+(e-b)^2+(f-c)^2}$$



Ex:-

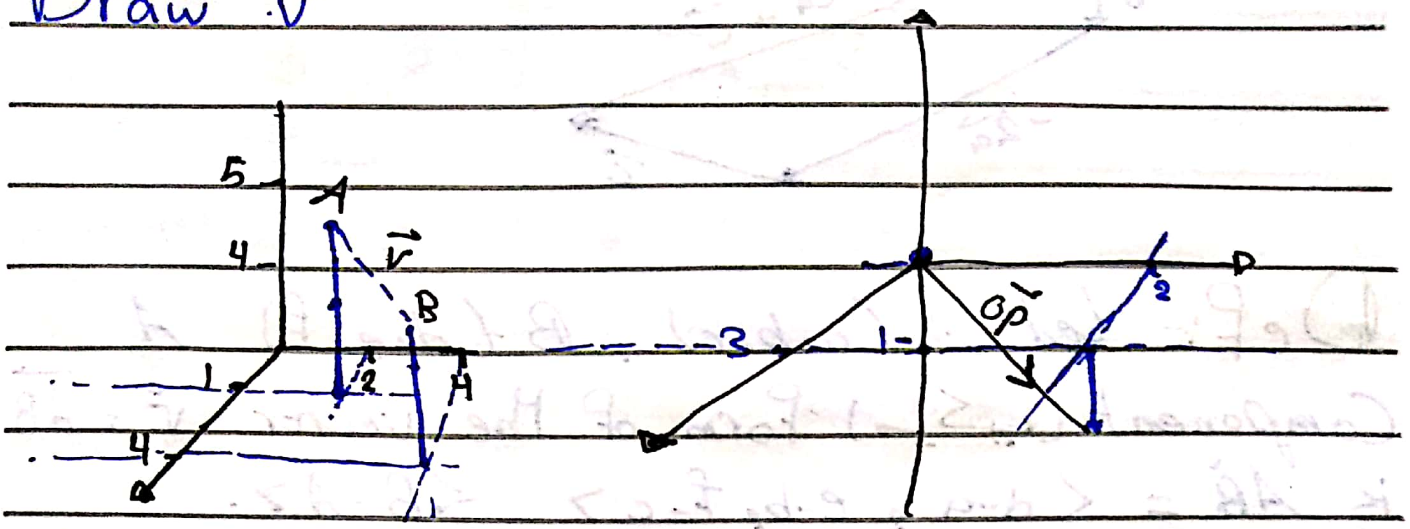
1) $\vec{v} = \langle 3,2,-1 \rangle \Rightarrow |\vec{v}| = \sqrt{3^2+2^2+(-1)^2} = \sqrt{14}$

2) $\vec{v} = \overrightarrow{AB}$, $A=(1,2,5)$, $B=(4,4,4)$.

$$|\vec{v}| = \sqrt{(4-1)^2+(4-2)^2+(4-5)^2} = \sqrt{14}$$

$$\vec{v} = \langle 4-1, 4-2, 4-5 \rangle = \langle 3,2,-1 \rangle$$

Draw $\vec{v}$



Rule:- $\vec{u} = \langle a, b, c \rangle$, $\vec{v} = \langle d, e, f \rangle$

$\boxed{1}$ $\alpha \vec{u} = \langle \alpha a, \alpha b, \alpha c \rangle$

$-\vec{u} = \langle -a, -b, -c \rangle$

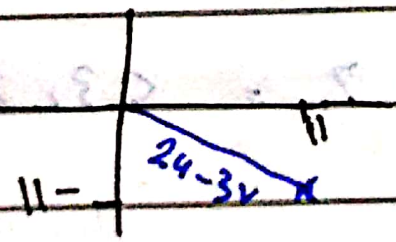
$2\vec{u} = \langle 2a, 2b, 2c \rangle$

$\boxed{2}$ $\vec{u} \pm \vec{v} = \langle a \pm d, b \pm e, c \pm f \rangle$.

Ex: $\vec{u} = \langle 1, 2 \rangle$, $\vec{v} = \langle -3, 5 \rangle$. Find

$|2\vec{u} - 3\vec{v}|$:- $2\vec{u} - 3\vec{v} = \langle 2(1) - 3(-3), 2(2) - 3(5) \rangle$

$\langle 11, -11 \rangle$



$|2\vec{u} - 3\vec{v}| = \sqrt{(11)^2 + (-11)^2} = \sqrt{242}$

Ex:- $\vec{u} = \langle 2, -1, 0 \rangle$, $\vec{v} = \langle -1, 5, 4 \rangle$

$$\vec{u} - \frac{1}{2}\vec{v} = \langle 2 - \tfrac{1}{2}(-1) , -1 - \tfrac{5}{2} , 0 - 2 \rangle$$

$$\langle \tfrac{5}{2} , -\tfrac{7}{2} , -2 \rangle$$

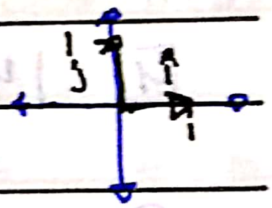
Rule:- $\vec{u}$, $\vec{v}$, $\vec{w}$

1. $\vec{u} + \vec{0} = \vec{u} = \vec{0} + \vec{u}$
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $(a+b)\vec{u} = a\vec{u} + b\vec{u}$
4. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
5. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
6. $\vec{u} + -\vec{u} = \vec{0}$

Def: Basis vector:

① in 2 space $i = \langle 1, 0 \rangle$

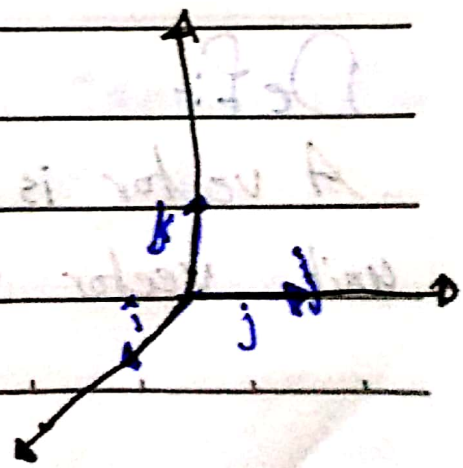
$j = \langle 0, 1 \rangle$



2. in 3 space: $i = \langle 1, 0, 0 \rangle$

$j = \langle 0, 1, 0 \rangle$

$k = \langle 0, 0, 1 \rangle$



Remark:-

1. $\langle a,b\rangle = a\langle 1,0\rangle + b\langle 0,1\rangle = a\hat{i} + b\hat{j}$

2. $\langle a,b,c\rangle = a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle$
   $= a\hat{i} + b\hat{j} + c\hat{k}$

Ex:-

1. $\langle 3,2,-5\rangle = 3\hat{i} + 2\hat{j} + 5\hat{k}$

2. $6\hat{i} - 7\hat{k} = \langle 6, 0, -7\rangle$

3. $2\hat{i} - 3\hat{j} + \langle 1,-5\rangle = \langle 2,-3\rangle + \langle 1,-5\rangle$
   $= \langle 3, -8\rangle$
   $= 3\hat{i} - 8\hat{j}$

4. $|4\hat{i} - 3\hat{k}| = \sqrt{(4)^2 + (0)^2 + (-3)^2} = \sqrt{25} = 5$

Rule:-

$$|a\hat{i} + b\hat{j} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$$

Def:-

A vector is a unit vector if its Length is 1.
unit vector written as $\hat{u}, \hat{v}, \hat{w}$

Ex:- [1] $\hat{i}, \hat{j}, \hat{k}$ unit vector

[2] $\hat{a} = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle \Rightarrow |\hat{a}| = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = 1$

$\therefore \hat{a}$ unit vector.

Ex:- $\vec{u} = \langle 2, -1, 0 \rangle \Rightarrow |\vec{u}| = \sqrt{5}$ $\vec{u}$ not unit vector

$\rightarrow \frac{\vec{u}}{|\vec{u}|} = \hat{u} = \frac{\vec{u}}{\sqrt{5}} = \langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 0 \rangle$ unit vector

$\hat{u}$ in the same direction of $\vec{u}$

$-\hat{u}$ " " oppsite " " "

Remark:-

$\frac{\vec{u}}{|\vec{u}|}$ unit vector in the same direction of $\vec{u}$

$\frac{-\vec{u}}{|\vec{u}|}$ " " " " oppsite " " "

Ex:- $\vec{u} = 2\hat{i} - 3\hat{j} + \hat{k}$

[1] Find 2 unit vector one in the same direction of $\vec{u}$ the second in the opposite direction

$|\vec{u}| = \sqrt{14}$

① $\frac{\vec{u}}{|\vec{u}|} = \frac{2}{\sqrt{14}}\hat{i} - \frac{3}{\sqrt{14}}\hat{j} + \frac{k}{\sqrt{14}}$ unit in the Same direction

B. $\frac{-\vec{u}}{|\vec{u}|} = \frac{-2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} - \frac{K}{\sqrt{14}}$ " " " opposite direction.

* [2] Find vector of length π in the same direction of $\vec{u}$?

$\frac{2\pi}{\sqrt{14}}\hat{i} - \frac{3\pi}{\sqrt{14}}\hat{j} + \frac{k\pi}{\sqrt{14}}$ of length π

[3] Find a vector of Length $\sqrt{2}$ in the opposite direction of $\vec{u}$

$\frac{-2\sqrt{2}}{\sqrt{14}}\hat{i} - \frac{3\sqrt{2}}{\sqrt{14}} + \frac{\sqrt{2}}{\sqrt{14}}k$

Sec 12.3 The Dot product

ضرب نقطي

Def:- The Dot product of $\vec{u} = \langle a, b, c \rangle$ and $\vec{v} = \langle d, e, f \rangle$ is

$$\vec{u} \cdot \vec{v} = ad + be + cf$$

Ex:-

1. $\langle 2, -1 \rangle \cdot \langle 3, 5 \rangle = 2(3) + (-1) \cdot (5) = 1$

2. $(2\hat{i} - 4\hat{j} + 5\hat{k}) \cdot (3\hat{i} + 2\hat{k}) = \langle 2, -4, 5 \rangle \cdot \langle 3, 0, 2 \rangle$
$= 6 + 0 + 10 = 16$

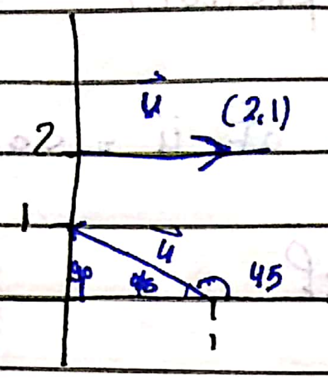
Rule:-

1. $a(\vec{u} \cdot \vec{v}) = (a\vec{u}) \cdot \vec{v} = \vec{u} \cdot (a\vec{v})$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
4. $|a\vec{u} + b\vec{v}|^2 = a^2 |\vec{u}|^2 + 2ab\, \vec{u} \cdot \vec{v} + b^2 |\vec{v}|^2.$
5. $|a\vec{u} - b\vec{v}|^2 = a^2 |\vec{u}|^2 - 2ab\, \vec{u} \cdot \vec{v} + b^2 |\vec{v}|^2.$

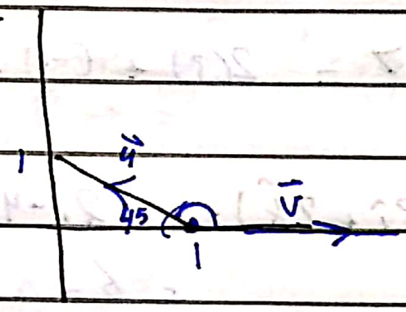
Def:- The angle θ between two vectors $\vec{u}, \vec{v}$ is the angle $\theta \in [0, \pi]$ between them when the vector have the same initial pt.

Ex:-

يجب ان تكون بداية الشعاعين واحدة.



Sol:-



$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Rule:-

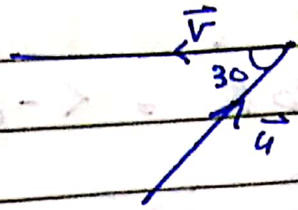
$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$$

$$\therefore \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|} \qquad \vec{u} \neq 0,\ \vec{v} \neq 0$$

$$\vec{u} \cdot \vec{v} > 0 \Rightarrow \theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

$$\vec{u} \cdot \vec{v} < 0 \Rightarrow \theta = \pi - \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$$

Ex:- If $|\vec{u}| = 3$ $|\vec{v}| = 2$ find $\vec{u} \cdot \vec{v}$



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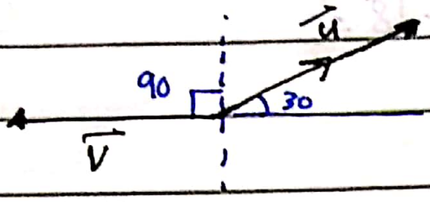
واحدة



$\theta = 150$.

$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos\theta = 3(2) \cos 150$

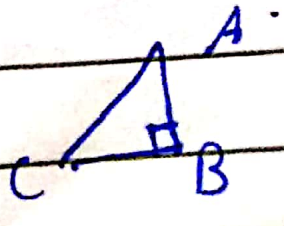
$6(-\cos 30) = -6\frac{\sqrt{3}}{2} = -3\sqrt{3}$

Rule:- let θ Be the angle between $\vec{u}, \vec{v}$

1. If $\vec{u} \cdot \vec{v} > 0 \Rightarrow \theta$ is acute angle.
2. If $\vec{u} \cdot \vec{v} < 0 \Rightarrow \theta$ " obtuse "
3. If $\vec{u} \cdot \vec{v} = 0 \Rightarrow \theta$ " right angle

(we say perpendicular or Orthogonal).

Ex:- Show that $A(1,2)$, $B(-2,3)$, $C(4, -)$ 21 are vectors of a right triangle at B?



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$\overrightarrow{AB} = \langle -3, 1 \rangle$

$\overrightarrow{CB} = \langle -6, -18 \rangle$

$\overrightarrow{AB} \cdot \overrightarrow{CB} = -3(-6) + -18 = 0$

$\overrightarrow{AB} \perp \overrightarrow{CB}$

$\triangle ABC$ is a right ~~ring~~ triangle

Exercise:-

Find x that make the vertices $A(1,2,3)$, $B(-1,0,2)$ $C(1,1,x)$ of a right triangle.

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Ex:- $|\vec{a}| = 4$, $|\vec{b}| = 3$, the angle between $\vec{a}, \vec{b}$ is $\frac{2\pi}{3}$

① Find $\vec{a} \cdot \vec{b}$

② Find $|2\vec{a} - 3\vec{b}|$

① $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\theta$

$= 4 \cdot 3 \cdot \cos \frac{2\pi}{3}$

$= 12 \cdot (-\cos 60)$

$= -12 * \frac{1}{2} = \boxed{-6}$

. ② $|2\vec{a} - 3\vec{b}|^2 = 4|\vec{a}|^2 - 2(2)(3)\vec{a} \cdot \vec{b} + 9|\vec{b}|^2$

ووضعنا التربيع لنحل ثم نأخذ الجذر

$= 4(16) - 12(-6) + 9(9)$.

$= 64 + 72 + 81$

$= 217$

$|2\vec{a} - 3\vec{b}| = \sqrt{217}$

Ex: If $|\vec{a} - 2\vec{b}| = 3$, $|\vec{a} + 2\vec{b}| = 4$ Find

① $\vec{a} \cdot \vec{b}$ ② $|\vec{a}|^2 + 4|\vec{b}|^2$.

Sol:-

$|\vec{a} - 2\vec{b}|^2 = |\vec{a}|^2 - 4\vec{a} \cdot \vec{b} + 4|\vec{b}|^2 = 9 \rightarrow$ ①

$|\vec{a} + 2\vec{b}|^2 = |\vec{a}|^2 + 4\vec{a} \cdot \vec{b} + 4|\vec{b}|^2 = 16 \rightarrow$ ②

① ①-② :- $-8\vec{a}\cdot\vec{b} = -7$

$\vec{a}\cdot\vec{b} = \frac{7}{8}$

② ①+② :- $2(|\vec{a}|^2 + 4|\vec{b}|^2) = 25$

$|\vec{a}|^2 + 4|\vec{b}|^2 = \frac{25}{2}$

Ex:- If $|2\vec{a} - 3\vec{b}| = \sqrt{45}$

$|\vec{a} + 2\vec{b}| = \sqrt{27}$

$|\vec{a}| = \sqrt{3}$

*Find ① $\vec{a}\cdot\vec{b}$ ② angle between $\vec{a}, \vec{b}$

③ $|2\vec{a} - \vec{b}|$.

$|2\vec{a} - 3\vec{b}|^2 = 45$

$4|\vec{a}|^2 - 6\times2\,\vec{a}\cdot\vec{b} + 9|\vec{b}|^2 = 45$

12 $\cancel{4(3)} - 12\,\vec{a}\cdot\vec{b} + 9|\vec{b}|^2 = 33 \div 3$

$\cancel{4} - 4\,\vec{a}\cdot\vec{b} + 3|\vec{b}|^2 = 11 \rightarrow$ ①

$|\vec{a} + 2\vec{b}|^2 = 27$

$|\vec{a}|^2 + 4\,\vec{a}\cdot\vec{b} + 4|\vec{b}|^2 = 27$

$4\,\vec{a}\cdot\vec{b} + 4|\vec{b}|^2 = 24$

$\rightarrow$

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$\boxed{1}$

① + ② : $7|\vec{b}|^2 = 35 \Rightarrow |\vec{b}|^2 = 5 \Rightarrow |\vec{b}| = \sqrt{5}$

② : $\vec{a}\cdot\vec{b} + 5 = 6 \Rightarrow \vec{a}\cdot\vec{b} = 1$ بالتعويض

② $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,.\,|\vec{b}|} = \dfrac{1}{\sqrt{3}\,.\,\sqrt{5}} = \dfrac{1}{\sqrt{15}}$

$\theta = \cos^{-1}\dfrac{1}{\sqrt{15}}$

لو أن $\cos\theta = \dfrac{-1}{\sqrt{15}}$

$\theta - \pi = -\cos^{-1}\dfrac{1}{\sqrt{15}}$

③ $|2\vec{a} - \vec{b}|^2 = 4|\vec{a}|^2 - 4\vec{a}\cdot\vec{b} + |\vec{b}|^2$

$= 4(3) - 4(1) + 5$

$= 13$

$|2\vec{a} - \vec{b}| = \sqrt{13}$

Ex:- Prove that

$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$

$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = (|\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2) + (|\vec{a}|^2 - 2\vec{a}\cdot\vec{b}$
$+ |\vec{b}|^2)$

$= 2(|\vec{a}|^2 + |\vec{b}|^2)$

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Def:-

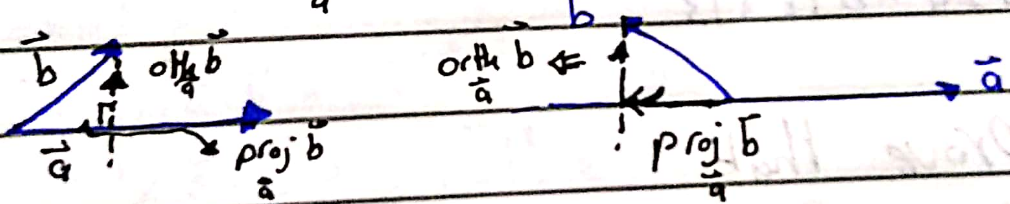
1. The scalar projection (إسقاط) of $\vec{b}$ onto $\vec{a}$ is $comp_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

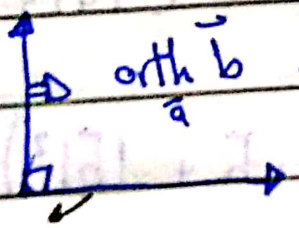
2. " vector " " $\vec{b}$ - onto $\vec{a}$ is

$$proj_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}$$

$$proj_{\vec{a}}\vec{b} = (comp_{\vec{a}}\vec{b})\,\frac{\vec{a}}{|\vec{a}|}$$

3. the orthogonal projection of $\vec{b}$ onto $\vec{a}$ is $orth_{\vec{a}}\vec{b} = \vec{b} - proj_{\vec{a}}\vec{b}$





$orth_{\vec{a}}\vec{b} = \vec{b}$ لأن $proj_{\vec{a}}\vec{b} = 0$

$0 = proj_{\vec{a}}\vec{b}$

$\vec{b} = proj_{\vec{a}}\vec{b}$

$orth_{\vec{a}}\vec{b} = 0$

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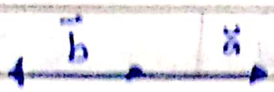


$\vec{b} = proj_{\vec a} \vec{b}$ $\qquad$ $orth_{\vec a} \vec{b} = 0$

Ex:- Find the Scalar projection, vector projection and orthogonal projection of $\vec{u} = 3\hat{i} - 2\hat{j} + \hat{k}$ onto $\vec{v} = 2\hat{i} - 3\hat{k}$

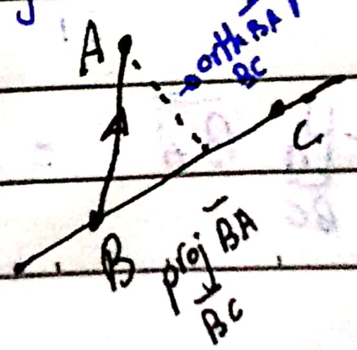
Sol:- $comp_{\vec v} \vec{u} = \dfrac{\vec{v} \cdot \vec{u}}{|\vec{v}|} = \dfrac{6+0-3}{\sqrt{13}} = \dfrac{3}{\sqrt{13}}$

$proj_{\vec v} \vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \dfrac{3}{13}(2\hat{i} - 3\hat{k}) = \dfrac{6}{13}\hat{i} - \dfrac{9}{13}\hat{k}$

$orth_{\vec r} \vec{u} = \vec{u} - proj_{\vec v} \vec{u} = (3\hat{i} - 2\hat{j} + \hat{k}) - (\frac{6}{13}\hat{i} - \frac{9}{13}\hat{k})$

$= \dfrac{33}{13}\hat{i} - 2\hat{j} + \dfrac{22}{13}\hat{k}$

Rule:- The distance from a pt A to a Line L that pass ([illegible]) through the pts B, C is



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$$\text{dist}(A, l) = \left|\text{orth}_{\vec{BC}} \vec{BA}\right|$$

Ex:- Find the distance from the pt A

(3, -4, 0) to the Line L that pass through

the pts B(1, 2, -1), C(0, -3, 5)

Sol:-

$$\text{dist}(A, l) = \left|\text{orth}_{\vec{BC}} \vec{BA}\right|$$

$$= \left|\vec{BA} - \text{proj}_{\vec{BC}} \vec{BA}\right|$$

$\vec{BA} = \langle 2, -6, 1 \rangle$

$\vec{BC} = \langle -1, -5, 6 \rangle$

$$\text{proj}_{\vec{BC}} \vec{BA} = \left(\frac{\vec{BA} \cdot \vec{BC}}{|\vec{BC}|^2}\right) \vec{BC}$$

$$= \frac{-2 + 30 + 6}{1 + 25 + 36} \langle -1, -5, 6 \rangle$$

$$= \frac{17}{31} \langle -1, -5, 6 \rangle = \left\langle \frac{-17}{31}, \frac{-85}{31}, \frac{102}{31} \right\rangle$$

$$\text{orth}_{\vec{BC}} \vec{BA} = \vec{BA} - \text{proj}_{\vec{BC}} \vec{BA} \quad \rightarrow \text{ط}$$

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$$= \langle 2, -6, 1\rangle - \langle -\frac{17}{31}, -\frac{85}{31}, \frac{102}{31}\rangle$$

$$\langle \frac{79}{31}, -\frac{201}{31}, -\frac{71}{31}\rangle$$

$$\text{dist} = \left|\langle \frac{79}{31}, \frac{-101}{31}, \frac{-71}{31}\rangle\right|$$

$$\frac{1}{31}\left|\langle 79, -101, -71\rangle\right|$$

$$= \frac{1}{31}\sqrt{(79)^2 + (-101)^2 + (-71)^2}$$

Ex: Show that $\text{orth}_{\vec{v}}\,\vec{u}$ orthogonal to $\vec{v}$

$$\left(\text{orth}_{\vec{v}}\,\vec{u}\right)\cdot\vec{v} = \left(\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}\right)\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{v} - \left(\text{proj}_{\vec{v}}\,\vec{u}\right)\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\,\vec{v}\right)\cdot\vec{v}$$

$$\vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\cancel{|\vec{v}|^2}}\,\cancel{|\vec{v}|^2}$$

$$= 0 \qquad \text{orth}_{\vec{v}}\,\vec{u} \perp \vec{v}$$

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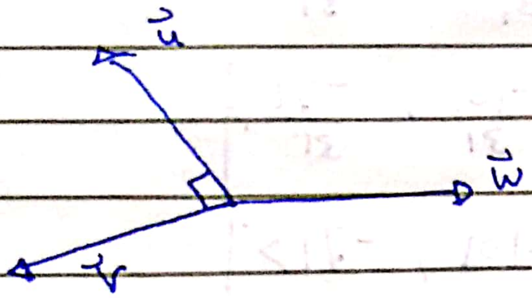
Exercise: If $\vec{u}, \vec{v}$ unit vector and

$\vec{u} + \vec{v} + \vec{w} = 0$ s.t



Find $|\vec{w}|$

$-w = \vec{u} + \vec{v}$

$|\vec{w}|^2 = |-\vec{w}|^2 = |\vec{u} + \vec{v}|^2$ ....

Sec 12.4 Cross product

Def: The Cross product of two vectors $\vec{u}, \vec{v}$ written $\vec{u} \times \vec{v}$ is a vector with

Length $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \sin \theta$

with direction identified using the right hand rule. $\rightarrow$

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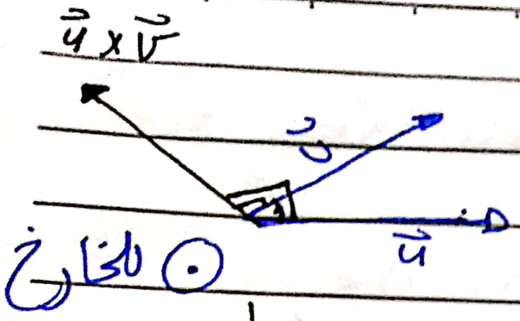

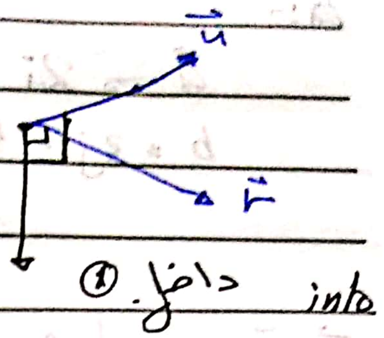

Remark:-

① $\vec{u} \times \vec{v} \perp \vec{u}$ and $\vec{u} \times \vec{v} \perp \vec{v}$

② $\vec{u} \times \vec{v} \perp$ plane containing (تحتوي) $\vec{u}, \vec{v}$

※ Rule:-

If $\vec{u} = \langle a, b, c \rangle$, $\vec{v} = \langle d, e, f \rangle$ then

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ a & b & c \\ d & e & f \end{vmatrix}$$

$a\ b\ c$ → قبل $\times$

$d\ e\ f$ → بعد $\times$

$$= i(bf - ec) - j(af - cd) + k(ae - bd).$$

Ex:-

$\vec{a} = 2i - 2j + 3k$

$b = 3j - 5\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & j & k \\ 2 & -2 & 3 \\ 0 & 3 & -5 \end{vmatrix}$$

$= \hat{i}(10 - 9) - j(-10 - 0) + k(6 - 0)$

$= \hat{i} + 10j + 6k.$

$$\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & j & k \\ 0 & 3 & -5 \\ 2 & -2 & 3 \end{vmatrix}$$

$= \hat{i}(9 - 10) - j(0 - 10) + k(0 - 6)$

$= -\hat{i} - 10\hat{j} - 6\hat{k}$

Rule:- let $\vec{u}, \vec{v}, \vec{w}$ be vector in 3D

1. $\vec{u} \times \vec{0} = \vec{0} = \vec{0} \cdot \vec{u}$

2. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

3. $\vec{v} \times \vec{v} = \vec{0}$

4. $a(\vec{u} \times \vec{v}) = (a\vec{u}) \times \vec{v} = \vec{u} \times (a\vec{v})$

5. $(\vec{u} \pm \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} \pm \vec{v} \times \vec{w}$

6. $\underset{1}{u} \times (\underset{2}{\vec{v}} \times \underset{3}{\vec{w}}) = (\underset{1}{\vec{u}} \cdot \underset{3}{\vec{w}}) \times \underset{2}{\vec{v}} - (\underset{1}{\vec{u}} \cdot \underset{2}{\vec{v}}) \times \underset{3}{\vec{w}}$

7. $|\vec{u} \times \vec{v}| = \sqrt{|\vec{u}|^2 \, |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2}$

Ex:- If $|\vec{a} + \vec{b}| = 3$ $|\vec{a}| = |\vec{b}| = 3$ find $|\vec{a} \times \vec{b}|$

Sol:-

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2$$

$$9 = 9 + 2\vec{a} \cdot \vec{b} + 9$$

$$\vec{a} \cdot \vec{b} = \frac{-9}{2}$$

$$|\vec{a} \times \vec{b}| = \sqrt{9 \times 9 - \frac{81}{4}} \quad \ldots$$

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Ex:- If $\vec{u}$, $\vec{v}$ orthogonal vector s.t

$|\vec{u}| = 2$, $|\vec{v}| = 3$ find $\vec{u} \times (\vec{v} \times \vec{u})$.

$\boxed{2}$ $(\vec{u} \times \vec{v}) \times \vec{v}$

$\boxed{1}$ $\underset{\boxed{1}}{\vec{u}} \times (\underset{\boxed{2}}{\vec{v}} \times \underset{\boxed{3}}{\vec{u}}) = (\vec{u} \cdot \vec{u})\vec{v} - (\vec{u} \cdot \vec{v})\vec{u}$

$= |\vec{u}|^2 \cdot \vec{v} - 0 \cdot \vec{u}$

$4\vec{v}$

$|\vec{u} \times (\vec{v} \times \vec{u})| = |4\vec{v}| = |4|\,|\vec{v}| = 12$

$=$

$\boxed{2}$

$(\vec{u} \times \vec{v}) \times \vec{v} = -\vec{v} \times (\vec{u} \times \vec{v}) = -[(\vec{v} \cdot \vec{v})\vec{u} - (\vec{v} \cdot \vec{u})\vec{v}] = -[|\vec{v}|^2 \vec{u} - 0\,\vec{u}] = -9\vec{u}$

$|\vec{u} \times (\vec{v} \times \vec{u})| = |-9\vec{u}| = 9\,|\vec{u}| = 18$

Ex:- let $\vec{a} = \langle 1,-1,0 \rangle$, $\vec{b} = \langle 2,3,4 \rangle$ Find two:

[1] vector orthogonal to both $\vec{a}$, $\vec{b}$ عمودي على $\vec{a}$ و $\vec{b}$

[2] unit " " " the plane Containing $\vec{a}$, $\vec{b}$.

Sol:- [1] $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 2 & 3 & 4 \end{vmatrix}$

$= i(-4-0) - j(4-0) + k(3+2) = -4\hat{i} - 4j + 5k$

Vectors:- $-4i - 4j + 5k$, $4\hat{i} + 4j - 5k$.

(2) $\vec{a} \times \vec{b}$, $-\vec{a} \times \vec{b}$ vector $\perp$ plane

$|\vec{a} \times \vec{b}| = \sqrt{16 + 16 + (25)} = \sqrt{57}$

unit vector: $\dfrac{-4i - 4j + 5k}{\sqrt{57}}$, $\dfrac{4i + 4j - 5k}{\sqrt{57}}$

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Def:- Two vector $\vec{u}, \vec{v}$ are parallel (توازي). written $\vec{u} \parallel \vec{v}$ if

1) $\vec{u} = c\vec{v}$ or $\vec{v} = d\vec{u}$

Ex:- $\vec{a} = \langle 2, -6, 4 \rangle$

$\vec{b} = \langle -1, 3, -2 \rangle$

$\Rightarrow \vec{a} = -2b \Rightarrow \vec{a} \parallel \vec{b}$

Ex:- $\vec{u} = 2i - 3j + 5k.$

$\vec{v} = 4i - 6j - 10k.$

$\vec{u} = \langle 2, -3, 5 \rangle$

$\vec{v} = \langle 4, -6, -10 \rangle$ $\Rightarrow$ $\vec{u} \neq c\vec{v}$, $\vec{v} \neq d\vec{u}$ for all Scalare

$\vec{u} \not\parallel \vec{v}$

Ex:-

$\vec{0}, \vec{v} \Rightarrow 0\vec{v} = \vec{0} \Rightarrow \vec{v} \parallel \vec{0}$

$\vec{0}$ توازي اي متجه

Ex:- $\vec{u} = \langle 6, 8, 0 \rangle$

$\vec{v} = \langle 9, 12, 0 \rangle$ $\quad \frac{3}{2}\vec{u} = \vec{v} \quad \vec{v} \parallel \vec{u}$

Remark:-

$$\vec{u} \parallel \vec{v} \iff \theta = 0 \text{ or } \pi$$
$$\iff |\vec{u} \times \vec{v}| = 0$$
$$\Rightarrow \vec{u} \times \vec{v} = \vec{0}$$

Ex:- $\vec{u} = \langle 6, 8, 0 \rangle$ $\quad \vec{v} = \langle 9, 12, 0 \rangle$.

$\vec{w} = \langle 9, 12, 1 \rangle$.

Is $\vec{u} \parallel \vec{v}$?

① $|\vec{u} \times \vec{v}| = \sqrt{|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2}$

$= \sqrt{100(225) - (150)^2} = \dots = 0$

$\vec{u} \parallel \vec{v}$

$\boxed{2}$ $|\vec{u} \times \vec{w}| = \sqrt{100(226) - (150)^2} = \dots \neq 0$

$\vec{u} \not\parallel \vec{w}$

Rule:
Let A, B, C be pts in 3D the area of the triangle with vertices A, B, C is
رؤوس

$$\text{Area} = \frac{1}{2} |\vec{AB} \times \vec{AC}|.$$

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Ex:- Find the area of triangle ① with vertices

$A(1,2)$, $B(-1,0)$, $C(3,5)$

② with edges (حواف) $\vec{u} = \langle -2,-2 \rangle$, $\vec{v} = \langle 2,3 \rangle$

$A \to A = (1,2,0)$
$B \to B = (-1,0,0)$ in 3D
$C \to C = (3,5,0)$

نفسها $\left[\begin{array}{l} \vec{AB} = \langle -2,-2,0 \rangle \\ \vec{AC} = \langle 2,3,0 \rangle \end{array} \right.$

$$\text{Area} = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

$$= \frac{1}{2} \sqrt{|\vec{AB}|^2 |\vec{AC}|^2 - (\vec{AB} \cdot \vec{AC})^2}$$

$$= \frac{1}{2} \sqrt{8(13) - (-4-6)^2} = \frac{1}{2} \sqrt{4} = \boxed{1}$$

② $\vec{u} = \langle -2,-2 \rangle \Rightarrow \vec{u} = \langle -2,-2,0 \rangle$
$\vec{v} = \langle 2,3 \rangle \Rightarrow \vec{v} = \langle 2,3,0 \rangle$ نفس نظرنا

$$\text{Area} = \frac{1}{2} |\vec{u} \times \vec{v}| = \dots\dots = 1 \text{ by part (1)}$$

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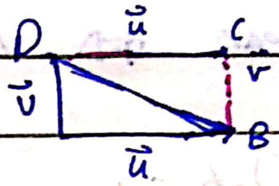
Rule :- The area of the parallelogram متوازي الاضلاع with adjacent (متجاور) edjes $\vec{u}$, $\vec{v}$ is

$$\text{Area} = |\vec{u} \times \vec{v}|$$



Ex :- Find the area of the parallelograme with

① adjacent side $\vec{u} = \langle 2, 3, 4 \rangle$

$\vec{v} = \langle 6, 1, 1 \rangle$

② with vertices $A(-2, 1, 5)$, $B(0, 4, 4)$

$C(4, 1, 6)$

Sol :- $\text{Area} = |\vec{u} \times \vec{v}| = \sqrt{(14)(38) - (16)^2} = \dots$

② $\text{Area} = |\vec{AB} \times \vec{AC}|$

$\vec{AB} = \langle 2, 3, -1 \rangle$

$\vec{AC} = \langle 6, 0, 1 \rangle$

$\sqrt{(14)(37) - (11)^2} =$

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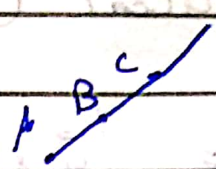
Def: 3 pts A, B, C are collinear if they are on the same line

Colinear $\Leftrightarrow$ $\vec{AB} \parallel \vec{AC}$



Ex:- Determine whether the pts A(2, 4, -3) B(3, -1, 1), C(4, -6, 5) Colinear or not?

Sol:-

$\vec{AB} = \langle 1, -5, 4 \rangle$

$\vec{AC} = \langle 2, -10, 8 \rangle$ نريد فرض التوازي

$|\vec{AB} \times \vec{AC}| = \sqrt{42(168) - (84)^2} = \dots = 0$ ∴

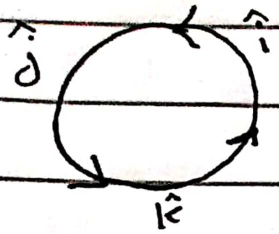
$\vec{AC} = 2\vec{AB}$

$\vec{AB} \parallel \vec{AC}$

∴ A, B, C Collinear

$\frac{\vec{AB}}{\vec{AC}} = \frac{1}{2}, \frac{-5}{-10}, \frac{4}{8} = \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ parallel

∴ A, B, C Colinear



$i \times j = k$

$j \times k = \hat{i}$

$k \times i = j$

$j \times i = -k$

$k \times j = -\hat{i}$

$\hat{i} \times k = -j$

$(2i - 3j) \times (j + 5k) = 2\hat{i} \times j + 2\hat{i} \times 5k - 3j \times j - 3j \times 5k$

$= 2(k) + 10(-j) - 3(0) - 15\hat{i}$

$= -15\hat{i} - 10j + 2k.$

Def:- The scalar triple of $\vec{a} = \langle a, b, c \rangle$, $\vec{v} = \langle d, e, f \rangle$

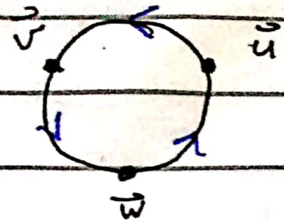
& نعوض أ لوجودها

$\vec{w} = \langle g, h, L \rangle$ نتيجة is

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & l \end{vmatrix}$$

$a(hf - el) - b(gf - dl) + c(dh - eg)$

Remark



$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{w}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = -\vec{w}\cdot(\vec{v}\times\vec{u})$$

Ex :- If $\vec{a}\cdot(\vec{b}\times\vec{c}) = 7$, find $\vec{b}\cdot(\vec{a}\times 2\vec{c})$

, $\vec{c}\cdot(5\vec{a}\times\vec{b})$

Sol :- $\vec{b}\cdot(\vec{a}\times 2\vec{c}) = 2(\vec{b}\cdot(\vec{a}\times\vec{c})) = 2(-7) = -14$

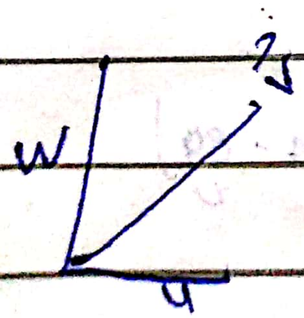
$\vec{c}\cdot(5\vec{a}\times\vec{b}) = 5\,\vec{c}\cdot(\vec{a}\times\vec{b}) = 5(7) = 35$.

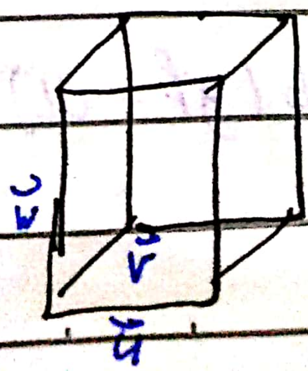
Rule :-

The volume (حجم) of the parallelpiped (متوازي السطوح) with adjacend (متجاورة) edges

is :

$$V = \left|\vec{u}\cdot(\vec{v}\times\vec{w})\right| \quad \text{مطلق}$$





Ex:- Find the volume of the parallelpiped:

[1] determined by the vectors

$\vec{a} = 6\hat{i} + 3\hat{j} - k$, $\vec{b} = j + 2k$, $\vec{c} = 4i - 2j + 5k$

[2] with adjecent edges PQ, PR, PS, where

$P(-2,1,0)$, $Q(4,4,-1)$, $R(-2,2,2)$, $S(2,-1,5)$

Sol:-

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix}$$

$$= 6(5+4) - 3(0-8) - (0-4)$$

$$54 + 24 + 4 = 82$$

$$V = |82| = 82$$

[2] $\overrightarrow{PQ} = \langle 6, 3, -1 \rangle$ $\overrightarrow{PR} = \langle 0, 1, 2 \rangle$

$\overrightarrow{PS} = \langle 4, -2, 5 \rangle$

تابع

$$\vec{PQ} \cdot (\vec{PR} \times \vec{PS}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 82$$

$V = 82$

في نفس المستوى

Def: ① 3 vector $\vec{u}, \vec{v}, \vec{w}$ are coplaner if they are in the same plane

$\vec{u}, \vec{v}, \vec{w}$ Coplaner $\Leftrightarrow$ $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$

② 4 pts A, B, C, D are complanar $\Leftrightarrow$

$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = 0$.

Ex:- determain wether

① the vectors $\vec{a} = \langle 1, 4, -7 \rangle$ $\vec{b} = \langle 2, -1, 4 \rangle$ $\vec{c} = \langle 0, -1, 2 \rangle$ Coplanar or not

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -1 & 2 \end{vmatrix}$$

$(-2-4) - 4(4-0) + -7(-2-0) = 2 - 16 + 14 = 0$

$\vec{a}, \vec{b}, \vec{c}$ Coplanar.

$\boxed{2}$ The pts $A(1,1,1)$, $B(2,5,-6)$, $C(3,0,5)$

$D(1,0,3)$ Contained in the same plane or not

$\vec{AB} = \langle 1, 4, -7 \rangle$
$\vec{AC} = \langle 2, -1, 4 \rangle$
$\vec{AD} = \langle 0, -1, 2 \rangle$

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -1 & 2 \end{vmatrix} = \cdots = 0$$

$\therefore$ A, B, C, D in the same plane.

$\boxed{3}$ Find the value of x such that ① the vectors $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, x \rangle$ $\vec{c} = \langle 0, -1, 2 \rangle$ are

ⓐ Coplanar $\boxed{b}$ not coplaner

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & x \\ 0 & -1 & 2 \end{vmatrix}$$

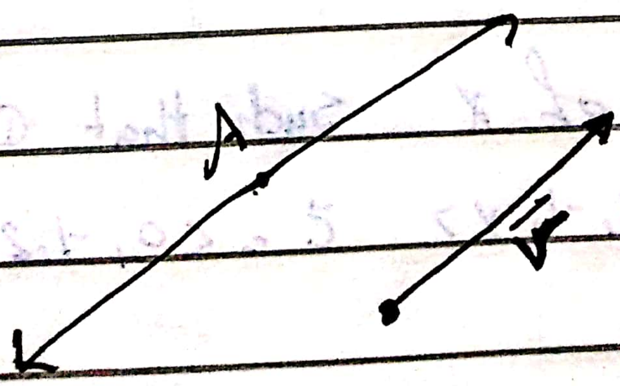
$\boxed{x=4}$

(B) not Coplaner $\rightarrow$ $\mathbb{R}-\{4\}$.

Sec 12.5

The eqs of Lines and planes

Def:- let L be a Line that passes throgh the pt $A(x_0, y_0, z_0)$ and parallel to a vector $\vec{v}=\langle a, b, c\rangle$



① The parametric (param) eq.s of L are:

$x = x_0 + at$
$y = y_0 + bt$ علاقات ثلاث للخط المستقيم $t \in \mathbb{R}$
$z = z_0 + ct$

[2] The symmetric (symm) تشابه eq of L

are = $\frac{x - x_0}{a} , \frac{y - y_0}{b} , \frac{z - z_0}{c}$ ذي محاذرة التشابه تُخذف ال t

If $a = 0$: $x = x_0$, $\frac{y - y_0}{b} = \frac{z - z_0}{c}$

If $b = 0$: $y = y_0$, $\frac{x - x_0}{a} = \frac{z - z_0}{c}$

If $c = 0$: $z = z_0$, $\frac{x - x_0}{a} = \frac{y - y_0}{b}$

[3] The vector eq of L is

$\langle x, y, z \rangle = \langle x_0 , y_0 , z_0 \rangle + t \langle a, b, c \rangle$

$= \langle x_0 + at , y_0 + bt , z_0 + ct \rangle$

Ex:- Find param eqs , symm eqs. and vector eq of the line l that pass through the pt $A(1, 2, 3)$ and parallel to the vector $6\hat{i} - 7\hat{k}$. Find two pts on l other than A

Sol:-

$$\left.\begin{aligned} x &= 1 + 6t \\ y &= 2 + 0t \\ z &= 3 + {-7}t \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 1 + 6t \\ y &= 2 \\ z &= 3 - 7t \end{aligned}$$

Symm eq:- $\dfrac{x-1}{6} = \dfrac{z-3}{-7}$, $y = 2$.

vector eq:- $\langle x, y, z \rangle = \langle 1+6t , 2, 3-7t \rangle$

$= \langle 1, 2, 3 \rangle + \langle 6, 0, -7 \rangle t$

to find 2 pts on l

In param eq let $t = 0$

نفس النقطة A عطانا $x = 1$ $y = 2$ $z = 3$

تأخذ t غير صفرية

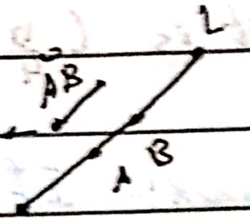
$t = 1$ $x = 7$ $y = 2$ $z = -4$.

pt $(7, 2, -4)$

let $t = 2$ $x = 13$ $y = 2$ $z = -11$ pts $(13, 2, -11)$.

Ex: Find param, symm, vector eq of the Line L that passes through the pt $A(1,2,3)$ $B(-2,5,7)$ and what pts that intersect the xy-plane.

توازي L



$\vec{AB} \parallel L$. $\boxed{t}$ المحور الموازي L والذي يضرب بـ

$\vec{AB} = \langle -3, 3, 4 \rangle$

من A or من B

Param eq: $x = 1 - 3t$ $\quad x = -2 - 3t$

$y = 2 + 3t$ $\quad y = 5 + 3t$

$z = 3 + 4t$ $\quad z = 7 + 4t$

B أو A مثل t لـ t ثابتة

$t = -1$ $(4, -1, -1)$ وتستخدم في الخارج A و B

$x = 4 - 3t$

$y = -1 + 3t$

$z = -1 + 4t$

Symm eq:

$$\frac{x-1}{-3} = \frac{y-2}{3} = \frac{z-3}{4}$$

vector eq : $\vec{v} = \langle 1,2,3 \rangle + \langle -3,3,4 \rangle t$

xy-plane $- z=0 \Rightarrow 3+4t=0 \quad t = \frac{-3}{4}$

$x = 1-3(\frac{-3}{4}) = \frac{13}{4}$

$y = 2+3(\frac{-3}{4}) = \frac{-1}{4}$

$z = 0$

pt is $(\frac{13}{4}, \frac{-1}{4}, 0)$.

Remark:- If $\vec{u} \parallel L_1$, $\vec{v} \parallel L_2$

$\therefore L_1 \parallel L_2 \Leftrightarrow \vec{u} \parallel \vec{v}$

Ex:- determian wther the Lines L_1, L_2 are parallel, intersects, Skew (متخالفة) or equal:

$\boxed{1}$ $L_1 \Rightarrow x = 2-3t$ $\quad L_2 \Rightarrow x = 5+9s$

$y = 2t \quad\quad y = 3-6s$

$z = 7 \quad\quad z = 1$

$\boxed{2}$ $L_1 \Rightarrow x = t \quad y = 3-t \quad , z = 2+3t \rightarrow$

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$L_2 \Rightarrow x = 7 + 2S$, $y = 2 + S$ $z = 5$.

[3] $L_1 \Rightarrow$ $x = 1 + t$, $y = -2 + 3t$, $z = 4t$
$L_2 \Rightarrow$ $x = 2S$, $y = 3 + S$, $z = -3$

[4] $L_1 \Rightarrow$ $x = 1 - 3t$, $y = 2 + 3t$, $z = 3 + 4t$
$L_2 \Rightarrow$ $x = -2 + \frac{3}{2}S'$, $y = 5 - \frac{3}{2}S'$, $z = 7 - 2S'$

Sol:- اول خطوة افحص التوازي . (معاملات t توازي)

① $\vec{u} = <-3, 2, 0> \parallel L_1$
$\vec{v} = <9, -6, 0> \parallel L_2$. $-3\vec{u} = \vec{v}$. $\Rightarrow \vec{u} \parallel \vec{v}$

Check (equality) on L_2 $\therefore L_2 \parallel L_1$

$5 + 9S = 2$ pt on L_1 $(2, 0, 7)$ عند تعويض $t = 0$

$3 - 6S' = 0$

$7 = 1$ impossible. $L_1 \neq L_2$.

② $\vec{u} = <1, -1, 3> \parallel L_1$
$\vec{v} = <2, 1, 0> \parallel L_2$ مش متوازنان $\vec{u} \not\parallel \vec{v}$

$L_1 \neq L_2$ اذا اكيد غير متساوين

Check Intersection

$L_1 = L_2$

$x = x$: $t = 1 + 2S$

$y = y$: $3 - t = 2 + S'$

$2, 2t = 5$

$t - 2s = 1 \rightarrow ①$

$-t - s = -1 \rightarrow ②$

$3t = 3 \rightarrow ③$

من ③ $\boxed{t=1}$

من ① $\boxed{s=0}$

نعوض في 2 $-1 - 0 \overset{?}{=} -1$ yes then

L_1, L_2 Interection.

To find the pt of Interection $t=1$ in L_1

$x = 1$

$y = 2$ pt $(1, 2, 5)$

$z = 5$

حل آخر: $s=0$ نعوض في L_2 نفس النقطة

3. $\vec{u} = \langle 1, 3, 4 \rangle \parallel L_1$

$\vec{v} = \langle 2, 1, 4 \rangle \parallel L_2$

$\vec{u} \nparallel \vec{v} \therefore L_1 \nparallel L_2 \rightarrow L_1 \neq L_2$

Intersection:

$1 + t = 2s \rightarrow t - 2s = -1$

$-2 + 3t = 3 + s \rightarrow 3t - s = 5$

$4 - t = -3 + 4s \quad -t - 4s = -7$

$1+3 \Rightarrow -6S = -8 \Rightarrow S = \frac{4}{3}$

$t - 2(\frac{4}{3}) = -1 \Rightarrow t - \frac{8}{3} = -1$

$\boxed{t = \frac{5}{3}}$

② نعوض في ③

$3(\frac{5}{3}) - \frac{4}{3} \overset{??}{=} 5$

$\frac{11}{3} = 5$ No

L_1, L_2 not intersection

L_1, L_2 skewed

$\boxed{4}$ $\vec{u} = \langle -3, 3, 4 \rangle \parallel L_1$ معاملات t

$\vec{v} = \langle \frac{3}{2}, \frac{-3}{2}, -2 \rangle \parallel L_2$

$-2\vec{v} = \vec{u} \therefore \vec{u} \parallel \vec{v}$

$\Rightarrow L_1 \parallel L_2$

Take pt on $L_2 \Rightarrow S = 0 \rightarrow A(-2, 5, 7)$

Check is A on L_1?

$1 - 3t = -2 \rightarrow t = 1$

$2 + 3t = 5 \rightarrow$ "

$3 + 4t = 7 \rightarrow$ "

A on L_1 when $t = 1$

$\therefore L_1 = L_2$

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معادلة المستوى

Def: The eq of the plane that passes through the pt $A(x_0, y_0, z_0)$ and with normal (عمودي) vector $\vec{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Leftrightarrow \quad ax + by + cz = ax_0 + by_0 + cz_0$$

$$\vec{n}\,\vec{v} = \vec{n} \cdot r_0 \qquad \vec{v} = \langle x, y, z \rangle$$

$$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$$

Ex: Find the eq. of the plane through the pt $A(0, -6, 7)$ and with normal vector $\vec{n} = \langle 2, 5, 4 \rangle$. Find the intercepts and sketch the plane.

$$eq = 2x + 5y + 4z = 2(0) + 5(-6) + 4(7)$$

$$2x + 5y + 4z = \cancel{2} -2$$

Intercepts:

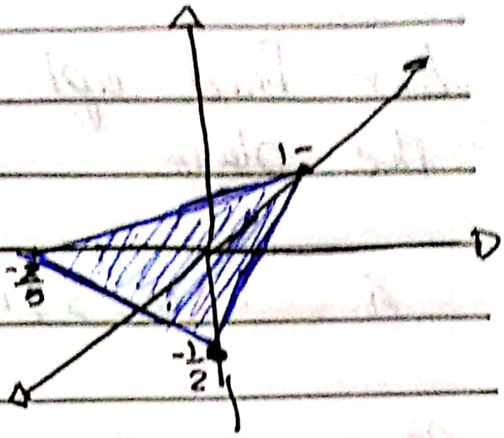
x - Intercept $\quad y = 0 \quad z = 0 \Rightarrow 2x = -2 \Rightarrow \boxed{x = -1}$

y - " $\quad z = 0 \quad x = 0 \Rightarrow 5y = -2 \Rightarrow y = \frac{-2}{5}$

z - " $\quad y = 0 \quad x = 0 \Rightarrow 4z = -2$

$$\boxed{z = -\frac{1}{2}}$$

Sketch:-



Ex**:- Find the eq of the plane that pass through the pts. $P(1,3,2)$ $Q(3,-1,6)$ & $R(5,2,0)$.

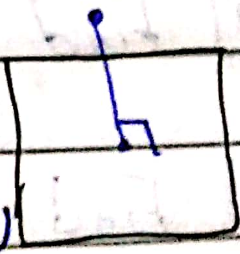
$\overrightarrow{PQ} = (2,-4,4)$

$\overrightarrow{PR} = (4,-1,-2)$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$= \hat{i}(8+4) - j(-4-16) + k(-2+16)$

$= 12\hat{i} + 20j + 14k.$



$eq = 12x + 20y + 14z \Rightarrow 12(1) + 20(3) + 14(2)$

$\div$ $\boxed{6x + 10y + 7z = 50}$

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Exr Find a pt and a normal vector to the plane $2x-3y+z=6$

$\vec{n} = \langle 2,-3,1\rangle \perp$ plane

pt : $y=0$, $z=0$ $2x=6$ $x=3$

pt $(3,0,0)$

Exr Let L be the Line of intersection of the two planes:-

$P_1 : x+y-2z=3$

$P_2 = x-y+z=1$

[1] Find param. eqs of the Line L

[2] Find the eq of the plane:-

[a] that pass through the pt $(1,2,3)$ and

(i) perpendicular to the Line L

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(ii) Contains the Line l

[b] that pass through the pts $(1,2,3)$, $(4,5,6)$

and parallel to l

[c] that pass through the pt $(1,2,3)$ and parallel

to the Lines $L_1: x=1$, $\frac{y-3}{-2}=z$

$L_2: \frac{x-1}{3}=\frac{y}{2}$, $z=1$ أكمل كل الأسئلة.

① we want to find two pts on l Take:-

$z=0$ $P_1 \Rightarrow x+y=3$ $\boxed{x=2}$

$P_2 \Rightarrow x-y=1$ $\boxed{y=1}$ بالتعويض

pt $A(2,1,0)$.

Take $x=0$ $P_1 \Rightarrow y-2z=3$

$P_2 \Rightarrow -y+z=1$

$-z=4$ $\boxed{z=-4}$

$y=-8$

pt $B(0,-5,-4)$.

A, B on l $\vec{AB}=\langle -2,-6,-4\rangle \parallel L$ $\div -2$

$\vec{AB}=\langle 1,3,2\rangle \parallel L$

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param eq :-
$$x = 2 + t$$
$$y = 1 + 3t$$
$$z = 0 + 2t$$

$\boxed{2}$ a → $\vec{AB} \parallel L$, $L \perp$ plane

$\therefore \vec{AB} \perp$ plane.

بعد $\langle -2, -6, -4 \rangle \perp$ plane

القسمة $\langle 1, 3, 2 \rangle \perp$ plane.

$$x + 3y + 2z = 1(1) + 3(2) + 2(3) = 13$$

$\boxed{2}$ ⇒ a ⇒ (ii)

A, B on plane. $A(2, 1, 0)$, $B(0, -5, -4)$

$C(1, 2, 3)$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ -2 & -6 & -4 \\ -1 & 1 & 3 \end{vmatrix}$$

$$i(-18+4) - j(-6-4) + k(-2-6)$$

$-14i + 10j - 8k \perp$ plan. (-2) نقسم على

$+7i - 5j + 4k \perp$ plane.

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eq of plane $7x - 5y + 4z = 7(2) - 5(1)$

$7x - 5y + 4z = 9$.

$\boxed{b}$ plan pass $A(1,2,3)$, $B(4,5,6)$.

plane $\parallel L$

$\Rightarrow \langle -2, -6, -4 \rangle \parallel L$

$\div 2 \Rightarrow \langle 1, 3, 2 \rangle \parallel L$

$\langle 1,3,2 \rangle \parallel$ plane.

$\vec{AB} = \langle 3,3,3 \rangle \parallel$ plane

$\div 3 \Rightarrow \langle 1,1,1 \rangle \parallel$ plane.

$\vec{n} = \langle 1,3,2 \rangle \times \langle 1,1,1 \rangle$

$$= \begin{vmatrix} \hat{i} & j & k \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i} + j - 2\hat{k}$$

eq of plane

$x + y - 2z = 1 + 2 + -6$

$\boxed{x + y - 2z = -3}$

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$\boxed{c}$ $A(1,2,3)$

$L_1: x=1, y=3-2t, z=t \Rightarrow \vec{u} = \langle 0,-2,1 \rangle \parallel L_1$

$L_1 \parallel$ plane

$\therefore \vec{u} \parallel$ "

$L_2: x=1+3t, y=2t, z=1 \Rightarrow \vec{v} = \langle 3,2,0 \rangle \parallel L_2$

$\vec{v} \parallel$ plane.

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & -2 & 1 \\ 3 & 2 & 0 \end{vmatrix} = -2i + 3j + 6k$$

eq of plane:

$$-2x + 3y + 6z = -2 + 6 + 18$$
$$= 22.$$

Ex: Find the eq of the plane that pass through the Line of intersection of the plane: $x+z=-1$, $y=z$ and parallel to the Line $x=1$, $\frac{y-3}{-2} = z$

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$P_1 : x + z = -1$ L Line of intersection.
$P_2 : y = z$

$z = 0 \Rightarrow x = -1 \Rightarrow y = 0 \Rightarrow A(-1,0,0)$ on L
$z = 1 \Rightarrow x = -2 \Rightarrow y = 1 \Rightarrow B(-2,1,1)$

$L_1 = x = 1 \; , \; \frac{y-3}{-2} \; , \; z \Rightarrow \vec{u} = \langle 0,-2,1 \rangle \parallel L_1$

$\Rightarrow \vec{u} \parallel$ plane

$\vec{AB} = \langle -1,1,1 \rangle \parallel L \Rightarrow \vec{AB} \parallel$ plane

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

$-3\hat{i} - j - 2k \perp$ plane.

A on plane.

eq is :-
$$-3(x+1) - (y-0) - 2(z-0) = 0$$

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Ex:-

Find all pts at which the line

$$\frac{x-2}{3} = \frac{y}{-2} = z-5$$

Sol:- Line : param eqs : $x = 2+3t$

$y = -2t$

$z = 5+t$

Surface :

$$4(2+3t) - 5(-2t)^2 - 2(5+t) = -22.$$

$$8 + 12t - 20t^2 - 10 - 2t = -22.$$

$$-20t^2 + 10t + 20 = 0 \quad \div -10$$

$$2t^2 - t - 2 = 0$$

القانون العام :- $t = \frac{1 \pm \sqrt{1 - 4(2)(-2)}}{4} = \frac{1 \mp \sqrt{17}}{4}$

pts : At $t = \frac{1 + \sqrt{17}}{4}$

$x = 2 + \frac{3}{2}(1 + \sqrt{17})$ $\left(2 + \frac{3}{4}(1+\sqrt{17}),\right.$

$y = -2\left(\frac{1 + \sqrt{17}}{4}\right)$ $-\frac{1+\sqrt{17}}{2},$

$z = 5 + \frac{1 + \sqrt{17}}{4}$ $\left.\frac{5+1+\sqrt{17}}{4}\right)$

النقطة 1

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At $t = \frac{1}{4}\sqrt{17}$

$x = \dots$

$y = \dots$

$z = \dots$

نعوض في
و نجد النقطة

Rule:- Let P_1, P_2 be two planes

$n_1 \perp P_1$, $n_2 \perp P_2$.

1. $P_1 \parallel P_2 \Leftrightarrow n_1 \parallel n_2$.

2. $P_1 \perp P_2 \Leftrightarrow n_1 \perp n_2$.

3. The angle θ between P_1, P_2 is the angle between $\vec{n}_1, \vec{n}_2$

$$\sin\theta = \frac{|\vec{n}_1 \times \vec{n}_2|}{|\vec{n}_1|\,|\vec{n}_2|}.$$

Ex:- Find the angle between $x+y+z=1$

, $x-2y+3z=1$ Are the plane perendcular or parallel or neither perpendicular nor parallel

Sol:- $\vec{n}_1 = \langle 1,1,1 \rangle \perp P_1$

$n_2 = \langle 1,-2,3 \rangle \perp P_2$

$n_1 \cdot n_2 = 1-2+3 \neq 0 \Rightarrow n_1, n_2$ not perpendcular

P_1, P_1 not $\perp$

Check parallel:-

$\frac{1}{1} \neq \frac{1}{-2} = \frac{1}{3} \Rightarrow$ not parallel

They are neither parallel nor perepandcular

$$\sin\theta = \frac{|\vec{n}_1 \times \vec{n}_2|}{|n_1|\,|n_2|} = \frac{\sqrt{|\vec{n}_1|^2 |\vec{n}_2|^2 - (n_1 \cdot n_2)^2}}{\sqrt{3}\ \sqrt{4}}$$

$$\frac{\sqrt{3(14)-4}}{\sqrt{3}\cdot\sqrt{2}\,\sqrt{7}} = \frac{\sqrt{38}}{\sqrt{3}\,\sqrt{14}} = \frac{\cancel{\sqrt{2}}\ \sqrt{19}}{\cancel{\sqrt{2}}\ \sqrt{3}\ \sqrt{7}} = \sqrt{\frac{19}{21}}$$

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$\theta = \sin^{-1}\left(\sqrt{\frac{14}{21}}\right)$

$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}}{|\vec{n}_1|\,|\vec{n}_2|}\right)$

$\cos^{-1}\left(\frac{2}{\sqrt{42}}\right)$

Rule:- The dist between a pt $A(x_0, y_0, z_0)$ and a plane $ax + by + cz + d = 0$ is

مطلق

$$\text{dist} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

Ex:- Find the dist between the pt $(1,2,3)$ and the plane $2x - 5y = 1$

Sol:- plane $\Rightarrow 2x - 5y - 1 = 0$

$$\text{dist} = \frac{|2(1) - 5(2) - 1|}{\sqrt{2^2+(-5)^2}} = \frac{9}{\sqrt{29}} = \frac{9\sqrt{29}}{29}.$$

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Rule : Let P_1, P_2 be two planes

[1] If $P_1 // P_2$, then $dist(P_1, P_2) = dist(A, P_2)$

where A is a pt on P_1

[2] If $P_1 \not\parallel P_2$, then P_1, P_2 intersected

$\Rightarrow dist(P_1, P_2) = 0$

Ex :- Find the distance between the 2 planes :-

[1] $P_1 : 2x - 4y + 8z = 1$ $\qquad P_2 : -x + 2y - 4z = 10$

[2] $P_1 : 2x - 4y + 8z = 1$ $\qquad P_2 : -x + 2y + 4z = 10$

Sol :- $\vec{n}_1 = \langle 2, -4, 8 \rangle \perp P_1$

$\vec{n}_1 = -2\vec{n}_2$

$\vec{n}_2 = \langle -1, 2, -4 \rangle \perp P_2$ $\qquad \Rightarrow \vec{n}_1 // \vec{n}_2$

$\therefore P_1 // P_2$.

$P_1 : y = 0 \quad z = 0 \Rightarrow x = \frac{1}{2}$

$A(\frac{1}{2}, 0, 0)$ on P_1

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$$dist(P_1, P_2) = dist(A, P_2) = \frac{\left| -\frac{1}{2} + 2(0) - 4(0) - 10 \right|}{\sqrt{(-1)^2 + (2)^2 + (-4)^2}}$$

$$= \frac{\left| \frac{21}{2} \right|}{\sqrt{21}} = \frac{21}{2\sqrt{21}}$$

$$= \frac{\sqrt{21}}{2}.$$

$\boxed{2}$ $\vec{n}_1 = \langle 2, -4, 8 \rangle \perp P_1$

$\vec{n}_2 = \langle -1, 2, 4 \rangle \perp P_2$

$n_1 \not\parallel P_2 \quad \therefore P_1 \not\parallel P_2$

P_1, P_2 intersection

$dist(P_1, P_2) = 0$

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Sec 12.6

Cylinders and Quadric surfaces.

Def:- A cylindrical surface is a surface obtained by moving a curve along a fixed axes

Ex:- The following are Cylindrical Surface

[1] $x^2+y^2=3$

[2] $z=\sqrt{y}$

[3] $x^2-3z^2-z=0$

[4] $x+2y=5$

[5] $x=4$

[6] $3x-2y+z=7$

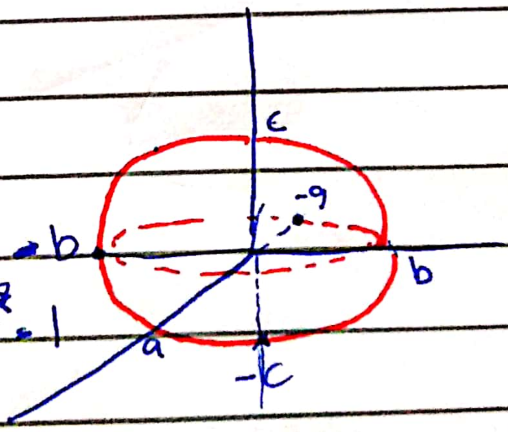
Ellipsoid . شكل بيضاوي

يوجد ثابت

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$a, b, c > 0$

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1$$



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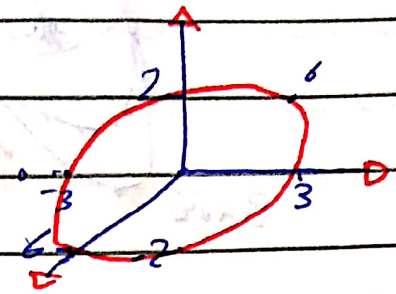
Ex:

① $x^2+4y^2+9z^2=36 \Rightarrow$ Ellipsoid

بالقسمة على 36 لكل من الطرفين

$$\frac{x^2}{36}+\frac{y^2}{9}+\frac{z^2}{4}=1$$



$\boxed{2}$ $x^2-6x+4y^2+24y+z^2=0$

$$x^2-6x+9+4(y^2+6y)+z^2=0+9$$

$$(x-3)^2+4(y^2+6y+9)+z^2=9+36$$

$$(x-3)^2+4(y+3)^2+z^2=45$$

$$\frac{(x-3)^2}{45}+\frac{(y+3)^2}{\frac{45}{4}}+\frac{z^2}{45}=1$$ Ellipsoide.

Center $(3,-3,0)$

الرسم فى الخلف

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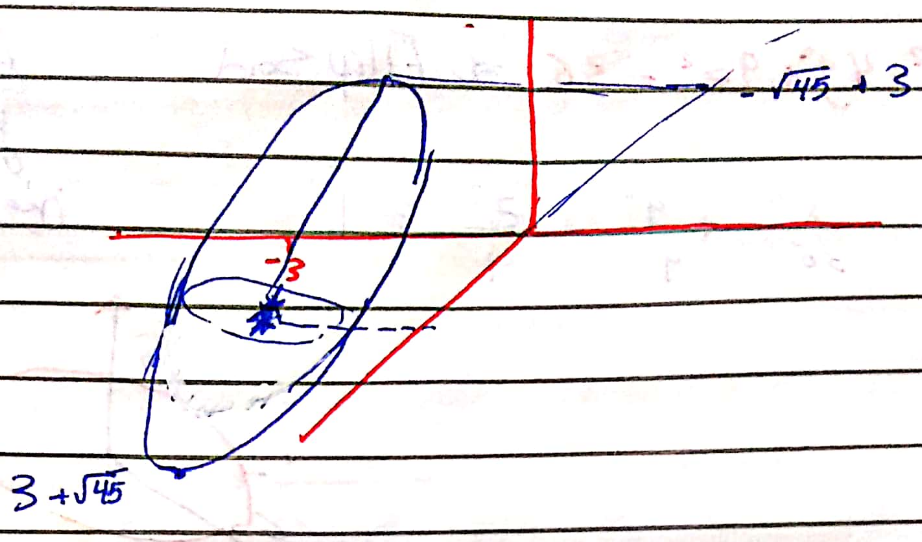
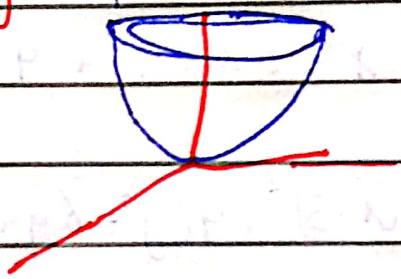


$\boxed{2}$ paraboloid:-

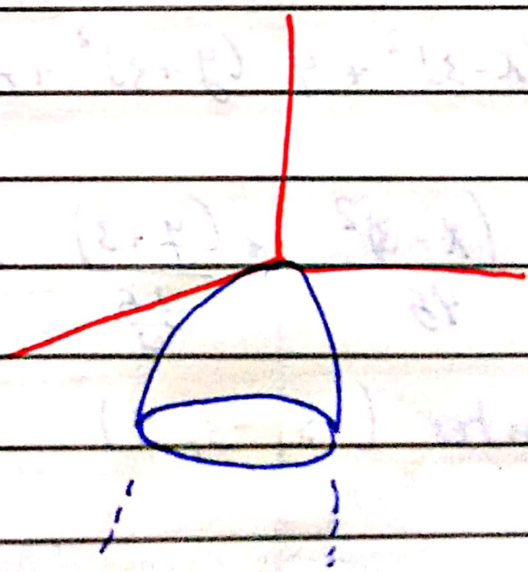
لا يوجد جذر .

$Z = x^2+y^2$



تربيعين موجودين في نفس الجهة وموجبين

$Z = -x^2 - y^2$



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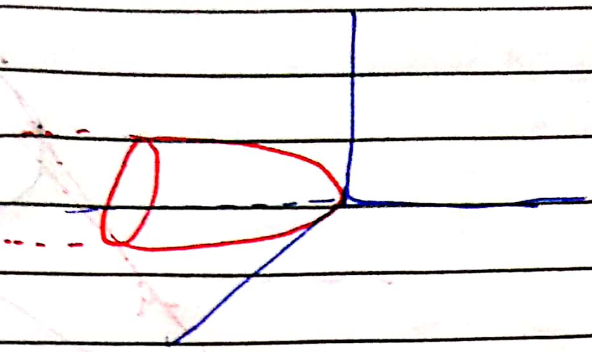
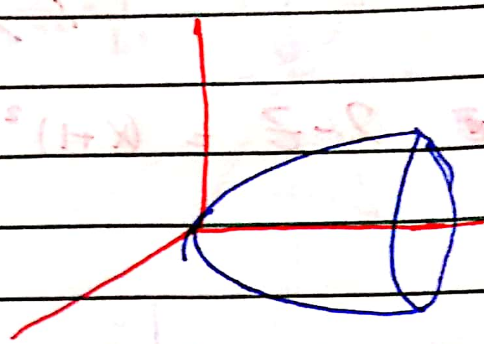
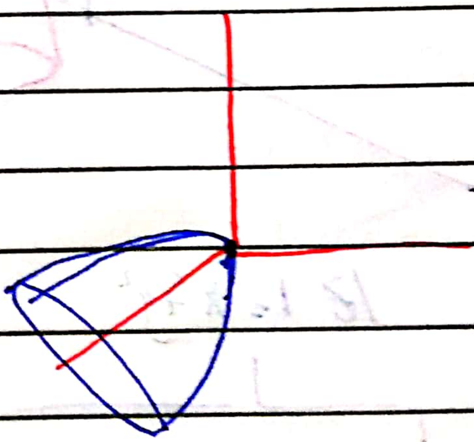
$-y = x^2 + z^2$



$y = 5x^2 + 7z^2$



$x = \frac{y^2}{8} + z^2$



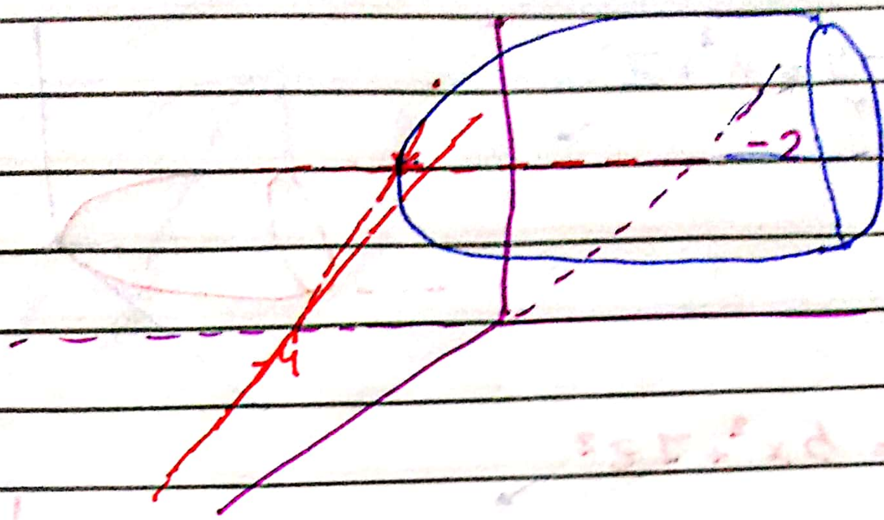
$y = x^2 + 4x + 5z^2$

$4 + y = x^2 + 4x + 4 + 5z^2$ $\qquad y + 4 = (x+2)^2 + (5z^2)$

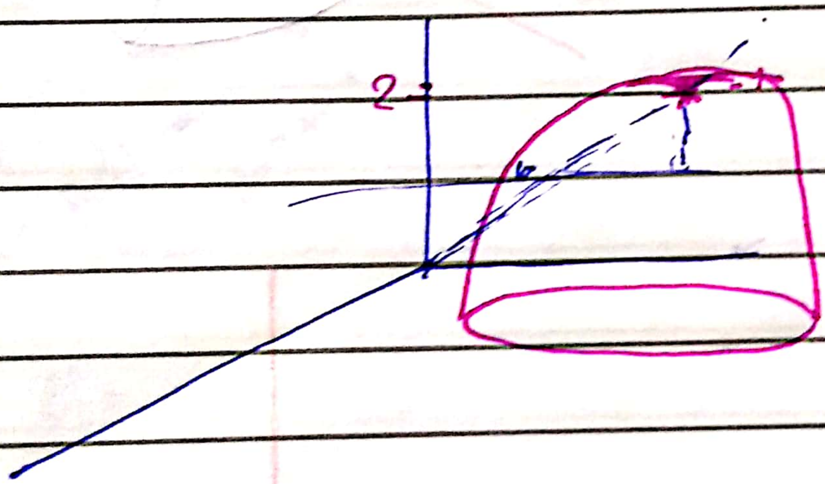
$x = -2 \qquad y = -4 \qquad z = 0$

$(-2, -4, 0)$

الرسم في الخلف

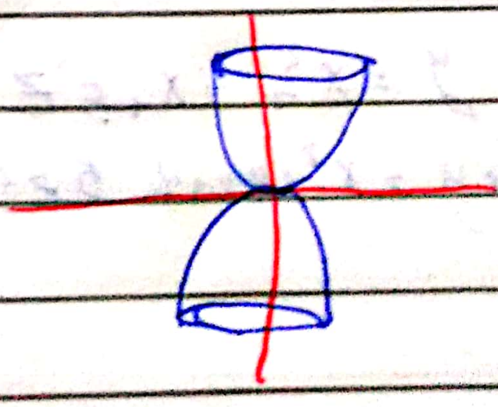


* $2 - z = (x+1)^2 + y^2$ paraboloid vertex $(-1, 0, 2)$



$|z| = x^2 + y^2$

$z = x^2 + y^2$ $\qquad$ $-z = x^2 + y^2$



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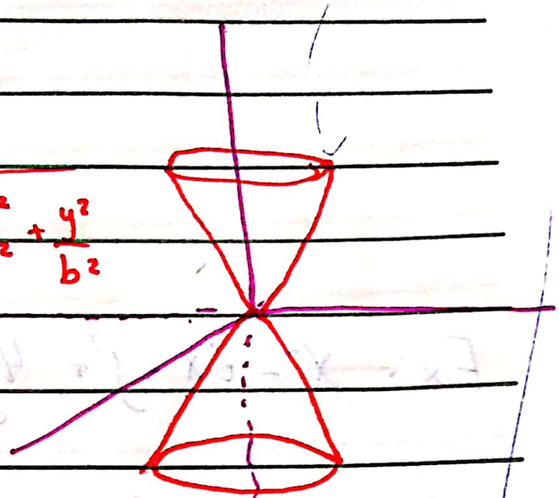
مخروط

3 Elliptic Cone.

لايوجد ثابت
لوحده .

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

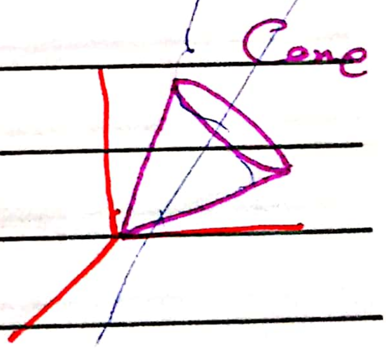
$$z = +\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$$



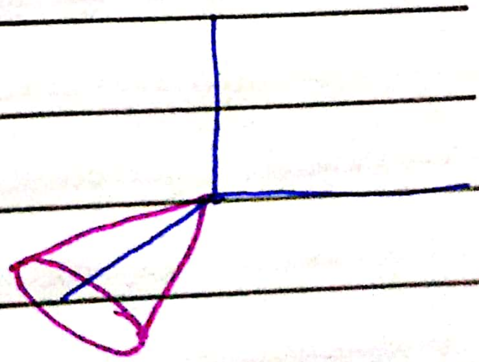
$$z = -\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$$

Ex: $x = -\sqrt{y^2 + \frac{z^2}{6}}$

Cone.



$$x = \sqrt{y^2 + 4z^2}$$



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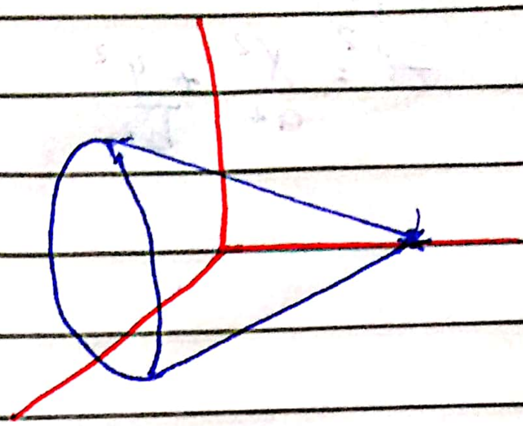
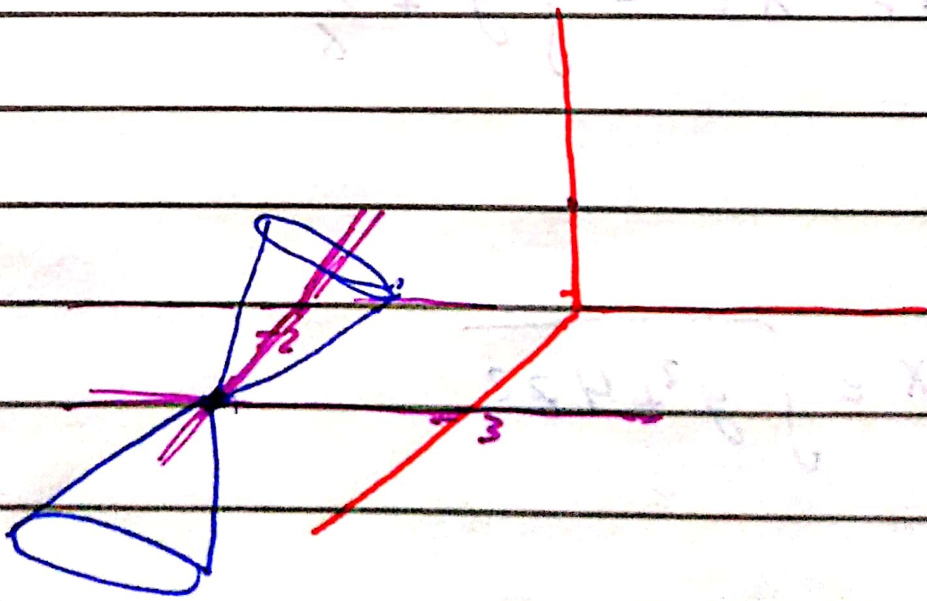
$1-y = \sqrt{x^2+z^2}$ Cone vertex $(0,1,0)$



Ex:- $x^2-6x+y^2+4y-z^2+13=0$ Cone

$(x-3)^2+(y+2)^2+\cancel{13}=\cancel{9+4}+z^2$

$(x-3)^2+(y+2)^2=z^2.$ vertex $(3,-2,0)$



Ex:- Identify (Give the name) of the trace of the surface $2x^2+y^2-z^2=16$ in the

$\boxed{1}$ plane $z=1$

$\boxed{2}$ " $y=1$

Sol:- ① $2x^2+y^2-1=16 \Rightarrow 2x^2+y^2=17$

Elipse parallel to the xy-plane.

In 2D (plane).

دائرة $(x)^2+(y)^2=$ موجب

ناقص ellipse $\frac{(x)^2}{a^2}+\frac{(y)^2}{b^2}=+$

$a \neq b$

$\boxed{2}$ eq $\Rightarrow 2x^2+1-z^2=16$

$2x^2-z^2=15$

hyparabola parallel to xz-plane.

hyparabola $\frac{(x)^2}{a^2}-\frac{(y)^2}{b^2}=+$

زائد

parabola متغير ليس $=(x)^2$

له تربيع مكافئ

Ex:- Give the name of the trace of the quadric surface $x^2-3y+z^2=10$ in the plane

$\boxed{1}$ $z=2$

$\boxed{2}$ $y=-1$

تابع اكمال $\rightarrow$

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Subject

$\boxed{1}$ eq $\Rightarrow x^2 - 3y + 4 = 10$

$x^2 - 3y = 6$

Parabola parallel xy-plane.

$\boxed{2}$ eq $\Rightarrow x^2 + 3 + z^2 = 16$

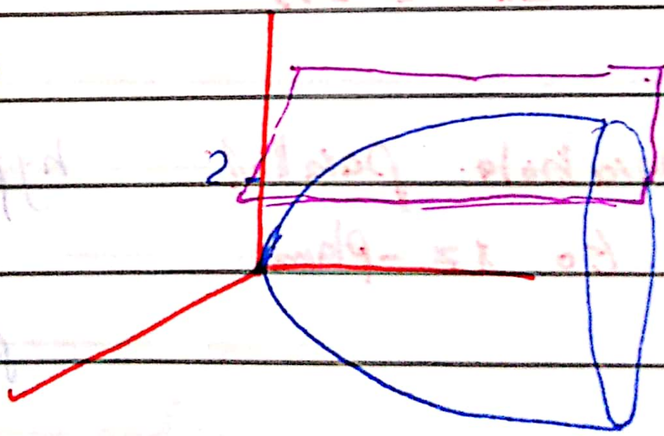
$x^2 + z^2 = 7$

Circle parallel to the xz-plane.

لأنه $z=2$ تقطعه

عند $z=2$.

حب التخيل .



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Vector Funcs. مساقته قد تكون متغيرات وقد تكون ثوابت.

$\vec{r}(x,y,z) = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$

Vector Func.

Ex:- $\vec{r}(t) = \langle \ln t, e^t, t^2 \rangle$ vect func

$\vec{r}(x,y) = \langle 1, \frac{2}{x}, 5 \rangle$ " "

$\vec{r}(t) = \langle 1, 0, -1 \rangle$ " "

Ex:- $\vec{r}(t) = \langle 2t^2, e^t, \ln(t-1) \rangle$

$\vec{r}'(t) = \langle 4t, e^t, \frac{1}{t-1} \rangle$ المشتقة

$\boxed{2}$ $\vec{r}(t) = \langle t^2, 5t \rangle$.

$$\int_0^1 \vec{r}(t)\,dt = \langle \int_0^1 t^2\,dt, \int_0^1 5t\,dt \rangle$$

$$\frac{t^3}{3}\Big]_0^1, \frac{5t^2}{2}\Big]_0^1 = \langle \frac{1}{3}, \frac{5}{2} \rangle.$$

$\boxed{3}$ $\vec{r}(t) = \langle f, g, h \rangle$

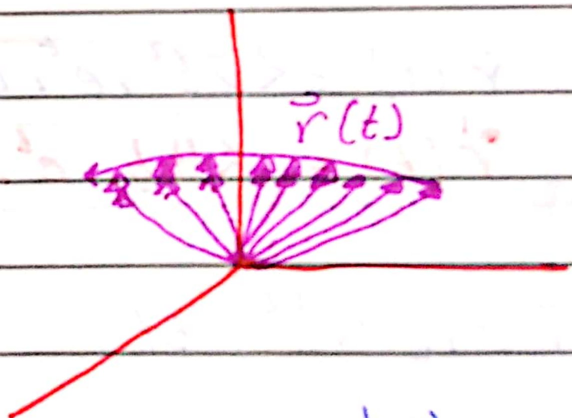
$$|\vec{r}(t)| = \sqrt{f^2 + g^2 + h^2}$$

Ex:- $\vec{r}(t) = \langle \cos t, \sin t, 3 \rangle$

$$|\vec{r}(t)| = \sqrt{\cos^2 t + \sin^2 t + 9}$$

$$= \sqrt{10}$$

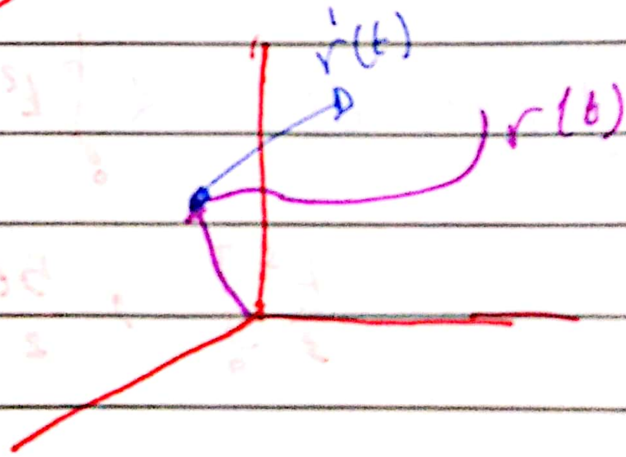
Geometrically: $\vec{r}(t)$



Rule:-

$\vec{r} \perp \vec{r}\,'$

$\vec{r} \cdot \vec{r}\,' = 0$



Def:- The arc length of $\vec{r}(t)$ form $t=a$ to $t=b$ is

$$L=\int_a^b |\vec{r}'(t)|\, dt$$

Ex:- Find the arc length of $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ from $t=0$, to $t=2$

Sol:-

$\vec{r}' = \langle -\sin t, \cos t, 1 \rangle$

$\Rightarrow |\vec{r}'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$L=\int_0^2 |\vec{r}'(t)| = \int_0^2 \sqrt{2} = 2\sqrt{2}.$$

Rule:-

لا يوجد زاوية

let $\vec{r}(t)$ be a smooth (انسيابي)

Curve:-

$\boxed{1}$ The tangent vector to $\vec{r}(t)$ is $\vec{r}\,'(t)$

$\boxed{2}$ " unit " " " $\vec{T}(t) = \dfrac{\vec{r}\,'}{|\vec{r}\,'|}$

tangent

$\boxed{3}$ The binormal vector of $\vec{r}(t)$ is

$\vec{B}(t) = \vec{T} \times \vec{N}$

$\boxed{4}$ The Unit normal vector of $\vec{r}(t)$ is

$\vec{N}(t) = \dfrac{T'}{|T'|}$

ثنائية التعامد

Find the binormal vector of the Circular

helix $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$

حلزون

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Sol:- $r' = \langle -\sin t, \cos t, 1 \rangle$

$|\vec{r}'| = \sqrt{2}$

$\vec{T} = \frac{\vec{r}'}{|\vec{r}'|} = \langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$T' = \langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \rangle \Rightarrow |T'| = \frac{1}{2}$

$\vec{N} = \frac{T'}{|T'|} = 2\langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0 \rangle$

$$\vec{B} = \vec{T} \times \vec{N} = 2\begin{vmatrix} i & j & k \\ \frac{-\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-\cos t}{\sqrt{2}} & \frac{-\sin t}{\sqrt{2}} & 0 \end{vmatrix}$$

$$2\left(\hat{i}\left(0 + \frac{\sin t}{2}\right) - j\left(0 + \frac{\cos t}{2}\right) + k\left(\frac{\sin^2 t}{2} + \frac{\cos^2 t}{2}\right)\right)$$

$\langle \sin t, -\cos t, 1 \rangle$

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Def:- The curvature of $\vec{r}(t)$ written $\kappa(t)$ is defined by:

$\boxed{1}$ $$\kappa(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$$

$\boxed{2}$ $$\kappa(t) = \frac{|\vec{T}\,'|}{|\vec{r}\,'|}$$

Ex:- Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at the origin

Sol:- $A(0,0,0)$ $\quad \langle t, t^2, t^3 \rangle$ pt $(0,0,0)$

$t = 0$

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle \Rightarrow \vec{r}\,'(0) = \langle 1, 0, 0 \rangle$

$\vec{r}\,'' = \langle 0, 2, 6t \rangle \Rightarrow \vec{r}\,''(0) = \langle 0, 2, 0 \rangle$

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$$K(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$$

$$= \frac{\sqrt{|\vec{r}\,'|^2\,|\vec{r}\,''|^2 - (\vec{r}\,' \cdot \vec{r}\,'')^2}}{|\vec{r}\,'|^3}$$

$$= \frac{\sqrt{1(4) - 0^2}}{(\sqrt{1})^3} = 2$$

Ex:- Find the Curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

SoL:- $K(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} =$

$$= \frac{\sqrt{(1+4t^2+9t^4)\,(4+36t^2) - (4t+18t^3)^2}}{\left(\sqrt{1+4t^2+9t^4}\right)^3}$$

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Show that the curvature of radius a is constant equals $\frac{1}{a}$

Sol:-

$x = x_0 + a\cos t$

$y = y_0 + a\sin t$

$\vec{r} = \langle x, y \rangle$

$\langle x_0 + a\cos t, y_0 + a\sin t \rangle$

$\vec{r}' = \langle -a\sin t, a\cos t \rangle$

$\vec{r}'' = \langle -a\cos t, -a\sin t \rangle$

$|\vec{r}'| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$

$|\vec{r}''| = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = a$

$$K(t) = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{\sqrt{|\vec{r}'|\,|\vec{r}''| - (\vec{r}' \cdot \vec{r}'')}}{|\vec{r}'|^3}$$

$$\frac{\sqrt{a^2 \cdot a^2 - (a^2\cos t\sin t - a^2\cos t\sin t)^2}}{a^3} = \frac{a^2}{a^3} = \frac{1}{a}$$

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Sec 14.1

Func of Several variables

$f(x,y)$ = Func in 2- variable x,y
$\therefore f$ func in Several variables.

$f(x,y,z)$ = Func in 3- variable x,y,z
$\therefore f$ in several variables.

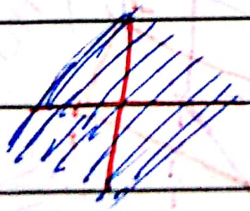
Ex:- Find and Sketch the Domain of the Func

1) $f(x,y)=3$ 2) $f(x,y)=\sqrt{|x|+|y|-1}$

3) $f(x,y,z)=\sqrt{|x|+|y|-1}$

تذكيرات
* المقام لا يساوي صفر
* ما تحت الجذر الزوجي ≥0
* ما داخل ال Log > 0

Sol:- Domain $=\{(x,y)\in R\}$
$=\mathbb{R}^2=$ plane



2) Domain $(f)=\{(x,y)\in\mathbb{R}^2 : |x|+|y|-1\geq 0\}$

$|x|+|y|-1=0$

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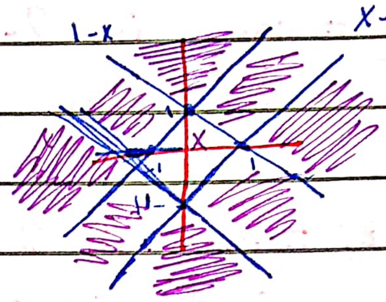
$|x| + |y| = 1$

$x \geq 0$, $y \geq 0 \Rightarrow x + y = 1$ $\qquad y = 1 - x$

$x \geq 0$ $\quad y < 0 \Rightarrow x - y = 1$

$y = x - 1$



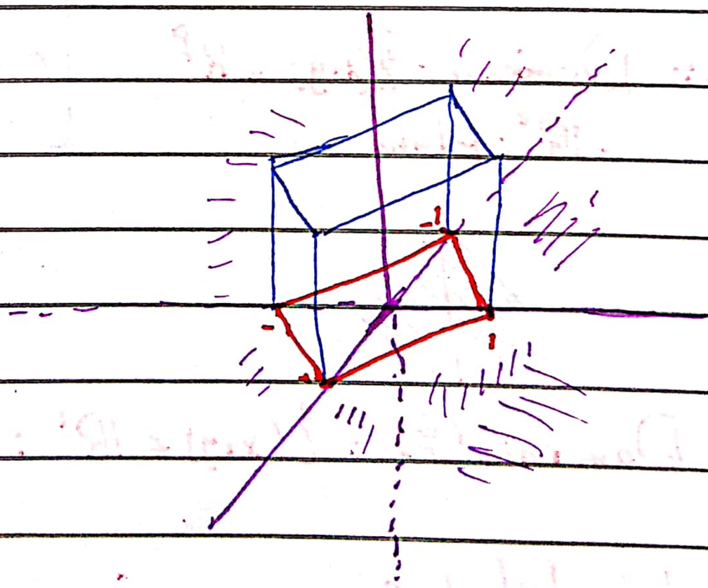
$x \leq 0$, $y \geq 0 \Rightarrow -x + y = 1$

$y = 1 + x$

$x \leq 0$, $y \leq 0 \Rightarrow -x - y = 1$

$y = -1 - x$

$\boxed{3}$ $Dom(f) = \{(x, y, z) \in \mathbb{R}^3 : \ |x| + |y| - 1 \geq 0$



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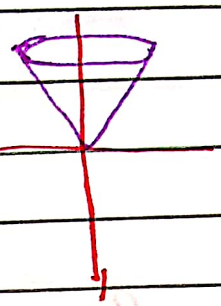
Ex:- Sketch the graph of the func

$f(x,y) = \sqrt{x^2+y^2}$

Sol:- $z = \sqrt{x^2+y^2}$

Surface is $f(x,y)$



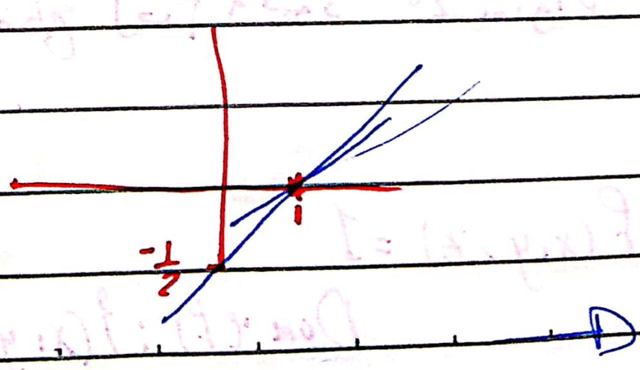
ممكن نرسم المجال عادي
لكن عندما يطلب رسم اقتران
ذا متغيرين نرسمه في 3D
لكن لا نستطيع رسم اقتران 3D
ممكن فقط رسم مجاله

Ex: $f(x,y,z) = \ln(1-x+2y)$

$Dom(f) = \{(x,y,z) \in \mathbb{R}^3 : 1-x+2y > 0\}$

Sketch Dom:- $1-x+2y=0 \Rightarrow y = \frac{x-1}{2}$

حل خاطئ لأنه
3 لكن 2D
اقتران في 3D لكن نحن خلال



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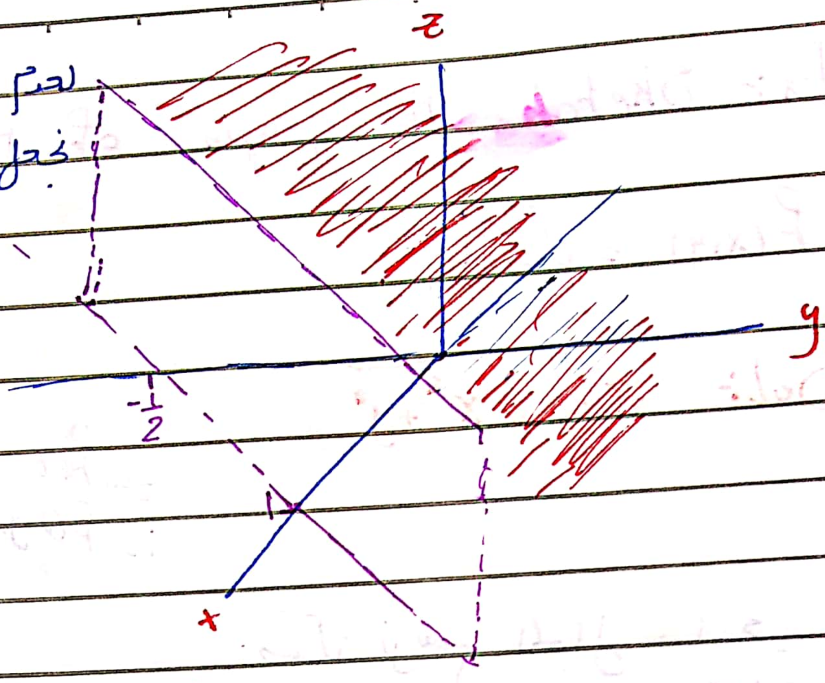
لعدم وجود المساواة
نجعل الخط متقطع.



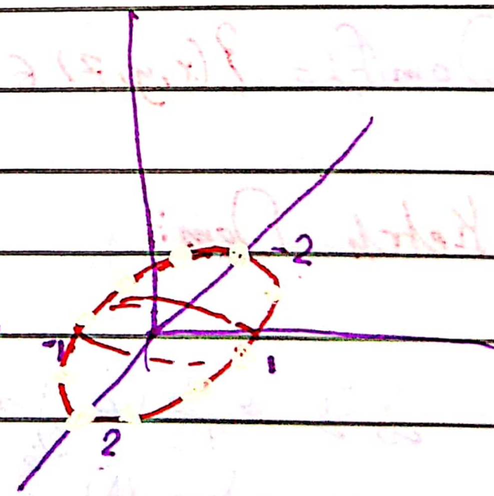
Ex:- $f(x,y,z) = \dfrac{1}{\sqrt{1-\frac{x^2}{4}-y^2-z^2}}$

$Dom(f) = \{(x,y,z) \in \mathbb{R}^3 : 1-\frac{x^2}{4}-y^2-z^2 > 0$

$\frac{x^2}{4}+y^2+z^2=1$

Ellipsoid

متقطع لعدم وجود المساواة.



* $f(x,y,z)=1$

$Dom(f) : \{(x,y,z) \in \mathbb{R}^3\}$ كله

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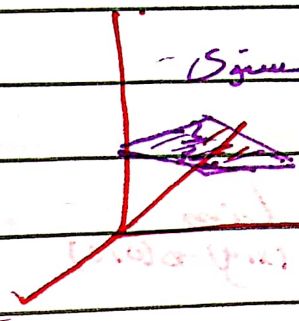
$f(x,y) = 3x+2$

$f(1,0) = 3(1)+2 = 5$

$f(0,1) = 3(0)+2 = 2$



سطح المستوى = Dom

Sec 14.2.

Limits and Continuity

Ex:- [1] $\lim\limits_{(x,y,z)\to(0,1,-1)} x e^{xyz}$

$x \to 0$

$y \to 1$

$z \to -1$

* نفشل التعويض عندما تكون نتيجة النهاية

$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, 1^{\infty}, 0^{0}$

$\infty - \infty, -\infty + \infty$

$\infty \pm 0$

* $DNE = \dfrac{\text{عدد ليس الصفر}}{\text{صفر}}$

تعويض $\Rightarrow 0e^{0(1)(-1)} = 0e^{0} = 0 * 1 = \boxed{0}$

[2] $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{xy^2} - 1}{x^2 - y + 1} = \dfrac{0}{1} = \boxed{0}$

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$\boxed{3}$ $\lim\limits_{(x,y)\to(0,1)} \frac{e^x + \sin x}{x^2 + 2y - 2} = \frac{e^0 + \sin 0}{0 + 2 - 2}$

$\frac{1}{0} = D.N.E$

$\boxed{4}$ $\lim\limits_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x + y} = \frac{(x-y)\cancel{(x+y)}}{\cancel{(x+y)}} = \boxed{0}$

Ex:- $\lim\limits_{(x,y)\to(2,2)} \frac{x^4 - 4x^2y + 4y^2}{x^2 - 2y} = \frac{0}{0}$

$\lim \frac{(x^2 - 2y)^{\cancel{2}}}{\cancel{x^2 - 2y}} = \lim\limits_{(2,2)} (x^2 - 2y) = 0$

$\boxed{2}$ $\lim\limits_{(x,y)\to(2,1)} \frac{x^2 + xy - 6y^2}{x - 2y} = \lim \frac{\cancel{(x-2y)}(x+3y)}{\cancel{(x-2y)}}$

$\lim\limits_{(2,1)} (x + 3y) = 5$

$\boxed{3}$ $\lim\limits_{(x,y)\to(0,-3)} \frac{\sin(2xy^2)}{x} = \lim \frac{\sin(2xy^2)}{xy^2} \cdot \frac{xy^2}{x}$ $\Rightarrow$

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$$\theta = xy^2 \quad (x,y)\to(0,-3), \quad \theta \to 0$$

$$= \lim_{\theta\to 0} \frac{\sin 2\theta}{\theta} \lim_{(x,y)\to(0,-3)} y^2$$

$$= 2(9) = 18$$

$$\boxed{\lim_{\theta\to 0} \frac{\sin k\theta}{\theta} = k}$$

ملاحظة

$\boxed{4}$ $$\lim_{(x,y)\to(0,0)} \frac{(4x-6y+1)^5-1}{(2x-3y-2)^3+8} = \frac{0}{0}$$

$$= \lim_{(0,0)} \frac{(2(2x-3y)+1)^5-1}{(2x-3y-2)^3+8}$$

$$z = 2x-3y$$
$$(x,y)\to(0,0)$$
$$\boxed{z=0}$$

$$\lim_{z\to 0} \frac{(2z+1)^5-1}{(z-2)^3+8}$$

* هنا يجوز الحل على L.H لأنه تحول لمتغير واحد.
لكن من بداية المسألة لم نطبق عليها لأنها متغيرين يحولوه لواحد.

$$\lim_{z\to 0} \frac{5(2z+1)^4(2)}{3(z-2)^2} = \frac{10}{12} = \boxed{\frac{5}{6}}$$

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تذكير :-

Polar Coordinates :-

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$(x,y) \to (0,0)$

$r \to 0^+$

$\theta \in \mathbb{R}$

$$\boxed{|a \pm b| \leq |a| + |b|}$$

Ex :- (1) $\lim\limits_{(0,0)} \dfrac{x^4 - y^5}{x^2+y^2}$

$$\frac{r^4\cos^4\theta - r^5\sin^5\theta}{r^2}$$

$$\lim_{r\to 0} \frac{r^4(\cos^4\theta - r\sin^5\theta)}{r^2}$$

$$\lim_{r\to 0} r^2(\cos^4\theta - r\sin^5\theta)$$

$= 0$ by the Sequeez Thm.

$|\cos^4\theta - r\sin^5\theta|$

$\leq |\cos^4\theta| + r|\sin^5\theta|$

$\leq 1 + 0$

$\therefore 0 \leq |r^2(\cos^4\theta - r\sin^5\theta)| \leq r^2(1+r)$

$\downarrow$ $\qquad\qquad\qquad\qquad \downarrow$

0 $\qquad$ صفر $\therefore$ $\qquad$ 0

Def :- let $C : x = f(t)$ $\quad y = g(t)$ $\quad z = h(t)$ be curve that pass through a pt (x_0, y_0, z_0) when $t = t_0$ $\boxed{\text{او بمعنى اخر عوض ب } t_0 \text{ تعطيني } x_0}$

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ \text{along } C}} F(x,y,z) = \lim_{t\to t_0} F(f(t), g(t), h(t))$$

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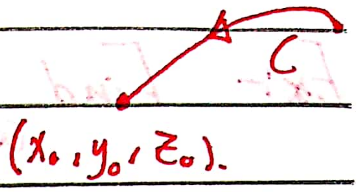
Rule:-



$\boxed{1}$ $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} F(x,y,z)$ does not Exist (DNE)

$$\iff \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ \text{along } C_1}} F \neq \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ \text{along } C_2}} F$$

C_1, C_2 pass through (x_0, y_0, z_0)

$\boxed{2}$ $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} F = L \in \mathbb{R}$ exist

$$\iff \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ \text{along } C}} F = L \text{ for all curves } C \text{ pass through } (x_0, y_0, z_0).$$

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Ex:- Find $\lim_{(x,y)\to(1,-1)} \frac{x^2-3y-4}{x+y}$ along the

Curves:

$\boxed{1}$

(a) $C_1 = y = 2x-3$

(b) $C_2 = x = 3t, \; y = t^2 - \frac{10}{9}$

$\boxed{2}$ Is $\lim_{(x,y)\to(1,-1)} \frac{x^2-3y-4}{x+y}$ exist?

$\boxed{1} \to \boxed{a}$

$$\lim_{\substack{\text{along} \\ C_1}} F = \lim_{x\to 1} \frac{x^2-3(2x-3)-4}{x+2x-3}$$

$$= \lim_{x\to 1} \frac{x^2-6x+5}{3x-3} = \frac{0}{0}$$ لوبتال *

$$\lim \frac{2x-6}{3} = \frac{-4}{3}$$

(b) $x \to 1$ $\quad y \to -1$ $\quad x = 3t$

* اذا x اعطيت قيمتان للt $\quad 1 = 3t \quad t = \frac{1}{3}$

وال y " " للt $\quad t \to \frac{1}{3} \Rightarrow$

ناخذ الاشتراك بينهم

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$$\lim_{\substack{\text{along} \\ C_2}} F = \lim_{t \to \frac{1}{3}} \frac{(3t)^2 - 3(t^2 - \frac{10}{9}) - 4}{3t + t^2 - \frac{10}{9}}$$

$$\text{لوبيتال} = \lim \frac{18t - 6t}{3 + 2t}$$

$$= \frac{4}{3 + \frac{2}{3}} = \frac{4}{\frac{11}{3}} = \frac{12}{11}$$

$\boxed{2}$ the $\lim_{\substack{\text{along} \\ C_1}} F \neq \lim_{\substack{\text{along} \\ C_2}} F \Rightarrow$

$$\lim \frac{x^2 - 3y - 4}{x + y} \quad \text{DNE}$$

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Ex:- Find the Limit if it exists

[1] $\lim\limits_{(x,y,z)\to(1,-2,3)} \dfrac{(x-1)^2(y+2)(z-3)^2}{(x-1)^3+3(y+2)^5+(z-3)^{15}}$

let: C_1 $x = \boxed{t} + \boxed{1}$, $y = \boxed{0} + \boxed{-2}$, $z = \boxed{0} + \boxed{3}$

$t \to 0 : (x,y,z) \Rightarrow (1,-2,3)$

$t = 0 : (x,y,z) = (1,-2,3)$.

$\lim\limits_{\substack{\text{along} \\ C_1}} F = \lim\limits_{t\to 0} \dfrac{(t+1-1)^2(-2+2)(3-3)^2}{(t+1-1)^3+3(-2+2)^5+(3-3)^{15}}$

$= \dfrac{0}{t^3} = \lim 0 = 0$.

let C_2 : $x = \boxed{t^{\frac{15}{3}}} + \boxed{1}$, $y = \boxed{t^{\frac{15}{5}}} + \boxed{-2}$, $z = \boxed{t^{\frac{15}{15}}} + \boxed{3}$

اسس موجودة في المقام $3, 5, 15$

م.م.أ $= 15$

مضاعف مشترك اصغر لكي نرفع اس

الـ t^* ونقسم العامل على الاسس الاصلية

$t^{\frac{15}{3}}$ x لهم لذا عند الـ x

$x = t^5 + 1$, $y = t^3 - 2$, $z = t + 3$

$\Rightarrow$

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When $t=0$: $(x,y,z) = (1,-2,3)$

$$\lim_{\substack{\text{along}\\ C_2}} F = \lim_{t \to 0} \frac{(t^5+1-1)^2 (t^3-2+2)(t+3-3)^2}{(t^5+1-1)^3 + 3(t^3-2+2)^5 + (t+3-3)^{15}}$$

$$\lim \frac{t^{10} \cdot t^3 \, t^2}{t^{15} + 3t^{15} + t^{15}} = \lim \frac{t^{15}}{5t^{15}} = \lim \frac{1}{5} = \frac{1}{5}$$

$$\lim_{\substack{\text{along}\\ C_1}} F \neq \lim_{\substack{\text{along}\\ C_2}} F \Rightarrow \lim F \text{ DNE.}$$

Ex: Find the limit if it exists

$$\lim_{(x,y,z)\to(-1,3,0)} \frac{(x+1)\,z^2}{(x+1)^2 + (y-3)^3 + z^4}$$

C_1: $x = \boxed{0} + \boxed{-1}$, $y = \boxed{t} + \boxed{3}$, $z = \boxed{0} + \boxed{0}$

$x = -1$ $\quad y = t+3 \quad z = 0$

$t=0$: $(x,y,z) = (-1,3,0)$

$F \Rightarrow$

$$\lim_{\substack{\text{along} \\ C_1}} F = \lim_{t \to 0} \frac{0}{0^2 + t^3} = 0$$

اسس المقام للمتغيرات

2, 4, 3

م.م.أ = 12

let C_2: $x = \boxed{t^6} + \boxed{-1}$, $y = \boxed{t^4} + \boxed{3}$, $z = \boxed{t^3} + \boxed{0}$

when $t = 0$: $(x, y, z) \to (-1, 3, 0)$

$$\lim_{\substack{\text{along} \\ C_2}} F = \lim_{t \to 0} \frac{(t^6 - 1 + 1)(t^3)^2}{(t^6 - 1 + 1)^2 + (t^4 + 3 - 3)^3 + (t^3)^4}$$

$$= \lim \frac{t^{12}}{t^{12} + t^{12} + t^{12}} = \frac{1}{3}$$

$$\lim_{\substack{\text{along} \\ C_1}} F \neq \lim_{\substack{\text{along} \\ C_2}} F \Rightarrow \lim \ \text{D.N.E}$$

Ex :- Find the Limit if it exists

1. $\lim_{(x,y)\to(0,0)} \frac{x^{2/3} y^2}{x^2+y^3}$

2. $\lim_{(x,y)\to(0,1)} \frac{x^2}{x^2+(y-1)^3}$

3. $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{\sqrt{x^2+y^2+1}-1}$

Solution :-

1. $C_1 : x=0 \,, y=t$

when $t=0 : (x,y)\to(0,0)$

$$\lim_{\substack{\text{along}\\ C_1}} F = \lim_{t\to 0} \frac{0^{2/3} t^2}{0^2+t^3} = 0$$

$C_2 : x=t^3 \quad y=t^2$

$$\lim_{\substack{\text{along}\\ C_2}} F = \lim_{t\to 0} \frac{t^2 t^4}{2t^6} = \frac{1}{2}$$

D.N.E

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$\boxed{2}$ $C_1: \ x = 0 \qquad y = t + 1$

when $t = 0$: $(x,y) = (0,1)$

$$\lim_{\substack{\text{along} \\ C_1}} F = \lim_{t \to 0} \frac{0^3}{0^2 + t^3} = 0$$

$C_2: \ x = \boxed{t} + \boxed{0} \ , \ y = \boxed{0} + \boxed{1}$

$$\lim_{\substack{\text{along} \\ C_2}} F = \lim_{t \to 0} \frac{t^2}{t^2} = 1$$

$\therefore$ DNE.

$\boxed{3}$ $x = r \cos\theta \qquad y = r \sin\theta$

$(x,y) \to (0,0)$

$r \to 0^+$

$$\lim_{(x,y) \to (0,0)} F = \lim_{r \to 0^+} \frac{r^2 \cos^2\theta \ r \sin\theta}{\sqrt{r^2+1} - 1}$$

$$\lim \left[\frac{r^3}{\sqrt{r^2+1} - 1} \right] (\cos^2\theta \ \sin\theta)$$

اذا بقيت θ تكون الاجابة

DNE لأنه مع اختلاف

الزاوية تختلف المنحنيات .

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*اذا كانت الاجابة تاعت [ ] صفر اذا كل وتساوي صفر اذا كان ما داخل [ ] اجابته ليست صفر نعوض المتغيرات

$$\lim_{r\to 0} \frac{r^3}{\sqrt{r^2+1}-1} * \frac{\sqrt{r^2+1}+1}{\sqrt{r^2+1}+1} = \lim \frac{r^3 * 2}{r^2+1-1}$$

$$\lim \frac{2r^3}{r^2} = \lim 2r = 0$$

Ex: Find limit if it exist

$$\lim \frac{xy}{\sqrt{x^2+y^2+1}-1}$$

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$$\lim F = \lim \boxed{\frac{r^2\cos\theta\sin\theta}{\sqrt{r^2+1}-1}} = \lim (2)\cos\theta\sin\theta$$

$= 2.$

Curves اشتق

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Def:- A func $f(x,y,z)$ is Conts (متصل) at $(x_0, y_0, z_0) \in Dom(f)$

if $\lim_{(x,y,z) \to (x_0, y_0, z_0)} f = f(x_0, y_0, z_0)$

$\boxed{1}$ $f(x,y) = \frac{x}{x^2 - y}$ is Conts on

$Dom(f) = \{(x,y) \in \mathbb{R}^2 ; y \neq x^2\}$

$\boxed{2}$ $f(x,y,z) = \sqrt{1-(x^2+y^2+3z^2)}$ is Conts on

$Dom(f) = \{(x,y,z) \in \mathbb{R}^3 : 1-(x^2+y^2+3z^2) \geq 0\}$

$\boxed{3}$ $f(x,y) = 5xe^y$ Conts on $Dom(f) = \mathbb{R}^2$

nada

Ex:- Find all values of k that make

$$f(x,y) = \begin{cases} \dfrac{k\,x^2 - 2y^2}{x^2+y^2} , & (x,y) \neq (0,0) \\ -2 , & (x,y) = (0,0) \end{cases}$$

is Cont at the origin

Sol:- $\lim\limits_{(x,y)\to(0,0)} f = f(0,0)$

$$\lim_{(0,0)} \frac{k\,x^2 - 2y^2}{x^2+y^2} = -2$$

$x = t \to 0$ $\quad y = 0 \to 0$

$t \to 0$

$\lim\limits_{\text{along } C_1} f = -2 \Rightarrow \lim\limits_{t\to 0} \dfrac{k\,t^2}{t^2} = -2$

$$\boxed{k = -2}$$

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Ex:- Find K that make the func

$$f(x,y) = \begin{cases} \dfrac{\sqrt{xy+8}-3}{xy-1}, & xy>1 \\ K, & xy \le 1 \end{cases}$$

Cont every where

Sol:- $\lim_{xy \to 1} \dfrac{\sqrt{xy+8}-3}{xy-1} = K$ $\quad xy = t$, $t \to 1$

$$K = \lim_{t \to 1} \frac{\sqrt{t+8}-3}{t-1} \overset{\frac{0}{0}}{=} \lim \frac{\frac{1}{2\sqrt{t+8}}}{1}$$

$$\lim \frac{1}{2\sqrt{t+8}} = \frac{1}{6}$$

$$f(t) = \begin{cases} \dfrac{\sqrt{t+8}-3}{t-1}, & t>1 \\ k, & t \le 1 \end{cases}$$

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Sec 14.3 :- Paratial Derivatives

Def :- The partial derivative of a func $z = f(x,y)$ with respect to (w.r.t)

$$z_x(x_0,y_0) = f_x(x_0,y_0) = \underbrace{\lim_{x \to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0}}_{\text{Def}} = \underbrace{\frac{d f(x,y_0)}{dx}\Big|_{x=x_0}}_{\text{قواعد الاشتقاق}}$$

$\boxed{2}$ y at apt (x_0,y_0) is

$$z_y(x_0,y_0) = f_y(x_0,y_0) = \lim \frac{f(x_0,y) - f(x_0,y_0)}{y - y_0} = \frac{d f(x_0,y)}{dy}\Big|_{y=y_0}$$

Ex :- Using def find $f_x(1,0)$, $f_y(1,0)$

where $f(x,y) = \sqrt{x^4 + y^3 + 3}$

$$f_x(1,0) = \lim_{x \to 1} \frac{f(x,0) - f(1,0)}{x - 1} \Rightarrow$$

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$$\lim_{x\to 1} \frac{\sqrt{x^4+3}-2}{x-1}$$

$$\lim \frac{4x^3}{2\sqrt{x^4+3}} = \lim \frac{2x^2}{\sqrt{x^4+3}} = \frac{2}{2} = 1$$

$$f_x(1,0) = \lim_{y\to 0} \frac{f(1,y) - f(1,0)}{y-0}$$

$$= \lim_{y\to 0} \frac{\sqrt{y^2+4}-2}{y} = \lim \frac{\frac{3y^2}{2\sqrt{y^2+4}}}{1}$$

$$\lim \frac{3y^2}{2\sqrt{y^2+4}} = \frac{0}{4} = 0.$$

Ex:- $f(x,y) = \sqrt{x^2+y^2}$ Show that $f_x(0,0)$

$f_y(0,0)$ DNE

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x-0}$$

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$$\lim_{x \to 0} \frac{\sqrt{x^2}}{x} = \lim_{x \to 0} \frac{|x|}{x}$$

$\to \lim_{x \to 0^+} \frac{x}{x} = 1$

DNE

$\to \lim_{x \to 0^-} \frac{-x}{x} = -1$

Ex:- Find $f_x(0,0)$ where $f(x,y) = 3x + \sqrt[3]{8x^2 + 27y^6}$

$f(x,y) = 3x + (8x^2 + 27y^6)^{\frac{1}{3}}$

$f_x = 3 + \frac{1}{3}(8x^2 + 27y^6)^{-2/3} * 24x^2$

$f(0,0) = 3 + \frac{1}{3} 0^{-2/3} (0) = \boxed{3}$

هنا الحل خاطئ وضعه الدكتور للانتباه

اصفر Δ

$f(x,0) = 3x + \sqrt[3]{8x^3 + 0} = 3x + 2x = 5x$

$f_x(x,0) = 5 \Rightarrow \boxed{f_x(0,0) = 5}$

* نعوض y صفر عندما نشتق بالنسبة للـ x لكي يكون الحل صحيح .

Ex:- If $f(x,y) = \sin \dfrac{x}{1+y^2}$

$$f_x = \cos\left(\frac{x}{1+y^2}\right) \cdot \frac{1}{1+y^2}$$

$$f_y = \cos\left(\frac{x}{1+y^2}\right) \cdot \frac{-2xy}{(1+y^2)^2}$$

رموز ٦

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

$$z_x = \frac{\partial z}{\partial x} \qquad z_y = \frac{\partial z}{\partial y}$$

Ex:- let $w = e^{xy^3} \ln(z^2+1)$

Find $\dfrac{\partial w}{\partial y}, \dfrac{\partial w}{\partial z}$

Sol:- $\dfrac{\partial w}{\partial y} = 3xy^2 \, e^{xy^3} \ln(z^2+1) = 3xy^2 w$

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$$\frac{\partial w}{\partial z} = e^{xy^3} \quad \frac{2z}{z^2+1}$$

Ex:- $f(x,y,z) = \sin^2(xyz^3)$

$$\frac{df}{dx} = 2\sin(xyz^3)\ \cos(xyz^3)\ yz^3$$

Ex:- $z = xe^{xy} \Rightarrow \frac{\partial z}{\partial x} = xy\,e^{xy} + e^{xy}$

$$= \boxed{yz + \frac{z}{x}}$$ [illegible]

~~Higher Derivatives~~ [illegible] Higher Derivatives:-

$z = f(x,y)$

[1] $f_{xx} = \frac{\partial^2 f}{\partial x^2}$

[2] $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

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$f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$

$f_{yy} = \frac{\partial^2 f}{\partial y^2}$

$w = f(x, y, z)$ مرتبة المشتقة.

(عدد المتغيرات). *

Ex: If $f(x,y) = x(8x^3 + 27y^3)^{\frac{1}{3}}$

Find $f_{xy}(0,0)$, $f_{yx}(0,0)$

Sol:- $f_{(x)} = \frac{x}{3}(8x^3+27y^3)^{-2/3}(24x^2) + (8x^3+27y^3)^{\frac{1}{3}}$

$f_x(0,y) = 0(27y^3)^{-2/3} \cdot 0 + (0+27y^3)^{\frac{1}{3}}$

$= 3y$

$f_{xy}(0,y) = 3 \Rightarrow f_{xy}(0,0) = 3$

$\Rightarrow$

[2] $f_{(y)} = \frac{x}{3}\left(8x^3+27y^3\right)^{-2/3}\left(3(27)y^2\right)$

$f_y(x,0) = x\left(8x^3+0\right)^{-2/3} \; 27(0)^2$

$f_{yx} = 0 \Rightarrow \boxed{f_{yx}(0,0) = 0}$

Rule:- Clairaut's Thm:-

$z = f(x,y)$ defined on a disk D

If ~~f(x,y)~~ f_{xy}, f_{yx} both Conts on D

The $f_{xy}(a,b) = f_{yx}(a,b)$ (a,b) pt inside D.

D

Ex:- $f(x,y,z) = y^2 \dfrac{e^{2xyz}}{8x^2}$ find f

[1] f_{yxxzx} انا لي الحرية في الترتيب at the pt (1,2,0)

لنا $\Rightarrow f_{zyxxx} \Rightarrow f_z = \dfrac{y^2}{8x^2}\, 2xy\, e^{2xyz}\Big|_{z=0} = \dfrac{y^3}{4x}$

لأنه x بالمقام وسيظل صفر لذا نلعب بالترتيب

$f_{zy} = \dfrac{3y^2}{4x}\Big|_{y=2} = \dfrac{3}{x}$ ونشتق الـ x

Ex:- $\dfrac{\partial^{103}}{\partial y^{63}\,\partial x^{40}}\left(x^{10}\sin(xy) + x^{50} \right)$

at pt $(-1,0)$.

Sol:- $f(x,y) = x^{10}\sin xy + x^{50}$

$$\frac{\partial^{103} f}{\partial y^{63}\,\partial x^{40}} = \frac{\partial^{103} f}{\partial x^{40}\,\partial y^{63}}$$

$$= \frac{\partial^{40}}{\partial x^{40}}\left(-x^{73}\right) = -\frac{(73)!}{(33)!}(-1)^{33}$$

$$= \frac{(73)!}{(33)!}$$

$f_y = x^{10+1}\cos(xy)$

$f_{yy} = -x^{10+2}\sin(xy)$

$f_{yyy} = -x^{10+3}\cos(xy)$

$f_{yyyy} = x^{10+4}\sin(xy)$

$\dfrac{\partial^8 f}{\partial y^8} = x^{10+8}\sin(xy)$

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$$\frac{\partial^{60} f}{\partial y^{60}} = x^{10+60} \sin(xy)$$

$$\frac{\partial^{63} f}{\partial y^{63}} = -x^{10+63} \cos(xy)$$

$$\left.\frac{\partial^{63} f}{\partial y^{63}}\right|_{y=0} = -x^{73}$$

$$g(x,y) = -x^{73}$$

$$g_x = -73\, x^{73-1}$$

$$g_{xx} = -(73)(72)\, x^{73-2}$$

$$g_{xxx} = -(73)(72)(71)\, x^{73-3}$$

$\vdots$

$$\frac{\partial^{40} g}{\partial x^{40}} = \frac{-(73)!}{(73-40)!} x^{73-40}$$

$$= \frac{-(73)!}{33!} x^{33}$$

Rule:- ما يهمني ال

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

Ex:- If $f(x,y) = \int_{y}^{xy} \cos e^{t}\, dt$

$f_{(x)} = \cos(e^{xy})\, y - \underbrace{\cancel{\cos(e^{y})\cdot 0}}_{\text{صفر}}$

$f(0,y) = \cos(e^{0y})\, y = (\cos 1)\, y$

$f_{xy} = \cos 1$

$f_{xy}(0,0) = \cos 1$

Ex:- let $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} , & (x,y) \neq (0,0) \\ 1 , & (x,y) = (0,0) \end{cases}$

Find $f_x(x,y)$, $f_y(x,y)$ $\Rightarrow$

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If $(x,y) \neq (0,0)$

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$$

$$f_x = \frac{2x(x^2+y^2)\cos(x^2+y^2) - 2x\sin(x^2+y^2)}{(x^2+y^2)^2}$$

If $(x,y) = (0,0) \Rightarrow$ find $f_x(0,0)$.

$$f_x(0,0) = \frac{d}{dx} f(x,0)\Big|_{x=0} = \frac{d}{dx} \sin \frac{x^2+0^2}{x^2+0^2}\Big|_{x=0}$$

$$\frac{d}{dx} \frac{\sin x^2}{x^2}\Big|_{x=0} = \frac{\cdots}{x^4} = \frac{0}{0}$$

نلجأ إلى التعريف

$$f(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{\frac{\sin x^3}{x^2} - 1}{x}$$

$$= \lim_{x \to 0} \frac{\sin x^2 - x^2}{x^3} = H'L = \lim \frac{2x\cos x^2 - 2x}{3x^2}$$

$\Rightarrow$

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$$= \frac{2}{3} \lim \frac{x(\cos x^2 - 1)}{x^2}$$

$$= \frac{2}{3} \lim \frac{\cos x^2 - 1}{x} = \frac{0}{0}$$ لو بتال

$$= \frac{2}{3} \lim \frac{-2x \sin x^2}{1}$$

$$= \frac{2}{3} * 0$$

$$= \boxed{0}$$

$$f(x,y) = \begin{cases} \dfrac{2x(x^2+y^2)\cos(x^2+y^2) - 2x\sin(x^2+y^2)}{(x^2+y^2)} \\ 0 \end{cases}$$

حسابات f_y نحن ما جربيها لكن سيكون نفس الجواب.

Sec 14.5 The Chain Rule.

Let $f(x, y, z, w)$

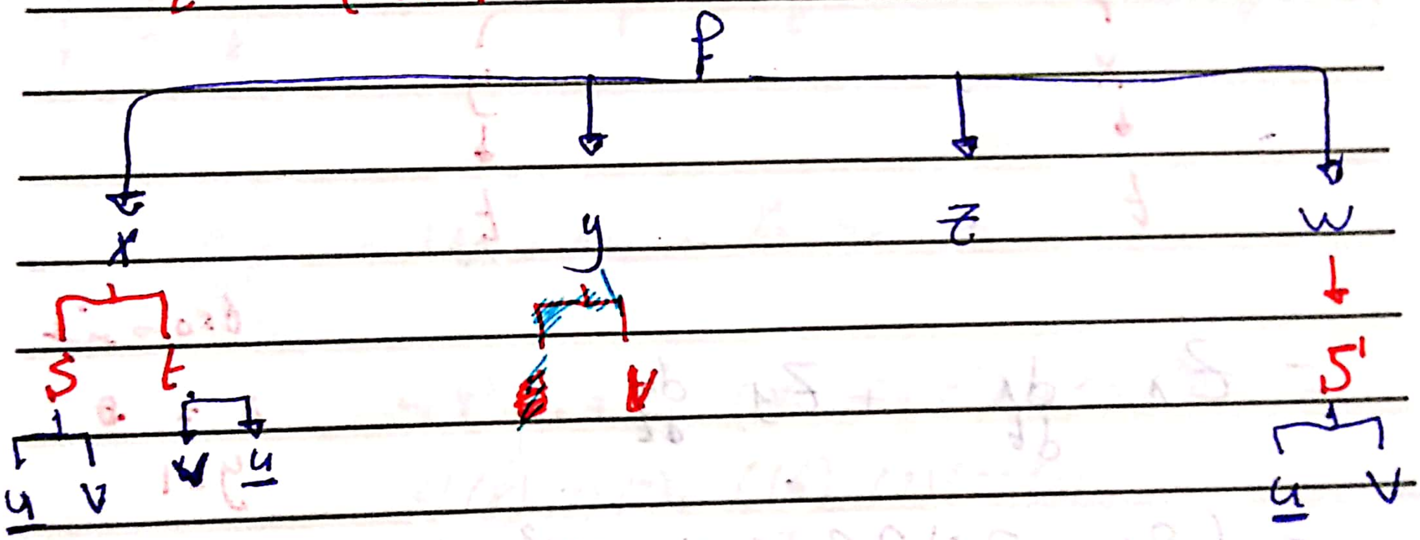
$x = x(s, t)$ $\qquad$ $w = w(s')$

x اقتران بدلالة s و t

$y = y(\cancel{s,t}\ v)$

$s = s(u, v)$

$t = t(u, v)$



$$\frac{\partial f}{\partial u} = f_x * x_s * s_u + f_x\, x_t\, t_u + f_w\, w'\, s_u$$

$$\frac{\partial f}{\partial s} = f_x * x_s + f_w \frac{dw}{ds'}$$

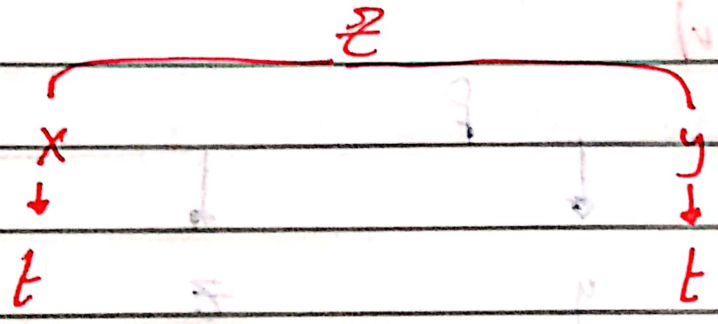
$$\frac{\partial f}{\partial v} = f_x \cdot x_s \cdot s_v + f_x\, x_t\, t_v + f_y \frac{dy}{dv} + f_w\, w' \cdot s_v$$

Ex = $z = x^2 y + 3xy^2$

$x = \sin 2t$

$y = \cos t$

Find $\frac{dz}{dt}$ when $t = 0$



[illegible]

$x = 0$

$y = 1$

$$= z_x \cdot \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$= (2xy + 3y^2)\, 2\cos(2t) + (x^2 + 6xy)(-\sin t).$$

zero.

$$= 2(2xy + 3y^2) + \cancel{(x^2 + 6xy)}\,(0)$$

$$4xy + (3y^2)2 = \boxed{3} * 2 = \boxed{6}$$

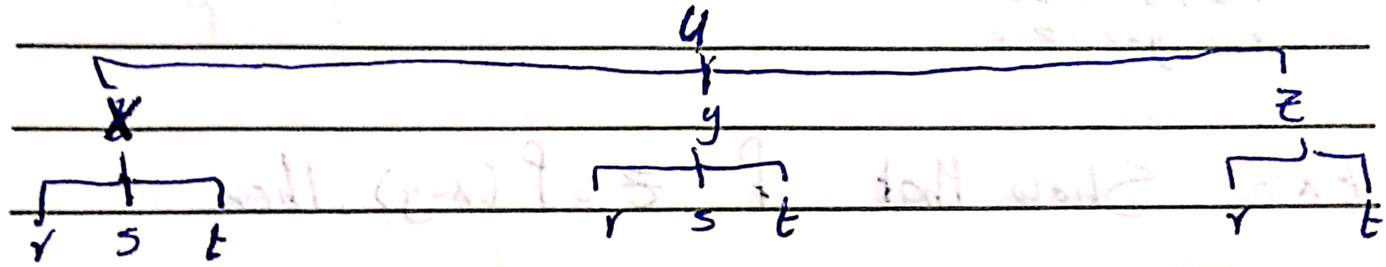
If $u = x^4 y + y^2 z^3$

$x = r s e^{t}$

$y = r s^2 e^{-t}$

$z = r^2 \sin t$

Find $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial t}$ when $r = 2$, $s = 1$, $t = 0$



$$\frac{\partial u}{\partial s} = u_x \cdot x_s + x_y \, y_s =$$

$$4x^3 y \cdot r e^{t} + (x^4 + 2y z^3)\, 2 r s e^{-t}$$

$$4(8)(2)(2) + (16)(2)(2)(1)$$

$$\left.\frac{\partial u}{\partial s}\right|_{r=2,\ s=1,\ t=0} \qquad 98 + 64 = 162$$

$x = 2$, $z = 0$, $y = 2$

$x = (2)(1)e^{0}$

$y = (2)(1)^2 e^{0}$

$z = 4 \sin 0 = 0$

[2] $\frac{\partial u}{\partial t} = u_x x_t + u_y y_t + u_z z_t$

$= 4x^3 y \, rse^{t} + (x^4 + 2yz^3)(-rs^2 e^{-t}) + 0.$

$= 4(8)(2) + (16)(-2) = 64 - 32 = \boxed{32}$

$\frac{\partial u}{\partial t}\Big|$ عند تعويضها بحد قيمة وتكون ثابتة

$r=2,\ s=1,\ t=0$

$x=2,\ y=2,\ z=0$

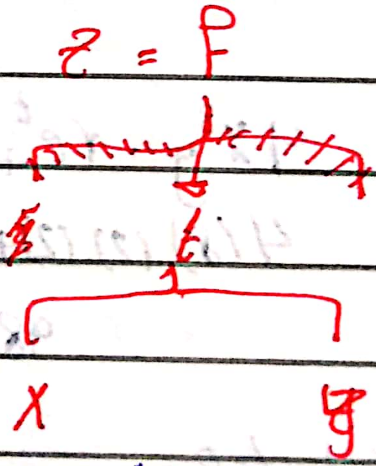
Ex:- Show that if $z = f(x-y)$, then

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

Sol:-

$z = f(t)$

$t = x-y$



$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = f'(t) \cdot t_x + f'(t)\, t_y$$

$$f'(t)(1) + f'(t)(-1) = f'(t) - f'(t)$$

$$= 0 \ \#$$

Ex:-

$g(s,t) = f(s^2-t^2,\ t^2-s^2)$ Show that

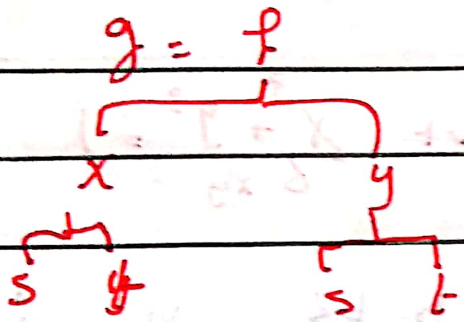
$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Sol:-

$$g(s,t) = f(x,y)$$

$x = s^2 - t^2$

$y = t^2 - s^2$



$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = t\,(f_x x_s + f_y y_s)$$

$$+\ s\,(f_x x_t + f_y y_t)$$

$$= t\,(2s f_x + -2s f_y) + s\,(-2t f_x + 2t f_y)$$

$$\cancel{2st f_x} - \cancel{2st f_y} - \cancel{2st f_x} + \cancel{2st f_y} = 0$$

اذا كان عندي علاقة طرفها الاول (الايمن) = صفر

Rule:-

$$F(x,y,z)=0$$

$$\frac{\partial z}{\partial x}=-\frac{F_{(x)}}{F_{(z)}}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

Ex:- Find y' If you Know: $\frac{x^3+y^3}{6xy}=1$

① نبسطها ونجعل طرفها صفر ودون كسور لكي نطول حل المسألة

Relation $\Rightarrow x^3+y^3=6xy$

$$\underbrace{x^3+y^3-6xy}_{F}=0$$

$$y'=\frac{dy}{dx}=-\frac{F_{(x)}}{F_{(y)}}=-\frac{(3x^2-6y)}{3y^2-6x}$$

نقسم على ٣ لنكمل الخطوة الاخيرة

$$\frac{-3x^2+6y}{3y^2-6x}=\frac{2y-x^2}{y^2-2x}$$

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Ex: If $\frac{x^3}{z} + \frac{y^2}{z} + z^2 = \frac{1}{z} - 6xy$

$\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial y \partial z}$ at the pt(0,3).

نضرب ب z وننقل الطرف اليمين الى اليسار.

$$x^3 + y^2 + z^3 + 6xyz - 1 = 0$$

(1) $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(3x^2 + 6yz)}{3z^2 + 6xy}$

$\left.\frac{\partial z}{\partial x}\right|_{(0,3,-2)} = \boxed{3}$

$x = 0$

$y = 3$

$z = -2$ من تعويض

في المعادلة x.y

(2) $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

$= -\frac{2y + 6xz}{3z^2 + 6xy}$

$\Big|_{(0,3,-2)} = -\frac{6+0}{12+0} = \boxed{\frac{-1}{2}}$

$$\frac{\partial^2 z}{\partial y\, \partial x}$$

$$\frac{\partial z}{\partial x} = -\frac{3x^2+6yz}{3z^2+6xy}$$

$$\left.\frac{\partial z}{\partial x}\right|_{x=3} = -\frac{6yz}{3z^2}$$

هنا نعوض X لانتا نشتق لل y

$$F = \frac{\partial z}{\partial x} = -2\frac{y}{z}$$

نكمل من هنا

*

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{-F_y}{F_z}$$

$$= -\frac{\frac{-\cancel{2}}{z}}{-\cancel{2}\left(\frac{-y}{z^2}\right)}$$

$$\Big|_{(0,3,-2)} = -\frac{\frac{-1}{2}}{\frac{-3}{4}} = \frac{-2}{3}$$

* نصل ع الطريقة في الصفحة التالية للضمان هذه منها غلط

Let $\frac{\partial z}{\partial x} = w$

$w = \frac{\partial z}{\partial x}$; ① $\Rightarrow w + \frac{2y}{z} = 0$

$\underbrace{z w + 2y}_{F} = 0$

$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial w}{\partial y}$

$- \frac{F_y}{F_w} = \frac{-2}{z} = \frac{-2}{-2} = 1$

Sec 14.6

The Dierctional Derivative and the Gradiant vector

Def : The Gradiant vector of a func $f(x, y, z)$ at a pt $A(x_0, y_0, z_0)$

Gradiant $\nabla f(x_0, y_0, z_0) = \langle f_x, f_y, f_z \rangle \big|_A$

قراءة

نبلا

Find $\nabla f(1,2)$, where $f(x,y) = \frac{x^2}{y} + \sin(\pi x)$

Sol :

$$\nabla f = \langle f_x , f_y \rangle =$$

$$\langle \frac{2x}{y} + \pi \cos \pi x , \frac{-x^2}{y^2} \rangle$$

$$\nabla f(1,2) = \langle 1 + \pi \cos \pi , \frac{-1}{4} \rangle = \langle 1 - \pi , \frac{-1}{4} \rangle$$

Def:- The directional derivative (D.D) of a func $f(x,y)$ at a pt (x_0, y_0) in the direction of unite vector $\hat{v} = \langle a, b \rangle$ is

$$D_{\hat{r}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

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Ex:- Using def. find the directional derivative of $F(x,y) = x^2 - xy + y^2$ at the pt $A(1,0)$ in the direction of the unit vector $\hat{v} = \frac{\hat{i}-\hat{j}}{\sqrt{2}}$

Sol:- $\hat{v} \Rightarrow a = \frac{1}{\sqrt{2}}$ $b = \frac{-1}{\sqrt{2}}$

$$D_{\hat{v}} F(1,0) = \lim_{h \to 0} \frac{F\left(1+\frac{1}{\sqrt{2}}h,\ 0+\frac{-1}{\sqrt{2}}h\right) - F(1,0)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\left(1+\frac{h}{\sqrt{2}}\right)^2 - \left(1+\frac{h}{\sqrt{2}}\right)\left(-\frac{h}{\sqrt{2}}\right) + \left(\frac{-h}{\sqrt{2}}\right)^2\right) - \left(1^2 - 1(0) + 0^2\right)}{h}$$

$$\lim_{h \to 0} \frac{\left(1+\frac{h}{\sqrt{2}}\right)^2 + \frac{h}{\sqrt{2}} + \frac{h^2}{2} + \frac{h^2}{2} - 1}{h}$$ لوبيتال

$$\lim 2\left(1+\frac{h}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + h + h$$

$$(2)(1)\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

Remark:-

Gradiant $f(x,y)$ is $\nabla f = \langle f_x, f_y \rangle$

Rule:-

$$D_{\hat{v}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \hat{v}$$

Ex:- Find the dieraction ~~derit~~ derivative of $f(x,y) = x^2 - xy + y^2$ at $(1,0)$ in the direction of $\frac{\hat{i} - \hat{j}}{\sqrt{2}}$

Sol:- $\left| \frac{i-j}{\sqrt{2}} \right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = \sqrt{1} = 1$

$\frac{i-j}{\sqrt{2}}$ unit vector

$\nabla f = \langle f_x, f_y \rangle = \langle 2x - y, -x + 2y \rangle$

$\nabla f(1,0) = \langle 2, -1 \rangle$

$$D_{\hat{v}} f(1,0) = \nabla f(1,0) \cdot \hat{v} = \langle 2, -1 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$$

$$= \frac{3}{\sqrt{2}}$$

Ex :-

If $\nabla f(2,3) = \langle 3,1 \rangle$

Find $\displaystyle\lim_{h \to 0} \frac{f\left(2+\frac{h}{\sqrt{2}},\ 3-\frac{h}{\sqrt{2}}\right) - f(2,3)}{h}$

Sol :-

$\hat{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$

h جوه   h جوه

$\displaystyle\lim_{h \to 0} \frac{f\left(2+\frac{h}{\sqrt{2}},\ 3-\frac{h}{\sqrt{2}}\right) - f(2,3)}{h} = D_{\hat{v}} f(2,3)$

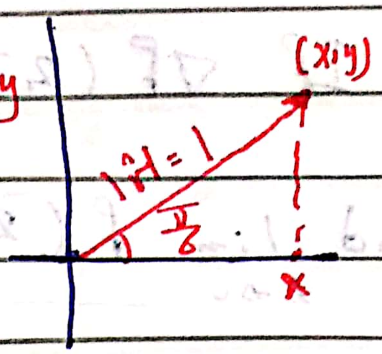
$= \nabla f(2,3) \cdot \hat{v}$

$= \langle 3,1 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle = \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$ ..

Ex :-

Find the directional derivative of $f(x,y) = x^3 - 3xy + 4y^2$ in the direction that makes the angle $\theta = \frac{\pi}{6}$

$x = \cos\theta = \frac{\sqrt{3}}{2}$



$y = \sin\theta = \frac{1}{2}$

$\hat{r} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$\nabla f = \langle f_x, f_y \rangle = \langle 3x^2 - 3y, -3x + 8y \rangle$

$D_{\hat{r}} f = \nabla f \cdot \hat{r} = \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x - 8y)$

Ex:- Find the directional derivative of $f(x,y,z)$

$= x\sin(yz)$ at the pt $(1,3,0)$ in the

direction of $\vec{v} = \hat{i} + 2\hat{j} - k$

* مش لانه unit

$|\vec{r}| = \sqrt{1+4+1} = \sqrt{6} \neq 1$

$\hat{v} = \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \rangle$

$\nabla f = \langle f_x, f_y, f_z \rangle$

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$\langle \sin(yz),\ xz\cos(yz),\ xy\cos(yz) \rangle$

$\nabla f(1,3,0) = \langle 0, 0, 3 \rangle$

$D_{\hat{v}} f(1,3,0) = 0 * \frac{1}{\sqrt{6}} + 0\ \frac{2}{\sqrt{6}} + 3\left(\frac{-1}{\sqrt{6}}\right) = \frac{-3}{\sqrt{6}}$

Ex:-

Find the dierection derivative of

$f(x,y,z) = \frac{x^2+y}{z}$ at the pt $A(1,0,2)$ in the

direction from A to $B(-1,2,1)$

$\vec{v} = \overline{AB} = \langle -2, 2, -1 \rangle$

$|\vec{v}| = \sqrt{9} = 3$

$\hat{v} = \langle \frac{-2}{3}, \frac{2}{3}, \frac{-1}{3} \rangle$

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle \frac{2x}{z}, \frac{1}{z}, -\frac{(x^2+y)}{z^2} \rangle$

$\nabla f\big|_A = \langle \frac{2(1)}{2}, \frac{1}{2}, -\frac{(1^2+0)}{2} \rangle = \langle 1, \frac{1}{2}, \frac{-1}{4} \rangle$ →

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$D_{\hat{v}} f \big|_A = \nabla f \big|_A \cdot \hat{v} = -\frac{2}{3} + \frac{2}{6} + \frac{1}{12}$

$= \frac{-8+4+1}{12} = \frac{-3}{12} = \frac{-1}{4}$

Ex:-

Let $\hat{u} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$, $\hat{v} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$

$D_{\hat{u}} f(1,2) = -5$, $D_{\hat{v}} f(1,2) = 10$

[1] Find $f_x(1,2)$, $f_y(1,2)$

[2] Find the dierichion derivative of $f(x,y)$ at the pt $A(1,2)$ in the direction of the vector from $A(1,2)$ to $B(0,0)$

Sol:-

[1] $\nabla f(1,2) = \langle f_x(1,2), f_y(1,2) \rangle$

$\langle a, b \rangle$

$a = f_x(1,2)$

$b = f_y(1,2)$

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$D_{\hat{u}} f(1,2) = -5 \Rightarrow \nabla f(1,2) \cdot \hat{u} = -5$

$\langle a, b \rangle \cdot \langle \frac{3}{5}, \frac{-4}{5} \rangle = -5$

$\frac{3a}{5} - \frac{4b}{5} = -5$

$3a - 4b = -25 \rightarrow (1)$

$D_{\hat{v}} f(1,2) = 10 = \nabla f(1,2) \cdot \hat{v}$

$= \langle a, b \rangle \cdot \langle \frac{4}{5}, \frac{3}{5} \rangle$

$\frac{4a}{5} + \frac{3b}{5} = 10$

$4a + 3b = 50 \rightarrow \boxed{2}$

$3 * (1) \Rightarrow 9a - 12b = -75$

$4 * (2) \Rightarrow 16a + 12b = 200$

$25a = 125 \Rightarrow \boxed{a = 5}$ نحوزها

$\boxed{b = 10}$

$f_x(1,2) = 5 \qquad f_y(1,2) = 10$

Subject

$\boxed{2}$

$\vec{w} = \overrightarrow{AB} = \langle -1, -2 \rangle \Rightarrow |\vec{w}| = \sqrt{5}$

$\hat{w} = \langle \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle$

$D_{\hat{w}} f(1,2) = \nabla f(1,2) \cdot \hat{w}$

$= \langle 5, 10 \rangle \cdot \langle \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{2}} \rangle$

$= \frac{-25}{\sqrt{5}} = -5\sqrt{5}$

Ex:-

If $f(x,y) = e^{2xy}$

$\vec{v} = \langle 1, 0 \rangle$

Find $\frac{\partial D_{\vec{v}} f}{\partial x \, \partial y}$ at the pt $(1,2)$.

Sol:-

$\nabla f = \langle f_x, f_y \rangle = \langle 2y e^{2xy}, 2x e^{2xy} \rangle$

$D_{\hat{v}} f = 2y e^{2xy} (1) + 2x e^{2xy} (0)$

$= 2y e^{2xy}$

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$$\frac{\partial D_{\vec{v}} f}{\partial x} = 4y^2 e^{2xy}$$

$$\frac{\partial D_{\hat{v}} f}{\partial x}\Big|_{x=1} = 4y^2 e^{2y}$$

$$\frac{\partial^2 D_{\vec{v}} f}{\partial y\, \partial x} = 8y^2 e^{2y} + 8y e^{2y}$$

$$\frac{\partial^2 D_{\vec{v}} f}{\partial y\, \partial x}\Big|_{(1,2)} = 32e^4 + 16e^4 = 48e^4$$

$$\frac{\partial^2 D_{\vec{v}} f}{\partial x\, \partial y}\Big|_{(1,2)} = 48e^4$$

Rule:- The maximum value of the Dierctional derivative of a func $f(x,y)$ at a pt (x_0, y_0) holds in the direction of $\nabla f(x_0, y_0)$ It's value is

$$D_{\vec{v}} f(x_0, y_0) = |\nabla f(x_0, y_0)|$$

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where $\hat{v}$ in the Same direction of

$\nabla f(x_0, y_0)$

Remark:- the rate of change of $f(x, o)$ at

(x_0, y_0) in the direction of $\hat{v}$ is

$D_{\hat{v}} f(x_0, y_0)$

Ex:- Find the maxmum rate of change

of the func $f(x,y) = xe^y$ at the pt $(\frac{1}{2}, 2)$

In what direction this max rate of

Change accures

Sol:- max. rate of Change $= \max D_{\hat{v}} f(\frac{1}{2}, 2)$

$= |\nabla f(\frac{1}{2}, 2)|$

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$\nabla f = \langle f_x, f_y \rangle$

$\langle e^y, xe^y \rangle$

$f(\frac{1}{2}, 2) = \langle e^2, \frac{1}{2}e^2 \rangle$

max rate of Change $= |\nabla f(\frac{1}{2}, 2)|$

$= \sqrt{e^4 + \frac{e^4}{4}} = \sqrt{\frac{4e^4}{4}}$

$= \frac{\sqrt{5}}{2} e^2$

The dierction is

$\nabla f(\frac{1}{2}, 2) = \langle e^2, \frac{1}{2}e^2 \rangle$

$\Rightarrow$ dierction $\langle 2, 1 \rangle$.

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Ex:-

Find the max directional derivative of $f(x,y) = x^2y + \sin y$ at the pt $(-2,1)$ and find the unit vector in the direction in ~~wich~~ which this max value occures. (ت . ك)

Sol:-

$$\max D_{\hat{v}} f(-2,1) = |\nabla f(-2,1)|.$$

$$\nabla f = \langle f_x, f_y \rangle = \langle 2xy, x^2 + \cos y \rangle$$

$$\nabla f(-2,1) = \langle -4, 4 + \cos 1 \rangle$$

$$\therefore \max D_{\hat{v}} f(-2,1) = \sqrt{(-4)^2 + (4 + \cos 1)^2}$$

$$= \sqrt{32 + 8\cos 1 + \cos^2 1}$$

Direction in which max $D_{\hat{v}} f(-2,1)$ occures in the direction of $\nabla f(-2,1) = \langle -4, 4 + \cos 1 \rangle$

$$\hat{v} = \frac{\nabla f}{|\nabla f|} = \frac{\langle -4, 4 + \cos 1 \rangle}{\sqrt{32 + 8\cos 1 + \cos^2 1}}$$

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Ex:- If the dierction derivative of $f(x,y,z)$ at the pt $(-3,-2,1)$ in the direction of $2i-j-2k$ is 5 and $|\nabla f(-3,-2,1)| = 5$ find $\nabla f(-3,-2,1)$

SoL:-

$\vec{v} = 2i - j - 2k \Rightarrow |\vec{v}| = \sqrt{4+1+4} = 3$

$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \rangle$

$D_{\hat{v}} f(-3,-2,1) = 5 = |\nabla f(-3,-2,1)| \Rightarrow$ This is the max. Direction derivative

$\hat{v}$, $\nabla f(-3,-2,1)$ in the same Direction.

$\frac{\nabla f(-3,-2,1)}{|\nabla f(-3,-2,1)|} = \hat{v} \Rightarrow \frac{\nabla f(-3,-2,1)}{5} = \langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \rangle$

$\nabla f(-3,-2,1) = \langle \frac{10}{3}, \frac{-5}{3}, \frac{-10}{3} \rangle$

Ex:- Find all pts at which the dierction of the max direction derivative of $f(x,y) = x^2+y^2-2x-4y$ is $\boxed{i+j}$

$\vec{v} = i+j \Rightarrow |\vec{v}| = \sqrt{2}$

$\hat{v} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

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$\max D_{\hat{v}} f = |\nabla f|$

Dierction $\hat{v}$, ∇f in the Same direction.

$$\frac{\nabla f}{|\nabla f|} = \hat{v}$$

$$\frac{\langle 2x-2, \; 2y-4 \rangle}{\sqrt{(2x-2)^2+(2y-4)^2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

① $$\frac{2x-2}{\sqrt{(2x-2)^2+(2y-4)^2}} = \frac{1}{\sqrt{2}}$$

* نربع الطرفين ونضرب تبادلي.
لكن هذه الطريقة طويلة.

② $$\frac{2y-4}{\sqrt{(2x-2)^2+(2y-4)^2}} = \frac{1}{\sqrt{2}}$$

لذا سوف نقسم ① على ②

$$\frac{2x-2}{2y-4} = 1$$

$y = x+1 \rightarrow$ ③

نلاحظ Observe that ①,②

$2x-2 > 0$ $\qquad x > 1$

$2y-4 > 0$ $\qquad y > 2$

pts $(x,y) = (x, x+1)$ where $x > 1$

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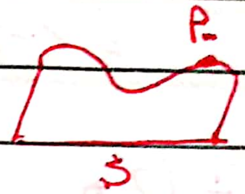
Rule:- let S be the surface $F(x,y,z)=0$
and $P_0(x_0, y_0, z_0)$ is pt on S



$\boxed{1}$ $\vec{n} = \nabla F(x_0, y_0, z_0)$ orthogonal
(⊥) to S at P_0

$\boxed{2}$ The eq of the tangent plane to S at P_0

$$(F_x\big|_{P_0})(x-x_0) + (F_y\big|_{P_0})(y-y_0) + (F_z\big|_{P_0})(z-z_0)=0$$

$\boxed{3}$ The parametric eqs of the normal Line to
S at P_0 is
$x = x_0 + at$ $\quad a = F_x \big|_{P_0}$
$y = y_0 + bt$ $\quad b = F_y \big|_{P_0}$
$z = z_0 + ct$ $\quad c = F_z \big|_{P_0}$

Ex:- Find the eqs of the tangent plane and normal Line at the pt $(-2, 1, -3)$ to the ellipsoid: $\frac{x^2}{4} + y^2 = 3 - \frac{z^2}{9}$

Surface: $\frac{x^2}{4} + y^2 + \frac{z^2}{9} - 3 = 0$

نضرب بالـ 36

$$\underbrace{9x^2 + 36y^2 + 4z^2 - 108}_{F} = 0$$

Subject

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 18x \cancel{+72y}, 8z \rangle$

$= \langle -36, 72, -24 \rangle \perp$ Surface.

$\div 12 \Rightarrow \langle -3, 6, -2 \rangle \perp$ Surface.

eq of tangent plane.

$-3x + 6y - 2z = -3(-2) + 6(1) - 2(-3) = 18$

Parametric eqs of normal Line:-

$x = -2 + -3t$

$y = 1 + 6t$

$z = -3 + -2t$

Find the eq of the tangent plane and symm eqs of the normal Line at the pt $(1,2,5)$ to the Paraboloid $z = x^2 + y^2$

Sol:- Surface $= \underbrace{x^2 + y^2 - z}_{f} = 0$

$\nabla f = \langle 2x, 2y, -1 \rangle$

$\nabla f(1,2,5) = \langle 2, 4, -1 \rangle$

eq of tangent plane:-

$2x + 4y - z = 2 + 8 - 5 = 5$

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Symm eqs of normal line :-

$$\frac{x-1}{2} = \frac{y-2}{4} = \left(\frac{z-5}{-1}\right) = (5-z).$$

Ex:- If the eq of the tangent plane of the surface $f(x,y,z)=0$ at $(1,2,3)$ is $2x-3y=5$ and $|\nabla f(1,2,3)| = 4$

Find :- ① $\nabla f(1,2,3)$

$\boxed{2}$ Find the directional derivative of $f(x,y,z)$ at $(1,2,3)$ in the direction of $\hat{i}-2\hat{k}$

Sol:-

plane $2x-3y=5 \Rightarrow \vec{n} = \langle 2,-3,0 \rangle \perp$ plane

$\vec{n} \parallel \nabla f(1,2,3)$ ~~in the same direction~~

$$\frac{\nabla f(1,2,3)}{|\nabla f(1,2,3)|} = \frac{\vec{n}}{|\vec{n}|} \Rightarrow \frac{\nabla f(1,2,3)}{4} = \frac{\pm\langle 2,-3,0 \rangle}{\sqrt{13}}$$

$$\nabla f(1,2,3) = \pm\left\langle \frac{8}{\sqrt{13}}, \frac{-12}{\sqrt{13}}, 0 \right\rangle$$

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$\boxed{2}$ $\vec{v} = \hat{i} - 2k \Rightarrow |\vec{v}| = \sqrt{5}$

$$\hat{v} = < \frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}} >$$

$$D_{\hat{v}} f(1,2,3) = \nabla f(1,2,3) \cdot \hat{v}$$

$$= \frac{8}{\sqrt{13}\sqrt{5}} = \frac{8}{\sqrt{65}} = \frac{8\sqrt{65}}{65}$$

If the symm eqs of the normal line to the surface $z = f(x,y)$ at the pt $(1,2,3)$ are

$$\frac{x-1}{3} = \frac{4-2y}{3} = 3 - z \quad \text{and}$$

$|\nabla f(1,2)| = 4$, find $\nabla f(1,2)$

السؤال غلط

المهم فكرته

X

هذا الحل خاطئ سوف يتم حله في الصفحة التالية

Sol:-

Surface $\underbrace{f(x,y) - z}_{F} = 0$

$$\nabla F < F_x, F_y, F_z > = < f_x, f_y, -1 >$$

eqs of line:

$$< 3, \frac{-3}{2}, -1 > \parallel \text{Line}$$

$$\times 2 \qquad < 6, -3, -2 > \parallel \text{Line}$$

$$< 6, -3, -2 > \perp \text{Surface}$$

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∇F , $\langle 6,-3,-2\rangle$ in the same direction.

$\frac{\nabla F}{|\nabla F|} = \frac{\langle 6,-3,-2\rangle}{|\langle 6,-3,-2\rangle|} \div \frac{\nabla F}{4} = \frac{\langle 6,-3,-2\rangle}{7}$

$\nabla F = \langle \frac{24}{7}, \frac{-12}{7}, \frac{-8}{7} \rangle$

$\boxed{\text{محذوف}}$

Sol:- Line $\Rightarrow$ vector $\vec{v} = \langle 3, \frac{-3}{2}, -1\rangle \parallel$ Line.

$\langle 6,-3,-2\rangle \parallel$ Line.

Surface $F(x,y) - z = 0$

$\nabla F = \langle F_x, F_y, -1\rangle \perp$ Surface

$\nabla F \parallel \langle 6,-3,-2\rangle \rightarrow ①$

$|\nabla F| = 4$, $\nabla F = \langle F_x, F_y\rangle \Rightarrow F_x^2 + F_y^2 = 16 \rightarrow \boxed{2}$

Case 1:- ∇F , $\langle 6,-3,-2\rangle$ in the same direction

$\frac{\nabla F}{|\nabla F|} = \frac{\langle 6,-3,-2\rangle}{7}$

$\frac{\nabla F}{\sqrt{F_x^2 + F_y^2 + 1}} = \frac{1}{7}\langle 6,-3,-2\rangle \Rightarrow \nabla F = \frac{\sqrt{17}}{7}\langle 6,-3,-2\rangle$

$\langle F_x, F_y, -1\rangle = \frac{\sqrt{17}}{7}\langle 6,-3,-2\rangle$

$F_x = \frac{6\sqrt{17}}{7}$, $F_y = -\frac{\sqrt{17}}{7}(3)$

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Ex: (10) Find the parametric eqs. of the Line through the pt $A(1,2,3)$ which is parallel to the normal Line of the surface $z = 2x^2y + 3xy^2$ at the pt $B(1,1,5)$

Sol:-

$\vec{v} \parallel$ Line , Line $\parallel$ normal Line , normal Line $\perp$ Surface

$\nabla f \perp$ surfac

$\Rightarrow \vec{v} \perp$ surface

$\therefore \nabla f \parallel$ normal Line

Surface $2x^2y + 3xy^2 - z = 0$

$\nabla f \parallel$ Line.

$$\vec{v} = \nabla f\,(1,1,5)$$

$$= \langle 4xy + 3y^2,\ 2x^2 + 6xy,\ -1 \rangle\ (1,1,5)$$

$$= \langle 7, 8, -1 \rangle$$

$\langle 7, 8, -1 \rangle \parallel$ Line.

eqs of Line

$$x = 1 + 7t$$

$$y = 2 + 8t$$

$$z = 3 - t$$

Ex:- Find the tangency pt of the tangent plane

$-2x + y + 2z + 3 = 0$ to the surface

$x^2 + y^2 + z^2 = 1$

Sol:-

$\vec{n} = \langle -2, 1, 2 \rangle \perp$ plane

Surface: $\underbrace{x^2 + y^2 + z^2 - 1}_{f} = 0$

$\nabla f = \langle 2x, 2y, 2z \rangle \perp$ surface

$\vec{n} \perp$ surface

$\nabla f \parallel \vec{n} \Rightarrow \nabla f = \alpha \vec{n}$, α Scalar

$\langle 2x, 2y, 2z \rangle = \langle -2\alpha, \alpha, 2\alpha \rangle$

$x = -\alpha$, $y = \frac{\alpha}{2}$, $z = \alpha$

$(x, y, z) = (-\alpha, \frac{\alpha}{2}, \alpha)$ Lies in the plane

$-2(-\alpha) + \frac{\alpha}{2} + 2\alpha + 3 = 0$

$2\alpha + \frac{\alpha}{2} + 2\alpha = -3$

$\frac{4\alpha + \alpha + 4\alpha}{2} = -3 \Rightarrow 9\alpha = -6$

$\alpha = \frac{-6}{9} = \frac{-2}{3}$ pt $(x, y, z) = (\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3})$

$(-\alpha, \frac{\alpha}{2}, \alpha)$

Maximum and Minimum Values :-

Def:- A func $f(x,y)$ is said to have :

$\boxed{1}$ a Local max (Local min) at a pt (x_0,y_0) if there is a disk D centered at (x_0,y_0) s.t $f(x_0,y_0) \geq f(x,y)$ ($f(x_0,y_0) \leq f(x,y)$) for all $(x,y) \in D$

$\boxed{2}$ a Local extremum (sys) at (x_0,y_0) if $f(x,y)$ has a Local max or a Local min at (x_0,y_0)

$\boxed{3}$ an absolute max (absolute min) at a pt (x_0,y_0) if $f(x_0,y_0) \geq f(x,y)$ ($f(x_0,y_0) \leq f(x,y)$) for all $(x,y) \in \text{Dom} f$

$\boxed{4}$ an absolute extremum at (x_0,y_0) if $\boxed{f}$ has absolute max or absolute min at (x_0,y_0)

* The value of $f(x_0,y_0)$ is the Local max (min)
  " " " absolute " (")

Def:- A func $f(x,y)$ is said to have a critical (or stationary) pt at $(x_0,y_0) \in \text{Dom}(f)$ if $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$ or $\rightarrow$

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$f_x(x_0, y_0)$ DNE or $f_y(x_0, y_0)$ DNE.

Thm :- If $f(x,y)$ has a relative extremum at (x_0, y_0), then (x_0, y_0), then critical pt of f

2^{nd} Derivative Test:-

Suppose the 2^{nd} partial derivative of $f(x,y)$ are Cont on a disk centered at a pt (a,b) and $f_x(a,b) = 0$, $f_y(a,b) = 0$

Let $D = f_{xx} f_{yy} - (f_{xy})^2 \big|_{(a,b)}$

$\boxed{1}$ If $D > 0$:

↳ $\boxed{1}$ $f_{xx}(a,b) > 0 \Rightarrow [f$ has a local min value at (a,b), This " " " is $f(a,b)]$

↳ $\boxed{2}$ $f_{xx}(a,b) < 0$
$\Downarrow$
f has local max value at (a,b), This local max value is $f(a,b)$

$\boxed{2}$ If $D < 0$, then f has neither local max nor a local min at (b,a), This pt is Called a saddle pt of f. (سرج الحصان)

$\boxed{3}$ If $D = 0$ the test fails

Subject

Ex:- Find and classify the critical pts of the func $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$ as local max, local min or saddle pts:-

Sol:- $f_x = 6x^2 + 6y^2 - 150 = 0$

$6x^2 + 6y^2 = 150 \leftarrow \div 6$

$x^2 + y^2 = 25 \rightarrow (1)$

$f_y = 12xy - 9y^2 = 0$

$3y(4x - 3y) = 0$

$y = 0$ or $y = \frac{4}{3}x \rightarrow (2)$

* $x^2 + y^2 = 25$ and $y = 0 \rightarrow (\pm 5, 0)$

$x^2 = 25 \quad x = \pm 5$

$x^2 + y^2 = 25$, $y = \frac{4}{3}x$

$x^2 + \frac{16}{9}x^2 = 25$ نضرب *9

$9x^2 + 16x^2 = 25 * 9 \Rightarrow \cancel{25}x^2 = \cancel{25} * 9$

$x = \pm 3$

$(\pm 3, \cancel{y})$

$(3, 4) \quad (-3, -4)$ من خلال التعويض

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The Critical pt of f are:

(5,0), (-5,0), (3,4), (-3,-4)

Classify:

$f_{xx} = 12x$

$f_{yy} = 12x - 18y$

$f_{xy} = 12y$

$D = f_{xx} * f_{yy} - (f_{xy})^2$

$12x(12x - 18y) - (12y)^2$

→ (5,0)

$12*5(12*5) - 0 \geq 0$ $\boxed{+}$ عندما يكون ⊕ نرجع

$f_{xx}(5,0) = 12(5) \geq 0$ لل f_{xx}

عندما D ⊖ نقول Saddle pt

f has a Local min value at (5,0)

* اذا كانت f_{xx} موجبة يجب f_{yy} تكون موجبة والعكس

صحيح

A local min of f(5,0) is = -500

اذا D⊕ ⊕ min ⊖ max نرجع لاحدى المشتقات

$(-5,0)$: $D = 12(-5)\,(12(-5)) - 0 > 0$

$f_{xx}\,(-5,0) \Rightarrow 12(-5)$

Local max value of $f\,(-5,0)$

Local max value of f is

$f(-5,0) = 500$.

* $(3,4) \Rightarrow 12(3)\,(12(3) - 18(4)) - (12(4))^2 < 0$

f has a saddle pt at $(3,4)$

$(-3,-4)$ $D = 12(-3)\,(12(-3) - 18(-4)) - (12(-4))^2 < 0$

f has a saddle

pt at $(-3,-4)$

Find and classify the critical pts of the func
$f(x,y) = x^4 + y^4 - 4xy + 1$ as local max, local min
or saddle pts:-

Sol:-

$\boxed{f_x = 4x^3 - 4y = 0}$ $\boxed{y = x^3}$

$f_y = 4y^3 - 4x = 0$ $\boxed{x = y^3}$

$x = (x^3)^3 = x^9 \Rightarrow x = 0, 1, -1$

$x = 0 \Rightarrow y = 0$ $(0,0), (1,1), (-1,-1)$

$x = 1 \Rightarrow y = 1$ critical pts of f

$x = -1 \Rightarrow y = -1$

Classify:-

$f_{xx} = 12x^2$

$f_{yy} = 12y^2$

$f_{xy} = -4$

$D = 12x^2(12y^2) - (f_{xy})^2$

$(0,0)$ $D = -16 < 0$ Saddle pt at $(0,0)$

$(1,1)$: $D = 12(12) - 16 > 0$

$f_{xx}(1,1) = > 0$ f has local min value at $(1,1)$

The local min value of f is $f(1,1) = -1$

$(-1,-1)$: $D = 12(-1)^2(12(-1)^2) - 16 > 0$

$f_{xx}(-1,-1) > 0$

f has a local min value of f is $f(-1,-1) = -1$

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تجب

وصى

Absolute Maximum and Minimum Values

For a function f of one variable, the Extreme Value Theorem says that if f is continuous on a closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value. According to the Closed Interval Method in Section 4.1, we found these by evaluating f not only at the critical numbers but also at the endpoints a and b.

There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, **a closed set** in $\mathbb{R}^2$ is one that contains all its boundary points. [A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.] For instance, the disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

which consists of all points on and inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$). But if even one point on the boundary curve were omitted, the set would not be closed. (See Figure 11.)

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk. In other words, it is finite in extent. Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if f has an extreme value at (x_1, y_1), then (x_1, y_1) is either a critical point of f or a boundary point of D. Thus we have the following extension of the Closed Interval Method.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

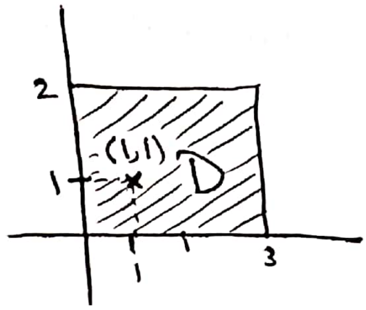
1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Ex1: Find the absolute max. and min. values of the func. $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y) : 0 \le x \le 3, 0 \le y \le 2\}$.

Solutions:

النقاط الحرجة

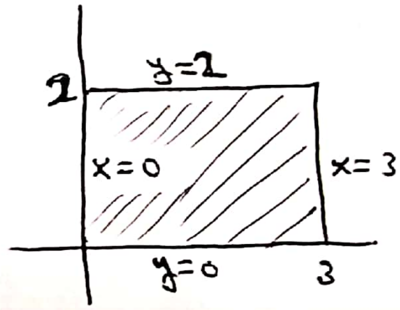
Step 1: Critical pts. in D:



$$f_x = 2x - 2y = 0 \to y = x$$
$$f_y = -2x + 2 = 0 \to x = 1$$
$$\to y = 1$$

$\therefore$ f has only 1 critical pt in D which is $(1,1)$ لأنها داخل المنطقة نأخذها

Step 2: Critical pts. on the boundary of D
الحدود



(1) On $y=0$:

$$f_1(x) = f(x,0) = x^2 \quad 0 \le x \le 3$$
$$f_1' = 0 \to 2x = 0 \to x = 0$$

critical pts. $(0,0)$, $(3,0)$

أطراف المجال نقاط حرجة

(2) On $x=0$: $f_2(y) = f(0,y) = 2y \quad 0 \le y \le 2$

$f_2' = 2 \ne 0 \Rightarrow$ Critical pts. $(0,0)$, $(0,2)$

(3) On $y=2$: $f_3(x) = x^2 - 4x + 4$, $0 \le x \le 3$

$f_3' = 0 \to 2x - 4 = 0 \to x = 2 \to$ Critical pts. $(2,2)$, $(0,2)$, $(3,2)$

(4) On $x=3$: $f_4(y) = 9 - 6y + 4y = 9 - 2y$, $0 \le y \le 2$

$f_4' = -2 \ne 0 \Rightarrow$ Critical pts. $(3,0)$, $(3,2)$

Step 3:

pts.	(1,1)	(0,0)	(3,0)	(0,2)	(2,2)	(3,2)	~~(2,0)~~
f	1	0	9	4	0	1	~~4~~

$\therefore$ The absolute max. of f is 9.

" " min of f is 0.

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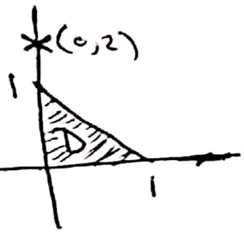
Ex2: Find the absolute max. and min. values of the func. $f(x,y)=x^2+y^2-4y$ on the closed triangular ~~...~~ region D with vertices $(0,0)$, $(1,0)$, $(0,1)$.

Sol: **Step 1:** Critical pts. in D



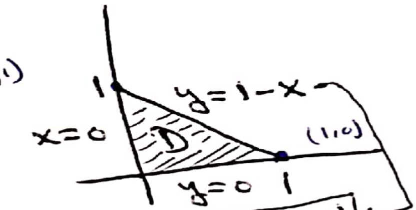
$f_x = 2x = 0 \rightarrow x=0$

$f_y = 2y-4=0 \rightarrow y=2$

$\rightarrow (0,2) \notin D$ نهمل خارج

$\therefore f$ has no critical pts. in D.

Step 2: Critical pts. on the boundary of D



(1) $y=0$: $f_1(x)=f(x,0)=x^2$, $0\le x\le 1$

$f_1' = 2x = 0 \rightarrow x=0 \rightarrow$ pts. $(0,0),(1,0)$

لايجاد المعادلة:

$m=\frac{\Delta y}{\Delta x}=\frac{1-0}{0-1}=-1$

$y-0=-1(x-1)$

$\boxed{y=1-x}$

(2) $x=0$: $f_2(y)=f(0,y)=y^2-4y$, $0\le y\le 1$

$f_2' = 2y-4=0 \rightarrow y=2 \notin [0,1]$ نهمل

pts. $(0,0),(0,1)$

(3) $y=1-x$. $f_3(x)=x^2+(1-x)^2-4(1-x)$ $0\le x\le 1$

$f_3' = 2x+2(1-x)(-1)-4(-1)=0 \rightarrow 4x+2=0$.

$\rightarrow x=-\frac{1}{2}\notin[0,1]$ نهمل

pts. $(0,1)$, $(1,0)$

Step3:

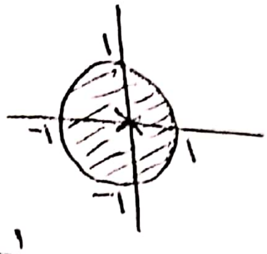
pts.	$(0,0)$	$(1,0)$	$(0,1)$
$f(x,y)$	0	1	-3

The absolute max. value of f is 1

" " min. " " f is -3

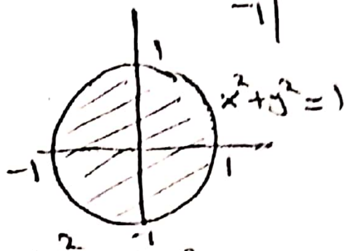
Ex 3: Find the absolute max. and min. values of the func. $f(x,y) = x^4 + 2y^3$ in the disk $D = \{(x,y) : x^2 + y^2 \le 1\}$.

Sol: Step 1: Critical pts. in D:



$f_x = 4x^3 = 0 \rightarrow x = 0$
$f_y = 6y^2 = 0 \rightarrow y = 0$
$\rightarrow (0,0)$ critical pt. in D

Step 2: On the boundary of D:



$x^2 + y^2 = 1 \rightarrow x^2 = 1 - y^2$

$f_1(y) = f\big|_{x^2 = 1-y^2} = (x^2)^2 + 2y^3 = (1-y^2)^2 + 2y^3$

$= y^4 + 2y^3 - 2y^2 + 1 \quad , \quad -1 \le y \le 1$

$f_1' = 4y^3 + 6y^2 - 4y = 0 \quad \div 2$

$y(2y^2 + 3y - 2) = 0 \rightarrow y(y+2)(2y-1) = 0$

$\Rightarrow y = 0$ or -2, or $\frac{1}{2}$.

$y = 0$: $x^2 = 1 - y^2 = 1 \rightarrow x = \pm 1 \rightarrow (\pm 1, 0)$

$y = -2$: is rejected since $-1 \le y \le 1$.

$y = \frac{1}{2}$: $x^2 = 1 - y^2 = \frac{3}{4} \rightarrow x = \pm \frac{\sqrt{3}}{2} \rightarrow (\pm \frac{\sqrt{3}}{2}, \frac{1}{2})$.

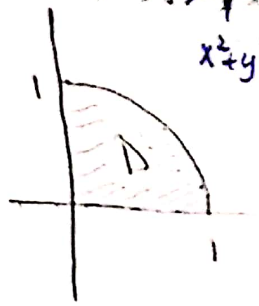
Step 3:

pts	$(0,0)$	$(1,0)$	$(-1,0)$	$(\frac{\sqrt{3}}{2}, \frac{1}{2})$	$(-\frac{\sqrt{3}}{2}, \frac{1}{2})$
$f(x,y)$	$\boxed{0}$	1	1	$\frac{13}{16}$	$\frac{13}{16}$

The absolute max. of f is 1
" " min. of f is 0.

Ex4: Find the absolute max. and min. values of the func. $f(x,y) = xy^2$ on the closed bounded region $D = \{(x,y) : x \geq 0, y \geq 0, x^2+y^2 \leq$

Sol:



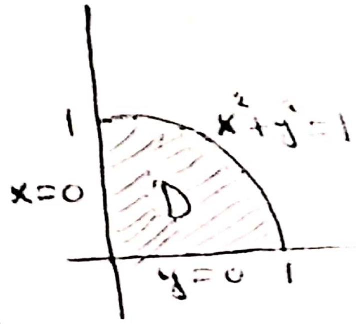
Step 1: In D:

$f_x = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0$

$f_y = 0 \Rightarrow 2xy = 0 \Rightarrow x = 0$ or $y = 0$

$\therefore$ Critical pts. $(0,0)$ and $(x,0)$, $x \in [0,1]$.

Step 2: on the boundary of D:



(1) $y=0$: $f_1(x) = 0$, $0 \leq x \leq 1$

$f_1' = 0$ for all $x \in (0,1)$

pts. $(x,0)$, for all $x \in [0,1]$

(2) $x=0$: $f_2(y) = 0$, $0 \leq y \leq 1$

$f_2' = 0$ for all $y \in (0,1)$

pts. $(0,y)$ for all $0 \leq y \leq 1$

(3) $x^2+y^2=1$: $f_3(x) = x(1-x^2) = x - x^3$, $0 \leq x \leq 1$

$f_3' = 1 - 3x^2 = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$; but $-\frac{1}{\sqrt{3}} \notin [0,1]$ drop $\rightarrow x = \frac{1}{\sqrt{3}}$

$x = \frac{1}{\sqrt{3}} \Rightarrow y^2 = 1 - x^2 = 1 - \frac{1}{3} = \frac{2}{3} \rightarrow y = \pm \frac{\sqrt{2}}{\sqrt{3}}$

but $y = -\frac{\sqrt{2}}{\sqrt{3}} \notin [0,1] \Rightarrow y = \frac{\sqrt{2}}{\sqrt{3}}$ $\therefore$ pt. $\left(\frac{1}{\sqrt{3}}, \right.$

Step 3:

pt.	$(0,0)$	$(x,0)$, $0 \leq x \leq 1$	$(0,y)$, $0 \leq y \leq 1$	$\left(\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$
f	0	0	0	$\frac{2}{3\sqrt{3}}$

$\therefore$ the absolute max. of f is $\frac{2}{3\sqrt{3}}$

" " min. " f is 0.

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Multiple Integrals

Sec 15.1 Double Integrals over rectangular regions

Sec 15.2 Iterated Integrals :-

Fubini's Thm:- Let $R = \{(x,y) : a \le x \le b, c \le y \le d\}$

$= [a,b] \times [c,d]$

Then

$$\underbrace{\iint_R f(x,y)\, dA}_{\text{Double integral}} = \underbrace{\int_a^b \int_c^d f(x,y)\, dy\, dx}_{\text{iterated integral}}$$

$$= \underbrace{\int_c^d \int_a^b f(x,y)\, dx\, dy}_{\text{iterated integrals}}$$

$dA = dy\,dx$ or $dA = dx\,dy$

Rule:

$\boxed{1}$ $\iint_R f+g = \iint_R f + \iint_R g$

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② $\iint_R cf = c\iint_R f$

③ If $f(x,y) \geq g(x,y)$, for all $(x,y) \in \mathbb{R}$ then

$$\iint_R f \geq \iint_R g$$

Ex:- $I = \iint_R x \sin(xy)\, dA$, $R = [0, \frac{\pi}{2}] \times [1, 2]$

$$= \int_0^{\pi/2} \int_1^2 x \sin xy \; dy\, dx$$

$$\int_0^{\pi/2} \left. \frac{-x \cos xy}{x} \right]_1^2 dx = -\int_0^{\pi/2} (\cos 2x - \cos x)\, dx$$

$$-\left[\frac{\sin 2x}{2} - \sin x \right]_0^{\pi/2}$$

$$-\left[\frac{\sin 2x}{2} - \sin x \right]_0^{\pi/2}$$

$$-\left[\left(\frac{\sin\pi}{2} - \sin\pi\right) - \left(\frac{\sin 0}{2} - \sin 0\right)\right] = 1$$

Ex:-

$$\int_{-2}^{2}\int_{-1}^{1} \sqrt{1-y^2}\, dy\, dx = \int_{-1}^{1}\int_{-2}^{2} \sqrt{1-y^2}\, dx\, dy$$

لأن المنطقة مستطيلة

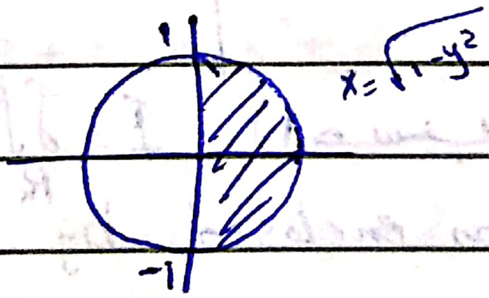
$$\int_{-1}^{1} \sqrt{1-y^2}\,(2+2)\, dy$$

$$= 4\int_{-1}^{1} \sqrt{1-y^2}\, dy$$

$$4\sin^{-1} y \Big]_{-1}^{1} = 4\left(\sin^{-1} 1 - \sin^{-1} -1\right)$$

$$= 4\left(\pi/2 + \frac{\pi}{2}\right) = 4\pi$$

X خطأ



$x = \sqrt{1-y^2}$

$x^2 = 1-y^2$

$x^2+y^2 = 1$

$y = \sqrt{1-x^2}$

4 * مساحة نصف الدائرة $= \frac{4\pi}{2}$

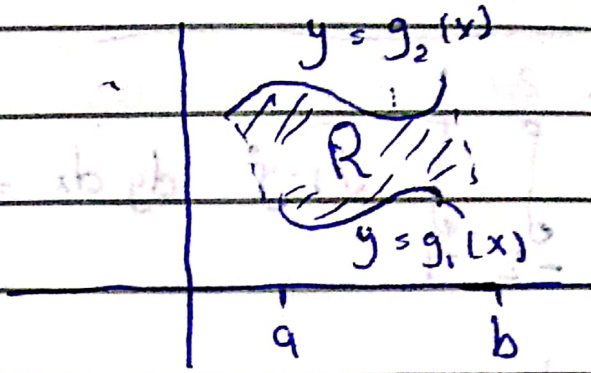
$= 2\pi$

Sec 15.3

Double Integral over Nonrectangular regions

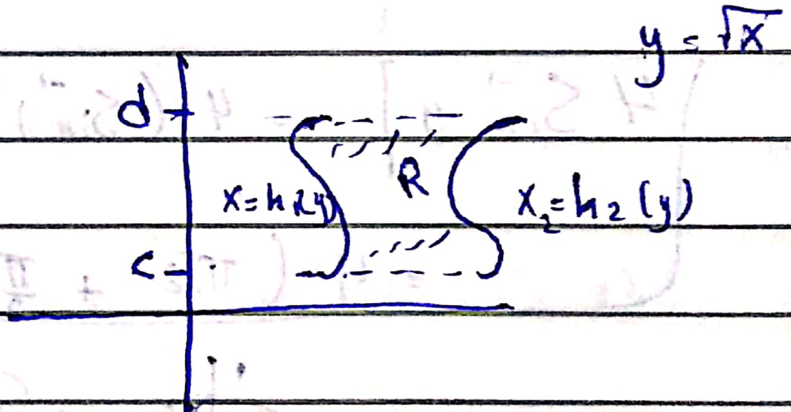
Type 1 regions

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1}^{g_2} f\,dy\,dx$$



Type 2 regions:-

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1}^{h_2} f\,dx\,dy$$



Ex:- Compute (سؤال) $I = \iint_R (x+2y)\,dA$

where R is the region enclosed by $y = 2x^2$

$y = 1 + x^2$

Sol:- $x = \pm\sqrt{\frac{y}{2}}$ $\quad x = \pm\sqrt{y-1}$

Inersection:-

$2x^2 = 1 + x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

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$$I = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$\int_{-1}^{1} \left[x(1 + x^2 - 2x^2) + (1+x^2)^2 - (2x^2)^2 \right] dx$$

$$\int_{-1}^{1} \left[x - x^3 + x^4 + 2x^2 + 1 - 4x^4 \right] dx$$

Ex :- Evaluate (احسب) $I = \iint_D xy\, dA$
Where D is the region bounded (bdd) by
$y = x - 1$ $y^2 = 2x + 6$

Curves: $x = y + 1$ $x = \dfrac{y^2 - 6}{2}$

Intersection :-

$$y + 1 = \frac{y^2 - 6}{2}$$

$$y^2 - 2y - 8 = 0 \Rightarrow (y + 2)(y - 4) = 0$$

$$y = -2, 4$$

نقطة اختبار $y = 0$ لنعلم مين اي الاقترانات تحت واليها فوق التكامل.

$$I = \int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} xy\, dx\, dy$$

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$$\int_{-2}^{4} \frac{x^2 y}{2} \Big]_{\frac{y^2-6}{2}}^{y+1} dx$$

$$= \frac{1}{2} \int_{-2}^{4} \left[(y+1)^2 - \left(\frac{y^2-6}{2}\right)^2 \right] y \cdot dy$$

$= \ldots$

مهم Ex:- Find $\Rightarrow$ $I_1 = \int_0^4 \int_{\sqrt{y}}^{2} e^{x^3} \, dx \, dy$

$\boxed{2}$ $I_2 = \int_0^1 \int_{2y}^{2} e^{x^2} \, dx \, dy$

$\boxed{3}$ $I_3 = \int_0^{\frac{1}{2}} \int_{2x}^{1} \sin(y^2) \, dy \, dx$

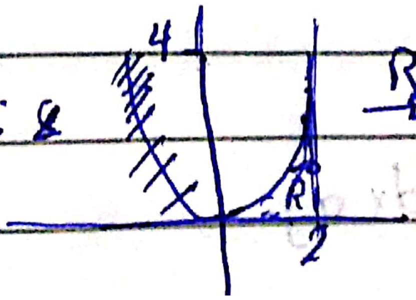
المنطقة المستطيلة يكتب اقترانها من عند الاعداد ومن عند الاعداد وتجوز فيها التبديل .

Sol:-

$\boxed{1}$ Curves : $x = \sqrt{y} \rightarrow x = 2$ $\quad 0 \le y \le 4$

$y = x^2$

& نأخذها حسب المجال



$R \rightarrow$ Type $\underline{2}$

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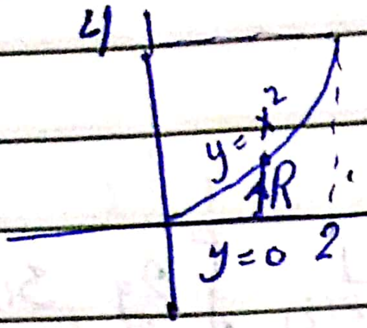
المتغيرات شمال ← يمين ⟶ بعدها $x = y$ ⟶ $dx\,dy$

R9.



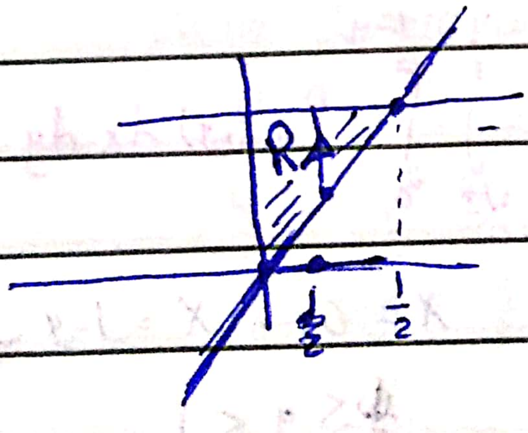
$$I_1 = \int_0^2\int_0^{x^2} e^{x^3}\,dy\,dx = \int_0^2 x^2 e^{x^3}\,dx = \frac{1}{3}\int_0^2 3x^2 e^{x^3}\,dx$$

$$\frac{1}{3}\, e^{x^3}\Big]_0^2 = \frac{1}{3}\left(e^8 - 1\right)$$

[2] Exercise.

[3] Curves : $y = 2x$ ⟶ $y = 1$ فوق

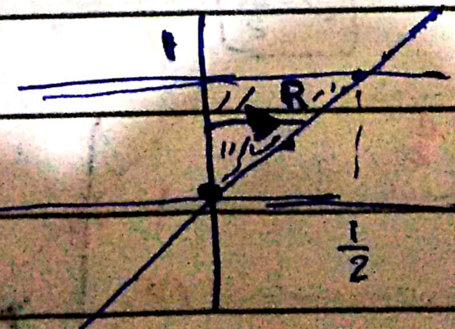
$0 \le x \le \frac{1}{2}$



$2x = 1$ $\quad \boxed{x = \frac{1}{2}}$

نعكس

Type. II



$$I_3 = \int_0^1 \int_0^{\frac{y}{2}} \sin(y^2)\,dx\,dy$$

$$= \int_0^1 \frac{y}{2} \sin y^2\,dy = \frac{1}{2(2)} \int_0^1 2y \sin y^2\,dy$$

$$= \frac{-1}{4} \cos y^2 \Big]_0^1 = \frac{-1}{4} (\cos 1 - \cos 0)$$

$$= \frac{1-\cos 1}{4}$$

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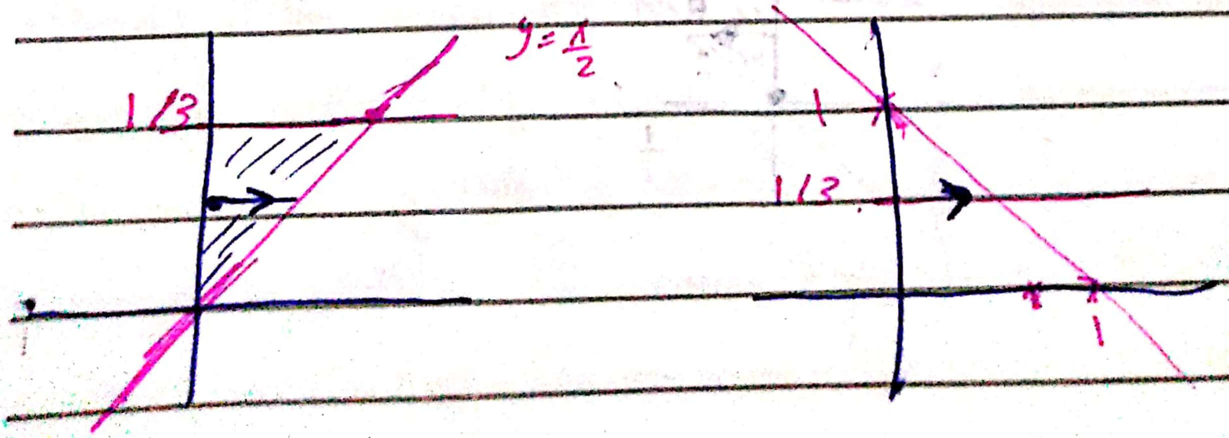
Ex:- Combine the sum of the two double integrals as a single double integral

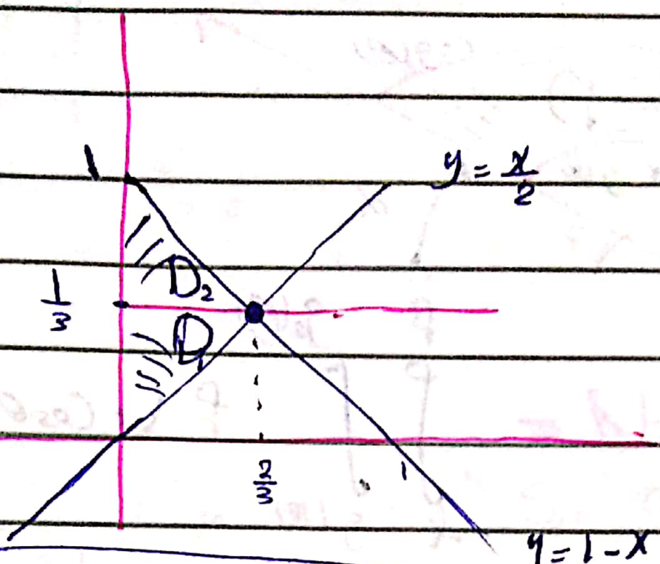
$$I = \int_0^{1/3} \int_0^{2y} f(x,y)\,dx\,dy + \int_{1/3}^{1} \int_0^{1-y} f(x,y)\,dx\,dy$$

$x = 0 \to x = 2y$ $\qquad$ $x = 0 \to x = 1-y \to y = 1-x$

$0 \le y \le \frac{1}{3}$ $\qquad$ $\boxed{y = \frac{x}{2}}$ $\qquad$ $\frac{1}{3} \le y \le 1$



$dy\,dx$ ← إلى $dx\,dy$ وندمج الرسمتان لنحول من



$\frac{1}{3} = \frac{x}{2}$

$\boxed{x = \frac{2}{3}}$

$$\boxed{I = \int_0^{2/3} \int_{x/2}^{1-x} f \, dy \, dx}$$

ها هي

Sec 15.3 Double integral in Polar Coordinates

polar Coordinate (r, θ)

$x = r\cos\theta$

$y = r\sin\theta$

$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\arctan\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{y}{x}\right)$

$x^2 + y^2 = r^2 \Leftrightarrow r = \sqrt{x^2 + y^2}$

$\Rightarrow$

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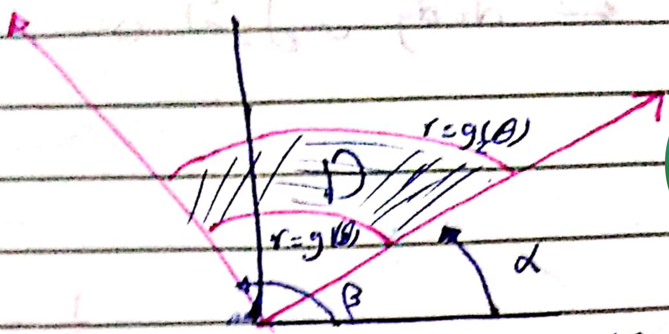


$$\iint_D f(x,y)\, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

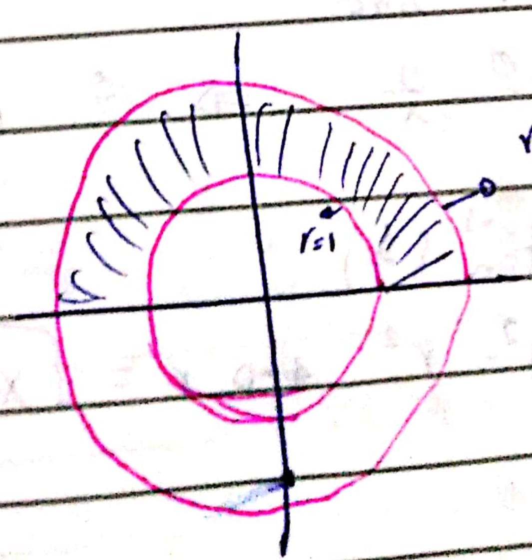
تستبدل dA بـ $r\,dr\,d\theta$

Ex:-

Compute $I = \iint_R (3x + 4y^2)\, dA$ where

R is the region in the upper half-plane

bdd by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$



$\Rightarrow$

$$I = \int_0^{\pi}\int_1^{2} (3r\cos\theta + 4r^2\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^{2} (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta$$

$$\int_0^{\pi} r^3\cos\theta + r^4\sin^2\theta \Big]_1^2 \, d\theta$$

$$\int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\,d\theta$$

$$7\sin\theta\Big]_0^{\pi} + 15\int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta$$

= .... كمل انت

Compute

$\boxed{1}$ $$I_1 = \int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx$$

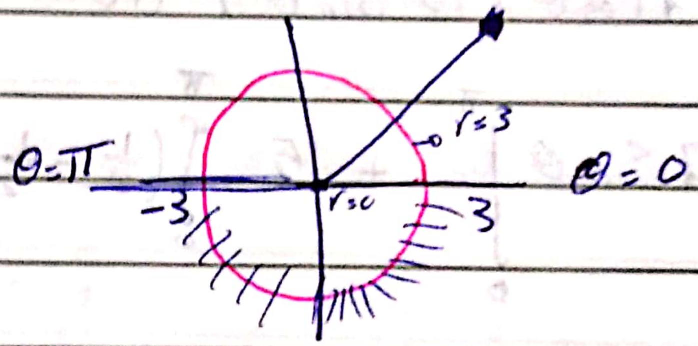
$\boxed{2}$ $I_2 = \int_{-2}^{0} \int_{-\sqrt{4-y^2}}^{0} e^{x^2+y^2}\, dx\, dy$

$\boxed{3}$ $I = \int_{0}^{2} \int_{x}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

Sol:-

$y=0 \longrightarrow y=\sqrt{9-x^2}$ Circul. $\boxed{x^2+y^2=9}$

$-3 \leq x \leq 3$



$I_1 = \int_{0}^{\pi} \int_{0}^{3} \sin(r^2)\, r\, dr\, d\theta$

$\frac{1}{2} \int_{0}^{\pi} \int_{0}^{3} 2r \sin(r^2)\, dr\, d\theta$

$\frac{1}{2} \int_{0}^{\pi} -\cos(r^2) \Big]_{0}^{3} = -\frac{1}{2} \int_{0}^{\pi} (\cos 9 - 1)\, d\theta$

$-\frac{1}{2} (\cos 9 - 1)(\pi) = \frac{\pi}{2} (1 - \cos 9)$

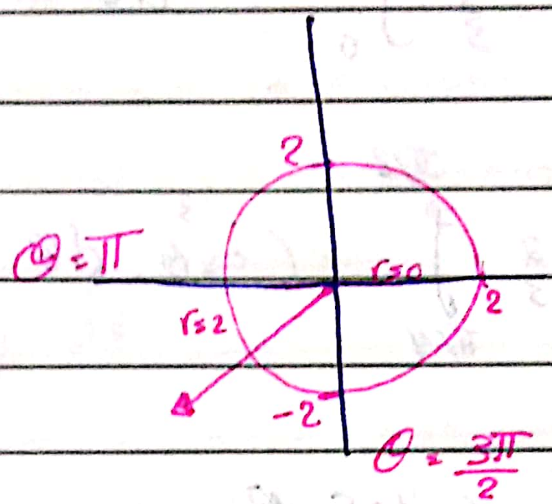
$\boxed{2}$ I_2

$$x = -\sqrt{4-x^2} \longrightarrow x = 0$$

$$-2 \le y \le 0$$

$$x^2 = 4 - y^2$$

$$x^2 + y^2 = 4$$



$$I = \int_{\pi}^{3\pi/2} \int_0^2 e^{r^2} \, r \, dr \, d\theta$$

$= \ldots$ أكمل

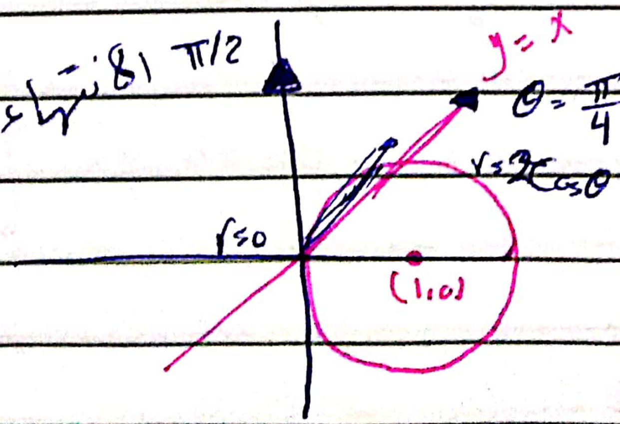
$\boxed{3}$ $y = x \longrightarrow y = \sqrt{2x - x^2} \longrightarrow$

$$0 \le x \le 2$$

$$y^2 = 2x - x^2$$ دائرة

$$(x-1)^2 + y^2 = 1$$

المركز $(1,0)$



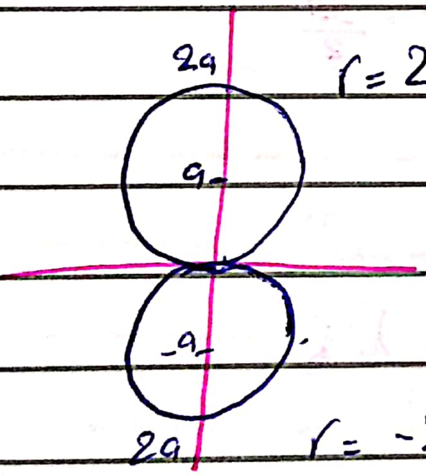
$$y = x$$

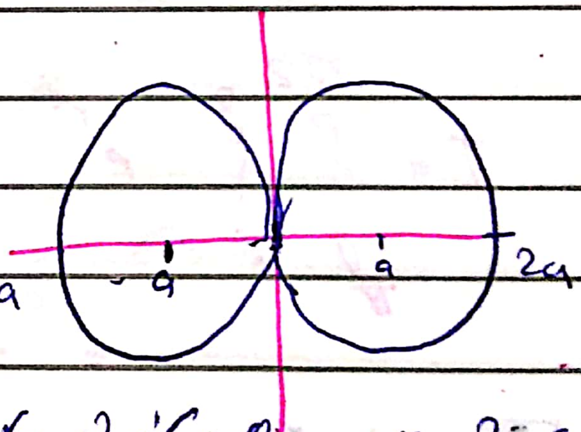
$$\frac{y}{x} = 1 \longrightarrow \tan\theta = 1$$

$$I_3 = \int_{\pi/4}^{\pi/2} \int_{0}^{2\cos\theta} r \, r \, dr \, d\theta$$

$$\int_{\frac{\pi}{4}}^{\pi/2} \frac{r^3}{3} \Big]_0^{2\cos\theta} d\theta$$

$$= \frac{8}{3} \int_{\pi/4}^{\pi/2} \cos^3\theta \, d\theta =$$ غلط





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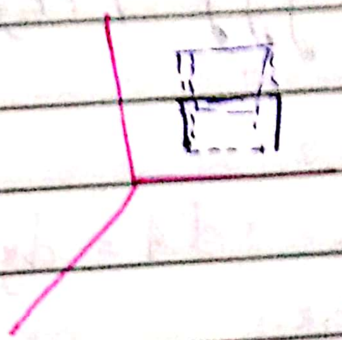
Triple Integrals:-

Fubini's Thm: Let $G = [a,b] \times [c,d] \times [e,f]$

$$\iiint_G F(x,y,z)\,dv = \int_c^d \int_a^b \int_e^f F \; dz\,dx\,dy$$



Ex:- $\displaystyle\iiint_S xy \; dv$

$$S = \{(x,y,z) : -1 \le x \le 2,\ 3 \le y \le 5,\ 0 \le z \le 1\}$$

Sol:-

$$\iiint_S xy \; dv = \int_3^5 \int_{-1}^2 \int_0^1 xy \; dz\,dx\,dy$$

$$\int_3^5 \int_{-1}^2 xy \; dx\,dy = \dots$$

[illegible]

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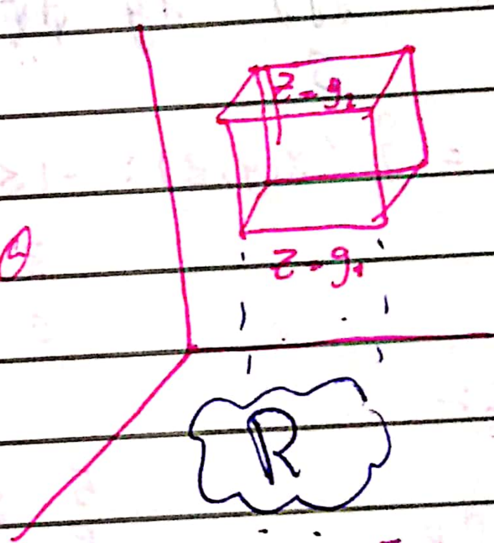
Rule:- let S' be the Solid bdd by the Surface $z = g_1(x,y)$ from below $z = g_2(x,y)$ form above and it's projection in the xy-plane is the region R Then

$$\iiint_{S'} f(x,y,z)\, dv = \iint_{R} \left[\int f\, dz\right] dA$$

$dA = dy\,dx$
or
$dx\,dy$ or $r\,dr\,d\theta$



Ex:- Evaluate $I = \iiint_E z\, dv$ where E is the Solid tetrahedron (هرم) bdd by the planes:-
$x=0$ $y=0$ $z=0$, $x+y+z=1$

Sol:-

E : Surfaces

$z=0$
$z=1-x-y$

تقاطع السطوح

Region in xy-plane

$x=0$
$y=0$
$1-x-y=0$
$y=1-x$

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$y = 1-x \Rightarrow y = 0 \rightarrow y = 1-x$ لمعرفة المنحنى الاصغر

Intersection $0 = 1-x$ $\quad x = 1$ والاكبر ناخذ نقطة

$0 \leq x \leq 1$ اختبار .$x = 0$ عند التعويض

$y = 1$ ينتج له

$$I = \iiint z \, dz \, dA = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx$$

السطح — تكون منطقة

نقطة اختبار $(x, y) = (0, 0)$

* عند تساوي z بال z في عند تعويض نقطة الاختبار, تمثل النقطة
* وعندما نعوض النقطة ال z الاصغر هي في الاسفل وال z الاكبر فوق في التكامل

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} \frac{z^2}{2} \Big]_0^{1-x-y} dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2 \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \frac{(1-x-y)^3}{3(-1)} \Big]_0^{(1-x)} dx$$

$$= -\frac{1}{6} \int_0^1 \left[\left(1-x-(1-x) \right)^3 - (1-x-0)^3 \right] dx$$

$$\frac{1}{6} \int_0^1 (1-x)^3 \, dx = \dots$$ وهكذا

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Rule:- The volume of the Solid G whose Lower surface is $z = g_1(x,y)$ and upper surface is $z = g_2(x,y)$ and the projection of G on the xy-plane is the region D is

$$(*)\quad V = \iiint_G 1\, dV = \iint_D \int_{g_1}^{g_2} 1\, dz\, dA$$

$$\iint_D (g_2 - g_1)\, dA$$

Ex:- Express the volume of the tetrahedron T bdd by $x + 2y + z = 2$, $x = 2y$, the yz-plane ($x = 0$ يعني) and the xy-plane as

1. triple integral
2. double integral and evaluate it

Surface.

$z = 2 - x - 2y$

$z = 0$

التقاطع

$2 - x - 2y = 0$

Curve.

$x = 2y$

$x = 0$

* المعادلة التي تحوي z Surface.

والتي لا تحوي z Curve.

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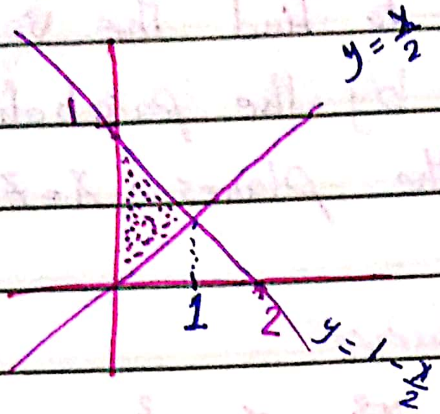

نأخذ المغير $y=$
عشانها تكررت مرتان في
المعادلة وما نأخذ $x=$
لانها تكررت ٣ مرات

$dA = dy\,dx$

$1-\frac{x}{2} = \frac{x}{2}$

$\boxed{1=x}$

① triple integral:-

$$V = \iiint_T 1\,dV = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} \int_0^{2-x-2y} 1\,dz\,dy\,dx$$

نقطة اختبار في المنطقة $(0,0)$

$z=0 \rightarrow \underset{(0,0)}{0}$ $\quad z=0$ اصغر $\quad z=2-x-2y \underset{(0,0)}{\downarrow} \rightarrow \boxed{z=2}$ اكبر

double integral,

$$V = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} (2-x-2y)\,dy\,dx$$

Evaluate $V = \int_0^1 \left[2y - xy - y^2 \right]_{\frac{x}{2}}^{1-\frac{x}{2}}$

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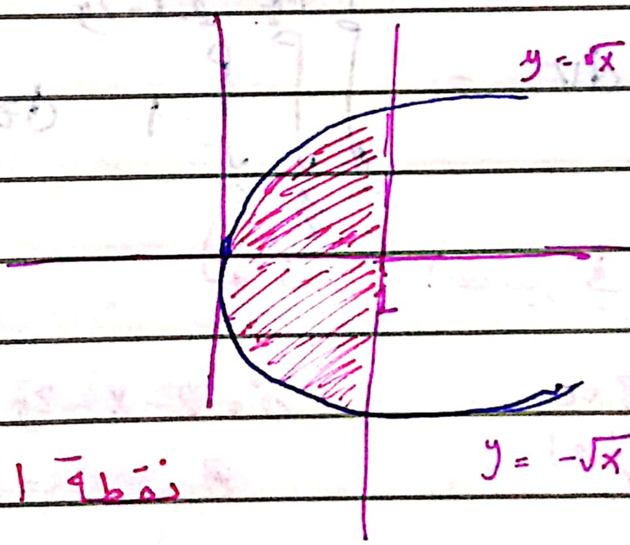
Ex:- Use double integrals to find the volume of the Solid that is bdd by the parabolic Cylinder $x = y^2$ and the planes $x = z$, $x = 1$ and the xy-plane.

Surface	Curve
$z = x$	$\cancel{x = y^2}$ $x = y^2 \Rightarrow y = \mp\sqrt{x}$ $y = \sqrt{x}$
$z = 0$	$x = 1$
التقاطع وصوموجود	$x = 0$

$dy\ dx$ لأن 3 مرات .

$0 \le x \le 1$



نقطة اختبار السطوح $(0,0)$

$z = x \underset{(0,0)}{\downarrow} \Rightarrow z = 0$

$z = 0 \underset{(0,0)}{\downarrow} \Rightarrow z = 0$

متشابه

$(1,0)$ $\quad z = x \underset{(1,0)}{\downarrow} \Rightarrow z = 1$ اكبر

$z = 0 \underset{(1,0)}{\downarrow} \Rightarrow z = 0$ اصغر

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$$V = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} x - 0 \; dy \, dx$$

$$= \int_0^1 x(2\sqrt{x}) \, dx = 2 \int_0^1 x^{3/2} \, dx = \dots$$

Ex :-

Express the volume as a triple integral of the Solid enclosed by $z = x^2 + y^2$ and the xy-plane and $y = 2x$, $y = x^2$

Surface | Curve

$z = 0$ | $y = x^2$

$z = x^2 + y^2$ | $y = 2x$

$x^2 + y^2 = 0$ تقاطع | $x^2 = 2x$ حدود X

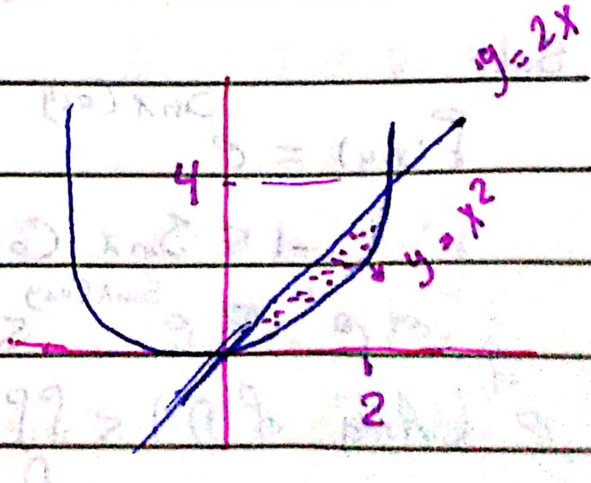
$(x = 0, \; y = 0)$ pt | $x(x-2) = 0$

تمثل | $x = 0 \, , \, 2$

$0 \leq x \leq 2$

$y = 2x \big|_{x=1}$ $\quad y = 2$ $\quad$ اكبر

$y = x^2 \big|_{x=1}$ $\quad y = 1$ $\quad$ اصغر



نقطة اختبار (2,4)

$z = 0 \downarrow_{(2,4)} \to z = 0$ اصغر

$z = x^2 + y^2 \downarrow_{(2,4)} \to z = 20$ اكبر

$$V = \int_0^2 \int_{x^2}^{2x} \int_0^{x^2+y^2} 1 \, dz \, dy \, dx$$

Estimation of Double Integrals:-

Let D be a closed bdd region in the xy-plane

Sot

$m \le f(x,y) \le M$, for all $(x,y) \in D$ then

$$m * (\text{Area of } D) \le \iint_D f \le M * (\text{Area of } D)$$

Ex:- Estimate

$I = \iint e^{\sin x \cos y} \, dA$, where D is the region inside the disk $x^2+y^2 \le 4$

Sol:-

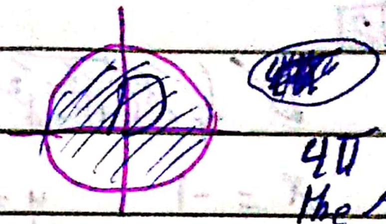
$$F(x,y) = e^{\sin x \cos y}$$

$$-1 \le \sin x \cos y \le 1$$

$$e^{-1} \le e^{\sin x \cos y} \le e^{1} \quad \text{For all } (x,y) \in D$$

$$e^{-1} (\text{Area of } D) \le \iint_D e^{\sin x \cos y} \, dA \le e \, (\text{Area of } D)$$

$\frac{4\pi}{e} \le I \le 4e\pi$



$I \in \left[\frac{4\pi}{e}, 4e\pi\right]$

4   36 " تقريباً

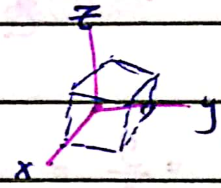
4π the Area of D

Sec 15.8

Triple Integral in Cylindrical Coordinates

let $A(x,y,z)$ be a pt in rectangular Coordinates

The Cylindrical Coordinates of A are $A(r,\theta,z)$



$x = r\cos\theta \qquad r^2 = x^2+y^2$

$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Ex:- If the pt is given in rectangular Coordinates then find it's Cylindrical Coordinates

(1) $A(1,-2,3)$ (2) $B(-1,1,-4)$ (3) $C(\sqrt{3},2,0)$

(4) $D(-2,\sqrt{3},1)$ (5) $E(0,2,1)$ $\boxed{6}$ $F(-1,0,-1)$

Sol:- (1)

$x=1 \quad y=-2 \quad z=3 \quad r=\sqrt{5} \quad \theta = \tan^{-1}\left(\frac{-2}{1}\right)$

$\theta = -\tan^{-1} 2$

$= 2\pi - \tan^{-1} 3$] equivalent

$A(\sqrt{5}, -\tan^{-1}(+2), 3)$

$\boxed{2}$

$B\ \cancel{(\sqrt{(-1)^2+1^2},\ \tan\frac{1}{-1}, -4)}$

$x=-1 \quad y=1 \quad z=-4$

الربع 2 في النقطة

$r=\sqrt{1+1}=\sqrt{2}$

$\tan\theta=\tan -1 \Rightarrow \theta=\pi-\tan^{-1}1=\pi-\frac{\pi}{4}$

$=\frac{3\pi}{4}$

$B(\sqrt{2}, \frac{3\pi}{2}, -4)$

$\boxed{3}$ $x=\sqrt{3}$, $y=2$, $z=0$

ر 1

$r=\sqrt{3+4}=\sqrt{7}$

$\tan\theta=\frac{y}{x}=\frac{2}{\sqrt{3}}$

$\theta=\tan^{-1}\frac{2}{\sqrt{3}}$

$C(\sqrt{7}, \tan^{-1}\frac{2}{3}, 0)$.

$\boxed{4}$ $x=-1 \quad y=-\sqrt{3} \quad z=1$

ر 3

$r=\sqrt{4}=2$

$\tan\theta=\frac{y}{x}=\sqrt{3}$

$\theta=\pi+\tan^{-1}\sqrt{3}=\pi+\frac{\pi}{3}=\frac{4\pi}{3}$

$D(2, \frac{4\pi}{3}, 1)$

⑤ $x=0$ $\quad \underbrace{y=2}_{\text{على محور الصادات الموجب}}$ $\quad z=1$

$r=\sqrt{4}=2$

$\theta=\frac{\pi}{2}$ $\qquad E\left(2, \frac{\pi}{2}, 1\right)$

$\boxed{6}$ $x=-1$ $\quad \underbrace{y=0}_{\text{ع محور السينات السالب}}$ $\quad z=-1$

$r=\sqrt{1+0}=1$

$\theta=\pi$ $\qquad F(1, \pi, -1)$

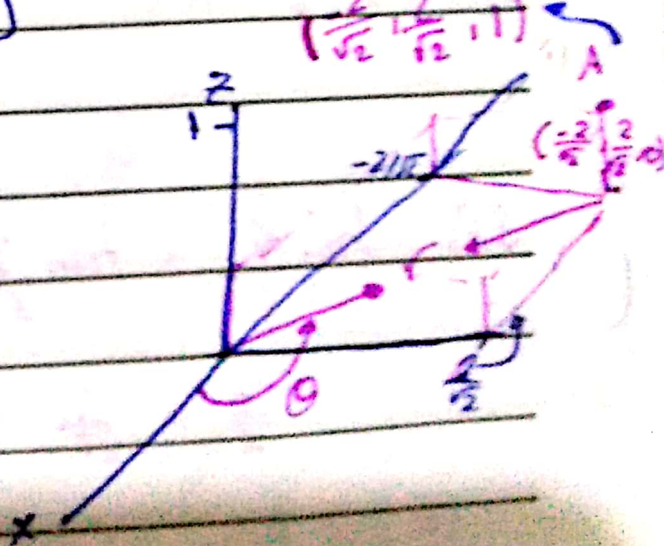
Ex:- If the pt is given in Cylindrical Coordinates then find its rectangular Coordinates and Sketch pt in Cylindrical Coordinate.

$A\left(2, \frac{3\pi}{4}, 1\right)$

$x=r\cos\theta = \frac{-2}{\sqrt{2}}$

$y=r\sin\theta=\frac{2}{\sqrt{2}}$ $\qquad \boxed{z=1}$

$pt\left(\frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, -1\right)$

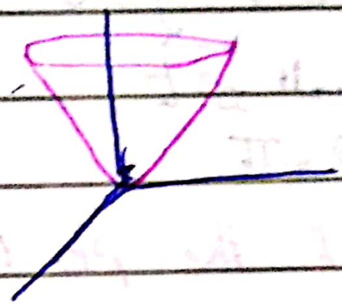


Ex: Describe (Iedntify) and Sketch the Surface whose eq in cylindrical coordinate is

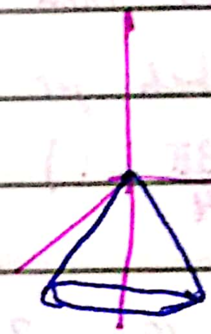
① $z = r$ ③ $\theta = +\frac{\pi}{6}$ ⑤ $\frac{4\pi}{3} = \theta$

② $z = 2r$ ④ $\theta = \frac{3\pi}{2}$

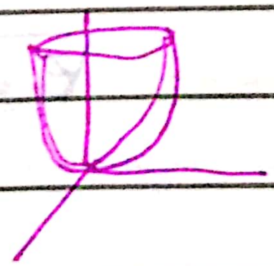
① $z = \sqrt{x^2+y^2}$



② $z = -2\sqrt{x^2+y^2}$ Cone



ضار جبية. ③ $z = r^2 = x^2+y^2$



paraboloid

[3] $\tan\theta = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$\frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow x - \sqrt{3}\,y = 0$

⇒ نكمل الحل

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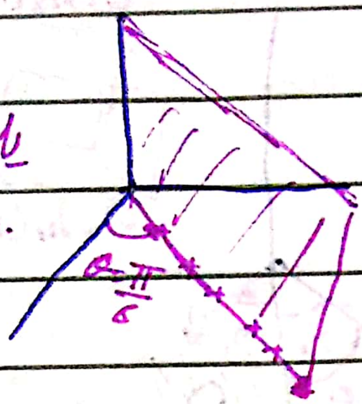
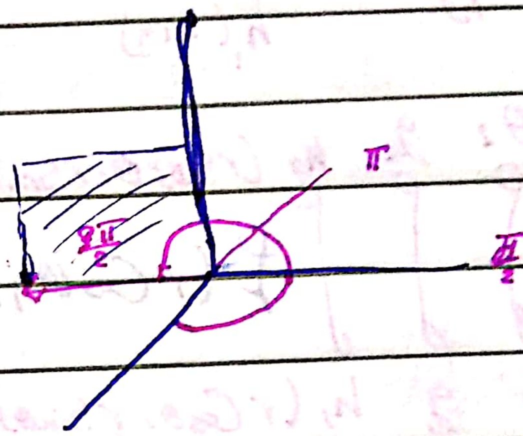
half Plane مس الـ z-axis.



نصف مستوى
دائماً عندما يكون $\theta =$
يكون الرسم نصف مستوى

$\boxed{4}$ $\tan\theta = \tan\frac{3\pi}{2}$



half plane

$x = r\cos\frac{3\pi}{2} = 0$

$y = r\cos\frac{3\pi}{2} = -r \le 0$

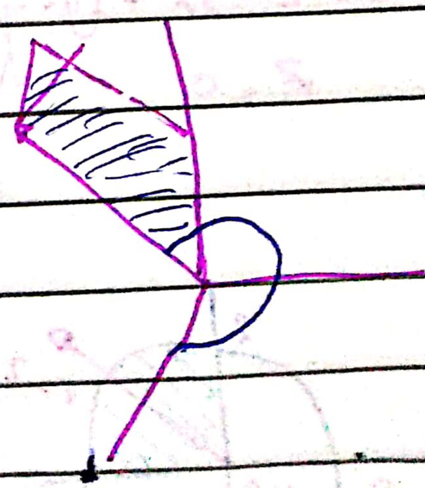
$\{(0, y, z) ; y \le 0, z \in \mathbb{R}\}$

$\boxed{5}$ $\tan\theta = \tan\frac{4\pi}{3}$

$\frac{y}{x} = \tan\frac{\pi}{3} = \sqrt{3}$

half plane.

$\sqrt{3}x - y = 0$



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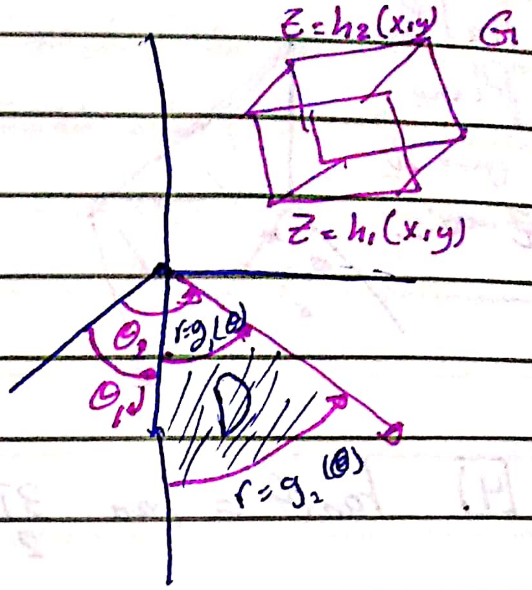
Rule:-



$$\iiint_G f(x,y,z)\,dv$$

$$= \iint_D \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\right] dA$$

$$\int_{\theta_1}^{\theta_2}\int_{g_1}^{g_2}\int_{h_1(r\cos\theta,\, r\sin\theta)}^{h_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta)\,dz\,dr\,d\theta\; r$$

Ex:- Evaluate $I = \iiint_E (x+y+z)\,dv$ where E is the Solid in the first octant that lies under the paraboloid $z = 12-3x^2-3y^2$

Surface | Curve

$z=12-3x^2-3y^2$ | $x=0$

$z=0$ | $y=0$

$3x^2+3y^2=12$ $\div 3$

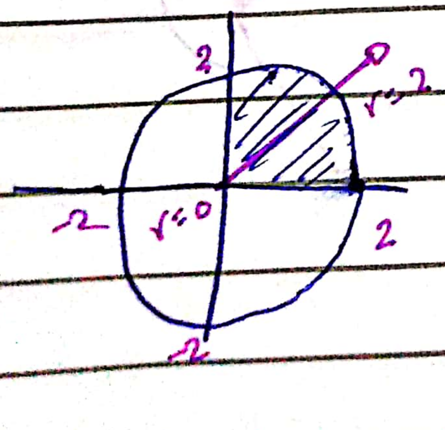
$x^2+y^2=4$

نقطة اختبار $(0,0)$

اصغر $z=0$

اكبر $z=12$

$\Rightarrow D$



$$I = \iint_D \int_0^{12-3(x^2+y^2)} (x+y+z)\, dz\, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2} \int_0^{12-3r^2} (r\cos\theta + r\sin\theta + z)\, dz\, r\, dr\, d\theta$$

Ex:- Evaluate $I = \iiint_G \sqrt{x^2+y^2}\, dV$ where G is the region bdd by $z = x^2+y^2$, $z = 4$

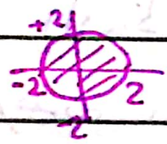
Surface

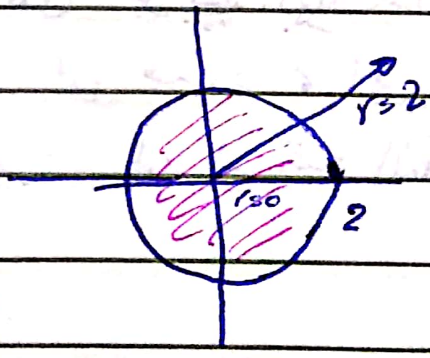
$z = x^2+y^2$

$z = 4$

Curves

$x^2+y^2 = 4$





$\theta = 0$

$= 2\pi$

$$I = \iint_D \int_{x^2+y^2}^{4} \sqrt{x^2+y^2}\, dz\, dA$$

$$\int_0^{2\pi} \int_0^{2} \int_{r^2}^{4} r\, dz\, r\, dr\, d\theta = \text{----}$$

Ex: Use cylindrical coordinate to find the volume of the solid that lies with in the cylinders

$x^2+y^2=1$ $z=4$ and above $z=1-x^2-y^2$

Surface

$z=4$

$z=1-x^2-y^2$

Curves

$x^2+y^2=1$

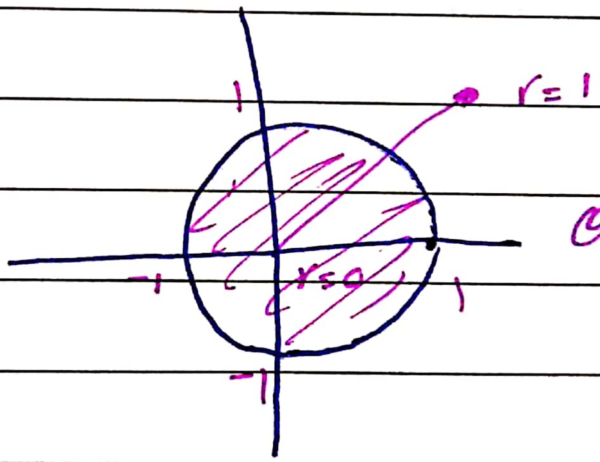
$\cancel{1-x^2-y^2=4}$

$\cancel{x^2+y^2=-3}$ نرفض

$V=\iiint 1\,dv$

$$\iint_D \int_{1-(x^2+y^2)}^{4} 1\,dz\,dA$$

$$\int_0^{2\pi}\int_0^{1}\int_{1-r^2}^{4} dz\; r\,dr\,d\theta$$



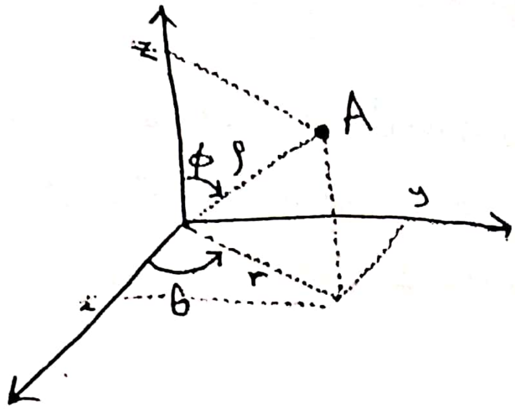
$\theta=0$

$=2\pi$

Sec 15.9. Triple Integrals in Spherical Coordinates.

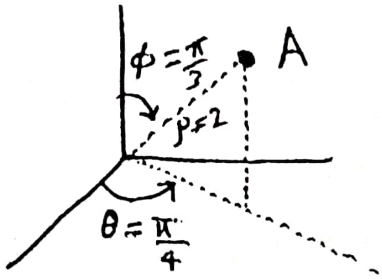
Let $A(x,y,z)$ be a pt. in rectangular coordinates. The spherical coordinates of A are $A(\rho,\theta,\phi)$, where

$$\rho = \sqrt{x^2+y^2+z^2}$$
$$x = \rho \sin\phi \cos\theta \qquad r = \rho\sin\phi$$
$$y = \rho \sin\phi \sin\theta \qquad \rho = \sqrt{r^2+z^2}$$
$$z = \rho\cos\phi \qquad \cos\phi = \frac{z}{\rho}$$
$$0 \le \theta < 2\pi \quad , \quad 0 \le \phi \le \pi$$



Ex 1: The pt. $A(2,\frac{\pi}{4},\frac{\pi}{3})$ is given in spherical coordinates. Plot the pt. showing its spherical coordinates and find its rectangular and cylindrical coordinates.

Sol:



$$\rho = 2, \quad \theta = \frac{\pi}{4}, \quad \phi = \frac{\pi}{3}$$
$$x = \rho\sin\phi\cos\theta = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{6}}{2}$$
$$y = \rho\sin\phi\sin\theta = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = 2\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{6}}{2}$$
$$z = \rho\cos\phi = 2\cos\frac{\pi}{3} = 2\,\frac{1}{2} = 1$$

$\therefore$ The rectangular coordinates of A are: $A(\frac{\sqrt{6}}{2},\frac{\sqrt{6}}{2},1)$

$$r = \rho\sin\phi = 2\sin\frac{\pi}{3} = 2\frac{\sqrt{3}}{2} = \sqrt{3}$$
$$\theta = \frac{\pi}{4}$$
$$z = \rho\cos\phi = 2\cos\frac{\pi}{3} = 1$$

$\therefore$ The cylindrical coordinates of A are: $A(\sqrt{3},\frac{\pi}{4},1)$

Ex 2: The pt. $A(\sqrt{3}, \frac{2\pi}{5}, -1)$ is in cylindrical coordinates. Find the spherical coordinates of A.

Sol: $r = \sqrt{3}$, $\theta = \frac{2\pi}{5}$, $z = -1$

$\rho = \sqrt{r^2 + z^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\theta = \frac{2\pi}{5}$

$\cos\phi = \frac{z}{\rho} = \frac{-1}{2} \Rightarrow \phi = \pi - \cos^{-1}\frac{1}{2} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore$ The spherical coordinates are $A(2, \frac{2\pi}{5}, \frac{2\pi}{3})$.

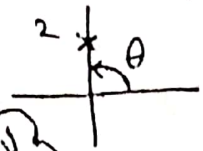
Ex 3: The pt. $B(0, 2, -2\sqrt{3})$ is in rectangular coordinates. Find the spherical coordinates of B.

Sol: $x = 0$, $y = 2$, $z = -2\sqrt{3}$

$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + 4 + 4(3)} = \sqrt{16} = 4$

θ : $(x, y) = (0, 2) \Rightarrow$ لاحظ احد الإحداثيات صفر لذا لا للقوانين بل الرسم

$\therefore \theta = \frac{\pi}{2}$.



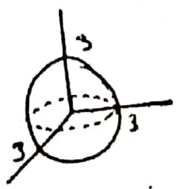
$\cos\phi = \frac{z}{\rho} = \frac{-2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2} \Rightarrow \phi = \pi - \cos^{-1}\frac{\sqrt{3}}{2} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore$ The spherical coordinates of B are $(4, \frac{\pi}{2}, \frac{5\pi}{6})$.

Ex 4: Identify and sketch the surface whose eq. is given in spherical:

(1) $\rho = 3$ (2) $\rho = 4\cos\phi$ (3) $\rho = \csc\phi \cot\phi$ (4) $\rho = 6\csc\phi$

(5) $\rho = 6\sin\phi \sin\theta$. (6) $\rho = \sec\phi$

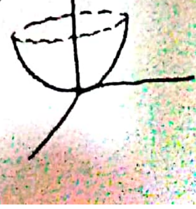


Sol: (1) $\sqrt{x^2+y^2+z^2} = 3 \Rightarrow x^2+y^2+z^2 = 9$

$\boxed{\text{Sphere centered at the origin of radius 3}}$

(2) $(\rho = 4\cos\phi) * \rho \Rightarrow \rho^2 = 4\rho\cos\phi \Rightarrow x^2+y^2+z^2 = 4z$

$\Rightarrow x^2+y^2+(z-2)^2 = 4$ $\boxed{\text{sphere centered at } (0,0,2) \text{ of radius 2.}}$

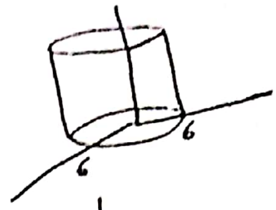


(3) $\rho = \frac{\cos\phi}{\sin^2\phi} \Rightarrow \rho\sin^2\phi = \cos\phi \Rightarrow \rho^2\sin^2\phi = \rho\cos\phi$

$r^2 = z \Rightarrow x^2 + y^2 = z$ $\boxed{\text{paraboloid.}}$

(4) $\rho = \frac{6}{\sin\phi} \Rightarrow \rho\sin\phi = 6 \Rightarrow r = 6 \Rightarrow r^2 = 36$

$\Rightarrow x^2 + y^2 = 36$ $\boxed{\text{cylinder}}$



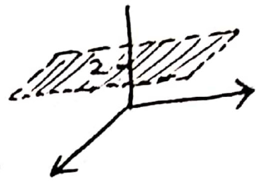
(5) $(\rho = 6\sin\phi\cos\theta) * \rho \Rightarrow \rho^2 = 6\rho\sin\phi\cos\theta$

$x^2 + y^2 + z^2 = 6x \Rightarrow (x-3)^2 + y^2 + z^2 = 9$

$\boxed{\text{sphere centered at } (3,0,0) \text{ of radius } 3.}$

~~Ex~~ (6) $\rho = \frac{1}{\cos\phi} \Rightarrow \rho\cos\phi = 1 \Rightarrow z = 1$

$\therefore$ $\boxed{\text{plane parallel to the xy-plane pass through } (0,0,1)}$



Ex5. Identify and sketch the surface whose eq. is in spherical:

(1) $\phi = 0$ (2) $\phi = \pi$ (3) $\phi = \frac{\pi}{2}$ (4) $\phi = \frac{\pi}{4}$ (5) $\phi = \frac{3\pi}{4}$

(6) $\phi = \frac{\pi}{6}$ (7) $\phi = \frac{5\pi}{6}$ (8) $\phi = \frac{\pi}{3}$ (9) $\phi = \frac{2\pi}{3}$.

Sol: (1) $\rho\cos\phi = \rho\cos 0 \Rightarrow z = \rho$ → لاحظ أن z موجبة

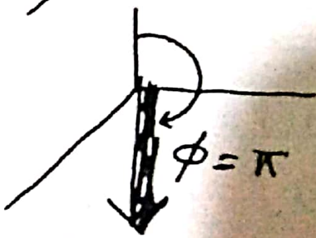
$z^2 = \rho^2 = x^2 + y^2 + z^2 \Rightarrow x^2 + y^2 = 0 \Rightarrow x = 0,\ y = 0,\ z \geq 0$

$\therefore$ $\boxed{\text{positive z-axis including the origin}}$.

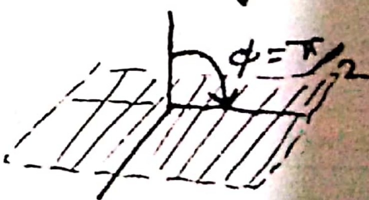
(2) $\rho\cos\phi = \rho\cos\pi \Rightarrow z = -\rho$ → لاحظ أن z ~~موجبة~~ سالبة

$z^2 = \rho^2 = x^2 + y^2 + z^2 \Rightarrow x^2 + y^2 = 0 \Rightarrow x = 0,\ y = 0,\ z \leq 0$

$\boxed{\text{negative z-axis including the origin}}$.



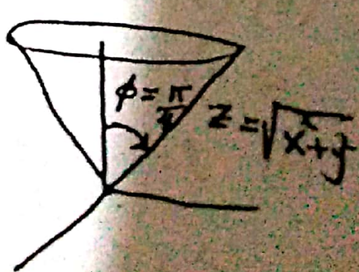
(3) $\rho\cos\phi = \rho\cos\frac{\pi}{2} \Rightarrow z = 0$ $\boxed{\text{xy-plane}}$



(4) $\rho\cos\phi = \rho\cos\frac{\pi}{4} \Rightarrow z = \frac{\rho}{\sqrt{2}}$ → لاحظ أن z موجبة

$\sqrt{2}\, z = \rho \Rightarrow 2z^2 = \rho^2 = x^2 + y^2 + z^2 \Rightarrow z^2 = x^2 + y^2$

~~...~~ $z = \sqrt{x^2 + y^2}$ $\boxed{\text{Cone}}$ نذكر z موجبة

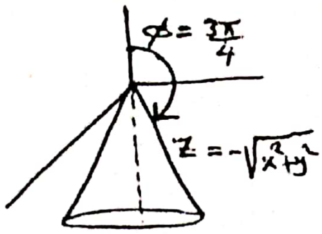


④

(5) $\rho\cos\phi = \rho\cos\frac{3\pi}{4} \Rightarrow z = -\frac{\rho}{\sqrt{2}}$ لاحظ أن z سالبة

$\sqrt{2}\,z = -\rho \Rightarrow 2z^2 = \rho^2 = x^2+y^2+z^2$

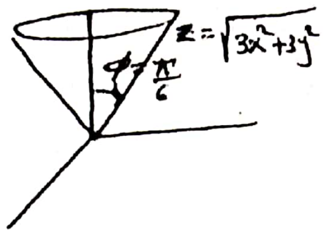
$\Rightarrow z^2 = x^2+y^2 \Rightarrow z = -\sqrt{x^2+y^2}$ تذكر أن z سالبة



(6) $\rho\cos\phi = \rho\cos\frac{\pi}{6} \Rightarrow z = \frac{\sqrt{3}}{2}\rho$ لاحظ أن z موجبة

$2z = \sqrt{3}\rho \Rightarrow 4z^2 = 3\rho^2 = 3x^2+3y^2+3z^2$

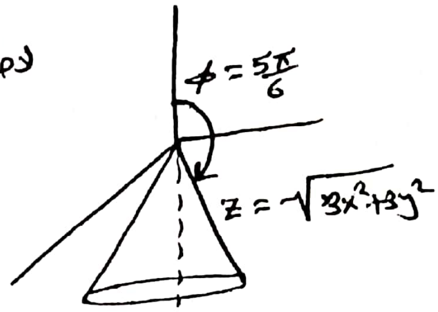
$\Rightarrow z^2 = 3x^2+3y^2 \Rightarrow z = \sqrt{3x^2+3y^2}$ تذكر z موجبة



(7) $\rho\cos\phi = \rho\cos\frac{5\pi}{6} \Rightarrow z = -\frac{\sqrt{3}}{2}\rho$ لاحظ أن z سالبة

$2z = -\sqrt{3}\rho \Rightarrow 4z^2 = 3\rho^2 = 3(x^2+y^2+z^2)$

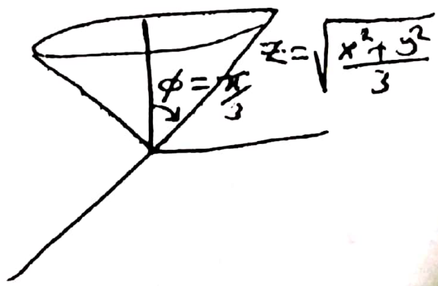
$\Rightarrow z^2 = 3x^2+3y^2 \Rightarrow z = -\sqrt{3x^2+3y^2}$



(8) $\rho\cos\phi = \rho\cos\frac{\pi}{3} \Rightarrow z = \frac{\rho}{2}$ لاحظ z موجبة

$2z = \rho \Rightarrow 4z^2 = \rho^2 = x^2+y^2+z^2$

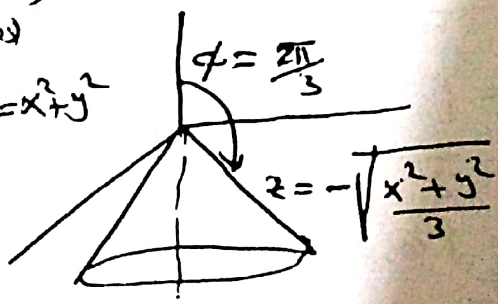
$3z^2 = x^2+y^2 \Rightarrow z = \sqrt{\frac{x^2+y^2}{3}}$ لأن z موجبة



(9) $\rho\cos\phi = \rho\cos\frac{2\pi}{3} \Rightarrow z = -\frac{\rho}{2}$ لاحظ z سالبة

$2z = -\rho \Rightarrow 4z^2 = \rho^2 = x^2+y^2+z^2 \Rightarrow 3z^2 = x^2+y^2$

$z = -\sqrt{\frac{x^2+y^2}{3}}$ لأن z سالبة



Remark 5.

(1) For $a>0$: $\rho = a$ sphere centered at the origin of radius a.

(2) $\phi = 0 \Leftrightarrow$ positive z-axis including the origin

$\phi = \pi \Leftrightarrow$ negative z-axis 〃 〃 〃

(3) $\phi = \frac{\pi}{2} \Leftrightarrow$ the xy-plane $\Leftrightarrow z = 0$.

(4) $0 < \phi_0 < \frac{\pi}{2} \Leftrightarrow \phi = \phi_0$ cone above the xy-plane

(5) $\frac{\pi}{2} < \phi_0 < \pi \Rightarrow \phi = \phi_0$ 〃 below 〃 〃 〃

in spheri[cal]

$\rho = 6\csc\phi$

Ex6. Write the eq: in spherical coordinates:

(1) $z=-3$ (2) $x^2+6x+y^2+z^2=0$ (3) $z=-\sqrt{\frac{x^2+y^2}{3}}$

(4) $z^2=x^2+y^2$ (5) $z=\sqrt{x^2+2y^2}$

Sol: (1) $\rho\cos\phi=-3 \Rightarrow \rho=-3\sec\phi$

(2) $x^2+y^2+z^2=-6x \Rightarrow \rho^2=-6\rho\sin\phi\cos\theta$

$\rho=-6\sin\phi\cos\theta$.

(3) $\rho\cos\phi=-\frac{\sqrt{r^2}}{\sqrt{3}}=-\frac{r}{\sqrt{3}}=-\frac{\rho\sin\phi}{\sqrt{3}} \Rightarrow \tan\phi=-\sqrt{3}$

$\Rightarrow \phi=\pi-\tan^{-1}\sqrt{3}=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$.

(4) $\rho^2\cos^2\phi=r^2=\rho^2\sin^2\phi \Rightarrow \tan^2\phi=1 \Rightarrow \tan\phi=1$ or $\tan\phi=-1$

$\Rightarrow \phi=\frac{\pi}{4}$ or $\phi=\pi-\tan^{-1}1=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$.

$\therefore \phi=\frac{\pi}{4}$ or $\frac{3\pi}{4}$

(5) $z=\sqrt{x^2+y^2+y^2} \Rightarrow \rho\cos\phi=\sqrt{r^2+y^2}=\sqrt{\rho^2\sin^2\phi+\rho^2\sin^2\phi\sin^2\theta}$

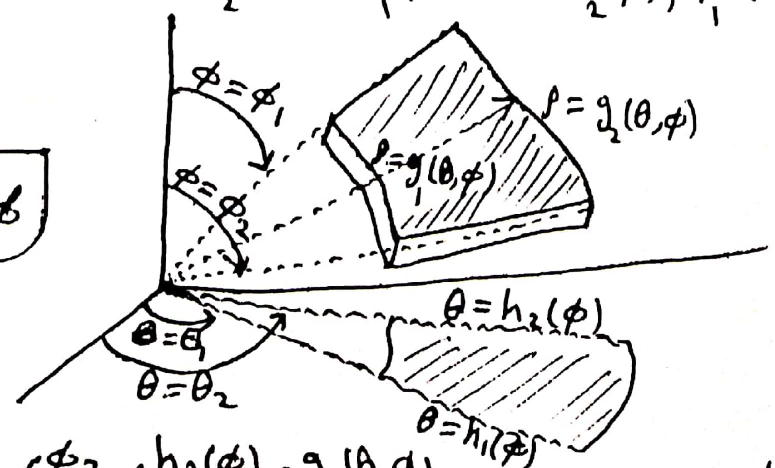
$\rho\cos\phi=\sqrt{\rho^2\sin^2\phi}\,\sqrt{1+\sin^2\theta}=\rho\sin\phi\,\sqrt{1+\sin^2\theta}$

$\tan\phi=\frac{1}{\sqrt{1+\sin^2\theta}}$.

Rule 7 Let G be the solid given in spherical coordinates as:

$G=\{(\rho,\theta,\phi): g_1(\theta,\phi)\le\rho\le g_2(\theta,\phi),\ h_1(\phi)\le\theta\le h_2(\phi),\ \phi_1\le\phi\le\phi_2\}$

$\boxed{dV=\rho^2\sin\phi\, d\rho\, d\theta\, d\phi}$



$$\iiint_G f(x,y,z)\,dV=\int^{\phi_2}\int^{h_2(\phi)}\int^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\ \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

لان الجسم كرة صالحة Sphe

Ex6: Evaluate $I = \iiint_B e^{-(x^2+y^2+z^2)^{3/2}}\, dV$, where

(1) $B = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 2\}$

(2) B is the region between $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$

(3) B " " " " $x^2+y^2+z^2=1$, $x^2+y^2+z^2=4$ and above $z=\sqrt{x^2+y^2}$

(1)

$$I = \int_0^{\pi}\int_0^{2\pi}\int_0^{\sqrt{2}} e^{-(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \rho^2 e^{-(\rho^2)^{3/2}} d\rho$$

$$= -\cos\phi \Big]_0^{\pi} * 2\pi * \frac{-1}{3} \int_0^{\sqrt{2}} -3\rho^2 e^{-\rho^3} d\rho$$

$$= (1+1) * 2\pi * \frac{-1}{3} e^{-\rho^3} \Big]_0^{\sqrt{2}}$$

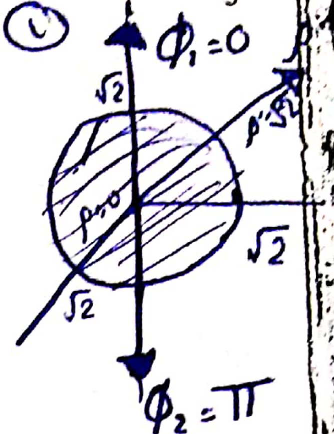
$$= \frac{4\pi}{3} \left(e^{-2\sqrt{2}} - 1\right)$$

$0 \le \phi \le \pi$

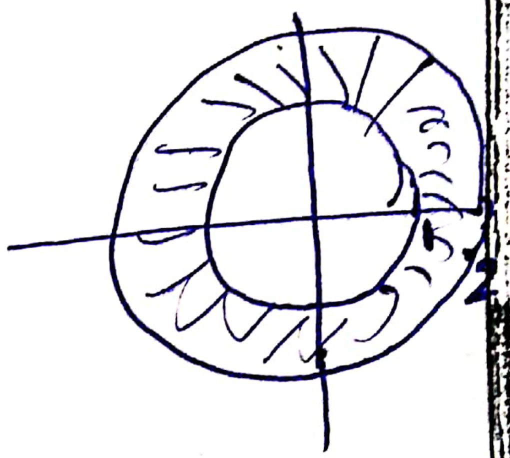
$0 \le \rho \le \sqrt{2}$

$0 \le \cancel{2\pi}\ \theta \le 2\pi$

(1)

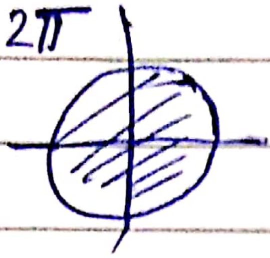


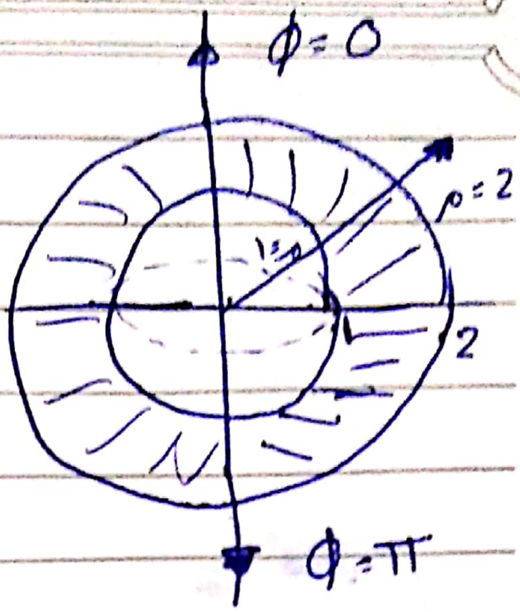
(2)



Ex 6

$\boxed{2}$ $0 \le \theta \le 2\pi$





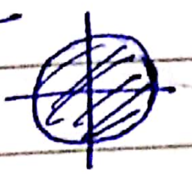
$$I = \int_0^{\pi}\int_0^{2\pi}\int_1^{2} e^{-(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \dots$$

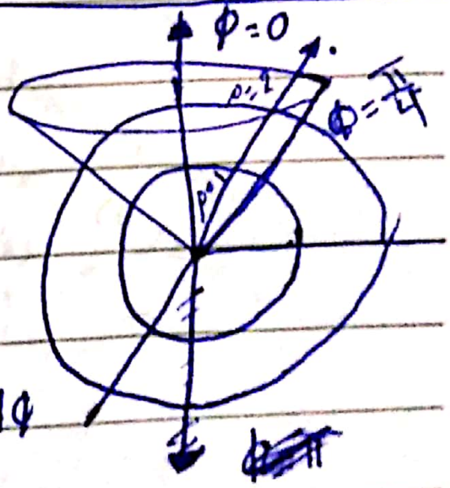
$$= -\frac{4\pi}{3}\left(e^{-8} - e^{-1}\right)$$

$\boxed{3}$

$z = \sqrt{x^2+y^2}$

$0 \le \theta \le 2\pi$





$$I = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_1^{2} e^{-(\rho^2)^{3/2}} \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$= \dots$

$z = r$

$\rho \cos\phi = \rho \sin\phi$

$\tan\phi = 1$

$\phi = \frac{\pi}{4}$

Ex7 Use spherical coordinates to find the volume of the solid: 7

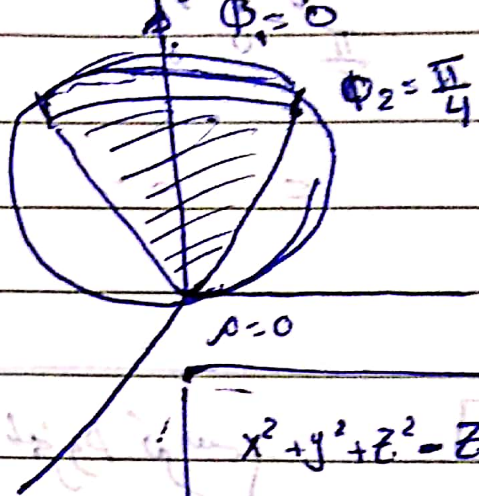
(1) that lies above the cone $z=\sqrt{x^2+y^2}$ and inside the sphere: $x^2+y^2+z^2=z$.

(2) that lies inside the sphere $x^2+y^2+z^2=4$, above the xy-plane and below the cone $z=\sqrt{3x^2+3y^2}$.

(3) that lies inside the sphere $x^2+y^2+z^2=4$ and above the cone: $z=-\sqrt{\frac{x^2+y^2}{3}}$.

Ex 7:-

① $\iiint_G 1 \, dv = \iiint \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$



above the Cone

داخل الكرة وفوق المخروط

$0 \le \theta \le 2\pi$

$x^2 + y^2 + z^2 - z = 0$

$x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}$

$(0, 0, \frac{1}{2}) \quad r = \frac{1}{2}$

$\rho^2 = \rho \cos\phi$

$\rho = \cos\phi$

$$V = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

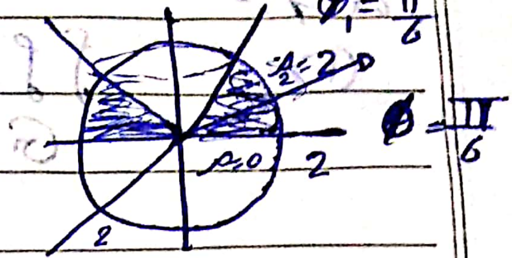
$$\int_0^{\pi/4} \int_0^{2\pi} \frac{\rho^3}{3} \sin\phi \Big]_0^{\cos\phi} d\theta \, d\phi$$

$$= \frac{1}{3} \int_0^{\pi/4} 2\pi \cos^3\phi \sin\phi \, d\phi$$

نكامل بالتعويض :-

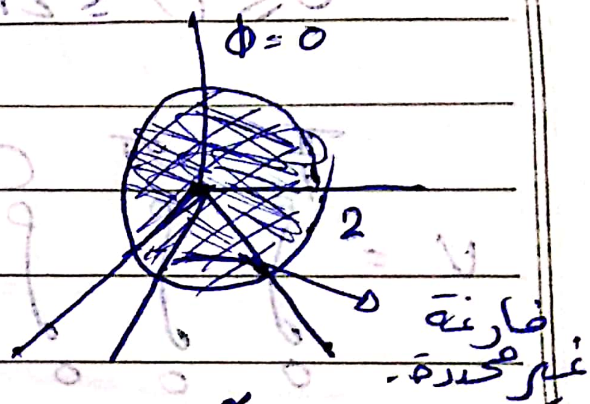
2

$$V = \int_{\pi/6}^{\pi/2}\int_{0}^{2\pi}\int_{0}^{2} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

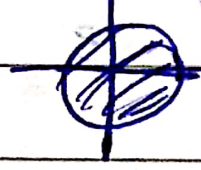


$0 \le \theta \le 2\pi$

3 داخل الكرة داخل المخروط كرسم هنا كذلك لان المخروط للاسفل.



$$V = \int_{0}^{2\pi/3}\int_{0}^{2\pi}\int_{0}^{2} \rho^2 \, d\rho \, d\theta \, d\phi$$



$0 \le \theta \le 2\pi$

in the first octant

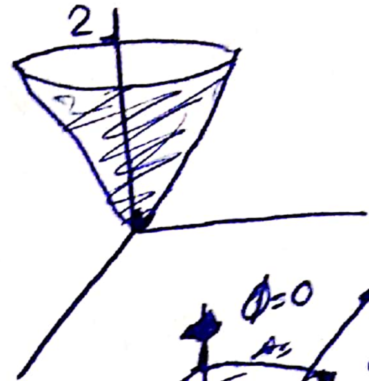
(8)

Ex8: Find the volume of the solid enclosed by $z=2$ and $z=\sqrt{x^2+y^2}$.

$$V=\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{2\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$= \dots\dots$

all octant



Polar والفرق هنا الحل به

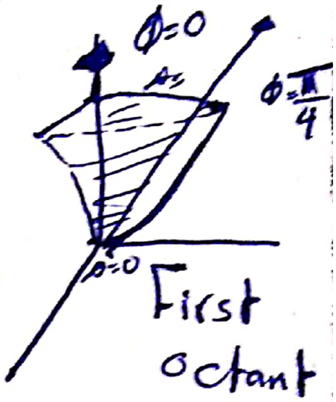
حل به spherical عندما تكون

كرات أو كرات ومخروط .

$z=2$

$\rho\cos\phi=2$

$\rho_2=2\sec\phi$



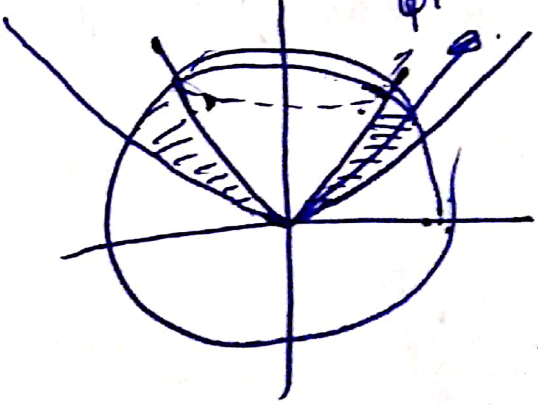
First octant

$0\leq\theta\leq\frac{\pi}{2}$

Ex9: Find the volume of the solid that lies between the cones $z=\sqrt{x^2+y^2}$, $z=\sqrt{\frac{x^2+y^2}{3}}$ and inside the sphere $x^2+y^2+z^2=4$.

$$V=\int_{\pi/4}^{\pi/3}\int_0^{2\pi}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$= \dots\dots$



$\rho_1=0$

$\rho_2=2$

$0\leq\theta\leq2\pi$

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موضوع الدرس سؤال مهم و مشهور اليوم التاريخ / /

$$I = \iiint_G -12 \, dv \text{ , where } G \text{ is hemi sphere}$$

$$x^2 + y^2 + z^2 \leq 4 \text{ , } z \geq 0$$

$$= -12 \iiint_G 1 \, dv = -12 \text{ Volum } G$$

نصف كرة .

$$-\frac{1}{2} * \frac{1}{2} \text{ Volum of Sphere}$$

$$-6 * \frac{4}{3} \pi (2)^3 = -64 \pi$$

(نصف القطر)