ElCoM
ELECTRICAL COMPUTER MECHATRONICS

تقدم لجنة ElCoM الاكاديمية

دفتر لمادة:

تفاضل و تكامل (3)

من شرح:

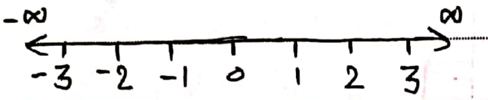
د.فادي عواودة

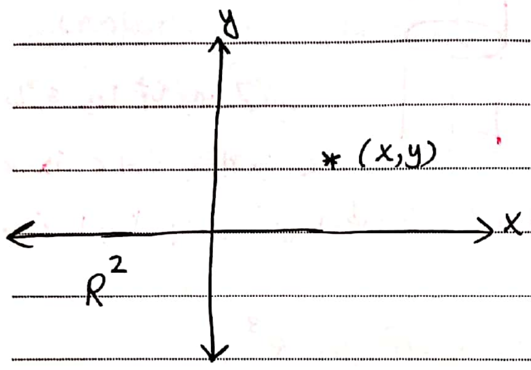
جزيل الشكر للطالبة:

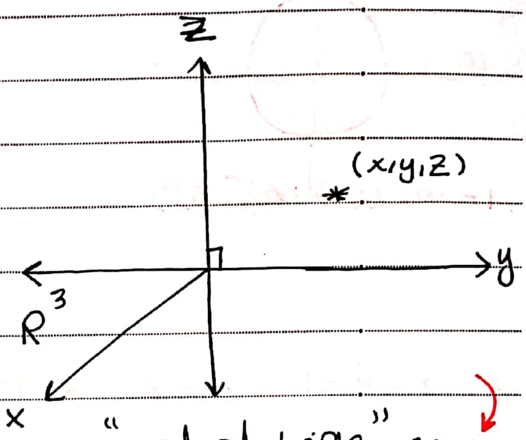
نتالي الكايد

Chapter (12) [12-1]

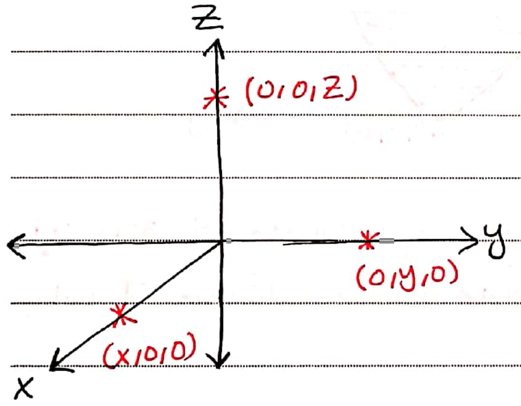
* Coordinat system *

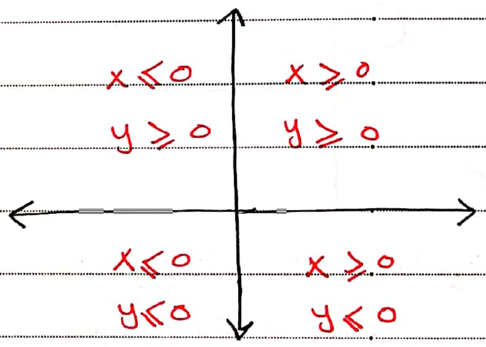






"ordered triple" اسمه

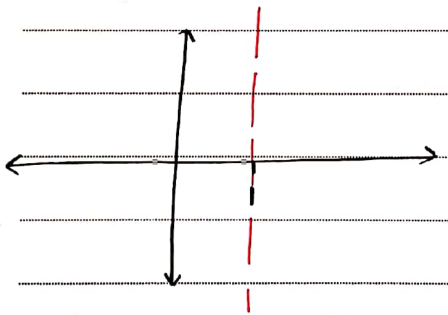


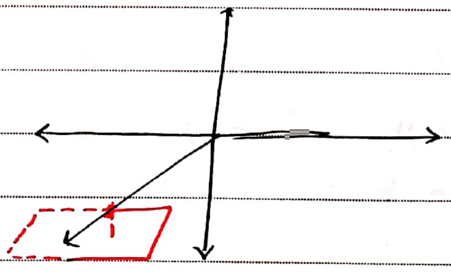


" الاختلافات بين R^2 و R^3 عند الرسم "

* $x = 1$ in R^2

* $x = 1$ in R^3

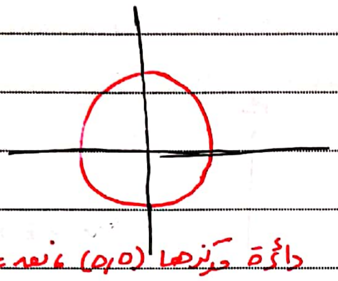




خط مستقيم يمر بالنقطة (1,0) على محور السينات .

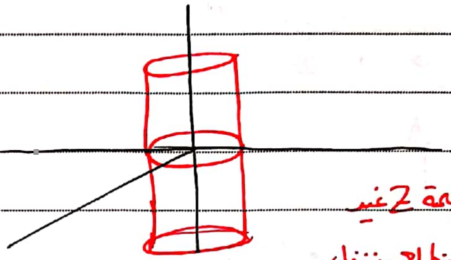
بما أن y, Z غير محددين إذن ، لا يكون خط مستقيم إنما صفحة (Plane) ويمكننا تحريك الورقة على Z و y دون تحديد

* $x^2+y^2=1$ in R^2



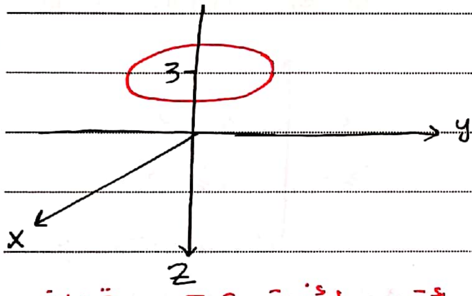
دائرة مركزها (0,0) ، نق = 1

* $x^2+y^2=1$ in R^3



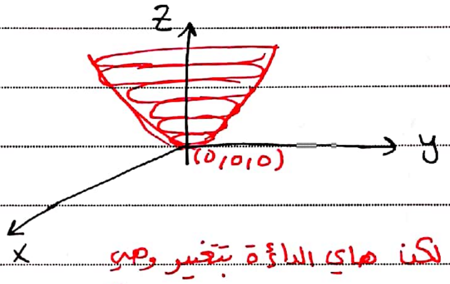
دائرة وبما أن قيمة Z غير محددة ، اذن بنطلع وبننزل حيث ما بدنا فبصير الشكل اسطوانة . (cylinder) .

* $x^2+y^2=1$, $Z=3$ in R^3



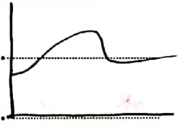
دائرة ، وبما أنه قيمة Z محددة ، إذن برسم دائرة عند $Z=3$.

* $Z=x^2+y^2$ in R^3



بما إنه دائرة لكن هاي الدائرة بتغير ونق بتساوي Z بتضل تكبر لحتى تعمل هادا الشكل .

* graph in R^2 is curve , but in R^3 is surface .



(curve) .

* sphere :-

$(x-a)^2+(y-b)^2+(Z-c)^2=r^2$. معادلة الكرة & لازم كل الأطراف عليها تربيع

اذا كانت (−) بتبطل معادلة كرة

center (a,b,c)

raduis $=r$

* $x^2+y^2+Z^2=3 \rightarrow$ Center $(0,0,0)$

$r=\sqrt{3}$

* $x^2 + 4x + y^2 - 2y + z^2 = 9$

$x^2 + 4x + 4 - 4 + y^2 - 2y + 1 - 1 + z^2 = 9$ .... by completed square

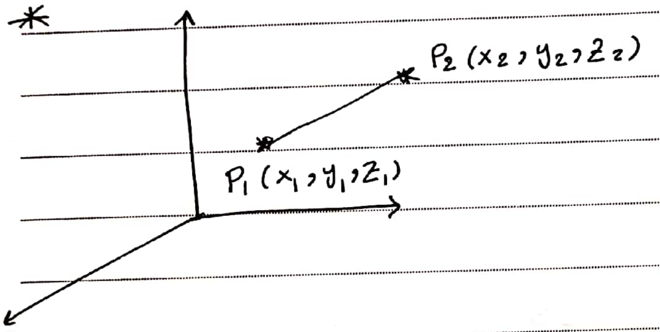
$(x+2)^2 + (y-1)^2 + z^2 = 14$

center → $(-2, 1, 0)$

$r \rightarrow \sqrt{14}$

للتذكير :- طريقة آكمال المربع
نأخذ معامل $(\frac{x}{2})^2$
ونضيفه ونطرحه

*

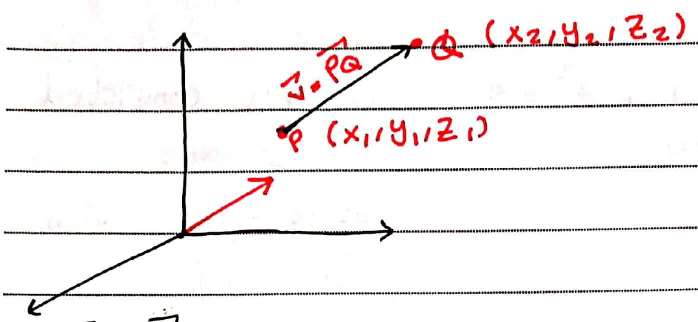


the distance between P_1 and P_2 $\Rightarrow |P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

* $P_1(2, 3, -1)$
  $P_2(0, 4, 5)$

$|P_1P_2| = \sqrt{(0-2)^2 + (4-3)^2 + (5+1)^2}$

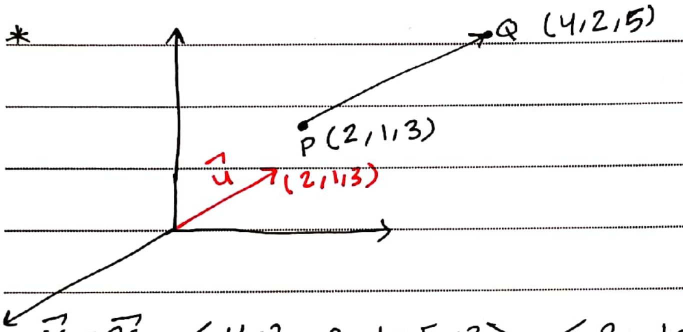
[12.2] vectors



كل متجه لا يبدأ من نقطة الأصل
سننقله ليبدأ من نقطة الأصل
مع الحفاظ على الطول والاتجاه
فالموقع غير مهم.

* $\vec{v} = \vec{PQ} = \langle x_2 - x_1 , y_2 - y_1 , z_2 - z_1 \rangle$

*



$\vec{v} = \vec{PQ} = \langle 4-2 , 2-1 , 5-3 \rangle = \langle 2 , 1 , 2 \rangle$

وبما أننا سننقله ليبدأ من نقطة الأصل سيصبح كالتالي

~~$\vec{u}$ = < [illegible] 2 [illegible] 3~~

$\vec{u} = \langle 2 , 1 , 2 \rangle$

* Length of $\vec{v}$ or magnitude of $\vec{v}$ $|\vec{v}|$

$\vec{v} = \langle v_1 , v_2 , v_3 \rangle$

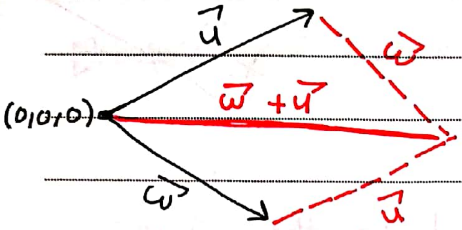
$|\vec{v}| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}$. من قانون المسافة بين نقطتين.

* Let $\vec{w} = \langle 1 , 5 , -2 \rangle$, what is the magnitude of $\vec{w}$?

$|\vec{w}| = \sqrt{(1)^2 + (5)^2 + (-2)^2}$

* let $\vec{u} = \langle \vec{u_1} , \vec{u_2} , \vec{u_3} \rangle$, $\vec{w} = \langle w_1 , w_2 , w_3 \rangle$,

$\vec{u} + \vec{w} = \langle \vec{u_1} + \vec{w_1} , \vec{u_2} + \vec{w_2} , \vec{u_3} + \vec{w_3} \rangle$.

*


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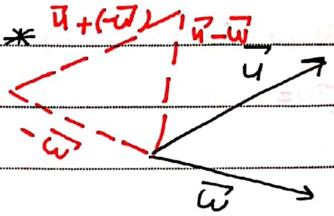
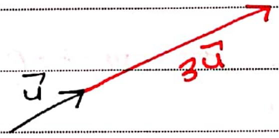


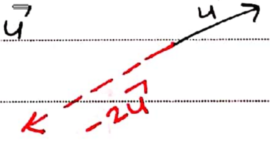
* Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$,
$\vec{u} - \vec{w} = \langle u_1 - w_1, u_2 - w_2, u_3 - w_3 \rangle$.

* $K \cdot \vec{u} = \langle K \cdot u_1, K \cdot u_2, K \cdot u_3 \rangle$
$3 \cdot \langle 2, 5, 7 \rangle = \langle 6, 15, 21 \rangle$

* $K \cdot \vec{u}$, where $K > 0 \Leftrightarrow 3\vec{u}$



* $K \cdot \vec{u}$, where $K < 0 \Leftrightarrow -2\vec{u}$



* $\vec{u} + (-\vec{w})$ يعني $\vec{w}$ بكون بالاتجاه المعاكس إله.

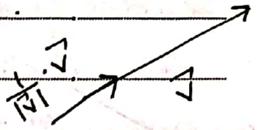
* Let $\vec{u} = \langle 2, 0, -3 \rangle$, $\vec{w} = \langle 1, 3, 5 \rangle$

$\boxed{1}$ $\vec{u} + 2\vec{w} = \langle 2, 0, -3 \rangle + \langle 2, 6, 10 \rangle = \langle 4, 6, 7 \rangle$.

$\boxed{2}$ $|\vec{u} + 2\vec{w}| = \sqrt{(4)^2 + (6)^2 + (7)^2}$ نفس الفرع السابق بس بزيد خطوة.

* $|\vec{v}| = 1$, then $\vec{v}$ is called a unit vector.

$\frac{1}{|\vec{v}|} \cdot \vec{v} = 1$



* find the vector of magnitude 3 and in the same direction as $\vec{u} = \langle 1, 3, 4 \rangle$?

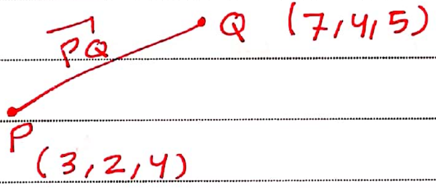
$|\vec{u}| = \sqrt{(1)^2 + (3)^2 + (4)^2} = \sqrt{26}$

$3 \cdot \frac{1}{|\vec{u}|} \cdot \vec{u} = \frac{3}{\sqrt{26}} \langle 1, 3, 4 \rangle = \langle \frac{3}{\sqrt{26}}, \frac{9}{\sqrt{26}}, \frac{12}{\sqrt{26}} \rangle$

لو كان عكس الاتجاه بضرب ب (−3) .

* find the vector of magnitude 2 direction opposite to $\vec{PQ}$?

عكس الاتجاه



$\vec{u} = \vec{PQ} = \langle 7-3, 4-2, 5-4 \rangle = \langle 4, 2, 1 \rangle$

$|\vec{u}| = \sqrt{(4)^2 + (2)^2 + (1)^2} = \sqrt{21}$

$-2 \cdot \frac{1}{\sqrt{21}} \cdot \vec{u} = \frac{-2}{\sqrt{21}} \langle 4, 2, 1 \rangle = \langle \frac{-8}{\sqrt{21}}, \frac{-4}{\sqrt{21}}, \frac{-2}{\sqrt{21}} \rangle$.

[12.3] dot product

* let $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then the dot product of $\vec{u}$ and $\vec{v}$ is denoted by :- $\vec{u} \cdot \vec{v} = u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3$

والجواب عبارة عن Scaler وليس vector .

* let $\vec{u} = \langle 2, 3, -4 \rangle$, $\vec{v} = \langle 1, -2, 5 \rangle$

$\vec{u} \cdot \vec{v} = (2*1 + 3*-2 + -4*5)$

$= 2 + -6 + -20$

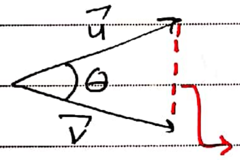
$= -24$

* $\vec{u} = 2\vec{i} + 3\vec{j} - \vec{k}$

$\vec{v} = 4\vec{i} - \vec{j} + 2\vec{k}$

$\vec{u} \cdot \vec{v} = (2)(4) + (3)(-1) + (-1)(2) = 3$

*



$\vec{u} - \vec{v} = \vec{w}$ / $0 \leq \theta \leq \pi$ الزاوية بين المتجهين صغيرة ومحصورة بين

✓ $|\vec{w}| = |\vec{u}| + |\vec{v}| - 2|\vec{u}||\vec{v}| \cos \theta$

$$\cos \theta = \frac{u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3}{|\vec{u}||\vec{v}|} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \Rightarrow \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$ ✓

✓ $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cdot \cos \theta$

* find the angle between $\vec{u} = \langle 2, 1, 5 \rangle$ and $\langle 3, -1, 2 \rangle$?

$\vec{u} \cdot \vec{v} = 6 - 1 + 10 = 15$.

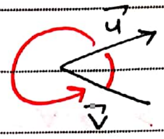
$|\vec{u}| = \sqrt{(2)^2 + (1)^2 + (5)^2} = \sqrt{30}$.

$|\vec{v}| = \sqrt{(3)^2 + (-1)^2 + (2)^2} = \sqrt{14}$

$\theta = \cos^{-1}\left(\frac{15}{\sqrt{30}\sqrt{14}}\right) \Rightarrow \theta = ,749$ [Radian].

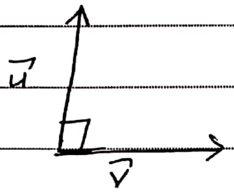
* ناتج الـ (dot product) ازا كانت موجبة يعني الزاوية صغيرة بالربع الاول وازا كانت سالبة
يعني الزاوية بالربع التاني .

*



بنوخد الزاوية الصغيرة بين المتجهين مو الكبيرة .

* if $\vec{u} \cdot \vec{v} = 0$, the $\vec{u}$ and $\vec{v}$ are orthogonal or prependicular
(متعامدين)



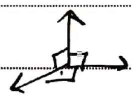
* determine wheather the two vectors are orthogonal or not?

$\vec{u} = \langle 7, 2, 1 \rangle$, $\vec{v} = \langle 2, 0, -3 \rangle$

$\vec{u} \cdot \vec{v} = 14 + 0 + -3 = 11 \neq 0$, they arent orthogonal.

* $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{k} = \langle 0, 0, 1 \rangle$

$\vec{i} \cdot \vec{k} = 0 + 0 + 0 = 0$, they are orthogonal.



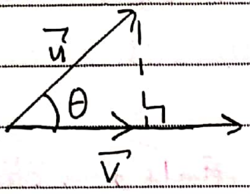
* $\vec{u} \cdot \vec{u} = u_1 \cdot u_1 + u_2 \cdot u_2 + u_3 \cdot u_3$

$= (u_1)^2 + (u_2)^2 + (u_3)^2$

$= |u|^2$

$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$

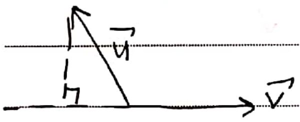
*



Projection of $\vec{u}$ onto $\vec{v}$.

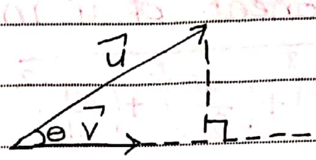
$\left(\mathrm{Proj}_{\vec{v}}\vec{u}\right)$.

*



$\mathrm{Proj}_{\vec{v}}\vec{u}$

*



نسقط عمود من $\vec{u}$ على امتداد $\vec{v}$.

$\mathrm{Proj}_{\vec{v}}\vec{u}$

$$\cos\theta = \frac{|\mathrm{Proj}_{\vec{v}}\vec{u}|}{|\vec{u}|} = \frac{\text{المجاور}}{\text{الوتر}}$$

$$\Leftrightarrow \cos\theta \cdot |\vec{u}| = |\mathrm{Proj}_{\vec{v}}\vec{u}|$$

scaler component of $\vec{u}$ in the direction of $\vec{v}$. هاد اسمه

$$\mathrm{Proj}_{\vec{v}}\vec{u} = |\vec{u}| \cdot \cos\theta \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$= |\vec{u}| \cdot \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \cdot \frac{\vec{v}}{|\vec{v}|}$$

$$\mathrm{Proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}$$

لما تكون الزاوية منفرجة بكون نفس الجواب بس سالب

* Let $\vec{u} = 2\vec{i} - 3\vec{j} + \vec{k}$ and $\vec{v} = \vec{i} + 4\vec{j} - 2\vec{k}$, find the $\mathrm{Proj}_{\vec{v}}\vec{u}$?

$\vec{u}\cdot\vec{v} = (2)(1) + (-3)(4) + (1)(-2) = -12$ اشارة سالب يعني زاوية منفرجة

$|\vec{v}| = \sqrt{1+16+4} = \sqrt{21}$

$|\vec{v}|^2 = 21$

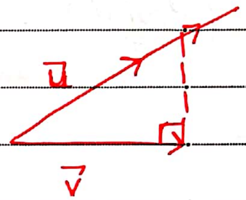
$\mathrm{Proj}_{\vec{v}}\vec{u} = \left(\frac{-12}{21}\right)\left(\vec{i} + 4\vec{j} + -2\vec{k}\right) = \frac{-12}{21}\vec{i} - \frac{48}{21}\vec{j} + \frac{24}{21}\vec{k}$

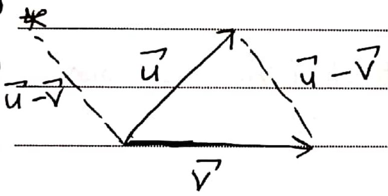
Scaler component of $\vec{u}$ in the direction of $\vec{v}$ is :

$$\left|\operatorname{Proj}_{\vec{v}} \vec{u}\right| = \sqrt{\left(\frac{-12}{21}\right)^2+\left(\frac{-48}{21}\right)^2+\left(\frac{24}{21}\right)^2}$$

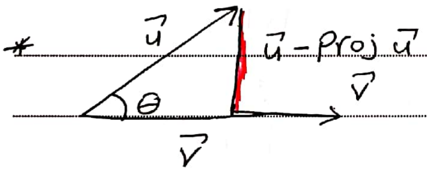
ملاحظة :- لو عكستهم ، ح يطلع الجواب مختلف مثلاً لو طلب $\operatorname{proj}_{\vec{u}} \vec{v}$ ، ح يكون جواب آخر

* $\operatorname{Proj}_{\vec{u}} \vec{v}$:- خط عمودي من $\vec{v}$ على امتداد $\vec{u}$





نلاحظ إنه أفضل طريقة لحتى أطلّع المسافة بين خط مستقيم ونقطة هي استعمال الـ (Projection) .



مهم

[illegible] ✓

[12.4] Cross product

* Let $\vec{u}, \vec{v}$ two vectors

→ cross product

$\vec{u} \times \vec{v}$ is a vector [direction and magnitude]

الناتج هو متجه
عمودي على
كلا المتجهين
أو عمودي على
المستوى اللي يجمعهم .

* نستخدم قاعدة اليد اليمنى ، أصابع اليد بيتجهوا من $\vec{u}$ إلى $\vec{v}$ $(\vec{u} \times \vec{v})$

* $\vec{u} \times \vec{v}$ تختلف عن $\vec{v} \times \vec{u}$ عند استخدام قاعدة اليد اليمنى .

* $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

* $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 \cdot v_3 - u_3 \cdot v_2)\vec{i} - (u_1 \cdot v_3 - u_3 \cdot v_1)\vec{j} + (u_1 \cdot v_2 - u_2 \cdot v_1)\vec{k}$$

* Let $\vec{u} = \langle 3, 2, 1 \rangle$ and $\vec{v} = \langle -2, 4, 1 \rangle$, find $\vec{u} \times \vec{v}$?

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ -2 & 4 & 1 \end{vmatrix} = (2*1 - 1*4)\vec{i} - (3*1 + 1*2)\vec{j} + (3*4 + 2*2)\vec{k}$$

$$= -2\vec{i} - 5\vec{j} + 16\vec{k}$$

$$= \langle -2, -5, 16 \rangle$$

$$\boxed{|\vec{u} \times \vec{v}| = -|\vec{u} \times \vec{v}|}$$

يعني نفس المقدار لكن
عكس الاتجاه

$$\vec{v}\times\vec{u}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 4 & 1\\ 3 & 2 & 1\end{vmatrix}=(4*1-1*2)\vec{i}-(-2*1-3*1)\vec{j}+(-2*2-4*\ \vec{k}$$

$$=2\vec{i}+5\vec{j}-16\vec{k}$$

$$=\langle 2,5,-16\rangle$$

نلاحظ انهم الاشارات عكس بعض بس بالمقدار متساويين .

* $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u}\times\vec{v}=\vec{0}$ [ننتبه نحط اشارة متجه فوق الصفر]

* نلاحظ أن (cross product) ينتج متجه .

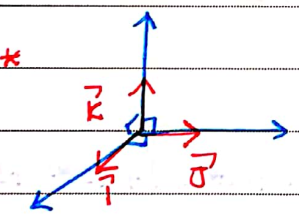
و (dot product) ينتج قيمة ليس متجه .

* نلاحظ أن (cross product) يكشف التوازي .

و (dot product) يكشف التعامد .

* $\vec{u}\times\vec{u}=\vec{0}$ المتجه أكيد يوازي نفسه فناتج الضرب يساوي صفر .

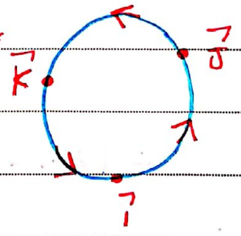
*



ناتج الضرب المتجهين

يا K أو -K

*



هاي الرسمة بحفظها ☺

$\vec{i}\times\vec{j}=\vec{k}$ ازا مشي مع الاتجاه بطلع جواب موجب

$\vec{j}\times\vec{i}=-\vec{k}$ ازا مشيت مع عكس الاتجاه بطلع جواب سالب

$\vec{k}\times\vec{j}=-\vec{i}$

* $\vec{J} \wedge \vec{J} = \vec{0}$

* $(\vec{I} \times \vec{J}) \cdot \vec{J} = 0$

من الرسمة :
ناتجهم هو $\vec{K}$
و $\vec{K}$ عمودي على
$\vec{J}$ ولمّا أعمل
dot product
رح يكون الناتج صفر.

* $(\vec{I} \times \vec{J}) \cdot \vec{K}$

$= \vec{K} \cdot \vec{K}$

$= |\vec{K}|^2$

$= (1)^2$

$= 1$

* $(\vec{J} \times \vec{I}) \cdot \vec{K}$

$= -\vec{K} \cdot \vec{K}$

$= -1 \xrightarrow{\text{للإثبات}}$

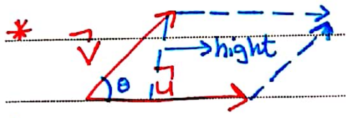
$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

$= -\vec{K} \cdot \vec{K} = |-\vec{K}||\vec{K}| \cos\underset{\pi}{\theta}$

$= -1$

*



* ازا عندي تنين vector بقدر أكمّل عليهم وأرسم متوازي أضلاع.

هاد اسمة (Parallel gram).

Area = base · hight

$= |\vec{u}||\vec{v}| \sin\theta$, $\sin\theta = \frac{\text{hight}}{|\vec{v}|}$

$= |\vec{u} \times \vec{v}|$

* example :- find the area of the Parallel gram determined by $\vec{u} = \langle 4, -1, 2 \rangle$ and $\vec{v} = \langle 1, 3, 5 \rangle$.

$$\text{area} = |\vec{u} \times \vec{v}| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 2 \\ 1 & 3 & 5 \end{vmatrix}$$

$(-5-6)\vec{i} - (20-2)\vec{j} + (12+1)\vec{k}$

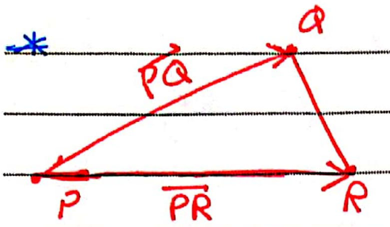
$= -11\,\vec{i} - 18\,\vec{j} + 13\vec{k}$

$= \sqrt{(-11)^2 + (-18)^2 + (13)^2}$

$= \sqrt{614}$

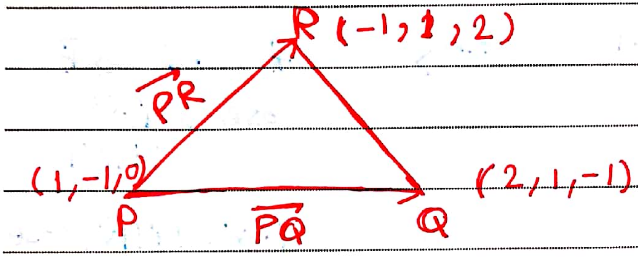
$\simeq 24.779$.

*



area tringle = $\frac{1}{2}|\vec{PQ} \times \vec{PR}|$

* example :- find the area of the tringle ?



$\vec{PR} = \langle -2, 2, 2 \rangle$

$\vec{PQ} = \langle 1, 2, -1 \rangle$

$$\vec{PR} \times \vec{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{vmatrix} = (2-4)\vec{i} - (2-2)\vec{j} + (-4-2)\vec{k}$$

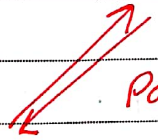
$= -6\vec{i} - 6\vec{k}$

$\therefore |\vec{PR} \times \vec{PQ}| = \sqrt{36+36} = \sqrt{72}$.

area of tringle = $\frac{1}{2} * |\vec{PR} \times \vec{PQ}| = \frac{1}{2}\sqrt{72}$.

* Recall :- $\vec{u} \times \vec{v} = |u||v| \sin\theta$

$\vec{u} \parallel \vec{v}$ so $\vec{u} \times \vec{v} = 0$



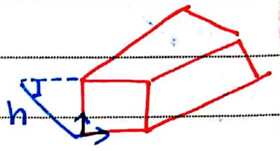
Parallel

area = $|\vec{u} \times \vec{v}|$..

* Parallel piped :- متوازي المستطيلات

V (الحجم) $= |\vec{u} \times \vec{v}| \, |\vec{w}| \cos\theta$

$= |(\vec{u} \times \vec{v}) \cdot \vec{w}|$



$\cos\theta = \frac{h}{|\vec{w}|}$, ~~hight = cos θ~~

$\text{hight} = \cos\theta \cdot |\vec{w}|$

* triple scaler Product.

$(\vec{u} \times \vec{v}) \cdot \vec{w} = |\vec{u} \times \vec{v}| \cdot |\vec{w}| \cdot \cos\theta$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (\quad)u_1 - (\quad)u_2 + (\quad)u_3$$

نفس حل المصفوفات بس الفرق الوحيد

بدل $\vec{i}$ بحط u_1 وبدل $\vec{j}$ بحط u_2 وهكذا.

example:- Find the volume of the Parallel Piped detirmiend by $\vec{u} = \langle 3, 2, 1 \rangle$

$\vec{v} = \langle -2, 4, 1 \rangle$

$\vec{w} = \langle 5, 0, -3 \rangle$

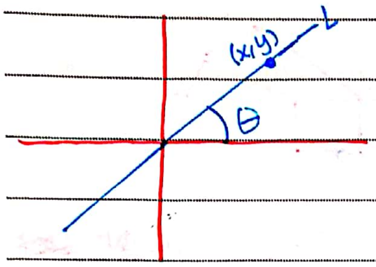
$$= \begin{vmatrix} 3 & 2 & 1 \\ -2 & 4 & 1 \\ 5 & 0 & -3 \end{vmatrix} = (-12-0)3 - (6-5)2 + (0-20)1$$

$= -36 - 2 - 20$

$= -58.$

[12.5]: Line and Plans in space.

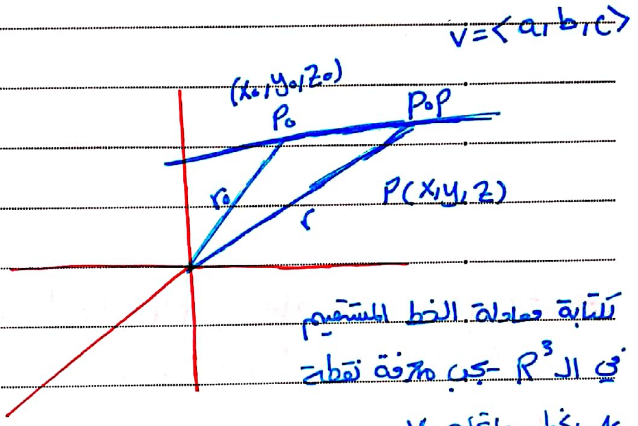
Line in space in R^2 :- $y = y_0 + m(x - x_0)$, $m = \text{slop} = \tan\theta$



Line in space in R^3 :-

$r_0 = \langle x_0 , y_0 , z_0 \rangle$

$r = \langle x , y , z \rangle$



لكتابة معادلة الخط المستقيم
في الـ R^3 يجب معرفة نقطة
على الخط واتجاه V

$(P_0P \parallel \vec{v})$

$\vec{r}$ ([illegible]) $= \vec{r_0} + \overline{P_0P}$

$\vec{r} = \vec{r_0} + t\vec{V}$ ⇝ equal on for the Line $\underline{L}$.

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$.

$$\left.\begin{array}{l} x = x_0 + at \\ y = y_0 + bt \\ z = \underline{\underline{z_0}} + \underline{c}t \end{array}\right] \Rightarrow \text{Parametric equation for the Line.}$$

Point vector

$t = \dfrac{x - x_0}{a}$, $t = \dfrac{y - y_0}{b}$, $t = \dfrac{z - z_0}{c}$

$\left(\dfrac{x - x_0}{a} , \dfrac{y - y_0}{b} , \dfrac{z - z_0}{c}\right)$ symetric equatōn for the Line.

example :- Consider the Line $\underline{L}$ Passes through $p(\overset{x_0}{2},\overset{y_0}{3},\overset{z_0}{-4})$ and Parallel the $\vec{v}=\langle \underset{a}{7},\underset{b}{2},\underset{c}{-4}\rangle$.

(1) vector equation for L.

(2) Parametric equation for L.

(3) symetric equation for L.

(1) $\vec{r}=\langle 2,3,-4\rangle+t\langle 7,2,-4\rangle$.

(2) $x=2+7t$

$y=3+2t$

$z=-4-4t$

(3) $\dfrac{x-2}{7}=\dfrac{y-3}{2}=\dfrac{z+4}{-4}$

example :- find Parametric equation for the Line through $p(3,-1,4)$ and $Q(-2,1,5)$?

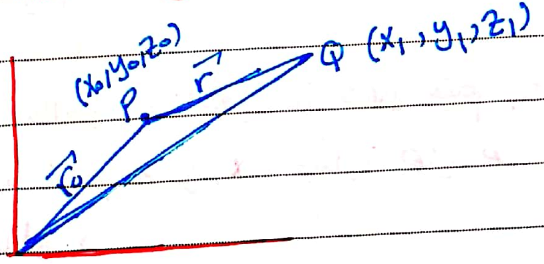
as a Point we choose $p(3,-1,4)$, $\vec{v}=\vec{PQ}=\langle -5,2,1\rangle$.

$x=3-5t$

$y=-1+2t$

$z=4+t$

* Line segment.



$\vec{r_0}=\langle x_0,y_0,z_0\rangle$

$\vec{r_1}=\langle x_1,y_1,z_1\rangle$

$\vec{r}=\vec{r_0}+\vec{v}$

$=\vec{r_0}+(\vec{r_1}-\vec{r_0})$

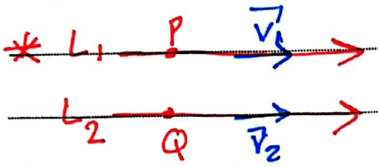
$\vec{r}=(1-t)\vec{r_0}+t\vec{r_1}$, $0\leq t\leq 1$

* find the equation of the Line sequent joining $P(3,4,-2)$ and $Q(1,0,5)$?

$\vec{r_0} = \langle 3,4,-2 \rangle$, $\vec{r_1} = \langle 1,0,5 \rangle$

$\vec{r} = (1-t)\vec{r_0} + \vec{r_1} \cdot t$ ∴ $0 \leq t \leq 1$

$\vec{r} = (1-t) \langle 3,4,-2 \rangle + t \langle 1,0,5 \rangle$.



كيفا بدي اعرف أنه هدول الخطية متوازين أولا ؟

باخذ فيكتور على كل خط وبعملهم cross Product

ازا كان صفر يعني متوازين .

* $L_1: \; x = 2 + 3t_1$, $y = 1 - t_1$, $z = 2 + t_1$

$L_2: \; x = 1 - 2t_2$, $y = 2 + t_2$, $z = 3t_2$

$\vec{v_1} = \langle 3,-1,1 \rangle$, $\vec{v_2} = \langle -2,1,3 \rangle$, $\vec{v_1} \times \vec{v_2} = \vec{0}$ Parallel

$\vec{v_1} \times \vec{v_2} \neq \vec{0}$ not Parallel .

$2 + 3t_1 = 1 - 2t_2$

$3(1 - t_1) = 2 + t_2$ +

$5 = 7 + t_2$

$$\boxed{\begin{array}{l} t_2 = -2 \\ t_1 = 1 \end{array}}$$

* بعد ما أطلع قيم t_1 ، t_2 بضل علي قيمة Z .

ازا كانوا زي بعض بعد التعويض يعني متقاطعين

وازا لأ يعني مو متقاطعين ،

* مشان أطلع نقاط التقاطع بساوي المعادلتين

ببعض وبستخدم الحذف والتعويض مشان أطلع قيم

المجاهيل .

$z = 2 + (1) = 3$

$z = 3(-2) = -6$

$3 \neq -6$

غير متقاطعين

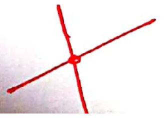
* ازا طلعوا متقاطعين من خلال قيم t_1 ، t_2

بوجد قيم x, y, z وبتكون هاي هي نقطة

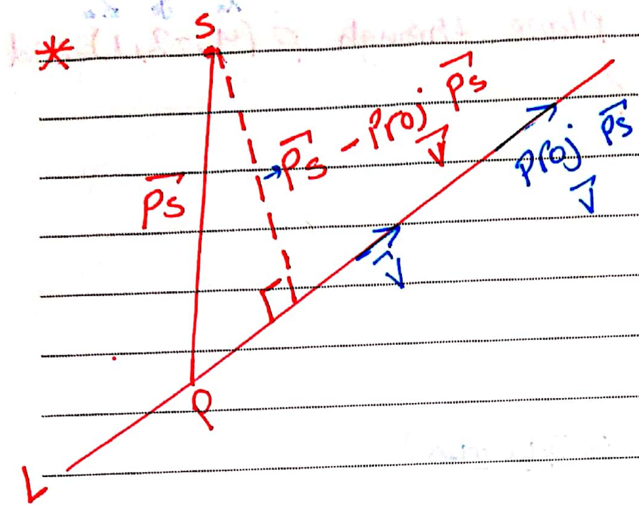
التقاطع ،

* بما إنهم مش متقاطعين ومش متوازيين إذن (SKEW)

يعني فوق بعض بس بدون تقاطع ←



*



المسافة بين نقطة وخط هي أقصر مسافة بينهما بإسقاط خط عمودي على ال (Line) من النقطة .

$$d = \left|\vec{Ps} - \text{Proj}_{\vec{v}}\,\vec{Ps}\right|$$

$$d = \frac{|\vec{Ps} \times \vec{v}|}{|\vec{v}|}$$ هاد الأسهل

* Find the ~~dis~~ distance between the Point s(2,3,-1) and the Line L : $x=1-2t$, $y=3+2t$, $z=1+t$.

$P(1,3,1)$, $\vec{v} = \langle -2,2,1 \rangle$

$\vec{Ps} = \langle 1,0,-2 \rangle$

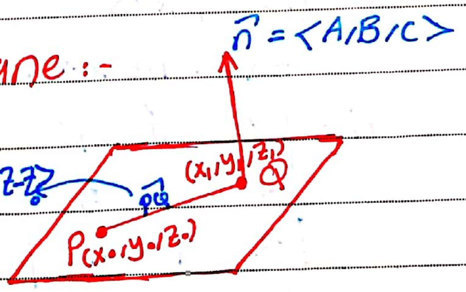
$$\vec{Ps} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ -2 & 2 & 1 \end{vmatrix} = 4\vec{i} + 3\vec{j} + 2\vec{k} .$$

$|4\vec{i} + 3\vec{j} + 2\vec{k}| = \sqrt{16+4+4} = \sqrt{29}$.

$|\vec{v}| = \sqrt{4+4+1} = 3$

$\therefore d = \frac{\sqrt{29}}{3}$.

* Plane :-



$\vec{PQ} = \langle x-x_0, y-y_0, z-z_0 \rangle$

$\vec{PQ} \perp \vec{n}$

$\vec{n} \cdot \vec{PQ} = 0$

$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

* find the equation of the plane through $P(\overset{x_0}{4},\overset{y_0}{-2},\overset{z_0}{1})$ and normal to $\vec{n} = \langle \underset{A}{2}, \underset{B}{3}, \underset{C}{5} \rangle$?

$2(x-4) + 3(y+2) + 5(z-1) = 0$

$2x - 8 + 3y + 6 + 5z - 5 = 0$

$2x + 3y + 5z = 7$

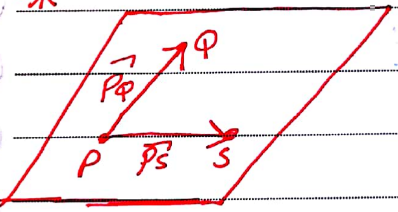
$ax + by + cz = d$ (هاي عبارة عن plane)

any linear equation in space is a plane.

$2x + y = 1$ → plane.

$z = 0$ → plane.

*



as a point choose P.

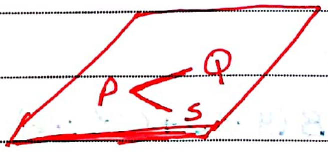
$\vec{n} = \vec{PQ} \times \vec{PS}$

ينتج متجه عمودي عليهم

* find the equation of the Plane Passing throught $P(3,0,-1)$, $Q(2,1,4)$ and $S(1,4,-2)$?

as a point we choose $P(3,0,-1)$, $\vec{n} = \vec{PQ} \times \vec{PS}$

$= \langle -1, 1, 5 \rangle \times \langle -2, 4, -1 \rangle$

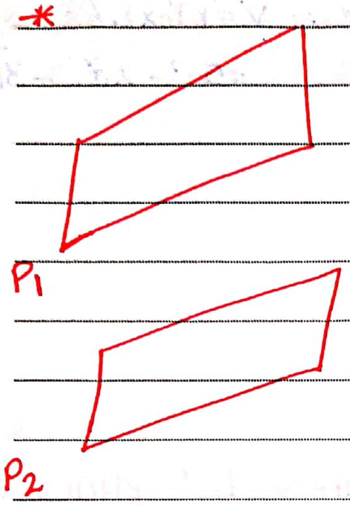


$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 5 \\ -2 & 4 & -1 \end{vmatrix}$$

$= -21\vec{i} - 11\vec{j} - 2\vec{k}$

the equation of the plane is:- $-21(x-3) + 11(y-0) - 2(z+1) = 0$

*



$P_1 \parallel P_2$ if and only if $\vec{n_1} \parallel \vec{n_2}$

يعني $\vec{n_1} \times \vec{n_2} = \vec{0}$.

* $P_1 : x+y+z=1$, $P_2 = x-2y+3z=1$

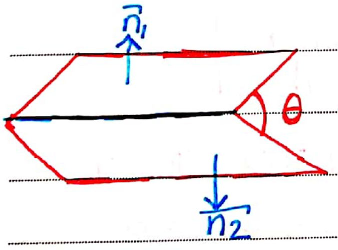
$\vec{n_1} = \langle 1,1,1 \rangle$, $\vec{n_2} = \langle 1,-2,3 \rangle$

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\vec{i} - 2\vec{j} - 3\vec{k} \neq 0$$

, P_1 not Parallel to P_2.

اذا مش متوازيين بكونوا متقاطعين بس ما في بال (plane) اشي اسمه (skew).

∴ P1 and P2 are intersect.



* لما يتقاطعوا أكيد بتقاطعوا ب (line).

find the equation of the Line of intersection?

to find a point on the line, set $z=0 \to x+y=1$

$$\begin{aligned} x+y&=1 \\ -\quad x-2y&=1 \\ \hline 3y&=0 \end{aligned}$$

$\boxed{y=0}$, $\boxed{x=1}$

$(x,y,z) = (1,0,0)$ is a point on L.

←

as a vector in the direction of the Line :- $\vec{V} = \vec{n_1} \times \vec{n_2}$

$= 5\vec{i} - 2\vec{j} - 3\vec{k}$

the equation of L is :-

$L: x = 1 + 5t$

$y = 0 - 2t \quad , -\infty < t < \infty$

$z = 0 - 3t$

how to find the angle between two intersected planes ?

it is Just between $\vec{n_1}$ and $\vec{n_2}$.

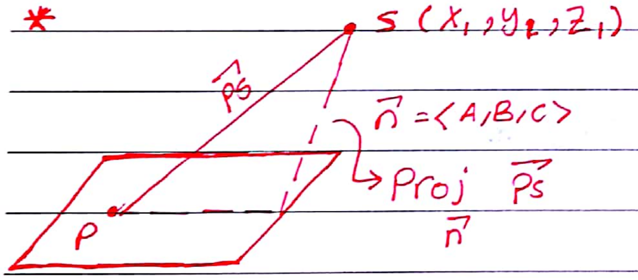
↳ the anlgle

$\theta = \cos^{-1}\left(\frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right) \Rightarrow \cos^{-1}\left(\frac{2}{\sqrt{3}\sqrt{14}}\right)$ الجواب بالراديان ..

$\vec{n_1} \cdot \vec{n_2} = 1 - 2 + 3 = 2$.

$|\vec{n_1}| = \sqrt{3} \quad , |\vec{n_2}| = \sqrt{14}$.

*



$d = \left|\text{Proj}_{\vec{n}} \vec{PS}\right|$

اذا كانت معادلة ال (Plane)

$Ax + By + CZ + D = 0$

$$d = \left|\frac{Ax_1 + By_1 + CZ_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

* Find the distance between the Point $(2,8,-1)$ and the Plane $x-2y+4z=1$?

$x-2y+4z-1=0$, $\vec{n}=\langle 1,-2,4\rangle$.

$$d=\left|\frac{2-2(3)+4(-1)-1}{\sqrt{1^2+(-2)^2+(4)^2}}\right|=\frac{-1}{\sqrt{21}}$$

*

$P_1 : x-y+2z=1$

$P_2 : -2x+2y-4z=3$

$\vec{n_1}=\langle 1,-1,2\rangle$, $\vec{n_2}=\langle -2,2,-4\rangle$

$$\vec{n_1}\times\vec{n_2}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 2\\ -2 & 2 & -4\end{vmatrix}=0\vec{i}+0\vec{j}+0\vec{k}=\vec{0} \quad \therefore P_1 \parallel P_2 .$$

Find the distance between the 2 Planes ?

باخذ نقطة على أحد الـ (plane) وبعدين بستخدم قانون حساب المسافة بين نقطة و (plane).

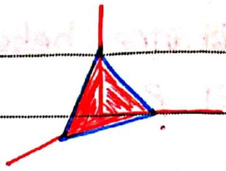
نعوض $x=0$ و $y=0$ بأحد معادلات الـ (plane) ونجد من خلالهم قيمة z.

وهكذا تم اعتماد نقطة على أحد الصفيحتان ، ثم نجد البعد بين هذه النقطة

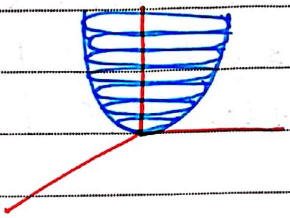
والصفيحة الثانية عند طريق التعويض المباشر في القانون.

[12.6]

* $ax + by + cZ = d \longrightarrow$ Plane.

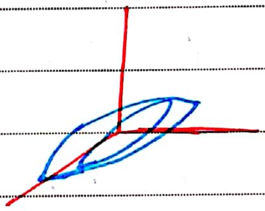


* $Z = x^2+y^2 \longrightarrow$ Parapolid



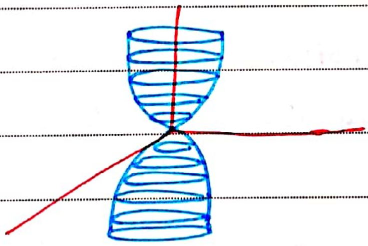
* $x^2+y^2+z^2=r^2 \longrightarrow$ splere of
center (0,0,0)
and raduis $\underline{r}$.

* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \longrightarrow$ ellips.

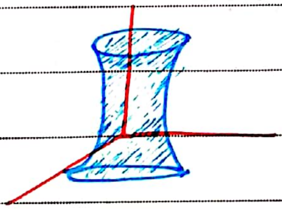


شكل بيضوي

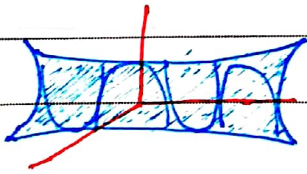
* $z^2 = x^2+y^2 \longrightarrow$ cone



* $\frac{x^2}{a^2} - \frac{y^2}{B^2} + \frac{z^2}{c^2} = 1 \longrightarrow$ hyperboloid



* $Z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \longrightarrow$ hyperbolic paraboloid



زي سرج الحصان

Chapter(13) , [13.1] : vector valued functions

let $\vec{r}(t)$ be the function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$= f(t).\vec{i} + g(t).\vec{j} + h(t).\vec{k}$

for each $t \in R$, $\vec{r}(t)$ is a vector in R^3.

for example ,

$\vec{r}(t) = \langle t^2+1 , 1-2t , \sqrt{t} \rangle$

$\vec{r}(1) = \langle 2,-1,1 \rangle$ تعويض مباشر

$\vec{r}:1 \to \langle 2,-1,1 \rangle$

the domain of $\vec{r}(t)$ is all Possible value of t that can be substituted in the rule of $\vec{r}(t)$.

example :- find the domain of $\vec{r}(t) = < t^2+1, Lnt , \sqrt{4-t^2} >$?

→ t^2+1 is valid for all $t \in R$.

→ Lnt is valid for $t>0$.

→ $\sqrt{4+t^2}$ is valid for $4-t^2 \geq 0$ → $(2-t)(2+t) \geq 0$

$-- \quad ++++ \quad --$

$-2 \qquad 2$

$-2 \leq t \leq 2$

∴ Domain is $(0,2]$.

example :- find the domain of $\vec{r}(t) = < \frac{\sin t}{t} , Ln(1-t) , \frac{1}{1+t^2} >$

→ $\frac{\sin t}{t}$ is valid for $t \neq 0$

→ $Ln(1-t)$ is valid for $1-t>0$ → or $t<1$

→ $\frac{1}{1+t^2}$ is valid for all $t \in R$,

Domain $= (-\infty,0) \cup (0,1) = (-\infty,1) / \{0\}$.

then, $\lim_{t \to t_0} \vec{r}(t) = \langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \rangle$.

example :- let $\vec{r}(t) = \langle \frac{\sin(t)}{t}, \frac{t}{t^2+t}, 1+t \rangle$, find $\lim_{t \to 0} \vec{r}(t) = ?$

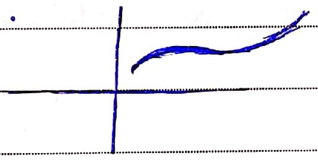
$$\lim_{t \to 0} \vec{r}(t) = \langle \lim_{t \to 0} \frac{\sin(t)}{t}, \lim_{t \to 0} \frac{t}{t^2+t}, \lim_{t \to 0} 1+t \rangle$$

$$= \langle 1, \lim_{t \to 0} \frac{\cancel{t}}{\cancel{t}(t+1)}, 1 \rangle$$

$$= \langle 1, 1, 1 \rangle$$

ازا كان وحدة منهم نهايتها غير موجود يعتبر كامل
المتجه نهايته غير موجود ،

the ~~curve~~ graph of a vector valued function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a curve.



example :- idenify the graph of $\vec{r}(t) = \langle 1+2t, 2-3t, 5+t \rangle$?

$x = 1 + 2t$

$y = 2 - 3t$

$z = 5 + t$

parametric equations for the line through $(1, 2, 5)$ are parallel to $\vec{v} = \langle 2, -3, 1 \rangle$.

example:- Consider the vector valued function $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$?

$x = \cos t$

$y = \sin t$

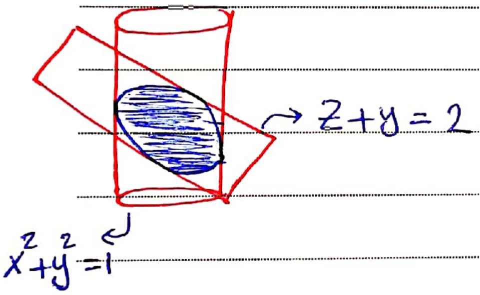
$z = t$

helix

$x^2+y^2=1$

the intersection of the tow serface is a caurve.

in an r example, the curve of intersection is an ellipse

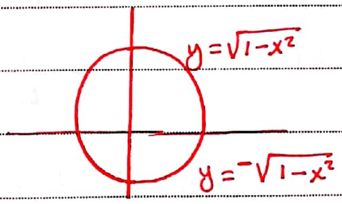


$x = \cos t$

$y = \sin t$, $0 \leq t \leq 2\pi$

Parametrization of the circle

$x^2+y^2=1$



الشكل الناتج عن تقاطع الاسطوانة مع الصفيحة هو دائرة .

$z = 2-y = 2-\sin t$

$\vec{r}(t) = \langle \cos t , \sin t , 2-\sin t \rangle$

curve of intersection

سمته هي ellipse .

[13.2]

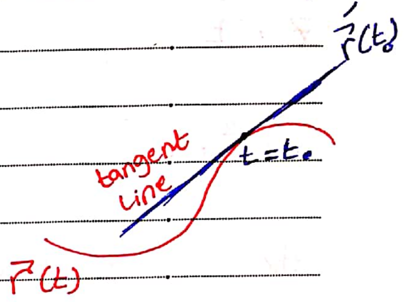
Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

example:- Let $\vec{r}(t) = \langle t^2+3, \sin 3t, e^{2t} \rangle$, find $\vec{v}'(t) = ?$

$\vec{v}'(t) = \langle 2t, 3\cos 3t, 2e^{2t} \rangle$

لو عوّضنا رقم بمشتقة المتجه t_0, ينتج متجه بنفس اتجاه (tangent Line)



So, $\vec{r}'(t_0)$ is called a tangent vector.
the tangent vector is in the direction of the tangent Line to the graph of $\vec{r}(t)$ at $t=t_0$.

a unit tangent vector $\vec{T}(t_0) = \dfrac{\vec{r}'(t_0)}{|\vec{r}'(t_0)|}$.

example:- Let $\vec{r}(t) = \langle t^3+1, \ln t, te^{t} \rangle$

① find $\vec{r}'(t) = \langle 3t^2, \frac{1}{t}, t.e^{t}+e^{t} \rangle$.

② find the tangent vector at $t=1$, $\vec{r}'(t) = \vec{r}'(1) = \langle 3, 1, 2e \rangle$

③ find the unit tangent vector at $t=1$?

$$\vec{T}(1) = \frac{\langle 3, 1, 2e \rangle}{\sqrt{(3)^2+(1)^2+(2e)^2}}.$$

$$= \left\langle \frac{3}{\sqrt{10+4e^2}}, \frac{1}{\sqrt{10+4e^2}}, \frac{2e}{\sqrt{10+4e^2}} \right\rangle.$$

④ Find the equation of the tangent Line to the graph of $\vec{r}(t)$ at $t = 1$.

the Point of the Line :-

$\vec{r}(t) = \langle 2,0,e \rangle$ so $\langle 2,0,e \rangle$ is apoint on the tangent Line $\vec{r}'(1)$ is in the direction of the Line.

the equation of the tangent Line is :-

$x = 2 + 3t$

$y = 0 + 1t$

$z = e + 2et$

* Dirivation Rule :-

1- $\vec{r}(t) = \vec{c}$, $\vec{r}'(t) = \vec{0}$

2- $(\vec{r}_1(t) \pm \vec{r}_2(t))' = \vec{r}_1'(t) \pm \vec{r}_2'(t)$.

3- $(c\vec{r}(t))' = c\vec{r}'(t)$

4- $(\vec{r}_1(t) \cdot \vec{r}_2(t))' = \vec{r}_1'(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \vec{r}_2'(t)$.

5- $(\vec{r}_1(t) \times \vec{r}_2(t))' = \vec{r}_1'(t) \times \vec{r}_2(t) + \vec{r}_1(t) \times \vec{v}_2'(t)$.

example :- Show that if $|\vec{r}(t)| = c$, then $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$ for all it .

لو عندي كرة ، وأخذت نقطة بالمركز ، فإنّ بُعد أي نقطة موجودة عالسطح تساوي مقدار ثابت ، فلو أخذنا متجه من المركز إلى أحد الأطراف [النقاط] على السطح ، فإنّ مشتقة المتجه هو عمودي على المتجه ،

Proof :- ^_^ مهم

$$\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2$$
$$(\vec{r}(t) \cdot \vec{r}(t))' = (c^2)'$$
$$\vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0$$
$$\vec{r}'(t) \cdot \vec{r}(t) + \vec{r}'(t) \cdot \vec{r}(t) = 0$$
$$2\vec{r}'(t) \cdot \vec{r}(t) = 0$$
$$\vec{r}'(t) \cdot \vec{r}(t) = 0$$

this means that $\vec{r}'(t)$ is orthogonal to $\vec{r}(t)$.

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\cdot\vec{i} + g(t)\cdot\vec{j} + h(t)\cdot\vec{k}$.

, then $\int \vec{r}(t)\cdot dt = \left(\int f(t)\cdot dt\right)\vec{i} + \left(\int g(t)\cdot dt\right)\vec{j} + \left(\int h(t)\cdot dt\right)\vec{k} + \vec{C}$

and $\int_a^b \vec{r}(t).dt = \left(\int_a^b f(t).dt\right)\vec{i} + \left(\int_a^b g(t).dt\right)\vec{j} + \left(\int_a^b h(t).dt\right)\vec{k}$

example :- Let $\vec{r}(t) = (t^2+3t)\vec{i} + (\sin 2t)\vec{j} + (\frac{1}{t})\vec{k}$

$$\int \vec{r}(t).dt = \int (t^2+3t)\,\vec{i} + \int(\sin 2t)\,\vec{j} + \int\left(\frac{1}{t}\right)\vec{k}$$

$$= \left(\frac{t^3}{3} + \frac{3t^2}{2}\right)\vec{i} + \left(\frac{-\cos 2t}{2}\right)\vec{j} + (\ln t)\,\vec{k} + \vec{C}$$

[13.3]

arc Length :-

$\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$.

$$\text{arc length} = L = \int_a^b |\vec{r}\,'(t)| \cdot dt = \int_a^b \sqrt{f'^2(t) + g'^2(t) + h'^2(t)} \cdot dt$$

example :- find the length of the arc of the helix $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\vec{k}$ from the Point (1,0,0) to the pont (1,0,2π) ?

$\vec{r}\,'(t) = (-\sin t)\vec{i} + (\cos t)\vec{j} + (1)\vec{k}$

$|\vec{r}\,'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}$

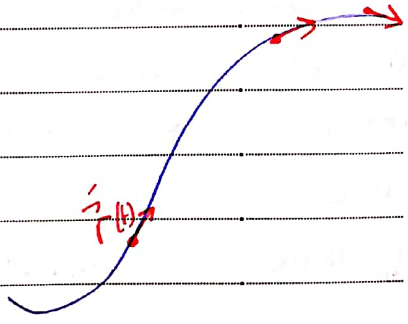
$\text{arc length} = \int_0^{2\pi} \sqrt{2} \cdot dt = \sqrt{2}\,t \Big]_0^{2\pi} = 2\sqrt{2}\,\pi$.

(1,0,0) results from t=0
(1,0,2π) results from t = 2π

عشان أعرف حدود
التكامل

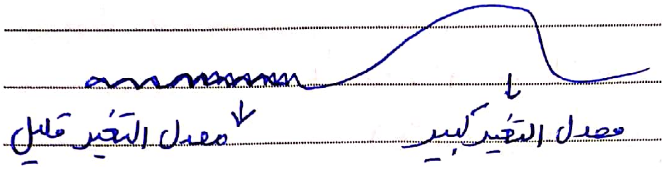
* Curvacture : يقيس معدل التغير في مقدار اتجاه الحركة

let $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$



لو عندي خط مستقيم ، فإن معدل التغير في
اتجاه الحركة يساوي صفر

* كلما كان المسار شبه مستقيم كان التغير في مقدار الحركة قليل ، وكلما كان انحراف المسار أكثر كان التغير أعلى :-



$$K(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} .$$

example :- find the curvacture of the twised $\vec{r}(t) = t\,\vec{i} + t^2\vec{j} + t^3\vec{k}$ at $(0,0,0)$.

(0,0,0) results from $t=0$

$\vec{r}'(t) = 1\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}$

$\vec{r}''(t) = 2\vec{j} + 6t\,\vec{k}$.

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = (6t^2)\,\vec{i} - (6t)\,\vec{j} + (2)\,\vec{k} = 6t^2\,\vec{i} - 6t\,\vec{j} + 2\vec{k} .$$

$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{36t^4 + 36t^2 + 4}$

$|\vec{r}'(t)| = \sqrt{1 + (2t)^2 + (3t^2)^2}$

$$K(t) = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\left(\sqrt{1 + 4t^2 + 9t^4}\right)^3}$$

$K(0) = 2$.

example:- paramchrization of the circle $x = \cos t$
$y = \sin t$

ت مهم

Proof:- $x^2 + y^2 = a^2 . \cos^2 t + a^2 . \sin^2 t$

$= a^2 (\cos^2 t + \sin^2 t)$

$= a^2 (1)$

$\vec{r}(t) = (a . \cos t)\vec{i} + (a \sin t)\vec{j}$

$\vec{r}'(t) = (-a \cdot \sin t)\vec{i} + (a \cancel{\sin} \cos t)\vec{j}$

$\vec{r}''(t) = (-a . \cos t)\vec{i} + (-a \sin t)\vec{j}$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & 0 \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (a^2 \sin^2 t + a^2 \cos^2 t)\vec{k}$$

$= a^2 \vec{k}$.

$|\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{0 + 0 + a^4} = a^2$

$|\vec{r}'(t)| = \sqrt{a^2 . \cos^2 t + a^2 . \sin^2 t} . = a$

$K(t) = \frac{a^2}{a^3} = \frac{1}{a}$

شو ما عوّضت قيمة t رح يضل

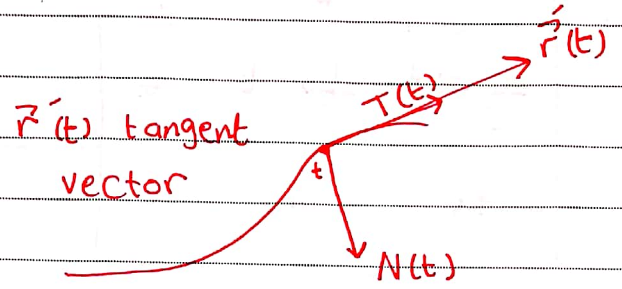
الجواب $\frac{1}{a}$

مقدار ثابت دائماً = $\frac{1}{\text{نصف القطر}}$

* Let $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$.

unit tangent vector $T(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

Since $|T(t)|$ alwyes $= 1$, then $T'(t)$ orthogonal to $T(t)$



* $N(t) = \dfrac{T'(t)}{|T'(t)|}$, Called unit normal vector

* $B(t) = T(t) \times N(t)$, Called Biomial vector.

ال T(t) يحسب معدل التغير في الحركة الحالية ، أما N(t) يتنبأ بتغير الاتجاه عند زيادة السرعة.

example :- let $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + \vec{k}$, Find $T(t)$, $N(t)$, $B(t)$?

$\vec{r}'(t) = (-\sin t)\vec{i} + (\cos t)\vec{j} + (1)\vec{k}$.

$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$.

$T(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left(\dfrac{-1}{\sqrt{2}} . \sin t\right)\vec{i} + \left(\dfrac{-1}{\sqrt{2}} \sin t\right)\vec{j} + \dfrac{1}{\sqrt{2}}\vec{k}$.

$T'(t) = \left(-\dfrac{1}{\sqrt{2}} . \cos t\right)\vec{i} + \left(\dfrac{-1}{\sqrt{2}} . \sin t\right)\vec{j} + 0\,\vec{k}$

←

$$|T'(t)| = \sqrt{\frac{\cos^2(t)}{2} + \frac{\sin^2 t}{2}} = \frac{1}{\sqrt{2}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = (-\cos t)\,\vec{i} + (-\sin t)\,\vec{j}$$

$$B(t) = T(t) \times N(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{-1}{\sqrt{2}} \cdot \sin t & \frac{1}{\sqrt{2}} \cdot \cos t & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

انتهت مادة الفيرست !!

* chapter(14):-

- the function $f(x_1, x_2, \dots, x_n)$, $n \geq 2$ is called a function of several variable.

$x_1, x_2, \dots, x_n$ are the independent variable.

For example :- $f(\underset{\downarrow}{x}, \underset{\downarrow}{y}) = xy^3 - x^2y + 3$

$f(1\ \ 2)$

* f is a function of two variable x,y are the independent variable

, we consider $z = f(x,y) = xy^3 - x^2y + 3$, z is the dependent variable.

* the graph of f is in R^3, $f(1,2) = (1)(2)^3 - (1)^2(2) + 3 = 9$

look to $f(\underset{\swarrow}{x}, \underset{\downarrow}{y}, \underset{\downarrow}{z}) = \sin(x+z) - yx^2 + z$, f is a function of

$f(0, 2, 0)$

3 variable x, y, z, x, y, z are the independent variable.

* the graph for this function is in R^4.

$f(0,2,0) = 0$

$w = f(x,y,z)$

$y = f(x)$

$z = f(x,y)$.

* the domain of $f(x,y)$ is all the pairs (x,y) that can be substituted in the rule of (f).

example :- let $f(x,y)=\sqrt{9-x^2-y^2}$, find :-

(1) domain f ?

$\sqrt{9-x^2-y^2}$ is valid for $9-x^2-y^2 \geq 0$

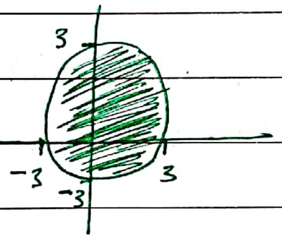
$$9 \geq x^2+y^2$$

$$x^2+y^2 \leq 9$$

Domain f = $\text{Dom}(f) = \{(x,y) \mid x^2+y^2 \leq 9\}$.

② Sketch the domain of (f) :-

$x^2+y^2=9$ Circle of center (0,0,0) and rduis 3.



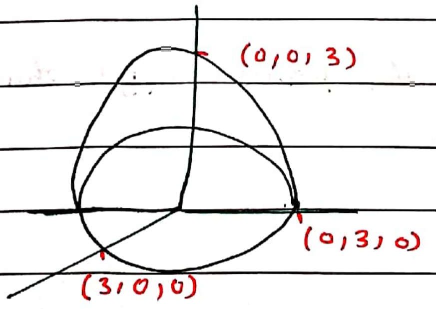
③ sketch $f(x,y)$?

$z = f(x,y) = \sqrt{9-x^2-y^2}$

$$z = \sqrt{9-x^2-y^2}$$

$$z^2 = 9-x^2-y^2$$

$x^2+y^2+z^2=9$, sphere with center (0,0,0) and radius 3.



example :- let $f(x,y) = \ln(x-y) + yx^2$, Find :-

(1) domain f :-

$\ln(x-y)$ is valid for $x - y > 0 \Rightarrow x > y$,

yx^2 is valid for all $(x,y) \in R^2$,

$x > y$

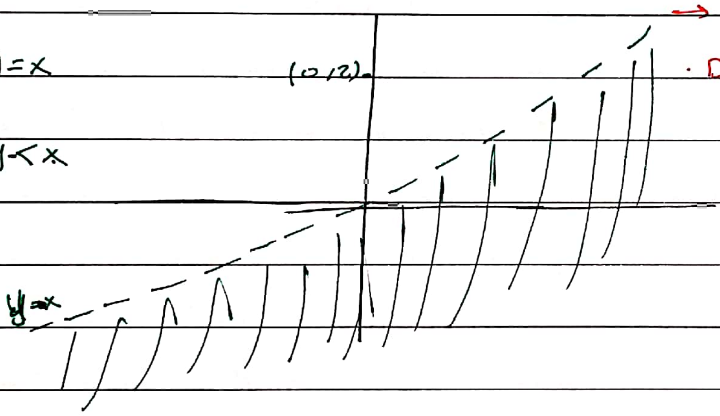
or . $y < x$.

$\text{Domain}(f) = \{(x,y) \mid y < x\}$

(2) Sketch the domain of (f) ?

$y = x$

$y < x$



→ الخط برسمو متقطع لأنه النقاط يلي على الخط مش داخلات بال Domain .

example :- let $f(x,y) = x^2 + y^2$.

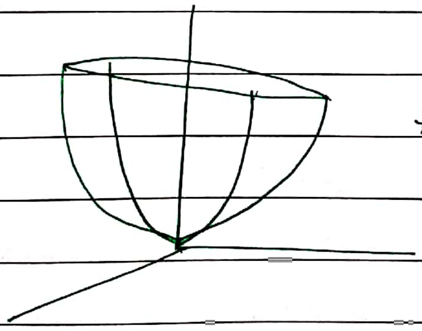
$\text{Dom}(f) = R^2$.

$z = x^2 + y^2$

لما أعوض $x = 0$ بطلع عندي منحنى

ولما أعوض $y = 0$ " " كمان منحنى

ولما أعوض $z =$ أي قيمة بطلع عندي دائرة.



the graph of $f(x,y)$.

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EX :- let $f(x,y) = \sqrt{4-x^2}$

① find domain f ? $\sqrt{4-x^2}$ is defined for $4-x^2 \geq 0$

Domain f = Dom(f) = $\{(x,y) \mid 4-x^2 \geq 0\}$.

② sketch the domain of f $4-x^2 \geq 0$?

$4-x^2 \geq 0$

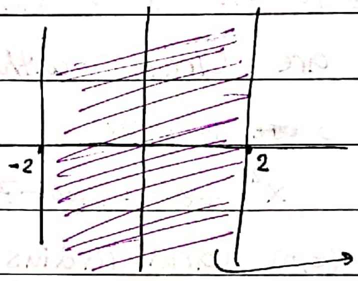
$(2-x)(2+x) \geq 0$

--- ++++ ---

-2 2

تغير الإشارة

تعويض أرقام



$-2 \leq x \leq 2$, y is free.

كل النقاط المحصورة بين $x = 2$, $x = -2$

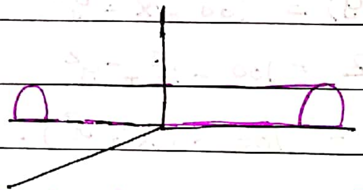
the domain consist for all point lie on and between $x = -2$ and $x = 2$.

③ skitch f(x,y) ?

$z = f(x,y) = \sqrt{4-x^2}$

$z \geq 0$, $z^2 = 4-x^2$.

$x^2 + z^2 = 4$



اسطوانة بس عند x الموجب فسحنا الجزء السالب .

range of function = $\{ z \mid 0 \leq z \leq 2 \} = [0,2]$.

من نصف قطر الدائرة .

* Identify the graph of $f(x,y) = 2x-3y$, $z = f(x,y) = 2x-3y$.

$2x - 3y - z = 0 \Rightarrow$ Plane.

a Plane through (0,0,0) and normal vector

$\vec{n} = \langle 2,-3,-1 \rangle$.

* level curves :-

the level curves of $h(x,y)$ are the curve.

$f(x,y) = k$,

k is a constant or we say k-level curve .

ex :- let $f(x,y) = 100 - x^2 - y^2$, find the level curves when $k = 75, 51$?

* the level curves are circle with center$(0,0)$ and radius $\sqrt{100-k}$.

* 75 - level curve , ~~at~~ $k = 75$

$$x^2 + y^2 = 100 - 75 = 25 .$$

* Circle of center $(0,0)$ and radius 5 .

* 51 - level curves , $k = 51$

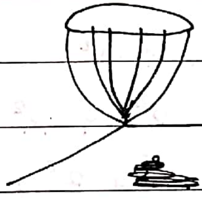
$$x^2 + y^2 = 100 - 51 = 49 .$$

Circle of center $(0,0)$ and radius 7 .

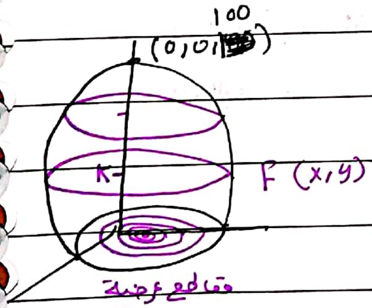
* $z = f(x,y) = 100 - x^2 - y^2$

$$z = 100 - x^2 + -y^2$$

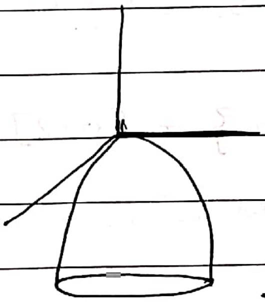
$$= 100 - (x^2 + y^2) .$$



$z = x^2 + y^2 .$



يكون شكلها بعد القطع هي دائرة .



$z = -(x^2 + y^2) .$

* we can analyze functions of more than 2 variable

For example :- $\underbrace{f(x,y,z)}_{\text{independent, variables}} = \ln(2x-3y+z) + yz^3 + xy$

$\ln(2x-3y+z)$ is defined for $2x-3y+z \geq 0$,

Domain of f $= \{(x,y,z) \mid \underbrace{2x-3y+z}_{\text{Plane}} > 0\}$

$2x-3y+z = 0$ لا بدي أرسم ، رح أخذ

كل النقاط يلي فوق ال Plane.

the Domain consists of all points above the plane : $2x-3y+z = 0$

* level surface :- the level surfaces for $f(x,y,z)$ are the surface $f(x,y,z) = k$, where k is a constant.

example :- let $f(x,y,z) = \sqrt{1-x^2-y^2-z^2}$, find the level surfaces at $k = \frac{1}{2}$, the level surfaces are in the form.

$f(x,y,z) = k$

$\sqrt{1-x^2-y^2-z^2} = k$

$1-x^2-y^2-z^2 = k^2$

$x^2+y^2+z^2 = 1-k^2$.

the level surfaces are sphere of center $(0,0,0)$ and radius $\sqrt{1-k^2}$

$\frac{1}{2}$ level surface , $k = \frac{1}{2}$

$x^2+y^2+z^2 = 1-\left(\frac{1}{2}\right)^2 = \frac{8}{4}$.

a sphere of center $(0,0,0)$ and radius $\frac{\sqrt{3}}{2}$.

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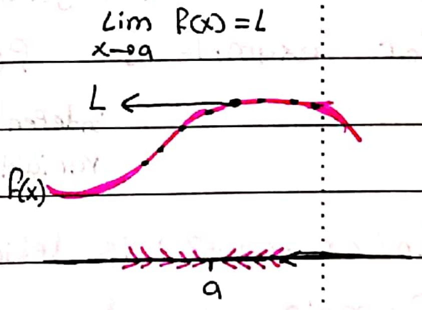
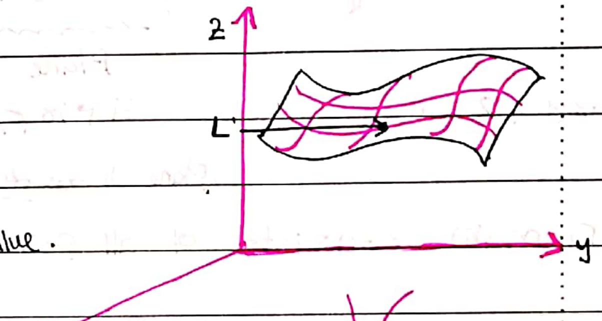
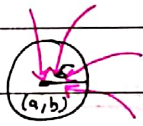
* (14.2) limits and continuity :-

$\lim\limits_{(x,y)\overset{z}{\to}(a,b)} f(x,y) = L$, if for any $\epsilon>0$, there is a disk D_s, such that for any $(x,y)\in D_s$, $|f(x,y)-L|<\epsilon$.





* if the limit exists, then using any Path we will get the same limit value.



يقترب من جميع الجهات:

* different paths of approach:-

If we can find two different Paths of approach along which f has two different limits, then it follows that $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ doesn't exist (d.n.e).

Now, let's look at limits that do exist :-

example (1) :- find $\lim\limits_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y}$.

* ما بصير أستعمل قاعدة لوبيتال ، لانها غير معرّفة على متغيرين x,y

$$\lim\limits_{(x,y)\to(1,1)} \frac{\cancel{(x-y)}(x+y)}{\cancel{(x-y)}} = \lim\limits_{(x,y)\to(1,1)} \frac{x+y}{1} = 2.$$

example :- Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\,\cancel{(\sqrt{x}-\sqrt{y})}(\sqrt{x}+\sqrt{y})}{\cancel{(\sqrt{x}-\sqrt{y})}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(\sqrt{x}+\sqrt{y})}{1}$$

$$= 0$$

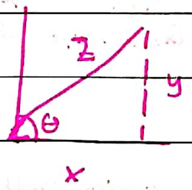
example :- $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy - y - 2x + 2}{x-1}$

$$= \lim_{(x,y)\to(1,1)} \frac{y(x-1) - 2(x-1)}{(x-1)}$$

$$= \lim_{(x,y)\to(1,1)} \frac{(y-2)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= -1$$

example :- $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$



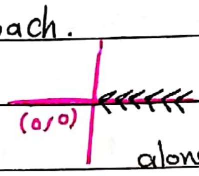
let $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$, $(x,y)\to(0,0)$ then $r\to 0$

$$= \lim_{r\to 0} \frac{3.\cancel{r^2}.\cos^2\theta \cdot r \cdot \sin\theta}{\cancel{r^2}}$$

= zero.

example :- show that the following limit doesn't exist using different paths of approach.

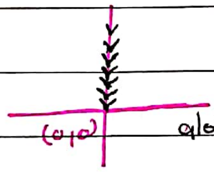
(1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$.



along the x-axis , $y=0$.

$\Rightarrow \lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } x\text{-axis}}} \dfrac{x^2-y^2}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2} = 1$.

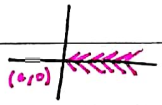
$\Rightarrow \lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } y\text{-axis}}} \dfrac{-y^2}{y^2} = -1$



along y-axis , $x=0$.

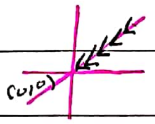
(2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{(x^2+y^2)}$.

$\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } x\text{-axis}}} \dfrac{xy}{x^2+y^2} = \dfrac{0}{x^2} = 0$



لو أخذت نهاية عند محور الصادات
ما رح أستفيد لأنه رح يطلعلي نفس
قيمة النهاية ، وأنا بدي أثبت انه
النهاية غير موجودة.

$\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } x=y}} = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{2x^2} = \dfrac{1}{2}$



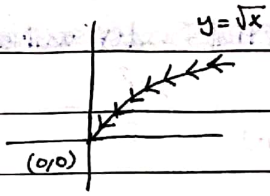
* different ~~[illegible]~~ Paths give different Limit value , so the limit (d.n.e) .

(3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$

$\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } x\text{-axis}}} \dfrac{xy^2}{x^2+y^4} = 0$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=\sqrt{x}}} \frac{xy^2}{x^2+y^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$$



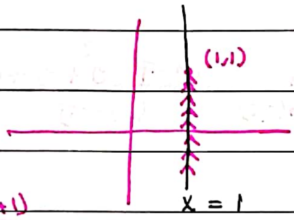
عوضناها من هنام.

two different paths give different limit, so the (d.n.e).

(4) $$\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$$

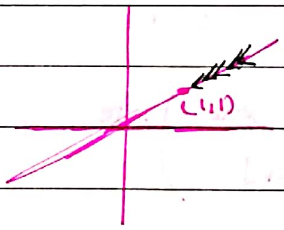
$$= \lim_{\substack{(x,y)\to(1,1)\\ \text{along } x=1}} \frac{y^2-1}{y-1} = \frac{\cancel{(y-1)}(y+1)}{\cancel{(y-1)}}$$

$$= 2.$$



along $x=1$

$$\lim_{\substack{(x,y)\to(1,1)\\ \text{along } y=x}} \frac{x^3-1}{x-1}$$



$$= \lim_{(x,y)\to(1,1)} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = 3.$$

two defferent paths, give different limits and so the element (d.n.e)

H.W
(5) $$\lim_{(x,y)\to(0,0)} \frac{y}{x}$$

Note:- $(A^3 - B^3) = (A - B)(A^2 + AB + B^2)$

نفس — عكس — دائماً +

* two different paths give different limit value, so the limit d.n.e

* Continuity الإتصال

→ we say that f(x,y) is continuous at (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)} = f(x_0, y_0)$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$; $\lim_{x\to 0} \frac{\tan x}{x} = 1$ $\left(\frac{\text{معامل}}{\text{معامل}}\right)$

$\lim_{x\to a} f(x) = f(a)$

إذا نهاية = موجودة ⟸ متصل

شروط الاتصال :-

1) صورة موجودة

٢) نهاية موجودة

→ a function contain terms of the form $x^n y^m$, where m,n Positive integer number are called Polynomial function, Polynomials are continuous every where.

* كثيرات الحدود دائماً متصلة.

* $f(x,y) = 2x^2y + 2y^3x + -x^2y^2$.

النهاية تعويض مباشر = صورة

∴ is continons every where.

For rational functions we have to be careful about zeros of denomenietar

مقام

* let $f(x,y) = \frac{xy}{x^2-y^2}$, it is rational function, Note that f isn't defined when $x^2 - y^2 = 0$

$(x-y)(x+y) = 0$

$x = y$ / $x = -y$

f is cont on $D = \{x,y \mid y \neq x \text{ or } y \neq -x\}$

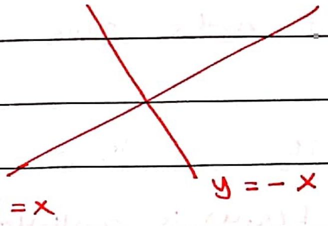
Subject 12 Date No.

* let $f(x,y) = \frac{\sin(x+y)}{x^2-y^2} = \sin\frac{(x+y)}{x^2-y^2}$

$x^2-y^2 \Rightarrow (x-y)(x+y)$

$x=y \quad , y=-x$

dis continous



sin x
متصل ع
$(-\infty,\infty)$.

f is continous on

$D = \{(x,y) \mid x \neq y \text{ or } x \neq \bar{y}\}$

* let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

$\lim_{(x,y)\to(0,0)} f(x,y)$ d.n.e (show that).

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$$

$\therefore$ limit d.n.e

So $f(x,y)$ is not continuous at $(0,0)$.

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* (14 : 3) Partial dervation :-

let $f = f(x,y)$ function of (f) of 2 variable

let us fix y, say $y=b$, and x varies fs $F(x,b) = f(x)$.

* the dervative of "f" with respect to x denoted by $\frac{\partial f}{\partial x} = f_x$ and defined by $f(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

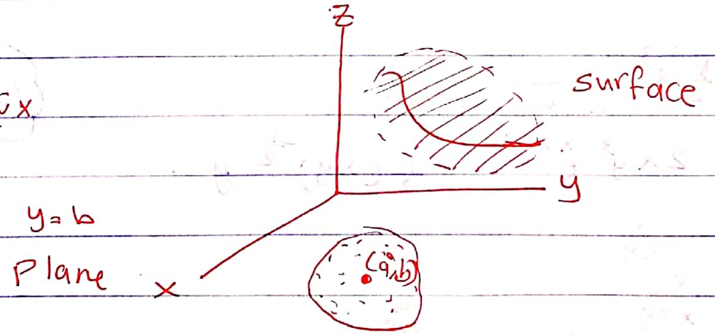
$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

Similary if we fix x, say $x=a$ then the partial derivative of "f" with respect to y is denoted by $\frac{\partial f}{\partial y} = f_y$, and defined by

$$\frac{\partial F}{\partial y} = f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

$f(x,y) = z$

$\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$ x تتغير، y ثابتة



* المشتقة تعني معدل تغير في الاقتران عند كل نقطة.

ex:- find $\frac{\partial F}{\partial x}$ in the following :-

$\boxed{1}$ $F(x,y) = x^3 y^2 + 3x^2 y^4 + 7y$.

$\frac{\partial F}{\partial x} = 3x^2 y^2 + 6xy$.

$\boxed{2}$ $f(x,y) = \sin(xy^2 + x^3)$

$$\frac{\partial F}{\partial x} = y^2 + 3x^2 \cos(xy^2 + x^3)$$

$(f^n)' = n f^{n-1} * f'$

$\boxed{3}$ $f(x,y) = \sqrt{x^4y^3 - 2x^2y^2 + 1} = (x^4y^3 - 2x^2y^2 + 1)^{\frac{1}{2}}$

$$\frac{\partial F}{\partial x} = \frac{1}{2}(x^4y^3 - 2x^2y^2 + 1)^{-\frac{1}{2}} * (4x^3y^3 - 4xy^2).$$

$\boxed{4}$ $f(x,y) = \dfrac{1}{x^3y + 3x^4y + 7}$

$$\frac{\partial F}{\partial x} = \frac{-(3x^2y + 12x^3y)}{(x^3y + 3x^4y + 7)^2} = -(3x^2 + 12x^3y)(x^3y + 3x^4y + 7)^{-2}.$$

$(\ln F)' = \dfrac{f'}{f}$

$\boxed{5}$ $F(x,y) = \ln(x^3 + y^2 + 2xy^7 + 1)$

$$\frac{\partial F}{\partial x} = \frac{3x^2y^2 + 2y^7}{x^3 + y^2 + 2xy^7 + 1}$$

$(e^u)' = u' e^u$

$\boxed{6}$ $f(x,y) = e^{x^2y^3} - \sqrt{xy} + 10y.$

$$\frac{\partial F}{\partial x} = 2xy^3 e^{x^2y^3} - \frac{1}{2}(xy)^{-\frac{1}{2}} y.$$

$\boxed{7}$ $f(x,y) = xe^{x^2y}$

$$\frac{\partial F}{\partial x} = xe^{x^2y} * 2xy + e^{x^2y}$$

$$= e^{x^2y}(2x^2y + 1).$$

$\boxed{8}$ $f(x,y) = \dfrac{y}{x^2 + y}$

$$\frac{\partial F}{\partial x} = \frac{x^2 + y(0) - y(2x)}{(x^2 + y)^2} = \frac{-2xy}{(x^2 + y)^2}.$$

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14.3 Partial derivatives :-

→ Recall :-

$f(x,y)$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

example :- let $f(x,y) = y \sin(xy)$.

$$\frac{\partial f}{\partial x} = f_x = y \cdot \cos(xy) \cdot y$$

$$= y^2 . \cos(xy).$$

$$\frac{\partial f}{\partial y} = f_y = y . \cos(xy) . x + \sin(xy)$$

$$= xy \cdot \cos(xy) + \sin(xy).$$

example :- let $f(x,y) = \cos\left(\frac{y}{1+x^2}\right)$, find $f_x(2,1)$ and $f_y(2,1)$.

$$f_x = -\sin\left(\frac{y}{1+x^2}\right) . \frac{0-y(2x)}{(1+x^2)^2}$$

$$= \frac{2xy}{(1+x^2)^2} . \sin\left(\frac{y}{1+x^2}\right)$$

$$f_x(2,1) = \frac{4}{25} . \sin\left(\frac{1}{5}\right).$$

$$f_y = -\sin\left(\frac{y}{1+x^2}\right) . \frac{1}{1+x^2}$$

$$f_y(2,1) = \frac{-1}{5} . \sin\left(\frac{1}{5}\right)$$

* we can also find partial derivative for functions of more that 2 variables.

example:- $F(x,y,z) = 6x^2y^3z + y^2z^4 + 10y$.

$\frac{\partial f}{\partial x} = f_x = 12xy^3z$.

$\frac{\partial f}{\partial y} = f_y = 18x^2y^2z + 2yz^4 + 10$.

$\frac{\partial f}{dz} = f_z = 6x^2y^3 + 4y^2z^3$.

* if z is defined implicitly as a function of x and y, then to find $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$ we need to use implicit differentiation.

* example :- let $x^3 + y^3 + z^3 + 6xy^2z^2 = 1$, find $\frac{\partial z}{\partial x}$ given that $z = f(x,y)$.

↳ إعطيت معادلة مو اقتران

we have to use implict differention, differentiate both sides with respect to x.

$$\frac{\partial}{\partial x}\left[x^3 + y^3 + z^3 + 6xy^2z^2\right] = \frac{d}{dx}[1].$$

$$= 3x^2 + 3z^2\frac{\partial z}{\partial x} + 6xy^2\,2z\frac{\partial z}{\partial x} + z^2\,6y^2 = 0.$$

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6z^2y^2}{3z^2 + 12xy^2z}.$$

* example :- let $x^2y + \ln(x+z)y = 3$, where $z = f(x,y)$, find $\left.\frac{\partial z}{\partial y}\right|_{(1,3,0)}$?

we use implicit differention . Differentiate both sides with reapeat to y .

$\frac{\partial}{\partial y}\left[x^2y + \ln(x+z)y\right] = \frac{\partial}{\partial y}[3]$

$x^2 + \ln(x+z) + \frac{y\,\frac{\partial z}{\partial y}}{x+z} = 0$

at the Point $\overset{x\ y\ z}{(1,3,0)}$ we get $1 + 3\left.\frac{\partial z}{\partial y}\right|_{(1,3,0)} = 0$

$\left.\frac{\partial z}{\partial y}\right|_{(1,3,0)} = \frac{-1}{3}$

*example :- let ~~#~~ $f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$, find $f_x(0,0)$.

$\boxed{\text{هون لازم استعمل التعريف العام للمشتقة}}$ *

$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$

$= \lim_{h\to 0} \frac{f(h,0) - 0}{h}$

$= \lim_{h\to 0} \frac{h-0}{h} = 1$

$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h+0) - f(0,0)}{h}$

$= \lim_{h\to 0} \frac{f(0,h) - 0}{h}$

$= \lim_{h\to 0} \frac{-h}{h} = -1$

* higher derivative :-

let $f = f(x,y)$

$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2}$

$(f_y)_y = f_{yy} = \frac{\partial^2 f}{\partial y^2}$.

$(f_x)_y = \underset{\longrightarrow}{f_{xy}} = \underset{\longleftarrow}{\frac{\partial^2 f}{\partial y \partial x}}$

اول شي بالنسبة
لx بعدين
بالنسبة لy

$f(y)_x = f_{yx} = \underset{\longleftarrow}{\frac{\partial^2 f}{\partial x \partial y}}$

اول شي بالنسبة
لy بعدين
بالنسبة لx

* f_{xy}, f_{yx} are called mixed Partial derivatives.

* example :- let $f(x,y) = 3x^2y^3 - 2xy^4 + 10x$.

$f_x = \frac{\partial f}{\partial x} = 6xy^3 - 2y^4 + 10$.

$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = 18xy^2 - 8y^3$.

$f_y = \frac{\partial f}{\partial y} = 9x^2y^2 - 8xy^3$.

{ Note that $f_{xy} = f_{yx}$ }

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6y^3$.

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 18x^2y - 24xy^2$.

$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = 18xy^2 - 8y^3$

Theorem :- let $f(x,y)$ be a function defined an an open disk D.
If f_x, f_y are continous at (a,b) containd in D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

example :- let $f(x,y) = y \sin(xy)$.

$f_x = \frac{\partial f}{\partial x} = y^2 \cos(xy)$

$f_y = \frac{\partial f}{\partial y} = y x \cos(xy) + \sin(xy)$.

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = -y^3 \sin(xy)$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = -y x \sin(xy) \cdot x + x \cos(xy) + x \cos(xy)$.

$= -y x^2 \sin(xy) + 2 x \cos(xy)$.

$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = x \cdot -y^2 \sin(xy) + 2\cos(xy) \cdot y$

$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = -y^2 x \cdot \sin(xy) + y \cos(xy) + y \cos(xy)$

$= -y^2 x \cdot \sin(xy) + 2 y \cos(xy)$.

Note that $f_{xy} = f_{yx}$

Is it always true that $f_{xy} = f_{yx}$?

the following theorem can answer this equation.

* Clairaut theorem :- Let $f(x,y)$ be a function defined on an open disk D. If f_x and f_y are continous at $(a,b) \in D$, then $f_{xy}(a,b) = f_{yx}(a,b)$.

example :- Let $f(x,y) = \begin{cases} \frac{x^3 y - x y^3}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

$\boxed{I}$ for $(x,y) \neq (0,0)$, then $f_x = \frac{(x^2+y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2}$

$f_x = \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2+y^2)^2}$

$$f_y = \frac{(x^2+y^2)(x^3-3x^2y)-(x^3y-xy^3)(2y)}{(x^2+y^2)^2} = \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}.$$

② $$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$ لازم أستعمل تعريف المشتقة

$$= \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h}$$

$$= \frac{0-0}{h} = 0$$

③ $$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,0+h)-f_x(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{-h-0}{h} = -1$$

أول اشي بشتق بالنسبة لـ x
بعدين بشتق (Fx) بالنسبة لـ y

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0)-f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h-0}{h} = 1$$

$f(xy(0,0) \neq f_{yx}(0,0)$.

this doesn't contradict Caliraut theorem because f_x, f_y are not continuous at (0,0).

* an equation that contains partial derivatives is called partial differential equations PDE.

example:- $u_t = u_{xx}$ heat equations.

$u_{xx} + u_{yy} = 0$ laplace's equations.

example:- show that $u(x,y) = e^x \sin y$ is a solution to $u_{xx} + u_{yy} = 0$?

we need to verify that $u(x,y)$ ~~solution~~ satisfies the equation.

$u_x = e^x \cdot \sin y$ $\quad$ $u_y = e^x \cdot \cos y$

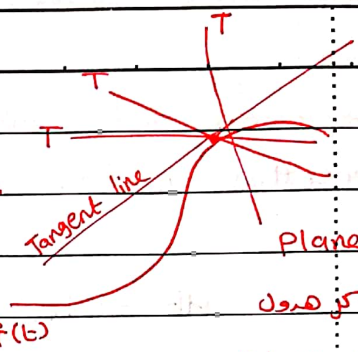
$u_{xx} = e^x \sin y$ $\quad$ $u_{yy} = -e^x \sin y$

$u_{xx} + u_{yy} = e^x \sin y + (-e^x \sin y) = 0$.

Subject Date No.

*(14.4) :- tangent Planes.

Let $z = f(x,y)$ has a graph the surface $\underline{S}$.
, the tangent Plane to this surface at (x_0, y_0, z_0) is the Plane Containing all tangent lines at Curves on $\underline{S}$ at the point



Plane فنعتبر
ال يضم كل هدول
ال (tangent line).

If the Partial derivatives of $f(x,y)$ are Continuouse at (x_0,y_0,z_0)
, then an equation to the tangent plane $z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$.

example :- find the tangent plane to the surface $z = 2x^3y^2 + 3xy^3 + 3x$.
at the Point $\underline{(1,1,8)}$?

نقطة على ال plane

$$f_x = \frac{\partial z}{\partial x} = 6x^2y^2 + 3y^3 + 3$$

$$f_x(1,1) = 12.$$

$$f_y = \frac{\partial z}{\partial y} = 4x^3y + 9xy^2 + 0$$

$$f_y(1,1) = 13.$$

an equation of the tangent plane is $z - 8 = 12(x-1) + 13(y-1)$.

$$12(x-1) + 13(y-1) - (z-8) = 0$$

$\longrightarrow$ vector عمودي على ال Plane :- $(12, 13, -1)$

$\longrightarrow$ نقطة عليها ال Plane :- $(1, 1, 8)$.

So, we can conclude that $\vec{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$ normal to the tangent Plane.

$f \circ g = f(g(x))$

$(f \circ g)' = f'(g(x)) * g'(x)$

(14.5) Chain Rule :-

Chain Rule usually used to find derivative of Composite functions. we will not memorize the rules but we will construct the rules using the tree diagram.

مش ظاهر بالمعادلة
بنعرف انه بنحل على Chain rule

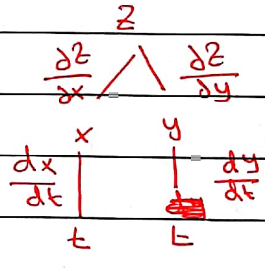
example :- let $z = x^2 y + y^3$, $x = \cos(2t+1)$, $y = \frac{1}{t}$, find $\frac{dz}{dt}$?

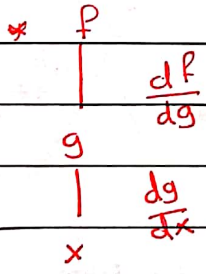
with the help of the tree diagram, we have the Chain Rule.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (\overset{2}{\cancel{3}}xy)(-2\sin(2t+1)) + (x^2 + 3y^2)(\frac{-1}{t^2})$$

$$= 2\cos(2t+1)\frac{1}{t}(-2\sin(2t+1)) + (\cos^2(2t+1) + 3\frac{1}{t^2})(\frac{-1}{t^2})$$

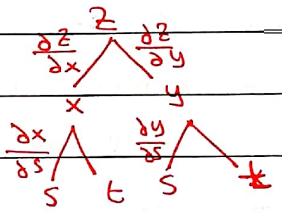




* example :- use the chain Rule to find $\frac{\partial z}{\partial s}$ where $z=\sqrt{x-3y}$,
$x = s^2+t^2$, $y = 1-2st$.

with the help of the tree diagram, we get the chain Rule.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$



$$= \frac{1}{2\sqrt{x-3y}}(2s) + \frac{-3}{2\sqrt{x-3y}}(-2t)$$

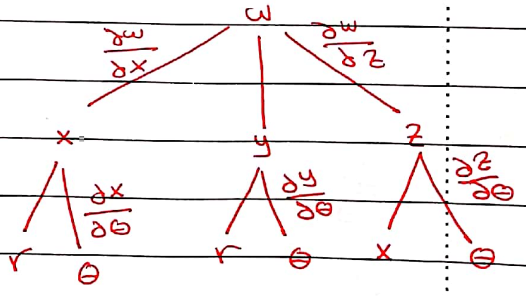
$$= \frac{s+3t}{\sqrt{x-3y}}$$

$$= \frac{s+3t}{\sqrt{s^2+t^2-3(1-2st)}}$$

* example :- use the Chain Rule to find $\frac{\partial w}{\partial \theta}$ at $r=2$, $\theta=\frac{\pi}{2}$
if $z = xy+xz+yz$, $x = r\cos\theta$, $y = r\sin\theta$,
$z = r\theta$?

with the help of the tree diagram, we get the Chain Rule

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial \theta}{\partial z}\frac{\partial z}{\partial \theta}$$



$$\frac{\partial w}{\partial \theta} = (y+z)(-r\sin\theta)+(x+z)(r\cos\theta)$$
$$+ (x+y)(r)$$

At $r = 2$, $\theta = \frac{\pi}{2}$ we have

$x = 2\cos\frac{\pi}{2} = 0$, $y = 2\sin\frac{\pi}{2} = 2$

, $z = 2\frac{\pi}{2} = \pi$.

$$\left.\frac{\partial w}{\partial \theta}\right|_{r=2,\theta=\frac{\pi}{2}} = (2+\pi)(-2)+0+4 = -2\pi .$$

$s = 1, t = 0$

example :- let $w(s,t) = F\big(u(s,t), v(s,t)\big)$ and $u(1,0) = 2$

$u_s(1,0) = -2$, $u_t(1,0) = 6$, $v(1,0) = 3$, $v_s(1,0) = 5$, $v_t(1,0) = 4$

$F_u(2,3) = -1$, $F_v(2,3) = 10$, Find $\left.\frac{\partial w}{\partial s}\right|_{(1,0)}$

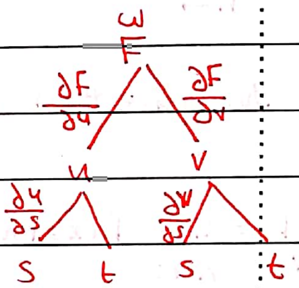
using the tree diagram, we have the chain Rule

$$\frac{\partial w}{\partial s} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial s}$$

$$\left.\frac{\partial w}{\partial s}\right|_{(1,0)} = F_u\big(u(1,0), u(1,0)\big)\, u_s(1,0) + F_v\big(u(1,0), u(1,0)\big)\, v_s(1,0)$$

$$= F_u(2,3)(-2) + F_v(2,3)(5)$$

$$= (-1)(-2) + 10(5) = 2 + 50 = 52 .$$



example:- use the Chain Rule to find $\frac{\partial T}{\partial p}$

when $p = 1, q = 1, r = 4$ if $T = 3u^2v^2 + 4uv$,

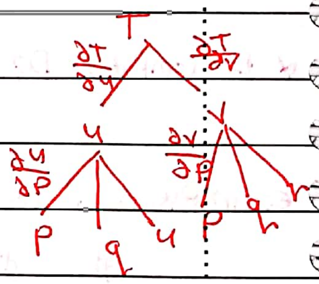
$u = pq\sqrt{r}$, $v = q\sqrt{p}\, r$.

using the tree diagram, we have the Chain Rule

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial p} = (6uv^2 + v)(q\sqrt{r}) + (6u^2v + u)\left(\frac{qr}{2\sqrt{p}}\right)$$

when $p = q = 1$, $r = 4$, we have $u = 2$, $v = 4$

$$\left.\frac{\partial T}{\partial p}\right|_{p=1, q=1, r=4} = \big(6(2)(4)^2 + 4\big)(2) + \big(6(2)^2(4) + 2\big)(2).$$

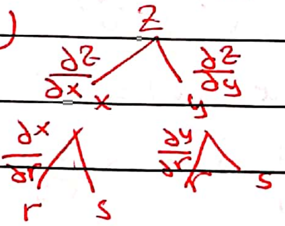


example :- if $z = f(x,y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$, $y = 2rs$, find $\frac{\partial^2 z}{\partial r^2}$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)$$

$$= 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2\frac{\partial z}{\partial x} + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

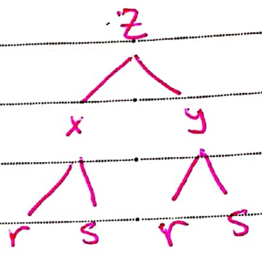


$z = f(x, y)$, $y = 2rs$

$x = r^2 + s^2$

$\frac{\partial^2 z}{\partial r^2} = ?$

الشجرة الأولى :-

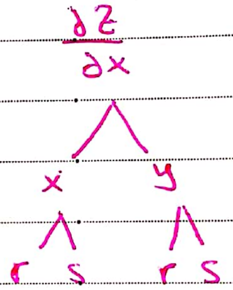


$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dr} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dr}$$

$$= \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial r} \cdot \frac{\partial z}{\partial x} \cdot 2r + \frac{\partial}{\partial r} \cdot \frac{\partial z}{\partial y} \cdot 2s$$

الشجرة الثانية :-

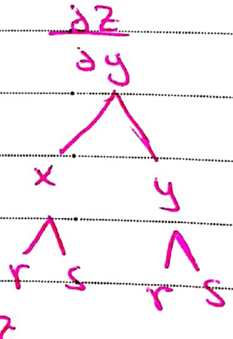


$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{\cancel{\ }}{\partial x} \cdot \frac{dx}{dr} + \frac{\partial z}{\partial x} \cdot \frac{1}{\partial y} \cdot \frac{dy}{dr}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} \cdot \frac{1}{\partial x} \cdot \frac{dx}{dr} + \frac{\partial z}{\partial y} \cdot \frac{1}{\partial y} \cdot \frac{dy}{dr}$$

نعود للمعادلة الطويلة ونقوم بتجميع الحدود و الأختصارات .

الشجرة الثالثة :-



$$= 2r\left[\frac{\partial z}{\partial x^2} \cdot 2r + \frac{\partial z}{\partial xy} \cdot 2s\right] +$$

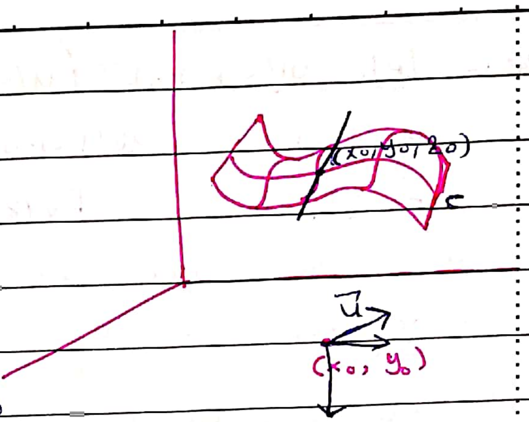
$$2s\left[\frac{\partial z}{\partial xy} \cdot 2r + \frac{\partial z}{\partial y^2} \cdot 2s\right]$$

$$= 4r^2 \cdot \frac{\partial z}{\partial x^2} + 4rs \cdot \frac{\partial z}{\partial xy} + 4rs \cdot \frac{\partial z}{\partial xy} + 4s^2 \cdot \frac{\partial z}{\partial y^2}$$

$$= 4r^2 \cdot \frac{\partial z}{\partial x^2} + 8rs \cdot \frac{\partial z}{\partial xy} + 4s^2 \cdot \frac{\partial z}{\partial y^2} .$$

Subject (الاحد 2019/12/8
في الساعة 3-2) Date No.

* (14:6) Directional Derivatives:-



let $z = f(x,y)$ is a surface.

The directional derivative of (f) at (x_0, y_0) in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is donated by $D_{\vec{u}} f$.

لازم انتبه انه ازا
ماكان unit لازم أقسمه على قيمته.

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}.$$

at the direction of the x-axis, $\vec{u} = \vec{i} = \langle \underset{a}{\underline{1}}, \underset{b}{\underline{0}} \rangle$.

$$D_{\vec{i}} f = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$= f_x(x_0, y_0).$$

at the direction of the y-axis, $\vec{u} = \vec{j} = \langle 0, 1 \rangle$

$$D_{\vec{j}} f = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} = f_y(x_0, y_0).$$

* to compute $D_{\vec{u}} f$ we use that $D_{\vec{u}} f = f(x_0, y_0) \cdot a + f_y(x_0, y_0) \cdot b$.

Example:- find the directional derivative of $f(x,y) = y e^{-xy}$ at $(0,1)$ in the direction of $\vec{v} = \langle 2, 3 \rangle$.

$f_x = -y^2 e^{-xy}$, $f_x(0,1) = -1$

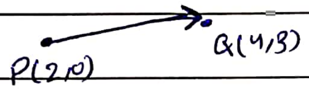
$f_y = -xy e^{-xy} + e^{-xy}$, $f_y(0,1) = 1$

Note that $|\vec{v}| = \sqrt{13}$, a unit vector in the direction of $\vec{v}$ is:

$$\vec{u} = \frac{1}{|\vec{v}|} \vec{v} = \left\langle \frac{3}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle.$$

$D_{\vec{u}} f = f_x(0,1)\cdot\frac{2}{\sqrt{13}} + f_y(0,1)\cdot\frac{3}{\sqrt{13}} = \frac{1}{\sqrt{13}}$.

example :- find the directional derivative of $f(x,y) = x\sin(xy)$. at the point $P(2,0)$ in the direction from P to $Q(4,3)$.



$\vec{v} = \vec{PQ} = \langle 2,3\rangle$.

$f_x = xy\cos(xy) + \sin(xy)$

$f_x(0,0) = 0$.

$f_y = x^2\cos(xy)$, $f_y(2,0) = 4$

here $|\vec{v}| = \sqrt{13}$, so a unit vector in the direction of $\vec{v}$ is

$\vec{u} = \frac{1}{|\vec{v}|}\cdot\vec{v} = \langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \rangle$.

$D_{\vec{u}} f = f_x(2,0)\cdot\frac{2}{\sqrt{13}} + f_y(2,0)\cdot\frac{3}{\sqrt{13}} = \frac{12}{\sqrt{13}}$.

, the vector $\langle f_x, f_y\rangle$ is called (the gradient vector of f) Df

$Df = \langle f_x, f_y\rangle$.

Note that ~~the directional~~ $D_{\vec{u}} f = f_x(x_0,y_0)\cdot a + f_y(x_0,y_0)\cdot b$

$= \nabla f\cdot\vec{u}$.

example :- let $f(x,y) = \sqrt{x-3y}$.

① find the gradient of f ?

② find the directional derrivative of f at (5,1) in the direction of $\vec{v} = \vec{i} - 2\vec{j}$.

$f_x = \frac{1}{2\sqrt{x-3y}}$, $f_y = \frac{-3}{2\sqrt{x-3y}}$

① $\nabla f = \langle f_x , f_y \rangle = \langle \frac{1}{2\sqrt{x-3y}} , \frac{-3}{2\sqrt{x-3y}} \rangle$

② at the point (5,1).

$\nabla f(5,1) = \langle \frac{1}{2\sqrt{2}} , \frac{-3}{2\sqrt{2}} \rangle$.

$|\vec{v}| = \sqrt{5}$, a unit vector in the direction of $\vec{v}$ is $\vec{u} = \frac{1}{\sqrt{5}} \langle 1, -2 \rangle = \langle \frac{1}{\sqrt{5}} , \frac{-2}{\sqrt{5}} \rangle$

$D_{\vec{u}} f = \nabla f(5,1) \cdot \vec{u}$.

$= \langle \frac{1}{2\sqrt{2}} , \frac{-3}{2\sqrt{2}} \rangle \cdot \langle \frac{1}{\sqrt{5}} , \frac{-2}{\sqrt{5}} \rangle$

$= \frac{1}{2\sqrt{10}} + \frac{6}{\sqrt{10}} = \frac{7}{2\sqrt{10}}$.

* for a function of 3 variables $f(x,y,z)$, then :- $\nabla f = \langle f_x , f_y , f_z \rangle$.

and $D_{\vec{u}} f = \nabla f \cdot \vec{u}$.

* example :- let $\overset{f(x,y,z)}{} = xy + xz + yz$.

(1) find the gradient of f ?

(2) find the directional derivative of f at (1,1,2) in the direction of $\vec{v} = \vec{i} + 2\vec{j} + \vec{k}$.

$f_x = y + z$, $f_y = x + z$, $f_z = x + y$.

$\nabla f = \langle y+z , x+z , x+y \rangle$.

$\nabla f(1,1,2) = \langle 3, 3, 2 \rangle$.

$|\vec{v}| = \sqrt{6}$, so a unit vector in the direction of $\vec{v}$ is $\vec{u} = \langle \frac{1}{\sqrt{6}} , \frac{2}{\sqrt{6}} , \frac{-1}{\sqrt{6}} \rangle$.

$D_{\vec{u}} f = \nabla f(1,1,2) \cdot \vec{u}$

$= \langle 3, 3, 2 \rangle \cdot \langle \frac{1}{\sqrt{6}} , \frac{2}{\sqrt{6}} , \frac{-1}{\sqrt{6}} \rangle = \frac{3}{\sqrt{6}} + \frac{6}{\sqrt{6}} - \frac{2}{\sqrt{6}} = \frac{7}{\sqrt{6}}$.

* Recall that :

$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$. → unit vector = 1

$= |\vec{\nabla} f| |\vec{u}| \cos\theta$

$= |\vec{\nabla} f| \cos\theta$

$-1 \leq \cos\theta \leq 1$

So, the maximum rate of change occurs in the direction $\vec{u} = \vec{\nabla} f$ and has the value $|\nabla f|$.

the minimum of rate of change accurs in the direction $\vec{u} = -\nabla f$ and has the value $-|\nabla f|$.

*example:- find the maximum and minimum rates of change of $f(x,y) = x^2y + 2xy^3$ at (1,2). and the directions which they occur.

first , we find ∇f $f_x = 2xy + 2y^3$, $f_y = x^2 + 6xy^2$.

$f_x = 2xy + 2y^3$, $f_y = x^2 + 6xy^2$.

$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2xy + 2y^3, x^2 + 6xy^2 \rangle$.

$\nabla f(1,2) = \langle 20, 25 \rangle$.

the maximum rate of change of f at (1,2) accurs in the direction of $\vec{u} = \nabla f(1,2) = \langle 20, 25 \rangle$.

which has the value $|\nabla f(1,2)| = \sqrt{(20)^2 + (25)^2}$.

the minimum rate of change of f at (1,2) occurs on the direction $\vec{u} = -\nabla f = \langle -20, -25 \rangle$.

, which has the value $-|\nabla f(1,2)| = -\sqrt{(20)^2 + (25)^2}$.

* example :- find the directions in which the function increase and decrease most rapidly at P_0, then find the derivatives of the function in there directions.

معدل التغير أعلى ما يمكن .

$$f(x,y) = x^2 y + e^{xy} \cdot \sin y \quad , P_0(1,0) \;?$$

$f_x = 2xy + y e^{xy} \sin y$

$f_x(1,0) = 0$

$f_y = x^2 + e^{xy} \cos y + x e^{xy} \sin y$

$f_y(1,0) = 2$

$\nabla f(1,0) = \langle f_x(1,0) , f_y(1,0) \rangle = \langle 0,2 \rangle.$

f increases most rapidly at P_0 in the direction $\vec{u} = \nabla f(1,0) = \langle 0,2 \rangle.$

$D_{\vec{u}} f = |\nabla f(1,0)| = 2$

f decrease most rapidly at P_0 in the direction $\vec{u} = -\nabla f(1,0) = \langle 0,-2 \rangle$

$D_{\vec{u}} f = -|\nabla f(1,0)| = -2$.

* example :- the temprature T at any point (x,y,z) in space is given by :

$$T(x,y,z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2} ,$$ in which direction does the temprature at $(1,1,-2)$ increase fastest?

first we find ∇T,

$$T_x = \frac{-80(2x)}{(1 + x^2 + 2y^2 + 3z^2)^2}$$

$$T_y = \frac{-80(4y)}{(1 + x^2 + 2y^2 + 3z^2)^2}$$

$$T_z = \frac{-48z}{(1 + x^2 + 2y^2 + 3z^2)^2}$$

$\nabla T(1,1,-2) = \langle T_x(1,1,-2), T_y(1,1,-2), T_z(1,1,-2) \rangle$

$= \langle \frac{-5}{8}, \frac{-10}{8}, \frac{30}{8} \rangle.$

the temprature at (1,1,-2) increases fastest in the diretion of $\vec{u} = \nabla T(1,1,-2) = \langle -\frac{5}{8}, \frac{-10}{8}, \frac{30}{8} \rangle.$

rate of change? the maximum rate of change of the temprature at (1,1,-2) is $|\nabla T(1,1,-2)| = |\langle \frac{-5}{8}, \frac{-10}{8}, \frac{30}{8} \rangle|$

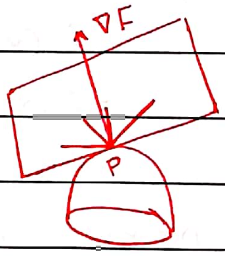
$= \frac{5}{8}\sqrt{41}$ °c/m.

$= \simeq 4$

★ tangent Plane and normal line :-

let S be a surface given in the form $F(x,y,z) = k$.

the tangent Plane of S at the Point $P_0(x_0, y_0, z_0)$ is the Plane contains all tangent line to curves on S at P_0.



A normal vector to the tangent Plane is

$\vec{n} = \nabla F(x_0, y_0, z_0).$

دايما عمودي على ال tangent Plane

the normal line is the line ~~[illegible]~~ through P_0 and Perpendicular to the tangent plane.

$\vec{J} = \nabla F(x_0, y_0, z_0)$.

example :- find the tangent Plane and normal line to the surface $z = x^2 + y^2 + 2$ at $(1, 1, 4)$.

first , we write surface equation in the form $F(x, y, z) = k$.

$z - x^2 - y^2 = 2$

$F(x, y, z) = z - x^2 - y^2$

$f_x = -2x$, $f_y = -2y$, $f_z = 1$

$\nabla F(1, 1, 4) = \langle F_x(1,1,4), F_y(1,1,4), F_z(1,1,4) \rangle$

$= \langle -2, -2, 1 \rangle$

~~For~~ the tangent plane at $(1,1,4)$ has a normal vector

$\vec{n} = \nabla f(1,1,4) = \langle -2, -2, 1 \rangle$.

the equation of tangent plane is :-

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$-2(x - 1) - 2(y - 1) + 1(z - 4) = 0$

يعني بنفس اتجاهه

the normal line through $(1,1,4)$ has a directional vector :-

$\vec{v} = \nabla F(1,1,4) = \langle -2, -2, 1 \rangle$.

$x = 1 - 2t$

$y = 1 - 2t$

$z = 4 + t$

(14.7) maxima and minima value .

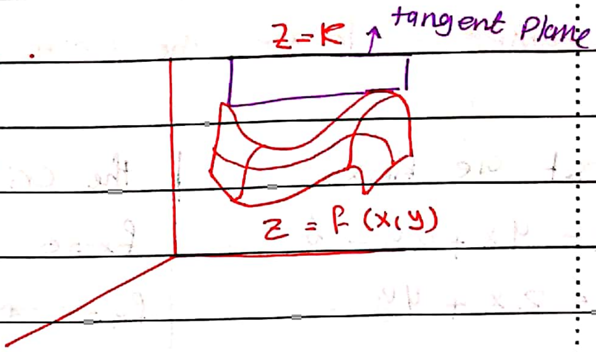
$z=f(x,y)$

$f(x,y)-z=0$

$F(x,y,z)=0$

$\langle f_x, f_y, -1\rangle$

$f_x(x-x_0)+f_y(y-y_0)-(z-z_0)=0$



~~* Definition~~

*Definition :- let $z=f(x,y)$, the critical point of f are the points (a,b) where $f_x(a,b)=0$ and $f_y(a,b)=0$ or $f_x(a,b)$, $f_y(a,b)$ do not exist.

example :- locate the critical points for $f(x,y)=(x+5)^2+(y-8)^2$;

~~the~~

the critical points occur when $f_x=0$, $f_y=0$

$f_x = 2(x+\underset{5}{\cancel{8}})=0 \Rightarrow x=-5$

$f_y = 2(y-8)=0 \Rightarrow y=8$.

*example :- locate the critical Points for $f(x,y) = 2x^2 + 2xy + 2y^2 - 6x$.

first we find :-

$f_x = 4x + 2y - 6$

$f_y = 2x + 4y$.

the critical Points accure when:

$f_x = 0$, $f_y = 0$

$f_x = 0 \Rightarrow 4x + 2y - 6 = 0 \Rightarrow 4x + 2y = 6$

$f_y = 0 \Rightarrow 2x + 4y = 0$

$$4x + 2y = 6$$
$$-2(\ 2x + 4y = 0\ )$$
$$-6y = 6 \Rightarrow y = -1 \ / \ x = 2$$

the only critical points for f is (2,-1).

*example :- locate the critical Points for $f(x,y) = 2x^2 - 4xy + y^4 + 2$.

first we find the 1^{st} partial dan

$f_x = 4x - 4y$, $f_y = -4x + 4y^3$

to find the critical Points we Solve

$f_x = 0 \Rightarrow 4x - 4y = 0 \Rightarrow x = y$

$f_y = 0 \Rightarrow -4x + 4y^3 = 0$

Plug the first equation in the second equation.

$$-4x + 4y^3 = 0$$
$$-4y + 4y^3 = 0$$
$$-4y(1 - y^2) = 0 .$$

$-4y(1-y)(1+y) = 0$

$y = 0$, $y = 1$, $y = -1$

when $y=0$, $x=0$

when $y=1$, $x=1$

when $y=-1$, $x=-1$

$(0,0)$, $(1,1)$, $(-1,-1)$ are the critical Points f.

* example :- locate the critical Points for $f(x,y)=3xy-x^3-y^3$.

$f_x = 3y-3x^2$, $f_y = 3x-3y^2$.

we solve

$f_x=0$, $f_y=0$

$f_x=0 \Rightarrow 3y-3x^2=0 \Rightarrow y=x^2$.

$f_y=0 \Rightarrow 3x-3y^2=0$

Plug the first equation in the second equation.

$3x-3x^4=0$

$\underbrace{3x(1-x^3)}=0$ → حل واحد لأن المعادلة التربيعية اللي بتطلع ٢ حلّ.

$x=0$ / $x=1$.

when $x=0$, $y=0^2=0$

when $x=1$, $y=1^2=1$

the critical Points are $(0,0)$, $(1,1)$.

*example :- locate the critical Points for $f(x,y)=e^{-\frac{1}{3}x^3+x-y^2}$.

$f_x = (-x^2+1)\,e^{(-\frac{1}{3}x^3+x-y^2)}$.

$f_y = -2y\,e^{(-\frac{1}{3}x^3+x-y^2)}$

$f_x=0 \Rightarrow (1-x^2)\,e^{(-\frac{1}{3}x^3+x-y^2)} = 0^2$

, but $e^x \neq 0 \Rightarrow 1-x^2=0$

$(1-x)(1+x)=0$

$x=1$, $x=-1$

$f_y = 0 \Rightarrow y = 0$

the critical points are $(1,0)$, $(-1,0)$

* locate the critida Points for $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

$f_x = 6xy - 6x$ $\quad$ $f_y = 3x^2 + 3y^2 - 6y$.

to find the critical point we solve

$f_x = 0$, $f_y = 0$

$f_x = 0 \Rightarrow 6xy - 6x = 0$

$6x(y-1) = 0$

$x = 0$ / $y = 1$

$f_y = 0 \Rightarrow 3x^2 + 3y^2 - 6y = 0$

when $x = 0 \Rightarrow 3y^2 - 6y = 0$

$3y(y-2) = 0$

$y = 0$ / $y = 2$.

when $y = 1 \Rightarrow 3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$3(x-1)(x+1) = 0$

$x = -1$ / $x = 1$

$(0,0)$, $(0,2)$, $(1,1)$, $(-1,1)$ are criticle Points for f.

example :- locate the Cirticale Points for

$f(x,y) = e^{(-\frac{1}{3}x^3+x-y^2)}$

$f_x = e^{-\frac{1}{3}x^3+x-y^2} (-x^2+1)$, $f_y = e^{(-\frac{1}{3}x^3+x-y^2)} (-2y)$

$f_x = 0 \hookrightarrow (1-x^2) e^{-\frac{1}{3}x^3+x-y^2} = 0$

-but $e^x \neq 0$, $x^2-1=0$

$(x-1)(x+1)=0$

$x=1 \,/\, x=-1$

$f_y = 0 \longrightarrow y = 0$

$(1,0),(-1,0)$ the critical Point.

* theorem :- if f has alocal maximum or a local minimum at (a,b), then (a,b) is acritiele Point of "f"

~~*~~ to classify the ciriticales Points we use the second dervative test

second derivative test :-

suppose ~~that~~ the second ~~se~~ partial derivatives of (f) are Continuous on a disk contain (a,b) and (a,b) is a critiele Point ~~for~~ for f. let

$D = f_{xx}(a,b) f_{yy}(a,b) - \left[f_{xy}(a,b) \right]^2$.

① $D>0$ and $f_{xx}(a,b)>0$, f has alocal minimum at (a,b).

② $D>0$ and $f_{xx}(a,b)<0$, f has a local maximum at (a,b).

③ $D<0$, f has a saddle Point at (a,b).

example:- locate and classify the critical Points for

$f(x,y)=2x^2+2xy+2y^2-6x+4y$.

first we find :- $f_x=4x+2y-6$, $f_y=2x+4y+4$

to find the critical Points we solve

$f_x=0 \rightarrow 4x+2y-6=0 \rightarrow (4x+2y=6)\div 2 \rightarrow 2x+y=3$

$f_y=0 \rightarrow 2x+4y+4=0 \rightarrow x+2y=-2$

$(2x+y=3)$

$(x+2y=-2)-2$

$-3y=7 \rightarrow y=-\frac{\cancel{2}7}{3}$

$x=2-2y=2-\frac{14}{3}=\frac{8}{3}$.

f has only one crititcle point $(\frac{8}{3},\frac{-7}{3})$

* to classify the criticle Point we find $D=f_{xx}f_{yy}-f_{xy}^2$

$D=(4)(4)-2^2=12$

$D(\frac{8}{3},\frac{-7}{3})=12>0$ and $f_{xx}(\frac{8}{3},\frac{-7}{3})=4>0$

f has locale minimum at $(\frac{8}{3},\frac{-7}{3})$.

* locate and classify the criticle Points for $f(x,y) = 3xy - x^3 - y^3$.

first we find $f_x = 3y - 3x^2$, $f_y = 3x - 3y^2$.

to find the criticle points we solve

$f_x = 0 \rightarrow 3y - 3x^2 = 0$

$y = x^2$ --- ①

$f_y = 0 \rightarrow 3x - 3y^2 = 0$ --- ②

plug the first equation in the second.

$3x - 3y^2 = 0 \rightarrow 3x - 3x^4 = 0$

$3x(1 - x^3) = 0$

$x = 0$, $x = 1$

$y = 0$ $y = 1$

f has two criticle points $(0,0)$ $(1,1)$

$f_{xx} = -6x$, $f_{yy} = -6y$, $f_{xy} = 3$

$D = f_{xx} f_{yy} - f^2_{xy} = 36xy - 9$

$D(0,0) = -9 < 0$, $(0,0)$ is a saddle point

$D(1,1) = 27 > 0$, $f_{xx}(1,1) = -6 < 0$

f has a local maximum at $(1,1)$.

* Determine the critick Point and locate any minimum and saddel Points of the function $f(x,y) = e^{(-\frac{1}{3}x^3 + x - y^2)}$

$f_x = (-x^2 + 1) e^{(-\frac{1}{3}x^2 + x - y^2)}$, $f_y = -2y e^{(-\frac{1}{3}x^3 + x - y^2)}$

to find the criticale Point

$f_x = 0 \rightarrow 1 - x^2 \cdot e^{(-\frac{1}{3}x^3 + x - y^2)} = 0$

$1 - x^2 = 0 \rightarrow x = 1 , x = -1$

$f_y = 0 \rightarrow -2y\, e^{-\frac{1}{3}x^3+x-y^2} = 0 \rightarrow y = 0$

f has two criticle point $(1,0)$, $(-1,0)$.

$f_{xx} = (-x^2+1)^2 e^{(-\frac{1}{3}x^3+x-y^3)} + -2x\, e^{(-\frac{1}{3}x^3+x-y^2)}$.

$= (x^4-2x^2-2x+1)\, e^{-\frac{1}{3}x^3+x-y^2}$.

$f_{yy} = (-2y)^2 e^{-\frac{1}{3}x^3+x-y^2} + -2\, e^{-\frac{1}{3}x^3+-y^2}$.

$=(4y^2-2)\, e^{-\frac{1}{3}x^3+x-y^2}$,

$f_{xy} = -2y(-x^2+1)\, e^{-\frac{1}{3}x^3+x-y^2}$

$D(1,0) = f_x x f_{yy} - f^2_{xy}$

$= -2e^{\frac{2}{3}}(-2e^{\frac{2}{3}}) - 0^2 = 4e^{\frac{4}{3}} > 0$

$f_{xx}(1,0) = -2e^{2/3} < 0$ f has local maximum at $(1,0)$.

$D(-1,0) = f_{xx}(-1,0)\, f_{yy}(-1,0) - f_{x^2y}(-1,0)$

$= 2\, e^{-2/3}(-2e^{-2/3}) - 0^2$

$= -4\, e^{-2/3} < 0$ it's a saddel Point.

minimum يعني

* find the Shortest distance from the Point $(1,0,-2)$ to the Plane $x+2y+z=4$

let (x,y,z) be a point on the Plane, the distanee between P and Q is :- $P\cdot(1,0,\bar{2})$

$D = \sqrt{(x-1)^2+y^2+(z+2)^2}$ $\boxed{Q\cdot(x,y,z)}$

(x,y,z) on the Plane.

$x+2y+z=4 \rightarrow z = 4-x-8y$

$D = \sqrt{(x-1)^2+y^2+(6-x-2y)^2}$

we want to minimize D.

we can minimize only the inside

$f(x,y) = (x-1)^2 + y^2 + (6-x-2y)^2$

$f_x = 2(x-1) + 2(6-x-2y)(-1)$

$f_y = 2y + 2(6-x-2y)(-2)$

find the criticle point we solve

$f_x = 0 \longrightarrow 2x - 2 + 12 + 2x + 2y = 0 \longrightarrow 4x + 4y = 14.$

$f_y = 0 \longrightarrow 2y - 24 + 4x + 8y = 0 \longrightarrow 4x + 10y = 24.$

$$\begin{array}{r} 4x + 4y = 14 \\ - \quad 4x + 10y = 24 \\ \hline -6y = -10 \end{array} \longrightarrow y = \frac{10}{6} = \frac{5}{3}.$$

$x = \frac{14 - 4y}{4} = \frac{22}{12} = \frac{11}{6}.$

$f_{xx} = 4$, $f_{yy} = 10$, $f_{xy} = 4$

$D\left(\frac{11}{6}, \frac{5}{3}\right) = 4(10) - (4)^2 = 24 > 0$

, but $f_{x\cancel{x}} = 4 > 0$

f has local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$

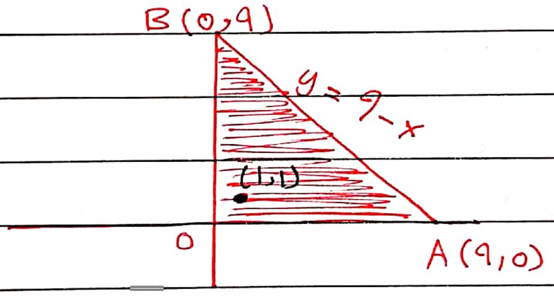
$D = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$ نعوضها

* absolute Maxima and Minima.

Note that if $f(x,y)$ continuaus on a closed bounded domain D the f attains its absolute extrema in D.

To find the absolute maximum and minimum for $f(x,y)$ you have to find the critical Points inside the domain D (Interior Points) and then the critical Points on the bondary of D (Boundary prints).

example: Find the absolute maximum and minimum value of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$. on the tringular region



sol:-

(I) Interior points

$f_x = 2 - 2x$, $f_y = 2 - 2y$.

The critica points inside the tringular region occur when

$f_x = 0 \Rightarrow 2 - 2x = 0 \Rightarrow x = 1$

$f_y = 0 \Rightarrow 2 - 2y = 0 \Rightarrow y = 1$

f has only one critical point (1,1).

(II) Boundary

on the segment OA. $y = 0$, $f(x,0) = 2 + 2x - x^2$ $0 \leq x \leq 9$

$f'(x,0) = 2 - 2x = 0 \Rightarrow x = 1$

$(1,0)$, $(0,0)$, $(9,0)$

نقطة حرجة ، أطراف فترة .

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on the segment OB . $x = 0$,

$f(0,y) = 2 + 2y - y^2$, $0 \leq y \leq 9$

$\bar{f}(0,y) = 2 - 2y = 0 \Rightarrow y = 1$

$(0,1)$, $\underbrace{(0,0),(0,9)}_{\text{اطراف الفترة}}$ $\}$ نقاط حرجة

on the segment BA , here $y = 9 - x$

$f(x, 9-x) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$.

$\bar{f} = -2x + 2(9-x) = -4x + 18 = 0$

$x = \frac{18}{4} = \frac{9}{2}$.

$y = 9 - x = 9 - \frac{9}{2} = \frac{9}{2}$.

$(\frac{9}{2}, \frac{9}{2})$, $(0,9)$, $(9,0)$.

we list all candidate

$(0,0), (0,9), (9,0), (0,1), (1,0), (1,1), (\frac{9}{2}, \frac{9}{2})$.

$f(0,0) = 2$

$f(0,9) = -61$

$f(9,0) = -61$

$f(0,1) = 3$

$f(1,0) = 3$

$f(1,1) = 4$ أكبر قيمة ل z يأخذها هي (4)

$f(\frac{9}{2}, \frac{9}{2}) = \frac{-41}{2}$

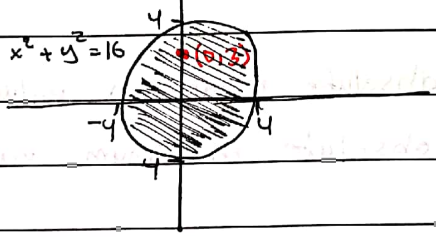
f has absolute maximum value 4 occur at $(1,1)$

f has absolute minimum value -61 occurs at $(0,9)$ and $(9,0)$

[redacted]

example :- Find the absolute menimum and the absolute maximum of $f(x,y) = 2x^2 - y^2 + 6y$ on the disk $y^2 + x^2 \leq 16$.

(I) Interior points.



$f_x = 4x$; $f_y = -2y + 6$

$f_x = 0 \Leftrightarrow 4x = 0 \Leftrightarrow x = 0$.

$f_y = 0 \Leftrightarrow -2y + 6 = 0 \Leftrightarrow y = 3$.

f has only one critical point inside the disk (0,3).

(II) boundary points. (يلي هو الدائرة)

the bondary is the circle $x^2+y^2=16$.

Here $x^2 = 16 - y^2$.

$f = 2(16 - y^2) - y^2 + 6y$

$= 32 - 3y^2 + 6y$, $-4 \leq y \leq 4$

$\bar{f} = -6y + 6 = 0 \Rightarrow y = 1$

$x^2 = 16 - y^2 = 16 - 1^2 = 15$

$x = \pm\sqrt{15}$

$(\sqrt{15}, 1), (-\sqrt{15}, 1), (\overset{0,-4}{\cancel{}}), (\overset{0,4}{\cancel{}})$.

list all candidate :-

$(0,3), (\sqrt{15}, 1), (-\sqrt{15}, 1), (0,4), (0,-4)$.

$f(0,3) = \cancel{8}\ 9$

$f(0,4)=8$

$f(0,-4)=-40$

$f(\sqrt{15},1)=35$

$f(-\sqrt{15},1)=35$

f has absolute mimimum value -40 occure at $(0,-4)$

f has obsolute maximum value 35 occure at $(\sqrt{15},1)$ and $(-\sqrt{15},1)$.

* يوجد بالدرايات طل سؤال على rectangel *

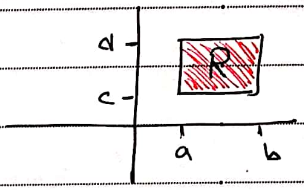
نهاية مادة الكند

CHaPter (15): Multiple integrals.

* (15-1) + (15.2) :- Double integrals :-

let $f(x,y)$ be a continuous function over a rectangler region $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$.

$= [a,b] \times [c,d]$.



the double integral

$$\iint_R f(x,y)\, dA = \int_A^b \int_c^d f(x,y)\, dy\, dx$$

عمرتكامل هذه الاقتران فوق هذه المنطقة

is called iterated integral.

to compute the iterated integral $\int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$ we first evaluate the integral $\int_c^d f(x,y) dy$

by cosidering x as a constant and the result is $g(x)$, then we integrate $\int_a^b g(x) \cdot dx$

$$\int_a^b \left[\int_c^d f(x,y)\, dy \right] dx \Rightarrow \int_a^b g(x) \cdot dx$$

example:- evaluate $\int_0^2 \int_1^3 x y^2 \cdot dx\, dy$

$$* \int_1^3 xy^2 \cdot dx = y^2 \frac{x^2}{2} \Big]_1^3 = \frac{9y^2}{2} - \frac{y^2}{2} = \frac{8y^2}{2} = 4y^2.$$

$$\int_0^2 4y^2 \cdot dy = \frac{4y^3}{3} \Big]_0^2 = \frac{4(8)}{3} - 0 = \frac{32}{3}.$$

what about if we start with $\int_1^3 \left[\int_0^2 x y^2\, dy \right] dx$?

$$= \int_0^2 xy^2 \cdot dy = \frac{xy^3}{3} \Big]_0^2 = \frac{x\,8}{3}$$

$$\int_1^3 \frac{8x}{3}\, dx = \frac{8x^2}{6} \Big]_1^3 = \frac{8(9)}{6} - \frac{8}{6} = \frac{64}{6} = \frac{32}{3}.$$

theorem :- If $f(x,y)$ is continous on a rectangular region $R=[a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b\int_c^d f(x,y)\,dy\,dx$$

رح يكون الناتج متساوي بس بشرط إنه rectangular

كيف بعرف انه rectangular ؟

تكون عندي حدود التكامل أعداد a,b,c,d يعني مبينة انها أطراف مستطيل.

example :- evaluate $\iint_R y\cdot\sin(xy)\,dA$ where $R=[1,2]\times[0,\pi]$.

If we use $\iint_R y\cdot\sin(xy)\cdot dA = \int_1^2\int_0^\pi y\cdot\sin(xy)\,dy\cdot dx$

we face difficult integrals.

~~, but if we use~~ $\iint_R y\cdot\sin(xy)\cdot dA = \int_0^\pi\int_1^2 y\cdot\sin(xy)\cdot dx\,dy$.

هيك أختار أسهل

, but it is easier to use ↑

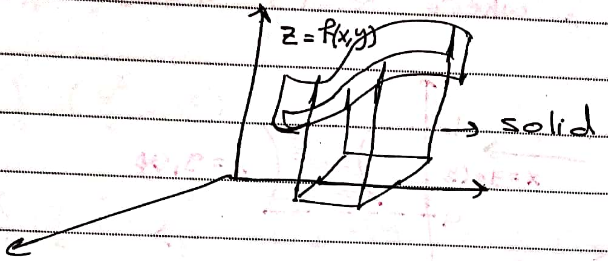
$$= \int_0^\pi \left. -\cancel{y}\cdot\frac{\cos(xy)}{\cancel{y}}\right]_1^2 dy$$

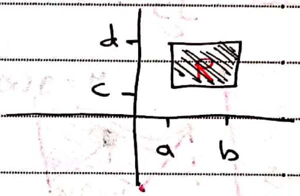
$$= -\int_0^\pi\left(\cos(2y) - \cos(y)\right)\cdot dy$$

$$= -\left.\left(\frac{\sin(2y)}{2} - \sin(y)\right)\right]_0^\pi$$

$$= -(0-0) \quad = \text{zero}.$$

* let $f(x,y) \geq 0$ for $(x,y) \in R$, where R is the rectangyler region $R = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\} = [a,b] \times [c,d]$.

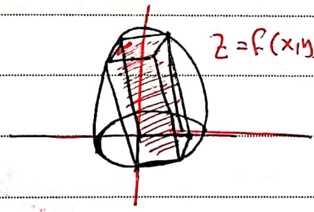




→ The volume of the solid bounded bebw by R and abore by the surface $z = f(x,y)$ is $V = \iint_R f(x,y)\,dA$.

* example: Find the volume of the solid that lies above the region ~~R~~ $R = [0,2] \times [0,2]$. and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$.

لو فرضنا انه ما في 16 - بتصير $z = (x^2 + 2y^2)$ تكون الشكل ellibs



$$V = \iint_R f(x,y)\,dA$$

$$= \int_0^2 \left[\int_0^2 (16 - x^2 - 2y^2)\, dx \right] dy$$

$$= \int_0^2 \left(16x - \frac{x^3}{3} - 2y^2x \right) \Big]_0^2 . dy$$

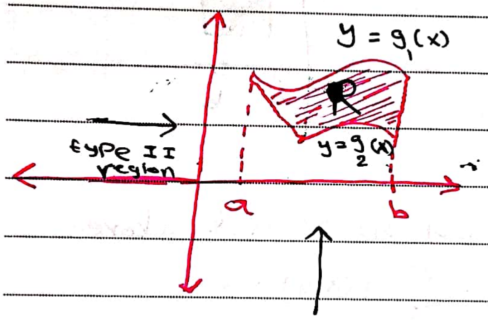
$$= \int_0^2 \left(32 - \frac{8}{3} - 4y^2\right) - 0 \ . dy$$

$$= \int_0^2 \left(\frac{38}{3} - 4y^2\right) . dy$$

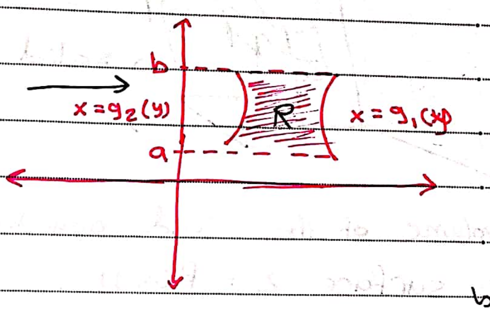
$$= \frac{38}{3} y - 4 \frac{y^3}{3} \Big]_0^2 = \left(\frac{176}{3} - \frac{32}{3}\right) - 0 = \frac{144}{3}$$

(15:3) Double Integrals over general Regions :-

Consider $\iint_R f(x,y)\cdot dA$, where R is a region in the xy-Plane



type I region

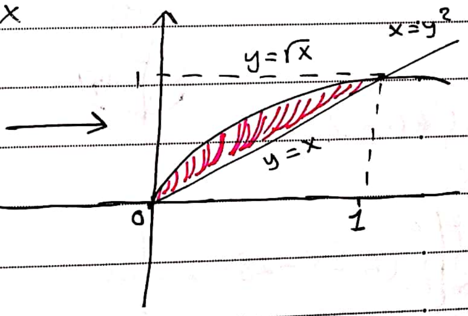


$\iint_R f(x,y) = \int_a^b \int_{g_2(x)}^{g_1(x)} f(x,y)\cdot dy\cdot dx$

$\iint_R f(x,y)\,dA = \int_a^b f(x,y)\,dx\,dy$.

example :- Evaluate $\iint_R (x+y)\,dt$ where R is the region in the xy-Plane bounded by $y=\sqrt{x}$ and $y=x$?

sketch the integration region R, to find the points of intersection we solve $\sqrt{x}=x$



$x = x^2 \rightarrow x^2 - x = 0 \rightarrow x(x-1)=0$

$x=0 \,/\, x=1$

R is a Type I region

$\iint_R (x+y)\,dA = \int_0^1 \int_x^{\sqrt{x}} (x+y)\,dy\,dx$

R is a type II region

$\iint_R (x+y)\,dA = \int_0^1 \int_{y^2}^{y} (x+y)\,dx\,dy$.

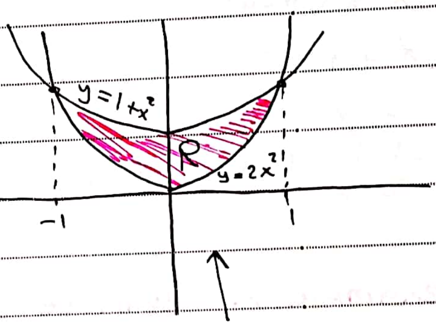
$$\int_0^1 \left[\int_{y^2}^{y} (x+y)\,dx \right] dy =$$

$$= \int_0^1 \frac{x^2}{2} + yx \Big]_{y^2}^{y} dy$$

$$= \int_0^1 \left(\frac{y^2}{2} + y^2\right) - \left(\frac{y^2}{2} + y^3\right) dy$$

$$= \int_0^1 \left(\frac{3}{2}y^2 - \frac{y^4}{2} - y^3\right) dy = \frac{1}{2}y^3 - \frac{y^3}{10} - \frac{y^4}{4} \Big]_0^1 = \frac{1}{2} - \frac{1}{10} - \frac{1}{4} = \frac{3}{20}$$

example :- evaluate $\iint_R xy^2 \,.\, dA$, where R is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



$2x^2 = 1 + x^2$

$x^2 = 1$

$x = \mp 1$

$$\iint_R xy^2 \,.dA = \int_{-1}^{1} \left[\int_{2x^2}^{1+x^2} xy^2 \,.dy \right] dx$$

$$= \int_{-1}^{1} \frac{x}{3} y^3 \Big]_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^{1} \frac{x}{3} \left((1+x^2)^3 - 8x^6 \right) dx$$

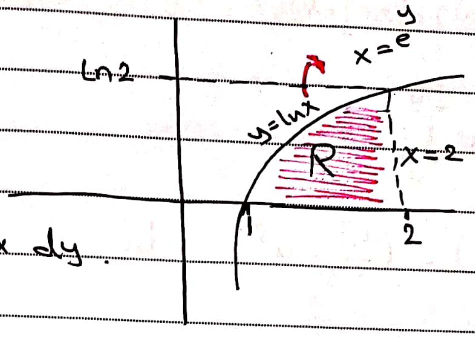
example :- sketch the region of integration and then change the order of integration

① $\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx$.

$= \iint_R f(x,y)\,dA$.

$0 \leq y \leq \ln x$

$1 \leq x \leq 2$



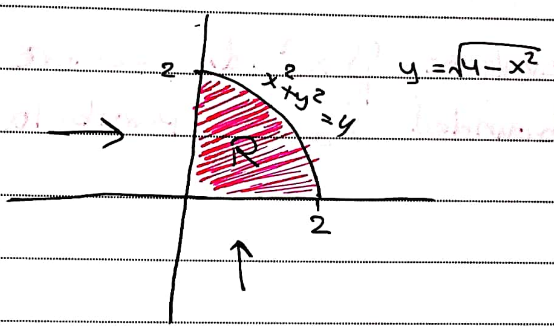
$y = \ln x$

$\iint_R f(x,y)\,dA = \int_0^{\ln 2}\int_{e^y}^{2} f(x,y)\,dx\,dy.$

② $\int_0^2\int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$

$0 \leq y \leq 2$

$0 \leq x \leq \sqrt{4-y^2}.$



$x = \sqrt{4-y^2}$

$x^2+y^2=2$

$= \int_0^2\int_0^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$

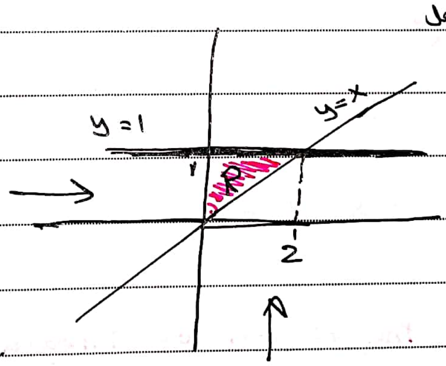
example :- evaluate $\int_0^1\int_x^1 \sin(y^2)\,dy\,dx$. هاد التكامل مستحيل يحل في الرياضيات

عشان احله لازم اعكس حدود التكامل

$= \iint_R \sin(y^2)\,dA$

$x \leq y \leq 1$

$0 \leq x \leq 1$



$= \int_0^1\int_0^y \sin(y^2)\,dx\,dy$

$= \int_0^1 \sin(y^2)\,x\Big]_0^y\,dy.$

$= \int_0^1 y\,.\sin(y^2)\,.dy$

let $u = y^2$

$du = 2y\,.dy$

$dy = \frac{du}{2y}$

$= \int_0^1 \cancel{y}\,\sin u\,\frac{du}{2\cancel{y}}$

$= -\frac{1}{2}\cos u\Big]_0^1 = -\frac{1}{2}(\cos 1 - 1)$

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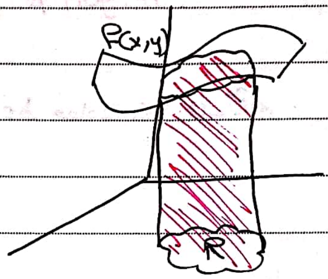
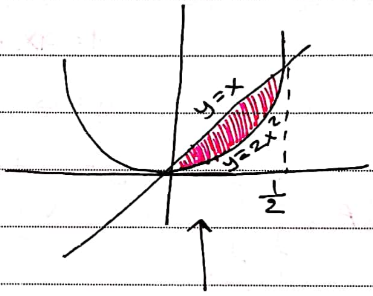
*example :- Find the volume of the solide below the Parboloid $z = x^2 + y^2$ and above the region in the xy-plane bounded by $y = 2x^2$ and $y = x$.

$$\text{Volume} = \iint_R f(x,y) \cdot dA$$



$$= \iint_R (x^2+y^2)\, dA$$

$$= \int_0^{1/2} \int_{2x^2}^{x} (x^2+y^2)\, dy\, dx.$$



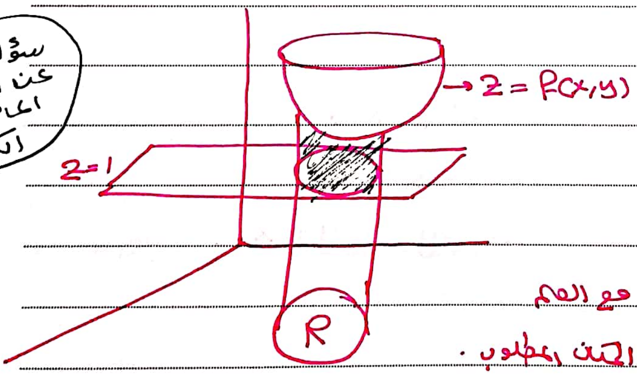
$$2x^2 = x$$

$$2x^2 - x = 0$$

$$2(2x-1) = 0$$

$$x=0 \,/\, x = \tfrac{1}{2}$$

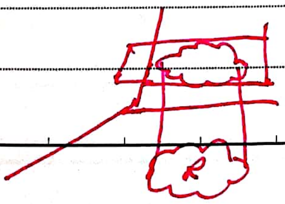
سؤال
عن الدرس
الخاص من
الكتاب



$$\iint_R (f(x,y) - 1)\, dA.$$

بدّه مساحة هاي المنطقة مع العلم
إنّ R مش واصلة لعند الجسم المطلوب .

سؤال
تاني

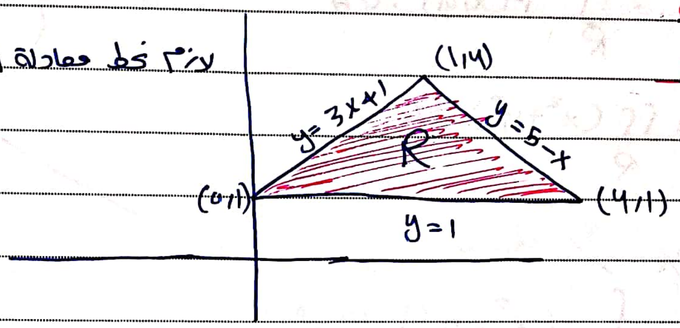


$$\iint_R 1 \cdot dA$$

example :- evaluate double integral over $\iint_R y^2\, dA$ where R is the tringular region with verticas (0,1), (1,4), (4,1)

بنعمل ال sketch reigon R.

بنرسم خط معادلة لكل المستقيم.



$a+b=4$

$\underline{4a+b=1}$

$3a=-3$

$a=-1$

$4+\frac{1}{3}=b$

$b=5$

$\boxed{y=-x+5}$

$y=ax+b$

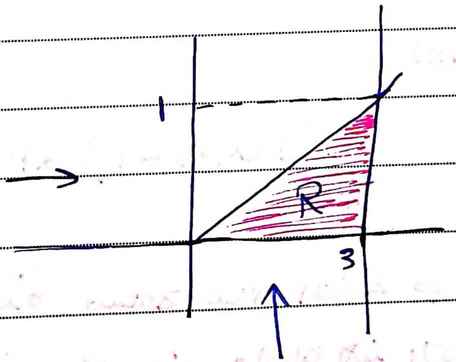
$y=b$ $\boxed{a=3}$ $\boxed{b=1}$

$y=3+1$

$$\iint_R y^2\, dA = \int_1^4 \int_{\frac{y-1}{3}}^{5-y} y^2\, dx\, dy.$$

example :- evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$.

$$= \iint_R e^{x^2}\, dA$$



$3y \leq x \leq 3$

$0 \leq y \leq 1$

$x = 3y \to y = \frac{x}{3}$.

$= \int_0^3 \int_0^{x/3} e^{x^2} \cdot dy\,dx$

$= \int_0^3 e^{x^2} y \Big]_0^{x/3} .dx$

let $u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$= \int_0^3 \frac{x}{3} e^{x^2} dx = \int_0^9 \frac{\cancel{x}}{3} e^{u} \cdot \frac{du}{2\cancel{x}}$

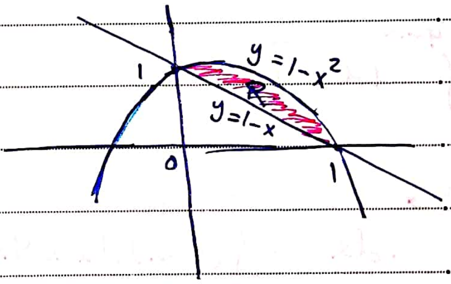
$= \int \frac{1}{6} e^{u} \Big]_0^9 = \frac{1}{6}(e^9 - 1)$

example :- Find the volume of the solid under the plane $x - 2y + z = 1$ and above the region bounded by $x + y = 1$ and $x^2 + y = 1$.

$\text{Volume} = \iint_R f(x,y)\,dA$

$z = 1 - x + 2y = f(x,y)$

To identify the region R,



$y = 1 - x$, $y = 1 - x^2$

$1 - x = 1 - x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$, $x = 1$

$\text{Volume} = \int_0^1 \int_{1-x}^{1-x^2} (1 - x + 2y)\,dy\,dx$

example :- find the volume of the solid enclosed by the cylenders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$.

$\text{Volume} = \iint_R f(x,y)\,dA = \iint_R x^2\,dA$

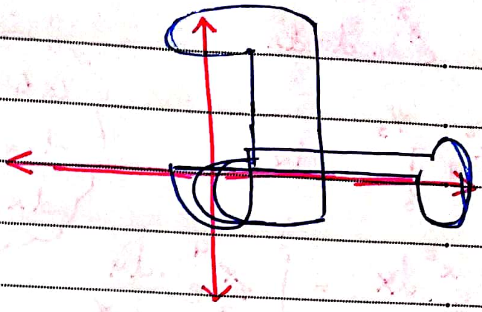
$= \int_{-2}^{2} \int_{x^2}^{4} x^2\,dy\,dx$

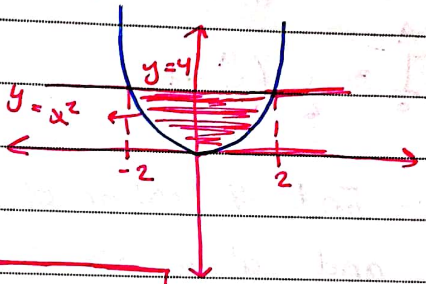
intersection with xy-Plane

$0 = x^2 \rightarrow x = 0$

$y = x^2$

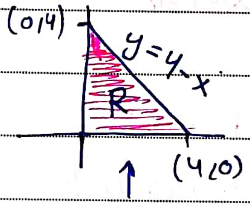
$y = 4$





Remark : $\iint_R 1 \, dA = $ area of R

example :- area of $R = \iint 1 \, dA$.



$= \int_0^4 \left[\int_0^{4-x} 1 \, dy \right] dx$

area of R = $\frac{1}{2}(4)(4) = 8$.

$= \int_0^4 y \Big]_0^{4-x} \cdot dx \quad = \int_0^4 (4-x)\, dx = 4x - \frac{x^2}{2} \Big]_0^4 = 16 - 8 = 8.$

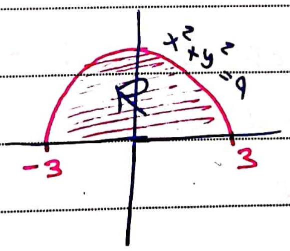
example: evaluate $\int_{-3}^{3} \int_0^{\sqrt{4-x^2}} 1 \, dy \, dx$.

$= \iint_R 1 \, dA$

= area of R

$= \frac{1}{2} \pi (3^2)$

$= \frac{4}{2} \pi$



$-3 \le x \le 3$

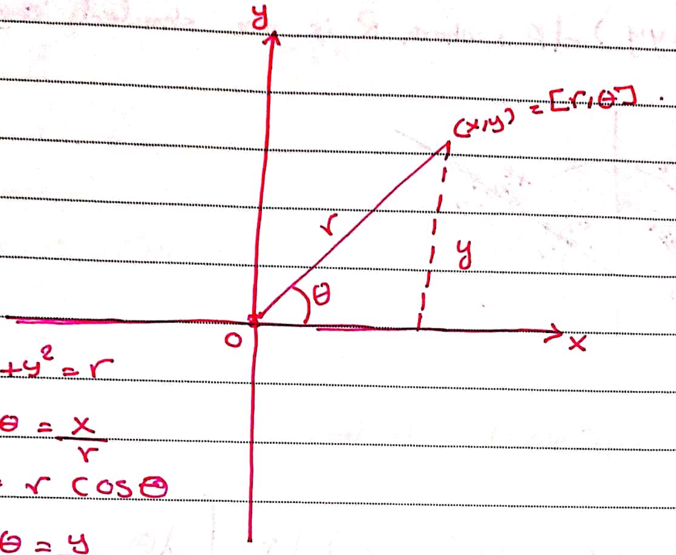
$0 \le y \le \sqrt{9-x^2}$

$y = \sqrt{9-x^2}$

$y^2 = 9 - x^2$

$x^2 + y^2 = 9$

15.4 : Polar Coordinate.



$x^2+y^2=r$

$\cos\theta = \frac{x}{r}$

$x = r\cos\theta$

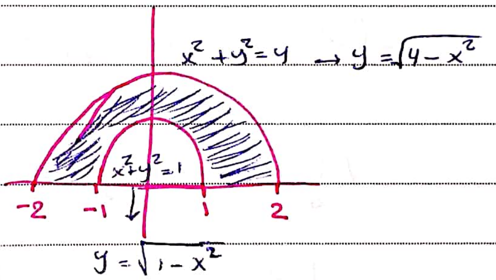
$\sin\theta = \frac{y}{r}$

$y = r\sin\theta$

$\tan\theta = \frac{y}{x}$

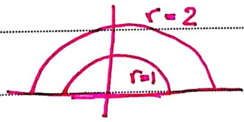
example:- $\iint_R f(x,y)\cdot dA$

$$\int_{-2}^{2}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$



R can be represenent

in Polar

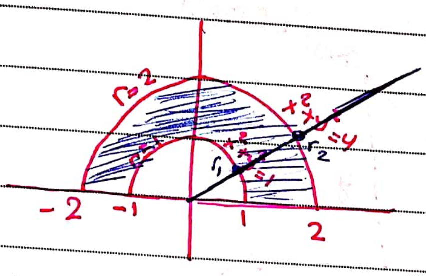


$R = \{(r,\theta) : 1\leq r\leq 2,\ 0\leq\theta\leq\pi\}$

* Theorem :-

$$\iint_R f(x,y)\cdot da = \int_{\alpha}^{\beta}\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\underbrace{r}\,dr\,d\theta$$

example:- evaluate $\iint_R (3x+4y)\,dA$, where R is the shaded region.



using Rolar Coordinate.

$$\iint_R (3x+4y)A = \int_0^{\pi}\int_1^{1}(3r\cos\theta + 4r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\left[\int_1^{2}(3r^2\cos\theta + 4r^2\sin\theta)\,dr\right]d\theta$$

$$= \int_0^{\pi} r^3\cos\theta + \frac{r}{3}r^2\sin\theta\Big]_1^2 \,d\theta$$

$$= \int_0^{\pi} r^3(\cos\theta + \frac{4}{3}\sin\theta)$$

$$= 7\int_0^{\pi}(\cos\theta + \frac{4}{3}\sin\theta)\,.d\theta$$

$$= 7\left[\sin\theta - \frac{4}{3}\cos\theta\right]_0^{\pi}$$

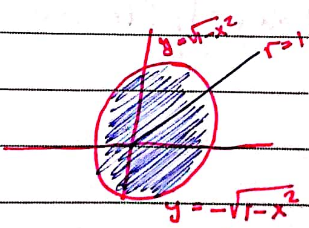
$$= 7\left[\frac{4}{3} + \frac{4}{3}\right]$$

$$= 7\,.\frac{8}{3}$$

* Change the integral from cartesion to Polar coordinate.

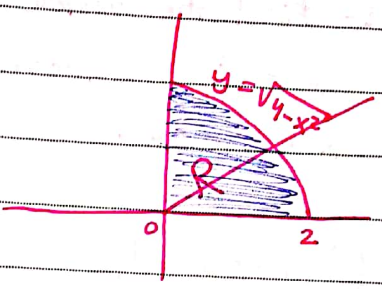
① $\int_{-1}^{1}\int_{-\sqrt{1-X^2}}^{\sqrt{1-X^2}} f(x,y)\,dy\,dx$

$x = \sqrt{1-y^2}$
$x^2 = 1 - y^2$
$x^2 + y^2 = 1$



$$= \int_0^{2\pi}\int_0^{1} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

② $\int_0^2 \int_0^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$



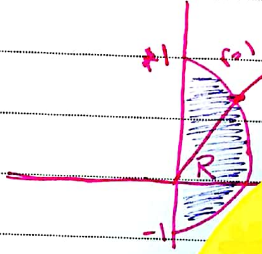
$y = \sqrt{4-x^2}$

$y^2 = 4-x^2$

$x^2+y^2=4 \rightarrow r^2=4 \rightarrow r=2.$

$= \int_0^{\pi/2} \int_0^2 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

③ $\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$



$x = \sqrt{1-y^2}$

$x^2 = 1-y^2$

$x^2+y^2=1$

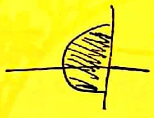
عَشان حل السؤال بطريقة سهلة

أكبر θ في الرسمة

$= \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^1 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$

← أقل θ في الرسمة

لو كانت الرسمة

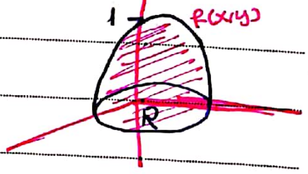


ستكون حدود θ

بالتالي من $\frac{\pi}{2}$ الى $\frac{3\pi}{3}$

example :- find the volume of the solid bounded by the Plane $z=0$ and the paraboloid $z=1-x^2-y^2$.

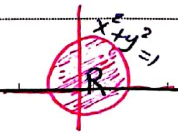
Volume $= \iint_R f(x,y)\,dA$



the Projection of Paraboloid on the xy-Plane is when $z=0$

$z = 1-x^2-y^2$

$0 = 1-x^2-y^2$



$x^2+y^2=1$

Volume $= \int_0^{2\pi} \int_0^1 (1-r^2)\,r\,dr\,d\theta$

* example :- Find the volume of the solid lies under the paraboloid $z = x^2+y^2$ above the xy-plane and inside the cylender $x^2+y^2 = 2x$.

$$\text{volume} = \iint_R f(x,y)\, dA$$

$f(x,y) = x^2+y^2$

the Projection of the cylender $x^2+y^2 = 2x$ on the xy-plane, $z=0$, given $x^2+y^2 = 2x$

~~ملاحظة~~

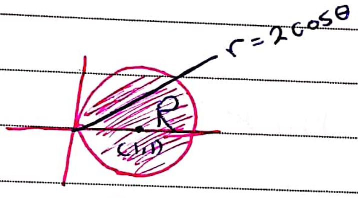
* يعني في أسطوانة ولما أعتبر x=0

ح يدخل منها بس الدائرة .

$x^2 - 2x + y^2 = 0$

$\underbrace{x^2 - 2x + 1}\, - 1 + y^2 = 0$

$(x-1)^2 + y^2 = 1$



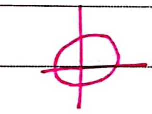
$$\text{Volume} = \int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2\, r\, dr\, d\theta$$

$x^2+y^2 = 2x$

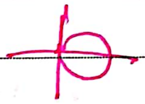
$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

ملاحظة :-



الدائرة يلي مركزها ع نقطة الأصل وباخدها كاملة $\left[\overset{0}{\cancel{}} , \frac{\pi}{2}\right]$.



الدائرة $\left[\frac{-\pi}{2} , \frac{\pi}{2}\right]$

(15.7) triple Integrals.

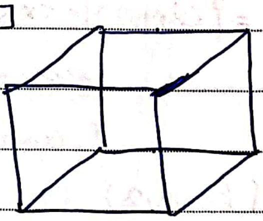
$$\iiint_E f(x,y,z)\,dV$$

E is a bounded region is three space R^3 [solid].

when E is a rectangular box, $E=\{(x,y,z)\,|\,a\le x\le b\ ,\ c\le y\le d\ ,\ r\le z\le s\}$

$= [a,b]\times[c,d]\times[r,s]$

$$\iiint_E f(x,y,z)\,dV = \int_a^b\int_c^d\int_r^s f(x,y,z)\,dz\,dy\,dx$$



{ عَين ترتيب هنا التكامل بـ (6) طُرق

وكله بيعطيني نفس الناتج . }

example: evalute $\iiint_E xyz^2\,dV$, where E is the rectangular box

$E=\{(x,y,z)\,|\,0\le x\le 1\ ,\ -1\le y\le 2\ ,\ 0\le z\le 3\}$.

$$\iiint_E xyz^2\,dV = \int_0^3\int_{-1}^2\left[\int_0^1 xyz^2\,dx\right]dy\,dz$$

$$= \int_0^2\int_{-1}^2 \frac{x^2}{2}\,z^2\Big]_0^1\,dy\,dz$$

$$= \frac{1}{2}\int_0^3\left[\int_{-1}^2 yz^2\,dy\right]dz$$

$$= \frac{1}{2}\int_0^3 \frac{y^2}{2}\,z^2\Big]_{-1}^2\,dz$$

$$= \frac{3}{4}\int_0^3 z^2\,dz$$

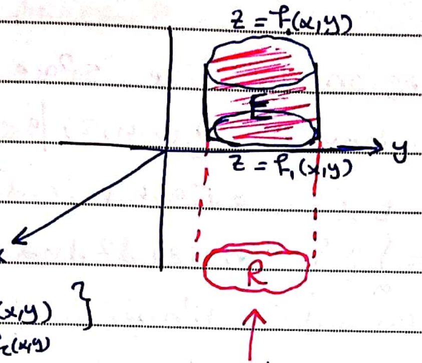
$$= \frac{3}{4} * \frac{z^3}{3}\Big]_0^3$$

$$= \frac{z^3}{4}\Big]_0^3$$

$$= \frac{27}{4}$$

Now, consider $\iiint_E f(x,y,z)\,dV$, where E is a bounded general region

*E is type 1 region.



$E = \{(x,y,z) | a \leq x \leq b ,$

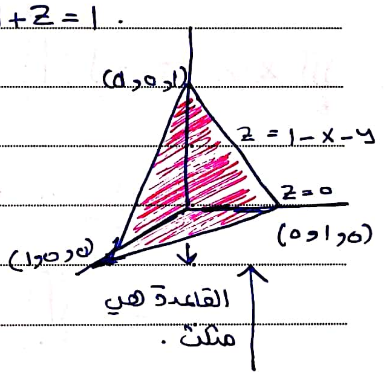
$g_1(x) \leq y \leq g_2(x)$

$f_1(x,y) \leq z \leq f_2(x,y) \}$

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz\,dy\,dx .$$

*example:- evaluate $\iiint_E z \cdot dV$, where E is the solid tetrahedron bounded by $x=0, y=0, z=0, x+y+z=1$.

منشور



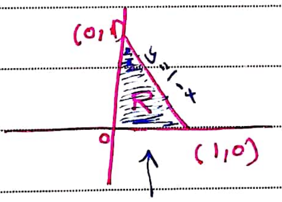
القاعدة هي مثلث .

• So it's clear that $0 \leq z \leq 1-x-y$.

* Projection of E on the xy-plane

When $z=0$, $x+y=1$

$y = 1-x$



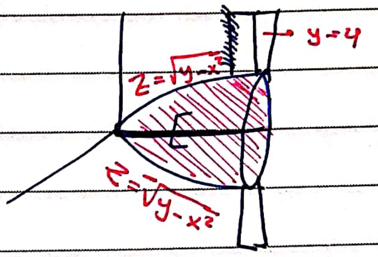
$\iiint_E z \cdot dV$

$$= \iint_R \int_0^{1-x-y} z \cdot dz \cdot dA$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

: $\frac{1}{24}$ الجواب النهائي .

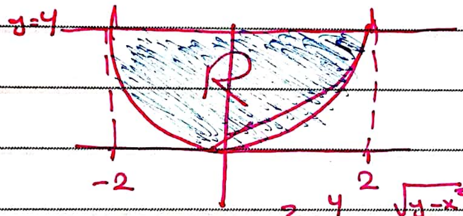
Example :- evaluate $\iiint_E \sqrt{x^2+z^2}\, dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the Plane $y = 4$.



Considering E as a type 1 region

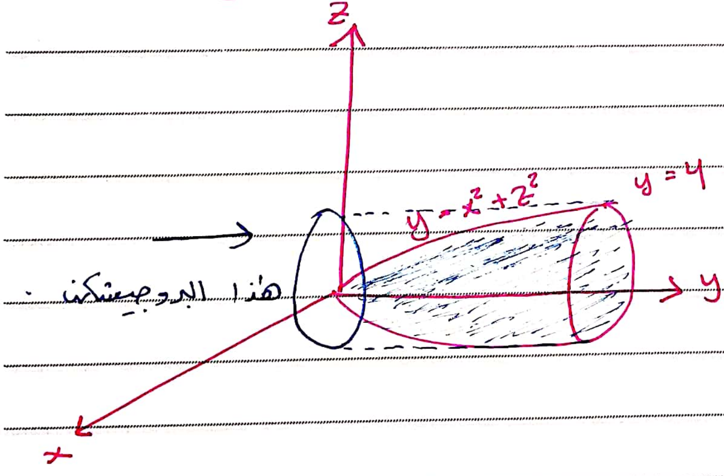
$-\sqrt{y-x^2} \leq z \leq \sqrt{y-x^2}$

the Projection of E on to the xy-Plane $z = 0$, $y = x^2$, $y = 4$

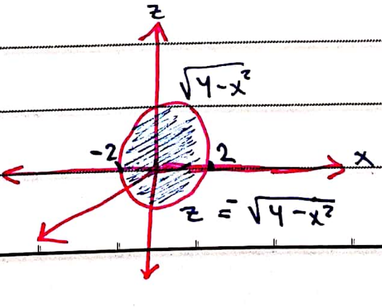


$$\iiint_E \sqrt{x^2+y^2}\, dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\, dz\, dy\, dx .$$

Considering E as ~~a~~ type II region.



The Projection of E onto the ~~xz~~ xy_plane y=0 .



يتبع ...

$$\iiint_E \sqrt{x^2+z^2}\, dV$$

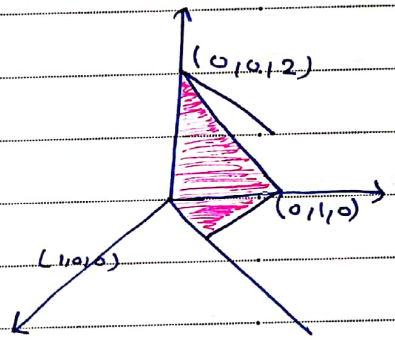
$$= \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r^2 \, dy\, dr\, d\theta$$

as an application of the triple integral, we can use it to evaluate volume .

$$V(E) = \iiint_E 1 \,.\, dV$$

example:- use a triple integral to find the volume of the tetrahedron T bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$, $z=0$?

$$0 \leq t \leq 2-x-2y$$



Projection of T.

onto the xy- plane ,

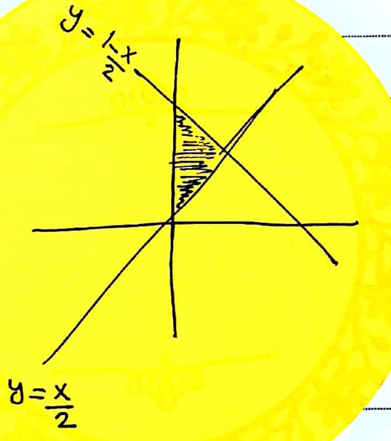
$z=0$,

$x+2y=2$, $x=2y$

$$y = \frac{2-x}{2} = 1-\frac{x}{2} \quad , \quad y = \frac{x}{2} .$$

$$1-\frac{x}{2} = \frac{x}{2}$$

$$x = 1$$



$$V = \iiint_E 1 \,.\, dV$$

$$V = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} \int_0^{2-x-2y} 1 \quad dz \,.\, dy \,.\, dx$$

dy

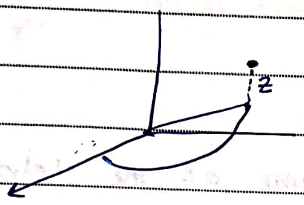
اكملوا الحل

$$= \frac{1}{3} .$$

(15 : 8) cylindrical coordinates.

the cylendrical coordinates uses

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$



Now that ,

$x^2 + y^2 = r^2$, $\tan\theta = \frac{y}{x}$

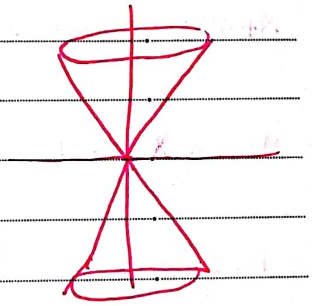
$z = z$ لا يوجد علاقة

example:- Describe the surface in cylendrical coordinate $z = r$.

$z = r$

$z^2 = r^2$

$z^2 = x^2 + y^2$



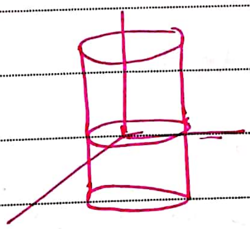
it is a cone .

* the surface $r = 2$ in clyendrical coordinate is :-

$r^2 = 4$

$x^2 + y^2 = 4$

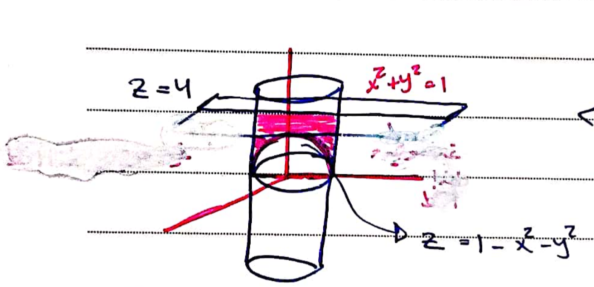
is a circular cylender



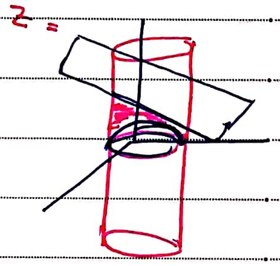
*theorem :-

$$\iiint_E f(x,y,z)\,dv = \int_{\alpha}^{\beta}\int_{r_1(\theta)}^{r_2(\theta)}\int_{z=f_1(r\cos\theta,r\sin\theta)}^{z=f_2(r\cos\theta,r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta .$$

*example :- A solid E line with in the cylender $x^2+y^2=1$ below the Plane $z=4$ and obove the Parabolid $z=1-x^2-y^2$, Find the volume of E?

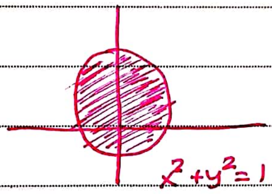


⟸ توضيح الرسمة



$$V(E) = \iiint_E 1\,dv = \int_0^{2\pi}\int_0^1\int_{1-r^2}^{4} 1\,r\,dz\,dr\,d\theta$$

using cylendrical coordinates, the Projection of E onto xy-Plane



$x^2+y^2=1$

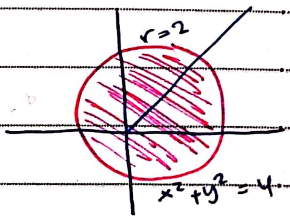
$r^2=1$

$r=1$

example :- evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$.

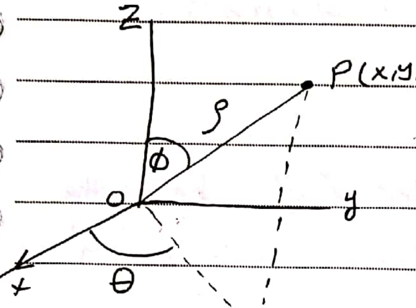
$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2 \cdot r\,dz \cdot dr \cdot d\theta$$

$$= \frac{16}{5}\pi .$$



(15.9) Spherical coordinate

the spherical coordinate (ρ, θ, ϕ) of a Point P in space.



ϕ : هي الزاوية التي بيصنعها (OP) مع محور (Z) الموجب .

$\rho \geq 0$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

$x = r\cos\theta$, $y = r\sin\theta$

$z = \rho\cos\phi$, $r = \rho\sin\phi$

$x = \rho\sin\phi.\cos\theta$

$y = \rho\ \sin\phi.\sin\theta$

$z = \rho\ \cos\phi$

$\rho^2 = x^2 + y^2 + z^2$.

example :- Describe the surface whose equation in spherical coordinates :-

① $\rho = 9$

$\rho^2 = 81$

$x^2 + y^2 + z^2 = 9^2$

a sphere with center (0,0,0) with raduis 9 .

$\rho =$ عدد [sphere دائماً]

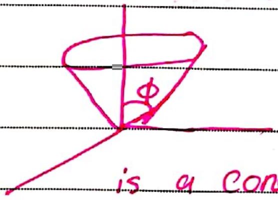
② $\phi = \frac{\pi}{3}$

[illegible]

لو تخيلنا الرسمة و لف حول Z

مع الحفاظ على الزاوية $\frac{\pi}{3}$.

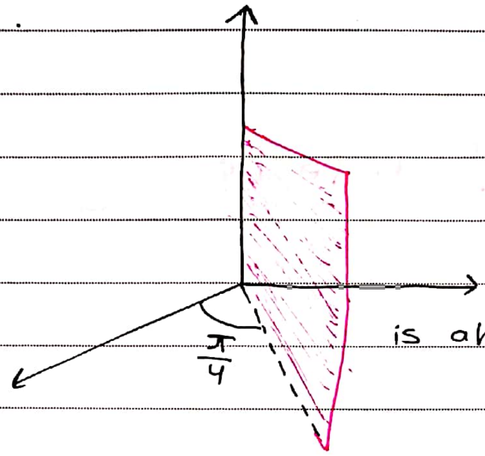
بيكون الشكل :-



is a cone .

$\phi =$ عدد [cone دائماً]

③ $\theta = \frac{\pi}{4}$.



بيكون عبارة عن plane
خارج من الورقة

is a half plane .

~~example~~ * evaluating triple integrals with spherical coordinates.

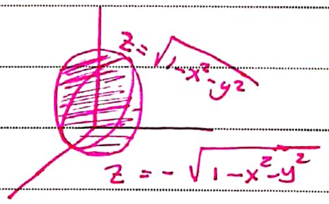
$\iiint_E f(x,y,z) \cdot dV$

$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta \, , \rho \sin\phi \overset{\sin}{\cancel{\cos}}\theta \, , \rho \cos\phi) \underbrace{\rho^2 \sin\phi}$

لا تحول
من xyz
إلى spherical

example :- evaluate $I = \iiint_E e^{(x^2+y^2+z^2)^{3/2}} \cdot dV$, where E is the unit ball $x^2+y^2+z^2 \leq 1$.

يعني كرة نصف قطرها (1)



$I = \iiint_E e^{(x^2+y^2+z^2)^{3/2}} \cdot dV$

$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z)^{3/2}} \, dz \, dy \, dx$.

سيكون التكامل صعب .

$x^2+y^2+z^2 \leq 1$

$\rho^2 \leq 1$

$0 \leq \rho \leq 1$

~~e~~ $0<\phi<\pi$

$0<\theta<2\pi$

$$I=\int_0^{\pi}\int_0^{2\pi}\int_0^{1}e^{\rho^3}\rho^2\sin\phi\; d\rho\, d\theta\, d\phi ,$$

تكامل كل شي لحاله:

$$* \int_0^1 e^{\rho^3}\rho^2\, d\rho$$

تستخدم الفرض

$u=\rho^2$

$du=3\rho^2\, d\rho$

$d\rho=\frac{du}{3\rho^2}$

$\int_0^1 e^u\cdot\rho^2\,\frac{du}{3\rho^2}$

$\frac{1}{3}e^u\Big]_0^1$

$=\frac{e-1}{3}$.

$$* \frac{e-1}{3}\int_0^{\pi}\int_0^{2\pi}\sin\phi\,.d\theta.d\phi$$

$$=\frac{2\pi(e-1)}{3}\int_0^{\pi}\sin\phi\,.d\phi$$

$$=\frac{4}{3}\pi(e-1) .$$

example :- Describe the surface whose equation in spherical Coordinates is :-

① $\rho = \sin\phi . \sin\theta$.

$\rho^2 = \rho \sin\phi \sin\theta$

$x^2+y^2+z^2 = y$

$x^2+y^2-y+z^2=0$

$x^2+y^2-y+\frac{1}{4}-\frac{1}{4}+z^2=0$

$x^2+(y-\frac{1}{2})^2+z^2=\frac{1}{4}$.

sphere of center $(0,\frac{1}{2},0)$ and raduis $\frac{1}{2}$.

$x = \rho\sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\theta$

$x^2+y^2+z^2 = \rho^2$

~~② $z = \sqrt{x^2+y^2}$~~

example :- convert this equation to spherical cordindte $z = \sqrt{x^2+y^2}$.

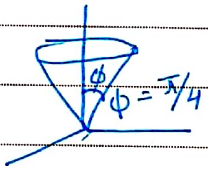
$z^2 = x^2+y^2$

$\rho^2 \cos^2\phi = \rho^2 \sin^2\phi . \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta$

$\rho^2 \cos^2\phi = \rho^2 \sin^2\phi$

$\cos\phi = \sin\phi$

$\phi = \frac{\pi}{4}$.



$\downarrow$

ما شِلت التربيع إلا لما تأكدت إنه $0-\pi = \phi$

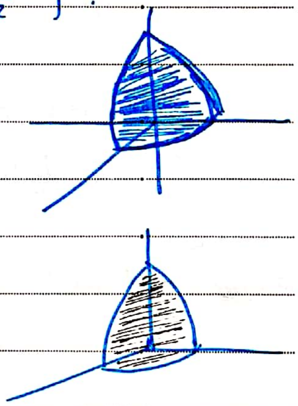
إذن شِلت التربيع وأنا متأكدة إنه الجواب

متساوي ولا يوجد عامل.

*example :- sketch the solid whose volume is given by the integral $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$.

volume $= \iiint_E 1 \, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$.

$E = \{ (\rho,\theta,\phi) \mid 0 \leq \rho \leq 1 , 0 \leq \theta \leq \frac{\pi}{2} , 0 \leq \phi \leq \frac{\pi}{2} \}$.



E is the solid bounded by the sphere $x^2+y^2+z^2=1$

and coordinate axis in the first octant

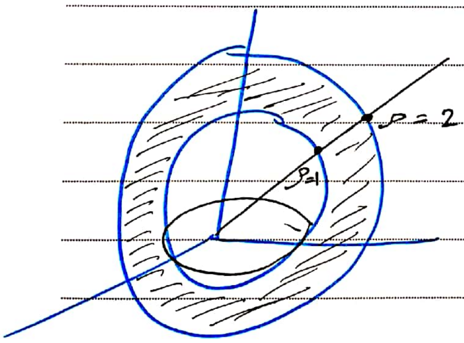
الثُمن الأول

*example :- evaluate $\iiint_E x \, e^{(x^2+y^2+z^2)^2} . dV$

, where E is the solid line between the sphere $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$.

the solid are spheres , so we ~~will~~ use spherical coordinate.

$E = \{ (\rho,\theta,\phi) \mid 1 \leq \rho \leq 2 , 0 \leq \theta \leq 2\pi , 0 \leq \phi \leq \pi \}$.



يتبع ...

$$\iiint_E x\, e^{(x^2+y^2+z^2)^2}\, dV$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_1^{2} \rho \sin\phi \cdot \cos\theta \; e^{\rho^4}\; \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi .$$

$$= \left(\int_0^{\pi} \sin^2\phi \, d\phi\right)\left(\int_0^{2\pi} \cos\theta \, d\theta\right)\left(\int_1^{2} e^{\rho^4} \rho^3 \, d\rho\right)$$

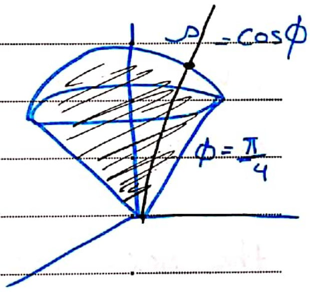
نستخدم المتطابقة

$\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$

نعوض $u = \rho^4$

example (مهم) :- use spherical coordinate to find the volume of the solid lies the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.

(ice cream cone) اسمه



* let us first convert the surfaces to spherical

$z = \sqrt{x^2+y^2}$

$z^2 = x^2+y^2$

$\rho^2 \cos^2\phi = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta .$

$\rho^2 \cos^2\phi = \rho^2 \sin^2\phi .$

$\cos\phi = \sin\phi$

$\phi = \frac{\pi}{4} .$

Now, $x^2+y^2+z^2 = z$

$\rho^2 = \rho \cos\phi$

$\rho = \cos\phi$

$$V = \iiint_E 1 \cdot dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos\phi} \rho^2 . \sin\phi \, . d\rho . d\theta \, d\phi \, .$$

$$= \cancel{\frac{2\pi}{8}} \qquad = \frac{\pi}{8} \, .$$

Nataly Al-Fayed ...