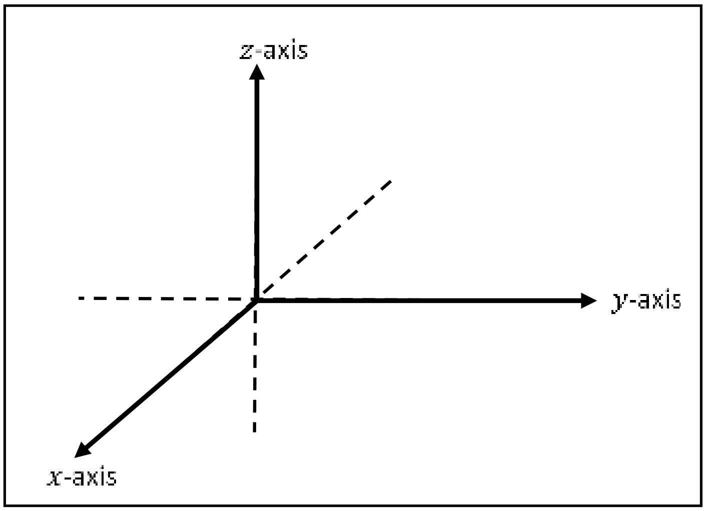
Chapter 12

Vectors and the Geometry of Space

Section 12.1

Three –Dimensional Coordinate Systems

12.1 Three –Dimensional Coordinate Systems



Definition 1:

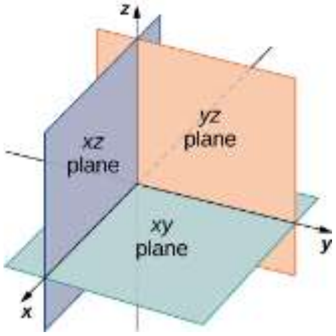
- ❖ The space can be represented by sketching thee perpendicular axes called: the x-axis, the y-axis, and the z-axis that are intersected at a point called the origin $\mathbf{0}$.
- ❖ These axes (x-axis, y-axis, z-axis) are called the coordinate axes (المحاور الاحداثية).
- ❖ The plane that contains the:

x-axis and y-axis is called the xy-plane

x-axis and z-axis is called the xz-plane

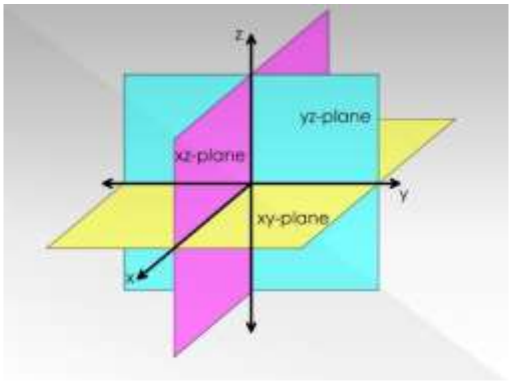
y-axis and z-axis is called the yz-plane

These planes are called the coordinate planes



Remark 2:

- ❖ We have 3 coordinate axes: x-axis, y-axis, z-axis
- ❖ We have 3 coordinate planes: xy-plane, xz-plane, yz-plane.
- ❖ The coordinate planes divide the space into 8 parts. Each part is called an octant. The first octant is the part that contains the positive parts of the coordinate axes.



Remark 3:

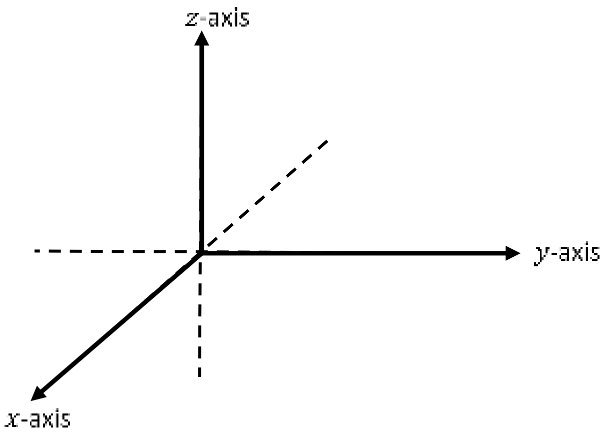
- A point (pt) P in the space is represented as $P(a, b, c)$, where:

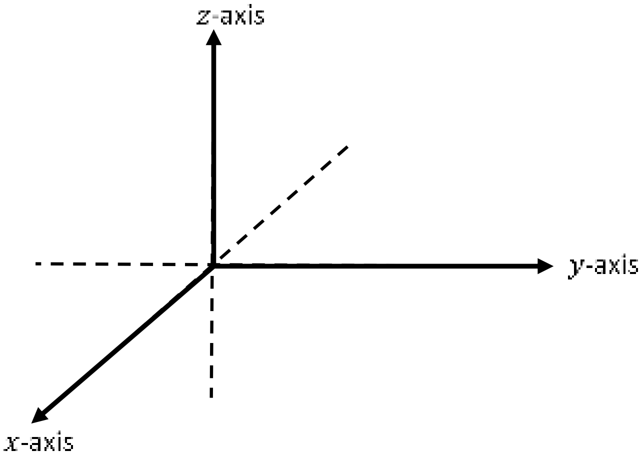
  $a = x$-coordinate of P, $b = y$-coordinate of P, $c = z$-coordinate of P

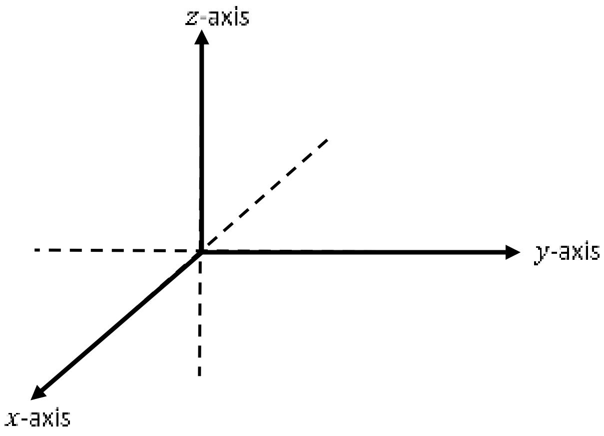
- The set of all numbers is $\mathbb{R} = (-\infty, \infty)$.
- The Cartesian product $\mathbb{R} \times \mathbb{R} = \{(x, y): x, y \in \mathbb{R}\}$ is called the 2-dimensional (2D or the plane) rectangular coordinate system. $\mathbb{R} \times \mathbb{R}$ is written as $\mathbb{R}^2$
- The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z): x, y, z \in \mathbb{R}\}$ is called the 3-dimensional (3D or space) rectangular coordinate system. $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is written as $\mathbb{R}^3$.

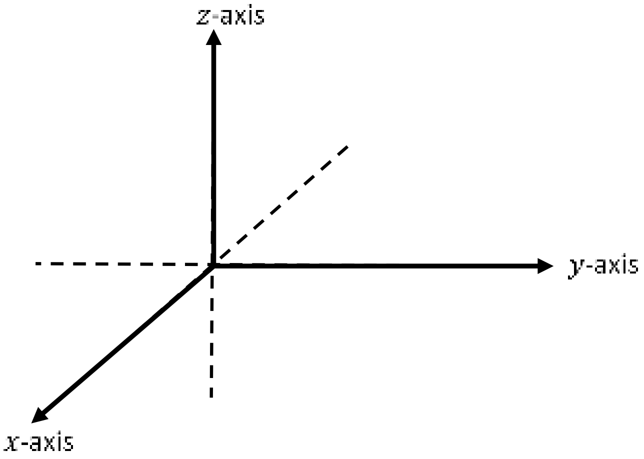
Example 4: Plot the following points in the space: $\boldsymbol{O}(0,0,0), A(1,0,0), B(0,2,0), C(0,0,3),$ $D(1,2,0), E(1,0,3), F(0,2,3),\ G(1,2,3), H(-1,2,3), I(1,-2,3), J(1,2,-3)$

Solution:









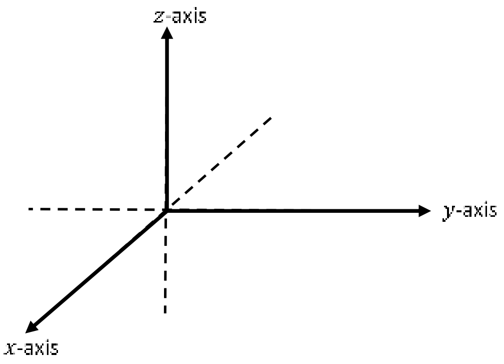
Remark 5:

- The graph of an equation in 2D (the plane $\mathbb{R}^2$) is a curve, for example if the equation $y = x^2$ is in the plane, then its graph is a curve.
- The graph of an equation in 3D (the space $\mathbb{R}^3$) is a surface, for example if the equation $y = x^2$ is in the space, then its graph is a surface.

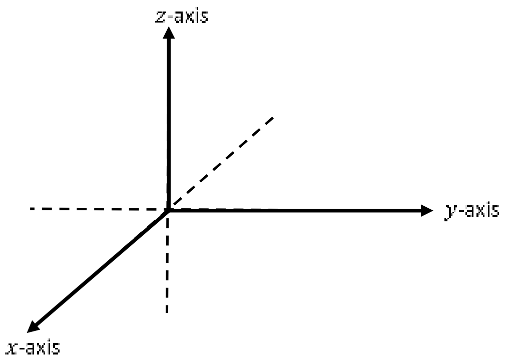
Example 6: Sketch the graph of the surface whose equation is given by:

(1) $y = x^2$ (2) $z = y$ (3) $z = 3$

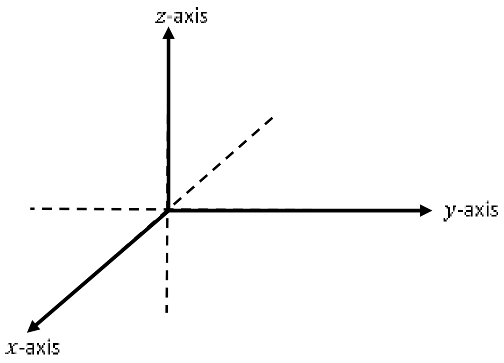
Solution: (1)



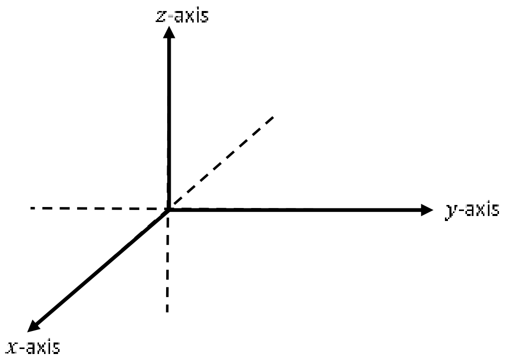
$\Rightarrow$



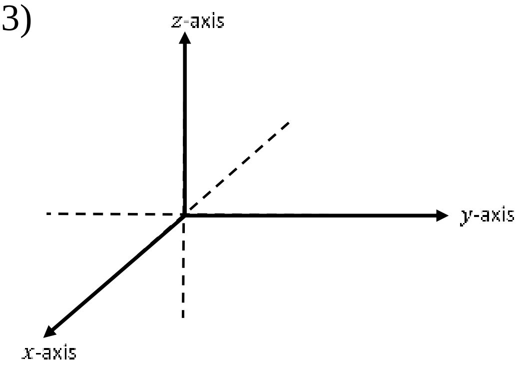
(2)



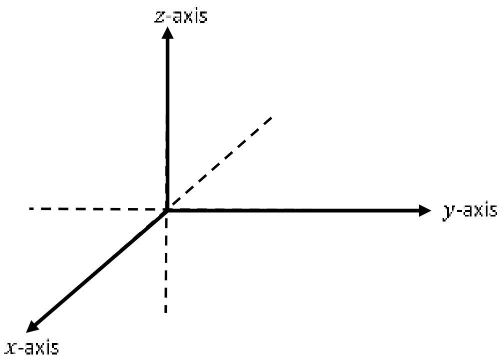
$\Rightarrow$



(3)



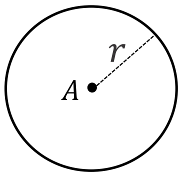
$\Rightarrow$



Remark 7:

(1) The equation of the xy-plane is $z = 0$
(2) The equation of the xz-plane is $y = 0$
(3) The equation of the yz-plane is $x = 0$
(4) In the plane, the equation of the circle centered at the pt. $A(a, b)$ of radius r is:

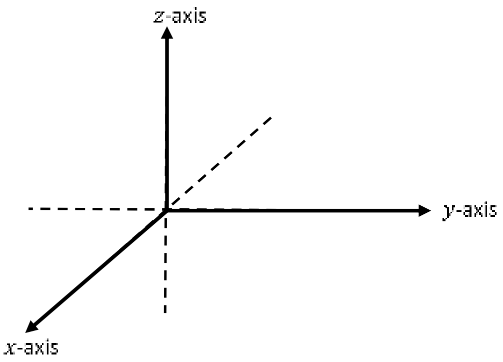
$$(x - a)^2 + (y - b)^2 = r^2$$



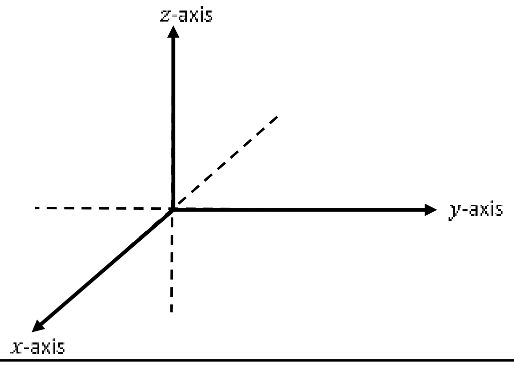
Example 8:

(1) Identify and sketch the graph of the equation $x^2 + y^2 = 4$ in $\mathbb{R}^3$.
(2) Which pts. (x, y, z) satisfy the equations $x^2 + y^2 = 4, z = 3$ in $\mathbb{R}^3$

Solution: (1)

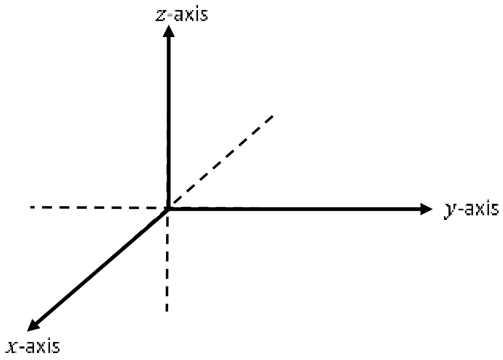


⇒

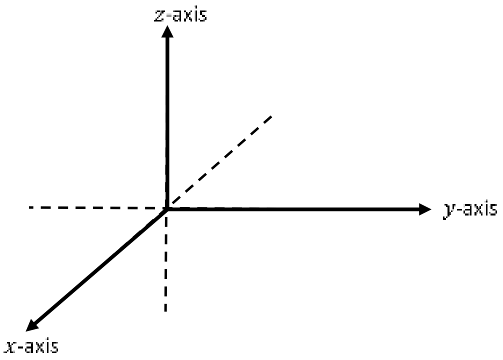


Cylinder of radius 2 with the z-axis as the axis of symmetry

(2)



⇒



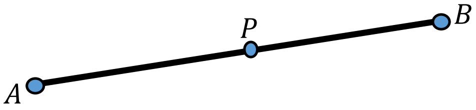
circle of radius 2 centered at the pt. (0,0,3) parallel to the xy-plane

Rule 9: Let $A(a,b,c)$ and $B(d,e,f)$ be two pts in $\mathbb{R}^3$.

(1) The distance between the pts A and B is $|AB| = \sqrt{(a-d)^2 + (b-e)^2 + (c-f)^2}$

(2) The midpoint (midpt.) of the **line segment** (قطعة مستقيمة) joining A and B is:

$$P\left(\frac{a+d}{2}, \frac{b+e}{2}, \frac{c+f}{2}\right)$$



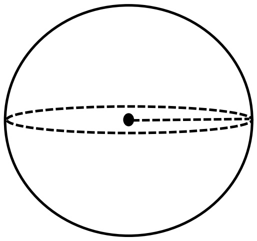
Example 10: Find the distance from the pt. $P(2,-1,4)$ to the pt. $Q(-2,0,1)$ and find the midpt. of the line segment joining P and Q.

Solution: The distance is $\text{dist}(P,Q) = \sqrt{(2-(-2))^2 + (-1-0)^2 + (4-0)^2} = \sqrt{33}$

The midpt. is $\left(\frac{2+(-2)}{2}, \frac{-1+0}{2}, \frac{4+1}{2}\right) = \left(0, \frac{-1}{2}, \frac{5}{2}\right)$

Rule 11: The standard form of the equation of the sphere centere the pt. $A(a,b,c)$ of radius r is:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$



When the center is the origin and the radius is 1, then the sphere

$x^2 + y^2 + z^2 = 1$ is called the unit sphere

Example 12: Which of the following is an equation of a sphere and write it in standard form and find its center and radius.

(1) $2x^2 - 12x + 3y^2 + 2z^2 + 8z = 1$
(2) $2x^2 - 12x + 2y^2 + 2z^2 + 8z = -30$
(3) $x^2 - 6x + y^2 + z^2 + 4z = -13$
(4) $2x^2 - 12x + 2y^2 + 2z^2 + 8z = 6$

Solution:

(1) The equation is not for a sphere.

(2) $2x^2 - 12x + 2y^2 + 2z^2 + 8z = -30 \Rightarrow x^2 - 6x + y^2 + z^2 + 4z = -15 \Rightarrow$

$$\underbrace{x^2 - 6x + 3^2 + y^2 + z^2 + 4z + 2^2 = -15 + 3^2 + 2^2}_{\text{اكمال المربع}}$$

$\Rightarrow (x-3)^2 + y^2 + (z+2)^2 = -2$ which is impossible.

The equation is not for any surface $\Rightarrow$ The equation is not for a sphere.

(3) $x^2 - 6x + y^2 + z^2 + 4z = -13 \Rightarrow$

$$\underbrace{x^2 - 6x + 3^2 + y^2 + z^2 + 4z + 2^2 = -13 + 3^2 + 2^2}_{\text{اكمال المربع}}$$

$\Rightarrow\ (x-3)^2 + y^2 + (z+2)^2 = 0 \Rightarrow x = 3, y = 0, z = -2$ which is the point $(3,0,-2)$

The equation is not for a sphere it is not a surface but it is the point $(3,0,-2)$

(4) $2x^2 - 12x + 2y^2 + 2z^2 + 8z = 6 \Rightarrow x^2 - 6x + y^2 + z^2 + 4z = 3 \Rightarrow$

$$\underbrace{x^2 - 6x + 3^2 + y^2 + z^2 + 4z + 2^2 = 3 + 3^2 + 2^2}_{\text{اكمال المربع}}$$

$\Rightarrow\ (x-3)^2 + y^2 + (z+2)^2 = 16$

The equation is for a sphere centered at the point $(3,0,-2)$ of radius 4

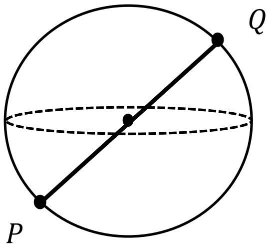
The standard form of the sphere is $(x-3)^2 + y^2 + (z+2)^2 = 16$

Example 13: Find the equation of the sphere centered at $A(0,-2,5)$ of radius $\sqrt{3}$

Solution: The equation is $x^2 + (y+2)^2 + (z-5)^2 = 3$

Example 14: Find the equation of the sphere if **one of its diameters** (أحد أقطارها) has end points $P(2,1,4)$ and $Q(2,-3,0)$.

Solution:



The radius is $r = \frac{1}{2}\text{dist}(P,Q)$

$= \frac{1}{2}\sqrt{(2-2)^2 + (-3-1)^2 + (0-4)^2} = \sqrt{32}$

The center is $midpt. = \left(\frac{2+2}{2}, \frac{1+(-3)}{2}, \frac{4+0}{2}\right) = (2,-1,2)$

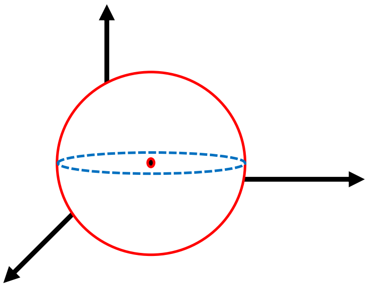
The equation is $(x-2)^2 + (y+1)^2 + (z-2)^2 = 32$

Example 15: Find the equation of the sphere in the first octant of radius 5 that touches the coordinate planes.

Solution: The center is $(5,5,5)$

$\Rightarrow$ The equation is:

$(x-5)^2 + (y-5)^2 + (z-5)^2 = 25$

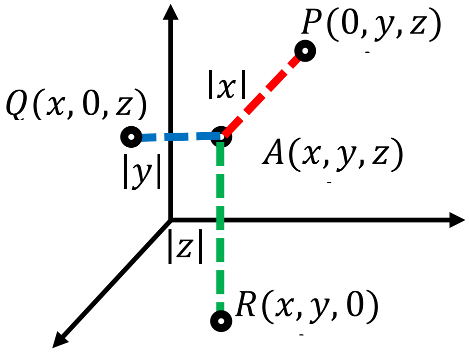


Rule 16: The distance from the pt. $A(x, y, z)$ to the:

(1) xy-plane is $|z|$

(2) xz-plane is $|y|$

(3) yz-plane is $|x|$



Example 17: Find the equation of the largest sphere in the first octant centered at the point $A(3,2,5)$.

Solution:

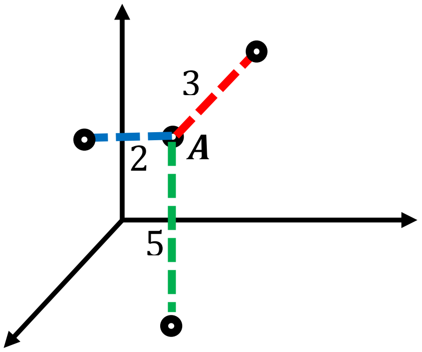
$D_1 = \text{Dist}(A, xy - \text{plane}) = 5$

$D_2 = \text{Dist}(A, xz - \text{plane}) = 2$

$D_3 = \text{Dist}(A, yz - \text{plane}) = 3$

The radius is $r = \min(D_1, D_2, D_3) = 2$

$\Rightarrow$ The equation is: $(x-3)^2 + (y-2)^2 + (z-5)^2 = 4$



Example 18: Find the equation of the sphere centered at $A(1, -2, -5)$ and touches the xz-plane.

Solution:

The radius is $r = \text{dist}(A, xz - \text{plane}) = |-2| = 2$

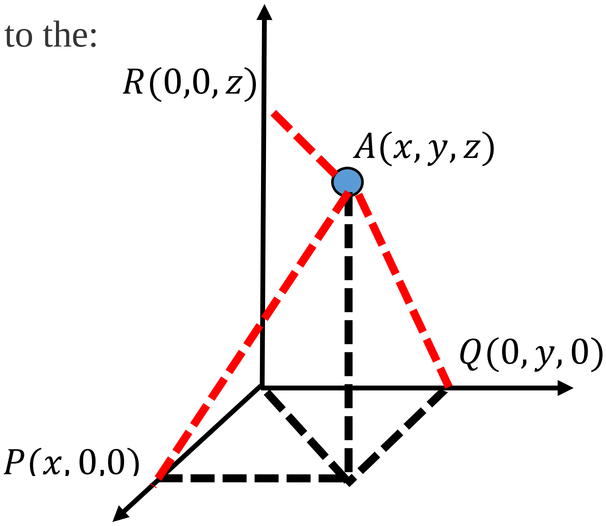
The equation is $(x-1)^2 + (y+2)^2 + (z+5)^2 = 4$

Rule 19: The distance from the pt. $A(x, y, z)$ to the:

(1) x-axis is $\sqrt{y^2 + z^2}$

(2) y-axis is $\sqrt{x^2 + z^2}$

(3) z-axis is $\sqrt{x^2 + y^2}$



Example 20: Find the distance from the pt. $A(1,4,-3)$ to the:

(1) x-axis (2) y-axis (3) z-axis

Solution:

(1) The distance $\text{dist}(A, x-axis) = \sqrt{4^2 + (-3)^2} = 5$

(2) The distance $\text{dist}(A, y-axis) = \sqrt{4^2 + (-3)^2} = \sqrt{10}$

(3) The distance $\text{dist}(A, z-axis) = \sqrt{1^2 + 4^2} = \sqrt{17}$

Example 21: What region in $\mathbb{R}^3$ is represented by the inequalities:

(1) $x^2 + y^2 + z^2 > 6z$ (2) $x^2 + y^2 + z^2 \le 6z$ (3) $y^2 \ge 1$

(4) $x \ge 1$ (5) $z < 0$ (6) $x^2 + z^2 < 4$

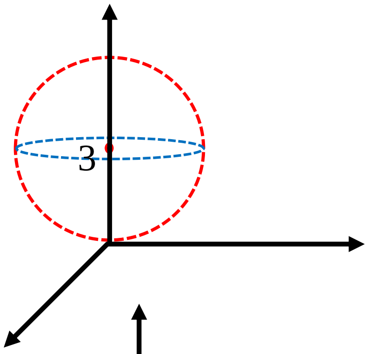
Solution:

(1) $x^2 + y^2 + z^2 > 6z: \Rightarrow\ x^2 + y^2 + z^2 = 6z$

$$\Rightarrow\ x^2 + y^2 + z^2 - 6z = 0$$

$$\Rightarrow\ x^2 + y^2 + \underbrace{z^2 - 6z + 9}_{\text{اكمال المربع}} = 0 + 9$$

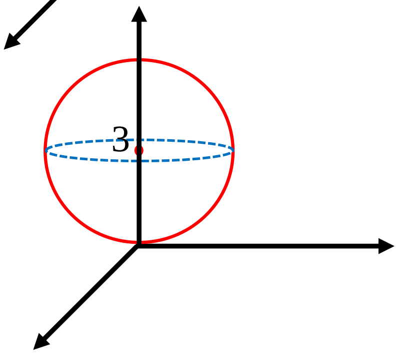
$$\Rightarrow\ x^2 + y^2 + (z-3)^2 = 9$$



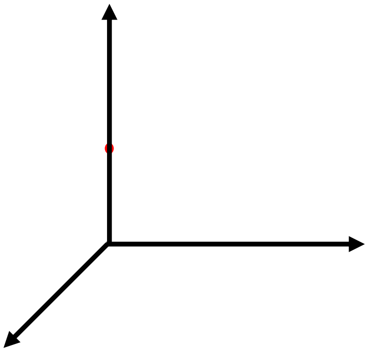
(2) $x^2 + y^2 + z^2 \le 6z: \Rightarrow\ x^2 + y^2 + z^2 = 6z$

$$\Rightarrow\ x^2 + y^2 + z^2 - 6z = 0$$

$$\Rightarrow\ x^2 + y^2 + \underbrace{z^2 - 6z + 9}_{\text{اكمال المربع}} = 0 + 9$$



(3) $y^2 \ge 1 \Rightarrow\ y^2 = 1\ \Rightarrow\ y = 1$ or $y = -1$



Chapter 12 Vectors and the Geometry of Space

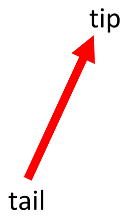
Section 12.2: Vectors

12.2: Vectors

Definition 1:

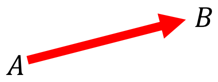
A vector $\vec{v}$ is a quantity that has both: $\underbrace{\text{magnitude}}_{①}$ (sometimes called length) written as $|\vec{v}|$ and $\underbrace{\text{direction}}_{②}$.

Remark 2:

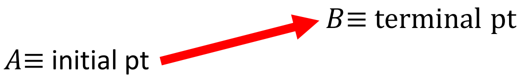


(1) A graph of a vector is given by a row:
   - ➢ The magnitude of a vector is the distance from its tail to its tip
   - ➢ The direction is indicated by the row.

(2) If we move from a pt. A to a pt. B, then the displacement vector (متجه الازاحة) $\vec{v}$ is given by $\vec{v} = \overrightarrow{AB}$. In this case $|\vec{v}| = \text{dist}(A, B)$



(3) When we write $\vec{v} = \overrightarrow{AB}$, then the point A is called the initial point and B is called the terminal point.



Remark 3: The zero vector $\vec{0}$ is be defined as a vector for which its initial and terminal points are the same

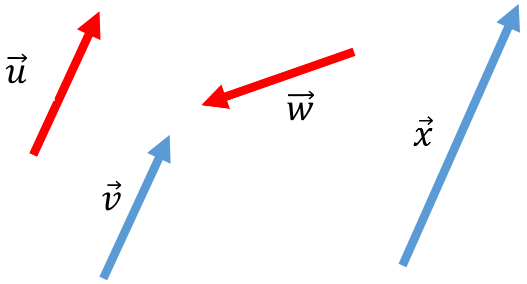
If $A, B,$ and C are pints, then $\vec{0} = \overrightarrow{AA} = \overrightarrow{BB} = \overrightarrow{CC} \Rightarrow \left|\overrightarrow{AA}\right| = \left|\overrightarrow{BB}\right| = \left|\overrightarrow{CC}\right| = 0$

Definition 4: The zero vector $\vec{0}$ is the vector of length 0 but in any direction

$$\Rightarrow \left|\vec{0}\right| = 0$$

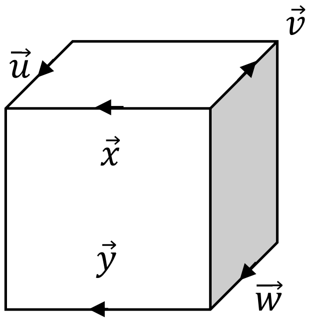
Definition 5: Two vectors $\vec{u}$ and $\vec{v}$ are equal, written as $\vec{u} = \vec{v}$, if they have the same magnitude and the same direction.

Example 6:



$\vec{u} = \vec{v}$ (the same length and the same direction)
$\vec{u} \neq \vec{w}$ (different directions)
$\vec{v} \neq \vec{x}$ (different length)

Example 7:



$\vec{u} = \vec{w}$ (the same length and the same direction)
$\vec{u} \neq \vec{v}$ (different directions)
$\vec{x} = y$ (the same length and the same direction)
$\vec{v} \neq \vec{x}$ (different directions)

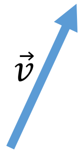
Definition 8: Let c be a scalar and $\vec{v}$ be a vector. Then $c\vec{v}$ is a vector of:

- ❖ $\underbrace{\text{length}}_{①}\ |c\vec{v}| = |c|\,|\vec{v}|$

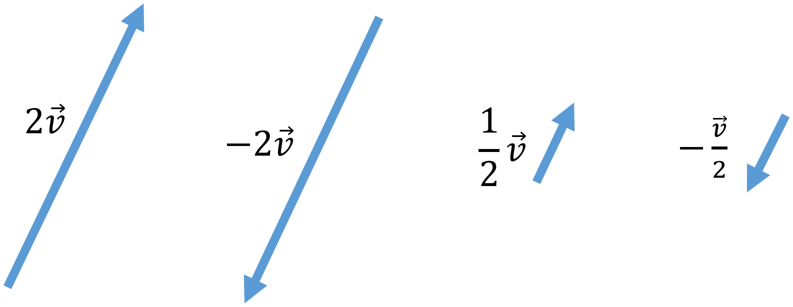
and its

- ❖ $\underbrace{\text{direction}}_{②}$ is: $\begin{cases} \text{in the same direction of } \vec{v}, \text{if } c > 0 \\ \text{in the opposite direction of } \vec{v}, \text{if } c < 0 \end{cases}$

Example 9: If $\vec{v}$ is the vector given in the figure with $|\vec{v}| = 3$. Plot the graph of the following vectors: $2\vec{v}$, $-2\vec{v}$, $\frac{1}{2}\vec{v}$, and $-\frac{\vec{v}}{2}$. Also, find the lengths of these vectors.

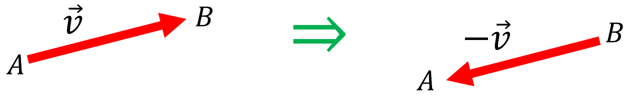


Solution:



$\Rightarrow |2\vec{v}| = 2|\vec{v}| = 6$, $|-2\vec{v}| = 2|\vec{v}| = 6$, $\left|\frac{1}{2}\vec{v}\right| = \frac{1}{2}|\vec{v}| = \frac{3}{2}$, and $\left|-\frac{\vec{v}}{2}\right| = \frac{1}{2}|\vec{v}| = \frac{3}{2}$

Remark 10: If $\vec{v} = \overrightarrow{AB}$, then $-\vec{v} = \overrightarrow{BA}$ that is $-\overrightarrow{AB} = \overrightarrow{BA}$. Also, $|\vec{v}| = |-\vec{v}|$

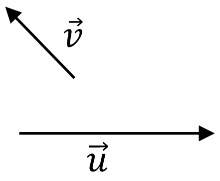


Rule 11: Let $\vec{u}$ and $\vec{v}$ be vectors in which the terminal point of $\vec{u}$ is the initial point of $\vec{v}$

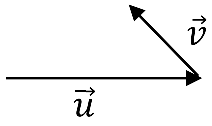
The sum of two vectors $\vec{u}$ and $\vec{v}$ written as $\vec{u} + \vec{v}$ is the vector with initial point as that of $\vec{u}$ and terminal point as that of $\vec{v}$, that is if $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{BC}$, then

$$\vec{u} + \vec{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

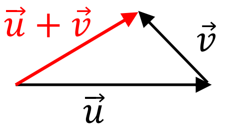
Remark 12: To plot the graph of



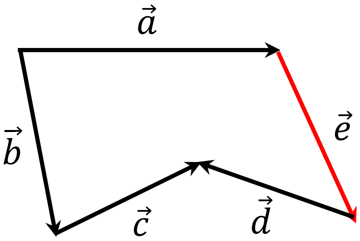
Step 1



Step 2



Example 13: Write the vector $\vec{e}$ as a sum of the vectors $\vec{a}, \vec{b}, \vec{c}$, and $\vec{d}$ given in the figure



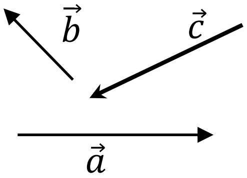
Solution: $\vec{e} = -\vec{a} + \vec{b} + \vec{c} - \vec{d}$

Example 14: Let A, B, and C be three points. Write $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC}$ in implicit form.

Solution: $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \overrightarrow{AC} - \overrightarrow{AC} = \overrightarrow{AC} + \left(-\overrightarrow{AC}\right) = \overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$

$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

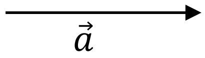
Example 15: Draw the vector $\vec{a} - 2\vec{b} - \frac{1}{2}\vec{c}$, where the vectors $\vec{a}, \vec{b}$, and $\vec{c}$ are given in the figure.



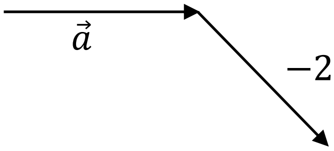
Solution: We deal with the vectors: $\vec{a},\ -2\vec{b},\ -\frac{1}{2}\vec{c}$:

Let $\vec{d} = \vec{a} - 2\vec{b} - \frac{1}{2}\vec{c}$

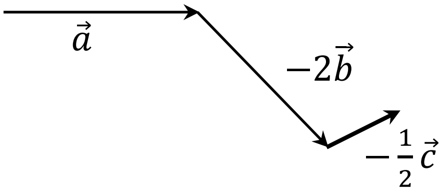
Step 1



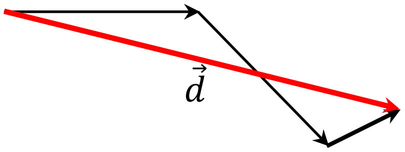
Step 2



Step 3



Step 4



Properties of Vectors 16: Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors and let c and d be salars. Then

(1) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ **(2)** $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$ **(3)** $\vec{u}-\vec{u}=\vec{0}$

(4) $(\vec{u}+\vec{v})+\vec{w}=\vec{v}+(\vec{u}+\vec{w})$ **(5)** $(c+d)\vec{u}=c\vec{u}+d\vec{u}$ **(6)** $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$

(7) $0\vec{u}=\vec{0}$

Component Form of Vectors:

Let $A(a,b,c)$ and $B(d,e,f)$ be points in $\mathbb{R}^3$. Then

$$\vec{v}=\overrightarrow{AB}=\langle B-A\rangle=\langle d-a,e-b,f-c\rangle$$

the numbers $-a,e-b$, and $f-c$ are called the components of $\vec{v}$

Let $P=(d-a,e-b,f-c)$ and $\boldsymbol{O}=(0,0,0)$. Then the position vector of $\vec{v}$ is $\vec{v}=\overrightarrow{\boldsymbol{O}P}$

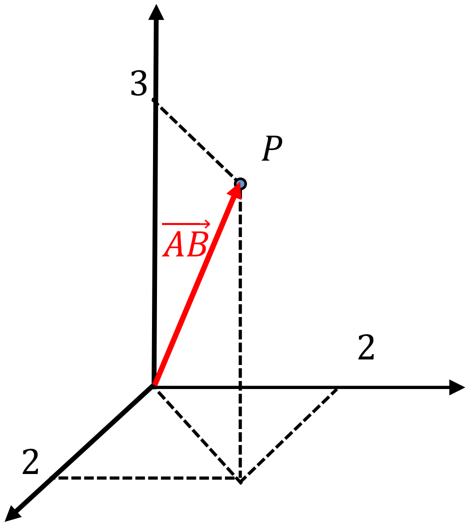
Example 17: Find and sketch the vector $\overrightarrow{AB}$ where $A(1,-1,-3)$ and $B(3,1,0)$

Solution:

$\overrightarrow{AB}=\langle B-A\rangle=\langle 3-1,1-(-1),0-(-2)\rangle=\langle 2,2,3\rangle$

To sketch $\overrightarrow{AB}$: we sketch it as a position vector

so let $P=(2,2,3)\Rightarrow\overrightarrow{AB}=\overrightarrow{\boldsymbol{O}P}$



Rule 18:

(1) Let $\vec{v}=\langle a,b,c\rangle$. Then $|\vec{v}|=\sqrt{a^2+b^2+c^2}$

(2) Let $\vec{v}=\langle a,b\rangle$. Then $|\vec{v}|=\sqrt{a^2+b^2}$

Example 19:

(1) Let $\vec{v}=\langle 2,-2,-1\rangle$. Then $|\vec{v}|=\sqrt{2^2+(-2)^2+(-1)^2}=\sqrt{9}=3$

(2) Let $\vec{v}=\langle -5,\sqrt{11}\rangle$. Then $|\vec{v}|=\sqrt{(-5)^2+\sqrt{11}^2}=\sqrt{36}=6$

Rule 20: Let $\vec{u} = \langle a, b, c\rangle$, $\vec{v} = \langle d, e, f\rangle$ and let α be a scalar.

(1) $\vec{u} + \vec{v} = \langle a+d, b+e, c+f\rangle$ (2) $\vec{u} - \vec{v} = \langle a-d, b-e, c-f\rangle$

(3) $\alpha\,\vec{u} = \langle \alpha a, \alpha b, \alpha c\rangle$ (4) $\vec{u} = \vec{v} \Leftrightarrow a = d, b = e, c = f$

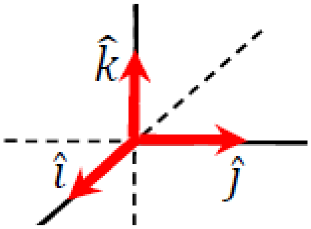
Example 21: Let $\vec{a} = \langle -1,0,3\rangle$ and $\vec{b} = \langle 2,-1,5\rangle$. Find $\left|2\,\vec{a} - \frac{\vec{b}}{3}\right|$.

Solution: First we find the vector $2\,\vec{a} - \frac{\vec{b}}{3}$:

$$2\,\vec{a} - \frac{\vec{b}}{3} = \langle 2(-1) - \frac{2}{3}, 2(0) - \frac{-1}{3}, 2(3) - \frac{5}{3}\rangle = \langle -\frac{8}{3}, \frac{1}{3}, -\frac{13}{3}\rangle$$

$$\left|2\,\vec{a} - \frac{\vec{b}}{3}\right| = \sqrt{\frac{64}{9} + \frac{1}{9} + \frac{169}{9}} = \frac{\sqrt{234}}{3}$$

Standard Basis Vectors:



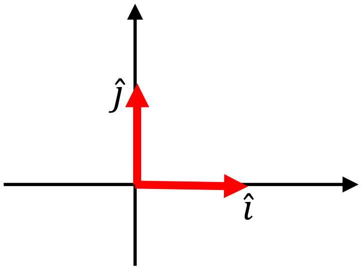
(1) In 3D, let

$\hat{\imath} = \langle 1,0,0\rangle$

$\hat{\jmath} = \langle 0,1,0\rangle$

$\hat{k} = \langle 0,0,1\rangle$

$\Rightarrow \langle a, b, c\rangle = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$



(2) $\hat{\imath} = \langle 1,0\rangle$

$\hat{\jmath} = \langle 0,1\rangle$

$\Rightarrow \langle a, b\rangle = a\hat{\imath} + b\hat{\jmath}$

Example 22:

(1) $5i - j - 7k = \langle 5,-1,-7\rangle$

(2) $\frac{i}{2} + 6k = \langle \frac{1}{2}, 0, 6\rangle$

Example 23: Let $\vec{a} = 5i - j$ and $\vec{b} = \langle 2,4,-1\rangle$. Find $\left|2\,\vec{a} + 3\vec{b}\right|$

Solution: First we find the vector $2\,\vec{a} + 3\vec{b}$:

$$2\,\vec{a} + 3\vec{b} = 2\langle 5,-1,0\rangle + 3\langle 2,4,-1\rangle = \langle 16,10,-3\rangle$$

$$\left|2\,\vec{a} + 3\vec{b}\right| = \sqrt{256 + 100 + 9} = \sqrt{365}$$

Definition 24: A vector $\vec{v}$ is called a unit vector if $|\vec{v}| = 1$

Example 25:

(1) $\vec{v} = \langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle \Rightarrow |\vec{v}| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1 \Rightarrow \vec{v}$ is a unit vector.

(2) $\vec{u} = 0.5i - 0.2j \Rightarrow |\vec{u}| = \sqrt{0.25 + 0.04} = \sqrt{0.29} \neq 1 \Rightarrow \vec{u}$ is a not unit vector.

(3) The zero vector $\vec{0}$ is not a unit vector.

Example 26: Find the values of a that make $\vec{v} = \langle -\frac{1}{2}, \frac{1}{\sqrt{3}}, a \rangle$ a unit vector

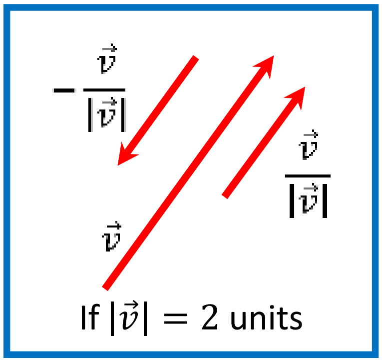
Solution:

$\vec{v}$ a unit vector $\Rightarrow |\vec{v}| = 1 \Rightarrow \sqrt{\frac{1}{4} + \frac{1}{3} + a^2} = 1 \Rightarrow \frac{7}{12} + a^2 = 1$

$\Rightarrow a^2 = 1 - \frac{7}{12} = \frac{5}{12} \Rightarrow a = \pm \frac{\sqrt{5}}{\sqrt{12}} = \pm \frac{\sqrt{5}}{2\sqrt{3}}$

Rule 27: If $\vec{v} \neq \vec{0}$, then $\frac{\vec{v}}{|\vec{v}|}$ and $-\frac{\vec{v}}{|\vec{v}|}$ are two unit vectors.

In fact: $\begin{cases} \frac{\vec{v}}{|\vec{v}|} \text{ is a unit vector in the same direction of } \vec{v} \\ -\frac{\vec{v}}{|\vec{v}|} \text{ is a unit vector in the opposit direction of } \vec{v} \end{cases}$



Rule 28: Let $\vec{v} = 2i - 2j + k$.

(1) Find a unit vector in the same direction of $\vec{v}$

(2) Find a vector of length $\frac{3}{2}$ in the same direction of $\vec{v}$

(3) Find a unit vector in the opposite direction of $\vec{v}$

(4) Find a vector of length $\sqrt{\pi}$ in the opposite direction of $\vec{v}$

Solution: $\vec{v} = 2i - 2j + k \;\Rightarrow\; |\vec{v}| = 3$

(1) a unit vector in the same direction of $\vec{v}$ is

$$\frac{\vec{v}}{|\vec{v}|} = \frac{2i - 2j + k}{3} = \frac{2}{3}i - \frac{2}{3}j + \frac{1}{3}k$$

(2) a vector of length $\frac{3}{2}$ in the same direction of $\vec{v}$ is

$$\frac{3}{2}\left(\frac{\vec{v}}{|\vec{v}|}\right) = \frac{3}{2}\left(\frac{2}{3}i - \frac{2}{3}j + \frac{1}{3}k\right) = i - j + \frac{1}{2}k$$

(3) a unit vector in the opposite direction of $\vec{v}$ is

$$-\frac{\vec{v}}{|\vec{v}|} = -\frac{2i - 2j + k}{3} = -\frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

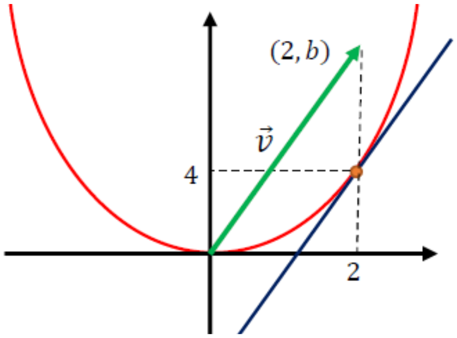
(4) a vector of length $\sqrt{\pi}$ in the opposite direction of $\vec{v}$ is

$$\sqrt{\pi}\left(-\frac{\vec{v}}{|\vec{v}|}\right) = \sqrt{\pi}\left(-\frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k\right) = -\frac{2\sqrt{\pi}}{3}i + \frac{2\sqrt{\pi}}{3}j - \frac{\sqrt{\pi}}{3}k$$

Example 29: Find all unit vector parallel to the tangent line to the parabola $y = x^2$ at the point $(2,4)$.

Solution: Let $\vec{v} = \langle 2, b\rangle$ be parallel to the tangent line

$\Rightarrow$ slope of $\vec{v}$ = slope of the tangent line



$$\frac{b}{2} = \left.\frac{dy}{dx}\right|_{x=2} \quad \Rightarrow \quad \frac{b}{2} = 4 \quad \Rightarrow \quad b = 8$$

$\Rightarrow \vec{v} = \langle 2,8\rangle$ is parallel to the tangent line

But we want unit vectors:

The unit vectors are:

$$\frac{\vec{v}}{|\vec{v}|} = \frac{\langle 2,8\rangle}{\sqrt{68}} = \frac{\langle 2,8\rangle}{2\sqrt{17}} = \langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\rangle$$

and

$$-\frac{\vec{v}}{|\vec{v}|} = \langle -\frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}}\rangle$$

Notations 30:

(1) The set of all vectors in $\mathbb{R}^2$ is written as V_2.
(2) The set of all vectors in $\mathbb{R}^3$ is written as V_3.

Example 31:

(1) $2i - 5j$ and $\langle -1,0.6\rangle$ are vectors in V_2.
(2) $-3i + 2k$ and $\langle 3,-2,7\rangle$ are vectors in V_3.

Chapter 12 Vectors and the Geometry of Space

Section 12.3: The Dot Product

12.3: The Dot Product

Definition 1:

Let $\vec{u} = \langle a, b, c\rangle$ and $\vec{v} = \langle d, e, f\rangle$. Then the dot product of $\vec{u}$ and $\vec{v}$ is defined by

$$\vec{u} \cdot \vec{v} = ad + be + cf$$

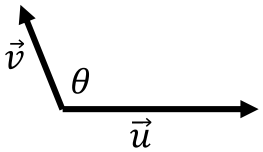
Example 2:

(1) $\langle 1, -2,3\rangle \cdot \langle 6,3,0\rangle = 1(5) + (-2)(3) + 3(0) = 0$
(2)$\langle 2,6\rangle \cdot \langle -5,2\rangle = 2(-5) + 6(2) = 2$
(3)$(3i - j) \cdot (-2i + 4k) = 3(-2) + (-1)(0) + 0(4) = -6$

Properties of Dot Product: Let $\vec{u}$,$\vec{v}$, and $\vec{w}$ be vectors in V_2 or V_3 and let a, b be a scalar. Then

(1) $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
(2) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
(3) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
(4) $(a\vec{u}) \cdot \vec{v} = \vec{u} \cdot (a\vec{v}) = a(\vec{u} \cdot \vec{v})$
(5) $\vec{0} \cdot \vec{v} = 0$
(6) $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$
(7) $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$
(8) $|a\vec{u} + b\vec{v}|^2 = a^2|\vec{u}|^2 + 2ab\vec{u} \cdot \vec{v} + b^2|\vec{v}|^2$
(9) $|a\vec{u} - b\vec{v}|^2 = a^2|\vec{u}|^2 - 2ab\vec{u} \cdot \vec{v} + b^2|\vec{v}|^2$

Definition 4: The angle θ between two vectors $\vec{u}$ and $\vec{v}$ is the angle between them when the vectors have the same initial point, where $0 \leq \theta \leq \pi$.



Rule 5: $\vec{u} \cdot \vec{v} = |\vec{u}|\ |\vec{v}\ |\cos\theta$

If $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0} \Rightarrow \cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \Rightarrow\ \theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\ |\vec{v}|}\right)$

Example 6: Find the angle between the two vectors $\vec{u} = -i + k$ and $\vec{v} = 3i + j + k$

Solution:

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \cos^{-1}\left(\frac{-1(3)+0(1)+1(1)}{\sqrt{2}\sqrt{11}}\right)$$

$$= \cos^{-1}\left(\frac{-2}{\sqrt{22}}\right) \cong \underbrace{115.2^{\circ}}_{\text{in Degrees}} \cong \underbrace{2.01}_{\text{in radian}}$$

Example 7: Find the value of x that makes the angle between the two vectors $\vec{u} = \langle 2,1,-1\rangle$ and $\vec{v} = \langle 1,x,0\rangle$ is $\frac{\pi}{4}$

Solution:

$$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}\,|\cos\theta \Rightarrow\ 2(1)+1(x)+(-1)(0) = \sqrt{6}\sqrt{1+x^2}\cos\frac{\pi}{4}$$

$$\Rightarrow 2+x = \frac{\sqrt{6}\sqrt{1+x^2}}{\sqrt{2}}\ \Rightarrow\ (2+x)^2 = 3(1+x^2) \Rightarrow x^2+4x+4 = 3+3x^2$$

$$\Rightarrow 2x^2-4x-1 = 0 \Rightarrow x = \frac{4\pm\sqrt{(-4)^2-4(2)(-1)}}{2(2)} = \frac{4\pm\sqrt{24}}{4} = \frac{4\pm2\sqrt{6}}{4} = 1\pm\frac{\sqrt{6}}{2}$$

Example 8: Find the angle between the two lines in $\mathbb{R}^2$: $2x-y=3$ and $3x+y=7$

Solution:

Let $\vec{u} = \langle 1,b\rangle$ parallel to the line L_1: $2x-y=3 \Rightarrow y = 2x+3$

$\Rightarrow$ Slope of $\vec{u}$ = Slope of $L_1 \Rightarrow \frac{b}{1} = 2 \Rightarrow b = 2$ $\vec{u} = \langle 1,2\rangle$

Let $\vec{v} = \langle 1,b\rangle$ parallel to the line L_2: $3x+y=7 \Rightarrow y = -3x+7$

$\Rightarrow$ Slope of $\vec{v}$ = Slope of $L_2 \Rightarrow \frac{b}{1} = -3 \Rightarrow b = -3$ $\vec{v} = \langle 1,-3\rangle$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right) = \cos^{-1}\left(\frac{-5}{\sqrt{5}\sqrt{10}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

Example 9: If $\vec{u}$ and $\vec{v}$ are vectors such that $|\vec{u}| = 4$, $|\vec{v}| = 3$ and the angle between $\vec{u}$ and $\vec{v}$ is $\frac{2\pi}{3}$.

(1) Find $\vec{u}\cdot\vec{v}$

(2) Find $|2\vec{u}-3\vec{v}|$

(3) Find $|3\vec{u}+\vec{v}|$

Solution:

$$(1)\,\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}\,|\cos\theta = 4(3)\cos\left(\frac{2\pi}{3}\right) = 12\left(-\cos\frac{\pi}{3}\right) = -12\left(\frac{1}{2}\right) = -6$$

$$(2)\,|2\vec{u}-3\vec{v}|^2 = 4|\vec{u}|^2 - 2(2)(3)\vec{u}\cdot\vec{v} + 9|\vec{v}|^2$$

$$= 4(16) - 12(-6) + 9(9) = 217$$

$$|2\vec{u} - 3\vec{v}| = \sqrt{217}$$

$$(3)\,|3\vec{u} + \vec{v}|^2 = 3^2|\vec{u}|^2 + 2(3)\vec{u}\cdot\vec{v} + |\vec{v}|^2 = 9(16) + 6(-6) + 9 = 117$$

$$|3\vec{u} + \vec{v}| = \sqrt{117}$$

Example 10: If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}| = \sqrt{3}$, $|2\vec{a} - 3\vec{b}| = \sqrt{45}$ and $|\vec{a} + 2\vec{b}| = \sqrt{27}$.

(1) Find $\vec{a}\cdot\vec{b}$

(2) Find the angle between $\vec{a}$ and $\vec{b}$

(3) Find $|\vec{a} + 2\vec{b}|$

Solution:

$$(1)\,|2\vec{a} - 3\vec{b}|^2 = \sqrt{45}^2 \;\Rightarrow\; 2^2|\vec{a}|^2 - 2(2)(3)\vec{a}\cdot\vec{b} + (-3)^2|\vec{b}|^2 = 45$$

$$\Rightarrow\; 4\sqrt{3}^2 - 12\vec{a}\cdot\vec{b} + 9|\vec{b}|^2 = 45 \;\Rightarrow\; -12\vec{a}\cdot\vec{b} + 9|\vec{b}|^2 = 33$$

$$\Rightarrow\; -4\vec{a}\cdot\vec{b} + 3|\vec{b}|^2 = 11 \;\ldots\ldots\text{①}$$

$$|\vec{a} + 2\vec{b}|^2 = \sqrt{27}^2 \;\Rightarrow\; |\vec{a}|^2 + 2(2)\vec{a}\cdot\vec{b} + (2)^2|\vec{b}|^2 = 27$$

$$\Rightarrow\; \sqrt{3}^2 + 4\vec{a}\cdot\vec{b} + 4|\vec{b}|^2 = 27 \;\Rightarrow\; 4\vec{a}\cdot\vec{b} + 4|\vec{b}|^2 = 24 \;\ldots\ldots\text{②}$$

① + ②: $7|\vec{b}|^2 = 35 \;\Rightarrow\; |\vec{b}|^2 = 5 \;\Rightarrow\; |\vec{b}| = \sqrt{5}$

②: $4\vec{a}\cdot\vec{b} + 4(5) = 24 \;\Rightarrow\; \vec{a}\cdot\vec{b} = 1$

$$(2)\,\theta = \cos^{-1}\left(\frac{\vec{a}\cdot b}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}(\sqrt{5})}\right) = \cos^{-1}\left(\frac{1}{\sqrt{15}}\right) \cong \underbrace{75.04^\circ}_{\text{in degrees}} \cong \underbrace{1.31}_{\text{in radian}}$$

$$(3)\,|\vec{a} + 3\vec{b}|^2 = |\vec{a}|^2 + 2(2)\vec{a}\cdot\vec{b} + (3)^2|\vec{b}|^2 = 3 + 4(1) + 9(5) = 52$$

$$\Rightarrow |\vec{a} + 3\vec{b}| = \sqrt{52}$$

Example 11: Prove that $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2(|\vec{a}|^2 + |\vec{b}|^2)$

Proof: $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = \left(|\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2\right) + (|\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2)$

$$= 2(|\vec{a}|^2 + |\vec{b}|^2)$$

Example 11.5: If $|\vec{a}| = 3$ and $|\vec{b}| = 4$, find $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2$

Solution:

$$|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2\left(|\vec{a}|^2 + |\vec{b}|^2\right) = 2(9 + 16) = 50$$

Remark 12:

Two vectors $\vec{u}$ and $\vec{v}$ are perpendicular (or orthogonal) written $\vec{u} \perp \vec{v} \Leftrightarrow \vec{u} \cdot \vec{v} = 0$

Example 13: Show that $2i + 2j - k$ is perpendicular to $5i - 4j + 2k$

Solution:

$$(2i + 2j - k) \cdot (5i - 4j + 2k) = 2(5) + 2(-4) + (-1)(2) = 0$$

$\Rightarrow (2i + 2j - k) \perp (5i - 4j + 2k)$

Example 14: Find the value of a that makes $ai - 2j + k$ perpendicular to $2i + j + ak$

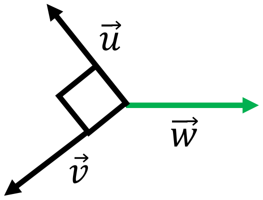
Solution:

$(ai - 2j + k) \cdot (2i + j + ak) = 0 \Rightarrow a(2) + (-2)(1) + 1(a) = 0 \Rightarrow 3a - 2 = 0$

$\Rightarrow a = \frac{2}{3}$

Example 15: If $\vec{u}$ and $\vec{v}$ are unit vectors such that $\vec{u} + \vec{v} + \vec{w} = 0$, then find $|\vec{w}|$

Solution:



$\vec{u}$ and $\vec{v}$ are unit vectors $\Rightarrow$ $|\vec{u}| = 1$ and $|\vec{v}| = 1$

$\vec{u} \perp \vec{v} \Rightarrow \vec{u} \cdot \vec{v} = 0$

$\vec{u} + \vec{v} + \vec{w} = 0 \Rightarrow \vec{w} = -(\vec{u} + \vec{v})$

$\Rightarrow |\vec{w}|^2 = |-(\vec{u} + \vec{v})|^2 = |\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 = 1 + 0 + 1 = 2$

$\Rightarrow |\vec{w}| = \sqrt{2}$

Remark 16: $\vec{u} \cdot \vec{v} = |\vec{u}|\, |\vec{v}\,| \cos\theta \;\Rightarrow \cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$

(1) $\vec{u} \cdot \vec{v} > 0 \;\Rightarrow \theta$ is a cute angle (زاوية حادة)
(2) $\vec{u} \cdot \vec{v} < 0 \;\Rightarrow \theta$ is an obtuse angle (زاوية منفرجة)
(3) $\vec{u} \cdot \vec{v} = 0 \;\Rightarrow \theta$ is a right angle (زاوية قائمة)

Example 17: The angle between the vectors $2i - k$ and $j + 2k$ is an obtuse angle since

$$(2i - k) \cdot (j + 2k) = -2 < 0$$

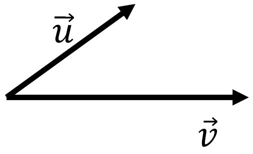
Definition 18:

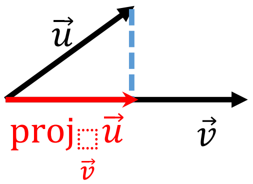
(1) The scalar projection of the vector $\vec{u}$ onto the vector $\vec{v}$ is:

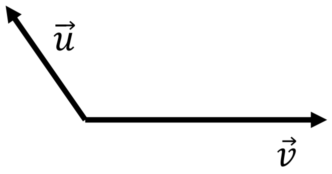
$$\text{comp}_{\vec{v}}\, \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

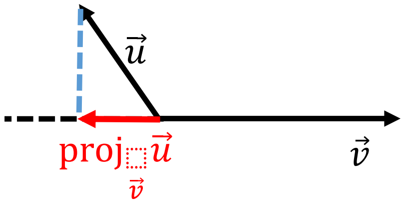
(2) The vector projection of the vector $\vec{u}$ onto the vector $\vec{v}$ is:

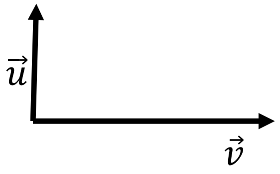
$$\text{proj}_{\vec{v}}\, \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right) \vec{v}$$

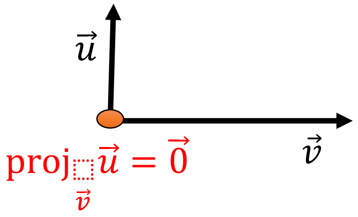










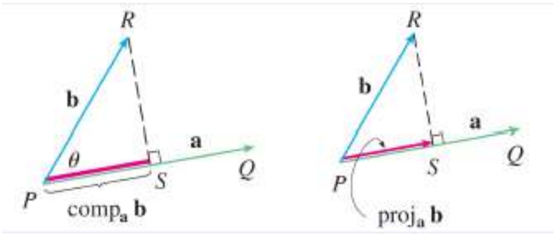


Observe that

- ❖ $\text{comp}_{\vec{v}}\, \vec{u} = |\vec{u}|\cos\theta$
- ❖ $\text{proj}_{\vec{v}}\, \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right) \vec{v}$

$$= \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}\right) \frac{\vec{v}}{|\vec{v}|} = \left(\text{comp}_{\vec{v}}\, \vec{u}\right) \frac{\vec{v}}{|\vec{v}|}$$

- ❖ $\left|\text{proj}_{\vec{v}}\, \vec{u}\right| = \left|\text{comp}_{\vec{v}}\, \vec{u}\right|$



Example 19: Find the scalar and vector projections of $\vec{v} = \langle 1,1,2 \rangle$ onto $\vec{u} = -2i + j - k$

Solution: The scalar projection of the vector $\vec{v}$ onto the vector $\vec{u}$ is:

$$\text{comp}_{\vec{u}}\,\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|} = \frac{1(-2) + 1(1) + 2(-1)}{\sqrt{4+1+1}} = \frac{-3}{\sqrt{6}}$$

The vector projection of the vector $\vec{v}$ onto the vector $\vec{u}$ is:

$$\text{proj}_{\vec{u}}\,\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u} = \frac{-3}{6}\vec{u} = -\frac{1}{2}(-2i + j - k) = i - \frac{1}{2}j + \frac{1}{2}k$$

Example 20:

(1) If $|\vec{u}| = 5$ and the angle between $\vec{u}$ and $\vec{v}$ is $\frac{5\pi}{6}$, then find the scalar projection of $\vec{u}$ onto $\vec{v}$

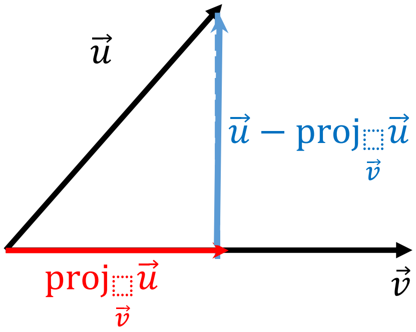
(2) If $\text{comp}_{\vec{v}}\,\vec{u} = -4$ and $\vec{v} = 3j - 4k$, then find the vector projection of $\vec{u}$ onto $\vec{v}$

Solution:

(1) $\text{comp}_{\vec{v}}\,\vec{u} = |\vec{u}|\cos\theta = 5\cos\frac{5\pi}{6} = 5\left(-\cos\frac{\pi}{6}\right) = -\frac{5\sqrt{3}}{2}$

(2) $\text{proj}_{\vec{v}}\,\vec{u} = \left(\text{comp}_{\vec{v}}\,\vec{u}\right)\frac{\vec{v}}{|\vec{v}|} = -4\,\frac{3j-4k}{\sqrt{9+16}} = \frac{-12i+16k}{5}$

Example 21: Show that $\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}$ is orthogonal to $\vec{v}$



Proof:

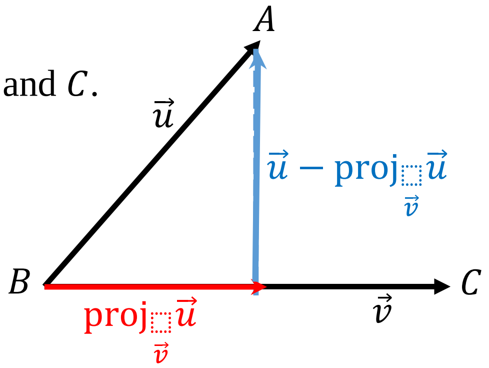
$$\begin{aligned}
\vec{v}\cdot\left(\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}\right) &= \vec{v}\cdot\vec{u} - \vec{v}\cdot\left(\text{proj}_{\vec{v}}\,\vec{u}\right) \\
&= \vec{u}\cdot\vec{v} - \vec{v}\cdot\left(\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\right) \\
&= \vec{u}\cdot\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)|\vec{v}|^2 = \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} = 0
\end{aligned}$$

$$\Rightarrow \left(\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}\right) \perp \vec{v}$$

Remark 22: Let L a line that pass through the points B and C.

Then the distance from the point A to the line L is:

$\left|\vec{u}-\text{proj}_{\vec{v}}\vec{u}\right|$ where $\vec{u}=\overrightarrow{BA}$ and $\vec{v}=\overrightarrow{BC}$



Example 23: Find the distance from the point $A(1,2,3)$ and the line that pass through the points $B(2,1,3)$ and $C(0,1,0)$

Solution: $\vec{u}=\overrightarrow{BA}=\langle A-B\rangle=\langle -1,1,0\rangle$ and $\vec{v}=\overrightarrow{BC}=\langle C-B\rangle=\langle -2,0,-3\rangle$

$$\vec{u}-\text{proj}_{\vec{v}}\vec{u}=\vec{u}-\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}=\vec{u}-\left(\frac{2}{13}\right)\vec{v}$$

$$=\langle -1-\frac{2}{13}(-2),1-\frac{2}{13}(0),0-\frac{2}{13}(-3)\rangle$$

$$=\langle -\frac{9}{13},1,\frac{6}{13}\rangle$$

$$\text{Distance}=\left|\vec{u}-\text{proj}_{\vec{v}}\vec{u}\right|=\sqrt{\frac{81}{169}+1+\frac{36}{169}}=\sqrt{\frac{286}{169}}=\frac{\sqrt{286}}{13}$$

هناك طريقة اسهل لحل هذا السؤال ستأتي في

Section 12.4: The Cross Product

Chapter 12 Vectors and the Geometry of Space

Section 12.4: The Cross Product

12.4: The Cross Product

Definition 1: The Cross product of two vectors $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$ is given by:

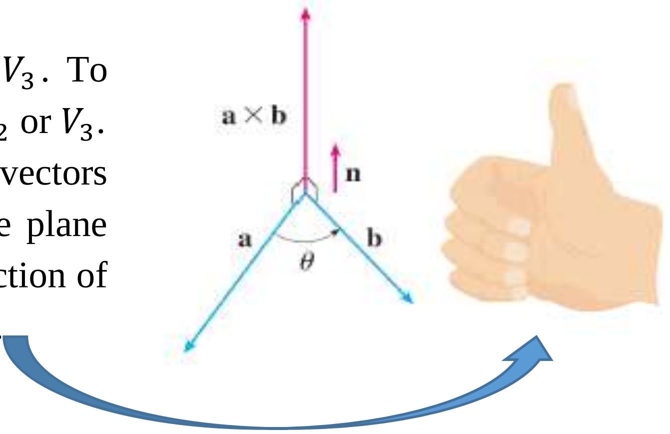
$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

$\Rightarrow$ $\vec{a} \times \vec{b}$ is a vector in V_3.

Remark 2:

(1) To find $\vec{a} \times \vec{b}$ we must have $\vec{a}$ and $\vec{b}$ in V_3. To find $\vec{a} \cdot \vec{b}$, the vectors $\vec{a}$ and $\vec{b}$ may be in V_2 or V_3.

(2) $\vec{a} \times \vec{b}$ is a vector orthogonal (يعامد) to the vectors $\vec{a}$ and $\vec{b}$ and so $\vec{a} \times \vec{b}$ is orthogonal to the plane containing both vectors $\vec{a}$ and $\vec{b}$. The direction of $\vec{a} \times \vec{b}$ is determined by the right hand rule.



Example 3: Let $\vec{a} = \langle 3,2,1\rangle$ and $\vec{b} = \langle -1,1,0\rangle$

(1) Find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$

(2) Find two vectors perpendicular (orthogonal) to both $\vec{a}$ and $\vec{b}$

(3) Find two unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$

(4) Find two unit vectors orthogonal to the plane that pass through the points $A(1,2,3), B(4,4,4)$, and $C(0,3,3)$

Solution:

(1) $$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = i\big(2(0) - 1(1)\big) - j\big(3(0) - 1(-1)\big) + k\big(3(1) - 2(-1)\big)$$
$$= -i - j + 5k$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = i\big(1(1) - 2(0)\big) - j\big(-1(1) - 3(0)\big) + k\big(-1(2) - 3(1)\big)$$
$$= i + j - 5k$$

(2) Two vectors orthogonal to both $\vec{a}$ and $\vec{b}$ are $\vec{a} \times \vec{b}$ and $-\vec{a} \times \vec{b}$

$\Rightarrow -i - j + 5k$ and $i + j - 5k$ are orthogonal to both $\vec{a}$ and $\vec{b}$

(3) Two unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$ are $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ and $-\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$\Rightarrow \frac{-i-j+5k}{|-i-j+5k|}$ and $\frac{i+j-5k}{|i+j-5k|}$ are unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$

$\Rightarrow \frac{-i-j+5k}{\sqrt{26}}$ and $\frac{i+j-5k}{\sqrt{26}}$ are unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$

(4) Let $\vec{a} = \overrightarrow{AB} = \langle B - A\rangle = \langle 3,2,1\rangle$ and $\vec{b} = \overrightarrow{AC} = \langle C - A\rangle = \langle -1,1,0\rangle$

$\Rightarrow \vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ are orthogonal to both $\vec{a}$ and $\vec{b}$

$\Rightarrow \vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ are orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

$\Rightarrow -i - j + 5k$ and $i + j - 5k$ are orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

$\Rightarrow \frac{-i-j+5k}{|-i-j+5k|}$ and $\frac{i+j-5k}{|i+j-5k|}$ are orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

$\Rightarrow \frac{-i-j+5k}{\sqrt{26}}$ and $\frac{i+j-5k}{\sqrt{26}}$ are unit vectors orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

Example 4: $\quad i\times j = k, \quad j\times k = i, \quad k\times i = j$

$j\times i = -k, \quad k\times j = -i, \quad i\times k = -j$

Properties of Cross Product: Let $\vec{u}$,$\vec{v}$, and $\vec{w}$ be vectors in V_2 or V_3 and let a be a scalar. Then

(1) $\vec{u}\times\vec{u} = \vec{0}$
(2) $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$
(3) $\vec{0}\times\vec{v} = \vec{v}\times\vec{0} = \vec{0}$
(4) $(a\vec{u})\times\vec{v} = \vec{u}\times(a\vec{v}) = a(\vec{u}\times\vec{v})$
(5) $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v}) + (\vec{u}\times\vec{w})$
(6) $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$

In general

$\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$

For Example:

$j\times(j\times k) = j\times(i) = -k$

$(j\times j)\times k = \vec{0}\times k = \vec{0}$

Rule 5: $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

Example 6: Let $\vec{a}$ and $\vec{b}$ be orthogonal such that $|\vec{a}| = 2$ and $|\vec{b}| = 3$.

Find $\vec{a}\times(\vec{b}\times\vec{a})$ and $|(\vec{b}\times\vec{a})\times\vec{a}|$

Solution:

$$(\vec{b}\times\vec{a})\times\vec{a} = -\vec{a}\times(\vec{b}\times\vec{a}) = -\left((\vec{a}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}\right) = -(|\vec{a}|^2\vec{b} - 0\vec{a}) = -4\vec{b}$$

$$|(\vec{b}\times\vec{a})\times\vec{a}| = |-4\vec{b}| = 4|\vec{b}| = 4(3) = 12$$

Example 7: Simplify $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$

Solution:

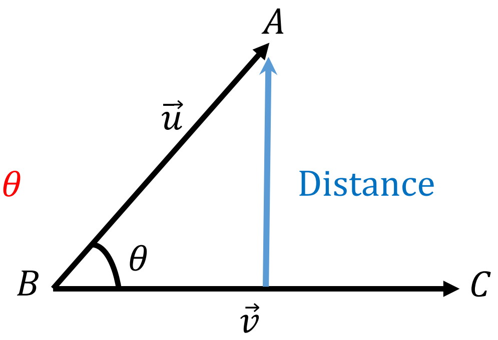
$$\begin{aligned}(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) &= \vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}+\vec{b}\times\vec{b}\\ &= \vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}+\vec{0}\\ &= 2\vec{a}\times\vec{b}\end{aligned}$$

Rule 8:

(1) The length of $\vec{a}\times\vec{b}$ is given by: $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$

(2) The length of $\vec{a}\times\vec{b}$ is given by:

$$|\vec{a}\times\vec{b}| = \sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}\ \text{(Lagrange identity)}$$



Remark 19: Let L a line that pass through the points B and C.

Then the distance from the point A to the line L is:

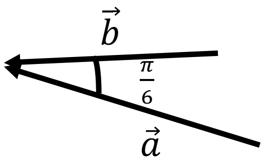
Distance $= \frac{|\vec{u}\times\vec{v}|}{|\vec{v}|}$ where $\vec{u}=\overrightarrow{BA}$ and $\vec{v}=\overrightarrow{BC}$

Example 9: Find the distance from the point $A(1,2,3)$ and the line that pass through the points $B(2,1,3)$ and $C(0,1,0)$

Solution: $\vec{u}=\overrightarrow{BA}=\langle A-B\rangle=\langle -1,1,0\rangle$ and $\vec{v}=\overrightarrow{BC}=\langle C-B\rangle=\langle -2,0,-3\rangle$

$$\text{Distance} = \frac{|\vec{u}\times\vec{v}|}{|\vec{v}|} = \frac{\sqrt{|\vec{u}|^2|\vec{v}|^2-(\vec{u}\cdot\vec{v})^2}}{|\vec{v}|} = \frac{\sqrt{2(13)-(2)^2}}{\sqrt{13}} = \frac{\sqrt{22}}{\sqrt{13}}$$

Example 10: Find $|\vec{a}\times\vec{b}|$, where $\vec{a}$ and $\vec{b}$ are given in the figure with $|\vec{a}|=8$, $|\vec{b}|=6$



Solution: $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta = 8(6)\sin\left(\frac{\pi}{6}\right) = 48\left(\frac{1}{2}\right) = 24$

Example 11: Find $|\vec{a}\times\vec{b}|$ and $\vec{a}\times\vec{b}$, where $|\vec{a}|=2$ and $|\vec{b}|=\frac{1}{2}$ and $|\vec{a}+2\vec{b}|=3$

Solution: $|\vec{a}+2\vec{b}|^2=3^2 \;\Rightarrow\; |\vec{a}|^2+4\vec{a}\cdot\vec{b}+4|\vec{b}|^2=9 \;\Rightarrow\; 4+4\vec{a}\cdot\vec{b}+1=9$

$\Rightarrow \vec{a}\cdot\vec{b}=1$

$$|\vec{a}\times\vec{b}|=\sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}=\sqrt{4\left(\frac{1}{4}\right)-(1)^2}=0 \;\Rightarrow\; \vec{a}\times\vec{b}=\vec{0}$$

Rule 12: Two vectors $\vec{a}$ and $\vec{b}$ are parallel written $\vec{a}//\vec{b}$ if $\vec{a}\times\vec{b}=\vec{0}$.

Observe the following:

(1) in **Example 11** we have $\vec{a}\times\vec{b}=\vec{0}$ so $\vec{a}//\vec{b}$.
(2) If $\vec{a}$ is any vector then $\vec{a}//\vec{0}$ since $\vec{a}\times\vec{0}=\vec{0}$

Remark 13: $\vec{a}//\vec{b} \;\Leftrightarrow\; \vec{a}=c\vec{b}$ or $\vec{b}=c\vec{a}$ for some scalar c.

Consequently: Let $\vec{a}=\langle a_1,a_2,a_3\rangle$ and $\vec{b}=\langle b_1,b_2,b_3\rangle$, Then $\vec{a}//\vec{b} \;\Leftrightarrow \frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$, where b_1,b_2,b_3 are nonzero scalars.

Example 14:

(1) $\langle 6,3,15\rangle//\langle 4,2,10\rangle$ since $\frac{6}{4}:\frac{3}{2}:\frac{15}{10} \;\Rightarrow$ are all equal

(2) $\langle 4,6,-28\rangle$ and $\langle 2,3,14\rangle$ are not parallel since the ratios $\frac{4}{2}:\frac{6}{3}:\frac{-28}{7}$ are not all equal

Example 15: Find the value of x that makes $\vec{a}=\langle 2,x-1,x\rangle$ and $\vec{b}=\langle x^2-1,0,x+1\rangle$ parallel.

Solution: $\frac{x^2-1}{2}=\frac{0}{x-1}=\frac{x+1}{x} \;\Rightarrow\; 0=\frac{x+1}{x} \;\Rightarrow\; x+1=0 \;\Rightarrow\; x=-1$

Check: Is there an error in the equations: $\underbrace{\frac{x^2-1}{2}=\frac{0}{x-1}=\frac{x+1}{x}}_{x=-1\text{ عوِّض}} \Rightarrow \frac{0}{2}=\frac{0}{-2}=\frac{0}{-1}$ (no error)

$\Rightarrow$ the value of x is $x=-1$.

Another solution: $\frac{x^2-1}{2}=\frac{0}{x-1} \;\Rightarrow\; \frac{x^2-1}{2}=0 \;\Rightarrow\; x^2-1=0 \;\Rightarrow\; x=\pm 1$

Check: Is there an error in the equations:

$\underbrace{\frac{x^2-1}{2}=\frac{0}{x-1}=\frac{x+1}{x}}_{x=-1\text{ عوِّض}} \Rightarrow \frac{0}{2}=\frac{0}{-2}=\frac{0}{-1}$ (no error)

$\underbrace{\frac{x^2-1}{2}=\frac{0}{x-1}=\frac{x+1}{x}}_{x=1\ \text{عوّض}} \Rightarrow \frac{0}{2}=\frac{0}{0}=\frac{2}{1}$ (there is an error in the equations)

$\Rightarrow x \neq 1 \ \Rightarrow \ x=-1$ only.

Exercise 16: Find the value of x that makes:

$$\vec{a}=\langle 3,1,x^2+2x+1\rangle \text{ and } \vec{b}=\langle 3x^2-3,3,3\rangle \text{ parallel.}$$

Answer is $x=-2$

Definition 17: Three points A,B,C are collinear (على استقامة واحدة) $\Leftrightarrow \overrightarrow{AB}//\overrightarrow{AC}$

Example 18: Determine whether the points $A(2,4,-3), B(3,-1,1), C(4,-6,5)$ are collinear or not.

Solution:

$\overrightarrow{AB}=\langle 1,-5,4\rangle$ and $\overrightarrow{AC}=\langle 2,-10,8\rangle \Rightarrow \frac{2}{1}=\frac{-10}{-5}=\frac{8}{4}$ are all equal $\Rightarrow \overrightarrow{AB}//\overrightarrow{AC}$

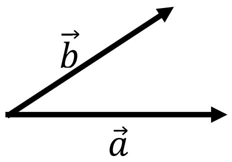
$\Rightarrow$ The points A,B,C are collinear

Another solution: $\overrightarrow{AC}=2\overrightarrow{AB} \Rightarrow \overrightarrow{AB}//\overrightarrow{AC} \Rightarrow$ The points A,B,C are collinear

Rule 19:

(1) The area (مساحة) of the parallelogram determined by the vectors $\vec{a}$ and $\vec{b}$ is $A=|\vec{a}\times\vec{b}|$

(2) The area of the triangle determined by the vectors $\vec{a}$ and $\vec{b}$ is $A=\frac{1}{2}|\vec{a}\times\vec{b}|$



Remark 20: Let A,B,C,D be points and let $\vec{a}=\overrightarrow{AB}$ and $\vec{b}=\overrightarrow{AC}$.

(1) The area of the parallelogram (متوازي اضلاع) with vertices A,B,C,D is $A=|\vec{a}\times\vec{b}|$

(2) The area of the triangle (مثلث) with vertices A,B,C is $A=\frac{1}{2}|\vec{a}\times\vec{b}|$

Example 21: let $\vec{a} = i + 2j - k$ and $\vec{b} = j + 3k$ and let $A(1,0,1), B(2,2,0), C(1,1,4), D$ be four points.

(1) Find the area of the parallelogram determined by the vectors $\vec{a}$ and $\vec{b}$.
(2) Find the area of the triangle determined by the vectors $\vec{a}$ and $\vec{b}$.
(3) Find the area of the parallelogram with vertices A, B, C, D
(4) Find the area of the triangle with vertices A, B, C

Solution:

(1) Area $= |\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2} = \sqrt{(6)(10) - (-1)^2} = \sqrt{59}$

(2) Area $= \frac{\sqrt{59}}{2}$

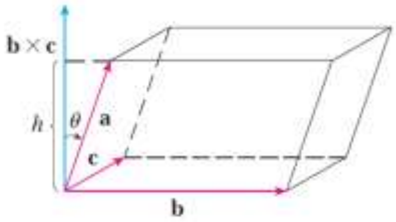
(3) $\vec{a} = \overrightarrow{AB} = \langle 1,2,-1 \rangle$ and $\vec{b} = \overrightarrow{AC} = \langle 0,1,3 \rangle \Rightarrow$ Area $= |\vec{a} \times \vec{b}| = \sqrt{59}$

(4) Area $= \frac{\sqrt{59}}{2}$

Definition 22: Let $\vec{a} = \langle a_1, a_2, a_3 \rangle, \vec{b} = \langle b_1, b_2, b_3 \rangle$, and $\vec{c} = \langle c_1, c_2, c_3 \rangle$ be vectors. The scalar triple of the vectors $\vec{a}, \vec{b}, \vec{c}$ written $\vec{a} \cdot (\vec{b} \times \vec{c})$ is defined by

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1(a_2 b_3 - a_3 b_2) - a_2(a_1 b_3 - a_3 b_1) + a_3(a_1 b_2 - a_2 b_1)$$



Rule 23: The volume of the parallelepiped determined by the vectors $\vec{a}, \vec{b}, \vec{c}$ is

$$V = \underbrace{|\vec{a} \cdot (\vec{b} \times \vec{c})|}_{\text{القيمة المطلقة}}$$

Remark 24: Let A, B, C, D be vertices of a parallelepiped and let $\vec{a} = \overrightarrow{AB}, \vec{b} = \overrightarrow{AC}$, $\vec{c} = \overrightarrow{AD}$. Then the volume of this parallelepiped is $V = \underbrace{|\vec{a} \cdot (\vec{b} \times \vec{c})|}_{\text{القيمة المطلقة}}$

Example 25: Find the volume of the parallelepiped:

(1) Determined by the vectors $\vec{a} = \langle 0,-2,5\rangle, \vec{b} = \langle 0,1,2\rangle, \vec{c} = \langle 6,3,-1\rangle$

(2) With adjacent edges PQ, PR, PS, where $P(-2,1,0), Q(2,-1,5), R(-2,2,2)$, and $S(4,4,-1)$.

Solution:

(1) $$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 0 & -2 & 5 \\ 0 & 1 & 2 \\ 6 & 3 & -1 \end{vmatrix}$$

$$= 0(-1-6) - (-2)(0-12) + 5(0-6) = 0 - 24 - 30 = -54$$

Volume $= |\vec{a}\cdot(\vec{b}\times\vec{c})| = |-54| = 54$

(2) Let $\vec{a} = \overrightarrow{PQ} = \langle 0,-2,5\rangle, \vec{b} = \overrightarrow{PR} = \langle 0,1,2\rangle, \vec{c} = \overrightarrow{PS} = \langle 6,3,-1\rangle$

$\Rightarrow\ \vec{a}\cdot(\vec{b}\times\vec{c}) = -54$ (by part (1))$\Rightarrow$ Volume $= |\vec{a}\cdot(\vec{b}\times\vec{c})| = |-54| = 54$

Rule 26:

(1) Three vectors $\vec{a}, \vec{b}$, and $\vec{c}$ in V_3 are coplanar (lie in the same plane) if $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$.

(2) Four points A, B, C, D in $\mathbb{R}^3$ are coplanar if $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$, where $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{AC}$, and $\vec{c} = \overrightarrow{AD}$

Example 27:

(1) Find the value of x that makes $\vec{a} = \langle 1,x,0\rangle, \vec{b} = \langle x,2,1\rangle, \vec{c} = \langle 0,1,1\rangle$ coplanar

(2) Find the value of x that makes the points $A(1,-1,2), B(2,x-1,2)$, $C(x+1,1,3)$, and $D(1,0,3)$ lie in the same plane.

Solution:

(1) $$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & x & 0 \\ x & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(2-1) - x(x-0) + 0(x-1) = 1 - x^2$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \ \Rightarrow\ 1 - x^2 = 0 \ \Rightarrow\ x = \pm 1$$

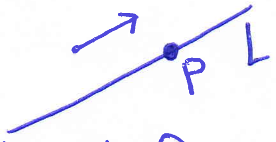
(2) $\vec{a} = \overrightarrow{AB} = \langle 1,x,0\rangle, \vec{b} = \overrightarrow{AC} = \langle x,2,1\rangle$, and $\vec{c} = \overrightarrow{AD} = \langle 0,1,1\rangle$

$\vec{a}\cdot(\vec{b}\times\vec{c}) = 1 - x^2$ (by part (1))

$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \ \Rightarrow\ 1 - x^2 = 0 \ \Rightarrow\ x = \pm 1$

Section 12.5 Equations of lines and planes

Def: Let L be the line that pass through the pt $P(x_0, y_0, z_0)$ and parallel to the vector $\langle a, b, c \rangle$.



1) The parametric (param.) eqs. of L are:

$x = x_0 + at \quad , y = y_0 + bt \quad , z = z_0 + ct$ where $t \in \mathbb{R}$.

2) The symmetric (symm.) eqs. of L are:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- * if $a = 0$: The symm. eqs. are: $\frac{y - y_0}{b} = \frac{z - z_0}{c}$, $x = x_0$
- * if $b = 0$: The symm. eqs. are: $\frac{x - x_0}{a} = \frac{z - z_0}{c}$, $y = y_0$
- * if $c = 0$: The symm. eqs. are: $\frac{x - x_0}{a} = \frac{y - y_0}{b}$, $z = z_0$

3) The vector eq. of L is

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t \quad , \text{where } t \in \mathbb{R}$$
$$= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle \quad , \text{where } t \in \mathbb{R}$$

Ex1: Find parametric, symm. and vector eqs. of the line through the pt. $A(1,2,3)$ and parallel to the vector $6\hat{i} - 7\hat{k}$. Also, find two pts. on the line other than A. At what pt. this line intersects the ① xy-plane, ② xz-plane.

Sol: param. eqs: $x = 1 + 6t \quad , y = 2 + 0t = 2 \quad , z = 3 - 7t$

symm. eqs.: $\frac{x-1}{6} = \cancel{\frac{y-2}{0}} = \frac{z-3}{-7} \quad , y = 2$

لا يجوز القسمة على صفر

vector eq: $\langle x, y, z \rangle = \langle 1 + 6t, 2, 3 - 7t \rangle$
$= \langle 1, 2, 3 \rangle + \langle 6, 0, -7 \rangle t$

Two pts. on line: Take $t = 1 \Rightarrow x = 1 + 6 = 7$
$y = 2$
$z = 3 - 7 = -4$

$\therefore (7, 2, -4)$ on line

Take $t=-1$: $\left.\begin{array}{l} x = 1-6 = -5 \\ y = 2 \\ z = 3+7 = 10 \end{array}\right] \rightarrow (-5,2,10)$ on line

At what pt the line intersects the xy-plane?

when $z = 0$: $3-7t = 0 \Rightarrow t = \frac{3}{7}$

$x = 1+6\left(\frac{3}{7}\right) = \frac{25}{7}$

$y = 2$

At the pt. $\left(\frac{25}{7}, 2, 0\right)$

At what pt. the line intersects the xz-plane?

when $y = 0$: $2 = 0$!!! impossible

$\therefore$ The line does not intersects the xz-plane.

Remark 3: let $\vec{u} \parallel L_1$ and $\vec{v} \parallel L_2$.

$\vec{u} \parallel \vec{v} \Leftrightarrow L_1 \parallel L_2$.

Ex 4: Determine whether the two lines L_1 and L_2 are parallel, intersects or skew. If they intersects, find the pt. of intersection:

(1) L_1: $x = 2-3t$, $y = 2t$, $z = 7$

L_2: $x = 5+9s$ $y = 3-6s$, $z = 1$

(2) L_1: $x = t$, $y = 3-t$, $z = 2+3t$

L_2: $x = 1+2s$, $y = 2+s$, $z = 5$

(3) L_1: $x = 1+t$, $y = -2+3t$, $z = 4-t$

L_2: $x = 2s$, $y = 3+s$, $z = -3+4s$.

Sol: (1) $\vec{u} = \langle -3, 2, 0 \rangle \parallel L_1$

$\vec{v} = \langle 9, -6, 0 \rangle \parallel L_2$

$-3\vec{u} = \vec{v} \Rightarrow \vec{u} \parallel \vec{v} \Rightarrow L_1 \parallel L_2 \Rightarrow L_1, L_2$ parallel.

(2) $\vec{u} = \langle 1,-1,3 \rangle \parallel L_1$

$\vec{v} = \langle 2,1,0 \rangle \parallel L_2$

$\vec{u} \not\parallel \vec{v}$ since $|\vec{u} \times \vec{v}| \neq 0 \Rightarrow L_1 \not\parallel L_2$

$\therefore L_1, L_2$ not parallel.

$\therefore L_1, L_2$ __may be__ intersector __or__ skewed

To check that: Assume (افترض) that L_1, L_2 intersected

on L_1 : on L_2

$$\left.\begin{aligned} x &= x \\ y &= y \\ z &= z \end{aligned}\right\} \Rightarrow \left.\begin{aligned} t &= 1+2s \\ 3-t &= 2+s \\ 2+3t &= 5 \end{aligned}\right\} \Rightarrow \begin{aligned} t-2s &= 1 \quad \text{—} ① \\ -t-s &= -1 \quad \text{—} ② \\ 3t &= 3 \quad \text{—} ③ \end{aligned}$$

لدينا ٣ معادلات نختار اثنين منها للحل لإيجاد t ، s وبالثالثة للتحقق

اخترت المعادلات ① ، ③ للحل و ② للتحقق

① $\Rightarrow t - 2s = 1$

③ $\Rightarrow 3t = 3 \Rightarrow t = 1 \Rightarrow 1 - 2s = 1 \Rightarrow s = 0$

$\therefore t = 1 , s = 0$

التحقق من ② :

② : $-t - s = -1 \Rightarrow -1 - 0 = -1$ Yes

بما أن معادلة التحقق صحيحة اذا لدينا تقاطع بين الخطين

$\therefore L_1, L_2$ intersects.

لإيجاد نقطة التقاطع نعوض $t = 1$ في L_1 (أو $s = 0$ في L_2)

L_1: $x = 1$, $y = 3 - 1 = 2$, $z = 2 + 3 = 5$

$\therefore$ pt of intersection is $(1, 2, 5)$

ملاحظة

يمكن التأكد من صحة الحل بحساب نقطة من خلال L_2 فيجب أن نحصل على نفس النقطة:

$\boxed{s = 0}$

L_2: $x = 1 + 2(0) = 1$, $y = 2 + 0 = 2$, $z = 5$

$\therefore$ pt. $(1, 2, 5)$

الفرع (3) $\Rightarrow$

(3) $\vec{u} = \langle 1,3,-1\rangle \parallel L_1$

$\vec{v} = \langle 2,1,4\rangle \parallel L_2$

$\vec{u} \not\parallel \vec{v}$ since $|\vec{u} \times \vec{v}| \neq 0 \Rightarrow L_1, L_2$ not parallel.

$\therefore L_1, L_2$ may be intersected or skewed.

To check : Assume that L_1, L_2 intersected:

on L_1 = on L_2

$$\left.\begin{aligned} x &= x \\ y &= y \\ z &= z \end{aligned}\right] \Rightarrow \left.\begin{aligned} 1+t &= 2s \\ -2+3t &= 3+s \\ 4-t &= -3+4s \end{aligned}\right] \Rightarrow \begin{aligned} t-2s &= -1 \text{ — ①} \\ 3t-s &= 5 \text{ — ②} \\ -t-4s &= -7 \text{ — ③} \end{aligned}$$

نختار معادلتين للحل لايجاد s، t والمعادلة الثالثة للتحقق :

اخترت ① ، ② للحل ورقم ③ للتحقق :

①: $t - 2s = -1$

②: $3t - s = 5$

$\Rightarrow$ نضرب ① بـ -3 $\Rightarrow$

$$\begin{aligned} -3t + 9s &= 3 \\ 3t - s &= 5 \\ \hline 8s &= 8 \Rightarrow s = 1 \end{aligned}$$

①: $t - 2(1) = -1 \Rightarrow \boxed{t = 1}$

$\therefore t = 1, s = 1$

نتحقق من خلال المعادلة ③ (معادلة التحقق)

③: $-t - 4s = -7 \Rightarrow -1 - 4 = -7$ No

∴ يوجد خلل ⇐ الخطان غير متقاطعان

$\therefore L_1, L_2$ not intersected

not parallel

$\therefore L_1, L_2$ skew.

* تذكر أن الخطوط المستقيمة إما متوازية أو متقاطعة او متخالفة

Ex5 Find param. eqs of the line that pass through the pts. $A(1,2,3)$, $B(-2,0,1)$.

Sol:

$\overrightarrow{AB} = \langle -3,-2,-2 \rangle \parallel$ line

نضرب المتجه بالعدد -1 يبقى المتجه الناتج يوازي الخط

$\langle 3,2,2 \rangle \parallel$ line

$\therefore$ parametric eqs. are : $x = 1+3t$, $y = 2+2t$, $z = 3+2t$

استخدمنا النقطة A والمتجه $\langle 3,2,2 \rangle$ لايجاد المعادلات

* لاحظ يمكن ان نحصل على جواب آخر باستخدام النقطة B ويكون مكافئ للجواب السابق:

$x = -2+3t$, $y = 2t$, $z = 1+2t$

* لاحظ ايضاً يمكن ايجاد نقطة على الخط بأخذ $t=2$ مثلاً واستخدامها لايجاد المعادلات:

$$t=2 : \left.\begin{array}{l} x = -2+6 = 4 \\ y = 4 \\ z = 1+4 = 5 \end{array}\right] \Rightarrow \text{pt. } C(4,4,5)$$

$\therefore$ eqs. $x = 4+3t$, $y = 4+2t$, $z = 5+2t$

* لاحظ ايضاً يمكن التلاعب بالمتجه الذي يوازي الخط بأخذ متجه مشتق منه بضربه بثابت مثلاً نضرب المتجه $\langle -3,-2,-2 \rangle$ بالعدد $-\frac{5}{\pi}$ مثلاً

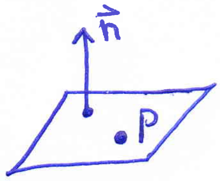
$\langle \frac{15}{\pi}, \frac{10}{\pi}, \frac{10}{\pi} \rangle \parallel$ line

سأستخدم النقطة B مع هذا المتجه:

$x = -2+\frac{15}{\pi}t$, $y = 0+\frac{10}{\pi}t$, $z = 1+\frac{10}{\pi}t$.

وهكذا ـــ ـ ـ ـــ

Def 5: The eq. of the plane that pass through the pt. $P(x_0, y_0, z_0)$ and with normal vector $\vec{n} = \langle a, b, c\rangle$ is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$



Equivalently:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

A vector eq. of the plane is

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

where $\vec{n} = \langle a, b, c\rangle$, $\vec{r} = \langle x, y, z\rangle$, $r_0 = \langle x_0, y_0, z_0\rangle$.

Ex 7: Find the eq. of the phane through the pt. $(0, -6, 7)$ with normal vector $\vec{n} = \langle 2, 5, 6\rangle$. Find the intercepts and sketch the plane.

Sol: The eq. of the plane: $2x + 5y + 6z = 2(0) + 5(-6) + 6(7)$

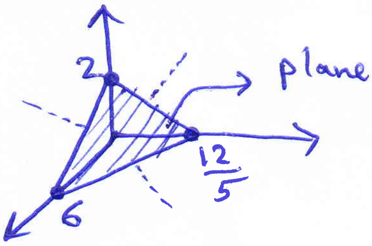
$\therefore 2x + 5y + 6z = 12$

The intercepts:

The x-intercept: $y=0, z=0$: $2x = 12 \Rightarrow x = 6$

" y- " : $x=0, z=0$: $5y = 12 \Rightarrow y = \frac{12}{5}$

" z- " : $x=0, y=0$: $6z = 12 \Rightarrow z = 2$



Ex 8: Find the pt. at which the line L intersects the plane P, where L: $x = 2 + 3t$, $y = -4t$, $z = 5 + t$

$P: 4x + 5y - 2z = 18$

Sol:

نعوض معادلات الخط المستقيم في معادلة المستوى

$4(2+3t) + 5(-4t) - 2(5+t) = 18 \Rightarrow 8 + 12t - 20t - 10 - 2t = 18$

$-10t = 20 \Rightarrow t = -2$

نعوض قيمة t في الخط

نحصل على نقاط التقاطع

$$\left.\begin{array}{l} x = 2 + 3(-2) = -4 \\ y = -4(-2) = 8 \\ z = 5 + -2 = 3 \end{array}\right] \Rightarrow \text{pt. of intersection is } (-4, 8, 3)$$

Ex 9: Find an eq. of the plane that pass through the pts. $A(1,3,2)$, $B(3,-1,6)$, $C(5,2,0)$.

Sol: $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$ $\quad \overrightarrow{AB} = \langle 2,-4,4 \rangle$, $\overrightarrow{AC} = \langle 4,-1,-2 \rangle$

$$\vec{n} = i(-4(-2) - -1(4)) - j(2(-2) - 4(4)) + k(2(-1) - 4(-4))$$
$$= 12i + 20j + 14k$$

$\vec{n} \perp$ plane

لاحظ أن $\vec{n}$ يمكن تبسيطه بالقسمة على 2
الناتج متجه يوازي $\vec{n}$ وبالتالي يعامد المستوى

$\therefore \langle \frac{12}{2}, \frac{20}{2}, \frac{14}{2} \rangle \perp$ plane

$\langle 6, 10, 7 \rangle \perp$ plane

$\therefore$ The eq. of the plane: $6x + 10y + 7z = \underbrace{6(1) + 10(3) + 7(2)}$

عوضنا النقطة A ولو عوضنا النقطة B أو C سيكون نفس العدد الناتج

$$\therefore \; 6x + 10y + 7z = 50$$

Ex 10: Find the eq. of the plane that pass through $\overset{\text{يحوي}}{\text{(contains)}}$ the line of intersection of the two planes $P_1: x + z = -1$ and $P_2: y = z$ and pass through the pt. $A(-3,1,1)$

Sol: عندما يكون في السؤال تقاطع مستويان الأفضل هو إيجاد نقطتين على خط التقاطع لهما وذلك (أولاً) بأخذ أحد المتغيرات الثلاثة x، y، z بقيمة من عندنا مثلاً z=0 وحساب x، y من المعادلتين

$$z = 0: \; \left.\begin{matrix} P_1 \Rightarrow & x = -1 \\ P_2 \Rightarrow & y = 0 \end{matrix}\right] \rightarrow B(-1,0,0) \text{ on line of intersection}$$

(ثانياً) تغيير قيمة المتغير z وإعادة العملية السابقة: مثلاً نأخذ z = 1

$$z = 1: \; \left.\begin{matrix} P_1: & x = -2 \\ P_2: & y = 1 \end{matrix}\right] \Rightarrow C(-2,1,1) \text{ on line of intersection}$$

أصبح لدينا الآن ثلاث نقاط على المستوى هي A(-3,1,1)، B(-1,0,0)، C(-2,1,1)
نعيد الحل كما في مثال 9 في الأعلى:

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & -1 & -1 \\ 1 & 0 & 0 \end{vmatrix} = 0i - 1j + 1k = -j + k$$

The eq. of plane is: $0x - y + z = 0(-3) - 1 + 1 \Rightarrow \boxed{-y + z = 0}$

Ex11: Find the eq. of the eq. of the plane of intersection of the two planes $P_1: x+y-z=1$, $P_2: 3x-3y+z=3$ and parallel to the line $L: x=1$, $y=3-2t$, $z=t$.

Sol:

أولا نجد نقطتين على خط تقاطع المستويان P_1 ، P_2 :

$y=0$: $P_1 \Rightarrow x-z=1$, $P_2 \Rightarrow 3x+z=3$ $\Rightarrow 4x=4 \Rightarrow x=1$

$x-z=1 \Rightarrow z=x-1=0$

$\therefore A(1,0,0)$

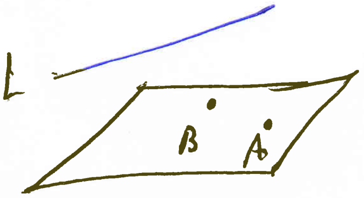
$y=1$: $P_1 \Rightarrow x-z=0$, $P_2 \Rightarrow 3x+z=6$ $\Rightarrow 4x=6 \Rightarrow x=\frac{6}{4}=\frac{3}{2}$

$x-z=0 \Rightarrow z=x=\frac{3}{2}$

$\therefore B(\frac{3}{2},1,\frac{3}{2})$

اصبح لدينا مستوى يحتوي النقطتان A ، B ويوازي الخط L

$\vec{u}=\overrightarrow{AB}=\langle \frac{1}{2},0,\frac{3}{2}\rangle$ // plane

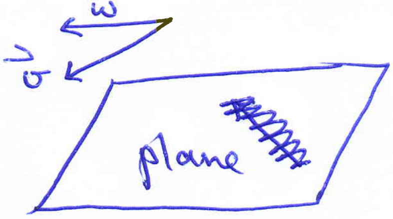


اضرب 2

$\vec{w}=\langle 1,0,3\rangle$ // plane

$\vec{v}=\langle 0,-2,1\rangle$ // L , L // plane $\Rightarrow \vec{v}=\langle 0,-2,1\rangle$ // plane

$\vec{w}$ // plane, $\vec{v}$ // plane



$\vec{v}\times\vec{w} \perp$ plane

$$\vec{v}\times\vec{w}=\begin{vmatrix} i & j & k \\ 0 & -2 & 1 \\ 1 & 0 & 3 \end{vmatrix}=-6i+j+2k \perp \text{plane}$$

$\therefore$ eq. of plane: $-6x+y+2z=\underbrace{-6(1)+0+2(0)}$

عوضنا النقطة A ويمكن تعويض B .

$$\therefore \boxed{-6x+y+2z=-6}$$

Ex 12: Find parametric eqs. of the line of intersection of the two planes $P_1: x+y-2z=8$, $P_2: x-y+z=2$

Sol:

نجد نقطتين على خط التقاطع:

$z=0$: $P_1 \Rightarrow x+y=8$, $P_2 \Rightarrow x-y=2$ $\Rightarrow 2x=10 \Rightarrow x=5$

$x+y=8 \Rightarrow 5+y=8 \Rightarrow y=3$

$A(5,3,0)$ on line

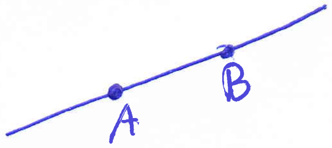
$x=0$: $P_1: y-2z=8$, $P_2: -y+z=2$ $\Rightarrow -z=10 \Rightarrow z=-10$

$y-2z=8 \Rightarrow y+20=8 \Rightarrow y=-12$

$\therefore B(0,-12,-10)$ on line

$\vec{v} = \overrightarrow{AB} = \langle -5,-15,-10 \rangle$ // line

بالقسمة على -5



$\therefore \langle 1,3,2 \rangle$ // line

param. eqs. of line are: $x = 5+t$, $y=3+3t$, $z=2t$

هناك جواب آخر باستخدام النقطة B:

$x=t$, $y=-12+3t$, $z=-10+2t$

لاحظ أيضاً أنه يمكن أيضاً التلاعب بالمتجه الذي يوازي الخط وذلك بضربه بعدد مثلاً. نضربه بالعدد $-\frac{2}{\sqrt{3}}$ مثلاً. فالجواب التالي أيضاً صحيح:

$x = 5-\frac{2}{\sqrt{3}}t$, $y = 3-\frac{6}{\sqrt{3}}t$, $z=-\frac{4}{\sqrt{3}}t$.

سنعتمد على حسابات هذا المثال في المسألة القادمة

Ex 13: Let L be the line of intersection of the two planes $P_1: x+y-2z=8$ and $P_2: x-y+z=2$.

(1) Find the eq. of the plane that pass the pt. $C(1,2,3)$ and contains the line L.

(2) Find the eq. of the plane that pass the pt. $C(1,2,3)$ and perpendicular to the line L.

Sol!

(1) From example ; The pts. $A(5,3,0)$ and $B(0,-12,-10)$ are on L $\Rightarrow A(5,3,0), B(0,-12,-10), C(1,2,3)$ are on the plane

$$\vec{n}=\vec{AB}\times\vec{AC}=\begin{vmatrix} i & j & k \\ -5 & -15 & -10 \\ -4 & -1 & 3 \end{vmatrix}=-35i+55j-55k$$

نكمل الحل في : example

$\div 5$

$\therefore -7i+11j-11k \perp$ plane $\Rightarrow$ eq. $-7x+11y-11z=-7(5)+11(3)$

$\therefore -7x+11y-11z=-2$

(2) $A(5,3,0), B(0,-12,-10)$ on L

$\vec{AB} \parallel L \Rightarrow \langle -5,-15,-10\rangle \parallel L$

$L \perp$ plane $\Rightarrow \langle -5,-15,-10\rangle \perp$ plane

$\Leftarrow -5$ بالقسمة على

$\langle 1,3,2\rangle \perp$ plane

eq. of plane: $x+3y+z=\underbrace{1+3(2)+3}$

* لاحظ استخدمنا النقطة C ولا يجوز استخدام A، B لأنها على الخط L وهذا الخط يعامد المستوى المطلوب وبالتالي فإن A، B ليستا في المستوى

$\therefore$ eq. $\boxed{x+3y+z=10}$

Remark 14: Let P_1, P_2 be two planes such that $\vec{n}_1 \perp P_1$ and $\vec{n}_2 \perp P_2$. Then

(1) P_1 parallel to P_2 : $P_1 \parallel P_2 \iff \vec{n}_1 \parallel \vec{n}_2$.

(2) If P_1, P_2 are not parallel and θ is the angle between P_1, P_2, then $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$.

(3) $P_1 \perp P_2 \iff \vec{n}_1 \cdot \vec{n}_2 = 0$.

Ex 15. Find the angle between $P_1: x-y=3$ and $P_2: x+2y-z=1$

Sol: $\vec{n}_1 = \langle 1,-1,0\rangle$, $\vec{n}_2 = \langle 1,2,-1\rangle$.

$$\cos\theta = \frac{1(1)+-1(2)+0(-1)}{\sqrt{1^2+(-1)^2}\sqrt{1^2+2^2+(-1)^2}} = \frac{-1}{\sqrt{2}\sqrt{6}} = \frac{-1}{\sqrt{12}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{-1}{\sqrt{12}}\right) \approx 106.7^\circ$$

Ex 16. Show that the two planes are orthogonal:

$P_1: 2x-3y+z=0$ and $P_2: x+2y+4z=3$

Sol: $\vec{n}_1 = \langle 2,-3,1\rangle \perp P_1$ and $\vec{n}_2 = \langle 1,2,4\rangle \perp P_2$.

$\vec{n}_1 \cdot \vec{n}_2 = 2(1)+-3(2)+1(4) = 0 \Rightarrow \vec{n}_1 \perp \vec{n}_2 \Rightarrow P_1 \perp P_2$.

Ex 17: Find a that makes P_1 and P_2 parallel:

$P_1: 2x+3ay-2z=1$ and $P_2: \frac{3}{2}ax+9y-3z=0$

Sol: $\vec{n}_1 = \langle 2,3a,-2\rangle \perp P_1$

$\vec{n}_2 = \langle \frac{3}{2}a,9,-3\rangle \perp P_2$

$$\frac{2}{\frac{3}{2}a} = \frac{3a}{9} = \frac{-2}{-3}$$

$\Downarrow$

$\frac{a}{3} = \frac{2}{3}$

$\Rightarrow a = 2$

$P_1 \parallel P_2 \Rightarrow \vec{n}_1 \parallel \vec{n}_2$ ~~$\Rightarrow \vec{n}_1 = b\vec{n}_2$~~

~~$\Rightarrow \langle 4,3a,-2\rangle = \langle \frac{3}{2}ab, 9b, -3b\rangle$~~

~~$\Rightarrow 4 = \frac{3}{2}ab$~~

~~$3a = 9b$~~ ~~$\to 3a = 9(\frac{2}{3}) \Rightarrow 3a = 6 \Rightarrow \boxed{a=2}$~~

~~$-2 = -3b \Rightarrow \boxed{b = \frac{2}{3}}$~~

Rule 18: The distance from the pt. $A(x_0, y_0, z_0)$ to the plane: $P: ax+by+cz+d=0$ is $\text{Dist.} = \dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$.

Ex19: Find the distance from the pt $A(-1,2,3)$ and the plane: $P: 2x-4y+z=1$.

Sol: $P: 2x-4y+z-1=0 \Rightarrow \vec{n} = \langle 2,-4,1 \rangle$

$$\text{Dist.} = \frac{|2(-1)-4(2)+3-1|}{\sqrt{2^2+(-4)^2+1^2}} = \frac{|-8|}{\sqrt{21}} = \frac{8}{\sqrt{21}}.$$

Rule 20: Let P_1, P_2 be two planes s.t. $\vec{n}_1 \perp P_1$ and $\vec{n}_2 \perp P_2$

(1) If $P_1 \not\parallel P_2 \Rightarrow$ The Distance between P_1, P_2 is 0

(2) If $P_1 // P_2$, $\Rightarrow$ The distance between P_1, P_2 is

$\text{Dist.}(P_1, P_2) = \text{Dist.}(A, P_2)$ where A is a pt. on P_1.

Ex: Find the distance between the planes P_1, P_2:

(1) $P_1: x-2y+z=3$ and $P_2: 2x-y+2z=6$

(2) $P_1: 10x+2y-2z=5$ and $P_2: 5x+y-z=1$

Sol: (1) $\vec{n}_1 = \langle 1,-2,1 \rangle \perp P_1$ and $\vec{n}_2 = \langle 2,-1,2 \rangle \perp P_2$

$\vec{n}_1 // \vec{n}_2$ since $|\vec{n}_1 \times \vec{n}_2| \neq 0 \Rightarrow P_1 \not\parallel P_2 \Rightarrow P_1, P_2$ intersected

$\therefore$ Distance $= 0$.

(2) $\vec{n}_1 = \langle 10,2,-2 \rangle \perp P_1$ and $\vec{n}_2 = \langle 5,1,-1 \rangle \perp P_2$.

$\vec{n}_1 // \vec{n}_2$ since $2\vec{n}_2 = \vec{n}_1$. $\Rightarrow P_1 // P_2$.

To find the distance: we find a pt. on P_1:

$P_1: 10x+2y-2z=5$: Take $x=0, y=0$: $10(0)+2(0)-2(z)=5$

$\Rightarrow z = -\frac{5}{2} \Rightarrow A(0,0,-\frac{5}{2})$ on P_1.

$$\text{Dist.}(P_1,P_2) = \text{Dist.}(A,P_2) = \frac{|5(0)+0 - -\frac{5}{2} - 1|}{\sqrt{5^2+1^2+(-1)^2}} = \frac{3/2}{\sqrt{27}} = \frac{1}{2\sqrt{3}}.$$

Sec. 12.6: Cylinders and Quadric Surfaces

Def1: Cylinders are surfaces that results by moving a curve in a direction of a fixed axis (line)

Ex2: (1) $z = x^2$, $x^2 + y^2 = 4$ are cylinders

(2) All planes are cylinders: $x - 2y = 1, z = 3$ are cylinders

(3) $x^2 - 3y + z = 5$, $x + 2y = \cos z$ are not cylinders.

Def3: A quadric surfaces is a graph of a second degree eq. in the variables x, y, z in the form:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where $A, B, C, D, \dots, J$ are scalars s.t. A, B, C not all zeros.

Ex4: Give the name of the trace of the quadric surface:

$2x^2 + y^2 - z^2 = 16$ ① in the plane $z = 1$ ② in the plane $y = 1$

Sol: (1) $z = 1$: $2x^2 + y^2 - 1^2 = 16 \Rightarrow 2x^2 + y^2 = 17 \Rightarrow \frac{x^2}{17/2} + \frac{y^2}{17} = 1$ *Ellipse*

(2) $y = 1$: $2x^2 + 1^2 - z^2 = 16 \Rightarrow 2x^2 - z^2 = 15$ Hyparabola.

Ex5: Identify the trace of the surface $x^2 + y^2 + z = 10$:

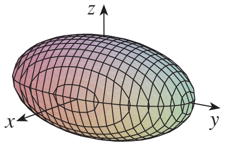
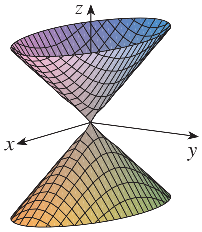
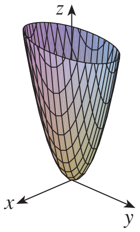
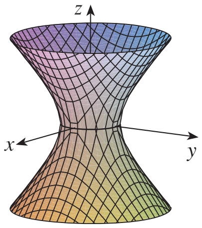
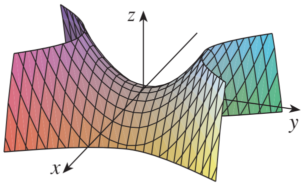
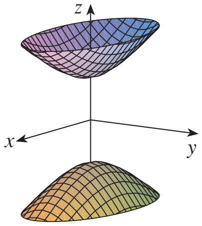
(1) in the plane $z = 1$ (2) in the plane $x = 2$.

Sol: (1) $z = 1$: $x^2 + y^2 + 1 = 10 \Rightarrow x^2 + y^2 = 9$ Circle.

(2) $x = 2$: $2^2 + y^2 + z = 10 \Rightarrow z = 6 - y^2$ Parabola.

14

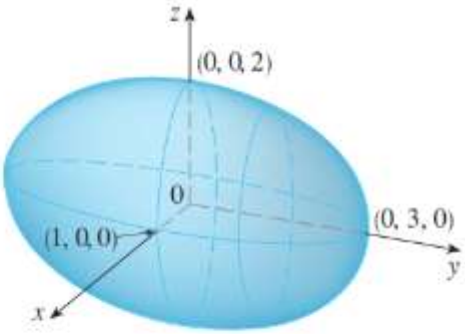
TABLE 1 Graphs of quadric surfaces

Surface	Equation	Surface	Equation
Ellipsoid  	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone  	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid  	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet  	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid  	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets  	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Ex 6: Use traces to sketch the surface $x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$. Show the intercepts and give the name.

Solution. Surface name is **Ellipsoid**

Intercepts:



x-intercept: $y=0, z=0$:

$$x^2+\frac{0^2}{9}+\frac{0^2}{4}=1 \Rightarrow x^2=1 \Rightarrow x=1,-1$$

y-intercept: $x=0, z=0$:

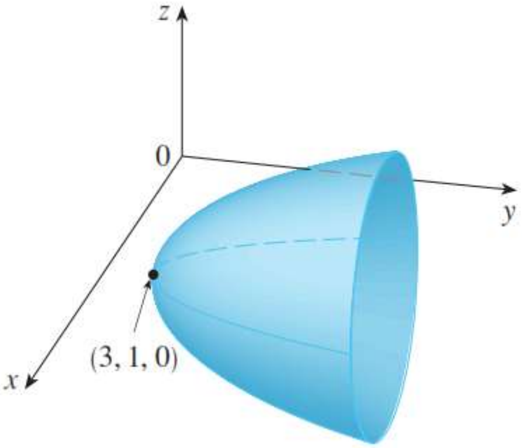
$$0^2+\frac{y^2}{9}+\frac{0^2}{4}=1 \Rightarrow y^2=9 \Rightarrow y=3,-3$$

z-intercept: $x=0, y=0$:

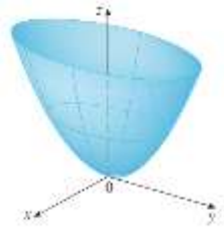
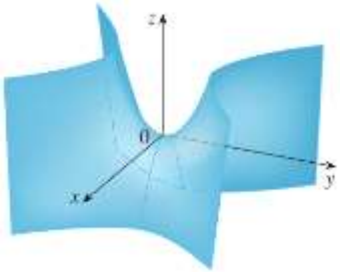
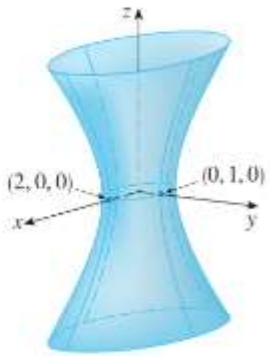
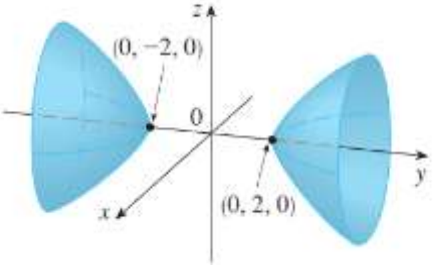
$$0^2+\frac{0^2}{9}+\frac{z^2}{4}=1 \Rightarrow z^2=4 \Rightarrow x=2,-2$$

Ex 8. Classify and sketch the surface $x^2+2z^2-6x-y+10=0$.

Solution.



<u>Ex 7</u>. Identify and sketch the surfaces:

(1) $z = 4x^2 + y^2$ 	(2) $x = 4y^2 + z^2$
(3) $z = y^2 - x^2$ 	(4) $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1.$ 
(5) $\frac{x^2}{4} - y^2 + \frac{z^2}{4} = 1.$	(6) $4x^2 - y^2 + 2z^2 + 4 = 0.$ 
(7) $z^2 = 2x^2 + y^2$	(8) $y^2 = x^2 + 4z^2$
(9) $z = \sqrt{2x^2 + y^2}$	(10) $z = -\sqrt{2x^2 + y^2}$
(11) $2 - y = \sqrt{2x^2 + z^2}$	(12) $x - 1 = \sqrt{(y-1)^2 + z^2}$

21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choice.

21. $x^2 + 4y^2 + 9z^2 = 1$

22. $9x^2 + 4y^2 + z^2 = 1$

23. $x^2 - y^2 + z^2 = 1$

24. $-x^2 + y^2 - z^2 = 1$

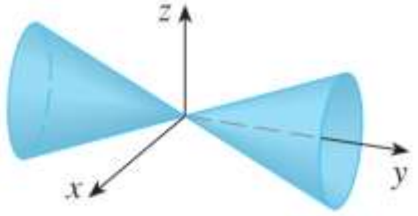
25. $y = 2x^2 + z^2$

26. $y^2 = x^2 + 2z^2$

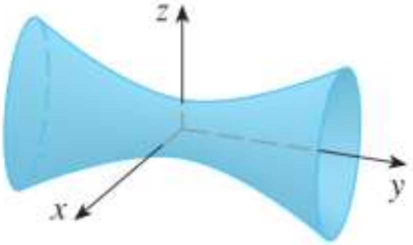
27. $x^2 + 2z^2 = 1$

28. $y = x^2 - z^2$

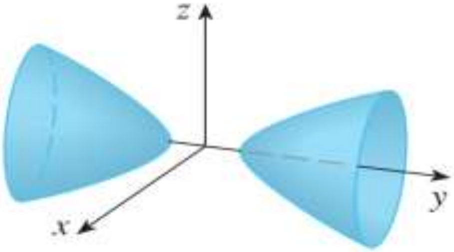
I



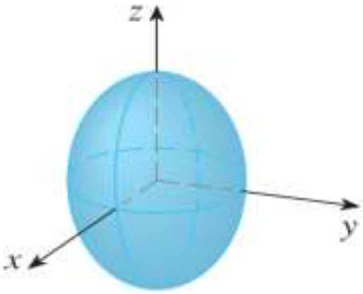
II



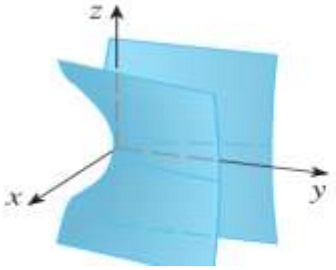
III



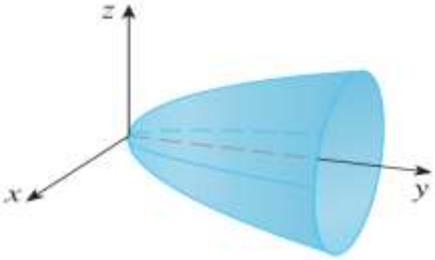
IV



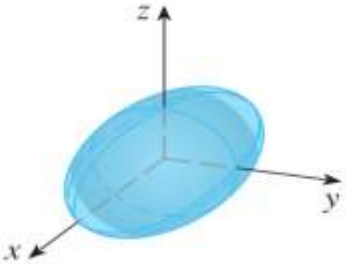
V



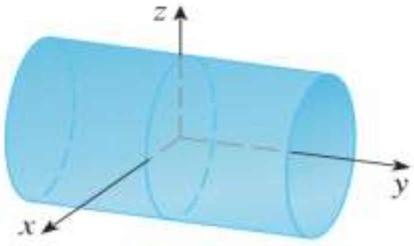
VI



VII



VIII



Ch.13: Vector Functions

Sec 13.1 Vector Functions and Vector curves

Def 1: A vector func., denoted by $\vec{r}(t)$, is a func. in the variable t with domain $A \subseteq \mathbb{R}$ and its range is a set of vectors

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k} \quad , t \in A$$

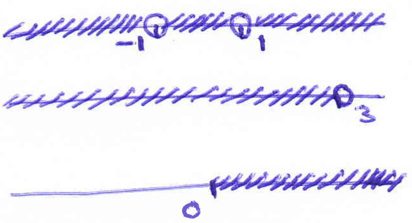
$\therefore \text{Dom}(\vec{r}) = \text{Dom}(f) \cap \text{Dom}(g) \cap \text{Dom}(h)$

Ex 2: Find the domain of $\vec{r}(t) = \langle \frac{t}{t^2-1}, \ln(3-t), \sqrt{t} \rangle$.

Sol: $\frac{t}{t^2-1}$: $t^2-1 \neq 0 \Rightarrow t \neq \pm 1$

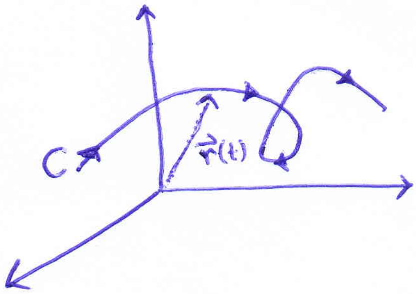
$\ln(3-t)$: $3-t > 0 \Rightarrow t < 3$

$\sqrt{t}$: $t \geq 0$



$\therefore \text{Dom}(\vec{r}) = [0,1) \cup (1,3)$.

Geometrically 3: The vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ defines a vector curve C traced out by the tip of the moving vector $\vec{r}(t)$. The direction of C is as the direction of the moving tip when t increases as shown in the figure. The vector $\vec{r}(t)$ is called a position vector.



Ex 4: Sketch and describe the curve defined by the vector func. :

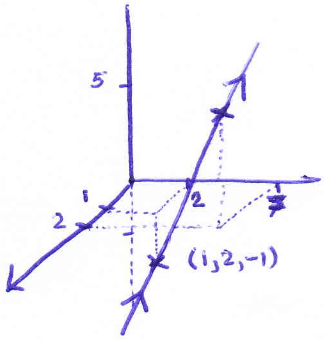
(1) $\vec{r}(t) = \langle 1+t, 2+5t, -1+6t\rangle$.

(2) $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$

(3) $\vec{r}(t) = \langle \sin t, t\rangle$.

Sol: (1) $x = 1+t$, $y = 2+5t$, $z = -1+6t$

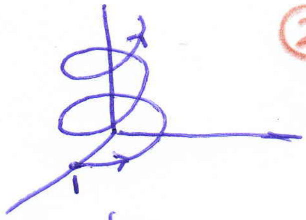
$\therefore$ It is a line

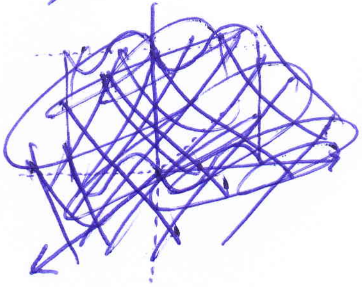


(2) $x = \cos t$, $y = \sin t$, $z = t$

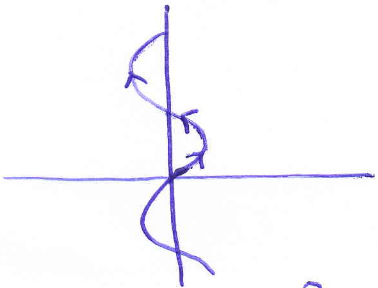
$\cos^2 t + \sin^2 t = 1 \Rightarrow x^2 + y^2 = 1$, $z = t$

circular helix زمبرک دائری





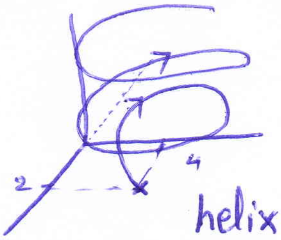
(3) $x = \sin t$, $y = t \Rightarrow x = \sin y$



Remark 5: $\vec{r}(t) = \langle \overset{3}{}\cos t - 1, 5 - \sin t , 3t \rangle$

$$\Rightarrow \left.\begin{array}{l} x = 3\cos t - 1 \\ y = 5 - \sin t \end{array}\right] \Rightarrow \left.\begin{array}{l} \cos t = \frac{x+1}{3} \\ \sin t = 5 - y \end{array}\right] \Rightarrow \begin{array}{l} \cos^2 t + \sin^2 t = 1 \\ \frac{(x+1)^2}{9} + (5-y)^2 = 1 \end{array}$$

$z = 3t$ $\qquad z = 3t$



$t = 0 \Rightarrow (2, 5, 0)$

Remark 6. If $a, b, c > 0$, the vector func.:
$\vec{r}(t) = \langle a\cos t, b\sin t, ct \rangle$ is called a helix. When $a = b$, then this vector func. is called a circular helix.

Rule 7: let $a \in \mathbb{R}$ or $a = \infty$ or $a = -\infty$. and $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

(1) If $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$, $\lim_{t\to a} h(t)$ all exist , then

$$\lim_{t\to a} \vec{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$

(2) If at least one of the limits: $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$, $\lim_{t\to a} h(t)$

does not exist , then $\lim_{t\to a} \vec{r}(t)$ does not exist (DNE).

Ex 8: If $\vec{r}(t) = \left\langle \frac{t^2}{\tan(2t)}, \frac{t-1}{t^2-1}, \frac{\sin \pi t}{\ln(t+1)} \right\rangle$, find $\lim_{t\to 0} \vec{r}(t)$ if exist.

Sol: $\lim_{t\to 0} \frac{t^2}{\tan(2t)} \overset{0/0}{=} \lim \frac{2t}{2\sec^2(2t)}$ نظرية لوبيتال

$$= \frac{2(0)}{2(1)} = 0$$

$$\lim_{t\to 0} \frac{t-1}{t^2-1} = \frac{0-1}{0-1} = 1$$

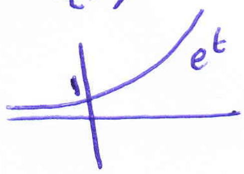
$$\lim \frac{\sin(\pi t)}{\ln(t+1)} \overset{0/0}{=} \lim \frac{\pi \cos(\pi t)}{\frac{1}{t+1}} = \lim_{t\to 0} \pi(t+1)\cos(\pi t) = \pi$$

$\therefore \lim \vec{r}(t) = \langle 0, 1, \pi \rangle$.

Ex 9: Find $\lim_{t\to\infty} \vec{r}(t)$, where $\vec{r}(t) = \left\langle t e^{-t}, \frac{t^3+t}{2t^3-1}, t\sin(\frac{1}{t}) \right\rangle$

Sol: $\lim_{t\to\infty} t e^{-t} \overset{\infty\cdot 0}{=} \lim \frac{t}{e^t} \overset{\infty/\infty}{=} \lim \frac{1}{e^t} = \frac{1}{\infty} = 0$

نظرية لوبيتال



$$\lim_{t\to\infty} \frac{t^3+t}{2t^3-1} = \lim \frac{t^3}{2t^3} = \frac{1}{2}$$

$$\lim t\sin(\tfrac{1}{t}) \overset{\infty\cdot 0}{=} \lim \frac{\sin(\frac{1}{t})}{(\frac{1}{t})} = \lim \frac{\cos(\frac{1}{t})\,(-\frac{1}{t^2})}{-\frac{1}{t^2}} = \lim \cos(\tfrac{1}{t}) = 1$$

$\therefore \lim \vec{r}(t) = \langle 0, \frac{1}{2}, 1 \rangle$.

Ex 10: $\lim_{t\to 0^-} \langle \sqrt{t}, \sin t \rangle$ DNE since $\lim_{t\to 0^-} \sqrt{t}$ DNE

Rule 11: Let $a \in Dom(\vec{r}(t))$. Then $\vec{r}(t)$ is conts. at a

$$\Leftrightarrow \lim_{t\to a} \vec{r}(t) = \vec{r}(a)$$

Ex 12: $\vec{r}(t) = \left\langle \frac{t^2}{\tan(2t)}, \frac{t-1}{t^2-1}, \frac{\sin \pi t}{\ln(t+1)} \right\rangle$ is conts. on Dom($\vec{r}$)

$$\boxed{\begin{array}{l} \tan(2t) \neq 0 \Rightarrow 2t \neq 0, \pm\pi, \pm 2\pi, \ldots \\ 2t \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots \end{array}}$$

$t \neq \pm 1$

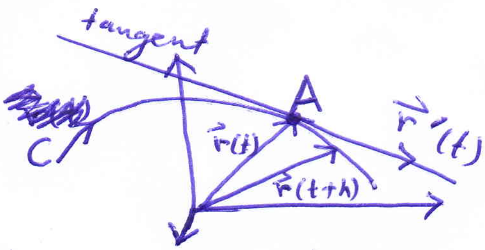
$t+1 > 0 \Rightarrow \boxed{t > -1}$

$t+1 \neq 1 \Rightarrow t \neq 0$

$Dom(\vec{r}) = (-1, \infty) \setminus \{0, -\frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots\}$.

$\therefore$ $\vec{r}$ conts. on Dom($\vec{r}$).

Sec 13.2 Derivative and Integral of Vector Funcs.



Def 1: let $\vec r(t) = \langle f(t), g(t), h(t)\rangle$ be a smooth curve. Then

$$\frac{d\vec r}{dt} = \lim_{h\to 0}\frac{\vec r(t+h)-\vec r(t)}{h}$$ if the limit exist

$$\boxed{\frac{d\vec r}{dt} = \vec r'(t).}$$

Rule 2: If $\vec r(t) = \langle f, g, h\rangle \Rightarrow \vec r' = \langle f', g', h'\rangle$

Ex 3: If $\vec r(t) = \langle 1+t^3, t e^{-t}, \sin 2t\rangle$

$\Rightarrow \vec r' = \langle 3t^2, t(-e^{-t}) + e^{-t}, 2\cos 2t\rangle$

Geometrically 4: Let C be the curve of the vector func. $\vec r(t)$ and A is a pt. on C. Then $\vec r'$ is a vector tangent to the tangent line to the vector curve C that pass through the pt. A (see the figure ⇑). The unit tangent vector to the curve C is

$$\vec T(t) = \frac{\vec r'(t)}{|\vec r'(t)|}$$

Ex5: Find the unit tangent vector of $\vec r(t) = (1+t^3)i + te^{-t}j$ when $t=0$ and at the pt. $A(1,0)$.

Sol: $\vec r' = \langle 3t^2, (1-t)e^{-t}\rangle$.

$\vec r'(0) = \langle 0,1\rangle \Rightarrow \vec T = \frac{\langle 0,1\rangle}{|\langle 0,1\rangle|} = \langle 0,1\rangle.$

At $A(1,0)$: $x=1, y=0$: $x = 1+t^3$, $y = te^{-t} \Rightarrow \begin{matrix} 1+t^3=1 \\ te^{-t}=0 \end{matrix} \Rightarrow t=0$

$\therefore \vec r'|_A = \vec r'(0) = \langle 0,1\rangle \Rightarrow \vec T = \frac{\langle 0,1\rangle}{|\langle 0,1\rangle|} = \langle 0,1\rangle.$

Ex6. Let $\vec{r}(t) = (1+t^3)\, i + te^{-t} j + \sin(2t) k$.

(1) Find a tangent vector of $\vec{r}(t)$.

(2) Find a unit tangent vector to $\vec{r}(t)$ at $t=0$

(3) Find a unit tangent vector to $\vec{r}(t)$ at the pt. $A(1,0,0)$.

Sol: (1) $\vec{r}'(t) = 3t^2 i + (1-t)e^{-t} j + 2\cos(2t) k$

(2) $\vec{r}'(0) = 0i + j + 2k = j + 2k$

$T'(0) = \frac{j+2k}{\sqrt{1^2+2^2}}$ ~~[illegible]~~ $= \frac{j}{\sqrt{5}} + \frac{2}{\sqrt{5}} k$

(3) At the pt. $A(1,0,0)$: $x=1,\ y=0;\ z=0$

$\vec{r}$: $x = 1+t^3,\ y = te^{-t},\ z = \sin 2t$

$\therefore$ $1+t^3 = 1$

$te^{-t} = 0 \Big] \longrightarrow t = 0 \Rightarrow \vec{r}'(0) = j + 2k \Rightarrow T(0) = \frac{j}{\sqrt{5}} + \frac{2}{\sqrt{5}} k$.

$\sin 2t = 0$

Ex7: Let $\vec{r}(t) = \sqrt{t}\, i + (2-t) j$, find $\vec{r}'(t)$ and sketch the position vector $\vec{r}'(1)$ and $\vec{r}(1)$

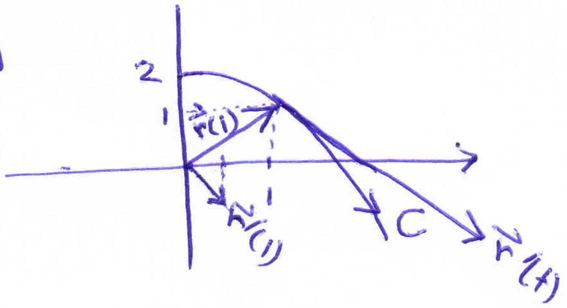
Sol: First we sketch $\vec{r}(t)$: $x = \sqrt{t}$, $y = 2-t \Rightarrow \underbrace{t = x^2;\ t = 2-y}$

~~[illegible]~~ $x^2 = 2-y \Rightarrow y = 2-x^2$

$\vec{r}'(t) = \frac{1}{2\sqrt{t}} i - j \Rightarrow \vec{r}'(1) = \frac{1}{2} i - j$

$\vec{r}(1) = i + j$

$\text{Dom}\ \vec{r}(t) = [0, \infty)$



Ex 8: Find parametric eqs. of the tangent line to the helix $x = 2\cos t$, $y = \sin t$, $z = t$ at the pt. $A(0, 1, \frac{\pi}{2})$

Sol: $\vec{r} = \langle 2\cos t, \sin t, t \rangle$, At A: $x=0 \Rightarrow 2\cos t = 0$

$y = 1$ $\quad \sin t = 1$

$z = \frac{\pi}{2}$ $\quad \boxed{t = \frac{\pi}{2}}$

$\vec{r}' = \langle -2\sin t, \cos t, 1 \rangle$

$r'(\frac{\pi}{2}) = \langle -2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1 \rangle = \langle -2, 0, 1 \rangle$

$r' \parallel$ tangent line $\Rightarrow \langle -2, 0, 1 \rangle \parallel$ tangent line:

$\therefore$ param. eqs: $x = 0 - 2t$, $y = 1 + 0t$, $z = \frac{\pi}{2} + t$

$x = -2t$, $y = 1$, $z = \frac{\pi}{2} + t$

Rule 9: let $\vec{u}(t), \vec{v}(t)$ be vector funcs, and $f(t)$ func. and a, b scalars. Then

(1) $\frac{d}{dt}(a\vec{u} \pm b\vec{v}) = a\frac{d\vec{u}}{dt} \pm b\frac{d\vec{v}}{dt}$

(2) $\frac{d}{dt}(f\vec{u}) = f\frac{d\vec{u}}{db} + f'\vec{u}$

(3) $\frac{d}{dt}(\vec{u}.\vec{v}) = \vec{u}.\frac{d\vec{v}}{dt} + \vec{v}.\frac{d\vec{u}}{dt}$

(4) $\frac{d}{dt}(\vec{u}\times\vec{v}) = \vec{u}\times\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\times\vec{v}$

(5) $\frac{d}{dt}\vec{u}(f(t)) = \vec{u}'(f)\, f'$ (chain Rule)

Ex 10: Let $\vec{u} = \langle t, e^{-t}, \sin(2t)\rangle$, $\vec{v} = \langle t-2, t^2+2, e^{2t}\rangle$

Then $\frac{d}{dt}(2\vec{u} - 3\vec{v}) = 2\frac{d\vec{u}}{dt} - 3\frac{d\vec{v}}{dt} =$

$= 2\langle 1, -e^{-t}, 2\cos t\rangle - 3\langle 1, 2t, 2e^{2t}\rangle$

$= \langle -1, -2e^{-t} - 6t, 4\cos t - 6e^{2t}\rangle.$

Thm 11: Let $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$

$\int \vec{r}(t)dt = \langle \int f(t)dt, \int g(t)dt, \int h(t)dt\rangle$

Ex 12: Let $\vec{r}(t) = 3\cos t\, i + \sin t\, j + 2t\, k$

(1) Find $\int \vec{r}(t)dt$ (2) Find $\int_0^{\pi/2} \vec{r}(t)\,dt$

Sol: (1) $\int \vec{r}(t)dt = \langle \int 3\cos t, \int \sin t, \int 2t\rangle$

$= \langle 3\sin t + C_1, -\cos t + C_2, t^2 + C_3\rangle$

$= \langle 3\sin t, -\cos t, t^2\rangle + \vec{C}$, where $\vec{C} = \langle C_1, C_2, C_3\rangle$

(2) $\int_0^{\pi/2} \vec{r}(t)\,dt = \langle \int_0^{\pi/2} 3\cos t, \int_0^{\pi/2} \sin t, \int_0^{\pi/2} 2t\rangle$

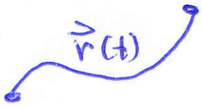
$= \langle 3\sin t, -\cos t, t^2\rangle \Big]_0^{\pi/2}$

$= \langle 3\sin\frac{\pi}{2}, -\cos\frac{\pi}{2}, \frac{\pi^2}{4}\rangle - \langle 3\sin 0, -\cos 0, 0^2\rangle$

$= \langle 3, 0, \frac{\pi^2}{4}\rangle - \langle 0, -1, 0\rangle$

$= \langle 3, 1, \frac{\pi^2}{4}\rangle$

Sec 13.3 Arc length and Curvature. ⑦



Thm 1: Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$

(1) The arc length of $\vec{r}(t)$ from $t=a$ to $t=b$ is $L = \int_a^b |\vec{r}'|\,dt$

(2) The arc length func. of $\vec{r}(t)$ $\boxed{\text{in the direction of increasing } t}$ is $s(t) = \int_a^t |\vec{r}'|\,dt$

(3) Parametrization of $\vec{r}(t)$ with respect to the arc length func. s is $\vec{r}(t(s))$

نحسب t بدلالة s من العلاقة، $s=\int_a^t |r'|$ ثم نعوض يعني نجعل t موضوع

القانون ثم نعوضها مكان t في صيغة $\vec{r}(t)$

Ex 2: Let $\vec{r}(t) = \cos t\, i + \sin t\, j + t\, k$

(1) Find the arc length of $\vec{r}(t)$ from the pts. $A(1,0,0)$ to $B(1,0,2\pi)$

(2) Find the arc length func. in the direction of increasing t from the pt. $A(1,0,0)$.

(3) reparametrize $\vec{r}(t)$ with respect to the arc length from the pt. $A(1,0,0)$ in the direction of increasing t.

Sol: (1) $\vec{r}' = \langle -\sin t, \cos t, 1 \rangle$

$A(1,0,0)$: $\left.\begin{array}{l} \cos t = 1 \\ \sin t = 0 \\ t = 0 \end{array}\right] \Rightarrow t = 0$ | $B(1,0,2\pi)$ $\left.\begin{array}{l} \cos t = 1 \\ \sin t = 0 \\ t = 2\pi \end{array}\right] \to t = 2\pi$

$\therefore \quad 0 \le t \le 2\pi$

(1) $L = \int_0^{2\pi} |\vec{r}'| = \int_0^{2\pi} \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}$

$= \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}(2\pi - 0) = 2\sqrt{2}\pi$

(2) $s = \int_0^t |\vec{r}'|\,dt = \int_0^t \sqrt{2} = \sqrt{2}\, t \Rightarrow s = \sqrt{2}\, t$

(3) $s = \sqrt{2} t \Rightarrow t = \frac{s}{\sqrt{2}}$

$\therefore \vec{r}(s) = \cos\left(\frac{s}{\sqrt{2}}\right) i + \sin\left(\frac{s}{\sqrt{2}}\right) j + \frac{s}{\sqrt{2}} k.$

Def 3: The curvature of a curve C, written as $K(t)$, is the magnitude of the rate of change of the unit tangent vector with respect to the arc length. Also, it is a measure of how quickly the curve changes direction at a pt.

Rule 4: Let C be a curve and let $\vec{r}(t)$ be its vector func.

Then (1) $K(t) = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$ (2) $K(t) = \dfrac{|\vec{T}\,'|}{|\vec{r}\,'|}$

(3) If C is a plane curve, and $\vec{r}(t) = x\,i + f(x)\,j$

then $K(t) = \dfrac{|f''|}{\left(1+(f')^2\right)^{3/2}}$

Ex 5: Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$

(1) at a general pt. (2) At the origin.

Sol: (1) $r' = \langle 1, 2t, 3t^2 \rangle$, $\vec{r}\,'' = \langle 0, 2, 6t \rangle$

$|\vec{r}\,'|^2 = 1 + 4t^2 + 9t^4$, $|\vec{r}\,''|^2 = 4 + 36t^2$

$$K(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{\sqrt{|\vec{r}\,'|^2\,|\vec{r}\,''|^2 - (\vec{r}\,' \cdot \vec{r}\,'')^2}}{|\vec{r}\,'|^3}$$

$$= \frac{\sqrt{(1+4t^2+9t^4)(4+36t^2) - (4t+18t^3)^2}}{(1+4t^2+9t^4)^{3/2}}$$

(2) At the origin: pt. $(0,0,0) \Rightarrow \left.\begin{matrix} t=0 \\ t^2=0 \\ t^3=0 \end{matrix}\right] \rightarrow t=0$

$$\therefore K(0) = \frac{\sqrt{1(4) - 0^2}}{1^{3/2}} = 2$$

Ex 6: Find the curvature of $y = x^2$ at the pt. $(2,4)$.

Sol: let $x = t \Rightarrow y = t^2 \Rightarrow \vec{r} = \langle t, t^2 \rangle \Rightarrow r' = \langle 1, 2t \rangle, \vec{r}\,'' = \langle 0, 2 \rangle$

pt. $(2,4) \Rightarrow x = t \Rightarrow t = 2 \Rightarrow \vec{r}\,' = \langle 1, 4 \rangle$, $\vec{r}\,'' = \langle 0, 2 \rangle$

$$K(2) = \frac{\sqrt{17(4) - 8^2}}{(17)^{3/2}} = \frac{\sqrt{68-64}}{17\sqrt{17}} = \frac{2}{17\sqrt{17}} = \frac{2\sqrt{17}}{289}$$

Ex 7: Show that the curvature of a circle of radius a is $\frac{1}{a}$.

Sol: Consider the circle $x^2+y^2=a^2$ in 2D.

∴ $x = a\cos t$, $y = a\sin t$.

∴ $\vec{r}(t) = \langle x, y\rangle = \langle a\cos t, a\sin t\rangle$

$\vec{r}\,' = \langle -a\sin t, a\cos t\rangle$

$\vec{r}\,'' = \langle -a\cos t, -a\sin t\rangle$

$|\vec{r}\,'| = \sqrt{(-a\sin t)^2+(a\cos t)^2} = \sqrt{a^2(\sin^2 t+\cos^2 t)} = \sqrt{a^2} = a$

$|\vec{r}\,''| = \sqrt{(-a\cos t)^2+(-a\sin t)^2} = \sqrt{a^2(\cos^2 t+\sin^2 t)} = \sqrt{a^2} = a$

$$K(t) = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \frac{\sqrt{a^2(a^2) - (a^2\sin t\cos t + -a^2\cos t\sin t)^2}}{a^3}$$

$$= \frac{\sqrt{a^4-0}}{a^3} = \frac{a^2}{a^3} = \frac{1}{a}.$$

Def 8: Let $\vec{r}(t)$ be a smooth curve. ① The unit normal vector to $\vec{r}(t)$ is $\vec{N}(t) = \frac{\vec{T}\,'}{|\vec{T}\,'|}$.

② The binormal vector is $\vec{B}(t) = \vec{T} \times N$.

Ex:9: Find the normal and binormal vectors of the circular helix $\vec{r}(t) = \cos t\, i + \sin t\, j + t k$.

Sol: $\vec{r}\,' = \langle -\sin t, \cos t, 1\rangle \Rightarrow |\vec{r}\,'| = \sqrt{\sin^2 t+\cos^2 t+1} = \sqrt{2}$

∴ $\vec{T} = \frac{r'}{|\vec{r}\,'|} = \langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

$\vec{T}\,' = \langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0\rangle \Rightarrow |\vec{T}\,'| = \sqrt{\frac{\cos^2 t}{2}+\frac{\sin^2 t}{2}} = \frac{1}{\sqrt{2}}$

∴ Normal vector: $\vec{N}(t) = \frac{\vec{T}\,'}{|\vec{T}\,'|} = \frac{1}{1/\sqrt{2}}\langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0\rangle$

∴ $N(t) = \langle -\cos t, -\sin t, 0\rangle$.

$$\vec{B}(t) = \vec{T}\times\vec{N} = \begin{vmatrix} i & j & k \\ -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{\sin t}{\sqrt{2}} i - \frac{\cos t}{\sqrt{2}} j + \frac{k}{\sqrt{2}}.$$

Remark 10: A vector equation ~~and parametric equations~~ for the line segement that joins $P(a,b,c)$ to $Q(d,e,f)$ is $\vec{r}(t) = \langle tP+(1-t)Q \rangle$, $0 \le t \le 1$

Ex 11: Find a vector equation and parametric equations for the line segement that joins $P(1,0,1)$ to $Q(2,3,1)$

Sol:
$$\vec{r}(t) = \langle tP+(1-t)Q \rangle$$
$$= \langle t(1)+(1-t)2 \,,\; t(0)+(1-t)3 \,,\; t(1)+(1-t)(1) \rangle$$
$$= \langle 2-t \,,\; 3-3t \,,\; 1 \rangle \,,\quad 0 \le t \le 1$$

parametric eqs:
$$x = 2-t \,,\; y = 3-3t \,,\; z = 1 \,,\quad 0 \le t \le 1.$$

Ex 12: Find a vector func. that represents the curve of intersection of the two surfaces:

(1) $x^2+y^2=4$ and $z=xy$ (2) $z=x^2-y^2$ and $x^2+y^2=1$

(3) $z=\sqrt{x^2+y^2}$, $z=1+y$

Sol: (1) $x = 2\cos t$, $y = 2\sin t$
$$z = 4\cos t \sin t = 2(2\cos t \sin t) = 2\sin 2t$$
$$\therefore \vec{r}(t) = \langle 2\cos t ,\, 2\sin t ,\, 2\sin 2t \rangle$$

(2) ~~$x = \cos t$~~ $x^2+y^2=1 \Rightarrow x = \cos t$, $y = \sin t$
$$z = x^2-y^2 \Rightarrow z = \cos^2 t - \sin^2 t = \cos 2t$$
$$\therefore \vec{r}(t) = \cos t \; i + \sin t \; j + \cos 2t \; k.$$

(3) $x = t$: $z = \sqrt{x^2+y^2}$, $z = 1+y \Rightarrow 1+y = \sqrt{t^2+y^2}$
$$(1+y)^2 = t^2+y^2 \Rightarrow 1+2y+\cancel{y^2} = t^2+\cancel{y^2} \Rightarrow y = \frac{t^2-1}{2}$$
$$\therefore z = 1+y = 1+\frac{t^2-1}{2} = \frac{t^2+1}{2}$$
$$\therefore \vec{r}(t) = t\,i + \frac{t^2-1}{2}\,j + \frac{t^2+1}{2}\,k.$$

Ex 13: show that the curve with parametric eqs. $x = t^2$, $y = 1 - 3t$, $z = 1 + t^3$ passes through the pt. $A(1, 4, 0)$ and not through $B(4, 7, -6)$

Sol: A: $t^2 = 1 \Rightarrow t = \pm 1$
$1 - 3t = 4 \Rightarrow t = -1$ $\Big] \Rightarrow \therefore t = -1$ $\Big] \Rightarrow \therefore t = -1$
$1 + t^3 = 0 \Rightarrow t^3 = -1 \Rightarrow t = -1$

when $t = -1$ the curve passes through A.

B: $t^2 = 4 \Rightarrow t = \pm 2$
$1 - 3t = 7 \Rightarrow t = -2$
$1 + t^3 = -6 \Rightarrow t^3 = -7 \Rightarrow t = -\sqrt[3]{7}$

لا يوجد قيمة مشتركة للمتغير t

$\therefore$ there is no ~~t, such that~~ value for t

$\therefore$ the curve does not pass through B.

Ex 14: sketch the curve of the vector func.

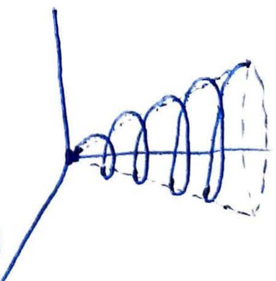
(1) $x = t\cos t$ $y = t$, $z = t\sin t$ $t \geq 0$

(2) $x = \cos^2 t$, $y = \sin^2 t$, $z = t$

Sol: (1) $(t\cos t)^2 + (t\sin t)^2 = t^2(\cos^2 t + \sin^2 t) = t^2$

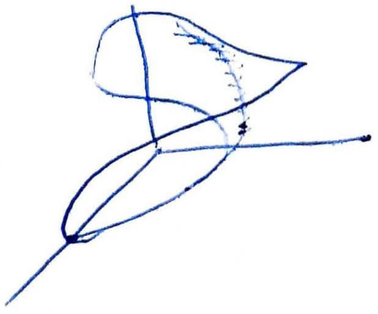
$\Rightarrow x^2 + z^2 = t^2 = y^2 \Rightarrow y^2 = x^2 + z^2$

$t \geq 0 \Rightarrow y = \sqrt{x^2 + z^2}$ $\Longrightarrow$



(2) $\cos^2 t + \sin^2 t = 1$
$x + y = 1$

$0 \leq \cos^2 t \leq 1$ $\Big] \Rightarrow 0 \leq x \leq 1$
$0 \leq \sin^2 t \leq 1$ $\Big] \Rightarrow 0 \leq y \leq 1$



Ex 15: Let $\vec{r}(t)$ be a smooth curve. Show that the unit tangent vector $\vec{T}(t)$ is orthogonal to the unit normal vector $\vec{N}(t)$ for all t.

Sol: $|\vec{T}|^2 = 1 \Rightarrow \vec{T}\cdot\vec{T} = 1 \Rightarrow \frac{d}{dt}(\vec{T}\cdot\vec{T}) = \frac{d}{dt}(1)$

$\vec{T}\cdot\vec{T}' + \vec{T}'\cdot\vec{T} = 0 \Rightarrow 2\vec{T}\cdot\vec{T}' = 0 \Rightarrow \vec{T}\cdot\vec{T}' = 0$

$\frac{1}{|\vec{T}'|}(\vec{T}\cdot\vec{T}') = 0 \Rightarrow \vec{T}\cdot\frac{\vec{T}'}{|\vec{T}'|} = 0 \Rightarrow \vec{T}\cdot\vec{N} = 0$

$\therefore \vec{T} \perp \vec{N}$.

Ex 16: let $\vec{r}(t)$ be a smooth curve such that $|\vec{r}(t)| = c$ (constant). Show that $\vec{r}(t)$ is orthogonal to $\vec{r}'(t)$

Sol: $|\vec{r}|^2 = c^2 \Rightarrow \vec{r}\cdot\vec{r} = c^2 \Rightarrow \frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{dc^2}{dt}$

$\Rightarrow \vec{r}\cdot\vec{r}' + \vec{r}'\cdot\vec{r} = 0 \Rightarrow 2\vec{r}\cdot\vec{r}' = 0 \Rightarrow \vec{r}\cdot\vec{r}' = 0$

$\therefore \vec{r} \perp \vec{r}'$.

Ex 17: Show that the curvature of any line is alway 0.

Sol: $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$

$\therefore \vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$

$\vec{r}' = \langle a, b, c\rangle \Rightarrow \vec{r}'' = \langle 0, 0, 0\rangle = \vec{0}$.

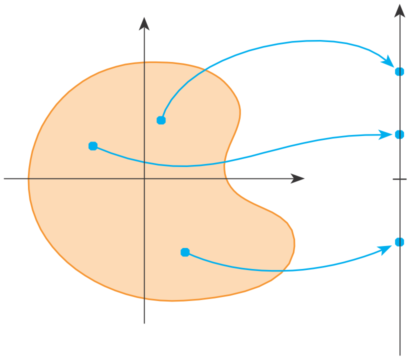
$\therefore \vec{r}' \times \vec{r}'' = \vec{0} \Rightarrow |\vec{r}' \times \vec{r}''| = 0$

$\therefore K(t) = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{0}{|\vec{r}'|^3} = 0$

Chapter 14: Partial Derivates

14.1 Functions of Several Variables

Definition 1: A **function f of two variables** is a rule that assigns to each ordered pair of real numbers x, y in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is,
$D = \{(x, y) \in \mathbb{R}^2 : f(x, y) \in \mathbb{R}\}$ and
$range = \{z \in \mathbb{R} : z = f(x, y), (x, y) \in D\}$.



Example 2: Let $f(x, y) = x + \ln(y^2 - x)$. Then $f(3,2) = 3 + \ln(2^2 - 3) = 3 + \ln 1 = 3$

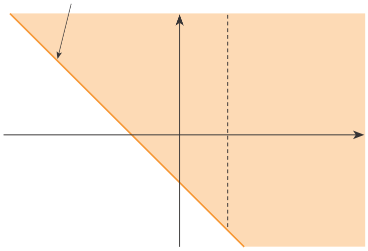
Example 3: Find and sketch the domain of the functions:

(a) $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$ (b) $f(x, y) = \ln(y^2 - x)$ (c) $f(x, y) = \sqrt{9 - x^2 - y^2}$

(d) $f(x, y) = \sqrt{xy}$ (e) $f(x, y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$

Solution:

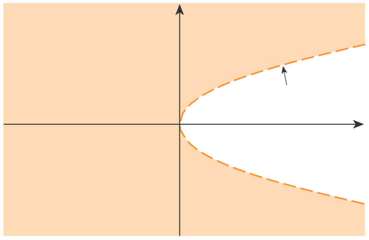
(a) $Dom(f) = \{(x, y) \in \mathbb{R}^2 : x + y + 1 \geq 0, x \neq 1\}$



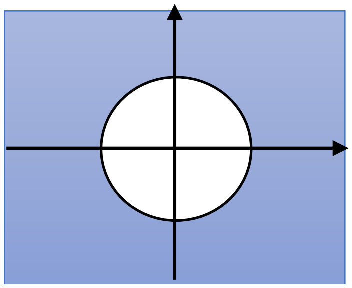
** $x + y + 1 = 0$
$x + y = -1$

(b) $Dom(f) = \{(x, y) \in \mathbb{R}^2 : y^2 - x > 0\}$
$= \{(x, y) \in \mathbb{R}^2 : y^2 > x\}$

** $y^2 = x$



(c) $Dom(f) = \{(x, y) \in \mathbb{R}^2 : 9 - x^2 - y^2 \geq 0\}$
$= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 9\}$

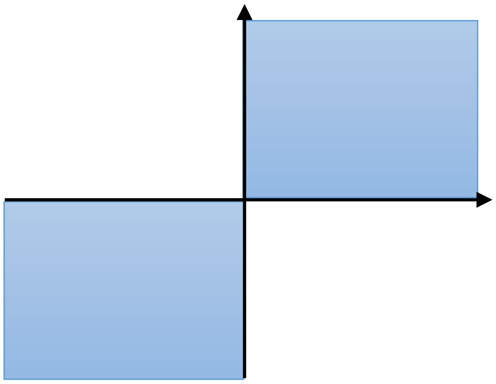


(d) $Dom(f) \quad = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$

$xy \geq 0$

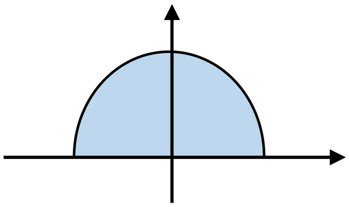
$\Rightarrow x \geq 0$ and $y \geq 0$ (first quadrant)

Or $x \leq 0$ and $y \leq 0$ (second quadrant)



(e) $Dom(f) \quad = \{(x, y) \in \mathbb{R}^2 : y \geq 0, 25 - x^2 - y^2 \geq 0\}$

$=\{(x, y) \in \mathbb{R}^2 : y \geq 0, x^2 + y^2 \leq 25\}$



Example 4: Find the domain and range of the function:

$$f(x, y) = 2 - 3\sqrt{9 - x^2 - y^2}$$

Solution: $Dom(f) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 9\}$.

For range f: Let $z = f(x, y) \implies z = 2 - 3\sqrt{9 - x^2 - y^2}$. So,

** $\sqrt{9 - x^2 - y^2} \geq 0 \implies 3\sqrt{9 - x^2 - y^2} \geq 0 \implies -3\sqrt{9 - x^2 - y^2} \leq 0$

$\implies 2 - 3\sqrt{9 - x^2 - y^2} \leq 2 \implies z \leq 2$

** $x^2 + y^2 \geq 0 \implies -x^2 - y^2 \leq 0 \implies 9 - x^2 - y^2 \leq 9$

$\implies \sqrt{9 - x^2 - y^2} \leq \sqrt{9} \implies \sqrt{9 - x^2 - y^2} \leq 3 \implies -3\sqrt{9 - x^2 - y^2} \geq -9$

$\implies 2 - 3\sqrt{9 - x^2 - y^2} \geq 2 - 9 \implies 2 - 3\sqrt{9 - x^2 - y^2} \geq -7 \implies z \geq -7$

So, $-7 \leq z \leq 2 \qquad \implies range(f) = [-7,2]$

Example 5: Find the domain and range of the function:

$$f(x, y) = x^2 + 2y^2$$

Solution: $Dom(f) = \{(x, y) \in \mathbb{R}^2\} = \mathbb{R}^2$.

For range f: Let $z = f(x, y) \implies \quad z = 2 + x^2 + 2y^2$. So,

** $x^2 + 2y^2 \geq 0 \qquad \implies 2 + x^2 + 2y^2 \geq 2 \qquad \implies z \geq 2$

$\implies \quad range(f) = [2, \infty)$

Definition 6: If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in D$.

Example 7: Sketch the graph of the functions:

(a) $f(x, y) = 6 - 3x - 2y$ (b) $f(x, y) = \sqrt{9 - x^2 - y^2}$

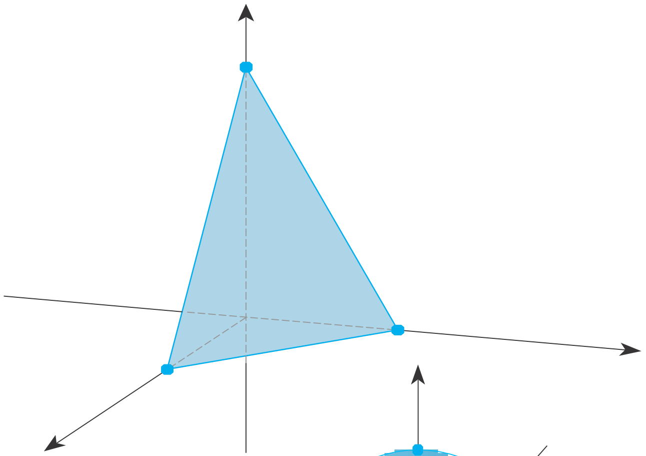
(c) $f(x, y) = 6 - \sqrt{x^2 + 2y^2}$ (d) $f(x, y) = x^2 + 2y^2$

Solution:

(a) $z = f(x, y) \quad \implies \quad z = 6 - 3x - 2y \quad \implies \quad 3x + 2y + z = 6$ (is a plane)

Intercepts:

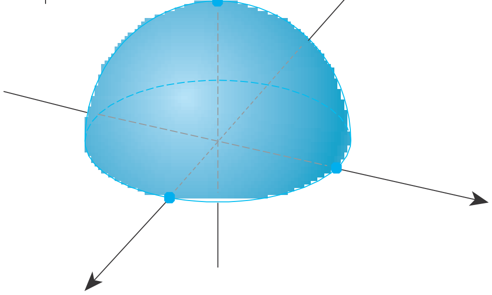
x-intercept: $y = z = 0$ $\implies 3x + 2(0) + 0 = 6 \implies 3x = 6$ $x = 2$
y-intercept: $x = z = 0$ $\implies 3(0) + 2y + 0 = 6 \implies 2y = 6$ $y = 3$
z-intercept: $x = y = 0$ $\implies 3(0) + 2(0) + z = 6 \implies z = 6$

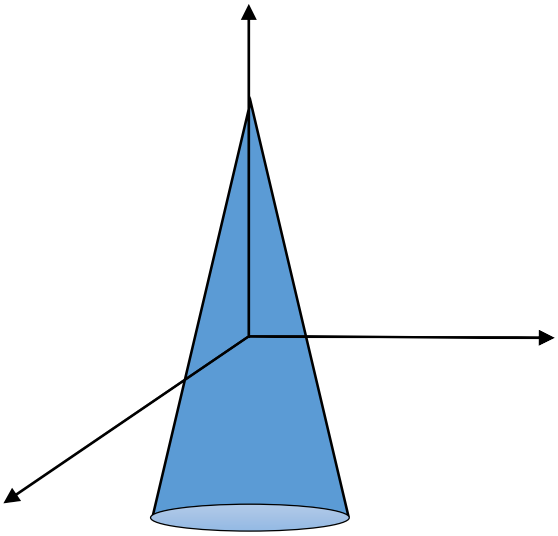
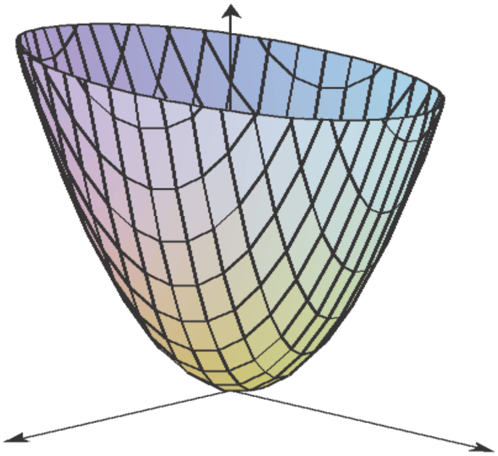


(b) $z = f(x, y) \quad \implies \quad z = \sqrt{9 - x^2 - y^2}$

$\implies \quad z^2 = 9 - x^2 - y^2$ with $z \geq 0$

$\implies \quad x^2 + y^2 + z^2 = 9$ with $z \geq 0$ (is a hemisphere)



(c) $z = f(x,y)$ $\Longrightarrow \quad z = 6 - \sqrt{x^2 + 2y^2}$ $\Longrightarrow \quad z - 6 = -\sqrt{x^2 + 2y^2}$ $\Longrightarrow \quad (z-6)^2 = x^2 + 2y^2$ (cone) 	(d) $z = f(x,y)$ $\Longrightarrow \quad z = x^2 + 2y^2$ (paraboloid) 

Remark 8: Describe how the graph of the function g can be obtained from the graph of the function f in each of the following cases:

(1) $g(x,y) = f(x,y) + 2$

(2) $g(x,y) = 2f(x,y)$

(3) $g(x,y) = -f(x,y)$

(4) $g(x,y) = 2 - f(x,y)$

(5) $g(x,y) = f(x-2,y)$

(6) $g(x,y) = f(x,y+2)$

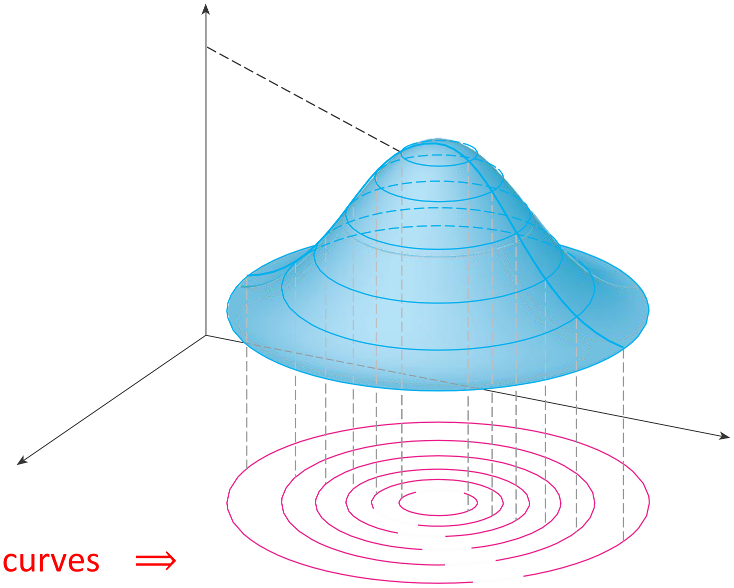
(7) $g(x,y) = f(x+3,y-2)$

Solution:

(1) Shift the graph of f upward 2 units.
(2) Stretch the graph of f vertically by a factor of 2 units.
(3) Reflect the graph of f about the xy-plane.
(4) Reflect the graph of f about the xy-plane and then shift it upward 2 units.
(5) Shift the graph of f in the direction of the positive x-axis 2 units.
(6) Shift the graph of f in the direction of the negative y-axis 2 units.
(7) Shift the graph of f in the direction of the negative x-axis 3 units and then shift it in the direction of the positive y-axis 2 units.

Definition 9: The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant ($k \in range(f)$).

- The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.


Level curves $\Longrightarrow$

- The graph of several level curves in the plane is called a contour map of the function f

Example 10: Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$.

Solution: The level curves are:

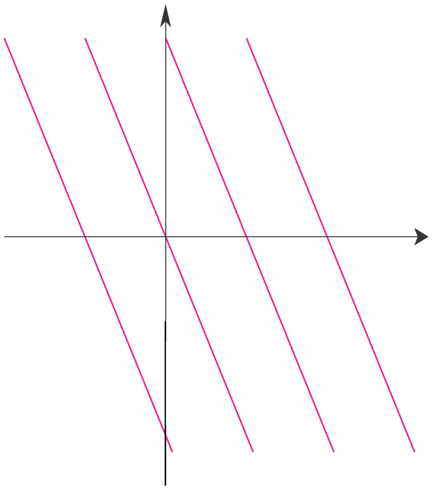
$$6 - 3x - 2y = k \quad \Longrightarrow \quad 3x + 2y = 6 - k$$

$$k = -6 \quad \Longrightarrow \quad 3x + 2y = 12 \ \text{(line with slope} -\frac{3}{2}\text{)}$$

$$k = 0 \quad \Longrightarrow \quad 3x + 2y = 6 \ \text{(line with slope} -\frac{3}{2}\text{)}$$

$$k = 6 \quad \Longrightarrow \quad 3x + 2y = 0 \ \text{(line with slope} -\frac{3}{2}\text{)}$$

$$k = 12 \quad \Longrightarrow \quad 3x + 2y = 12 \ \text{(line with slope} -\frac{3}{2}\text{)}$$


Contour map

Example 11: Sketch the level curves of the function $f(x,y)=\sqrt{9-x^2-y^2}$ for the values $k=0,1,2,3$

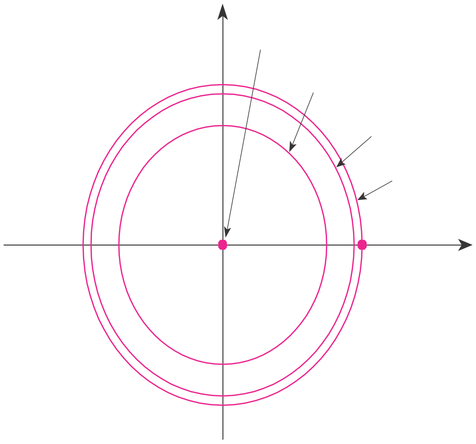
Solution: The level curves are: $f(x,y)=k$

$\Rightarrow \sqrt{9-x^2-y^2}=k \quad \text{for } k=0,1,2,3$

$\Rightarrow 9-x^2-y^2=k^2 \quad \text{for } k=0,1,2,3$

$\Rightarrow x^2+y^2=9-k^2 \quad \text{for } k=0,1,2,3$

$k=0 \implies x^2+y^2=9$ (circle)
$k=1 \implies x^2+y^2=8$ (circle)
$k=2 \implies x^2+y^2=5$ (circle)
$k=3 \implies x^2+y^2=0$ $\implies x=0, y=0 \implies$ A point $(0,0)$



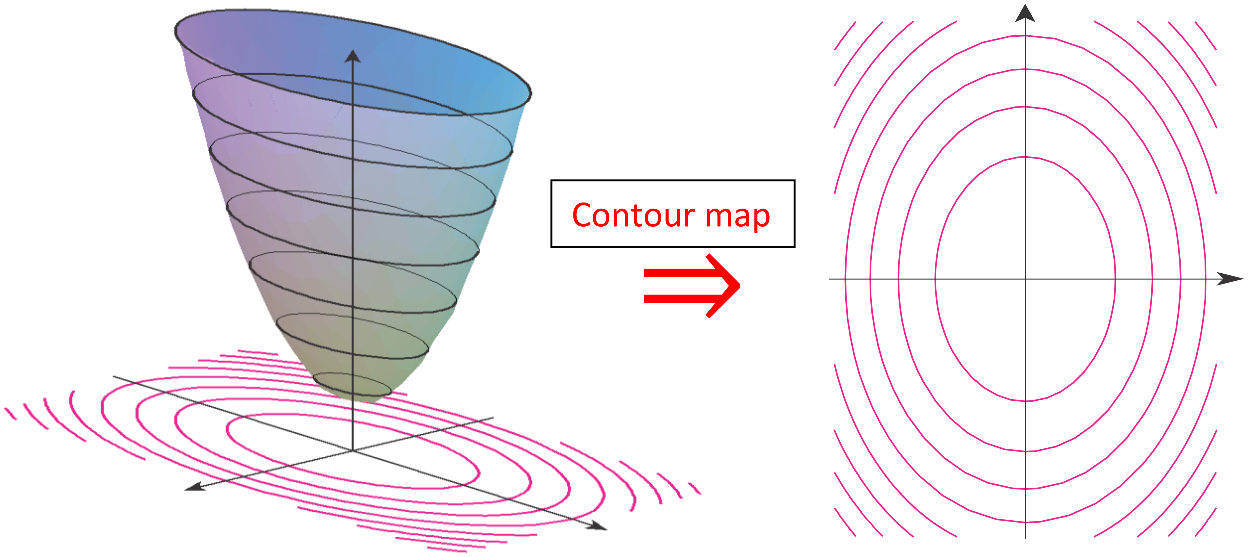
Contour map

Example 12: Draw a contour map (Sketch some level curves) of the function $f(x,y)=4x^2+y^2+1$

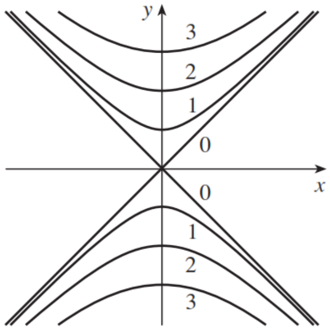
Solution: The level curves are: $f(x,y)=k \Rightarrow 4x^2+y^2+1=k$

$\implies 4x^2+y^2=k-1 \implies k-1\geq 0 \implies k\geq 1$

$k=1 \implies 4x^2+y^2=0$ $\implies x=0, y=0 \implies$ point $(0,0)$	$k=2 \implies 4x^2+y^2=1$ (ellipse) $\implies \frac{x^2}{1/4}+y^2=1$
$k=3 \implies 4x^2+y^2=2$ (ellipse) $\implies \frac{x^2}{1/2}+\frac{y^2}{2}=1$	$k=4 \implies 4x^2+y^2=3$ (ellipse) $\implies \frac{x^2}{3/4}+\frac{y^2}{3}=1$



Example 13: Draw a contour map (Sketch some level curves) of the function $f(x,y)=\sqrt{y^2-x^2}$.



Solution: The level curves are: the lines $y=x$ or $y=-x$

$f(x,y)=k \Rightarrow \sqrt{y^2-x^2}=k \quad \Longrightarrow \quad k\geq 0$

❖ For $k=0$: $\sqrt{y^2-x^2}=0 \Rightarrow y^2-x^2=0$

$\Rightarrow \ y^2=x^2 \ \Rightarrow \ y=x$ or $y=-x$

- The level curves are the lines: $y=x$ or $y=-x$

❖ For $k>0$: $\sqrt{y^2-x^2}=k \Rightarrow y^2-x^2=k^2$

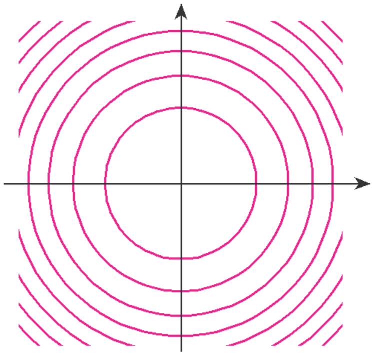
$$\Rightarrow \frac{y^2}{k^2}-\frac{x^2}{k^2}=1 \Rightarrow \text{ hyparabolas}$$

- The level curves are hyperbolas

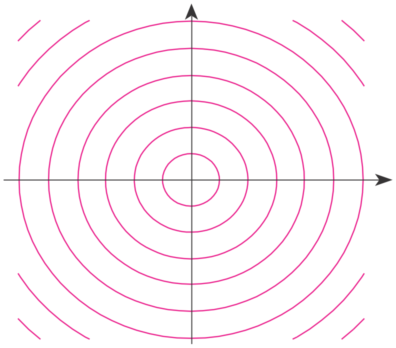
$$\boxed{\text{The level curves are two lines and hyperbolas}}$$

Remark 14: The surface is steep (حاد) where the level curves are close together. It is somewhat flatter (منبسط) where they are farther apart (متباعد).

Example 15: Two contour maps are shown in the figures. One is for a function f whose graph is a cone. The other is for a function g whose graph is a paraboloid. Which is which, and why?



I



II

Solution: Figure I is for the paraboloid which is the function g

Figure II is for the cone which is the function f.

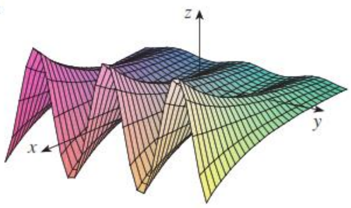
❖ Because the paraboloid is steep when x or y is very large so its level curves are close together (this appears in figure I) while on a cone the surface is never steep it stays steady.

Example 16: Match the function:
(a) with its graph (labled A-F below)
(b) with its contour map (labled I-VI below)
Give reasons for your choice.

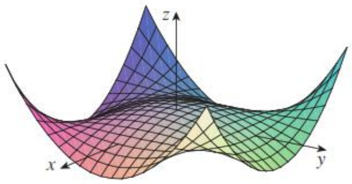
(a) $z = \sin(xy)$ (b) $z = e^x \cos y$ (c) $z = (1 - x^2)(1 - y^2)$

(d) $z = \sin(x - y)$ (e) $z = \sin x - \sin y$ (f) $z = \dfrac{x - y}{1 + x^2 + y^2}$

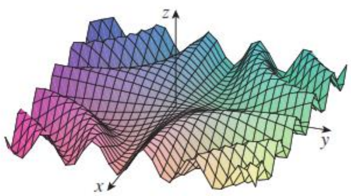
A



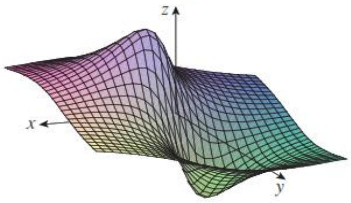
B



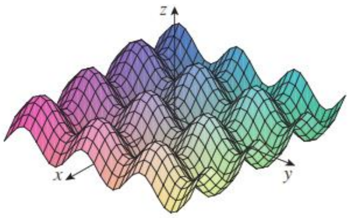
C



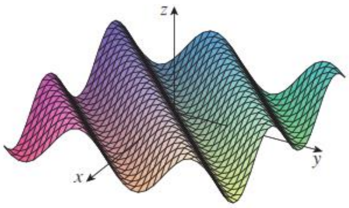
D



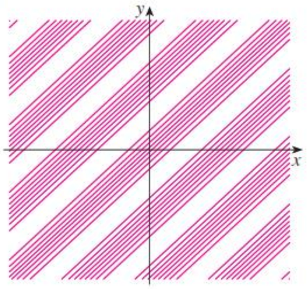
E



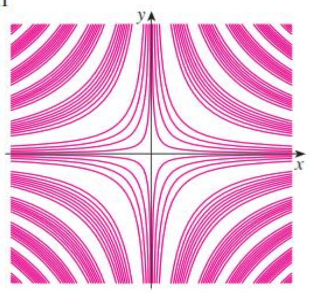
F



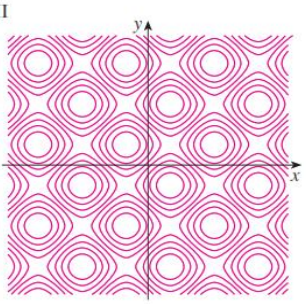
I



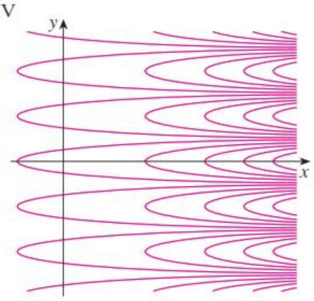
II



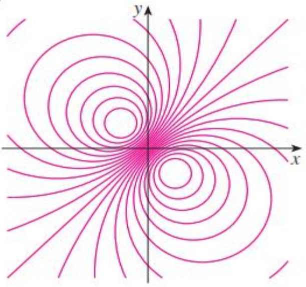
III



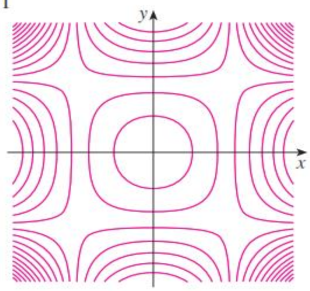
IV



V



VI



Solution:

(a)	(b)	(c)	(d)	(e)	(f)
C	A	B	F	E	D
II	IV	VI	I	III	V

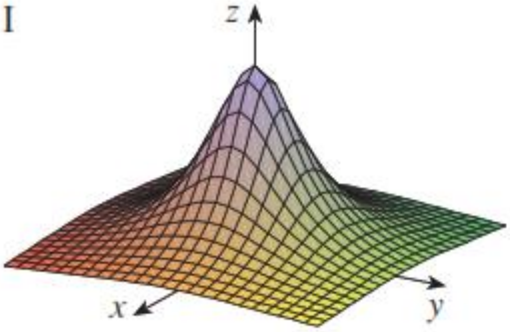
Example 17: Match the function with its graph (labeled I–VI). Give reasons for your choices.

(a) $f(x, y) = |x| + |y|$ (b) $f(x, y) = |xy|$

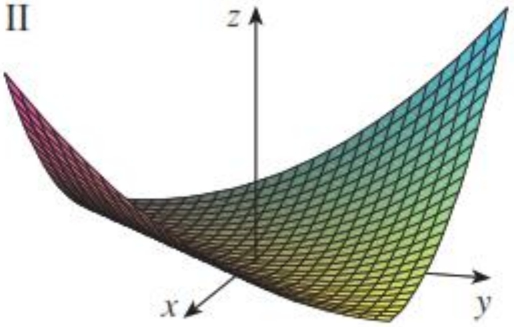
(c) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$ (d) $f(x, y) = (x^2 - y^2)^2$

(e) $f(x, y) = (x - y)^2$ (f) $f(x, y) = \sin(|x| + |y|)$

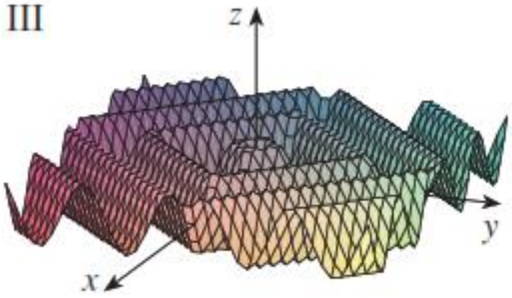
I



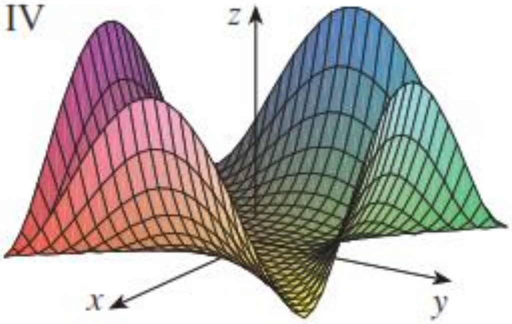
II



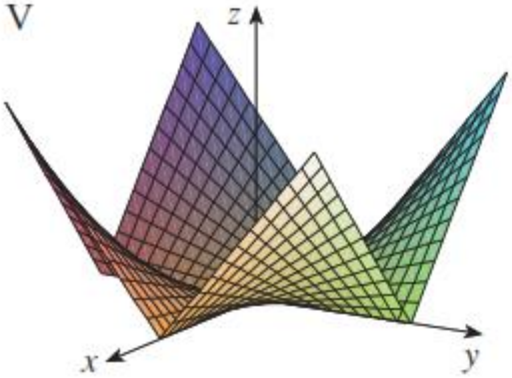
III



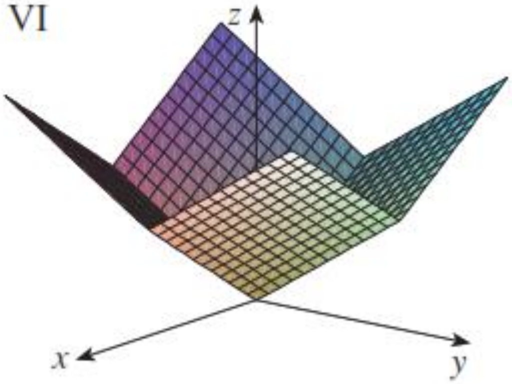
IV



V



VI



Solution:

(a)	(b)	(c)	(d)	(e)	(f)
VI	V	I	IV	II	III

Functions of Three or More Variables

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain D in $\mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

Example 18: Find and sketch the domain of the function:

(a) $f(x, y, z) = \ln(z - y) + xy\sin z$.

(b) $f(x, y, z) = \sqrt{z - x^2 - 2y^2}$

Solution:

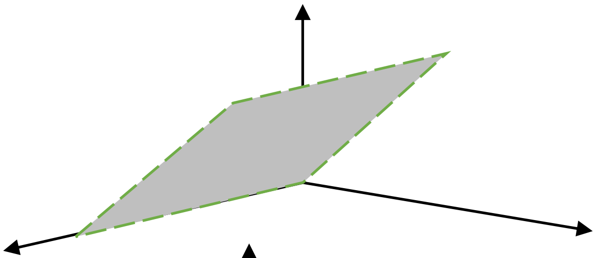
(a) $Dom(f) = \{(x, y, z) \in \mathbb{R}^3 : z - y > 0\}$

$= \{(x, y, z) \in \mathbb{R}^3 : z > y\}$

To sketch $Dom(f)$:

$z > y$: $\Rightarrow$ $z = y$ (plane)

So, Dom(f) is a **half-space** consisting of all points that lie above the plane $z = y$.



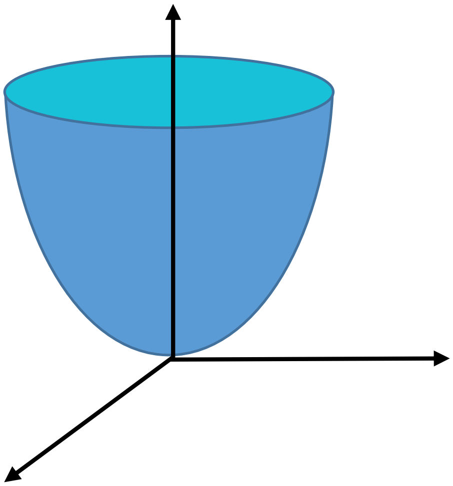
(b) $Dom(f) = \{(x, y, z) \in \mathbb{R}^3 : z - x^2 - 2y^2 \geq 0\}$

$= \{(x, y, z) \in \mathbb{R}^3 : z \geq x^2 + 2y^2\}$

To sketch $Dom(f)$:

$z \geq x^2 + 2y^2$: $\Rightarrow$ $z = x^2 + 2y^2$(paraboloid)

So, Dom(f) is the region inside and on the paraboloid $z = x^2 + 2y^2$



Example 19:

(a) $f(x, y, z) = \frac{1}{x}$ $\Rightarrow$ $Dom(f) = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0\}$

(b) $f(x, y, z) = 2e^{xyz}$ $\Rightarrow$ $Dom(f) = \{(x, y, z) \in \mathbb{R}^3\} = \mathbb{R}^3$

(c) $f(x, y, z) = x + y$ $\Rightarrow$ $Dom(f) = \{(x, y, z) \in \mathbb{R}^3\} = \mathbb{R}^3$

(d) $f(x, y, z, w) = \sqrt{w - z}$:

$\Rightarrow$ $Dom(f) = \{(x, y, z, w) \in \mathbb{R}^4 : w - z \geq 0\} = \{(x, y, z, w) \in \mathbb{R}^4 : w \geq z\}$

(e) $f(x, y, z) = \frac{1}{x}$ $\Rightarrow$ $Dom(f) = \{(x, y, z) \in \mathbb{R}^3 : x \neq 0\}$

Example 20: Find the domain and range of the function $f(x,y,z) = 2 + \sqrt{x^2+3}$

Solution:

$f(x,y,z) = 2 + \sqrt{x^2+3} \quad \Rightarrow \; Dom(f) = \{(x,y,z) \in \mathbb{R}^3\} = \mathbb{R}^3$

To find $range(f)$: Let $w = f(x,y,z)$

$$\Rightarrow range(f) = \{w \in \mathbb{R}: w = f(x,y,z), (x,y,z) \in Dom(f)\}$$

$\sqrt{x^2+3} \geq 0$	$\Rightarrow$	$2 + \sqrt{x^2+3} \geq 2$	$\Rightarrow$	$w \geq 2$
$x^2 \geq 0$	$\Rightarrow$	$x^2 + 3 \geq 3$	$\Rightarrow$	$w \geq 2 + \sqrt{3}$
$\sqrt{x^2+3} \geq \sqrt{3}$	$\Rightarrow$	$2 + \sqrt{x^2+3} \geq 2 + \sqrt{3}$		

So, $w \geq 2$ and $w \geq 2 + \sqrt{3} \;\; \Rightarrow w \geq 2 + \sqrt{3} \Rightarrow range(f) = [2 + \sqrt{3}, \infty)$.

Definition 21: The **level surfaces** of a function $f(x,y,z)$ for the value k are the surfaces given by the equation $f(x,y,z) = k$, where k is a constant, that is if the point (x,y,z) moves along a level surface, the value of $f(x,y,z)$ remains fixed.

Example 22: Find the level surfaces of the function $f(x,y,z) = x^2 + y^2 + z^2$.

Solution: Observe that $f(x,y,z) = x^2 + y^2 + z^2 \geq 0$

$\Rightarrow$ the values of k are $k \geq 0$ since for the level surfaces we have $f(x,y,z) = k$

For $k = 0$: $f(x,y,z) = 0 \Rightarrow x^2 + y^2 + z^2 = 0 \Rightarrow x = 0, y = 0, z = 0$

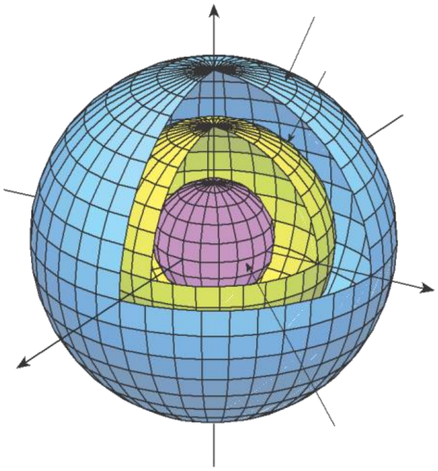
$\Rightarrow$ we have a point $(0,0,0)$

For $k = 1$: $f(x,y,z) = 1 \Rightarrow x^2 + y^2 + z^2 = 1$

$\Rightarrow$ A sphere of radius 1 centered at $(0,0,0)$

In general, for $k > 0$: : $f(x,y,z) = k \Rightarrow x^2 + y^2 + z^2 = k$

$\Rightarrow$ A sphere of radius $\sqrt{k}$ centered at $(0,0,0)$



Example 23: Describe the level surfaces of the function.

(a) $f(x,y,z) = 2x - 3y + 5z - 3$

(b) $f(x,y,z) = 2x^2 + 3y^2 + 5z^2$

(c) $f(x,y,z) = y^2 + z^2$

(d) $f(x,y,z) = z^2 - y^2 - x^2$

Solution:

(a) $f(x,y,z) = k \Rightarrow 2x - 3y + 5z - 3 = k \Rightarrow k \in \mathbb{R}$

$\Rightarrow 2x - 3y + 5z = k + 3$

The level surfaces are planes

(b) $f(x,y,z) = k \Rightarrow 2x^2 + 3y^2 + 5z^2 = k \Rightarrow k \geq 0$

For $k = 0$: $2x^2 + 3y^2 + 5z^2 = 0 \Rightarrow x = 0, y = 0, z = 0$

The level surface is a point which is the origin $(0,0,0)$

For $k > 0$: $2x^2 + 3y^2 + 5z^2 = k \Rightarrow \frac{x^2}{k/2} + \frac{y^2}{k/3} + \frac{z^2}{k/5} = 1$

$\Rightarrow$ The level surfaces are ellipsoids

❖ The level surfaces are the origin and ellipsoids

(c) $f(x,y,z) = k \Rightarrow y^2 + z^2 = k \Rightarrow k \geq 0$

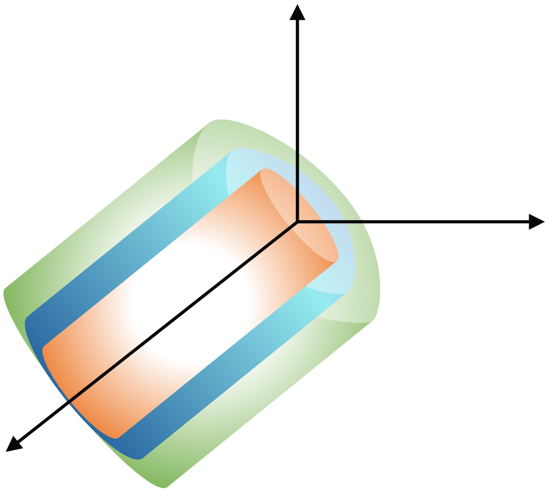
For $k = 0$: $y^2 + z^2 = 0$

$\Rightarrow y = 0, z = 0, x \in \mathbb{R}$

$\Rightarrow$ The level surface is the x-axis

For $k > 0$: $y^2 + z^2 = k$

$\Rightarrow$ The level surface is a cylinder



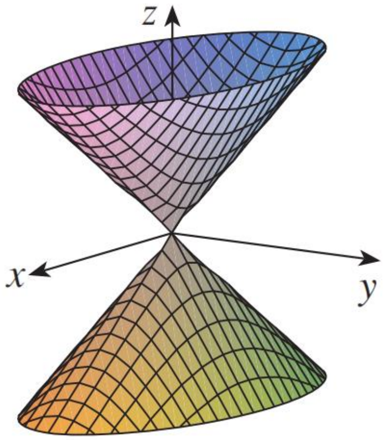
❖ The level surfaces are the x-axis and cylinders

(d) $f(x, y, z) = k \Rightarrow z^2 - x^2 - y^2 = k \Rightarrow k \in \mathbb{R}$

For $k = 0$: $z^2 - x^2 - y^2 -= 0$

$\Rightarrow z^2 = x^2 + y^2$

- The level surfaces are cones

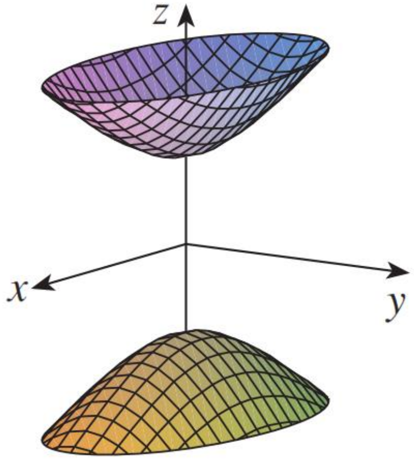


For $k > 0$: Take $k = 2 \Rightarrow z^2 - x^2 - y^2 = 2$

$\Rightarrow -\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = 1$

(1 positive and 2 negatives)

- The level surfaces are hyperboloid of two sheets

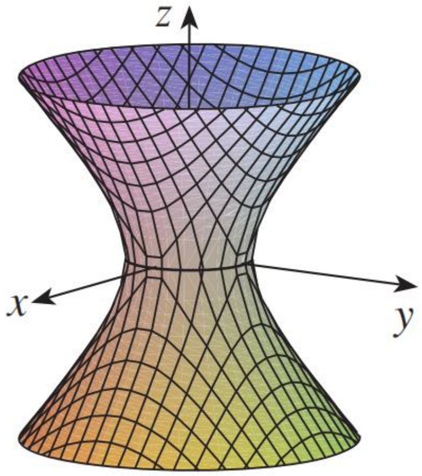


For $k < 0$: Take $k = -2 \Rightarrow y^2 - x^2 - z^2 = -2$

$\Rightarrow -\frac{y^2}{2} + \frac{x^2}{2} + \frac{z^2}{2} = 1$

(2 positives and 1 negatives)

- The level surfaces are hyperboloid of one sheet



- The level surfaces are:
  cones, hyperboloids of two sheets, and hyperboloids of one sheet

Chapter 14 Partial Derivatives

Section 14.2:
Limits and Continuity

14.2 Limits and Continuity

Definition 1: Let $C: x = f(t),\ y = g(t)$ be a path (curve) in the xy-plane. Then

C passes through that pass through the point $P_0(a,b)$ in $\mathbb{R}^2$ $\Longleftrightarrow$ **(يكافيء)** there exists $t_0 \in \mathbb{R}$ such $f(t_0) = a$ and $g(t_0) = b$.

Definition 2: Let $P_0(a,b)$ in $\mathbb{R}^2$ and let C be a path that pass through the point $P_0(a,b)$ when $t = t_0$. Then

$$\lim_{\substack{(x,y)\to P_0 \\ along\ C}} F(x,y) = \lim_{t\to t_0} F(f(t), g(t))$$

Definition 3: Let $P_0(a,b)$ be a point in $\mathbb{R}^2$ and let $L \in \mathbb{R}$.

(a) $\lim\limits_{(x,y)\to P_0} F(x,y) = L$ (exists) $\Longleftrightarrow$ $\lim\limits_{\substack{(x,y)\to P_0 \\ along\ C}} F(x,y) = L$ for **all** paths C in Dom$(F(x,y))$ that pass through the point P_0.

لاثبات ان النهاية موجودة: علينا ان نأخذ **كل** المسارات paths (المنحنيات curves) المارة بالنقطة P_0 وحساب النهاية من خلالها، **وهذا مستحيل وذلك لأن عدد المسارات لانهائي**. لذلك إن كانت النهاية موجودة واردنا حسابها فإننا لا نستخدم المسارات في حسابها **بل نلجأ الى طرق أخرى مثل:**

التعويض المباشر أو **التحليل والإختصار** أو **الضرب بالمرافق** أو **استخدام تعويضات خاصة**

(b) Let C_1 and C_2 be two paths in Dom$(F(x,y))$ that pass through a point P_0. If

$$\lim_{\substack{(x,y)\to P_0 \\ along\ C_1}} F(x,y) \neq \lim_{\substack{(x,y)\to P_0 \\ along\ C_2}} F(x,y), \text{ then } \lim_{(x,y)\to P_0} F(x,y) \text{ dose not exist (DNE)}$$

لاثبات ان النهاية غير موجودة: علينا ان نجد **مسارين** كليهما مار بالنقطة P_0 وحساب النهاية من خلال كل واحد من المسارين بحيث يكون جوابا النهايتين **مختلفين**.

Example 4: Find each of the following limit, if it exists:

(1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4+x^2y^2-6y}{x^2+3y^2}$ (2) $\lim\limits_{(x,y)\to(-1,1)} \dfrac{y^6-x^2}{y^3+x}$

(3) $\lim\limits_{(x,y)\to(4,2)} \dfrac{x^2-5xy^2+4y^4}{\sqrt{x}-2y}$

Solution:

(1)
$$\lim_{(x,y)\to(0,0)} \frac{x^4+x^2y^2-6y}{x^2+3y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2+3y^2)(x^2-2y^2)}{x^2+3y^2}$$
$$= \lim_{(x,y)\to(0,0)} (x^2-2y^2)$$
$$= \mathbf{0}$$

---

(2)
$$\lim_{(x,y)\to(-1,1)} \frac{y^6-x^2}{y^3+x} = \lim_{(x,y)\to(-1,1)} \frac{(y^3+x)(y^3-x)}{y^3+x}$$
$$= \lim_{(x,y)\to(-1,1)} (y^3-x)$$
$$= \mathbf{2}$$

---

(3)
$$\lim_{(x,y)\to(4,2)} \frac{x^2-5xy^2+4y^4}{\sqrt{x}-2y} = \lim_{(x,y)\to(4,2)} \frac{x^2-5xy^2+4y^4}{\sqrt{x}-2y} \times \frac{\sqrt{x}+2y}{\sqrt{x}+2y}$$
$$= \lim_{(x,y)\to(4,2)} \frac{x^2-5xy^2+4y^4}{x-4y^2} \times 6$$
$$= 6 \lim_{(x,y)\to(4,2)} \frac{(x-4y^2)(x-y^2)}{x-4y^2}$$
$$= 6 \lim_{(x,y)\to(4,2)} (x-y^2)$$
$$= 6(-14)$$
$$= \mathbf{-84}$$

Remark 5: Recall that

(1) $\lim\limits_{\theta\to 0}\dfrac{\sin(\theta)}{\theta}=1$ (2) $\lim\limits_{\theta\to 0}\dfrac{\theta}{\sin(\theta)}=1$

(3) $\lim\limits_{\theta\to 0}\dfrac{\tan(\theta)}{\theta}=1$ (4) $\lim\limits_{\theta\to 0}\dfrac{\theta}{\tan(\theta)}=1$

Example 6: Find each of the following limit, if it exists:

(1) $\lim\limits_{(x,y)\to(0,-3)}\dfrac{\sin(2xy^2)}{x}$ (2) $\lim\limits_{(x,y)\to(1,2)}\dfrac{4x^2-y^2}{\tan(4x-2y)}$

Solution:

(1) Let $\theta=2xy^2$. When $(x,y)\to(0,-3)$ we have $x\to 0$ and $y\to -3$. Then $\theta\to 0$. So,

$$\lim_{(x,y)\to(0,-3)}\frac{\sin(2xy^2)}{x}=\lim_{(x,y)\to(0,-3)}\frac{\sin(2xy^2)}{1}\times\frac{1}{x}$$

$$=\lim_{(x,y)\to(0,-3)}\frac{\sin(2xy^2)}{2xy^2}\times\frac{2xy^2}{x}$$

$$=\lim_{(x,y)\to(0,-3)}\frac{\sin(2xy^2)}{2xy^2}\times\lim_{(x,y)\to(0,-3)}\frac{2xy^2}{x}$$

$$=\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\times\lim_{(x,y)\to(0,-3)}2y^2$$

$$=\mathbf{1\times 2(-3)^2=18}$$

(2) Let $\theta=4x-2y$. When $(x,y)\to(1,2)$ we have $x\to 1$ and $y\to 2$. Then $\theta\to 0$. So,

$$\lim_{(x,y)\to(1,2)}\frac{4x^2-y^2}{\tan(4x-2y)}=\lim_{(x,y)\to(1,2)}\frac{1}{\tan(4x-2y)}\times\frac{4x^2-y^2}{1}$$

$$=\lim_{(x,y)\to(1,2)}\frac{4x-2y}{\tan(4x-2y)}\times\frac{4x^2-y^2}{4x-2y}$$

$$=\lim_{(x,y)\to(1,2)}\frac{4x-2y}{\tan(4x-2y)}\times\lim_{(x,y)\to(1,2)}\frac{4x^2-y^2}{4x-2y}$$

$$\lim_{(x,y)\to(1,2)}\frac{\theta}{\tan(\theta)}\times\lim_{(x,y)\to(1,2)}\frac{(2x-y)(2x+y)}{2(2x-y)}$$

$$=\lim_{\theta\to 0}\frac{\theta}{\tan(\theta)}\times\lim_{(x,y)\to(1,2)}\frac{(2x+y)}{2}$$

$$=\mathbf{1\times\frac{4}{2}=2}$$

Remark 7: When $(x, y) \to (0,0)$ and we have the terms $x^2 + y^2$ in the limit, we must think of the using of the substitutions: $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then, $x^2 + y^2 = r^2$ and when $(x, y) \to (0,0)$ we have $x \to 0$ and $y \to 0$. So, $r \to 0^+$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \boldsymbol{F}(x, y) = \lim_{r\to 0^+} \boldsymbol{F}(r\cos(\theta), r\sin(\theta)),$$ where $0 \le \theta < 2\pi$.

Example 8: Find the following limit, if it exists:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{\sqrt{x^2 + y^2 + 1} - 1}$$

Solution: Let $x = r\cos(\theta)$ and $y = r\sin(\theta) \Rightarrow x^2 + y^2 = r^2$ and $(x, y) \to (0,0)$ $\Rightarrow r \to 0^+$. So,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{\sqrt{x^2 + y^2 + 1} - 1}$$

$$= \lim_{r\to 0^+} \frac{\left(r\cos(\theta)\right)^2 (r\sin(\theta))}{\sqrt{r^2 + 1} - 1}$$

$$= \lim_{r\to 0^+} \frac{r^3 \cos^2(\theta)\sin(\theta)}{\sqrt{r^2 + 1} - 1}$$

$$= \lim_{r\to 0^+} \left(\frac{r^3}{\sqrt{r^2 + 1} - 1} \times \cos^2(\theta)\sin(\theta)\right)$$

$$= \lim_{r\to 0^+} \left(\frac{r^3}{\sqrt{r^2 + 1} - 1} \times \frac{\sqrt{r^2 + 1} + 1}{\sqrt{r^2 + 1} + 1} \times \cos^2(\theta)\sin(\theta)\right)$$

$$= \lim_{r\to 0^+} \left(\frac{r^3}{\sqrt{r^2 + 1} - 1} \times \frac{\sqrt{r^2 + 1} + 1}{\sqrt{r^2 + 1} + 1} \times \cos^2(\theta)\sin(\theta)\right)$$

$$= \lim_{r\to 0^+} \left(\frac{r^3}{(r^2 + 1) - 1} \times \frac{\sqrt{r^2 + 1} + 1}{1} \times \cos^2(\theta)\sin(\theta)\right)$$

$$= \lim_{r\to 0^+} \left(\frac{r^3}{r^2} \times \cos^2(\theta)\sin(\theta)\right) \times \lim_{r\to 0^+} \left(\left(\sqrt{r^2 + 1} + 1\right)\right)$$

$$= \lim_{r\to 0^+} \left(r\cos^2(\theta)\sin(\theta)\right) \times 2$$

$$= 0 \times 2 = \mathbf{0}$$

Observe that:

$$-1 \le \cos^2(\theta)\sin(\theta) \le 1$$

$\Rightarrow$

$$-r \le r\cos^2(\theta)\sin(\theta) \le r$$

$\Rightarrow$

$$\lim_{r\to 0^+} (-r) = 0$$

and

$$\lim_{r\to 0^+} r = 0$$

So,

by the squeeze theorem:

$$\lim_{r\to 0^+} r\cos^2(\theta)\sin(\theta)) = 0$$

Example 9: Find the following limit, if it exists $\displaystyle\lim_{(x,y)\to(1,1)} \frac{(6y-4x-1)^5-1}{(2x-3y)^8-1}$

Solution: Observe that

$$\lim_{(x,y)\to(1,1)} \frac{(6y-4x-1)^5-1}{(2x-3y)^8-1} = \lim_{(x,y)\to(1,1)} \frac{(-2(2x-3y)-1)^5-1}{(2x-3y)^8-1}$$

So, let $\theta = 2x-3y$. When $(x,y) \to (1,1)$. Then $x \to 1$ and $y \to 1$.

So, $\theta \to -1$. Then

$$\lim_{(x,y)\to(1,1)} \frac{(6y-4x-1)^5-1}{(2x-3y)^8-1} = \lim_{\theta\to-1} \frac{(-2\theta-1)^5-1}{\theta^8-1}$$

$$= \lim_{\theta\to-1} \frac{5(-2\theta-1)^4(-2)}{8\theta^7} \quad (by\ L'Hopital's\ Rule)$$

$$= \frac{\mathbf{5(1)^4(-2)}}{\mathbf{8(-1)^7}} = \frac{\mathbf{5}}{\mathbf{4}}$$

Remark 10: Recall the following:

- Let $C: x = f(t),\ y = g(t)$ be a path (curve) in the xy-plane. Then

C passes through that pass through the point $P_0(a,b)$ in $\mathbb{R}^2$

$\Longleftrightarrow$ **(يكافيء)**

there exists $t_0 \in \mathbb{R}$ such $f(t_0) = a$ and $g(t_0) = b$.

- Let $P_0(a,b)$ in $\mathbb{R}^2$ and let C be a path that pass through the point $P_0(a,b)$ when $t = t_0$. Then $\displaystyle\lim_{\substack{(x,y)\to P_0\\ along\ C}} F(x,y) = \lim_{t\to t_0} F(f(t),g(t))$

- Let C_1 and C_2 be two curves that pass through a point $P_0(a,b)$. If $\displaystyle\lim_{\substack{(x,y)\to P_0\\ along\ C_1}} F(x,y) \neq \lim_{\substack{(x,y)\to P_0\\ along\ C_2}} F(x,y)$, then $\displaystyle\lim_{(x,y)\to P_0} F(x,y)$ dose not exist (DNE)

لاثبات ان النهاية غير موجودة: علينا ان نجد **مسارين** (C_1) و (C_2) كليهما مار بالنقطة P_0 وحساب النهاية من خلال كل واحد من المسارين بحيث يكون جوابا النهايتين **مختلفين**.

$$\lim_{\substack{(x,y)\to P_0\\ along\ C_1}} F(x,y) \neq \lim_{\substack{(x,y)\to P_0\\ along\ C_2}} F(x,y)$$

Example 11:

(1) Find $\lim\limits_{(x,y)\to(1,-2)} \frac{x^2+3y+5}{2x+y}$ along the path $C_1: y = 2x-4$

(2) Find $\lim\limits_{(x,y)\to(1,-2)} \frac{x^2+3y+5}{2x+y}$ along the path $C_2: x = 3t, y = 1-9t.$

(3) Is $\lim\limits_{(x,y)\to(1,-2)} \frac{x^2+3y+5}{2x+y}$ exists? Justify.

Solution:

(1)

$$\lim_{\substack{(x,y)\to(1,-2)\\ along\ C_1}} \frac{x^2+3y+5}{2x+y} = \lim_{x\to1} \frac{x^2+3(2x-4)+5}{2x+(2x-4)} = \lim_{x\to1} \frac{x^2+6x-7}{4x-4}$$

$$= \lim_{x\to1} \frac{(x-1)(x+7)}{4(x-1)} = \lim_{x\to1} \frac{(x+7)}{4}$$

$$= \mathbf{2}$$

(2) We must find the value of t_0 when the point $(1,-2)$ is on C_2:

The point $(1,-2)$: $x = 1, \quad y = -2$

On C_2: $x = 3t_0, \ y = 1-9t_0$

x (on path) $= x$ (in point) $\Rightarrow 3t_0 = 1 \ \Rightarrow t_0 = \frac{1}{3}$

Observe that we can find t_0 from y component: y (on path) $= y$ (in point) $\Rightarrow 1-9t_0 = -2 \ \Rightarrow \ t_0 = \frac{1}{3}$	We have the same value for t_0

$$\lim_{\substack{(x,y)\to(1,-2)\\ along\ C_2}} \frac{x^2+3y+5}{2x+y} = \lim_{t\to\frac{1}{3}} \frac{(3t)^2+3(1-9t)+5}{2(3t)+(1-9t)}$$

$$= \lim_{t\to\frac{1}{3}} \frac{9t^2-27t+8}{1-3t} = \frac{0}{0} \quad (Indeterminate\ form)$$

$$= \lim_{t\to\frac{1}{3}} \frac{18t-27}{-3} \qquad (by\ L'Hopital's\ Rule)$$

$$= \mathbf{7}$$

(3) $\lim\limits_{(x,y)\to(1,-2)} \frac{x^2+3y+5}{2x+y}$ DNE (does not exist), because:

$$\lim_{\substack{(x,y)\to(1,-2)\\ along\ C_1}} \frac{x^2+3y+5}{2x+y} \neq \lim_{\substack{(x,y)\to(1,-2)\\ along\ C_2}} \frac{x^2+3y+5}{2x+y}$$

Example 12: Find the limit, if it exists $\lim\limits_{\substack{(x,y)\to(1,0)\\ along\ y=\ln(x)}} \dfrac{x\sin(e^y-1)}{x^2+y^2-x}$

Solution:

$$\lim_{\substack{(x,y)\to(1,0)\\ along\ y=\ln(x)}} \frac{x\sin(e^y-1)}{x^2+y^2-x} = \lim_{x\to 1}\frac{x\sin\left(e^{\ln(x)}-1\right)}{x^2+(\ln(x))^2-x} = \lim_{x\to 1}\frac{x\sin(x-1)}{x^2+(\ln(x))^2-x} = \frac{0}{0}$$

$$= \lim_{x\to 1}\frac{x\cos(x-1)+\sin(x-1)}{2x+\dfrac{2\ln(x)}{x}-1} \quad (by\ L'Hopital's\ Rule)$$

$$= \frac{1\cos 0+\sin 0}{2+\dfrac{2\ln(1)}{1}-1} = \mathbf{1}$$

Example 13: Find the limit, if it exists:

$$\lim_{(x,y)\to(2,-1)} \frac{x^2-2x-y^2-2y-1}{(x-2)^2+(y+1)^2}$$

Solution:

C_1: $\boldsymbol{x} = \boxed{\boldsymbol{t}}\boxed{+}\boxed{\mathbf{2}}$ **,** $\boldsymbol{y} = \boxed{\mathbf{0}}\boxed{+}\boxed{\mathbf{-1}}$

$\Rightarrow \quad C_1\colon\ x = t-2,\ y=-1$

$(x,y)\to(2,-1) \ \Rightarrow\ t\to 0$:

$$\lim_{(x,y)\to(2,-1)} \frac{x^2-2x-y^2-2y-1}{(x-2)^2+(y+1)^2}$$

$$= \lim_{t\to 0}\frac{(t+2)^2-2(t+2)-(-1)^2-2(-1)-1}{(t+2-2)^2+(-1+1)^2}$$

$$= \lim_{t\to 0}\frac{t^2+4t+4-2t-4-1+2-1}{t^2}$$

$$= \lim_{t\to 0}\frac{t^2+4t+4-2t-4-1+2-1}{t^2}$$

$$= \lim_{t\to 0}\frac{t^2+2t}{t^2} = \lim_{t\to 0}\frac{t(t+2)}{t^2} = \lim_{t\to 0}\frac{t+2}{t} = \frac{2}{0} \ \Rightarrow\ \text{DNE}$$

Example 14: Find the following limit, if it exists:

$$(1)\quad \lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2}{x^6-y^3+2z^4} \qquad\qquad (2)\quad \lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$$

Solution:

$$\textbf{(1)}\quad C_1:\ \boldsymbol{x}=\boxed{\boldsymbol{t}}\boxed{+}\boxed{\mathbf{0}}\,,\qquad \boldsymbol{y}=\boxed{\mathbf{0}}\boxed{+}\boxed{\mathbf{0}}\,,\qquad \boldsymbol{z}=\boxed{\mathbf{0}}\boxed{+}\boxed{\mathbf{0}}$$

$$\Rightarrow\quad C_1:\ x=t,\ \ y=0, z=0$$

$$(x,y),z\to(0,0,0)\ \ \Rightarrow\ \ t\to 0:$$

$$\lim_{\substack{(x,y)\to(0,0,0)\\ along\ C_1}} \frac{x^2y^2}{x^6-y^3+2z^4} = \lim_{t\to0}\frac{t^2 0^2}{t^6-0^3+2(0^4)} = = \lim_{t\to0}\frac{0}{t^6} = \lim_{t\to0} 0 = 0$$

$lcm((6,3,4)=12$	المضاعف المشترك الأصغر

Variable $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$	$=$	$\boldsymbol{t}^{\frac{\boldsymbol{lcm}}{\boldsymbol{exponent}}}$	$+$	**Coordinate of the variable in the point**

$$C_2:\qquad \boldsymbol{x}=\boxed{\boldsymbol{t}^{\frac{\mathbf{12}}{\mathbf{6}}}}\boxed{+}\boxed{\mathbf{0}}\,,\quad \boldsymbol{y}=\boxed{\boldsymbol{t}^{\frac{\mathbf{12}}{\mathbf{3}}}}\boxed{+}\boxed{\mathbf{0}}\,,\quad \boldsymbol{z}=\boxed{\boldsymbol{t}^{\frac{\mathbf{12}}{\mathbf{4}}}}\boxed{+}\boxed{\mathbf{0}}$$

$$\Rightarrow\quad C_2:\ x=t^2,\ \ y=t^4, z=t^3$$

$$(x,y)\to(0,0)\ \ \Rightarrow\ \ t\to 0:$$

$$\lim_{\substack{(x,y)\to(0,0,0)\\ along\ C_2}} \frac{x^2y^2}{x^6-y^3+2z^4} = \lim_{t\to0}\frac{(t^2)^2(t^4)^2}{(t^2)^6-(t^4)^3+2(t^3)^4}$$

$$= \lim_{t\to0}\frac{t^4t^8}{t^{12}-t^{12}+2t^{12}} = \lim_{t\to0}\frac{t^{12}}{2t^{12}} = \lim_{t\to0}\frac{1}{2} = \mathbf{\frac{1}{2}}$$

$$\Rightarrow\qquad \lim_{\substack{(x,y)\to(0,0,0)\\ along\ C_1}} \frac{x^2y^2}{x^6-y^3+2z^4} \neq \lim_{\substack{(x,y)\to(0,0,0)\\ along\ C_2}} \frac{x^2y^2}{x^6-y^3+2z^4}$$

$$\text{So,}\qquad \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^6+y^3+z^4}\qquad \text{Does not exist.}$$

Example 15: Find the following limit, if it exists:

$$\lim_{(x,y,z)\to(1,-2,3)} \frac{(x-1)^2(y+2)(z-3)^2}{(x-1)^3+3(y+2)^5+(z-3)^{15}}$$

Solution:

(1) C_1: $\boldsymbol{x} = \boxed{\boldsymbol{t}}\boxed{+}\boxed{\mathbf{1}}$, $\boldsymbol{y} = \boxed{\mathbf{0}}\boxed{+}\boxed{\mathbf{-2}}$, $\boldsymbol{z} = \boxed{\mathbf{0}}\boxed{+}\boxed{\mathbf{3}}$

$\Rightarrow \quad C_1\colon\ x = t+1,\ y = -2, z = 3$

$(x,y,z) \to (1,-2,3) \ \Rightarrow\ t \to 0$:

$$\lim_{\substack{(x,y,z)\to(1,-2,3)\\ along\ C_1}} \frac{(x-1)^2(y+2)(z-3)^2}{(x-1)^3+3(y+2)^5+(z-3)^{15}}$$

$$= \lim_{t\to 0} \frac{(t+1-1)^2(-2+2)(3-3)^2}{(t+1-1)^3+3(-2+2)^5+(3-3)^{15}}$$

$$= \lim_{t\to 0} \frac{0}{t^3} = \lim_{t\to 0} 0 = 0$$

$lcm((3,5,15) = 15$	المضاعف المشترك الأصغر

C_2: $\boldsymbol{x} = \boxed{\boldsymbol{t}^{\frac{\mathbf{15}}{\mathbf{3}}}}\boxed{+}\boxed{\mathbf{1}}$, $\boldsymbol{y} = \boxed{\boldsymbol{t}^{\frac{\mathbf{15}}{\mathbf{5}}}}\boxed{+}\boxed{\mathbf{-2}}$, $\boldsymbol{z} = \boxed{\boldsymbol{t}^{\frac{\mathbf{15}}{\mathbf{15}}}}\boxed{+}\boxed{\mathbf{3}}$

$\Rightarrow \quad C_2\colon\ x = t^5+1,\ y = t^3-2, z = t+3$

$(x,y,z) \to (1,-2,3) \ \Rightarrow\ t \to 0$:

$$\lim_{\substack{(x,y,z)\to(1,-2,3)\\ along\ C_2}} \frac{(x-1)^2(y+2)(z-3)^2}{(x-1)^3+3(y+2)^5+(z-3)^{15}}$$

$$= \lim_{t\to 0} \frac{(t^5+1-1)^2(t^3-2+2)(t+3-3)^2}{(t^5+1-1)^3+3(t^3-2+2)^5+(t+3-3)^{15}}$$

$$= \lim_{t\to 0} \frac{(t^5)^2t^3t^2}{(t^5)^3+3(t^3)^5+(t)^{15}} = \lim_{t\to 0} \frac{t^{15}}{t^{15}+3t^{15}+t^{15}} = \lim_{t\to 0} \frac{t^{15}}{5t^{15}} = \frac{\mathbf{1}}{\mathbf{5}}$$

$$\Rightarrow \quad \lim_{\substack{(x,y,z)\to(1,-2,3)\\ along\ C_1}} F(x,y,z) \neq \lim_{\substack{(x,y,z)\to(1,-2,3)\\ along\ C_2}} F(x,y,z)$$

So, $\displaystyle \lim_{(x,y,z)\to(1,-2,3)} \frac{(x-1)^2(y+2)(z-3)^2}{(x-1)^3+3(y+2)^5+(z-3)^{15}}$ Does not exist.

Example 16: Find the following limit, if it exists: $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$

Solution:

(1) C_1: $\boldsymbol{x =} \boxed{\mathbf{0}}\boxed{+}\boxed{\mathbf{0}}$ **,** $\boldsymbol{y =} \boxed{\boldsymbol{t}}\boxed{+}\boxed{\mathbf{0}}$

$\Rightarrow \; C_1\!: \; x = 0, \; y = t$

$(x,y) \to (0,0) \;\; \Rightarrow \;\; t \to 0$:

$$\lim_{\substack{(x,y)\to(0,0)\\ along\ C_1}} \frac{x^3y^2}{x^6+y^4} = \lim_{t\to 0}\frac{0^3t^2}{0^6+t^4} = = \lim_{t\to 0}\frac{0}{t^4} = \lim_{t\to 0} 0 = 0$$

$lcm((6,4) = 12$	المضاعف المشترك الأصغر

C_2: $\boldsymbol{x =} \boxed{\boldsymbol{t}^{\frac{\mathbf{12}}{\mathbf{6}}}}\boxed{+}\boxed{\mathbf{0}}$ **,** $\boldsymbol{y =} \boxed{\boldsymbol{t}^{\frac{\mathbf{12}}{\mathbf{4}}}}\boxed{+}\boxed{\mathbf{0}}$

$\Rightarrow \;\; C_2\!: \; x = t^2, \; y = t^3$

$(x,y) \to (0,0) \;\; \Rightarrow \;\; t \to 0$:

$$\lim_{\substack{(x,y)\to(0,0)\\ along\ C_2}} \frac{x^3y^2}{x^6+y^4} = \lim_{t\to 0}\frac{(t^2)^3(t^3)^2}{(t^2)^6+(t^3)^4} = \lim_{t\to 0}\frac{t^6t^6}{t^{12}+t^{12}} = \lim_{t\to 0}\frac{t^{12}}{2t^{12}} = \frac{\mathbf{1}}{\mathbf{2}}$$

$$\Rightarrow \qquad \lim_{\substack{(x,y)\to(0,0)\\ along\ C_1}} \frac{x^3y^2}{x^6+y^4} \neq \lim_{\substack{(x,y)\to(0,0)\\ along\ C_2}} \frac{x^3y^2}{x^6+y^4}$$

So, $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$ Does not exist.

Definition 17:

(1) A function $f(x,y)$ is said to be continuous at a point (a,b) in $Dom(f)$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$

(2) A function $f(x,y)$ is said to be continuous on a set $S \subseteq Dom(f)$ if $f(x,y)$ is continuous at every point in S.

(3) A function $f(x,y)$ is said to be continuous everywhere if it is continuous on $\mathbb{R}^2$

(4) A function $f(x,y,y)$ is said to be continuous everywhere if it is continuous on $\mathbb{R}^3$.

Example 18:

(1) $f(x,y) = \frac{x^4+x^2y-5}{y^2+1}$ is continuous on $Dom(f) = \mathbb{R}^2 \Rightarrow f$ is continuous everywhere.

(2) $f(x,y) = \frac{ye^x-5}{x^2+y^2}$ is continuous on $Dom(f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \neq 0\}$

$\Rightarrow f$ is continuous on $Dom(f) = \mathbb{R}^2 \backslash \{(0,0)\}$.

(3) $f(x,y,z) = \frac{z\ln(y)-5x}{x-2y-z}$ is continuous on:

$$Dom(f) = \{(x,y,z) \in \mathbb{R}^3 : x - 2y - z \neq 0, \qquad y > 0\}$$

(4) $f(x,y,z) = 2$ is continuous on $Dom(f) = \mathbb{R}^3 \Rightarrow f$ is continuous everywhere.

Example 19: Find the region on which the function f is continuous.

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2} & , \quad (x,y)\neq(0,0)\\ \quad 0 & , \quad (x,y)=(0,0)\end{cases}$$

Solution:

When $(x,y)\neq(0,0)$ the function f is continuous since $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$

To check whether the function f is continuous at $(0,0)$ or not we must study:

(1) Is $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists or not?

(2) If $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists, is $\lim\limits_{(x,y)\to(0,0)} f(x,y)=f(0,0)$

So, take C_1: $x=\boxed{t}\boxed{+}\boxed{0}$, $y=\boxed{0}\boxed{+}\boxed{0.}$ Then

$(x,y)\to(0,0) \Rightarrow t\to 0$:

$$\lim_{\substack{(x,y)\to(0,0)\\ along\ C_1}} f(x,y)=\lim_{\substack{(x,y)\to(0,0)\\ along\ C_1}}\frac{x^2-y^2}{x^2+y^2}=\lim_{t\to 0}\frac{t^2-0^2}{t^2+0^2}=1$$

So, take C_2: $x=\boxed{0}\boxed{+}\boxed{0}$, $y=\boxed{t}\boxed{+}\boxed{0.}$ Then

$(x,y)\to(0,0) \Rightarrow t\to 0$:

$$\lim_{\substack{(x,y)\to(0,0)\\ along\ C_2}} f(x,y)=\lim_{\substack{(x,y)\to(0,0)\\ along\ C_2}}\frac{x^2-y^2}{x^2+y^2}=\lim_{t\to 0}\frac{0^2-t^2}{0^2+t^2}=-1$$

$$\Rightarrow \quad \lim_{\substack{(x,y)\to(0,0)\\ along\ C_1}} f(x,y)\neq\lim_{\substack{(x,y)\to(0,0)\\ along\ C_2}} f(x,y)$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) \quad \text{Does not exist}$$

$\Rightarrow f(x,y)$ is discontinuous at $(0,0)$

$\Rightarrow$ $f(x,y)$ is continuous only on $\mathbb{R}^2\backslash\{(0,0)\}$

Remark 20:

(1) Recall that:

$$\boxed{\lim_{(x,y)\to(a,b)} F(x,y) = L \text{ (exists)}} \Leftrightarrow \boxed{\left.\lim_{\substack{(x,y)\to(a,b)\\ \text{along } C}} F(x,y) = L \right\} \begin{array}{c}\text{for } \textbf{all} \text{ paths } C\\ \text{that pass through}\\ \text{the point } (a,b)\end{array}}$$

(2) If a function $f(x,y)$ is continuous at a point (a,b), then

$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ exists

$\Rightarrow$ If C is a given path that passes through the point (a,b), then

$$\lim_{\substack{(x,y)\to(a,b)\\ \text{along } C}} F(x,y) = f(a,b)$$

Example 21: Find the value of k such that the function f is continuous at the origin, where

$$f(x,y) = \begin{cases} \dfrac{1-\cos\left(\sqrt{x^2+y^2}\right)}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ k & , \quad (x,y) = (0,0) \end{cases}$$

Solution: f is continuous at the origin $\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = k$ (the limit exists)

$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}} f(x,y) = k$, where C is any path in $Dom(f)$ passing through $(0,0)$.

So, take C: $x = \boxed{t}\boxed{+}\boxed{0}$, $y = \boxed{0}\boxed{+}\boxed{0}$ with $t > 0$. Then

$(x,y) \to (0,0) \Rightarrow t \to 0$:

$$k = \lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}} \frac{1-\cos\left(\sqrt{x^2+y^2}\right)}{x^2+y^2}$$

$$= \lim_{t\to 0} \frac{1-\cos\left(\sqrt{t^2+0^2}\right)}{t^2+0^2} = \lim_{t\to 0} \frac{1-\cos(t)}{t^2} = \lim_{t\to 0} \frac{\sin(t)}{2t} \quad (by\ L'Hopital's\ Rule)$$

$$= \lim_{t\to 0} \frac{\cos(t)}{2} \quad (by\ L'Hopital's\ Rule)$$

$$= \frac{1}{2} \Rightarrow k = \frac{1}{2}$$

Example 22: Find the value of k such that the function f is continuous everywhere, where

$$f(x,y)=\begin{cases}\dfrac{kx^2-2y^2}{x^2+y^2} & , \quad (x,y)\neq(0,0)\\ -2 & , \quad (x,y)=(0,0)\end{cases}$$

Solution: f is continuous everywhere $\Rightarrow$ f is continuous at the origin

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)=f(0,0)=-2 \text{ (the limit exists)}$$

$\Rightarrow \lim\limits_{\substack{(x,y)\to(0,0)\\ along\ C}} f(x,y)=-2$, where C is any path in $Dom(f)$ passing through $(0,0)$.

So, take C: $x=\boxed{t}\boxed{+}\boxed{0}$, $y=\boxed{0}\boxed{+}\boxed{0.}$ Then $(x,y)\to(0,0) \Rightarrow t\to 0$:

$$-2 = \lim_{\substack{(x,y)\to(0,0)\\ along\ C}} f(x,y) = \lim_{\substack{(x,y)\to(0,0)\\ along\ C}} \frac{kx^2-2y^2}{x^2+y^2} = \lim_{t\to0}\frac{kt^2-2(0^2)}{t^2+0^2}=k$$

$\Rightarrow k=-2$

Example 23: Find the value of a such that the function f is continuous at the point $(0,2)$, where

$$f(x,y)=\begin{cases}\dfrac{\sin(xy)}{x} & , \quad x\neq0\\ a & , \quad x=0\end{cases}$$

Solution: f is continuous at the point $(0,2)$ $\Rightarrow \lim\limits_{(x,y)\to(0,2)} f(x,y)=f(0,2)=a$

$\Rightarrow \lim\limits_{\substack{(x,y)\to(0,0)\\ along\ C}} f(x,y)=-2$, where C is any path in $Dom(f)$ passing through the point $(0,2)$.

So, take C: $x=\boxed{t}\boxed{+}\boxed{0}$, $y=\boxed{0}\boxed{+}\boxed{2.}$ Then $(x,y)\to(0,2) \Rightarrow t\to 0$:

$$a=\lim_{\substack{(x,y)\to(0,2)\\ along\ C}} f(x,y)=\lim_{\substack{(x,y)\to(0,2)\\ along\ C}}\frac{\sin(xy)}{x}=\lim_{t\to0}\frac{\sin(t(2))}{t}=\lim_{t\to0}\frac{\sin(2t)}{t}=2$$

$\Rightarrow a=2$

Example 24: Find the value of k such that the function f is continuous at the point $(1,1)$, where

$$f(x,y)=\begin{cases}\dfrac{\sqrt{xy+8}-3}{xy-1} & , \quad xy\neq 1\\ k & , \quad xy=1\end{cases}$$

Solution: f is continuous at the point $(1,1)$ $\Rightarrow$ $\lim\limits_{(x,y)\to(1,1)} f(x,y)=f(1,1)=k$

$\Rightarrow$ $\lim\limits_{\substack{(x,y)\to(1,1)\\ along\ C}} f(x,y)=k$, where C is any path in $Dom(f)$ passing through the point $(1,1)$.

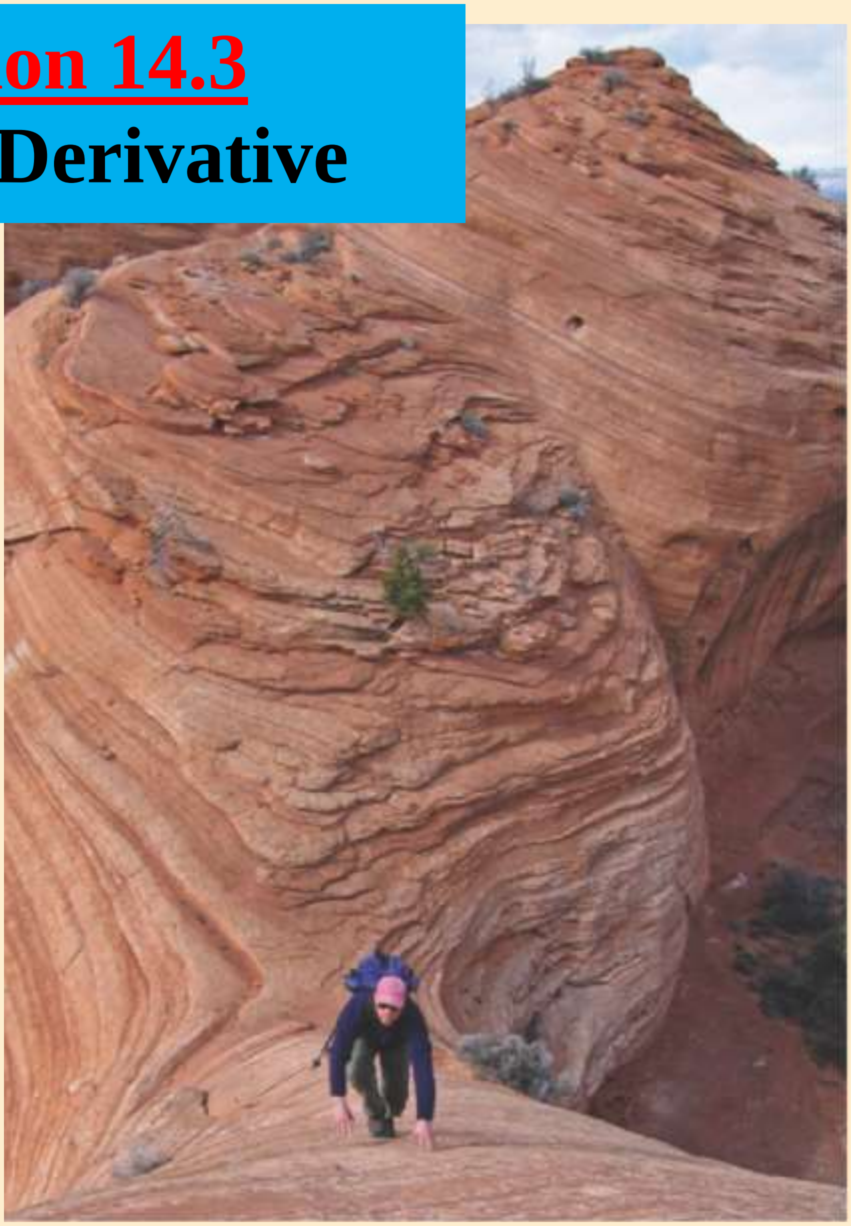
So, take C: $x=\boxed{t}\boxed{+}\boxed{1}$, $y=\boxed{0}\boxed{+}\boxed{1.}$ Then

$(x,y)\to(1,1) \Rightarrow t\to 0$:

$$\begin{aligned}
k &= \lim_{\substack{(x,y)\to(1,1)\\ along\ C}} f(x,y)\\
&= \lim_{\substack{(x,y)\to(1,1)\\ along\ C}} \frac{\sqrt{xy+8}-3}{xy-1}\\
&= \lim_{t\to 0}\frac{\sqrt{(t+1)+8}-3}{(t+1)-1}\\
&= \lim_{t\to 0}\frac{\sqrt{t+9}-3}{t}\times\frac{\sqrt{t+9}+3}{\sqrt{t+9}+3}\\
&= \lim_{t\to 0}\frac{(t+9)-9}{t}\times\frac{1}{\sqrt{t+9}+3}\\
&= \lim_{t\to 0}\frac{1}{\sqrt{t+9}+3}\\
&= \frac{1}{6} \quad \Rightarrow \quad k=\frac{1}{6}
\end{aligned}$$

Chapter 14 Partial Derivatives

Section 14.3
Partial Derivative



14.3 Partial Derivative

Definition 1: The **partial derivative of f:**

(a) **with respect to x** at a point $(\boldsymbol{a},\boldsymbol{b})$ written as $f_x(a,b)$ is defined by

$$f_x(a,b)=\lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a}=\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}.$$

(b) **with respect to y** at a point $(\boldsymbol{a},\boldsymbol{b})$ written as $f_y(a,b)$ is defined by

$$f_y(a,b)=\lim_{y\to b}\frac{f(a,y)-f(a,b)}{y-b}=\lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}$$

Remark 2: **(a)** $f_x(a,b)=g'(a)$, where $g(x)=f(x,b)$.
(b) $f_y(a,b)=h'(b)$, where $h(y)=f(a,y)$.

Example 3: Find $f_x(1,0)$ and $f_y(1,0)$, where $f(x,y)=\sqrt{x^4+y^3+3}$

Solution:

$$f_x(1,0)=g'(1),\quad \text{where } g(x)=f(x,0)=\sqrt{x^4+0^3+3}=\sqrt{x^4+3}$$

$$\Rightarrow\quad f_x(1,0)=\mathbf{1},\quad \text{since } g'(x)=\frac{4x^3}{2\sqrt{x^4+3}}$$

Also,

$$f_y(1,0)=h'(0),\quad \text{where } h(y)=f(1,y)=\sqrt{1^4+y^3+3}=\sqrt{y^3+4}$$

$$\Rightarrow\quad f_y(1,0)=\mathbf{0}\quad \text{since } h'(y)=\frac{3y^2}{2\sqrt{y^3+4}}$$

Example 4: Find $f_x(0,0)$, where $f(x,y)=3x+\sqrt[3]{8x^3+27y^6}$

Solution:

$$f_x(0,0)=g'(0),\quad \text{where } g(x)=f(x,0)=3x+\sqrt[3]{8x^3}=5x$$

$$\Rightarrow\quad f_x(0,0)=\mathbf{5},\quad \text{since } g'(x)=5$$

Example 5: Find $f_x(0,0)$ if it exists, where $f(x,y)=\sqrt{x^2+y^2}$

Solution:

$$f(x,0)=\sqrt{x^2+(0)^2}=\sqrt{x^2}=|x|$$

Since $|x|$ is not differentiable at $x=0$, then $f_x(0,0)$ does not exists
That is: the partial derivative of f with respect to x does not exists at $(0,0)$.

Example 6: Find $f_y(0{,}0)$, where $f(x,y)=\begin{cases}\dfrac{3x^2+xy+y^3}{x^2+y^2} & , \quad (x,y)\neq(0{,}0)\\ 0 & , \quad (x,y)=(0{,}0)\end{cases}$

Solution:

First Method: $f(0,y)=\begin{Bmatrix}\frac{3(0)^2+(0)y+y^3}{(0)^2+y^2} & , & y\neq 0\\ 0 & , & y=0\end{Bmatrix}=\begin{Bmatrix}y & , & y\neq 0\\ 0 & , & y=0\end{Bmatrix}=y$

$\Rightarrow\ f(0,y)=y\ \Rightarrow\ h(y)=f(0,y)\Rightarrow h(y)=y\Rightarrow\ f_y(0{,}0)=h'(0)=1$

Second Method: By definition:

$$f_y(0{,}0)=\lim_{y\to 0}\frac{f(0,y)-f(0{,}0)}{y-0}=\lim_{y\to 0}\frac{\frac{3(0)^2+(0)y+y^3}{(0)^2+y^2}-0}{y-0}$$

$$=\lim_{y\to 0}\frac{\left(\frac{y^3}{y^2}\right)}{y}=\lim_{y\to 0}\frac{y^3}{y^3}=1\ \Rightarrow\ f_y(0{,}0)=1$$

Example 7: Find $f_y(0{,}0)$, where $f(x,y)=\begin{cases}\dfrac{3x^2+xy+y^3}{x^2+y^2} & , \quad (x,y)\neq(0{,}0)\\ 1 & , \quad (x,y)=(0{,}0)\end{cases}$

Solution:

First Method: $f(0,y)=\begin{Bmatrix}\frac{3(0)^2+(0)y+y^3}{(0)^2+y^2} & , & y\neq 0\\ 1 & , & y=0\end{Bmatrix}=\begin{Bmatrix}y & , & y\neq 0\\ 1 & , & y=0\end{Bmatrix}$

Observe that $f(0,y)$ is discontinuous at $y=0$ $\Rightarrow$ $f_y(0{,}0)$ does not exist

Second Method: By definition:

$$f_y(0{,}0)=\lim_{y\to 0}\frac{f(0,y)-f(0{,}0)}{y-0}=\lim_{y\to 0}\frac{\frac{3(0)^2+(0)y+y^3}{(0)^2+y^2}-1}{y-0}$$

$$=\lim_{y\to 0}\frac{\left(\frac{y^3}{y^2}\right)-1}{y}=\lim_{y\to 0}\frac{y-1}{y}=\frac{-1}{0}\ \Rightarrow\ f_y(0{,}0)\ \text{ does not exist}$$

Example 8: Find $f_x(x,y)$ and $f_y(x,y)$, where $f(x,y)=xy^4e^{3x}+\cos(2y)$

Solution:

$$f_x(x,y)=xy^4(3e^{3x})+e^{3x}(y^4)+0=3xy^4e^{3x}+y^4e^{3x}$$

$$f_y(x,y)=xe^{3x}(4y^3)+\big(-\sin(2y)\,(2)\big)=4xy^3e^{3x}-2\sin(2y)$$

Example 9: Find $f_x(1,0)$ and $f_y(1,-1)$, where $f(x,y) = \begin{cases} \frac{3x^3}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$

Solution: At $(1,0)$ and $(1,-1)$, the function $f(x,y) = \frac{3x^3}{x^2+y^2}$. So,

$$f_x(x,y) = \frac{(x^2+y^2)9x^2 - 3x^3(2x)}{(x^2+y^2)^2} = \frac{3x^4 + 9x^2y^2}{(x^2+y^2)^2} \quad \Rightarrow f_x(1,0) = 3$$

$$f_y(x,y) = \frac{(x^2+y^2)(0) - 3x^3(2y)}{(x^2+y^2)^2} = \frac{-6x^3y}{(x^2+y^2)^2} \quad \Rightarrow f_y(1,-1) = \frac{6}{4}$$

Example 10: Find $f_x(x,y)$, where

$$f(x,y) = \begin{cases} \dfrac{3x^3+xy}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

Solution:

- If $(x,y) \neq (0,0) \Rightarrow f(x,y) = \frac{3x^3+xy}{x^2+y^2} \Rightarrow$

$$f_x = \frac{(x^2+y^2)(9x^2+y) - (3x^3+xy)(2x)}{(x^2+y^2)^2}$$

$$\Rightarrow \quad f_x = \frac{3x^4 + 9x^2y^2 - 2x^2y}{(x^2+y^2)^2}$$

First Method for finding $f_x(0,0)$:

If $(x,y) = (0,0) \Rightarrow$

$$g(x) = f(x,0) = \left\{ \begin{matrix} \dfrac{3x^3 + x(0)}{x^2+(0)^2} & , & x \neq 0 \\ 0 & , & x = 0 \end{matrix} \right\} = \left\{ \begin{matrix} 3x & , & x \neq 0 \\ 0 & , & x = 0 \end{matrix} \right\} = 3x$$

$$\Rightarrow \quad f_x(0,0) = g'(0) = 3$$

Second Method for finding $f_x(0,0)$: By Definition:

$$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x-0} = \lim_{x\to 0} \frac{\dfrac{3x^3 + x(0)}{x^2+(0)^2} - 0}{x-0}$$

$$= \lim_{x\to 0} \frac{3x}{x} = 1 \quad \Rightarrow \quad f_x(0,0) = 3$$

So,

$$f_x(x,y) = \begin{cases} \dfrac{(x^2+y^2)(9x^2+y) - (3x^3+xy)(2x)}{(x^2+y^2)^2} & , \quad (x,y) \neq (0,0) \\ 3 & , \quad (x,y) = (0,0) \end{cases}$$

Example 11: Find $\lim_{h\to 0}\frac{f(1+h,-1)-f(1,-1)}{h}$, where $f(x,y)=\sqrt[5]{x^7-y^2+1}$

Solution: By the definition of the partial derivatives, we have

$\lim_{h\to 0}\frac{f(1+h,-1)-f(1,-1)}{h}=f_x(1,-1)$ But $f_x=\frac{1}{5}(x^7-y^2+1)^{-\frac{4}{5}}(7x^6)$

$$\Rightarrow\ \lim_{h\to 0}\frac{f(1+h,-1)-f(1,-1)}{h}=\frac{7}{5}$$

Example 12: Let $f(x,y)$ be a function such that $f_x=2xy$, $f_y=x^2+2y$, and $f(1,1)=8$. Find $f(0,2)$

Solution: $f_x=2xy\ \Rightarrow\ f(x,y)=\int 2xy\,dx+G(y)$

$$f(x,y)=x^2y+G(y)\quad \dots\dots\dots(1)$$

Now, we find $G(y)$:

Differentiating equation (1) with respect to y:

$f_y=x^2+G'(y)$ but $f_y=x^2+2y\ \Rightarrow\ x^2+G'(y)=x^2+2y\ \Rightarrow\ G'(y)=2y$

$G(y)=\int 2ydy=y^2+C$, where C is a constant.

Equation (1) implies that: $f(x,y)=x^2y+G(y)=x^2y+y^2+C$.

$$\Rightarrow\ f(x,y)=x^2y+y^2+C\ \dots\dots\dots\dots.(2)$$

Now, we find C:

$f(1,1)=8\ \Rightarrow\ 1^2(1)+1^2+C=8\ \Rightarrow\ C=6.$

Equation (2) implies that:

$$f(x,y)=x^2y+y^2+6$$

Finally, $f(0,2)=(0)^2(2)+(2)^2+6=10$

Example 13: Find $f_z(-1,1,e^2)$, where $f(x,y,z)=e^{2xy}\ln(z)$.

Solution: $f_z=\frac{e^{2xy}}{z}\ \Rightarrow\ f_z(-1,1,e^2)=\frac{e^{2(-1)(1)}}{e^2}=\frac{e^{-2}}{e^2}=e^{-4}$

Notations 14: There are several forms of partial derivatives of $z=f(x,y)$:

$$f_x(x,y)=f_x=z_x=\frac{\partial}{\partial x}f(x,y)=\frac{\partial f}{\partial x}=\frac{\partial z}{\partial x}=D_xf=D_1f$$

and

$$f_y(x,y)=f_y=z_y=\frac{\partial}{\partial y}f(x,y)=\frac{\partial f}{\partial y}=\frac{\partial z}{\partial y}=D_yf=D_2f$$

Example 15: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(\pi, \sqrt{3})$, where $f(x,y) = \sin\left(\frac{x}{y^2+1}\right)$

Solution:

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{y^2+1}\right)\left(\frac{1}{(y^2+1)^2}\right)$$

$$\Rightarrow \quad \left.\frac{\partial f}{\partial x}\right|_{(\pi,\sqrt{3})} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\pi \\ y=\sqrt{3}}} = \cos\left(\frac{\pi}{\sqrt{3}^2+1}\right)\left(\frac{1}{\left(\sqrt{3}^2+1\right)^2}\right) = \frac{\pi\sqrt{3}}{16\sqrt{2}}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{y^2+1}\right)\left(\frac{-2xy}{(y^2+1)^2}\right)$$

$$\Rightarrow \quad \left.\frac{\partial f}{\partial y}\right|_{(\pi,\sqrt{3})} = \left.\frac{\partial f}{\partial y}\right|_{\substack{x=\pi \\ y=\sqrt{3}}} = \cos\left(\frac{\pi}{\sqrt{3}^2+1}\right)\left(\frac{-2\pi\sqrt{3}}{\left(\sqrt{3}^2+1\right)^2}\right) = -\frac{\pi\sqrt{3}}{8\sqrt{2}}$$

Interpolations of partial derivatives 16:

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x,y)$ represents a surface S (the graph of f). If $f(a,b) = c$, then the point $P(a,b,c)$ lies on S. By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S.

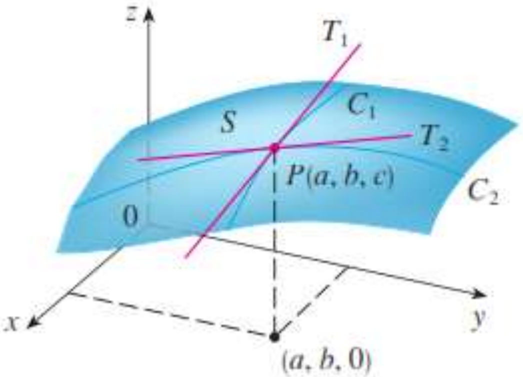


Figure 1

(In other words, C_1 is the trace of S in the plane $y = b$.) Likewise, the vertical plane $x = a$ intersects S in a curve C_2 . Both of the curves C_1 and C_2 pass through the point P. (See Figure 1)

Notice that the curve C_1 is the graph of the function $g(x) = f(x,b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a,b)$. The curve C_2 is the graph of the function $h(y) = f(a,y)$, so the slope of its tangent T_2 at P is $h'(b) = f_y(a,b)$.

Thus, the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a,b,c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

Example 17: If $f(x,y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers as slopes.

Solution: $f_x = -2x \Rightarrow f_x(1,1) = -2$ and $f_y = -4y \Rightarrow f_y(1,1) = -4$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2, y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.) The slope of the tangent line to this parabola at the point $(1,1,1)$ is $f_x(1,1) = -2$. Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2, x = 1$, and the slope of the tangent line at $(1,1,1)$ is $f_y(1,1) = -4$. (See Figure 3.)

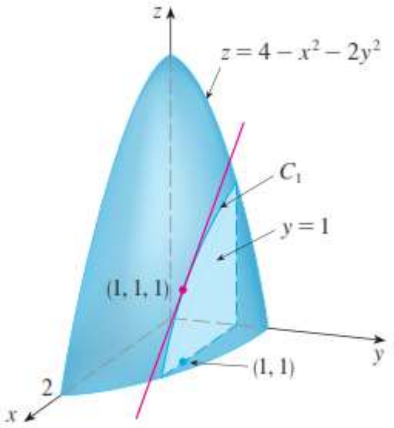


Figure 2

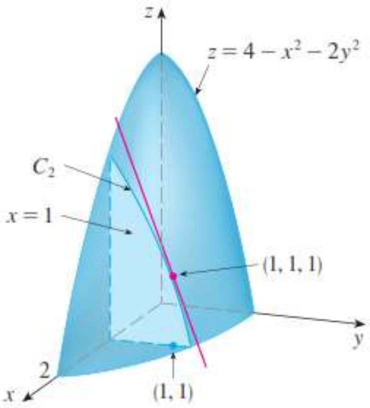
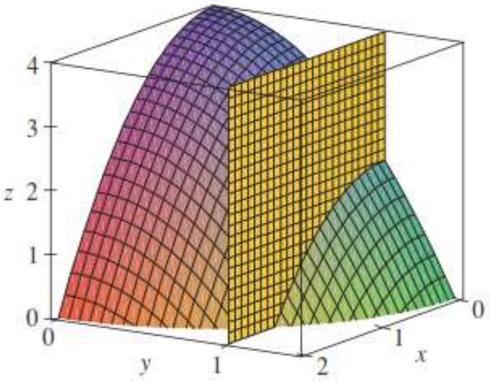
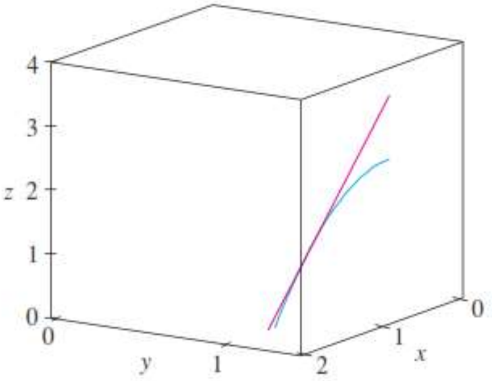
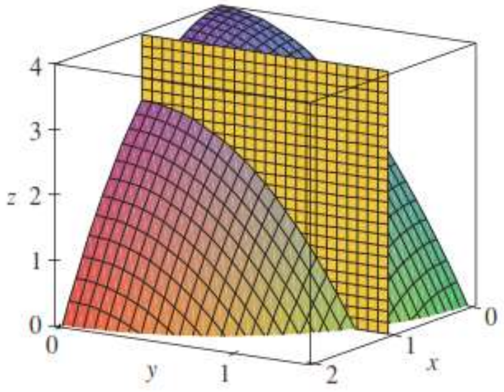
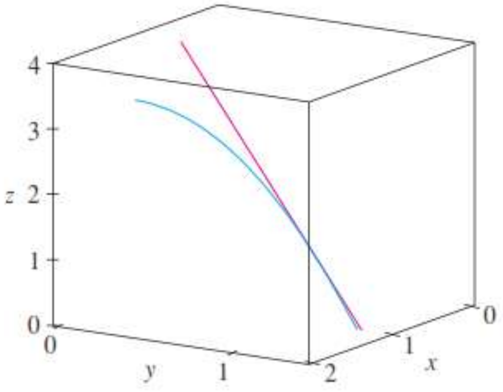


Figure 3

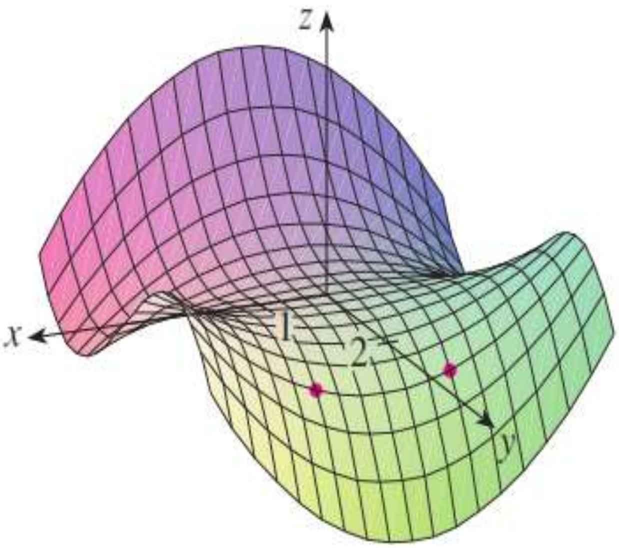








Remark 18: Determine the signs of the partial derivatives for the function whose graph is shown in the figure:



(1) $f_x(1,2)$

(2) $f_y(1,2)$

(3) $f_x(-1,2)$

(4) $f_y(-1,2)$

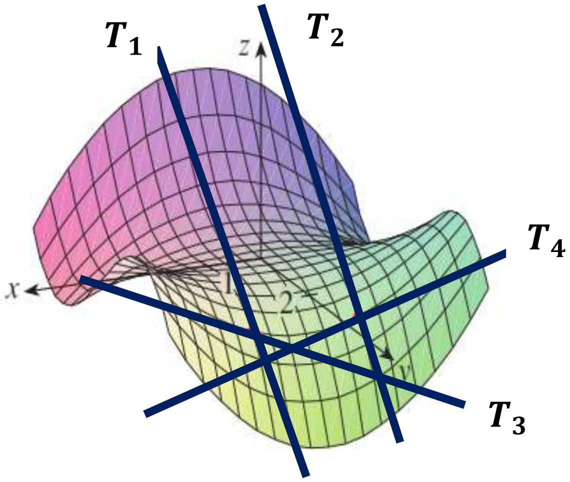
Solution:

(1) $f_x(1,2)$ positive, (See T_3)

(2) $f_y(1,2)$ negative, (See T_1)

(3) $f_x(-1,2)$ negative, (See T_4)

(4) $f_y(-1,2)$ negative, (See T_2)



Example 19: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is a function in x and y which is defined implicitly by the equation $\frac{x^3-y^4+z^3}{1-6xyz} = 1$

Solution: $\frac{x^3-y^4+z^3}{1-6xyz} = 1 \quad \Rightarrow x^3 - y^4 + z^3 = 1 - 6xyz$

$$\Rightarrow\ x^3 - y^4 + z^3 + 6xyz = 1 \quad \dots\dots\dots(1)$$

❖ Differentiating both sides of equation (1) with respect to x:

$\Rightarrow\ 3x^2 + 3z^2 z_x + 6xyz_x + 6yz = 0 \ \Rightarrow\ z_x(3z^2 + 6xy) = -(3x^2 + 6yz)$

$$\Rightarrow \quad z_x = -\frac{3x^2 + 6yz}{3z^2 + 6xy}$$

❖ Differentiating both sides of equation (1) with respect to y:

$\Rightarrow\ -4y^3 + 3z^2 z_y + 6xyz_y + 6xz = 0 \ \Rightarrow\ z_y(3z^2 + 6xy) = 4y^3 - 6xy$

$$\Rightarrow \quad z_y = \frac{4x^3 - 6xy}{3z^2 + 6xy}$$

Higher Derivatives 20: Let $z = f(x,y)$. Then the second partial derivatives of f are:

$$z_{xx} = f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} \qquad z_{yy} = f_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2}$$

$$z_{xy} = f_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} \qquad z_{yx} = f_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

Where $f_{xy} = (f_x)_y$, $f_{yx} = \left(f_y\right)_x$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

Example 21: Find the second order partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$

Solution: $f_x = 3x^2 + 2xy^3$ and $f_y = 3x^2y^2 - 4y$. So,

$$f_{xx} = 6x + 2y^3 \qquad f_{yy} = 6x^2y - 4$$

$$f_{xy} = (f_x)_y = (3x^2 + 2xy^3)_y = 6xy^2 \qquad f_{yx} = (f_y)_x = (3x^2y^2 - 4y)_x = 6xy^2$$

Remark 22: Observe that in the Example 18: $f_{xy} = f_{yx} = 6xy^2$ which is not always true

Example 23: Find $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ at the origin, where $f(x,y) = 2x\sqrt[3]{x^3 - 27y^3}$

Solution: $f_x = 2x\frac{1}{3}(x^3 - 27y^3)^{-\frac{2}{3}}(3x^2) + 2\sqrt[3]{x^3 - 27y^3}$

$$\Rightarrow \quad f_x = \frac{2x^3}{(x^3 - 27y^3)^{\frac{2}{3}}} + 2\sqrt[3]{x^3 - 27y^3}$$

$$\Rightarrow \quad f_x(0,y) = \frac{2(0^3)}{((0)^3 - 27y^3)^{\frac{2}{3}}} + 2\sqrt[3]{(0)^3 - 27y^3} \quad \Rightarrow \quad f_x(0,y) = -6y$$

$$\Rightarrow \quad f_{xy}(0,y) = -6 \qquad \Rightarrow \quad f_{xy}(0,0) = -6$$

Also,

$$f_y = 2x\frac{1}{3}(x^3 - 27y^3)^{-\frac{2}{3}}(-27(3)y^2) = \frac{-54xy^2}{(x^3 - 27y^3)^{\frac{2}{3}}}$$

$$\Rightarrow \quad f_y(x,0) = \frac{-54x(0^2)}{(x^3 - 27(0^3))^{\frac{2}{3}}} \quad \Rightarrow \quad f_y(x,0) = 0$$

$$\Rightarrow \quad f_{yx}(x,0) = 0 \qquad \Rightarrow \quad f_{yx}(0,0) = 0$$

Observe that in this example: $f_{xy}(0,0) \neq f_{yx}(0,0)$

Clairaut's Theorem 24: Suppose $f(x,y)$ is defined on a disk D that contains the point (a,b) . If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Example 25: If $f(x,y,z) = \dfrac{y^2e^{3xyz}}{8x^2}$, Find $f_{yxxzx}(1,2,0)$

Solution:

Since all partial derivatives of f of all orders are continuous near the point $(1,2,0)$, then Clairaut's Theorem implies that $f_{yxxzx} = f_{zyxxx}$. So,

$$f_z = \frac{y^2(3xy)e^{3xyz}}{8x^2} \quad \Rightarrow f_z(x,y,0) = \frac{y^2(3xy)e^{3xy(0)}}{8x^2} = \frac{3xy^3}{8x^2} = \frac{3y^3}{8x}$$

$$\Rightarrow f_{zy}(x,y,0) = \frac{9y^2}{8x} \quad \Rightarrow \quad f_{zy}(x,2,0) = \frac{9(2)^2}{8x} = \frac{9}{2x}$$

$$\Rightarrow \quad f_{zy}(x,2,0) = \frac{9}{2}x^{-1} \quad \Rightarrow \quad f_{zyx}(x,2,0) = -\frac{9}{2}x^{-2}$$

$$\Rightarrow \quad f_{zyxx}(x,2,0) = 9x^{-3} \quad \Rightarrow \quad f_{zyxxx}(x,2,0) = -27x^{-4}$$

$$\Rightarrow \quad f_{yxxzx}(1,2,0) = f_{zyxxx}(1,2,0) = -27(1)^{-4}$$

$$\Rightarrow \quad f_{yxxzx}(1,2,0) = -27$$

Example 26: If $f(x,y) = x^3y^5 - \dfrac{xy^2}{x+\ln(x)}$, Find $\left.\dfrac{\partial^6 f}{\partial y^4 \partial x^2}\right|_{(e,2)}$

Solution:

Observe that $\left.\dfrac{\partial^6 f}{\partial y^4 \partial x^2}\right|_{(e,2)} = f_{xxyyyy}(e,2)$

Since all partial derivatives of f of all orders are continuous near the point $(e,2)$, then Clairaut's Theorem implies that $f_{xxyyyy}(e,2) = f_{yyyyxx}(e,2)$. So,

$$f_y = 5x^3y^4 - \frac{2xy}{x+\ln(x)} \quad \Rightarrow f_{yy} = 20x^3y^3 - \frac{2x}{x+\ln(x)} \quad \Rightarrow f_{yyy} = 60x^3y^2$$

$$\Rightarrow \quad f_{yyyyxx} = 120x^3y \quad \Rightarrow \quad f_{yyyy}(x,2) = 240x^3 \ \Rightarrow \ f_{yyyyx}(x,2) = 720x^2$$

$$\Rightarrow f_{yyyyxx}(x,2) = 1440x \quad \Rightarrow \quad \left.\frac{\partial^6 f}{\partial y^4 \partial x^2}\right|_{(e,2)} = 1440e$$

Example 27: Find $\dfrac{\partial^{103}}{\partial y^{63}\partial x^{40}}(x^{10}\sin(xy)+x^{50})$ at the point $(-1,0)$

Solution: Let $f(x,y)=x^{10}\sin(xy)+x^{50}$

Then $\dfrac{\partial^{103}}{\partial y^{63}\partial x^{40}}(x^{10}\sin(xy)+x^{50})=f_{\underbrace{x\ldots\ldots x}_{40-times}\underbrace{y\ldots\ldots y}_{63-times}}$

Since all partial derivatives of f of all orders are continuous near the point $(-1,0)$, then Clairaut's Theorem implies that $f_{\underbrace{x\ldots\ldots x}_{40-times}\underbrace{y\ldots\ldots y}_{63-times}}(-1,0)=f_{\underbrace{y\ldots\ldots y}_{63-times}\underbrace{x\ldots\ldots x}_{40-times}}(-1,0)$.

So,

$$\left\{\begin{array}{l}\Rightarrow f_y=x^{10}[x\cos(xy)]+0\\ \Rightarrow f_{yy}=x^{10}[-x^2\sin(xy)]\\ \Rightarrow f_{yyy}=x^{10}[-x^3\cos(xy)]\\ \Rightarrow f_{yyyy}=x^{10}[x^4\sin(xy)]\end{array}\right\}\Rightarrow\left\{\begin{array}{l}\Rightarrow f_{\underbrace{y\ldots\ldots y}_{60-times}}=x^{10}[x^{60}\sin(xy)]\\ \Rightarrow f_{\underbrace{y\ldots\ldots y}_{61-times}}=x^{10}[x^{61}\cos(xy)]\\ \Rightarrow f_{\underbrace{y\ldots\ldots y}_{62-times}}=x^{10}[-x^{62}\sin(xy)]\\ \Rightarrow f_{\underbrace{y\ldots\ldots y}_{63-times}}=x^{10}[-x^{63}\cos(xy)]\end{array}\right\}$$

$$\Rightarrow f_{\underbrace{y\ldots\ldots y}_{63-times}}(x,0)=x^{10}\left[-x^{63}\cos\left(x(0)\right)\right]=-x^{73}$$

$$\Rightarrow f_{\underbrace{y\ldots\ldots y}_{63-times}\underbrace{x\ldots\ldots x}_{40-times}}(x,0)=-(73)(72)(71)\ldots(73-39)x^{73-40}$$

$$\Rightarrow f_{\underbrace{y\ldots\ldots y}_{63-times}\underbrace{x\ldots\ldots x}_{40-times}}(x,0)=-(73)(72)(71)\ldots(34)x^{33}=-\frac{73!}{33!}x^{33}$$

$$\Rightarrow f_{\underbrace{y\ldots\ldots y}_{63-times}\underbrace{x\ldots\ldots x}_{40-times}}(-1,0)=-\frac{73!}{33!}(-1)^{33}=\frac{73!}{33!}$$

Remark 28: Recall that: $\dfrac{d}{dx}\displaystyle\int_{g(x)}^{h(x)}F(t)dt=F\big(h(x)\big)h'(x)-F\big(g(x)\big)g'(x)$

Example 29: If $f(x,y)=\displaystyle\int_{y}^{xy}\cos(e^t)dt,$ find $f_{xy}(0,0)$

Solution:

$$f_x=\cos(e^{xy})\frac{\partial}{\partial x}(xy)-\cos(e^y)\frac{\partial}{\partial x}(y)=y\cos(e^{xy})-\cos(e^y)(0)$$

$$\Rightarrow\ f_x=y\cos(e^{xy})\ \Rightarrow\ f_x(0,y)=y\cos\left(e^{(0)y}\right)=y\cos(1)$$

$$\Rightarrow f_{xy}=\cos(1)\ \Rightarrow\ f_{xy}(0,0)=\cos(1)$$

Remark 30:

(1) The Laplace equation is the partial differential equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{where } u(x,y) \text{ is a function in } x \text{ and } y.$$

(2) The wave equation is the partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \qquad \text{where } u(x,t) \text{ is a function in } x \text{ and } t.$$

Example 31:

(1) Show that $u(x,y) = e^x \sin(y)$ is a solution to the Laplace equation.

(2) Show that $u(x,t) = \sin(x - at)$ is a solution to the wave equation.

Solution:

(1) $u(x,y) = e^x \sin(y) \Rightarrow u_x = e^x \sin(y) \Rightarrow u_{xx} = e^x \sin(y)$

$u(x,y) = e^x \sin(y) \Rightarrow u_y = e^x \cos(y) \Rightarrow u_{yy} = -e^x \sin(y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin(y) + -e^x \sin(y) = 0.$$

So $u(x,y) = e^x \sin(y)$ satisfies the Laplace equation

$\Rightarrow$ $u(x,y) = e^x \sin(y)$ is a solution to the Laplace equation

(2) $u(x,t) = \sin(x - at) \Rightarrow u_t = -a\cos(x - at) \Rightarrow u_{tt} = -a^2 \sin(x - at)$

$u(x,t) = \sin(x - at) \Rightarrow u_x = \cos(x - at) \Rightarrow u_{xx} = -\sin(x - at)$

$$\left\{ \begin{array}{l} \Rightarrow \dfrac{\partial^2 u}{\partial t^2} = -a^2 \sin(x - at) \\ \Rightarrow a^2 \dfrac{\partial^2 u}{\partial x^2} = -a^2 \sin(x - at) \end{array} \right\} \Rightarrow \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

So $u(x,t) = \sin(x - at)$ satisfies the wave equation

$\Rightarrow$ $u(x,t) = \sin(x - at)$ is a solution to the wave equation

Chapter 12 Vectors and the Geometry of Space

Section 12.4: The Cross Product

12.4: The Cross Product

Definition 1: The Cross product of two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is given by:

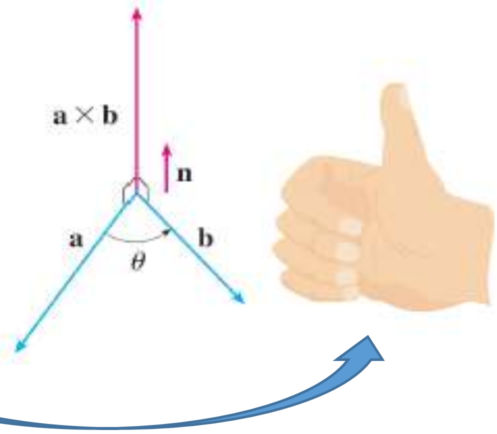
$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k$$

$\Rightarrow$ $\vec{a} \times \vec{b}$ is a vector in V_3.

Remark 2:

(1) To find $\vec{a} \times \vec{b}$ we must have $\vec{a}$ and $\vec{b}$ in V_3. To find $\vec{a} \cdot \vec{b}$, the vectors $\vec{a}$ and $\vec{b}$ may be in V_2 or V_3.

(2) $\vec{a} \times \vec{b}$ is a vector orthogonal (يعامد) to the vectors $\vec{a}$ and $\vec{b}$ and so $\vec{a} \times \vec{b}$ is orthogonal to the plane containing both vectors $\vec{a}$ and $\vec{b}$. The direction of $\vec{a} \times \vec{b}$ is determined by the right hand rule.



Example 3: Let $\vec{a} = \langle 3,2,1 \rangle$ and $\vec{b} = \langle -1,1,0 \rangle$

(1) Find $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$

(2) Find two vectors perpendicular (orthogonal) to both $\vec{a}$ and $\vec{b}$

(3) Find two unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$

(4) Find two unit vectors orthogonal to the plane that pass through the points $A(1,2,3), B(4,4,4)$, and $C(0,3,3)$

Solution:

(1) $$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = i\big(2(0) - 1(1)\big) - j\big(3(0) - 1(-1)\big) + k\big(3(1) - 2(-1)\big)$$
$$= -i - j + 5k$$

$$\vec{b} \times \vec{a} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = i\big(1(1) - 2(0)\big) - j\big(-1(1) - 3(0)\big) + k\big(-1(2) - 3(1)\big)$$
$$= i + j - 5k$$

(2) Two vectors orthogonal to both $\vec{a}$ and $\vec{b}$ are $\vec{a} \times \vec{b}$ and $-\vec{a} \times \vec{b}$

$\Rightarrow -i - j + 5k$ and $i + j - 5k$ are orthogonal to both $\vec{a}$ and $\vec{b}$

(3) Two unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$ are $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ and $-\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$\Rightarrow \frac{-i-j+5k}{|-i-j+5k|}$ and $\frac{i+j-5k}{|i+j-5k|}$ are unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$

$\Rightarrow \frac{-i-j+5k}{\sqrt{26}}$ and $\frac{i+j-5k}{\sqrt{26}}$ are unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$

(4) Let $\vec{a} = \overrightarrow{AB} = \langle B - A\rangle = \langle 3,2,1\rangle$ and $\vec{b} = \overrightarrow{AC} = \langle C - A\rangle = \langle -1,1,0\rangle$

$\Rightarrow \vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ are orthogonal to both $\vec{a}$ and $\vec{b}$

$\Rightarrow \vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ are orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

$\Rightarrow -i - j + 5k$ and $i + j - 5k$ are orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

$\Rightarrow \frac{-i-j+5k}{|-i-j+5k|}$ and $\frac{i+j-5k}{|i+j-5k|}$ are orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

$\Rightarrow \frac{-i-j+5k}{\sqrt{26}}$ and $\frac{i+j-5k}{\sqrt{26}}$ are unit vectors orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

Example 4: $i\times j = k,\quad j\times k = i,\quad k\times i = j$

$j\times i = -k,\quad k\times j = -i,\quad i\times k = -j$

Properties of Cross Product: Let $\vec{u}$,$\vec{v}$, and $\vec{w}$ be vectors in V_2 or V_3 and let a be a scalar. Then

(1) $\vec{u}\times\vec{u} = \vec{0}$
(2) $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$
(3) $\vec{0}\times\vec{v} = \vec{v}\times\vec{0} = \vec{0}$
(4) $(a\vec{u})\times\vec{v} = \vec{u}\times(a\vec{v}) = a(\vec{u}\times\vec{v})$
(5) $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v}) + (\vec{u}\times\vec{w})$
(6) $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$

In general

$\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$

For Example:

$j\times(j\times k) = j\times(i) = -k$

$(j\times j)\times k = \vec{0}\times k = \vec{0}$

Rule 5: $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

Example 6: Let $\vec{a}$ and $\vec{b}$ be orthogonal such that $|\vec{a}| = 2$ and $|\vec{b}| = 3$.
Find $\vec{a}\times(\vec{b}\times\vec{a})$ and $|(\vec{b}\times\vec{a})\times\vec{a}|$

Solution:

$$(\vec{b}\times\vec{a})\times\vec{a} = -\vec{a}\times(\vec{b}\times\vec{a}) = -\left((\vec{a}\cdot\vec{a})\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}\right) = -(|\vec{a}|^2\vec{b} - 0\vec{a}) = -4\vec{b}$$

$$|(\vec{b}\times\vec{a})\times\vec{a}| = |-4\vec{b}| = 4|\vec{b}| = 4(3) = 12$$

Example 7: Simplify $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$

Solution:

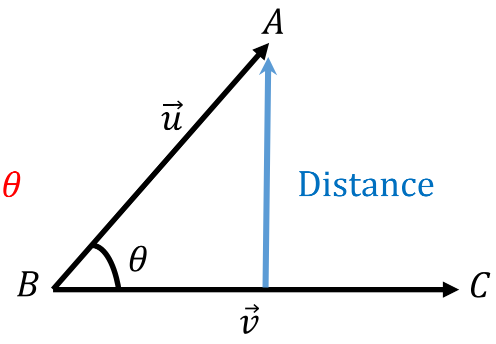
$$\begin{aligned}(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) &= \vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}+\vec{b}\times\vec{b}\\ &= \vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}+\vec{0}\\ &= 2\vec{a}\times\vec{b}\end{aligned}$$

Rule 8:

(1) The length of $\vec{a}\times\vec{b}$ is given by: $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$

(2) The length of $\vec{a}\times\vec{b}$ is given by:

$$|\vec{a}\times\vec{b}| = \sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2} \text{ (Lagrange identity)}$$



Remark 19: Let L a line that pass through the points B and C.

Then the distance from the point A to the line L is:

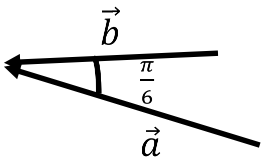
Distance $= \frac{|\vec{u}\times\vec{v}|}{|\vec{v}|}$ where $\vec{u}=\overrightarrow{BA}$ and $\vec{v}=\overrightarrow{BC}$

Example 9: Find the distance from the point $A(1,2,3)$ and the line that pass through the points $B(2,1,3)$ and $C(0,1,0)$

Solution: $\vec{u}=\overrightarrow{BA}=\langle A-B\rangle=\langle -1,1,0\rangle$ and $\vec{v}=\overrightarrow{BC}=\langle C-B\rangle=\langle -2,0,-3\rangle$

$$\text{Distance} = \frac{|\vec{u}\times\vec{v}|}{|\vec{v}|} = \frac{\sqrt{|\vec{u}|^2|\vec{v}|^2-(\vec{u}\cdot\vec{v})^2}}{|\vec{v}|} = \frac{\sqrt{2(13)-(2)^2}}{\sqrt{13}} = \frac{\sqrt{22}}{\sqrt{13}}$$

Example 10: Find $|\vec{a}\times\vec{b}|$, where $\vec{a}$ and $\vec{b}$ are given in the figure with $|\vec{a}|=8$, $|\vec{b}|=6$



Solution: $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta = 8(6)\sin\left(\frac{\pi}{6}\right) = 48\left(\frac{1}{2}\right) = 24$

Example 11: Find $|\vec{a}\times\vec{b}|$ and $\vec{a}\times\vec{b}$, where $|\vec{a}|=2$ and $|\vec{b}|=\frac{1}{2}$ and $|\vec{a}+2\vec{b}|=3$

Solution: $|\vec{a}+2\vec{b}|^2=3^2 \;\Rightarrow\; |\vec{a}|^2+4\vec{a}\cdot\vec{b}+4|\vec{b}|^2=9 \;\Rightarrow\; 4+4\vec{a}\cdot\vec{b}+1=9$

$\Rightarrow \vec{a}\cdot\vec{b}=1$

$$|\vec{a}\times\vec{b}|=\sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}=\sqrt{4\left(\frac{1}{4}\right)-(1)^2}=0 \;\Rightarrow\; \vec{a}\times\vec{b}=\vec{0}$$

Rule 12: Two vectors $\vec{a}$ and $\vec{b}$ are parallel written $\vec{a}//\vec{b}$ if $\vec{a}\times\vec{b}=\vec{0}$.

Observe the following:

(1) in **Example 11** we have $\vec{a}\times\vec{b}=\vec{0}$ so $\vec{a}//\vec{b}$.
(2) If $\vec{a}$ is any vector then $\vec{a}//\vec{0}$ since $\vec{a}\times\vec{0}=\vec{0}$

Remark 13: $\vec{a}//\vec{b} \;\Leftrightarrow\; \vec{a}=c\vec{b}$ or $\vec{b}=c\vec{a}$ for some scalar c.

Consequently: Let $\vec{a}=\langle a_1,a_2,a_3\rangle$ and $\vec{b}=\langle b_1,b_2,b_3\rangle$, Then $\vec{a}//\vec{b} \;\Leftrightarrow\; \frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$, where b_1,b_2,b_3 are nonzero scalars.

Example 14:

(1) $\langle 6,3,15\rangle//\langle 4,2,10\rangle$ since $\frac{6}{4}:\frac{3}{2}:\frac{15}{10} \;\Rightarrow$ are all equal

(2) $\langle 4,6,-28\rangle$ and $\langle 2,3,14\rangle$ are not parallel since the ratios $\frac{4}{2}:\frac{6}{3}:\frac{-28}{7}$ are not all equal

Example 15: Find the value of x that makes $\vec{a}=\langle 2,x-1,x\rangle$ and $\vec{b}=\langle x^2-1,0,x+1\rangle$ parallel.

Solution: $\frac{x^2-1}{2}=\frac{0}{x-1}=\frac{x+1}{x} \;\Rightarrow\; 0=\frac{x+1}{x} \;\Rightarrow\; x+1=0 \;\Rightarrow\; x=-1$

Check: Is there an error in the equations: $\underbrace{\frac{x^2-1}{2}=\frac{0}{x-1}=\frac{x+1}{x}}_{x=-1\ \text{عوِّض}} \;\Rightarrow\; \frac{0}{2}=\frac{0}{-2}=\frac{0}{-1}$ (no error)

$\Rightarrow$ the value of x is $x=-1$.

Another solution: $\frac{x^2-1}{2}=\frac{0}{x-1} \;\Rightarrow\; \frac{x^2-1}{2}=0 \;\Rightarrow\; x^2-1=0 \;\Rightarrow\; x=\pm 1$

Check: Is there an error in the equations:

$\underbrace{\frac{x^2-1}{2}=\frac{0}{x-1}=\frac{x+1}{x}}_{x=-1\ \text{عوِّض}} \;\Rightarrow\; \frac{0}{2}=\frac{0}{-2}=\frac{0}{-1}$ (no error)

$\underbrace{\frac{x^2-1}{2}=\frac{0}{x-1}=\frac{x+1}{x}}_{x=1\ \text{عوّض}} \Rightarrow \frac{0}{2}=\frac{0}{0}=\frac{2}{1}$ (there is an error in the equations)

$\Rightarrow x \neq 1 \ \Rightarrow \ x=-1$ only.

Exercise 16: Find the value of x that makes:

$$\vec{a}=\langle 3,1,x^2+2x+1\rangle \text{ and } \vec{b}=\langle 3x^2-3,3,3\rangle \text{ parallel.}$$

Answer is $x=-2$

Definition 17: Three points A,B,C are collinear (على استقامة واحدة) $\Leftrightarrow \overrightarrow{AB}//\overrightarrow{AC}$

Example 18: Determine whether the points $A(2,4,-3),B(3,-1,1),C(4,-6,5)$ are collinear or not.

Solution:

$\overrightarrow{AB}=\langle 1,-5,4\rangle$ and $\overrightarrow{AC}=\langle 2,-10,8\rangle \Rightarrow \frac{2}{1}=\frac{-10}{-5}=\frac{8}{4}$ are all equal $\Rightarrow \overrightarrow{AB}//\overrightarrow{AC}$

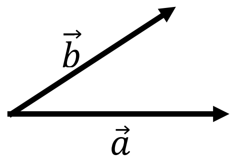
$\Rightarrow$ The points A,B,C are collinear

Another solution: $\overrightarrow{AC}=2\overrightarrow{AB} \Rightarrow \overrightarrow{AB}//\overrightarrow{AC} \Rightarrow$ The points A,B,C are collinear

Rule 19:

(1) The area (مساحة) of the parallelogram determined by the vectors $\vec{a}$ and $\vec{b}$ is $A=|\vec{a}\times\vec{b}|$

(2) The area of the triangle determined by the vectors $\vec{a}$ and $\vec{b}$ is $A=\frac{1}{2}|\vec{a}\times\vec{b}|$



Remark 20: Let A,B,C,D be points and let $\vec{a}=\overrightarrow{AB}$ and $\vec{b}=\overrightarrow{AC}$.

(1) The area of the parallelogram (متوازي اضلاع) with vertices A,B,C,D is $A=|\vec{a}\times\vec{b}|$

(2) The area of the triangle (مثلث) with vertices A,B,C is $A=\frac{1}{2}|\vec{a}\times\vec{b}|$

Example 21: let $\vec{a} = i + 2j - k$ and $\vec{b} = j + 3k$ and let $A(1,0,1), B(2,2,0), C(1,1,4), D$ be four points.

(1) Find the area of the parallelogram determined by the vectors $\vec{a}$ and $\vec{b}$.
(2) Find the area of the triangle determined by the vectors $\vec{a}$ and $\vec{b}$.
(3) Find the area of the parallelogram with vertices A, B, C, D
(4) Find the area of the triangle with vertices A, B, C

Solution:

(1) Area $= |\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2} = \sqrt{(6)(10) - (-1)^2} = \sqrt{59}$

(2) Area $= \frac{\sqrt{59}}{2}$

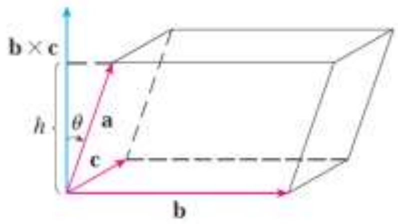
(3) $\vec{a} = \overrightarrow{AB} = \langle 1,2,-1 \rangle$ and $\vec{b} = \overrightarrow{AC} = \langle 0,1,3 \rangle \Rightarrow$ Area $= |\vec{a} \times \vec{b}| = \sqrt{59}$

(4) Area $= \frac{\sqrt{59}}{2}$

Definition 22: Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, and $\vec{c} = \langle c_1, c_2, c_3 \rangle$ be vectors. The scalar triple of the vectors $\vec{a}, \vec{b}, \vec{c}$ written $\vec{a} \cdot (\vec{b} \times \vec{c})$ is defined by

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1(a_2 b_3 - a_3 b_2) - a_2(a_1 b_3 - a_3 b_1) + a_3(a_1 b_2 - a_2 b_1)$$



Rule 23: The volume of the parallelepiped determined by the vectors $\vec{a}, \vec{b}, \vec{c}$ is

$$V = \underbrace{|\vec{a} \cdot (\vec{b} \times \vec{c})|}_{\text{القيمة المطلقة}}$$

Remark 24: Let A, B, C, D be vertices of a parallelepiped and let $\vec{a} = \overrightarrow{AB}, \vec{b} = \overrightarrow{AC}$, $\vec{c} = \overrightarrow{AD}$. Then the volume of this parallelepiped is $V = \underbrace{|\vec{a} \cdot (\vec{b} \times \vec{c})|}_{\text{القيمة المطلقة}}$

Example 25: Find the volume of the parallelepiped:

(1) Determined by the vectors $\vec{a} = \langle 0,-2,5\rangle, \vec{b} = \langle 0,1,2\rangle, \vec{c} = \langle 6,3,-1\rangle$

(2) With adjacent edges PQ, PR, PS, where $P(-2,1,0), Q(2,-1,5), R(-2,2,2)$, and $S(4,4,-1)$.

Solution:

(1) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 0 & -2 & 5 \\ 0 & 1 & 2 \\ 6 & 3 & -1 \end{vmatrix}$

$$= 0(-1-6) - (-2)(0-12) + 5(0-6) = 0 - 24 - 30 = -54$$

Volume $= |\vec{a}\cdot(\vec{b}\times\vec{c})| = |-54| = 54$

(2) Let $\vec{a} = \overrightarrow{PQ} = \langle 0,-2,5\rangle, \vec{b} = \overrightarrow{PR} = \langle 0,1,2\rangle, \vec{c} = \overrightarrow{PS} = \langle 6,3,-1\rangle$

$\Rightarrow\ \vec{a}\cdot(\vec{b}\times\vec{c}) = -54$ (by part (1))$\Rightarrow$ Volume $= |\vec{a}\cdot(\vec{b}\times\vec{c})| = |-54| = 54$

Rule 26:

(1) Three vectors $\vec{a}, \vec{b}$, and $\vec{c}$ in V_3 are coplanar (lie in the same plane) if $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$.

(2) Four points A, B, C, D in $\mathbb{R}^3$ are coplanar if $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$, where $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{AC}$, and $\vec{c} = \overrightarrow{AD}$

Example 27:

(1) Find the value of x that makes $\vec{a} = \langle 1,x,0\rangle, \vec{b} = \langle x,2,1\rangle, \vec{c} = \langle 0,1,1\rangle$ coplanar

(2) Find the value of x that makes the points $A(1,-1,2), B(2,x-1,2), C(x+1,1,3)$, and $D(1,0,3)$ lie in the same plane.

Solution:

(1) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & x & 0 \\ x & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(2-1) - x(x-0) + 0(x-1) = 1 - x^2$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \ \Rightarrow\ 1 - x^2 = 0 \ \Rightarrow\ x = \pm 1$$

(2) $\vec{a} = \overrightarrow{AB} = \langle 1,x,0\rangle, \vec{b} = \overrightarrow{AC} = \langle x,2,1\rangle$, and $\vec{c} = \overrightarrow{AD} = \langle 0,1,1\rangle$

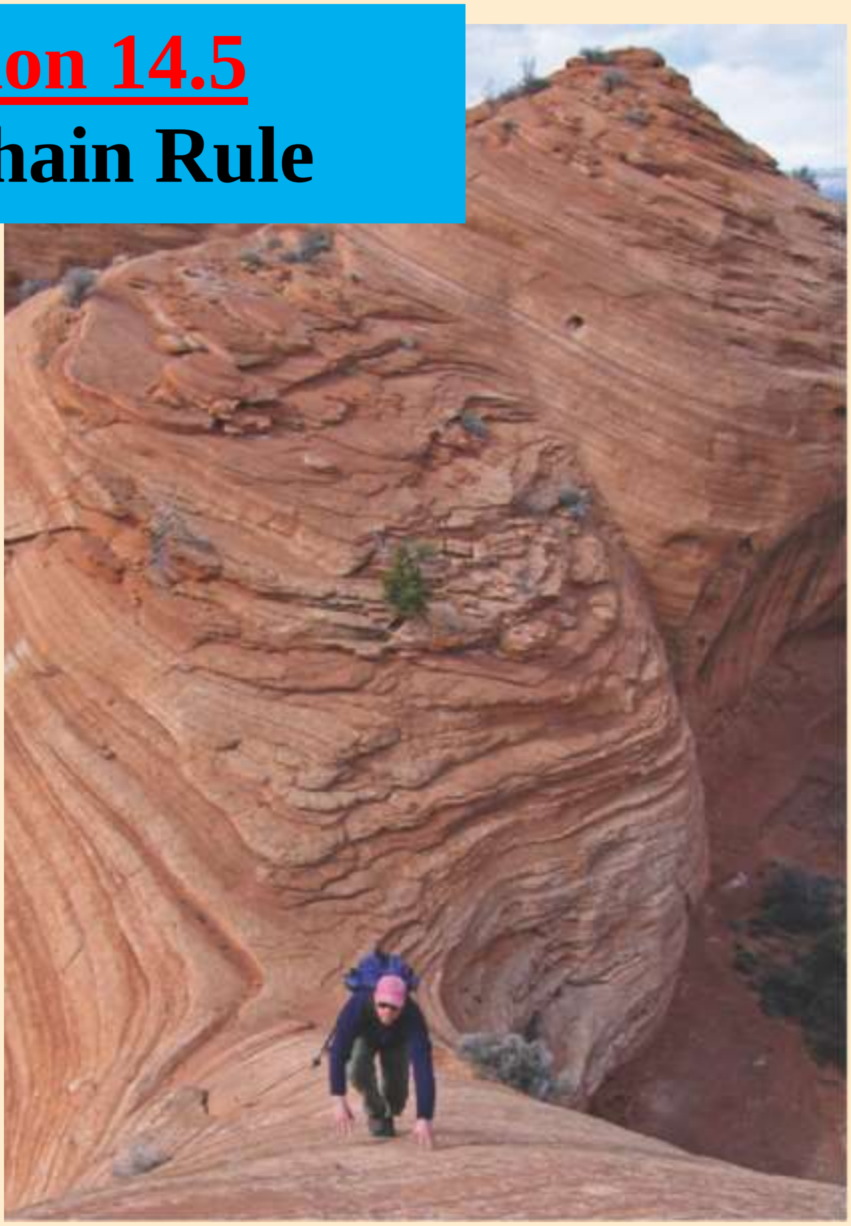
$\vec{a}\cdot(\vec{b}\times\vec{c}) = 1 - x^2$ (by part (1))

$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \ \Rightarrow\ 1 - x^2 = 0 \ \Rightarrow\ x = \pm 1$

Chapter 14 Partial Derivatives

Section 14.5

The Chain Rule

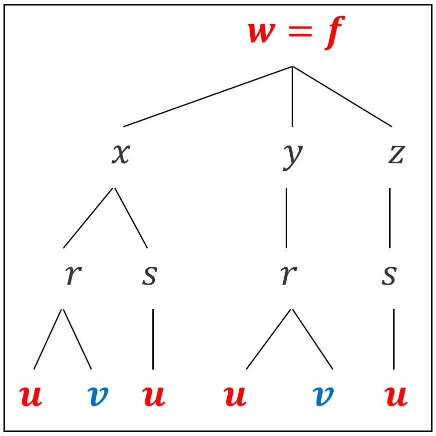


14.5 The Chain Rule

Rule 1: Let $w = f(x, y, z), x = x(r, s), y = y(r), z = z(s), r = r(u, v), and\ s = s(u)$.

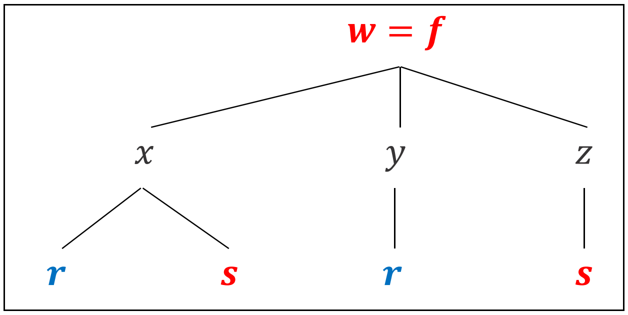
- $\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}\frac{ds}{du} + \frac{\partial f}{\partial y}\frac{dy}{dr}\frac{\partial r}{\partial u} + \frac{\partial f}{\partial z}\frac{dz}{ds}\frac{ds}{du}$

- $\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r}\frac{\partial r}{\partial v} + \frac{\partial f}{\partial y}\frac{dy}{dr}\frac{\partial r}{\partial v}$



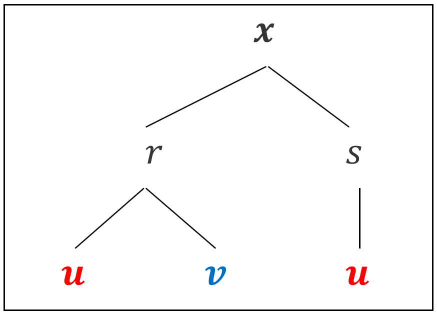
Tree Diagram

- $\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial z}\frac{dz}{ds}$



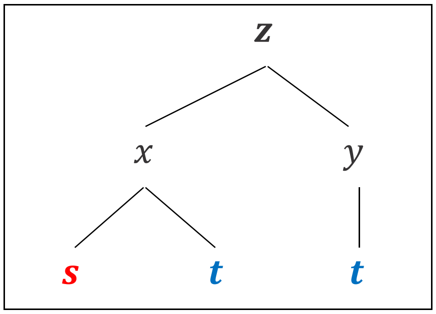
- $\frac{\partial x}{\partial u} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial x}{\partial s}\frac{ds}{du}$

- $\frac{\partial x}{\partial v} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial v}$



Example 2: Let $z = e^{2x}\sin(y), x = st^2, y = t^3$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution:



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}$$

$$= \left(2e^{2x}\sin(y)\right)t^2 = 2t^2e^{2x}\sin(y)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \left(2e^{2x}\sin(y)\right)(2st) + \left(e^{2x}\cos(y)\right)(3t^2)$$

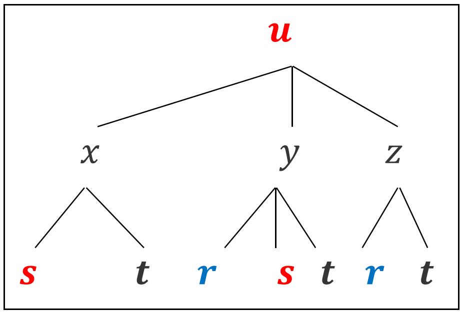
$$= 4ste^{2x}\sin(y) + 3t^2e^{2x}\cos(y) = 4stz + 3t^2e^{2x}\cos(y) =$$

Example 3: Let $u = x^4y + y^2z^3, x = se^{2t}, y = r^2se^{-t}, z = r\cos(t)$.

Find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ when $r = 2, s = 1, t = 0$.

Solution:

First we have to find x, y, z when $r = 2, s = 1, t = 0$.



$$\left.\begin{aligned} x &= 1e^{2(0)} = 1 \\ y &= (2)^2(1)e^{-(0)} = 4 \\ z &= 2\cos(0) = 2 \end{aligned}\right\} \Rightarrow \begin{cases} x = 1, y = 4, z = 2 \\ \text{when } r = 2, s = 1, t = 0 \end{cases}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = (4x^3y)(e^{2t}) + (x^4 + 2yz^3)(r^2e^{-t})$$

$$\Rightarrow \left.\frac{\partial u}{\partial s}\right|_{\substack{r=2,s=1,t=0 \\ x=1,y=4,z=2}} = (16)(1) + 65(4) = 276$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$= (4x^3y)(2se^{2t}) + (x^4 + 2yz^3)(-r^2se^{-t}) + (3y^2z^2)\left(-r\sin(t)\right)$$

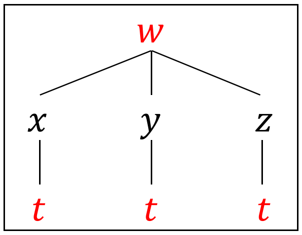
$$\Rightarrow \left.\frac{\partial u}{\partial s}\right|_{\substack{r=2,s=1,t=0 \\ x=1,y=4,z=2}} = 16(2) - 65(4) + 48(2) = -132$$

Example 4: Let $w = \ln\sqrt{x^2+y^2+z^2}$, $x=\sin(t), y=\cos(t), z=\tan(t)$. Find $\frac{dw}{dt}$.

Solution: Observe that:

$$w = \ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2)$$

$$\Rightarrow \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

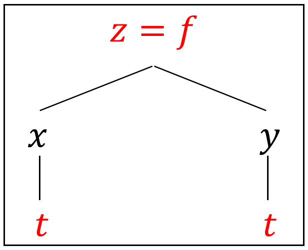


$$\begin{aligned}\frac{dw}{dt} &= \frac{1}{2}\frac{2x}{x^2+y^2+z^2}\cos(t) + \frac{1}{2}\frac{2y}{x^2+y^2+z^2}(-\sin(t)) + \frac{1}{2}\frac{2z}{x^2+y^2+z^2}\sec^2(t)\\ &= \frac{x\cos(t) - y\sin(t) + z\sec^2(t)}{x^2+y^2+z^2}\\ &= \frac{\sin(t)\cos(t) - \sin(t)\cos(t) + \tan(t)\sec^2(t)}{\sin^2(t)+\cos^2(t)+\tan^2(t)}\\ &= \frac{\tan(t)\sec^2(t)}{1+\tan^2(t)}\\ &= \frac{\tan(t)\sec^2(t)}{\sec^2(t)} = \tan(t)\end{aligned}$$

Example 5: Let $z=f(x,y), x=g(t), y=h(t), g(3)=2, h(3)=7, g'(3)=5,$ $h'(3)=-4, f_x(2,7)=6$ and $f_y(2,7)=-8$. Find $\frac{dz}{dt}$ when $t=3$.

Solution: First we have to find x, y when $t=3$:

$$\left.\begin{matrix} x = g(3) = 2 \\ y = h(3) = 7\end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x = 2, y = 7 \\ \text{when } t = 3\end{matrix}\right.$$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = f_x(x,y)g'(t) + f_y(x,y)h'(t)$$

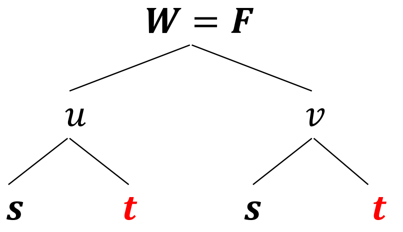
$$\Rightarrow \left.\frac{dz}{dt}\right|_{t=3,x=2,y=7} = f_x(2,7)g'(3) + f_y(2,7)h'(3) = 6(5) + (-8)(-4) = 62$$

Example 6: Let $W(s,t)=F\big(u(s,t),v(s,t)\big), F_u(2{,}3)=-1, F_v(2{,}3)=10,$

$u(1{,}0)=2, v(1{,}0)=3, u_s(1{,}0)=-2, v_s(1{,}0)=5, u_t(1{,}0)=6, v_t(1{,}0)=4.$

Find $W_t(1{,}0)$ and $W_s(1{,}0)$.

Solution: Observe that $W(s,t)=F(u,v)$ with $u=u(s,t), v=v(s,t)$.
Also, observe that to find $W_t(1{,}0)$ we have to find W_t when $s=1, t=0$:
Also, we need to find u, v when $s=1, t=0$:



$$\left.\begin{array}{l} u=u(1{,}0)=2 \\ v=v(1{,}0)=3 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} u=2, v=3 \\ \text{when } s=1, t=0 \end{array}\right.$$

Now,
$W_t=F_u u_t+F_v v_t$
$W_t(1{,}0)=F_u(2{,}3)u_t(1{,}0)+F_v(2{,}3)v_t(1{,}0)=-1(6)+10(4)=34$
Finding $W_s(1{,}0)$ is an exercise.

Example 7: Suppose that $f(x,y)$ is differentiable. Find $g_u(0{,}0)$ and $g_v(0{,}0)$, where

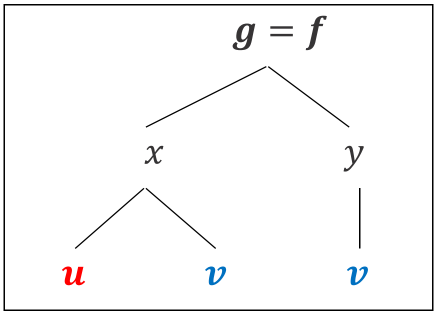
$$g(u,v)=f(e^u+\cos(v)\,,1+\sin(v))$$

	f	g	f_x	f_y
(0,0)	3	6	5	8
(2,1)	6	3	2	7

Solution: Let $x=e^u+\cos(v),\ \ y=1+\sin(v)$. So, the function is:

$$g(u,v)=f(x,y)$$

To find $g_u(0{,}0)$ means: to find g_u when $u=0, v=0$.
So, first we have to find x, y when $u=0, v=0$:

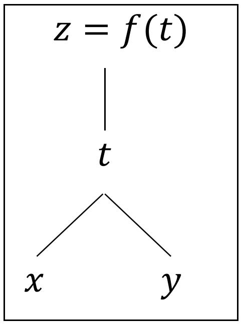


$$\left.\begin{array}{l} x=e^0+\cos(0)=2 \\ y=1+\sin(0)=1 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x=2, y=1 \\ \text{when } u=0, v=0 \end{array}\right.$$

Now,
$g_u=f_x x_u=f_x(x,y)(e^u)$
$\Rightarrow\ g_u(0{,}0)=f_x(2{,}1)(e^0)=2$
$g_v=f_x x_v+f_y\dfrac{dy}{dv}=f_x(x,y)(-\sin(v))+f_y(x,y)(\cos(v))$
$\Rightarrow\ g_v(0{,}0)=f_x(2{,}1)(-\sin(0))+f_y(2{,}1)\cos(0)=2(0)+7(1)=7$

Example 8: Let $z = f(x-y)$. Show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

Solution: Observe that $f(\dots)$ is a function in 1-variable, so, let



$$z = f(t), t = x - y$$

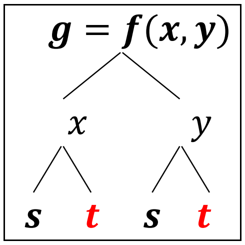
$$\frac{\partial z}{\partial x} = f'(t)t_x = f'(t)(1) = f'(t)$$

and $\frac{\partial z}{\partial y} = f'(t)t_y = f'(t)(-1) = -f'(t)$

$$\Rightarrow \quad \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = f'(t) + \left(-f'(t)\right) = 0 \qquad \Rightarrow \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

Example 9: Let $g(s,t) = f(s^2 - t^2, t^2 - s^2)$. Show that $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$.

Solution: Observe that $f(\dots,\dots)$ is a function in 2-variable:



Let $g(s,t) = f(x,y), x = s^2 - t^2, y = t^2 - s^2$

$$\frac{\partial g}{\partial s} = f_x x_s + f_y y_s = 2sf_x + (-2s)f_y$$

$$\frac{\partial g}{\partial t} = f_x x_t + f_y y_t = -2tf_x + 2tf_y$$

$$\Rightarrow \quad t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = t\left(2sf_x - 2sf_y\right) + s\left(-2tf_x + 2tf_y\right)$$

$$= 2stf_x - 2stf_y - 2stf_x + 2stf_y$$

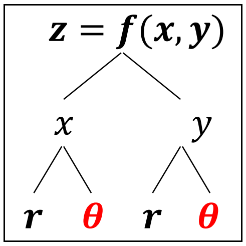
$$= 0$$

$$\Rightarrow \quad t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Example 10: Let $z = f(x, y)$ be with continuous second order partial derivatives such that $x = r\cos(\theta), y = r\sin\theta$. Show that:

(1) $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

(2) $$\frac{\partial^2 z}{\partial r^2} = \cos^2\theta \frac{\partial^2 f}{\partial x^2} + 2\sin\theta\cos\theta \frac{\partial^2 z}{\partial y \partial x} + \sin^2\theta \frac{\partial^2 z}{\partial y^2}$$



Solution:

(1) $$\frac{\partial z}{\partial r} = f_x x_r + f_y y_r = \cos\theta f_x + \sin\theta f_y$$

$$\frac{\partial z}{\partial \theta} = f_x x_\theta + f_y y_\theta = -r\sin\theta f_x + r\cos\theta f_y = -r(\sin\theta f_x - \cos\theta f_y)$$

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\cos\theta f_x + \sin\theta f_y\right)^2 + \frac{1}{r^2}\left(-r(\sin\theta f_x - \cos\theta f_y)\right)^2$$

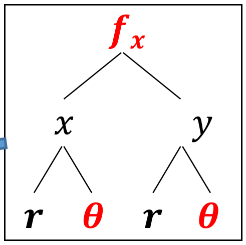
$$= \cos^2\theta (f_x)^2 + 2\sin\theta\cos\theta f_x f_y + \sin^2\theta \left(f_y\right)^2 + \frac{1}{r^2}\left(r^2(\sin^2\theta (f_x)^2 - 2\sin\theta\cos\theta f_x f_y + \cos^2\theta \left(f_y\right)^2)\right)$$

$$= (f_x)^2 + \left(f_y\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

(2) From part **(1)**: $\frac{\partial z}{\partial r} = \cos\theta f_x + \sin\theta f_y \Rightarrow \frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\cos\theta f_x + \sin\theta f_y\right)$

$$\Rightarrow \frac{\partial^2 z}{\partial r^2} = \cos\theta \frac{\partial f_x}{\partial r} + \sin\theta \frac{\partial f_y}{\partial r} \ldots\ldots\ldots\ldots (1)$$

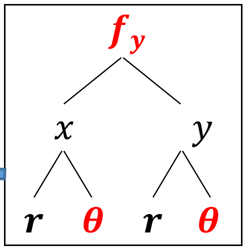
So, we have to find: $\frac{\partial f_x}{\partial r}$ and $\frac{\partial f_y}{\partial r}$:



$$\frac{\partial f_x}{\partial r} = f_{xx} x_r + f_{xy} y_r = \cos\theta f_{xx} + \sin\theta f_{xy} \ldots\ldots\ldots (2)$$

$$\frac{\partial f_y}{\partial r} = f_{yx} x_r + f_{yy} y_r = \cos\theta f_{yx} + \sin\theta f_{yy}$$

$$= \cos\theta f_{xy} + \sin\theta f_{yy} \quad \ldots\ldots\ldots\ldots (3)$$



($f_{yx} = f_{xy}$ since $f(x, y)$ is with continuous second order partial derivatives)

$$\Rightarrow \frac{\partial^2 z}{\partial r^2} = \cos\theta \frac{\partial f_x}{\partial r} + \sin\theta \frac{\partial f_y}{\partial r} \quad \text{(by (1))}$$

$$= \cos\theta\left(\cos\theta f_{xx} + \sin\theta f_{xy}\right) + \sin\theta\left(\cos\theta f_{xy} + \sin\theta f_{yy}\right) \quad \text{(by (1) and (2))}$$

$$= \cos^2\theta f_{xx} + 2\sin\theta\cos\theta f_{xy} + \sin^2\theta f_{yy}$$

Implicit Differentation:

Implicit Function Theorem 11:

(1) Let $y = f(x)$ is a function defined implicitly by the relation $F(x, y) = 0$, where F is a differentiable function with F_y is nonzero. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

(2) Let $z = f(x, y)$ is a function defined implicitly by the relation $F(x, y, z) = 0$, where F is a differentiable function with F_z is nonzero. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

انتبه لاشارة السالب

Example 12: Find y' at $x = 0$ if $\dfrac{x^3 + y^3 + 1}{6y} = x$

Solution: First, we have to find the value of y when $x = 0$:

$$x = 0: \frac{x^3 + y^3 + 1}{6y} = x \Rightarrow \frac{(0)^3 + y^3 + 1}{6y} = 0 \Rightarrow y^3 = -1 \Rightarrow y = -1$$

So, $x = 0 \Rightarrow y = -1$.

Second, we simplify (نبسط) the equation $\frac{x^3+y^3+1}{6y} = x$ if possible

$$\text{Equation: } \frac{x^3 + y^3 + 1}{6y} = x \Rightarrow x^3 + y^3 + 1 = 6xy \Rightarrow x^3 + y^3 - 6xy + 1 = 0$$

$\Rightarrow$ Let $F(x, y) = x^3 + y^3 - 6xy + 1$

$$y' = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x}$$

$$y'|_{x=0,y=-1} = -\frac{3(0)^2 - 6(-1)}{3(-1)^2 - 6(0)} = -\frac{6}{3} = -2$$

Example 13: If $x^3 + y^3 + z^3 + 6xyz = 9$, find

$$(1)\ \frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y} \qquad (2)\ \frac{\partial z}{\partial x} \text{ and } \frac{\partial z}{\partial y} \text{ at the point } (0,1)$$

Solution: $x^3 + y^3 + z^3 + 6xy = 1 \Rightarrow x^3 + y^3 + z^3 + 6xyz - 9 = 0$

Let: $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 9$

(1) $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{3x^2 + 6yz}{3z^2 + 6xy}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{3y^2 + 6xz}{3z^2 + 6xy}$

(2) At the point $(0,1) \Rightarrow x = 0, y = 1$. So, we have to find the value of z:

When $x = 0, y = 1$:

$x^3 + y^3 + z^3 + 6xyz - 1 = 0 \Rightarrow (0)^3 + (1)^3 + z^3 + 6(0)(1)z - 9 = 0$

$\Rightarrow z^3 = 8 \ \ z = 2$

$\Rightarrow x = 0, y = 1, z = 2$.

From part (1):

$$\frac{\partial z}{\partial x} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{x=0,y=1,z=2} = -\frac{3(0)^2 + 6(1)(2)}{3(2)^2 + 6(0)(1)} = -1$$

$$\frac{\partial z}{\partial y} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{x=0,y=1,z=2} = -\frac{3(1)^2 + 6(0)(2)}{3(2)^2 + 6(0)(1)} = -\frac{1}{4}$$

Example 14: Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x, y, z as a function of the other two. If F_x, F_y, F_z are nonzero, show that

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1$$

Solution: By the Implicit Function Theorem we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial x}{\partial y} = -\frac{F_y}{F_x} \qquad \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$$

$$\Rightarrow \frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = \left(-\frac{\cancel{F_x}}{\cancel{F_z}}\right)\left(-\frac{\cancel{F_y}}{\cancel{F_x}}\right)\left(-\frac{\cancel{F_z}}{\cancel{F_y}}\right) = -1$$

Example 15: Suppose that the equation $F(x, y) = 0$ implicitly defines y as a function of x and defines x as a function of y. If F_x, F_y are nonzero, show that

$$\frac{dy}{dx}\frac{dx}{dy} = 1$$

Solution: By the Implicit Function Theorem we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \text{ and } \frac{dx}{dy} = -\frac{F_y}{F_x} \Rightarrow \frac{dy}{dx}\frac{dx}{dy} = \left(-\frac{F_x}{F_y}\right)\left(-\frac{F_y}{F_x}\right) = 1$$

Example 16: Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x, y, z as a function of the other two. If F_x and F_z are nonzero, find

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}$$

Solution: By the Implicit Function Theorem we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial x}{\partial y} = -\frac{F_y}{F_x} \qquad \frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}\frac{\partial x}{\partial y} = \left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right) = -\left(-\frac{F_y}{F_z}\right) = -\frac{\partial z}{\partial y}$$

Chapter 14 Partial Derivatives

Section 14.6

The Directional Derivative and the Gradient Vector

14.6 The Directional Derivative and the Gradiant Vector

Definition 1: The gradient vector of the function $f(x,y)$ at the point (x_0,y_0) is defined by

$$\nabla f(x_0,y_0)=\langle f_x(x_0,y_0),f_y(x_0,y_0)\rangle$$

- Observe that: $f(x,y,z)\ \Rightarrow$

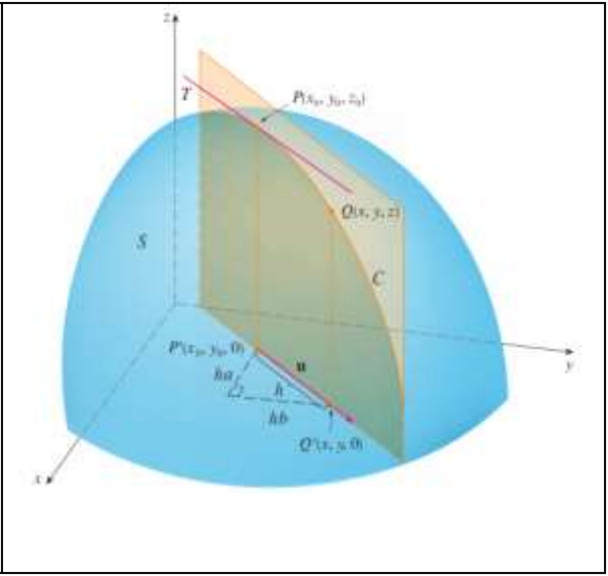
$$\nabla f(x_0,y_0,,z_0)=\langle f_x(x_0,y_0,z_0),f_y(x_0,y_0,z_0),f_z(x_0,y_0,z_0)\rangle$$

Definition 2: The Directional derivative (or the rate of change) of the function $f(x,y)$ at the point (x_0,y_0) in the direction of the unit vector $\hat{v}=\langle a,b\rangle$ is:

$$D_{\hat{v}}f(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+ah,y_0+bh)-f(x_0,y_0)}{h}$$

Interpolation of the Directional Derivative 3:

Suppose that we now wish to find the directional derivative (the rate of change) of the function $f(x,y)$ at a point $P'(x_0,y_0)$ in the direction of a unit vector $\boldsymbol{u}$ To do this we consider the surface S with the equation $z=f(x,y)$ (the graph of f) and we let $z_0=f(x_0,y_0)$. Then the point $P(x_0,y_0,z_0)$ lies on S. The vertical plane that passes through P in the direction of $\boldsymbol{u}$ intersects S in a curve C. The slope of the tangent line T to C at the point P is the directional derivative (rate of change) of f in the direction of $\boldsymbol{u}$.



Theorem 4: The Directional derivative (or the rate of change) of the function $f(x,y)$ at the point (x_0,y_0) in the direction of the unit vector $\hat{v}=\langle a,b\rangle$ is:

$$D_{\hat{v}}f(x_0,y_0)=\nabla f(x_0,y_0)\cdot\hat{v}\ \text{ (dot product)}$$

Example 5: Find the directional derivative of the function $f(x,y)=x^2y^3$ at the point $(-2,3)$ in the direction of the vector $\vec{v}=2i-5j$.

Solution:

Gradient: $\nabla f=\langle f_x,f_y\rangle=\langle 2xy^3,3x^2y^2\rangle$

$\Rightarrow \nabla f(-2,3)=\langle 2(-2)(3)^3,3(-2)^2(3)^2\rangle \Rightarrow \nabla f(-2,3)=\langle -36,108\rangle$

Unit vector: $\vec{v}=2i-5j \Rightarrow \hat{v}=\frac{\vec{v}}{|\vec{v}|}=\frac{2i-5j}{\sqrt{2^2+5^2}}=\frac{2i-5j}{\sqrt{29}}=\langle\frac{2}{\sqrt{29}},-\frac{5}{\sqrt{29}}\rangle$

$$D_{\hat{v}}f(-2,3)=\nabla f(-2,3)\cdot\hat{v} \quad =\langle -36,108\rangle\cdot\langle\frac{2}{\sqrt{29}},-\frac{5}{\sqrt{29}}\rangle$$

$$=(-36)\frac{2}{\sqrt{29}}+108\left(-\frac{5}{\sqrt{29}}\right)=-\frac{612}{\sqrt{29}}$$

Example 6: Find the rate of change of the function $f(x,y)=\frac{x^2-y}{y^2}$ at the point $(2,1)$ in the direction indicated by the angle $\theta=\frac{\pi}{3}$ (that is in the direction that makes the angle $\theta=\frac{\pi}{3}$ with the positive direction of the x-axis.

Solution:

Gradient: $\nabla f=\langle f_x,f_y\rangle=\langle\frac{2x}{y^2},\frac{y^2(-1)-(x^2-y)(2y)}{y^4}\rangle \Rightarrow \nabla f(2,1)=\langle 4,-7\rangle$

Unit vector: $\vec{v}=\langle a,b\rangle : a=\cos(\theta), b=\sin(\theta)$

$$\Rightarrow \left.\begin{array}{r} a=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2} \\ b=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}\end{array}\right]\!\!\Bigg] \Rightarrow \vec{v}=\langle\frac{1}{2},\frac{\sqrt{3}}{2}\rangle,\ |\vec{v}|=1 \Rightarrow \vec{v} \text{ is a unit vector}$$

$$D_{\vec{v}}f(2,1)=\nabla f(2,1)\cdot\vec{v} \quad =\langle 4,-7\rangle\cdot\langle\frac{1}{2},\frac{\sqrt{3}}{2}\rangle$$

$$=(4)\frac{1}{2}+(-7)\left(\frac{\sqrt{3}}{2}\right)=\frac{4-7\sqrt{3}}{2}$$

Example 7: Find the rate of change of the function $f(x,y,z)=x^2-3yz^3$ at the point $P(2,-1,1)$ in the direction from P to the point $Q\left(3,1,\frac{1}{2}\right)$.

Solution:

Gradient: $\nabla f=\langle f_x,f_y,f_z\rangle=\langle 2x,-3z^3,-9yz^2\rangle \Rightarrow \nabla f(2,-1,1)=\langle 4,-3,9\rangle$

Unit vector: $\vec{v}=\overrightarrow{PQ}=\langle Q-P\rangle=\langle 1,2,-\frac{1}{2}\rangle \Rightarrow |\vec{v}|=\frac{\sqrt{21}}{2} \Rightarrow \vec{v}$ not a unit vector

$$\hat{v}=\frac{\vec{v}}{|\vec{v}|}=\frac{\langle 1,2,-\frac{1}{2}\rangle}{\sqrt{21}/2}=\frac{2\,\langle 1,2,-\frac{1}{2}\rangle}{\sqrt{21}}=\langle\frac{2}{\sqrt{21}},\frac{4}{\sqrt{21}},-\frac{1}{\sqrt{21}}\rangle$$

$$D_{\hat{v}}f(2,-1,1) \quad =\nabla f(2,,-1,1)\cdot\hat{v}=\langle 4,-3,9\rangle\cdot\langle\frac{2}{\sqrt{21}},\frac{4}{\sqrt{21}},-\frac{1}{\sqrt{21}}\rangle$$

$$=\frac{8}{\sqrt{21}}-\frac{12}{\sqrt{21}}-\frac{9}{\sqrt{21}}=-\frac{13}{\sqrt{21}}$$

Remark 8: Recall that: The definition of the directional derivative (or the rate of change) of the function $f(x,y)$ at the point (x_0,y_0) in the direction of the unit vector $\hat{v}=\langle a,b\rangle$ is:

$$D_{\hat{v}}f(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+ah,y_0+bh)-f(x_0,y_0)}{h}$$

and $D_{\hat{v}}f(x_0,y_0)=\nabla f(x_0,y_0)\cdot\hat{v}$

Example 9: Let $f(x,y)=\ln(x^2+2y)-\sqrt{x}$. Find

$$\lim_{h\to 0}\frac{f\left(4-\frac{h}{3},\frac{\sqrt{8}h}{3}\right)-f(4,0)}{h}$$

Solution: By the definition of the directional derivative we have:

$$\lim_{h\to 0}\frac{f\left(4-\frac{h}{3},\frac{\sqrt{8}h}{3}\right)-f(4,0)}{h}=\nabla f(4,0)\cdot\langle-\frac{1}{3},\frac{\sqrt{8}}{3}\rangle \;\ldots\ldots\ldots..(1)$$

So, we have to find $\nabla f(4,0)$:

$$\nabla f=\langle\frac{2x}{x^2+2y}-\frac{1}{\sqrt{x}},\frac{2}{x^2+2y}\rangle \Rightarrow \nabla f(4,0)=\langle 0,\frac{1}{8}\rangle$$

$$\text{By (1)}\Rightarrow \lim_{h\to 0}\frac{f\left(4-\frac{h}{3},\frac{\sqrt{8}h}{3}\right)-f(4,0)}{h}=\langle 0,\frac{1}{8}\rangle\cdot\langle-\frac{1}{3},\frac{\sqrt{8}}{3}\rangle=\frac{\sqrt{8}}{24}$$

Example 10: Let $\hat{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$, $\hat{v} = \langle \frac{2\sqrt{2}}{3}, \frac{1}{3} \rangle$, $D_{\hat{u}}f(2,1) = 2$ and $D_{\hat{v}}f(2,1) = \frac{1}{3}$.

(a) Find the gradient vector of f at the point $(2,1)$.

(b) Find the directional derivative of f at the point $(2,1)$ in the direction of $i - 2j$.

Solution:

(a) Let $\nabla f(2,1) = \langle a, b \rangle$. Then

$$D_{\hat{u}}f(2,1) = 2 \;\Rightarrow\; \nabla f(2,1) \cdot \hat{u} = 2 \;\Rightarrow$$

$$\langle a, b \rangle \cdot \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle = 2 \;\Rightarrow\; \frac{a-b}{\sqrt{2}} = 2 \;\Rightarrow\; a - b = 2\sqrt{2} \;\ldots\ldots\ldots..(1)$$

$$D_{\hat{v}}f(2,1) = \frac{1}{3} \;\Rightarrow\; \nabla f(2,1) \cdot \hat{v} = \frac{1}{3} \;\Rightarrow$$

$$\langle a, b \rangle \cdot \langle \frac{2\sqrt{2}}{3}, \frac{1}{3} \rangle = \frac{1}{3} \;\Rightarrow\; \frac{2\sqrt{2}a + b}{3} = \frac{1}{3} \;\Rightarrow\; 2\sqrt{2}a + b = 1 \;\ldots\ldots\ldots..(2)$$

$$(1) + (2) \Rightarrow \; \left(1 + 2\sqrt{2}\right)a = 2\sqrt{2} + 1 \;\Rightarrow\; a = 1$$

The equation (1) $\Rightarrow\; 1 - b = 2\sqrt{2} \;\Rightarrow\; b = 1 - 2\sqrt{2}$

$$\nabla f(2,1) = \langle a, b \rangle = \langle 1, 1 - 2\sqrt{2} \rangle$$

(b) $\nabla f(2,1) = \langle 1, 1 - 2\sqrt{2} \rangle$ (by part (a))

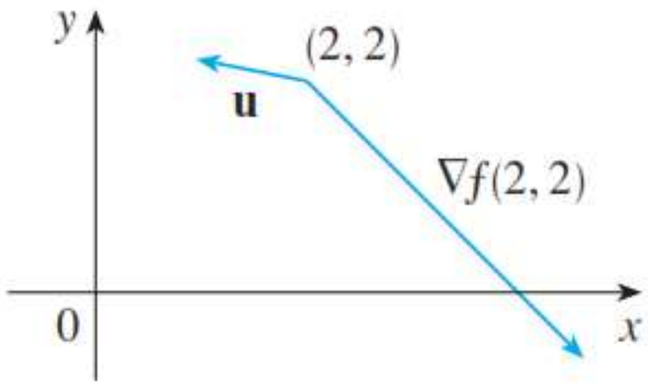
Unit vector: $\vec{w} = i - 2j \;\Rightarrow\; |\vec{w}| = \sqrt{5} \;\Rightarrow\; \widehat{w} = \langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle$ unit vector.

$$\Rightarrow\; D_{\widehat{w}}f(2,1) = \nabla f(2,1) \cdot \widehat{w} = \langle 1, 1 - 2\sqrt{2} \rangle \cdot \langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle = \frac{4\sqrt{2} - 1}{\sqrt{5}}$$

Remark 11: Since i, j, and k are unit vectors, then:

- $D_i f = \nabla f \cdot i = \langle f_x, f_y, f_z \rangle \cdot \langle 1,0,0 \rangle = f_x \;\Rightarrow\; D_i f = f_x$
- $D_j f = \nabla f \cdot j = \langle f_x, f_y, f_z \rangle \cdot \langle 0,1,0 \rangle = f_y \;\Rightarrow\; D_j f = f_y$
- $D_k f = \nabla f \cdot i = \langle f_x, f_y, f_z \rangle \cdot \langle 0,0,1 \rangle = f_z \;\Rightarrow\; D_k f = f_z$

Example 12: Use the figure to estimate $D_u f(2{,}2)$.



Solution:

$|u| = 1,\ |\nabla f(2{,}2)| \cong 3.7$

$\Rightarrow\ u = \langle \cos(150), \sin(150)\rangle = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle$

$\Rightarrow\ \nabla f(2{,}2) = \langle |\nabla f|\cos(315), |\nabla f|\sin(315)\rangle = \langle \frac{4}{\sqrt{2}}, -\frac{4}{\sqrt{2}}\rangle$

$$D_u f(2,-1{,}1) = |\nabla f(2{,}2)|\ |u|\ \cos\theta \cong 3.7\cos(150) = -\frac{3.7\sqrt{3}}{2} \cong -3.2$$

Remark 13: Let $\hat{v} = \langle a, b\rangle$ be a unit vector. Then

$$\begin{aligned} D_{\hat{v}} f(x_0, y_0) &= \nabla f(x_0, y_0)\cdot \hat{v} \\ &= |\nabla f(x_0, y_0)|\cdot|\hat{v}|\ \cos\theta \\ &= |\nabla f(x_0, y_0)|\cos\theta\ \ (\text{since } \hat{v} \text{ is a unit vector}) \end{aligned}$$

$$-1 \le \cos\theta \le 1\ \Rightarrow\ -|\nabla f(x_0, y_0)| \le |\nabla f(x_0, y_0)|\cos\theta \le |\nabla f(x_0, y_0)|$$

$$\Rightarrow\ -|\nabla f(x_0, y_0)| \le D_{\hat{v}} f(x_0, y_0) \le |\nabla f(x_0, y_0)|$$

Theorem 14: Suppose that f is a differentiable function of two or three variables

- ❖ The maximum value of the directional derivative $D_{\hat{v}} f(x_0, y_0)$ is $|\nabla f(x_0, y_0)|$ which occurs in the direction of $\nabla f(x_0, y_0)$.

$$\boxed{D_{\hat{v}} f = |\nabla f|} \Longleftrightarrow \boxed{\hat{v} \text{ and } \nabla f \text{ are in the same direction}} \Longleftrightarrow \boxed{\hat{v} = \frac{\nabla f}{|\nabla f|}}$$

- ❖ The minimum value of the directional derivative $D_{\hat{v}} f(x_0, y_0)$ is $-|\nabla f(x_0, y_0)|$ which occurs in the direction of $-\nabla f(x_0, y_0)$.

$$\boxed{D_{\hat{v}} f = -|\nabla f|} \Longleftrightarrow \boxed{\hat{v} \text{ and } -\nabla f \text{ are in the same direction}} \Longleftrightarrow \boxed{\hat{v} = -\frac{\nabla f}{|\nabla f|}}$$

Remark 15: Observe that:

The maximum value of the directional derivative of f	$\Longleftrightarrow$	The maximum rate of change of f	$\Longleftrightarrow$	The function f increases fastest.
The minimum value of the directional derivative of f	$\Longleftrightarrow$	The minimum rate of change of f	$\Longleftrightarrow$	The function f decreases fastest.

Example 16: Find the maximum directional derivative (or maximum rate of change) of the function $f(x,y) = 2y^2\sqrt{x}$ at the point $(9,-3)$ and find the direction in which it occurs.

Solution: $\nabla f = \langle \frac{y^2}{\sqrt{x}}, 4y\sqrt{x} \rangle \;\Rightarrow\; \nabla f(9,-3) = \langle 3,-36 \rangle$

the maximum directional derivative $= |\nabla f(9,-3)| = \sqrt{3^2 + (36)^2} = \sqrt{1305}$

The direction in which the maximum directional derivative occurs is in the direction of the vector $\nabla f(9,-3) = \langle 3,-36 \rangle$, that is in the direction of $\langle 1,-12 \rangle$

Example 17: Find the direction in which the function $f(x,y,z) = xe^{x-yz}$ decreases fastest at the point $(2,1,2)$.

Solution: $\nabla f = \langle xe^{x-yz} + e^{x-yz}, -xze^{x-yz}, -xye^{x-yz} \rangle \Rightarrow \;\nabla f(2,1,2) = \langle 3,-4,-2 \rangle$

$\Rightarrow$ The direction in which the function f decreases fastest is $-\nabla f(2,1,2) = \langle -3,4,2 \rangle$

Example 18: Find the unit vector $\hat{v}$, if $\nabla f(1,2) = \langle 3,-4 \rangle$ and $D_{\hat{v}} f(1,2) = 5$.

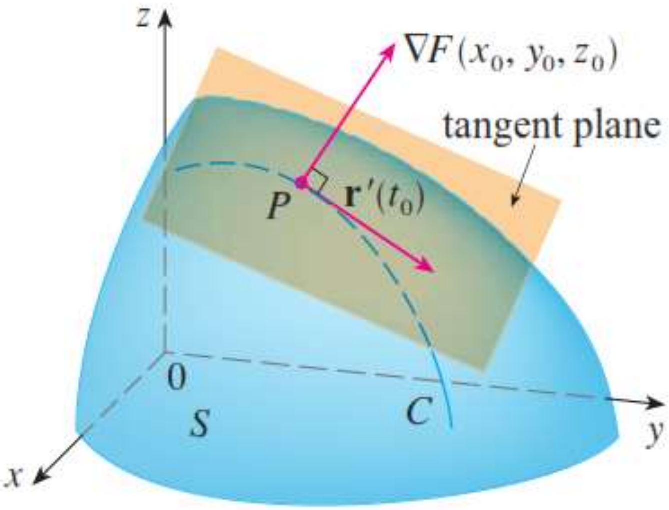
Solution: Since $|\nabla f(1,2)| = \sqrt{9+16} = \sqrt{25} = 5 \Rightarrow \; D_{\hat{v}} f(1,2) = |\nabla f(1,2)|$

$\Rightarrow \; D_{\hat{v}} f(1,2)$ has its maximum value. $\Rightarrow$ $\hat{v}$ and $\nabla f(1,2)$ are in the same direction:

$$\Rightarrow \hat{v} = \frac{\nabla f(1,2)}{|\nabla f(1,2)|} \;\Rightarrow\; \hat{v} = \frac{\langle 3,-4 \rangle}{5} = \langle \frac{3}{5}, -\frac{4}{5} \rangle.$$

Rule 19: Let $S: F(x,y,z) = k$ be a surface and $P(x_0, y_0, z_0)$ be a point on S.

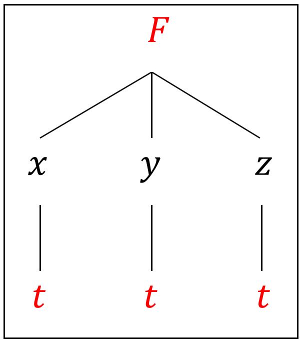
Let $C: \vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a curve on S that passes through P. Prove that ∇F is perpendicular to the tangent vector $\vec{r}\,'(t)$ of C at the point P.



Proof: The curve C is on S

$\Rightarrow$ C satisfies the equation of S

$$\Rightarrow \quad F\big(x(t), y(t), z(t)\big) = k \quad \dots\dots\dots\dots\dots (1)$$



Differentiating both sides of the equation (1) with respect to t: $\Rightarrow \dfrac{dF}{dt} = 0 \Rightarrow$

$$F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} = 0$$

$$\Rightarrow \langle F_x, F_y, F_z\rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\rangle = 0$$

$$\Rightarrow \nabla F \cdot \vec{r}\,'(t) = 0 \quad \dots\dots\dots\dots\dots (2)$$

The point $P(x_0, y_0, z_0)$ is on the curve $C \Rightarrow \vec{r}(t_0) = \langle x_0, y_0, z_0\rangle$

The equation (2) at the point $P(x_0, y_0, z_0) \Rightarrow \nabla F(x_0, y_0, z_0) \cdot \vec{r}\,'(t_0) = 0$

Which means that:

∇F is perpendicular to the tangent vector $\vec{r}\,'(t)$ of C at the point P

Remark 20: Rule 19 says the following:

∇F is normal to the surface $S: F(x,y,z) = k$ at any point on S.

So, we have the following Theorem:

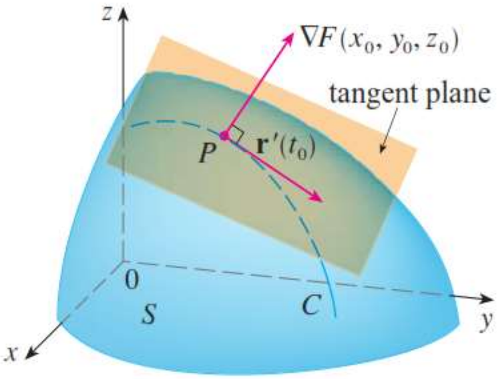
Theorem 21: Let $S: F(x,y,z) = k$ be a surface, $P(x_0,y_0,z_0)$ be a point on S. And let $\nabla F(x_0,y_0,z_0) = \langle a,b,c\rangle$. Then

(1) The equation of the tangent plane to the surface S at the point P is:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

(2) The parametric equations of the normal line to the surface S at the point P are:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$



Example 22: Find the equations of the tangent plane and the normal line at the point $(-2,1,3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$

Solution:

(1) The equation $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \Rightarrow \left(\frac{x^2}{4} + y^2 + \frac{z^2}{9} - 3 = 0\right) * 36$

$\Rightarrow 9x^2 + 36y^2 + 4z^2 - 108 = 0$

Let $F(x,y,z) = 9x^2 + 36y^2 + 4z^2 - 108$

$\Rightarrow \nabla F = \langle 18x, 72y, 8z\rangle \Rightarrow \nabla F(-2,1,3) = \langle -36,72,24\rangle$

$\Rightarrow$ vector is $\langle -3,6,2\rangle$

لاحظ يمكن تبسيط المتجه
بقسمته على 12

- ❖ The equation of the tangent plane at $(-2,1,3)$ is:
  $-3x + 6y + 2z = -3(-2) + 6(1) + 2(3) = 18 \Rightarrow -3x + 6y + 2z = 18$
- ❖ The equations of the normal line at $(-2,1,3)$ are:

$$x = -2 - 3t, \qquad y = 1 + 6t, \qquad z = 3 + 2t$$

ملاحظة: يجوز استخدام متجه موازي للمتجه $\langle -3,6,2\rangle$ مثلاً بقسمته على 3 فيصبح المتجه $\langle -1,2,\frac{2}{3}\rangle$ وبالتالي تصبح معادلات الخط العامودي (normal line) كما يلي:

$$x = -2 - t, \qquad y = 1 + 2t, \qquad z = 3 + \frac{2}{3}t$$

Example 22: Find the equations of the tangent plane and the normal line at the point $(-2,1,5)$ to the surface $z = x^2 + y^2$

Solution:

The equation $z = x^2 + y^2 \Rightarrow \; x^2 + y^2 - z = 0$. Let $F(x, y, z) = x^2 + y^2 - z$

$$\Rightarrow \quad \nabla F = \langle 2x, 2y, -1 \rangle \; \Rightarrow \; \nabla F(-2,1,3) = \langle -4,2,-1 \rangle$$

- ❖ The equation of the tangent plane at $(-2,1,5)$ is:

$$-4x + 2y - z = -4(-2) + 2(1) - (5) = 5 \quad \Rightarrow \quad -4x + 2y - z = 5$$

- ❖ The equations of the normal line at $(-2,1,5)$ are:

$$x = -2 - 4t, \qquad y = 1 + 2t, \qquad z = 5 - t$$

Example 23: At what point the surface $y = x^2 + z^2$ is tangent to the plane parallel to the plane $x + 2y + 3z = 1.$

Solution:

Surface: $y = x^2 + z^2 \Rightarrow \; x^2 - y + z^2 = 0$

$$\Rightarrow \; F(x, y, z) = x^2 - y + z^2 \Rightarrow \; \nabla F = \langle 2x, -1, 2z \rangle$$

Plane: $x + 2y + 3z = 1 \quad \Rightarrow \quad \vec{v} = \langle 1,2,3 \rangle$

$$\Rightarrow \; \langle 2x, -1, 2z \rangle \, // \, \langle 1,2,3 \rangle \; \Rightarrow \; \langle 2x, -1, 2z \rangle = \alpha \langle 1,2,3 \rangle$$

$$\Rightarrow \quad 2x = \alpha, 2\alpha = -1, 2z = 3\alpha \quad \Rightarrow \; x = \frac{\alpha}{2}, \qquad \alpha = -\frac{1}{2}, \qquad z = \frac{3\alpha}{2}$$

$$\left.\begin{array}{l} \alpha = -\frac{1}{2} : \Rightarrow \begin{cases} x = \frac{\alpha}{2} \quad \Rightarrow \quad x = -\frac{1}{4} \\ z = \frac{3\alpha}{2} \quad \Rightarrow \quad z = -\frac{3}{4} \end{cases} \\ y = x^2 + z^2 \; :\Rightarrow \; y = \left(-\frac{1}{4}\right)^2 + \left(-\frac{3}{4}\right)^2 = \frac{10}{16} \end{array}\right\} \Rightarrow \text{The point is } \left(-\frac{1}{4}, \frac{10}{16}, -\frac{3}{4}\right)$$

<u>يمكن الحل باسلوب اخر كما يلي:</u>

$$\langle 2x, -1, 2z \rangle \, // \, \langle 1,2,3 \rangle \quad \Rightarrow \; \frac{2x}{1} = \frac{-1}{2} = \frac{2z}{3} : \begin{cases} \Rightarrow \; \frac{2x}{1} = \frac{-1}{2} \; \Rightarrow \; x = -\frac{1}{4} \\ \Rightarrow \; \frac{-1}{2} = \frac{2z}{3} \; \Rightarrow \; z = -\frac{3}{4} \end{cases}$$

$$y = x^2 + z^2 \; :\Rightarrow \; y = \left(-\frac{1}{4}\right)^2 + \left(-\frac{3}{4}\right)^2 = \frac{10}{16} \; \text{The point is } \left(-\frac{1}{4}, \frac{10}{16}, -\frac{3}{4}\right)$$

Example 24: At what point the surface $x^2 - y^2 + z^2 - 2x = 1$ has a normal line parallel to the line $x = 4t,\ y = 1 - 2t,\ z = 2t.$

Solution:

$x^2 - y^2 + z^2 - 2x = 1 \ \Rightarrow\ x^2 - y^2 + z^2 - 2x - 1 = 0$

$\Rightarrow\ F(x, y, z) = x^2 - y^2 + z^2 - 2x - 1 \ \Rightarrow\ \nabla F = \langle 2x - 2, -2y, 2z\rangle$

normal line // line $(x = 4t,\ y = 1 - 2t,\ z = 2t)$ $\Rightarrow$ $\langle 4, -2, -2\rangle$ // normal line

But ∇F // normal line $\Rightarrow \nabla F$ // $\langle 4, -2, -2\rangle \Rightarrow$ $\langle 2x - 2, -2y, 2z\rangle$ // $\langle 4, -2, -2\rangle$

$\Rightarrow\ \langle 2x - 2, -2y, 2z\rangle = \alpha\langle 4, -2, -2\rangle \ \Rightarrow\ 2x - 2 = 4\alpha, -2y = -2\alpha, 2z = -2\alpha$

$\Rightarrow\ x = 2\alpha + 1\ , y = \alpha\ , z = -\alpha$

substituting in the surface $x^2 - y^2 + z^2 - 2x = 1$

$\Rightarrow (2\alpha + 1\ )^2 - \alpha^2 + (-\alpha)^2 - 2(2\alpha + 1) = 1 \Rightarrow 4\alpha^2 - 4\alpha - 3 = 0$

$$\alpha = \frac{-(-)4 \pm \sqrt{(-4)^2 - 4(4)(-3)}}{2(4)} = \frac{4 \pm \sqrt{64}}{8} = \frac{4 \pm 8}{8} \ \Rightarrow \alpha = -\frac{1}{2} \text{ or } \frac{3}{4}$$

$$\alpha = -\frac{1}{2} : \Rightarrow \left\{\begin{array}{l} x = 2\alpha + 1 \ \Rightarrow\ x = 0 \\ y = \alpha \ \Rightarrow\ y = -\dfrac{1}{2} \\ z = -\alpha \ \Rightarrow\ z = \dfrac{1}{2} \end{array}\right\}$$

$$\alpha = \frac{3}{4} : \Rightarrow \left\{\begin{array}{l} x = 2\alpha + 1 \Rightarrow\ x = \dfrac{5}{2} \\ y = \alpha \ \Rightarrow\ y = \dfrac{3}{4} \\ z = -\alpha \ \Rightarrow\ z = -\dfrac{3}{4} \end{array}\right\}$$

The points are:

$\left(0, -\frac{1}{2}, \frac{1}{2}\right)$

$\left(\frac{5}{2}, \frac{3}{4}, -\frac{3}{4}\right)$

Example 24: At what points does the normal line through the point $(1,1,2)$ on the ellipsoid $4x^2+y^2+4z^2=21$ intersects the sphere $x^2+y^2+z^2=6$

Solution:

$4x^2+y^2+4z^2=21 \Rightarrow\ 4x^2+y^2+4z^2-21=0$

$\Rightarrow\ F(x,y,z)=4x^2+y^2+4z^2-21 \Rightarrow\ \nabla F=\langle 8x,2y,8z\rangle$

$\Rightarrow\ \nabla F(1,1,2)=\langle 8,2,16\rangle \Rightarrow \langle 8,2,16\rangle$ // normal line $(\div 2)$ $\Rightarrow \langle 4,1,8\rangle$ // normal line

The equations of the normal line are: $x=1+4t, y=1+t, z=2+8t$

The normal line intersects the sphere $x^2+y^2+z^2=6$:

Substitute ($x=1+4t, y=1+t, z=2+8t$) in the equation $x^2+y^2+z^2=6$:

$$(1+4t)^2+(1+t)^2+(2+8t)^2=6$$

$$\Rightarrow\ 1+8t+16t^2+1+2t+t^2+4+32t+64t^2=6$$

$\Rightarrow\ 81t^2+42t=0\ \Rightarrow\ t(80t+42)=0\ \Rightarrow\ t=0$ or $t=-\frac{42}{81}$:

When $t=0$: $\Rightarrow \begin{cases} x=1+4t\ \Rightarrow\ x=1+0\ \Rightarrow x=1 \\ y=1+t\ \Rightarrow\ y=1+0\ \Rightarrow y=1 \\ z=2+8t\ \Rightarrow\ z=2+0\ \Rightarrow z=2 \end{cases}$

When $t=-\dfrac{42}{81}$: $\Rightarrow \begin{cases} x=1+4t\ \Rightarrow\ x=1-4\left(\dfrac{42}{81}\right) \Rightarrow x=-\dfrac{87}{81} \\ y=1+t\ \Rightarrow\ y=1-\left(\dfrac{42}{81}\right) \Rightarrow y=\dfrac{39}{81} \\ z=2+8t\ \Rightarrow\ z=2-8\left(\dfrac{42}{81}\right) \Rightarrow z=-\dfrac{174}{81} \end{cases}$

The points are: $(1,1,2)$ and $\left(-\frac{87}{81},\frac{39}{81},-\frac{174}{81}\right)$

Example 25: Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1,1,2)$ intersects the paraboloid a second time.

Solution:

$z = x^2 + y^2 \Rightarrow\ x^2 + y^2 - z = 0 \Rightarrow\ F(x,y,z) = x^2 + y^2 - z$

$$\Rightarrow\ \nabla F = \langle 2x, 2y, -1\rangle\ \Rightarrow\ \nabla F(1,1,2) = \langle 2,2,-1\rangle$$

The equations of the normal line are: $x = 1 + 2t, y = 1 + 2t, z = 2 - t$

The normal line intersects the paraboloid $z = x^2 + y^2$:

Substitute ($x = 1 + 2t, y = 1 + 2t, z = 2 - t$) in the equation $z = x^2 + y^2$:

$2 - t = (1 + 2t)^2 + (1 + 2t)^2\ \ \Rightarrow\ 2 - t = 2(1 + 4t + 4t^2)$

$\Rightarrow\ \ 8t^2 + 9t = 0\ \Rightarrow t(8t + 9) = 0\ \ \Rightarrow t = 0$ or $t = -\frac{9}{8}$

When $t = 0$: $\Rightarrow \begin{cases} x = 1 + 2t \ \Rightarrow\ x = 1 + 0\ \Rightarrow x = 1 \\ y = 1 + 2t\ \Rightarrow\ y = 1 + 0\ \Rightarrow y = 1 \\ z = 2 - t\ \Rightarrow\ z = 2 - 0\ \Rightarrow z = 2 \end{cases}$

When $t = -\dfrac{9}{8}$: $\Rightarrow \begin{cases} x = 1 + 2t\ \Rightarrow\ x = 1 + 2\left(-\dfrac{9}{8}\right) \Rightarrow x = -\dfrac{10}{8} \\ y = 1 + 2t\ \Rightarrow\ y = 1 + 2\left(-\dfrac{9}{8}\right) \Rightarrow y = -\dfrac{10}{8} \\ z = 2 - t\ \Rightarrow\ z = 2 - \left(-\dfrac{9}{8}\right) \Rightarrow z = \dfrac{25}{8} \end{cases}$

The points are: $(1,1,2)$ and $\left(-\dfrac{10}{8}, -\dfrac{10}{8}, \dfrac{25}{8}\right)$

$\Rightarrow$ the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1,1,2)$ intersects the paraboloid a second time at $\left(-\frac{10}{8}, -\frac{10}{8}, \frac{25}{8}\right)$.

Example 25: Show that every plane that is tangent to the cone $z^2 = x^2 + y^2$ passes through the origin.

Solution: Let (a, b, c) be a point on the cone $z^2 = x^2 + y^2$

$$\Rightarrow c^2 = a^2 + b^2 \quad \ldots\ldots\ldots\ldots..(1)$$

Now, we find the equation of the tangent plane to the cone:

$z^2 = x^2 + y^2 \;\; \Rightarrow \;\; x^2 + y^2 - z^2 = 0.$

Let $F(x, y, z) = \; x^2 + y^2 - z^2 \Rightarrow \nabla F = \langle 2x, 2y, -2z\rangle \Rightarrow \nabla F(a, b, c) = \langle 2a, 2b, -2c\rangle$

$\nabla F(a, b, c) \perp$ tangent plane $\Rightarrow \langle 2a, 2b, -2c\rangle \perp$ tangent plane $\quad \div 2$

$\langle a, b, -c\rangle \perp$ tangent plane and (a, b, c) is a point on the tangent plane

The equation of the tangent plane: $ax + by - cz = a^2 + b^2 - c^2 = 0$ (by equation (1))

$$\Rightarrow \; ax + by - cz = 0 \; \ldots\ldots\ldots\ldots(2)$$

At the origin $x = 0, y = 0, z = 0$: substituting in the equation (2):

$a(0) + b(0) - c(0) = 0 \Rightarrow$ The origin satisfies the equation (2)

The origin lies on the tangent plane that is the tangent plane passes through the origin.

Example 25: Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

Solution: Let (a, b, c) be a point on the sphere $x^2 + y^2 + z^2 = r^2$.

First, we find the equations of the normal line to the sphere $x^2 + y^2 + z^2 = r^2$ at (a, b, c):

$x^2 + y^2 + z^2 = r^2 \Rightarrow x^2 + y^2 + z^2 - r^2 = 0.$

Let $F(x, y, z) = x^2 + y^2 + z^2 - r^2 \Rightarrow \nabla F = \langle 2x, 2y, 2z\rangle \Rightarrow \nabla F(a, b, c) = \langle 2a, 2b, 2c\rangle$

$\nabla F(a, b, c)$ // normal line: $\Rightarrow \langle 2a, 2b, 2c\rangle$ // normal line $\div 2$

$\Rightarrow \langle a, b, c\rangle$ // normal line and (a, b, c) is a point on the normal line

The equations of the normal line: $x = a + at, y = b + bt, z = c + ct$

To show that the normal line passes through the center of $x^2 + y^2 + z^2 = r^2$:

Observe that the center of the sphere is (0,0,0):

$$\text{So, taking } t = -1 \Rightarrow \left\{\begin{matrix} x = a + at & \Rightarrow & x = a - a = 0 \\ y = b + bt & \Rightarrow & y = b - b = 0 \\ z = c + ct & \Rightarrow & z = c - c = 0 \end{matrix}\right\}$$

$\Rightarrow$ the normal line passes through the origin which is the center of the sphere.

Definition 26: Define $D_{\hat{v}}^2 f$ by $D_{\hat{v}}^2 f = D_{\hat{v}}(D_{\hat{v}} f)$, that is $D_{\hat{v}}^2 f = D_{\hat{v}} g$, where $g = D_{\hat{v}} f$

Example 27: Find $D_{\hat{v}}^2 f(0, -3)$, where $f(x, y) = x^2 y^3$ and $\hat{v} = \frac{2}{3} i - \frac{\sqrt{5}}{3} j$

Solution: $\nabla f = \langle 2xy^3, 3x^2y^2\rangle$

$\Rightarrow$ Let $g = D_{\hat{v}} f = \nabla f \cdot \hat{v} = \langle 2xy^3, 3x^2y^2\rangle \cdot \langle \frac{2}{3}, -\frac{\sqrt{5}}{3}\rangle = \frac{4}{3} xy^3 - \sqrt{5} x^2 y^2$

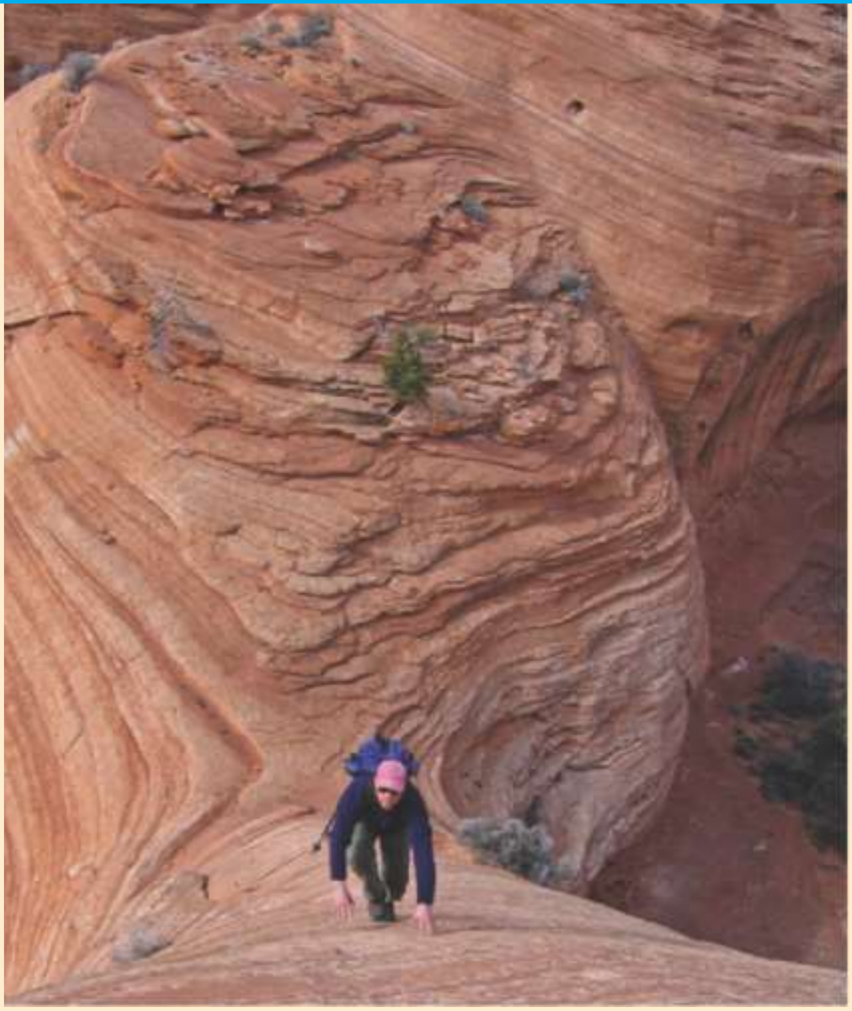
$$\nabla g = \langle \frac{4}{3} y^3 - 2\sqrt{5} xy^2, 4xy^2 - 2\sqrt{5} x^2 y\rangle \Rightarrow \nabla g(0, -3) = \langle -36, 0\rangle$$

$$D_{\hat{v}}^2 f(1,2) = D_{\hat{v}} g = \nabla g \cdot \hat{v} = \langle -36, 0\rangle \cdot \langle \frac{2}{3}, -\frac{\sqrt{5}}{3}\rangle = -24$$

Chapter 14 Partial Derivatives

Section 14.7

Maximum and Minimum Values



14.7 Maximum and Minimum Values

Definition 1: A function $f(x,y)$ is said to have:

(1) a local maximum value at a point $(a,b) \in Dom(f)$ if $f(a,b) \geq f(x,y)$ for all $(x,y) \in D$, where D is a disk in Domain f centered at (a,b). The number $f(a,b)$ is called a local maximum value of f.

---

(2) a local minimum value at a point $(a,b) \in Dom(f)$ if $f(a,b) \leq f(x,y)$ for all $(x,y) \in D$, where D is a disk in Domain f centered at (a,b). The number $f(a,b)$ is called a local minimum value of f.

---

(3) an absolute maximum value at a point $(a,b) \in Dom(f)$ if $f(a,b) \geq f(x,y)$ for all $(x,y) \in Dom(f)$. The number $f(a,b)$ is called the absolute maximum value of f.

---

(4) an absolute minimum value at a point $(a,b) \in Dom(f)$ if $f(a,b) \leq f(x,y)$ for all $(x,y) \in Dom(f)$. The number $f(a,b)$ is called the absolute minimum value of f.

---

(5) a local extremum at a point (a,b) if f has a local maximum or minimum value at (a,b).

---

(6) an absolute extremum at a point (a,b) if f has an absolute maximum or minimum value at (a,b).

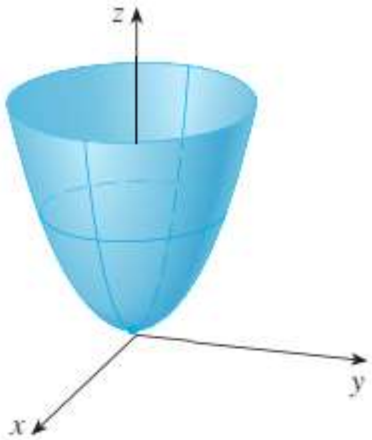
Example 2: Find the absolute and local extrema of the function $f(x,y) = 2x^2 + y^2$

Solution: First, we give a graph of the function f:

From the graph we see that:

f has a local minimum value at $(0,0)$. This local minimum value is $f(0,0) = 0$

Also, f has an absolute minimum value at $(0,0)$. This absolute minimum value is $f(0,0) = 0$



Part 1: Local Maximum and Minimum Values

Definition 3: A function $f(x,y)$ is said to have a critical point at $(a,b) \in Dom(f)$ if:

- $f_x(a,b) = 0$ and $f_y(a,b) = 0$

or

- $f_x(a,b)$ does not exist

or

- $f_y(a,b)$ does not exist

Example 4: Find the values of a and b that makes the function f has a critical point at $(1,-1)$, where $f(x,y) = x^2y + 3axy^2 - bxy$.

Solution: $f_x = 2xy + 3ay^2 - by$ and $f_y = x^2 + 6axy - bx$

$(1,-1)$ is a critical point $f_x(-1,1) = 0$ and $f_y(-1,1) = 0$

$f_x(-1,1) = 0 \Rightarrow -2 + 3a + b = 0 \Rightarrow 3a + b = 2 \ldots\ldots\ldots\ldots\ldots (1)$

$f_y(-1,1) = 0 \Rightarrow \ 1 - 6a - b = 0 \ \Rightarrow 6a + b = 1 \ldots\ldots\ldots\ldots\ldots (2)$

$$(2) - (1): \Rightarrow \ 3a = -1 \ \Rightarrow \ a = -\frac{1}{3}$$

$(1) \Rightarrow \ 3a + b = 2: 1 + b = 2 \ \Rightarrow \ b = 1$

Theorem 5: If a function $f(x,y)$ has a local maximum or minimum value at (a,b) and $f_x(a,b), f_y(a,b)$ both exist, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$

The Second Derivative Test 6: Suppose that the second partial derivatives of the function $f(x,y)$ are continuous on a disk centered at a point (a,b) and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Let

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

(1) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum value at (a,b). This local minimum value equals to $f(a,b)$.

(2) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum value at (a,b). This local maximum value equals to $f(a,b)$.

(3) If $D(a,b) < 0$, then f has neither a local maximum value nor a local minimum value at (a,b). In this case we say that f has a saddle point at (a,b).

Example 7: Find and classify the critical points of the function f as local maximum, local minimum, or saddle point, where $f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$. Moreover find the local maximum and minimum values of f.

Solution: $f_x = 6x^2 + 6y^2 - 150$ and $f_y = 12xy - 9y^2$

$f_x = 0 \Rightarrow 6x^2 + 6y^2 - 150 = 0 \Rightarrow (6x^2 + 6y^2 = 150) \div 6$

$$\Rightarrow x^2 + y^2 = 25 \ldots\ldots\ldots\ldots(1)$$

$f_y = 0 \Rightarrow (12xy - 9y^2 = 0) \div 3 \Rightarrow 4xy - 3y^2 = 0 \Rightarrow y(4x - 3y) = 0$

$$\Rightarrow y = 0 \text{ or } y = \frac{4}{3}x$$

Case1: If $y = 0$: Equation (1) $\Rightarrow x^2 = 25 \Rightarrow x = \pm 5 \Rightarrow$ two critical points $(\pm 5,0)$

Case2: If $y = \frac{4}{3}x$: Equation (1) $\Rightarrow x^2 + \frac{16}{9}x^2 = 25 \Rightarrow \frac{25}{9}x^2 = 25 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

- If $x = -3$: $y = \frac{4}{3}x \Rightarrow y = -4 \Rightarrow$ one point $(-3,-4)$
- If $x = 3$: $y = \frac{4}{3}x \Rightarrow y = 4 \Rightarrow$ one point $(3,4)$

❖ f has four critical points: $(-5,0), (5,0), (-3,-4), (3,4)$

$f_{xx} = 12x, \quad f_{yy} = 12x - 18y,$ and $f_{xy} = 12y$

$$D = f_{xx}f_{yy} - [f_{xy}]^2 \quad \Rightarrow \quad D = 12x(12x - 18y) - (12y)^2$$

At $(-5,0)$: $D(-5,0) = 12(-5)\big(12(-5)\big) > 0$ and $f_{xx}(-5,0) = 12(-5) < 0$:

$$\Rightarrow f \text{ has a local maximum value at the point } (-5,0)$$

$f(-5,0) = 2(-5)^3 - 150(-5) = 500$ is a local maximum value of f.

At $(5,0)$: $D(-5,0) = 12(5)\big(12(5)\big) > 0$ and $f_{xx}(5,0) = 12(5) > 0$:

$$\Rightarrow f \text{ has a local minimum value at the point } (5,0)$$

$f(5,0) = 2(5)^3 - 150(5) = -500$ is a local minimum value of f.

At $(-3,-4)$: $D(-3,-4) = 12(-3)\big(12(-3) - 18(-4)\big) - \big(12(-4)\big)^2 < 0$

$$\Rightarrow f \text{ has a saddle point at } (-3,-4)$$

At $(3,4)$: $D(3,4) = 12(3)\big(12(3) - 18(4)\big) - \big(12(4)\big)^2 < 0$

$$\Rightarrow f \text{ has a saddle point at } (3,4)$$

Example 8: Find the local maximum and local minimum values of the function $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution:

$f_x = 4x^3 - 4y$ and $f_y = 4y^3 - 4x$

$f_x = 0 \Rightarrow 4x^3 - 4y = 0 \Rightarrow \qquad y = x^3 \ldots\ldots\ldots\ldots(1)$

$f_y = 0 \Rightarrow 4y^3 - 4x = 0 \Rightarrow \qquad x = y^3 \ldots\ldots\ldots\ldots(2)$

Substitute equation (1) in (2): $x = (x^3)^3 \Rightarrow x = x^9 \Rightarrow x = 0, -1, 1$

Case1: If $x = 0$: Equation (1) $\Rightarrow y = 0^3 = 0 \Rightarrow$ one critical points $(0,0)$

Case2: If $x = -1$: Equation (1) $\Rightarrow y = (-1)^3 = -1 \Rightarrow$ one critical points $(-1,-1)$

Case3: If $x = 1$: Equation (1) $\Rightarrow y = (1)^3 = 1 \Rightarrow$ one critical points $(1,1)$

- ❖ f has three critical points: $(0,0), (-1,-1), (1,1)$

$f_{xx} = 12x^2, \quad f_{yy} = 12y^2,$ and $f_{xy} = -4$

$$D = f_{xx}f_{yy} - [f_{xy}]^2 \quad \Rightarrow \quad D = 12x^2(12y^2) - (-4)^2$$

At $(0,0)$: $D(0,0) = -16 < 0$: $\Rightarrow f$ has a saddle point at $(0,0)$

$\Rightarrow f$ has neither a local maximum nor a local minimum at $(0,0)$

At $(-1,-1)$: $D(-1,-1) = 12(12) - 16 > 0$ and $f_{xx}(-1,-1) = 12 > 0$:

$\Rightarrow f$ has a local minimum value at the point $(-1,-1)$

$\Rightarrow f(-1,-1) = -1$ is a local minimum value of f

At $(1,1)$: $D(1,1) = 12(12) - 16 < 0$ and $f_{xx}(1,1) = 12 > 0$:

$\Rightarrow f$ has a local minimum value at the point $(-1,-1)$

$f(1,1) = -1$ is a local minimum value of f

Example 9: Find and classify the critical points of the function f as local maximum, local minimum, or saddle point, where $f(x,y) = x^2 + y^2 - 2x - 6y + 12$.

Solution: $f_x = 2x - 2$ and $f_y = 2y - 6$

$f_x = 0 \Rightarrow 2x - 2 = 0 \Rightarrow \qquad x = 1 \ldots\ldots\ldots\ldots.(1)$

$f_y = 0 \Rightarrow 2y - 6 = 0 \Rightarrow \qquad y = 3$ ………....(2)

❖ f has only one critical point: $(1,3)$

$f_{xx} = 2, \quad f_{yy} = 2,$ and $f_{xy} = 0$

$$D = f_{xx}f_{yy} - [f_{xy}]^2 \Rightarrow D = 4$$

At $(1,3)$: $D(1,3) = 4 > 0$ and $f_{xx}(1,3) = 2 > 0$:

$\Rightarrow f$ has a local minimum value at the point $(1,3)$

Part 2: Absolute Maximum and Minimum Values of Functions with Only One Critical point

Theorem 10: Let $f(x,y)$ be with domain $\mathbb{R}^2$ and has only one critical point at (a,b).

(1) If $f(a,b)$ is a local maximum value of the function f, then $f(a,b)$ is an absolute maximum value of f.

(2) If $f(a,b)$ is a local minimum value of the function f, then $f(a,b)$ is an absolute minimum value of f.

Example: Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + y^2 - 2x - 6y + 12$.

Solution: From Example 9 we see that this function has only one critical point at $(1,3)$ and at this point f has a local minimum value. So by Theorem 10, f has an absolute minimum value at $(1,3)$ $\Rightarrow$ The absolute minimum value of f is $f(1,3) = 2$.

Also, observe that the function f has no absolute maximum value because it has only one critical point.

Example 11: Find the shortest distance from the point $(1,0,-2)$ to the plane

$$x+2y+z=4$$

First Solution: Let (x,y,z) be a point on the plane $x+2y+z=4$. The distance from the point $(1,0,-2)$ to the point (x,y,z) is

$$d=\sqrt{(x-1)^2+(y-0)^2+(z-(-2))^2}$$

$$\Rightarrow\ d^2=(x-1)^2+y^2+(z+2)^2\ \ldots\ldots\ldots..(1)$$

Since $z=4-x-2y$ substituting this in the equation (1) we have:

$$\Rightarrow\ d^2\ =(x-1)^2+y^2+(4-x-2y+2)^2$$

$$=(x-1)^2+y^2+(6-x-2y)^2$$

Let $f(x,y)=d^2\Rightarrow f(x,y)=(x-1)^2+y^2+(6-x-2y)^2$

$f_x=2(x-1)+2(6-x-2y)(-1)\ \Rightarrow\ f_x=4x+4y-14$

$f_y=2y+2(6-x-2y)(-2)\ \Rightarrow\ f_y=4x+10y-24$

$$f_x=0\ \Rightarrow(4x+4y-14=0)\div 2\ \Rightarrow 2x+2y=7\ \ldots\ldots(2)$$

$$f_y=0\ \Rightarrow\ (4x+10y-24=0)\div 2\ \Rightarrow 2x+5y=12\ \ldots\ldots..(3)$$

Eq.(3)-Eq.(2) $\Rightarrow 3y=5\ \Rightarrow\ y=\frac{5}{3}\ \Rightarrow\ 2x+2\left(\frac{5}{3}\right)=7\ \Rightarrow x=\frac{11}{6}$

The function f has only on critical point at $\left(\frac{11}{6},\frac{5}{3}\right)$

$f_{xx}=4, f_{yy}=10,$ and $f_{xy}=4\Rightarrow D=(f_{xx})(f_{yy})-[f_{xy}]^2$

$D=40-16=24>0$ and $f_{xx}=4>0\Rightarrow f$ has a local minimum value at $\left(\frac{11}{6},\frac{5}{3}\right)$

Since f has only on critical point at $\left(\frac{11}{6},\frac{5}{3}\right)$ and $f\left(\frac{11}{6},\frac{5}{3}\right)$ is a local minimum value of f, then $f\left(\frac{11}{6},\frac{5}{3}\right)$ is an absolute minimum value of f

The absolute minimum value of f is $f\left(\frac{11}{6},\frac{5}{3}\right)$

$\Rightarrow$ The shortest distance is $d=\sqrt{f\left(\frac{11}{6},\frac{5}{3}\right)}=\sqrt{\left(\frac{5}{6}\right)^2+\left(\frac{5}{3}\right)^2+\left(\frac{5}{6}\right)^2}=\frac{5\sqrt{6}}{6}$

Second Solution: We use the law of distance from a point and a plane:

Equation of plane: $x+2y+z=4 \Rightarrow \quad x+2y+z-4=0$, point $(1,0,-2)$

$$\text{The shortest distance is } d=\frac{|1+2(0)+(-2)-4|}{\sqrt{1^2+2^2+1^2}}=\frac{|-5|}{\sqrt{6}}=\frac{5}{\sqrt{6}}=\frac{5\sqrt{6}}{6}$$

Remark: Let $f(x,y)$ be a function with $\text{range}(f)=\text{S}$, where S is a set in $\mathbb{R}$, then

- The absolute maximum value of f = maximum value in S
- The absolute minimum value of f = minimum value in S

Example 11: Find the absolute maximum and minimum values of the function

$$f(x,y)=5-\sqrt{9-x^2-y^2}$$

Solution: First we find the range of f: Let $z=3-\sqrt{9-x^2-y^2}$

$$\sqrt{9-x^2-y^2}\geq 0 \ \Rightarrow -\sqrt{9-x^2-y^2}\leq 0$$

$$5-\sqrt{9-x^2-y^2}\leq 5 \ \Rightarrow \ z\leq 5 \quad \dots\dots\dots..(1)$$

$$z=5-\sqrt{9-x^2-y^2}=5-\sqrt{9-(x^2+y^2)}:$$

$$(x^2+y^2)\geq 0 \Rightarrow \ -(x^2+y^2)\leq 0 \ \Rightarrow \ 9-(x^2+y^2)\leq 9$$

$$\sqrt{9-(x^2+y^2)}\leq\sqrt{9} \ \Rightarrow \ \sqrt{9-(x^2+y^2)}\leq 3$$

$$-\sqrt{9-(x^2+y^2)}\geq -3 \ \Rightarrow \ 5-\sqrt{9-(x^2+y^2)}\geq 5-3$$

$$z\geq 2 \dots\dots\dots\dots.(2)$$

$\Rightarrow$ from (1) & (2) $range(f)=[2,5]$

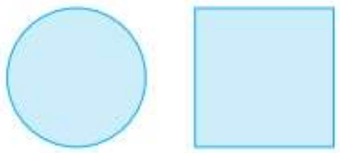
- The absolute maximum value of f = maximum value in $[2,5]=5$
- The absolute minimum value of f = minimum value in $[2,5]=2$

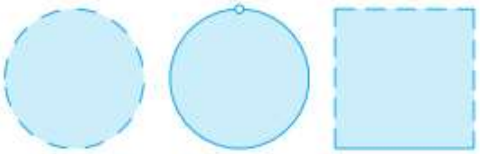
Part 3: Absolute Maximum and Minimum Values of Functions over closed bounded sets

Definition 12:

(1) A closed set in $\mathbb{R}^2$ is a set that contains all its boundary points, where a boundary point of a set D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.

(2) A bounded set in $\mathbb{R}^2$ is a set that is contained within some disk.


(a) Closed sets


(b) Sets that are not closed

Extreme Value Theorem for Functions of two variables 13:

If $f(x, y)$ is continuous on a closed, bounded set D in $\mathbb{R}^2$, then attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

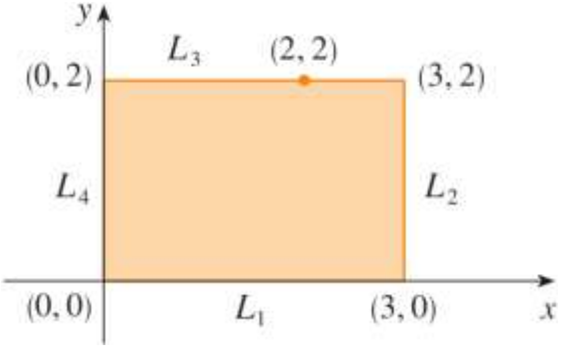
Remark 14: To find the absolute maximum and minimum values of a continuous function $f(x, y)$ on a closed, bounded set D:

Step 1. Find the values of at the critical points of $f(x, y)$ in D.

Step 2. Find the extreme values of $f(x, y)$ on the boundary of D.

Step 3. The largest of the values from steps 1 and 2 is the absolute maximum value and the smallest of these values is the absolute minimum value.

Example 15: Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y): 0 \le x \le 3, 0 \le y \le 2\}$.



Solution:

Step 1: We find the critical points of f in D:

$f_x = 2x - 2y$ and $f_y = -2x + 2$

$$\left.\begin{array}{l} f_x = 0 \Rightarrow 2x - 2y = 0 \Rightarrow y = x \\ f_y = 0 \Rightarrow -2x + 2 = 0 \Rightarrow x = 1 \end{array}\right\}$$

$\Rightarrow y = 1 \Rightarrow (1,1)$ is a critical point of f

Check: Is $(1,1) \in D$?

Yes since $x = 1 \in \underbrace{(0,3)}_{\text{interval}}$ and $y = 1 \in \underbrace{(0,2)}_{\text{interval}}$ $\Rightarrow$ we have a point $(1,1)$

Step 2: We find the extreme values of f on the boundary of D:

Observe that the boundary of D consists of 4 line segments: L_1, L_2, L_3, L_4:

- On L_1 $(y = 0)$: $g_1(x) = f(x,0) = x^2 - 2x(0) + 2(0) = x^2, 0 \le x \le 3$.
  $g_1'(x) = 2x, 0 < x < 3$.
  $g_1'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0 \notin \underbrace{(0,3)}_{\text{interval}}$
  We have two points when $x = 0$ and $x = 3 \Rightarrow$ the points are $(0,0), (3,0)$
- On L_2 $(x = 3)$: $h_1(y) = f(3,y) = (3)^2 - 2(3)y + 2y = 9 - 4y, 0 \le y \le 2$.
  $h_1'(y) = -4, 0 < y < 2$.
  $h_1'(y) \ne 0, \forall y \in \underbrace{(0,2)}_{\text{interval}} \Rightarrow$ We have two points when $y = 0$ and $y = 2$:
  $\Rightarrow$ the points are $(3,0), (3,2)$.
- On L_3 $(y = 2)$: $g_2(x) = f(x,2) = x^2 - 2x(2) + 2(2) = x^2 - 4x + 4, 0 \le x \le 3$.
  $g_2'(x) = 2x - 4, 0 < x < 3$.
  $g_2'(x) = 0 \Rightarrow 2x - 4 = 0 \Rightarrow x = 2 \in \underbrace{(0,3)}_{\text{interval}}$
  We have three points when $x = 2$, $x = 0$ and $x = 3$
  $\Rightarrow$ the points are $(2,2), (0,2), (3,2)$
- On L_4 $(x = 0)$: $h_2(y) = f(0,y) = (0)^2 - 2(0)y + 2y = 2y, 0 \le y \le 2$.
  $h_1'(y) \ne 0, \forall y \in \underbrace{(0,2)}_{\text{interval}} \Rightarrow$ We have two points when $y = 0$ and $y = 2$:
  $\Rightarrow$ the points are $(0,0), (0,2)$.

Step 3:

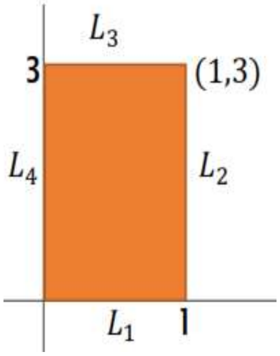
Points	(1,1)	(0,0)	(3,0)	(3,2)	(2,2)	(0,2)
$f(x,y)$	1	0	9	1	0	4

The absolute maximum value of f is 9

The absolute minimum value of f is 0

Example 16: Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - 2xy - y^2 + 8y - 1$ on the rectangle $D = \{(x,y): 0 \leq x \leq 1, 0 \leq y \leq 3\}$.

Solution:



Step 1: We find the critical points of f in D:
$f_x = 2x - 2y$ and $f_y = -2x + 2y + 8$

$$\left.\begin{aligned} f_x &= 0 \Rightarrow 2x - 2y = 0 && \Rightarrow x - y = 0 \\ f_y &= 0 \Rightarrow -2x - 2y + 8 = 0 && \Rightarrow x + y = 4 \end{aligned}\right\}$$

$\Rightarrow x = 2, y = 2 \Rightarrow (2,2)$ is a critical point of f

Check: Is $(2,2) \in D$? No since $x = 2 \notin \underbrace{(0,1)}_{\text{interval}}$

We do not have any critical point of f in step 1

Step 2: We find the extreme values of f on the boundary of D:
Observe that the boundary of D consists of 4 line segments: L_1, L_2, L_3, L_4:

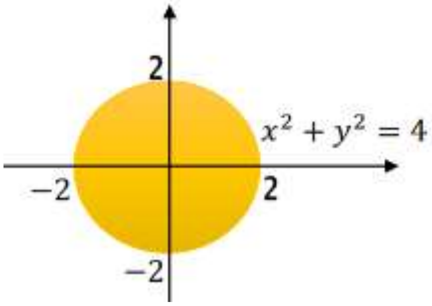
- On L_1 $(y = 0)$: $g_1(x) = f(x,0) = x^2, 0 \leq x \leq 1 \Rightarrow g_1'(x) = 2x, 0 < x < 1$.
  $g_1'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0 \notin \underbrace{(0,1)}_{\text{interval}}$
  We have two points when $x = 0$ and $x = 1 \Rightarrow$ the points are $(0,0), (1,0)$
- On L_2 $(x = 1)$: $h_1(y) = f(1,y) = -y^2 + 6y + 1, 0 \leq y \leq 3$.
  $h_1'(y) = -2y + 6, 0 < y < 3$: $h_1'(y) = 0 \Rightarrow -2y + 6 = 0 \Rightarrow y = 3 \notin \underbrace{(0,3)}_{\text{interval}}$
  We have two points when $y = 0$ and $y = 3$: $\Rightarrow$ the points are $(1,0), (1,3)$.
- On L_3 $(y = 3)$: $g_2(x) = f(x,3) = x^2 - 6x + 15, 0 \leq x \leq 1$:
  $g_2'(x) = 2x - 6, 0 < x < 1$ $g_2'(x) = 0 \Rightarrow 2x - 6 = 0 \Rightarrow x = 3 \notin \underbrace{(0,1)}_{\text{interval}}$:
  $\Rightarrow$ We have two points when $x = 0$ and $x = 1 \Rightarrow$ the points are $(0,3), (1,3)$
- On L_4 $(x = 0)$: $h_2(y) = f(0,y) = -y^2 + 8y, 0 \leq y \leq 3$.
  $h_2'(y) = -2y + 8, 0 < y < 2$: $h_2'(y) = 0 \Rightarrow -2y + 8 = 0 \Rightarrow y = 4 \notin \underbrace{(0,3)}_{\text{interval}}$
  We have two points when $y = 0$ and $y = 3 \Rightarrow$ the points are $(0,0), (0,3)$

Step 3:

Points	(0,0)	(1,0)	(1,3)	(0,3)
$f(x,y)$	-1	0	9	14

The absolute maximum value of f is 14
The absolute minimum value of f is -1

Example 17: Find the absolute maximum and minimum values of the function $f(x,y) = xy^2$ on the disk $D = \{(x,y): x^2 + y^2 \leq 4\}$.



Solution:

Step 1: We find the critical points of f in D:

$f_x = y^2$ and $f_y = 2xy$

$$f_x = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0 \ldots\ldots\ldots (1)$$

$$f_y = 0 \Rightarrow 2xy = 0 \Rightarrow \begin{cases} x = 0 \ldots . (2) \\ y = 0 \ldots . (3) \end{cases}$$

Case 1: Equations (1) & (2): $y = 0$ & $x = 0 \Rightarrow$ we have one critical point $(0,0)$ in D

Case 2: Equations (1) & (3): $y = 0$ & $y = 0 \Rightarrow y = 0, \forall x \in \underbrace{(-2,2)}_{\text{interval}}$

$$\Rightarrow \text{we have infinitely many critical points } (x,0), x \in \underbrace{(-2,2)}_{\text{interval}} \text{ in } D$$

Step 2: We find the extreme values of f on the boundary of D:

Observe that the boundary of D is the circle: $x^2 + y^2 = 4$

❖ On $x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2$

$$g(x) = f(x,y)|_{y^2=4-x^2} = x(4 - x^2) = 4x - x^3$$

$$\Rightarrow g(x) = 4x - x^3, -2 \leq x \leq 2 \Rightarrow g'(x) = 4 - 3x^2, -2 < x < 2$$

$$g'(x) = 0 \Rightarrow 4 - 3x^2 = 0 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm\frac{2}{\sqrt{3}} \in \underbrace{(-2,2)}_{\text{interval}}$$

- If $x = \frac{2}{\sqrt{3}}$: $y^2 = 4 - \left(\frac{2}{\sqrt{3}}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3} \Rightarrow y = \pm\frac{\sqrt{8}}{\sqrt{3}}$

  $\Rightarrow$ We have two points: $\left(\frac{2}{\sqrt{3}}, -\frac{\sqrt{8}}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{3}}, \frac{\sqrt{8}}{\sqrt{3}}\right)$

- If $x = -\frac{2}{\sqrt{3}}$: $y^2 = 4 - \left(-\frac{2}{\sqrt{3}}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3} \Rightarrow y = \pm\frac{\sqrt{8}}{\sqrt{3}}$

  $\Rightarrow$ We have two points: $\left(-\frac{2}{\sqrt{3}}, -\frac{\sqrt{8}}{\sqrt{3}}\right), \left(-\frac{2}{\sqrt{3}}, \frac{\sqrt{8}}{\sqrt{3}}\right)$

Step 3: $f(x,y) = xy^2$

Point	$(x,0), y \in \underbrace{(-2,2)}_{\text{interval}}$	$\left(\frac{2}{\sqrt{3}}, -\frac{\sqrt{8}}{\sqrt{3}}\right)$	$\left(\frac{2}{\sqrt{3}}, \frac{\sqrt{8}}{\sqrt{3}}\right)$	$\left(-\frac{2}{\sqrt{3}}, -\frac{\sqrt{8}}{\sqrt{3}}\right)$	$\left(-\frac{2}{\sqrt{3}}, \frac{\sqrt{8}}{\sqrt{3}}\right)$
$f(x,y)$	0	$\frac{16}{3\sqrt{3}}$	$\frac{16}{3\sqrt{3}}$	$\frac{16}{3\sqrt{3}}$	$\frac{16}{3\sqrt{3}}$

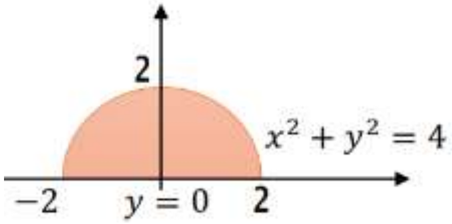
The absolute maximum value of f is $\frac{16}{3\sqrt{3}}$

The absolute minimum value of f is 0

Example 18: Find the absolute maximum and minimum values of the function $f(x,y) = x^4 + 2x^2y^2 + y^4 + 8y - 1$ on the half disk $D = \{(x,y): x^2 + y^2 \leq 4, y \geq 0\}$.

First Solution:

Step 1: We find the critical points of f in D:
$f_x = 4x^3 + 4xy^2$ and $f_y = 4x^2y + 4y^3 + 8$



$f_x = 0 \Rightarrow 4x^3 + 4xy^2 = 0 \Rightarrow 4x(x^2 + y^2) = 0$

$$\Rightarrow \begin{cases} x = 0 \ldots\ldots\ldots..(1) \\ x^2 + y^2 = 0 \quad \Rightarrow x = 0 \text{ and } y = 0 \ldots\ldots\ldots.(2) \end{cases}$$

$f_y = 0 \Rightarrow 4x^2y + 4y^3 + 8 = 0$ $\ldots\ldots\ldots\ldots..(3)$

Case 1: Equations (1) & (3): $x = 0$ & $4x^2y + 4y^3 + 8 = 0 \Rightarrow 4(0)^2y + 4y^3 + 8 = 0$

$\Rightarrow 4y^3 + 8 = 0 \Rightarrow y^3 = -2 \Rightarrow y = -\sqrt[3]{2}$

we have one critical points $(0, -\sqrt[3]{2})$ but $(0, -\sqrt[3]{2}) \notin D$

Case 2: Equations (2) & (3): $x = 0, y = 0$ & $4x^2y + 4y^3 + 8 = 0$

$\Rightarrow 4(0)^2(0) + 4(0)^3 + 8 = 0 \Rightarrow 8 = 0$ which is impossible

We do not have any critical point of f in Step 1

Step 2: We find the extreme values of f on the boundary of D:

The boundary of D consists of two parts: $C: x^2 + y^2 = 4$ and $L: y = 0$

- On $C: x^2 + y^2 = 4$: Observe that $f(x,y) = (x^2 + y^2)^2 + 8y - 1$

  Let $h(y) = f(x,y)|_{x^2 = 4 - y^2} = (4)^2 + 8y - 1 = 8y + 15$

  $\Rightarrow h(y) = 8y + 15, 0 \leq y \leq 2 \Rightarrow h'(y) = 8, 0 < y < 2 \Rightarrow h'(y) \neq 0, \forall y \in \underbrace{(0,2)}_{\text{interval}}$

  We have two critical points when $y = 0$ and $y = 2$
  $y = 0: x^2 = 4 - y^2 \Rightarrow x^2 = 4 - (0)^2 = 4 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$
  We have Two Points $(-2,0), (2,0)$.
  $y = 2: x^2 = 4 - y^2 \Rightarrow x^2 = 4 - (2)^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$
  We have One Point $(0,2)$.

- On $L: y = 0$: $g(x) = f(x,0) = x^4 + 2x^2(0)^2 + (0)^4 + 8(0) - 1$
  $\Rightarrow g(x) = x^4, -2 \leq x \leq 2 \Rightarrow g'(x) = 4x^3, -2 < x < 2$

$g'(x) = 0 \Rightarrow 4x^3 = 0 \Rightarrow x = 0$ We have one point (0,0)

Step 3: $f(x,y) = x^4 + 2x^2y^2 + y^4 + 8y - 1$

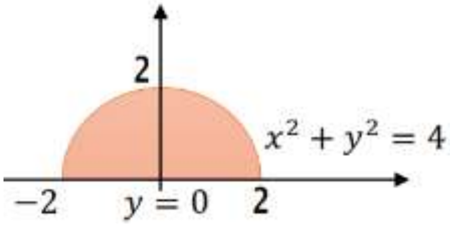
point	$(-2,0)$	$(2,0)$	$(0,2)$	$(0,0)$
$f(x,y)$	15	15	31	-1

⇒ The absolute maximum of f is 31

⇒ The absolute minimum of f is -1

Second Solution:

Observe that $f(x,y) = (x^2 + y^2)^2 + 8y - 1$



Since $0 \le x^2 + y^2 \le 4 \Rightarrow 0 \le (x^2 + y^2)^2 \le 16$

$\Rightarrow 0 + 8y - 1 \le (x^2 + y^2)^2 + 8y - 1 \le 16 + 8y - 1$

$\Rightarrow 8y - 1 \le f(x,y) \le 8y + 15, \ \forall y \in [0,2]$

The maximum value of $(8y + 15)$ when $y \in [0,2]$ is 31

The minimum value of $(8y - 1)$ when $y \in [0,2]$ is -1

⇒ The absolute maximum of f is 31

⇒ The absolute minimum of f is -1

15

Multiple Integrals

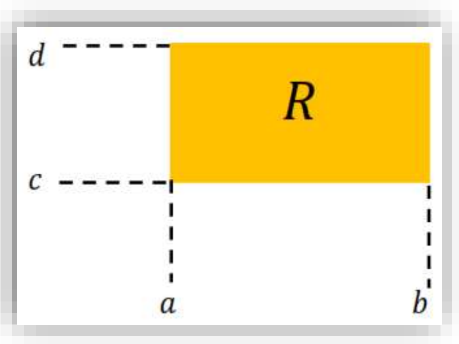
Section 15.1: Double Integrals over Rectangles

Section 15.2: Iterated Integrals

15.1 Double Integrals over Rectangles
15.2 Iterated Integrals

Fubini's Theorem 1:

Let $R = \{(x,y): a \le x \le b, c \le y \le d\}$ be a rectangle and let $f(x,y)$ be a continuous function on R. Then the double integral $\iint\limits_R f(x,y)dA$ can be expressed as an iterated integral as:



$$\boxed{dA = dydx \quad \text{or} \quad dA = dxdy}$$

$$\iint\limits_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dy\,dx = \int_c^d \int_a^b f(x,y)dx\,dy$$

Remark 2:

(1) $\int_a^b \int_c^d f(x,y)dy\,dx = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$:

Means: Compute $g(x) = \int_c^d f(x,y)dy$ by taking x as a constant, then compute $\int_a^b g(x)dx$.

(2) $\int_c^d \int_a^b f(x,y)dx\,dy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$:

Means: Compute $h(x) = \int_a^b f(x,y)dx$ by taking y as a constant, then compute $\int_c^d h(y)dy$.

(3) The rectangle $R = \{(x,y): a \le x \le b, c \le y \le d\}$ can be expressed as:

$$R = \{(x,y): a \le x \le b, c \le y \le d\} = [a,b] \times [c,d].$$

Properties of Double Integrals 3:

Let $f(x,y)$ and $g(x,y)$ be continuous functions on a rectangle R o. Then

(1) $\displaystyle\iint_R (f+g)dA = \iint_R f dA + \iint_R g dA$

(2) $\displaystyle\iint_R (f-g)dA = \iint_R f dA - \iint_R g dA$

(3) $\displaystyle\iint_R cf dA = c\iint_R f dA$, where c is a constant.

(4) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then $\displaystyle\iint_R f dA \geq \iint_R g dA$

- ❖ If the variables x and y in $f(x,y)$ are separated, that is $f(x,y) = g(x)h(y)$ and f is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f dA = \int_a^b \int_c^d g(x)h(y)\, dy\, dx = \left(\int_a^b g(x)\, dx\right)\left(\int_c^d h(y)\, dy\right)$$

Example 4: Evaluate the double integral $\displaystyle\iint_R (x-3y^2)dA$, where

$R = \{(x,y): 0 \leq x \leq 2, -1 \leq y \leq 3\}$.

Solution: $dA = dxdy$ or $dA = dydx$

$$\iint_R (x-3y^2)dA = \int_{-1}^{3}\left(\int_0^2 (x-3y^2)\, dx\right) dy = \int_{-1}^{3}\left(\frac{x^2}{2} - 3xy^2\Big]_0^2\right) dy$$

$$= \int_{-1}^{3} (2-6y^2)\, dy = 2y - \frac{6y^3}{3}\Big]_{-1}^{3} = -48$$

Example 5: Compute the following double integral: $\iint_R \; x\sin(xy)dA$, where $R = [1,2] \times [0, \frac{\pi}{2}]$.

Solution: $dA = dydx$ or $dA = dxdy$

$dA = dydx$ ⇒ we integrate $\int \; y\sin(xy)\,dy$ by substitution

$dA = dxdy$ ⇒ if we use we integrate $\int \; y\sin(xy)\,dx$ by parts

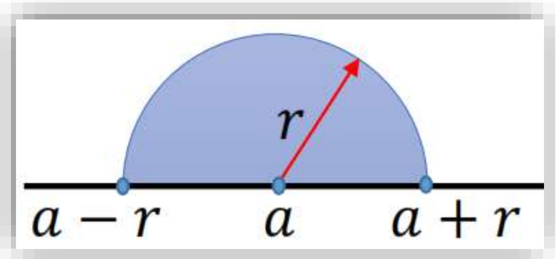
$$\iint\limits_R x\sin(xy)dA \;=\int_1^2\left(\int_0^{\frac{\pi}{2}} x\sin(xy)\,dy\right)dx \quad =\int_1^2\left(x\left(-\frac{\cos(xy)}{x}\right)\bigg]_0^{\frac{\pi}{2}}\right)dx$$

$$=-\int_1^2\left(\cos\left(\frac{\pi}{2}x\right)-1\right)dx=-\left(\frac{\sin\left(\frac{\pi}{2}x\right)}{\frac{\pi}{2}}-x\right)\bigg]_1^2$$

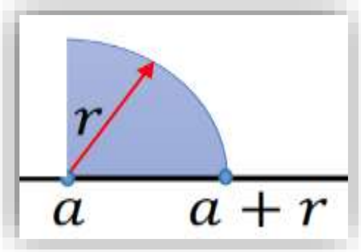
$$=1+\frac{2}{\pi}$$

Remark 6: Recall that $y = \sqrt{r^2-(x-a)^2}$ is the equation of a semicircle:

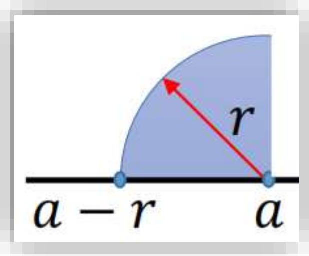
$$\int_{a-r}^{a+r}\sqrt{r^2-(x-a)^2}\,dx=\frac{1}{2}\pi r^2$$



$$\int_{a}^{a+r}\sqrt{r^2-(x-a)^2}\,dx=\frac{1}{4}\pi r^2$$



$$\int_{a-r}^{a}\sqrt{r^2-(x-a)^2}\,dx=\frac{1}{4}\pi r^2$$



Example 7: Evaluate the double integral $\iint_R \sqrt{4-x^2}\,dA$, where $R = [-2,2] \times [0,3]$.

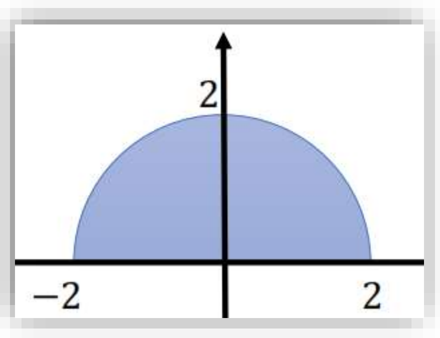
Solution: $dA = dydx$ or $dA = dxdy \Rightarrow$ we choose $dA = dydx$ (why?)

$y = \sqrt{4-x^2}$ is an equation of a semicircle:

$$\iint_R \sqrt{4-x^2}\,dA = \int_{-2}^{2}\left(\int_0^3 \sqrt{4-x^2}\,dy\right)dx$$

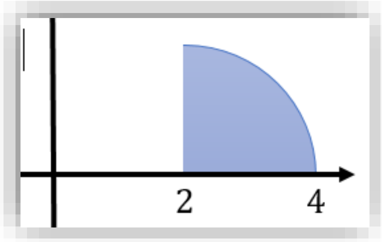
$$= 3\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$= 3\left(\frac{1}{2}\right)\pi(2)^2 = 6\pi$$



Example 8: Evaluate the double integral $\iint_R \sqrt{4x-x^2}\,dA$, where $R = [2,4] \times [0,3]$.

Solution: $dA = dydx$ or $dA = dxdy \Rightarrow$ we choose $dA = dydx$ (why?)

<table>
<tr>
<td>$y = \sqrt{4x-x^2} = \underbrace{\sqrt{4-(x-2)^2}}_{\text{اكمال مربع}}$ is an equation of a semicircle:</td>
<td>
</td>
</tr>
<tr>
<td colspan="2">$$\iint_R \sqrt{4x-x^2}\,dA = \int_{2}^{4}\left(\int_0^3 \sqrt{4-(x-2)^2}\,dy\right)dx = 3\int_2^4 \sqrt{4-(x-2)^2}\,dx$$ $$= 3\left(\frac{1}{4}\right)\pi(2)^2 = 3\pi$$</td>
</tr>
</table>

Example 9: Evaluate the iterated integral $\int_0^{\pi}\int_0^{\pi/12} \sin\left(\frac{x}{3}\right)\cos(2y)\,dydx$.

Solution: Observe that the variables in $\sin\left(\frac{x}{3}\right)\cos(2y)$ are separated, so

$$\int_0^{\pi}\int_0^{\pi/12} \sin\left(\frac{x}{3}\right)\cos(2y)\,dydx = \left(\int_0^{\pi} \sin\left(\frac{x}{3}\right)dx\right)\left(\int_0^{\pi/12} \cos(2y)\,dy\right)$$

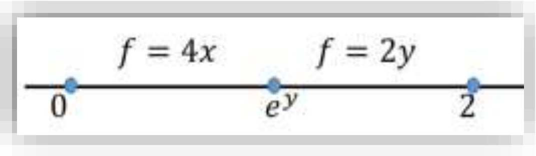
$$= -3\cos\left(\frac{x}{3}\right)\rrbracket_0^{\pi} \frac{\sin(2y)}{2}\rrbracket_0^{\pi/12} = \cdots\cdots\cdots\cdots$$

Example 10: Evaluate the iterated integral $\int_0^2\int_0^1 f(x,y)\,dydx$, where

$$f(x,y) = \begin{cases} 2y & ,x \geq e^y \\ 4x & ,x < e^y \end{cases}$$

Solution: Since the function f is defined when $x \geq e^y$ and $x < e^y$ so we first integrate with respect to x and then y:

$$\Rightarrow \int_0^2\int_0^1 f(x,y)\,dydx = \int_0^1\left(\int_0^2 f(x,y)\,dx\right)dy$$



$$= \int_0^1\left(\int_0^{e^y} f\,dx + \int_{e^y}^2 f\,dx\right)dy$$

$$= \int_0^1\left(\int_0^{e^y} 4x\,dx + \int_{e^y}^2 2y\,dx\right)dy = \int_0^1\left(2x^2]\!]_0^{e^y} + (2-e^y)2y\right)dy$$

$$= \int_0^1 (2e^{2y} + 4y - 2ye^y)\,dy$$

$$= e^{2y} + 2y^2 - (2ye^y - 2e^y)]\!]_0^1$$

$$= e + 1$$

Differentiate	Integrate
$2y$	e^y
2	e^y
0	e^y

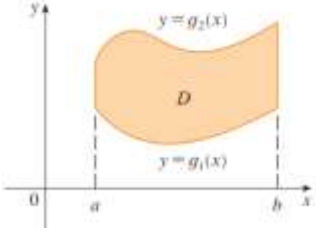
$$\int 2ye^y\,dy = 2ye^y - 2e^y$$

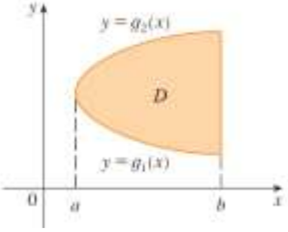
15 Multiple Integrals

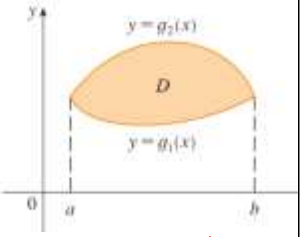
Section 15.3: Double Integrals over General Regions

15.3 Double Integrals over General Regions

Type 1 Regions: Let $D = \{(x,y): a \le x \le b, g_1(x) \le y \le g_2(x)\}$





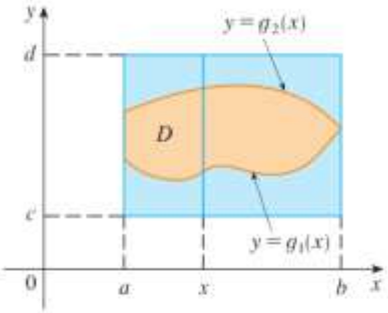


لاحظ أن المنحنيات التي تحد المنطقة تكتب بجعل y موضوع القانون: $y = f(x)$

and let $f(x,y)$ be a continuous function on D.

Then the double integral: $dA = dydx$

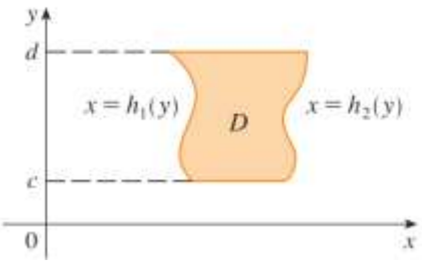
$$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$$

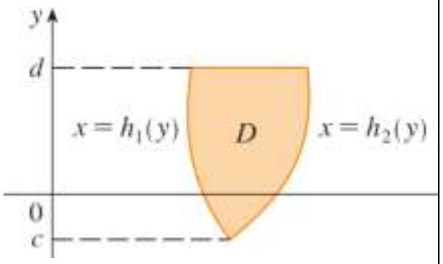


Type 2 Regions: Let $D = \{(x,y): h_1(y) \le x \le h_2(y), c \le y \le d\}$

لاحظ أن المنحنيات التي تحد المنطقة تكتب بجعل x موضوع القانون:

$x = h(y)$

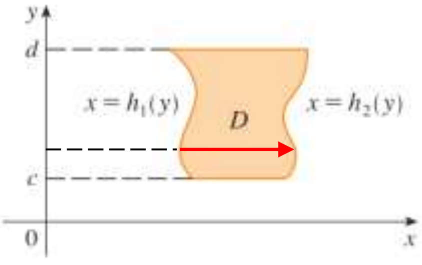




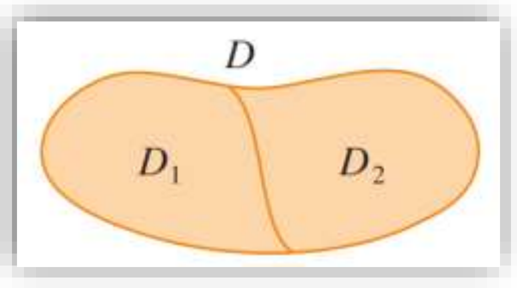
and let $f(x,y)$ be a continuous function on D.

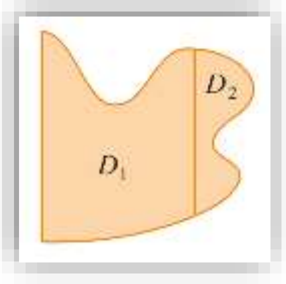
Then the double integral: $dA = dxdy$

$$\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy$$



Remark 3: If the region D consists of two (or more) regions of type 1 (or Type 2) as in the figures:





$$\text{Then } \iint\limits_{D} f(x,y)dA = \iint\limits_{D_1} f(x,y)dA + \iint\limits_{D_2} f(x,y)dA$$

Example 4: Sketch the region and change the order of integration:

(1) $\int_{1}^{2}\int_{0}^{\ln(x)} f(x,y)dy\,dx$ (2) $\int_{-\sqrt{2}}^{\sqrt{2}}\int_{y^2}^{2} f(x,y)dx\,dy$

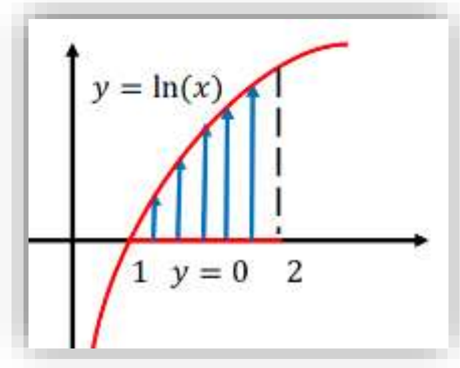
Solution:

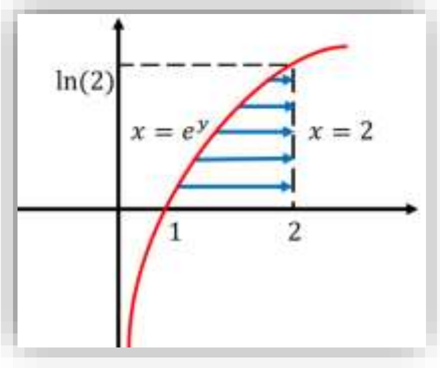
(1) $dA = dydx$ Type 1 Region

Curves: $y = 0 \;\rightarrow\; y = \ln(x)$
$x = 1 \rightarrow x = 2$

$dA = dxdy$ Type 2 Region

Curves: $x = e^y \;\rightarrow\; x = 2$
$y = 0 \rightarrow y = \ln(2)$





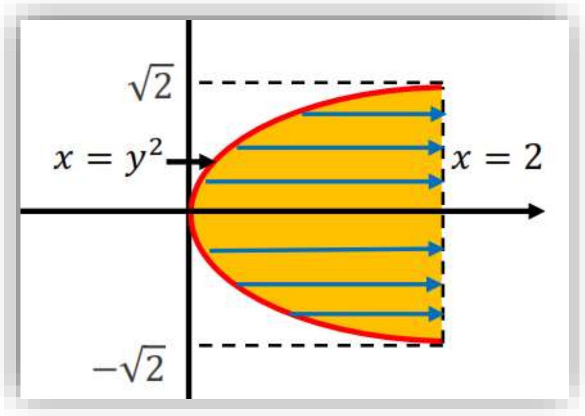
$$\int_{1}^{2}\int_{0}^{\ln(x)} f(x,y)dy\,dx = \int_{0}^{\ln(2)}\int_{e^y}^{2} f(x,y)dx\,dy$$

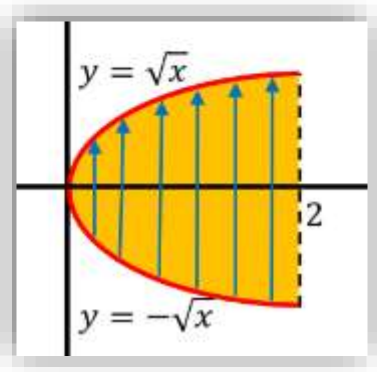
(1) $dA = dxdy$ Type 2 Region

Curves: $x = y^2 \;\rightarrow\; x = 2$
$y = -\sqrt{2} \rightarrow y = \sqrt{2}$

$dA = dydx$ Type 1 Region

Curves: $y = -\sqrt{x} \;\rightarrow\; y = \sqrt{x}$
$y = 0 \rightarrow y = 2$





$$\int_{-\sqrt{2}}^{\sqrt{2}}\int_{y^2}^{2} f(x,y)dx\,dy = \int_{0}^{2}\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)dy\,dx$$

Example 5: Sketch the region and change the order of integration:

$$\int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} f(x,y)dx\,dy$$

Solution:

$dA = dxdy$ Type 2 Region

Curves: $x = \frac{y^2}{2} - 3 \rightarrow x = y + 1$

$y = -2 \rightarrow y = 4$

To find the pints of intersection of

$x = \frac{y^2}{2} - 3$ and $x = y + 1$:

$\frac{y^2}{2} - 3 = y + 1 \Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow (y-4)(y+2) = 0$

$\Rightarrow y = -2$ or $y = 4$

$dA = dydx$ Type 1 Region

$x = \frac{y^2}{2} - 3 \Rightarrow y^2 = 2x + 6$

$\Rightarrow y = \pm\sqrt{2x+6}$ …….(1)

$x = y + 1 \Rightarrow y = x - 1$

Intersection: $y = -2 \Rightarrow x = -1$

$y = 4 \Rightarrow x = 5$

We have 2 Regions:

Region D_1:

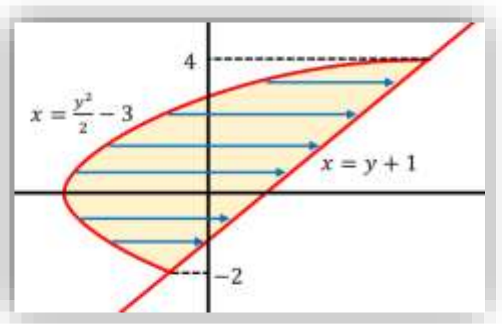
$y = -\sqrt{2x+6} \rightarrow y = \sqrt{2x+6}$

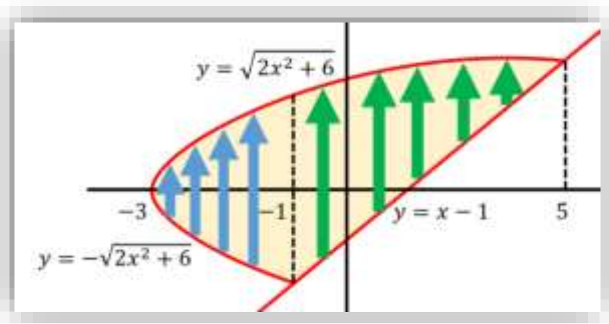
$x = -3 \rightarrow x = -1$

Region D_2:

$y = x - 1 \rightarrow y = \sqrt{2x+6}$

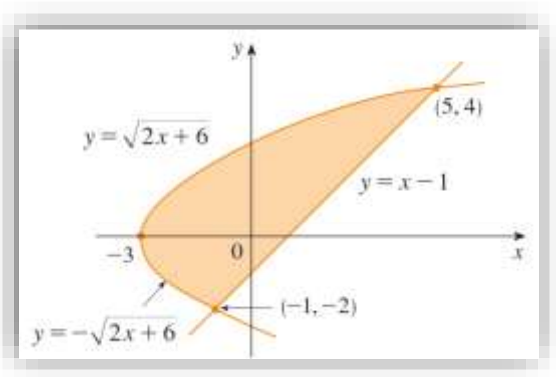
$x = -1 \rightarrow x = 5$



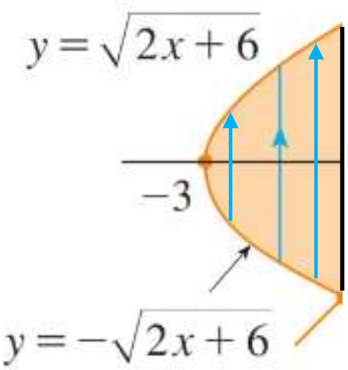


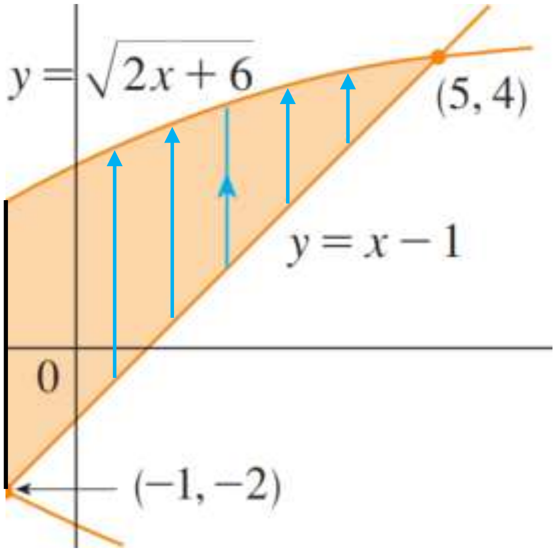
$$\int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} f(x,y)dx\,dy = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y)dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} f(x,y)dy\,dx$$

Example 6: Evaluate $\iint_D xydA$, where D is the shaded region in the figure:



Solution: Curves are written as in Type 1 regions. So we have two regions:



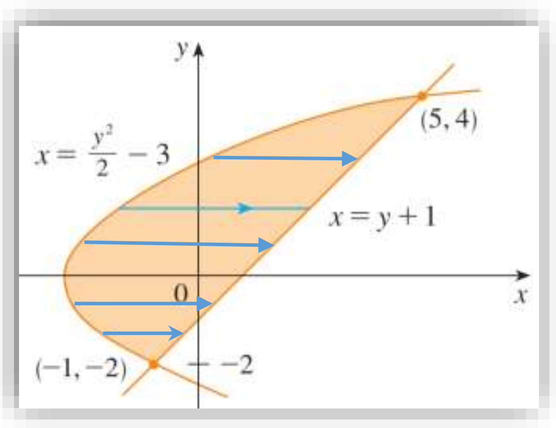


Look to the region as Type 2 regions:
We have one region and the curves are:

$x=\frac{y^2}{2}-3$ and $x=y+1$

⇒ Using Type 2 region is better than using Type 1 regions:

$$\Rightarrow \text{Take } dA=dxdy$$



$$\Rightarrow \iint_D xydA=\int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xy\,dxdy$$

$$=\int_{-2}^{4}\frac{x^2}{2}y\Big]_{\frac{y^2}{2}-3}^{y+1}dy=\frac{1}{2}\int_{-2}^{4}(y+1)^2y-\left(\frac{y^2}{2}-3\right)^2 y\,dy=36$$

Example 7: Evaluate $\iint_D xy dA$, where D is the region bounded by $y = x - 1$ and $y^2 = 2x + 6$

Solution: Curves: $y = x - 1$ and $y^2 = 2x + 6 \Rightarrow x = y + 1$ and $x = \frac{y^2-6}{2} = \frac{y^2}{2} - 3$

$$\Rightarrow \; dA = dxdy \text{ (Type 2 Regions)}$$

Intersection of curves: $y + 1 = \frac{y^2}{2} - 3 \;\Rightarrow y^2 - 2y - 8 = 0 \Rightarrow \; (y-4)(y+2) = 0$

$\Rightarrow y = -2$ or $y = 4 \Rightarrow \; -2 \le y \le 4$

ملاحظة : $\int_{-2}^{4} \int_{\text{منحنى الحد الأدنى}}^{\text{منحنى الحد الأعلى}} xydxdy$

لتحديد المنحنى في الحد الأدنى والمنحنى في الحد الأعلى في حدود التكامل الأول وبدون رسم: بما أن $-2 \le y \le 4$ نأخذ نقطة اختبار في الفترة $[-2,4]$ ولتكن مثلاً $y = 0$ ثم نعوضها في المنحنيين فالذي قيمتة أصغر يكون المنحنى في الحد الأدنى والذي قيمته أكبر يكون المنحنى في الحد الأعلى

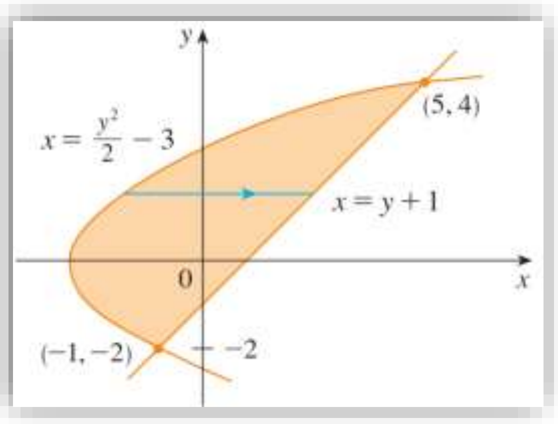
$$\Rightarrow x = y + 1 \Rightarrow x = 0 + 1 = 1$$

$$\Rightarrow \; x = \frac{y^2}{2} - 3 \;\Rightarrow x = \frac{(0)^2}{2} - 3 \;=\; -3$$

$$\Rightarrow \; x = \frac{y^2}{2} - 3 \text{ (lower curve in integral)}$$

$$\Rightarrow \; x = y + 1 \;\text{ (upper curve in integral)}$$

$$\Rightarrow \iint\limits_D xydA = \int_{-2}^{4} \int_{\frac{y^2}{2}-3}^{y+1} xy\, dxdy = \int_{-2}^{4} \frac{x^2}{2} y \Big]_{\frac{y^2}{2}-3}^{y+1} dy = 36$$



$y \in [-2,4] \Rightarrow$ Take $y = 0$:
$x = y + 1 \Rightarrow x = 0 + 1 = 1$
$x = \frac{y^2}{2} - 3 \Rightarrow x = \frac{(0)^2}{2} - 3 \;= -3$
$x = y + 1$ (upper curve)
$x = \frac{y^2}{2} - 3$ (lower curve)

Example 8: Compute $\iint_D (x+2y)dA$, where D is the region enclosed by $y=2x^2$ and $y=1+x^2$.

Solution: Curves: $y=2x^2$ and $y=1+x^2 \Rightarrow dA = dydx$ (Type 1 Regions)

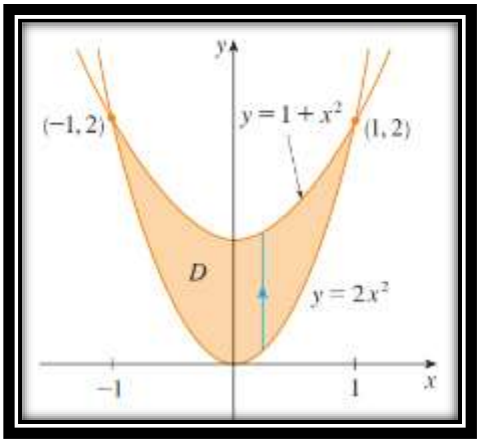
Intersection of curves: $2x^2 = 1+x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow -1 \le x \le 1$

ملاحظة : $\int_{-1}^{1}\int_{\text{منحنى الحد الأدنى}}^{\text{منحنى الحد الأعلى}} (x+2y)dydx$

لتحديد المنحنى في الحد الأدنى والمنحنى في الحد الأعلى في حدود التكامل الأول وبدون رسم:
بما أن $-1 \le x \le 1$ نأخذ نقطة اختبار في الفترة $[-1,1]$ ولتكن مثلاً $x = 0$ ثم نعوضها
في المنحنيين فالذي قيمتة أصغر يكون المنحنى في الحد الأدنى والذي قيمته أكبر يكون المنحنى
في الحد الأعلى

$y = 2x^2 \Rightarrow y = 2(0)^2 = 0$
$y = 1+x^2 \Rightarrow y = 1+(0)^2 = 1$
$\Rightarrow y = 2x^2$ (lower curve in integral)
$\Rightarrow y = 1+x^2$ (upper curve in integral)

$$\Rightarrow \iint_D (x+2y)dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)dy\,dx = \int_{-1}^{1} (xy+y^2)\Big]_{2x^2}^{1+x^2} dx = \frac{32}{15}$$



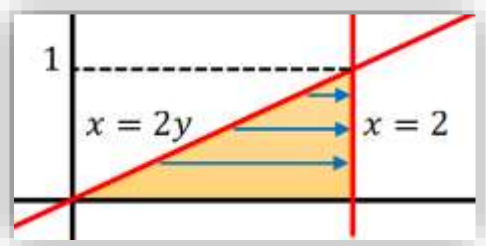
$x \in [-1,1] \Rightarrow$ Take $x = 0$:
$y = 2x^2 \Rightarrow y = 2(0)^2 = 0$
$y = 1+x^2 \Rightarrow y = 1+(0)^2 = 1$
$\Rightarrow y = 2x^2$ (lower curve)
$\Rightarrow y = 1+x^2$ (upper curve)

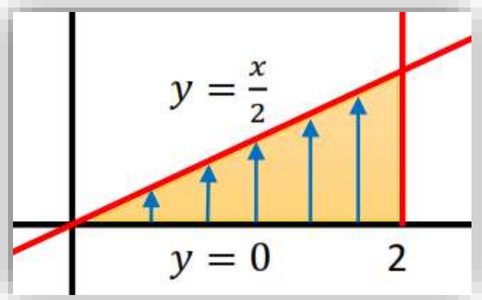
Example 9: Compute the following iterated integrals:

$$(1)\ \int_0^1\int_{2y}^2 e^{x^2}\,dx\,dy \qquad (2)\ \int_0^{\frac{1}{2}}\int_{2x}^1 \sin(y^2)\,dy\,dx$$

$$(3)\ \int_0^4\int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$$

Solution:





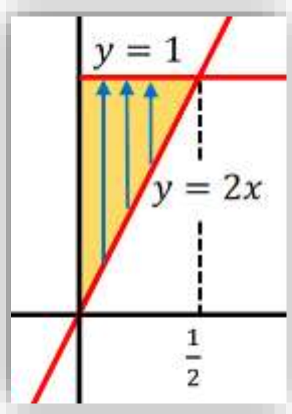
(1) $dA = dxdy \Rightarrow$ We have Type 2 Region:
Curves: $x = 2y \rightarrow x = 2$
$y = 0 \rightarrow y = 1$

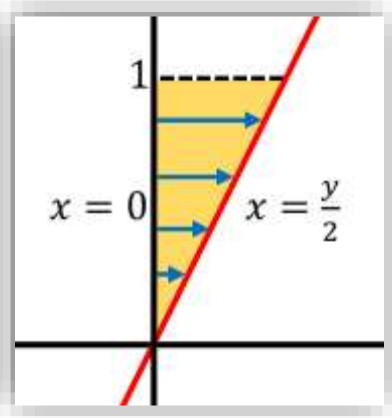
➡ Go to type 1 Regions:
$dA = dydx$
Curves: $y = 0 \rightarrow y = \frac{x}{2}$
$x = 0 \rightarrow x = 2$

$$\int_0^1\int_{2y}^2 e^{x^2}\,dx\,dy = \int_0^2\int_0^{\frac{x}{2}} e^{x^2}\,dy\,dx$$

$$= \int_0^2 \frac{x}{2}e^{x^2}\,dx = \frac{e^{x^2}}{4}\Big]_0^2$$

$$= \frac{e^4-1}{4}$$





(2) $dA = dydx \Rightarrow$ We have Type 1 Region
Curves: $y = 2x \rightarrow y = 1$
$x = 0 \rightarrow x = \frac{1}{2}$

➡ Go to type 2 Regions:
$dA = dxdy$
Curves: $x = 0 \rightarrow x = \frac{y}{2}$
$y = 0 \rightarrow y = 1$

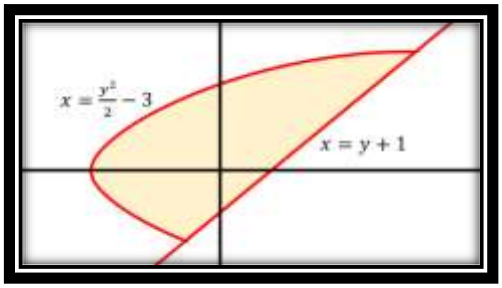
$$\int_0^{\frac{1}{2}}\int_{2x}^1 \sin(y^2)\,dy\,dx = \int_0^1\int_0^{\frac{y}{2}} \sin(y^2)\,dx\,dy$$

$$= \int_0^1 \frac{y}{2}\sin(y^2)\,dy = -\frac{\cos(y^2)}{4}\Big]_0^2 = -\frac{\cos(1)-1}{4} = \frac{1-\cos(1)}{4}$$

(3) Exercise

Rule 10: Let D be a region in the xy-plane. Then the area of D $= \iint_D 1dA$

Example 11: Find the area of the shaded region in the figure:



Solution:

First we find the points of intersections

$$\frac{y^2}{2} - 3 = y + 1 \Rightarrow y^2 - 2y - 8 = 0$$

$$\Rightarrow (y-4)(y+2) = 0$$

$$\Rightarrow y = -2 \text{ or } y = 4$$

$$Area = \iint_D 1dA = \int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} 1\,dx\,dy = \int_{-2}^{4}\left(y + 1 - \frac{y^2}{2} + 3\right)dy = \underbrace{\cdots\cdots\cdots\cdots}_{\text{أكمل الحل}}$$

Example 12: Find the area of the region bounded by the curves $y = e^{2x}, y = 2$, and $x = 4$.

Solution: Observe that the curves are:

$y = e^{2x}, y = 2$

$x = 4 \ \rightarrow \ x = ?????$ (we find it from intersection of curves):

$e^{2x} = 2 \Rightarrow 2x = \ln 2 \Rightarrow x = \frac{\ln 2}{2}$

Ask your self: which is bigger: $x = 4$ or $x = \frac{\ln 2}{2}$ $\left(\frac{\ln 2}{2} \cong 0.34\right)$

$x = \frac{\ln 2}{2} \rightarrow x = 4$

Ask your self: which curve is upper and which is lower:

Take a value of x between $\frac{\ln 2}{2}$ and $4 \Rightarrow$ Take $x = 3$

$$\left.\begin{array}{r} y = e^{2x} \Rightarrow y = e^6 \cong (2.7)^6 \\ y = 2 \Rightarrow y = 2 \end{array}\right\} \Rightarrow \begin{array}{l} y = 2 \text{ (lower)} \\ y = e^{2x} \text{ (upper)} \end{array}$$

$$Area = \iint_D 1dA = \int_{\frac{\ln 2}{2}}^{4}\int_{2}^{e^{2x}} 1\,dy\,dx = \int_{\frac{\ln 2}{2}}^{4}(e^{2x} - 2)dx$$

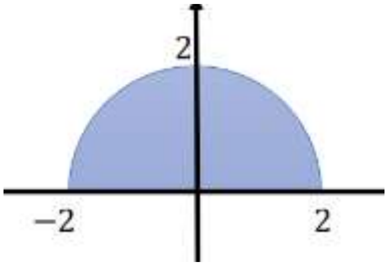
$$= \frac{e^{2x}}{2} - 2x\Big]_{\frac{\ln 2}{2}}^{4} = \underbrace{\cdots\cdots\cdots\cdots}_{\text{أكمل الحل}}$$

Example 12: Find the following:

(1) $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} -3\,dy\,dx$

(2) $\iint_D 2\,dA,$ where $D = \{(x,y): x^2+y^2 \le 9, x \ge 0, y \ge 0\}$

Solution:

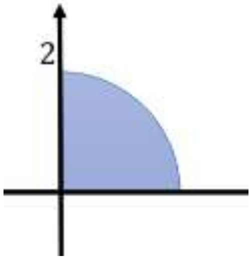


(1) Curves: $y = 0 \ \rightarrow y = \sqrt{4-x^2}$

$x = -2 \ \ \rightarrow \ \ x = 2$

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} -3\,dy\,dx = -3\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 1\,dy\,dx = -3Area = -3\left(\frac{\pi 2^2}{2}\right) = -6\pi$$

(2) $\iint_D 2\,dA = 2\iint_D 1\,dA = 2(\text{Area of } D) = 2\left(\frac{\pi 2^2}{4}\right) = 2\pi$



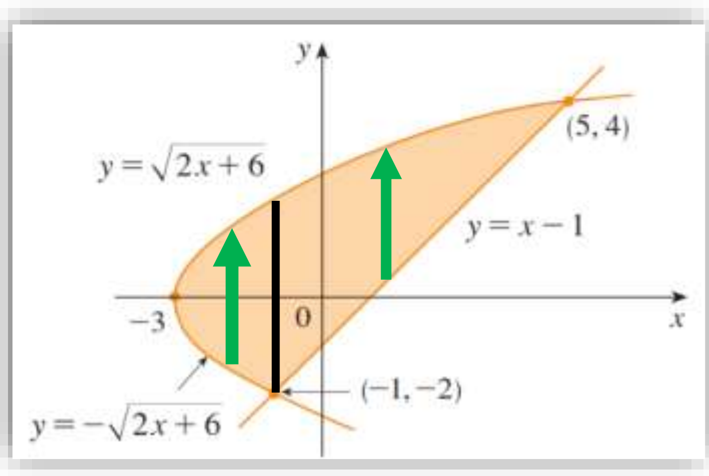
Example 13: Combine the sum of the two double integrals as a single double integral:

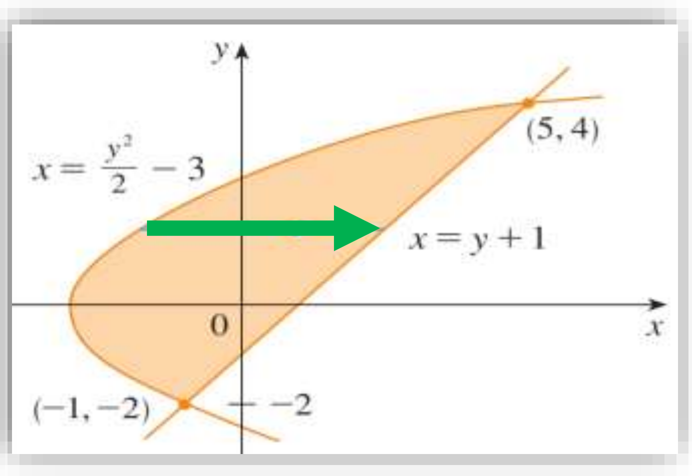
$$\int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y)\,dydx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} f(x,y)\,dydx$$

Solution:

$$\int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y)\,dydx:\ y = -\sqrt{2x+6} \rightarrow y = \sqrt{2x+6}, -3 \le x \le -1$$

$$\int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} f(x,y)\,dydx:\ y = x-1 \rightarrow y = \sqrt{2x+6}, -1 \le x \le 5$$





$$\int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y)\,dydx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} f(x,y)\,dydx = \int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} f(x,y)\,dxdy$$

Estimation of Integrals 14: Let D be a closed bounded region in the xy-plane such that $m \leq f(x,y) \leq M$ for all (x,y) in D. Then

$$m(\text{Area of } D) \leq \iint\limits_D f\,dA \leq M(\text{Area of } D)$$

Example 15: Estimate the value of $\iint\limits_D e^{\sin(x)\cos(y)}dA$, where

D is the region enclosed by the disk $x^2 + y^2 \leq 4$.

Solution: $-1 \leq \sin(x)\cos(y) \leq 1 \quad \Rightarrow e^{-1} \leq e^{\sin(x)\cos(y)} \leq e^{1}$

$$\Rightarrow e^{-1} \times (\text{Area of } D) \leq \iint_D e^{\sin(x)\cos(y)}dA \leq e \times (\text{Area of } D) \cdots\cdots\cdots\cdots (1)$$

$$\text{Area of } D = \pi(2)^2 = 4\pi\text{: } (1) \Rightarrow e^{-1} \times 4\pi \leq \iint_D e^{\sin(x)\cos(y)}dA \leq e \times 4\pi$$

$$\Rightarrow \iint\limits_D e^{\sin(x)\cos(y)}dA \in [\,4\pi e^{-1}, 4\pi e]$$

Example 15: Estimate the value of $\iint\limits_D e^{\sin(x)\cos(x)}dA$, where

D is the region enclosed by the disk $x^2 + y^2 \leq 4$.

Solution: $-1 \leq \sin(2x) \leq 1 \Rightarrow -1 \leq 2\sin(x)\cos(x) \leq 1$

$$\Rightarrow -\frac{1}{2} \leq \sin(x)\cos(x) \leq \frac{1}{2} \Rightarrow e^{-\frac{1}{2}} \leq e^{\sin(x)\cos(x)} \leq e^{\frac{1}{2}}$$

$$\Rightarrow e^{-\frac{1}{2}} \times (\text{Area of } D) \leq \iint\limits_D e^{\sin(x)\cos(x)}dA \leq e^{\frac{1}{2}} \times (\text{Area of } D)$$

$$\Rightarrow e^{-\frac{1}{2}} \times 4\pi \leq \iint\limits_D e^{\sin(x)\cos(x)}dA \leq e^{\frac{1}{2}} \times 4\pi \quad \Rightarrow \iint\limits_D e^{\sin(x)\cos(x)}dA \in \left[\frac{4\pi}{\sqrt{e}}, 4\pi\sqrt{e}\right]$$

15

Multiple Integrals

Section 15.7: Triple Integrals

15.7. Triple Integrals

Fubini's Theorem for Triple integrals 1: If $f(x,y,z)$ is continuous on the rectangular box $B=\{(x,y,z): a\le x\le b, c\le y\le d, r\le z\le s\}=[a,b]\times[c,d]\times[r,s]$, then

$$\underbrace{\iiint\limits_B f\,dV}_{\text{called triple integral}} = \underbrace{\int_a^b\int_c^d\int_r^s f\,dzdydx}_{\text{called iterated integrals}} = \underbrace{\int_r^s\int_c^d\int_a^b f\,dx\,dy\,dz}_{\text{called iterated integrals}} = \underbrace{\int_a^b\int_r^s\int_c^d f\,dy\,dz\,dx}_{\text{called iterated integrals}} = \cdots\cdots$$

Observe that $dV=dzdydx=dzdxdy=dxdydz=dxdzdy=dydxdz=dydzdx$

Example 2: Evaluate $\iiint_B\ xyz^2dV$

where $B=\left\{(x,y,z): \sqrt{2}\le x\le 2, 0\le y\le 4, -1\le z\le 1\right\}$

Solution: Take $dV=dxdydz$

$$\iiint\limits_B xyz^2dV=\int_{-1}^{1}\int_0^4\int_{\sqrt{2}}^{2} xyz^2\,dx\,dy\,dz=\int_{-1}^{1}\int_0^4 \frac{x^2}{2}\Bigg]_{\sqrt{2}}^{2} yz^2\,dy\,dz$$

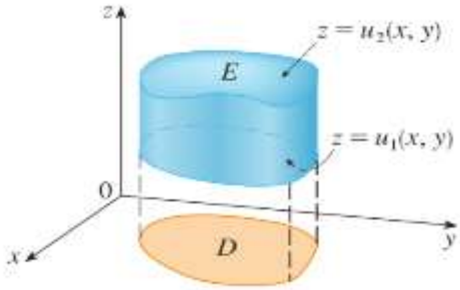
$$=3\int_{-1}^{1}\frac{y^2}{2}\Bigg]_0^4 z^2\,dz=24\int_{-1}^{1}z^2\,dz=24\frac{z^3}{3}\Bigg]_{-1}^{1}=16$$

Observe that this example can be solved faster as:

$$\iiint\limits_B xyz^2dV=\int_{-1}^{1}\int_0^4\int_{\sqrt{2}}^{2} xyz^2\,dx\,dy\,dz=\left(\int_{\sqrt{2}}^{2}x\,dx\right)\left(\int_0^4 y\,dy\right)\left(\int_{-1}^{1}z^2\,dz\right)=\cdots\cdots$$

Triple Integrals for Non-Rectangular Box Regions 3:

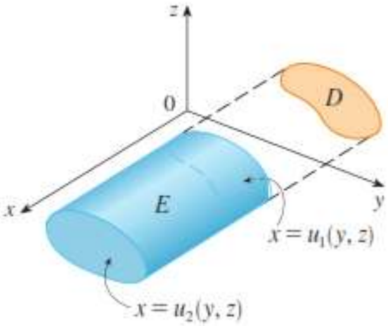
(1) Let E be the solid in $3D$ such that $u_1(x,y) \leq z \leq u_2(x,y)$ and the region D is the projection of S on the xy-plane.



$$\text{Then: } \iiint_E f dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f\, dz \right) dA$$

We may take dA as: $dA = dydx$ or $dA = dxdy$ or $dA = rdrd\theta$

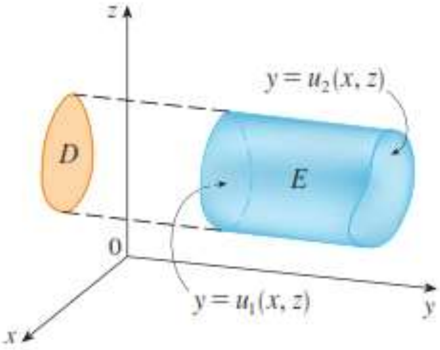
(2) Let E be the solid in $3D$ such that $u_1(y,z) \leq x \leq u_2(y,z)$ and the region D is the projection of S on the yz-plane.



$$\text{Then: } \iiint_E f dV = \iint_D \left(\int_{u_1(y,z)}^{u_2(y,z)} f\, dx \right) dA$$

We may take dA as: $dA = dydz$ or $dA = dzdy$ or $dA = rdrd\theta$

(3) Let E be the solid in $3D$ such that $u_1(x,z) \leq y \leq u_2(x,z)$ and the region D is the projection of S on the xy-plane.



$$\text{Then: } \iiint_E f dV = \iint_D \left(\int_{u_1(x,z)}^{u_2(x,z)} f\, dy \right) dA$$

We may take dA as: $dA = dxdz$ or $dA = dzdx$ or $dA = rdrd\theta$

Example 4: Evaluate $\iiint_E zdV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution: $dV = dzdA$

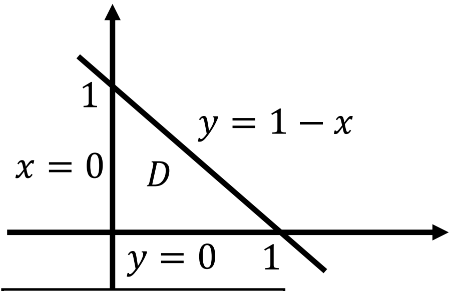
Surfaces: $z = 0$ and $z = 1 - x - y$

Region D: bounded by $x = 0$, $y = 0$ (لاحظ أن الحدود لا تعطي منطقة مغلقة)

&& Add (if possible) Intersection

of surfaces to region D: $1 - x - y = 0$

We have to sketch the region D:



$dA = dydx$
$dV = dzdydx$

Region D: $y = 0$, $y = 1 - x$, $1 - x - y = 0$

$$\iiint_E f\,dV = \iint_D \left(\int_0^{1-x-y} z\,dz \right) dA$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{z^2}{2}\Big]_0^{1-x-y} dy\,dx$$

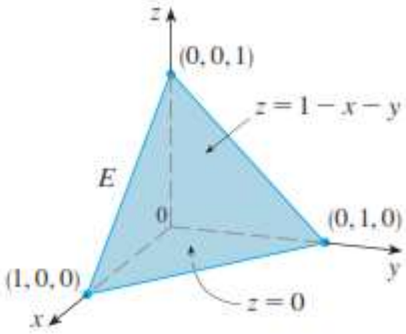
لتحديد السطح في الحد الأدنى والسطح في الحد الأعلى في
حدود التكامل الأول وبدون رسم: نأخذ نقطة اختبار في المنطقة
D ولتكن مثلاً $(0,0)$ ثم نعوضها في معادلتي السطحين
فالذي قيمة z أصغر يكون السطح في الحد الأدنى والذي قيمة
z له أكبر يكون السطح في الحد الأعلى

$z = 0$ at $(x, y) = (0,0) \Rightarrow z = 0$
$z = 1 - x - y$ at $(x, y) = (0,0) \Rightarrow z = 1$

- ❖ $z = 0$ (lower)
- ❖ $z = 1 - x - y$ (upper)

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (1 - x - y)^2\,dy\,dx = \frac{1}{2} \int_0^1 \frac{(1 - x - y)^3}{-3}\Big]_0^{1-x} dx$$

$$= \frac{1}{6} \int_0^1 (1 - x)^3\,dx = \frac{1}{6} \frac{(1 - x)^4}{-4}\Big]_0^1 = \frac{1}{24}$$



Example 5: Find $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Solution: $dV = dy\,dA$

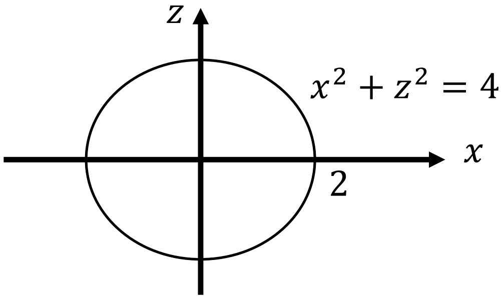
Surfaces: $y = x^2 + z^2$ and $y = 4$

Region D: bounded by … … ….

&& Add (if possible) Intersection

of surfaces to region D: $x^2 + z^2 = 4$

We must sketch the region D: $x^2 + z^2 = 4$



$$\iiint_E f\,dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy \right) dA$$

$$= \int_0^{2\pi} \int_0^{2} \int_{r^2}^{4} r\,dy\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^{2} r^2(4-r^2)\,dr\,d\theta$$

$$= \underbrace{\cdots\cdots\cdots}_{\text{أكمل الحل}} = \frac{128\pi}{15}$$

لتحديد السطح في الحد الأدنى والسطح في الحد الأعلى في حدود التكامل الأول وبدون رسم: نأخذ نقطة اختبار في المنطقة D ولتكن مثلاً $(0,0)$ ثم نعوضها في معادلتي السطحين فالذي قيمة y أصغر يكون السطح في الحد الأدنى والذي قيمة y له أكبر يكون السطح في الحد الأعلى

$y = x^2 + z^2$ at $(x,z) = (0,0) \Rightarrow y = 0$

$y = 4$ at $(x,z) = (0,0) \Rightarrow y = 4$

- ❖ $y = x^2 + z^2$ (lower)
- ❖ $y = 4$ (upper)

Example 6: Let $I = \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$

(1) Express the iterated integral I as a triple integral

(2) Rewrite the iterated integral I in a different order, integrating first with respect to x, then z, and then y.

(3) Rewrite the iterated integral I in a different order, integrating first with respect to y, then x, and then z.

Solution: (1) First we give the boundary equations of the solid E given in the iterated integral I. We do this by one of two ways:

(a) by sketching the solid E

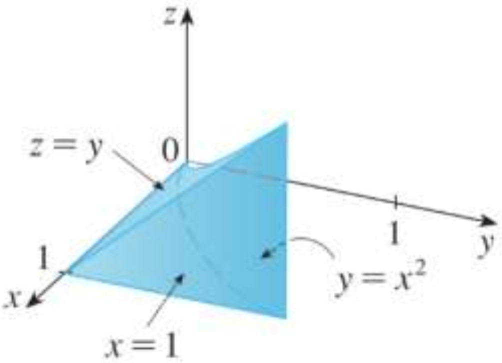
(b) by deleting the equations that results from surface or curve intersections.

$$E\text{quations: } \underbrace{z=0, z=y}_{\substack{\text{Surfaces} \\ \text{لا يلغى أي منهم}}},\ \underbrace{y=0, y=x^2}_{\substack{\text{Curves} \\ \text{ممكن يلتغي بعضهم}}} \&\& \underbrace{x=0,\ x=1}_{\text{ممكن يلتغي بعضهم أو كلهم}}$$

Let E be the solid bounded by:

$z = 0, z = y, y = x^2$, and $x = 1$

The triple integral is: $I = \iiint_E f\,dV$



(2) Let E be the solid bounded by:

$z = 0, z = y, y = x^2$, and $x = 1$

$dV = dxdzdy$ (from question)

$\Rightarrow x =$ for surfaces, & region D in the yz-plane

Observe that $y = x^2 \Rightarrow x = \pm\sqrt{y}$

But $0 \le x \le 1 \Rightarrow x = \sqrt{y}$

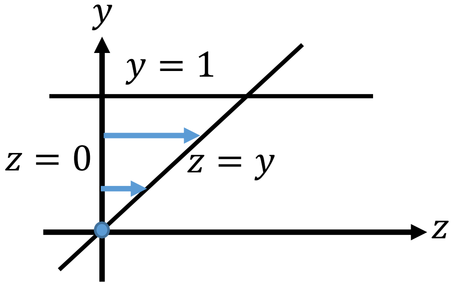
Surfaces: $x = \sqrt{y}$ and $x = 1$

Region D: $z = 0, z = y$ (لاحظ أن الحدود لا تعطي منطقة مغلقة)

&& Add (if possible) Intersection

of surfaces to region ($\sqrt{y} = 1 \Rightarrow y = 1$)

Region D bounded by: $z = 0, z = y$ and $y = 1$.



Lower and upper Surfaces:

$$\begin{Bmatrix} x = \sqrt{y} \text{ at } (0,0) \Rightarrow & x = 0 \\ x = 1 \text{ at } (0,0) \Rightarrow & x = 1 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} x = \sqrt{y} \text{ (lower)} \\ x = 1 \text{ (upper)} \end{Bmatrix}$$

$$I = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$$

(3) Let E be the solid bounded by: $z = 0, z = y, y = x^2$, and $x = 1$

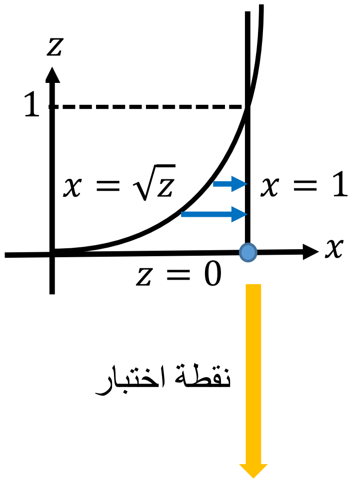
$dV = dydxdz$ (from question) ⇒ $y =$ for surfaces & region D in the xz-plane

Surfaces: $y = z$ and $y = x^2$

Region D: $z = 0, x = 1$ (لاحظ أن الحدود لا تعطي منطقة مغلقة)

Add (if possible) Intersection of surfaces to region ($z = x^2$)

Region D bounded by: $z = 0, x = 1$ and $z = x^2$.



نقطة اختبار

Lower and upper Surfaces:

$$\left\{\begin{matrix} y = z \text{ at } (x,z) = (1,0) \ \Rightarrow \ \ y = 0 \\ y = x^2 \text{ at } (x,z) = (1,0) \Rightarrow \ \ y = 1 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} y = z \text{ (lower)} \\ y = x^2 \text{ (upper)} \end{matrix}\right\}$$

$$I = \int_{0}^{1} \int_{\sqrt{z}}^{1} \int_{z}^{x^2} f(x,y,z)\, dy\, dx\, dz$$

Example 7: Rewrite the iterated integral $\int_0^4 \int_{\frac{x}{2}}^{\sqrt{x}} \int_0^{2-y} f(x,y,z)\,dz\,dy\,dx$ in a different order, integrating first with respect to x, then y, and then z.

Solution: Solid E: $\underbrace{z=0, z=2-y}_{\text{لا يلغى أي منهم}}$, $\underbrace{y=\frac{x}{2}, y=\sqrt{x}}_{\text{ممكن يلتغي بعضهم}}$ && $\underbrace{x=0, x=4}_{\text{ممكن يلتغي بعضهم أو كلهم}}$

We delete equations of intersections:

- $z=0, z=2-y \Rightarrow 2-y=0 \;\; \Rightarrow y=2$ (is not a boundary of E)
- $y=\frac{x}{2}, y=\sqrt{x} \Rightarrow \frac{x}{2}=\sqrt{x} \;\Rightarrow x=2\sqrt{x} \;\Rightarrow x^2=4x \;\Rightarrow x^2-4x=0$
  $\Rightarrow x(4-x)=0$
  $\Rightarrow x=0, x=4$ (are not boundaries of E so, should be deleted).

The solid E is bounded by $z=0, z=2-y, y=\frac{x}{2}, y=\sqrt{x}$

$dV = dxdydz$: $x =$ for surfaces & & region D in the yz-plane

Surfaces: $x=2y, x=y^2$

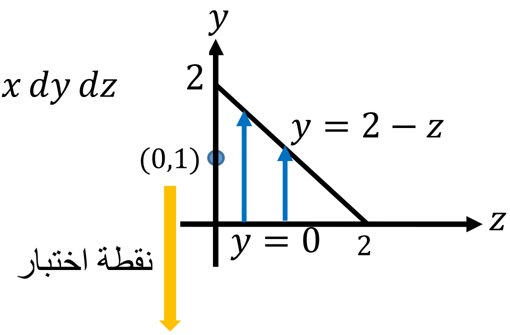
Region D:

$z=0, z=2-y$ (لاحظ أن الحدود لا تعطي منطقة مغلقة)

Add (if possible) Intersection of surfaces to region $\Rightarrow y^2=2y \Rightarrow y=0, \underbrace{y=2}_{\text{not a boundary curve of } D}$)

$\Rightarrow$ Region D: $z=0, z=2-y, y=0.$

$$\int_0^4 \int_{\frac{x}{2}}^{\sqrt{x}} \int_0^{2-y} f\,dz\,dy\,dx = \int_0^2 \int_0^{2-z} \int_{y^2}^{2y} f\,dx\,dy\,dz$$



Lower and upper Surfaces:

$$\left\{\begin{matrix} x=2y \text{ at } (z,y)=(0,1) \;\Rightarrow\; x=2 \\ x=y^2 \text{ at } (z,y)=(0,1) \Rightarrow\; y=1 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x=2y \text{ (upper)} \\ x=y^2 \text{ (lower)} \end{matrix}\right\}$$

Example 8: Express the iterated integral $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f\,dy\,dz\,dx$ in a different order, integrating first with respect to z, then y, and then x.

Solution: Solid E: $\underbrace{y=0, y=1-x}_{\text{لا يلغي أي منهم}}$, $\underbrace{z=0, z=1-x^2}_{\text{ممكن يلتقي بعضهم}}$ && $\underbrace{x=0, x=1}_{\text{ممكن يلتقي بعضهم أو كلهم}}$

We delete equations of intersections:

- $y=0, y=1-x \Rightarrow 1-x=0 \;\Rightarrow\; x=1$ (is not a boundary of E)
- $z=0, z=1-x^2 \Rightarrow 1-x^2=0 \;\Rightarrow x=\pm 1$ (are not boundaries of E)

The solid E is bounded by: $y=0, y=1-x, z=0, z=1-x^2, x=0$

$dV = dzdydx$: $z=$ for surfaces & & region D in the xy-plane

Surfaces: $z=0, z=1-x^2$

Region D:

$y=0, y=1-x, x=0$ (لاحظ أن الحدود تعطي منطقة مغلقة)

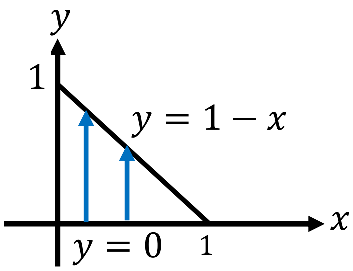
لا داعي لتقاطع السطوح

$\Rightarrow$ Region D: $y=0, y=1-x, x=0$

Lower and upper Surfaces:

$$\left\{\begin{matrix} z=0 \text{ at } (x,y)=(0,0) \Rightarrow \;\; z=0 \\ z=1-x^2 \text{ at } (x,y)=(0,0) \Rightarrow \;\; z=1 \end{matrix}\right\}$$

$$\Rightarrow \left\{\begin{matrix} z=0 \text{ (upper)} \\ z=1-x^2 \text{ (lower)} \end{matrix}\right\}$$



$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f\,dy\,dz\,dx = \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f\,dz\,dy\,dx$$

Exercise 9:

(a) Let $\int_0^{\sqrt{\pi}} \int_0^{x} \int_0^{xz} f\, dy\, dz\, dx$.

(1) Express the iterated integral I as a triple integral

(2) Rewrite the iterated integral I in a different order, integrating first with respect to z, then x, and then y.

(b) Express the iterated integral $\int_0^1 \int_{\sqrt{x}}^{1} \int_0^{1-y} f\, dz\, dy\, dx$ in different order:

(1) First integrate with respect to x, then y, then z

(2) First integrate with respect to y, then x, then z

Volumes 10: Let E be the solid in $3D$ such that $u_1(x,y) \leq z \leq u_2(x,y)$ and the region D is the projection of S on the xy-plane.

(1) The volume of the solid E can expressed as a triple integral as:

$$V = \iiint_E 1\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} 1\, dz \right) dA$$

(2) The volume of the solid E can expressed as a duble integral as:

$$V = \iint_D (u_2(x,y) - u_1(x,y))\, dA$$

Example 11:

(1) Use triple integral to find the volume of the solid enclosed by the parabolic cylinder $y = z^2$ and the planes $y = x, z = 2 - x$.

(2) Use double integral to find the volume of the solid enclosed by the parabolic cylinder $y = z^2$ and the planes $y = x, z = 0, z = 2 - x$.

Solution:

Surfaces: $y = z^2, y = x$

$$\Rightarrow \; dV = dy dA$$

Region D: $z = 2 - x$

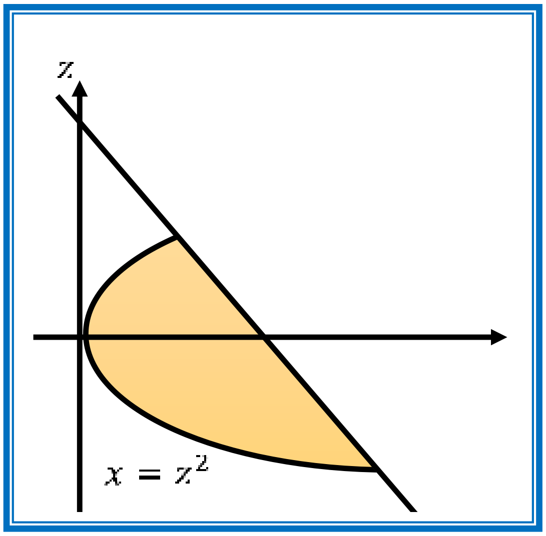
لاحظ أن الحدود لا تعطي منطقة مغلقة

علينا أن نقاطع السطوح:

Surfaces: $y = z^2, y = x \;\Rightarrow\; z^2 = x$

Region D: $z = 2 - x, z^2 = x$

$$dA = dxdz$$



Intersection of curves: $(x = 2 - z, x = z^2)$

$$z^2 = 2 - z \;\Rightarrow z^2 + z - 2 = 0$$

$$\Rightarrow (z + 2)(z - 1) = 0 \Rightarrow z = -2 \;\rightarrow z = 1$$

Lower and upper Surfaces:

$$\begin{Bmatrix} y = z^2 \text{ at } (x, z) = (2,0) \Rightarrow \; y = 0 \\ y = x \text{ at } (x, z) = (2,0) \Rightarrow \; y = 2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} y = z^2 \text{ (lower)} \\ y = x \text{ (upper)} \end{Bmatrix}$$

$$(1)\; V = \iiint_E 1\, dV = \iint_D \left(\int_{z^2}^{x} 1\, dy \right) dA$$

$$= \int_{-2}^{1} \int_{z^2}^{2-z} \int_{z^2}^{x} 1\, dy\, dx\, dz = \int_{-2}^{1} \int_{z^2}^{2-z} (x - z^2) dx\, dz = \underbrace{\cdots\cdots\cdots\cdots}_{\text{أكمل الحل}}$$

$$(2)\; V = \iint_D (x - z^2)\, dA = \int_{-2}^{1} \int_{z^2}^{2-z} (x - z^2) dx\, dz = \underbrace{\cdots\cdots\cdots\cdots}_{\text{أكمل الحل}}$$

Example 12: Use triple integral to find the volume of the tetrahedron T enclosed by the planes $x + 2y + z = 2, x = 2y, x = 0, z = 0$.

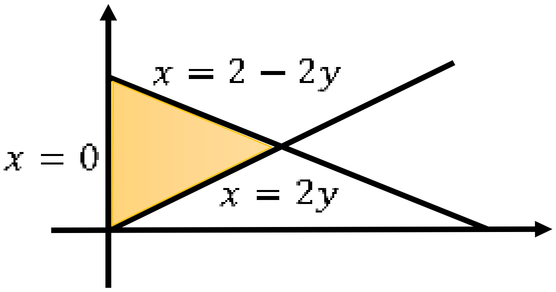
Solution: $dV = dzdA$

Surfaces: $z = 2 - x - 2y$, $z = 0$.

Region D: $x = 2y$, $x = 0$ (لاحظ أن الحدود لا تعطي منطقة مغلقة ⇐ يجب ان نقاطع السطوح)

Intersection of surfaces: $2 - x - 2y = 0 \Rightarrow x = 2 - 2y$

Region D: : $x = 2y$, $x = 0$, $x = 2 - 2y$.



$dA = dydx$

Intersection of lines: $2y = 2 - 2y$

$\Rightarrow y = \frac{1}{2} \Rightarrow x = 2\left(\frac{1}{2}\right) = 1$

$$\text{Volume} = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} \int_0^{2-x-2y} 1\, dz\, dy\, dx$$

$$= \underbrace{\cdots\cdots}_{\text{أكمل الحل}}$$

Example 13: Use double integral to find the volume of the tetrahedron T enclosed by the planes $x + 2y + z = 2, x = 2y, x = 0, z = 0$.

Solution: …………..

$$\text{Volume} = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} (2 - x - 2y - 0)\, dy\, dx = \underbrace{\cdots\cdots}_{\text{أكمل الحل}}$$

Example 14: Set up (do not evaluate) as a double integral the volume of the solid bounded by $z = 1$ and the paraboloid $z = 5 - x^2 - y^2$.

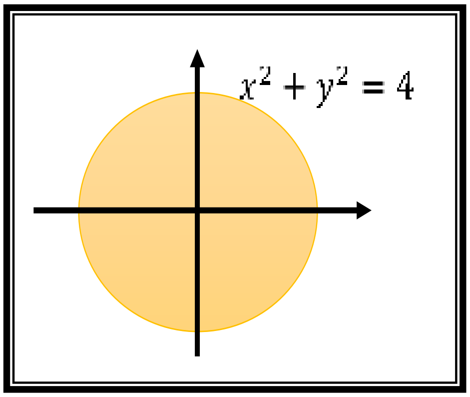
Solution:

Surfaces: $z = 1, z = 5 - x^2 - y^2$

Region D: !!!!!!!!! &&& Intersections of surfaces $\Rightarrow 5 - x^2 - y^2 = 1 \Rightarrow x^2 + y^2 = 4$

$$\text{Volume} = \iint_D (5 - x^2 - y^2 - 1)\, dA = \iint_D \left(4 - (x^2 + y^2)\right) dA$$

$$= \int_0^{2\pi} \int_0^2 (5 - r^2) r dr d\theta$$



Example 15: Set up (do not evaluate) as a triple integral the volume of the solid lies under the cone $z = \sqrt{x^2+y^2}$ and above the xy-plane and inside the cylinder $x^2 + y^2 = -6y$.

Solution: The solid is bounded by $z = \sqrt{x^2+y^2}, z = 0, x^2+y^2 = -6x$.

Surfaces: $z = \sqrt{x^2+y^2}, z = 0$

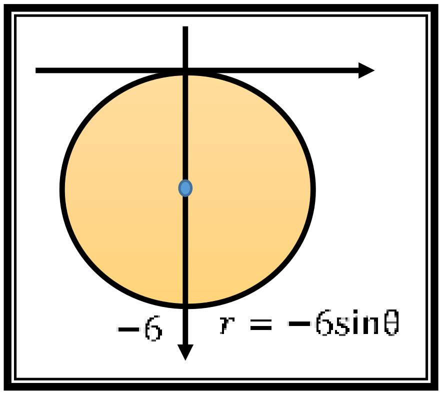
Region D: $x^2 + y^2 = -6y$ (لاحظ أن المنحنيات تعطي منطقة مغلقة ⇐ لا داعي لتقاطع السطوح)

Sketch the region D: $x^2 + y^2 = -6y \Rightarrow x^2 + \underbrace{y^2 + 6y}_{\text{إكمال مربع}} = 0$

$\Rightarrow x^2 + \underbrace{y^2 + 6y + 9}_{\text{إكمال مربع}} = \underbrace{\quad 9 \quad}_{\text{بسبب إكمال المربع}} \Rightarrow \Rightarrow x^2 + (y+3)^2 = 9$

$$\text{Volume} = \iiint_E 1 dV = \iint \int_0^{\sqrt{x^2+y^2}} 1\, dz\, dA$$

$$= \int_{-\pi}^{0} \int_0^{-6\sin\theta} \int_0^{r} r\, dz\, dr\, d\theta$$



Example 16:

(1) Use triple integral to find the volume of the solid enclosed by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0, x = 1$.

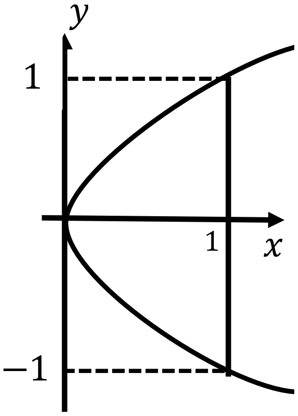
(2) Use double integral to find the volume of the solid enclosed by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0, x = 1$.

Solution: $dV = dzdA$

Surfaces: $z = x$, $z = 0$.

Region D: $x = y^2$, $x = 1$ (لاحظ أن المنحنيات تعطي منطقة مغلقة ⇐ لا داعي لتقاطع السطوح)

Intersection of curves: $y^2 = 1 \Rightarrow y = -1$ or $y = 1$



(1) Volume $= \iiint_E 1dV = \int_{-1}^{1}\int_{y^2}^{1}\int_0^x 1\, dz\, dx\, dy$

$$= \int_{-1}^{1}\int_{y^2}^{1}(x-0)\, dx\, dy = \int_{-1}^{1}\frac{x^2}{2}\Big]_{y^2}^{1} dy = \frac{1}{2}\int_{-1}^{1}(1-y^4)dy$$

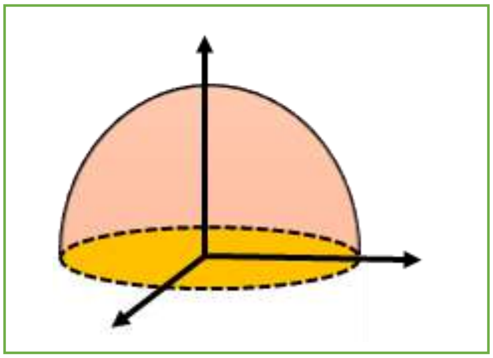
$$= \frac{4}{5}$$

(2) Volume $= \iint_D (x-0)dA = \int_{-1}^{1}\int_{y^2}^{1}(x-0)\, dx\, dy = \cdots = \frac{4}{5}$

Example 16: Compute $\iiint_E -12dV$, where $E = \{(x,y,z): x^2 + y^2 + z^2 \leq 9, z \geq 0\}$

Solution: $\iiint_E -12dV = -12\iiint_E 1dV = -12 \times \text{Volume of } E$

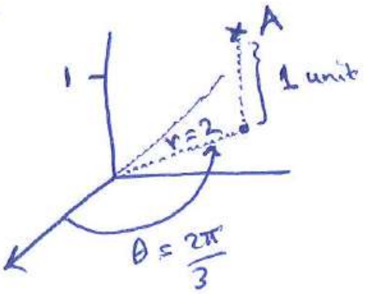
$$= -12\,\frac{1}{2}\,\frac{4}{3}\pi(3)^3 = -12(9)(2) = -216\pi$$



Section 15.8 : Triple Integrals in Cylindrical Coordinates

Let $A(x,y,z)$ be a pt in rectangular (Cartesian) coordinates. The cylindrical coordinates of A are $A(r,\theta,z)$ where
$r=\sqrt{x^2+y^2}$, $\tan\theta=\frac{y}{x} \leftrightarrow x=r\cos\theta$, $y=r\sin\theta$

Ex 1: Plot the pt with cylindrical coordinates $A(2,\frac{2\pi}{3},1)$ and find its rectangular coordinates.



Sol: $r=2$, $\theta=\frac{2\pi}{3}$, $z=1$

$x=r\cos\theta=2\cos\frac{2\pi}{3}=2(-\cos\frac{\pi}{3})=-2(\frac{1}{2})=-1$

$y=r\sin\theta=2\sin\frac{2\pi}{3}=2(+\sin\frac{\pi}{3})=2\frac{\sqrt{3}}{2}=\sqrt{3}$

$z=1$

$\therefore A(-1,\sqrt{3},1)$.

Ex 2: Find the cylindrical coordinates of the pts. whose Cartesian coordinates are: $A(3,-3,-7)$, $B(-3,3,7)$, $C(-3,-3,1)$
$D(2,0,-1)$, $E(-2,0,1)$, $F(0,2,-1)$, $G(0,-2,1)$

Sol: A: $x=3$, $y=-3 \Rightarrow$ pt. in 4th quadrant ربع رابع

$\tan\theta=\frac{y}{x}=\boxed{-}\boxed{1}$ → ربع رابع
→ زاوية مرجع → $\frac{\pi}{4}$

$\therefore \theta=2\pi-\frac{\pi}{4}=7\frac{\pi}{4}$ او $\theta=\frac{-\pi}{4}$

$\therefore r=\sqrt{x^2+y^2}=\sqrt{18}=3\sqrt{2}$

$\therefore A(3\sqrt{2},\frac{7\pi}{4},-7)$ or $(3\sqrt{2},\frac{-\pi}{4},-7)$

B: $x=-3$, $y=3 \longrightarrow$ pt. in 2nd quadrant

$\left.\begin{array}{l}\tan\theta=-1 \longrightarrow \theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4} \\ r=\sqrt{x^2+y^2}=3\sqrt{2}\end{array}\right] B(3\sqrt{2},\frac{3\pi}{4},7)$

C: $x=-3$, $y=-3 \Rightarrow$ pt. in 3rd quadrant
$r=3\sqrt{2}$, $\tan\theta=1 \Rightarrow \theta=\pi+\frac{\pi}{4}=5\frac{\pi}{4}$
$\therefore C(3\sqrt{2},\frac{5\pi}{4},1)$.

D: $x = 2 \quad y = 0 \rightarrow$ pt. on +ve x-axis $\rightarrow \theta = 0$

$r = \sqrt{x^2+y^2} = 2 \rightarrow D(2, 0, -1)$

E: $x = -2, \; y = 0 \rightarrow$ pt. on the -ve x-axis $\rightarrow \theta = \pi$

$r = \sqrt{x^2+y^2} = 2 \rightarrow E(2, \pi, 1)$

F: $x = 0, \; y = 2 \rightarrow$ pt. on +ve y-axis $\rightarrow \theta = \frac{\pi}{2}$

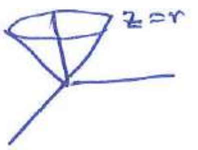
$r = \sqrt{x^2+y^2} = 2 \rightarrow F(2, \frac{\pi}{2}, -1)$

G: $x = 0, \; y = -2 \rightarrow$ pt. on -ve - yaxis $\rightarrow \theta = \frac{3\pi}{2}$

$r = \sqrt{x^2+y^2} = 2 \rightarrow G(2, \frac{3\pi}{2}, 1)$.

Ex3 Describe and sketch whose eq. in cylindrical coordinate is

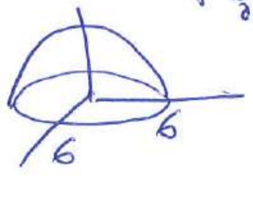
(1) $z = r$ (2) $z = -r$ (3) $z = \sqrt{6^{\cancel{2}} - r^2}$.



Sol: (1) $z = r \rightarrow z = \sqrt{x^2+y^2}$ cone

(2) $z = -r \rightarrow z = -\sqrt{x^2+y^2}$ cone

(3) $z = \sqrt{6 - r^2} \rightarrow r^2 + z^2 = 6 \rightarrow x^2+y^2+z^2 = 6$ hemi sphere نصف كرة



Ex4: Write the eq. $z = x^2 - y^2$ in cylindrical coordinates.

Sol: $z = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$

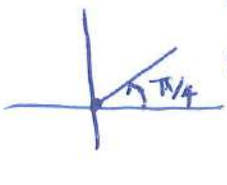
Ex 5: Identify (Give the name) the surface and write its eq. in Cartesian coordinates.

(1) $r = 5$ (2) $z = 4 - r^2$ (3) $\theta = \frac{\pi}{4}$ (4) $\theta = 0$

(5) $\theta = \frac{\pi}{2}$, (6) $\theta = \pi$ (7) $\theta = \frac{2\pi}{3}$

Sol: (1) $r^2 = 25 \rightarrow x^2+y^2 = 25$ cylinder

(2) $z = 4 - r^2 \rightarrow z = 4 - (x^2+y^2)$ paraboloide



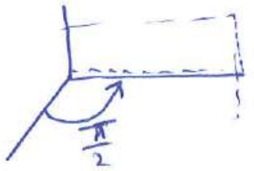
(3) $\tan\theta = \tan\frac{\pi}{4} \Rightarrow \frac{y}{x} = 1 \rightarrow y = x$ plane انتبه جواب خطأ

$y = x, \; x \geq 0$ half plane

(4) $\tan\theta = \tan 0 \Rightarrow \frac{y}{x} = 0 \to y = 0 \to xz\text{-plane}$ انتبه جواب خطأ

$y = 0,\ x \geq 0 \Rightarrow$ half xz-plane (with $x \geq 0$) $\Rightarrow y = 0, x \geq 0$.

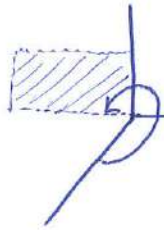
(5) $\tan\theta = \tan\frac{\pi}{2} \Rightarrow \frac{y}{x} = \frac{1}{0}$??? نحاول نفهم السطح بالرسم



$\Rightarrow \underbrace{\text{half } yz\text{-plane (with } y \geq 0)}_{x = 0,\ y \geq 0}$

(6) $\tan\theta = \tan\pi \Rightarrow \frac{y}{x} = 0 \to y = 0,\ x \leq 0$. half xz-plane with $x \leq 0$

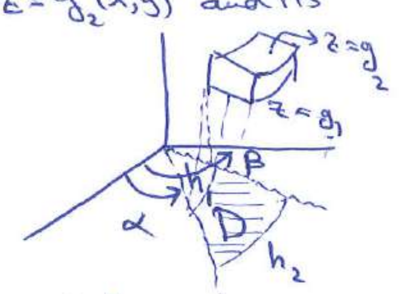
(7) $\tan\theta = \overset{\tan}{\wedge}\frac{2\pi}{3} \to \frac{y}{x} = \frac{-1}{0}$??? نحاول نفهم السطح بالرسم



$\Rightarrow$ half yz-plane with $y \leq 0$

$= x = 0,\ y \leq 0$.

Rule 6: Let $f(x,y,z)$ be conts. on the Solid S' in $\mathbb{R}^3$ whose lower ~~upper~~ surface is $z = g_1(x,y)$ and upper surface $z = g_2(x,y)$ and its projection on the xy-plane is the region:



$$D = \{(r,\theta) : h_1(\theta) \leq r \leq h_2(\theta),\ \alpha \leq \theta \leq \beta\}$$

where $0 \leq \beta - \alpha < 2\pi$, then

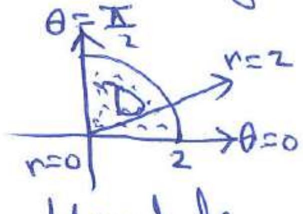
$$\iiint_{S'} f\, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r\cos\theta,\, r\sin\theta)}^{g_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, dz\, r\, dr\, d\theta$$

Ex7: Evaluate $I = \iiint_E (x+y+z)\, dV$ where E is the solid in the first octant that lies under the paraboloid $z = 12 - 3x^2 - 3y^2$

sol: Surfaces: $z = 12 - 3x^2 - 3y^2$, $z = 0$

region D: $x = 0$, $y = 0$ and $12 - 3x^2 - 3y^2 = 0$

$x^2 + y^2 = 4$



$dA = r\,dr\,d\theta$

$$I = \iint_D \int_0^{12-3x^2-3y^2} (x+y+z)\, dz\, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{12-3r^2} (r\cos\theta + r\sin\theta)\, r\, dz\, dr\, d\theta = \cdots$$

Ex 8: Use triple integrals to find the volume of the solid S that lies within the cylinder $x^2+y^2=1$ below the plane $z=1$ and above the paraboloide $z=1-x^2-y^2$.

Sol: Surfaces: $z=1$, $z=1-x^2-y^2$

region D: $x^2+y^2=1$ and ~~$1-x^2-y^2=1$~~ $\rightarrow$ $x=0$, $y=0$ pt.

$dA = r\,dr\,d\theta$

$$V=\iiint_S 1\,dV=\iint_D\int_{1-x^2-y^2}^{1} 1\,dz\,dA$$

$$=\int_0^{2\pi}\int_0^1\int_{1-r^2}^{1} r\,dz\,dr\,d\theta = \dots$$

Ex 9: Evaluate: (1) $I_1=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$

(2) $I_2=\int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{0}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$

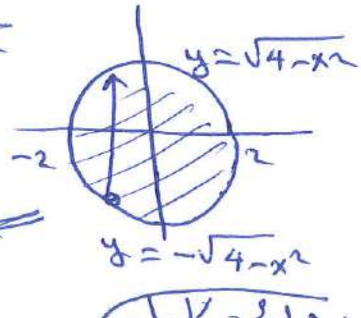
Sol:

لاحظ أنّ تكاملات المنطقة (التكامل 2 والثالث) فيها:

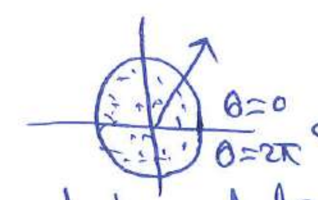
$y=-\sqrt{4-x^2}$ وهذه معادلة دائرة وبالتالي افضل وضع لحل السؤال التحويل الى polar لكن في 3D يسمى cylindrical :

(1) region D: $y=-\sqrt{4-x^2} \rightarrow y=\sqrt{4-x^2}$

$-2 \le x \le 2$



$\Longleftarrow$



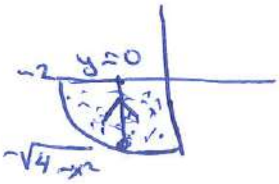
$dy\,dx = r\,dr\,d\theta$

$$I_1=\int_0^{2\pi}\int_0^2\int_r^2 r^3\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2 r^3(2-r)\,dr\,d\theta = \frac{16\pi}{5}$$

(2) region D: $y=-\sqrt{4-x^2} \rightarrow y=0$

$-2 \le x \le 0$



$$I_2=\int_{\pi}^{3\pi/2}\int_0^2\int_r^2 r^3\,dz\,dr\,d\theta = \dots$$