CHAPTER 1

1.1. Given the vectors $\mathbf{M} = -10\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$ and $\mathbf{N} = 8\mathbf{a}_x + 7\mathbf{a}_y - 2\mathbf{a}_z$, find:

a) a unit vector in the direction of $-\mathbf{M} + 2\mathbf{N}$.

$-\mathbf{M} + 2\mathbf{N} = 10\mathbf{a}_x - 4\mathbf{a}_y + 8\mathbf{a}_z + 16\mathbf{a}_x + 14\mathbf{a}_y - 4\mathbf{a}_z = (26, 10, 4)$

Thus

$$\mathbf{a} = \frac{(26,10,4)}{|(26,10,4)|} = \underline{(0.92, 0.36, 0.14)}$$

b) the magnitude of $5\mathbf{a}_x + \mathbf{N} - 3\mathbf{M}$:

$(5,0,0) + (8,7,-2) - (-30,12,-24) = (43,-5,22)$, and $|(43,-5,22)| = \underline{48.6}$.

c) $|\mathbf{M}||2\mathbf{N}|(\mathbf{M}+\mathbf{N})$:

$|(-10,4,-8)||(16,14,-4)|(-2,11,-10) = (13.4)(21.6)(-2,11,-10)$

$= \underline{(-580.5, 3193, -2902)}$

1.2. Vector $\mathbf{A}$ extends from the origin to (1,2,3) and vector $\mathbf{B}$ from the origin to (2,3,-2).

a) Find the unit vector in the direction of $(\mathbf{A} - \mathbf{B})$: First

$$\mathbf{A} - \mathbf{B} = (\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z) - (2\mathbf{a}_x + 3\mathbf{a}_y - 2\mathbf{a}_z) = (-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)$$

whose magnitude is $|\mathbf{A} - \mathbf{B}| = [(-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)\cdot(-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)]^{1/2} = \sqrt{1+1+25} = 3\sqrt{3} = 5.20$. The unit vector is therefore

$$\mathbf{a}_{AB} = \underline{(-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)/5.20}$$

b) find the unit vector in the direction of the line extending from the origin to the midpoint of the line joining the ends of $\mathbf{A}$ and $\mathbf{B}$:

The midpoint is located at

$$P_{mp} = [1 + (2-1)/2,\ 2 + (3-2)/2,\ 3 + (-2-3)/2)] = (1.5, 2.5, 0.5)$$

The unit vector is then

$$\mathbf{a}_{mp} = \frac{(1.5\mathbf{a}_x + 2.5\mathbf{a}_y + 0.5\mathbf{a}_z)}{\sqrt{(1.5)^2 + (2.5)^2 + (0.5)^2}} = \underline{(1.5\mathbf{a}_x + 2.5\mathbf{a}_y + 0.5\mathbf{a}_z)/2.96}$$

1.3. The vector from the origin to the point A is given as $(6,-2,-4)$, and the unit vector directed from the origin toward point B is $(2,-2,1)/3$. If points A and B are ten units apart, find the coordinates of point B.

With $\mathbf{A} = (6,-2,-4)$ and $\mathbf{B} = \frac{1}{3}B(2,-2,1)$, we use the fact that $|\mathbf{B} - \mathbf{A}| = 10$, or
$|(6 - \frac{2}{3}B)\mathbf{a}_x - (2 - \frac{2}{3}B)\mathbf{a}_y - (4 + \frac{1}{3}B)\mathbf{a}_z| = 10$
Expanding, obtain
$36 - 8B + \frac{4}{9}B^2 + 4 - \frac{8}{3}B + \frac{4}{9}B^2 + 16 + \frac{8}{3}B + \frac{1}{9}B^2 = 100$
or $B^2 - 8B - 44 = 0$. Thus $B = \frac{8 \pm \sqrt{64-176}}{2} = 11.75$ (taking positive option) and so

$$\mathbf{B} = \frac{2}{3}(11.75)\mathbf{a}_x - \frac{2}{3}(11.75)\mathbf{a}_y + \frac{1}{3}(11.75)\mathbf{a}_z = \underline{7.83\mathbf{a}_x - 7.83\mathbf{a}_y + 3.92\mathbf{a}_z}$$

1.4. A circle, centered at the origin with a radius of 2 units, lies in the xy plane. Determine the unit vector in rectangular components that lies in the xy plane, is tangent to the circle at $(-\sqrt{3}, 1, 0)$, and is in the general direction of increasing values of y:

A unit vector tangent to this circle in the general increasing y direction is $\mathbf{t} = -\mathbf{a}_\phi$. Its x and y components are $\mathbf{t}_x = -\mathbf{a}_\phi \cdot \mathbf{a}_x = \sin\phi$, and $\mathbf{t}_y = -\mathbf{a}_\phi \cdot \mathbf{a}_y = -\cos\phi$. At the point $(-\sqrt{3}, 1)$, $\phi = 150°$, and so $\mathbf{t} = \sin 150°\mathbf{a}_x - \cos 150°\mathbf{a}_y = \underline{0.5(\mathbf{a}_x + \sqrt{3}\mathbf{a}_y)}$.

1.5. A vector field is specified as $\mathbf{G} = 24xy\mathbf{a}_x + 12(x^2 + 2)\mathbf{a}_y + 18z^2\mathbf{a}_z$. Given two points, $P(1, 2, -1)$ and $Q(-2, 1, 3)$, find:

a) $\mathbf{G}$ at P: $\mathbf{G}(1, 2, -1) = \underline{(48, 36, 18)}$

b) a unit vector in the direction of $\mathbf{G}$ at Q: $\mathbf{G}(-2, 1, 3) = (-48, 72, 162)$, so

$$\mathbf{a}_G = \frac{(-48, 72, 162)}{|(-48, 72, 162)|} = \underline{(-0.26, 0.39, 0.88)}$$

c) a unit vector directed from Q toward P:

$$\mathbf{a}_{QP} = \frac{\mathbf{P} - \mathbf{Q}}{|\mathbf{P} - \mathbf{Q}|} = \frac{(3, -1, 4)}{\sqrt{26}} = \underline{(0.59, 0.20, -0.78)}$$

d) the equation of the surface on which $|\mathbf{G}| = 60$: We write $60 = |(24xy, 12(x^2 + 2), 18z^2)|$, or $10 = |(4xy, 2x^2 + 4, 3z^2)|$, so the equation is

$$\underline{100 = 16x^2y^2 + 4x^4 + 16x^2 + 16 + 9z^4}$$

1.6. Find the acute angle between the two vectors $\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ by using the definition of:

a) the dot product: First, $\mathbf{A} \cdot \mathbf{B} = 2 - 3 + 6 = 5 = AB\cos\theta$, where $A = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$, and where $B = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$. Therefore $\cos\theta = 5/14$, so that $\theta = \underline{69.1°}$.

b) the cross product: Begin with

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & 1 & 3 \\ 1 & -3 & 2 \end{vmatrix} = 11\mathbf{a}_x - \mathbf{a}_y - 7\mathbf{a}_z$$

and then $|\mathbf{A} \times \mathbf{B}| = \sqrt{11^2 + 1^2 + 7^2} = \sqrt{171}$. So now, with $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta = \sqrt{171}$, find $\theta = \sin^{-1}\left(\sqrt{171}/14\right) = \underline{69.1°}$

1.7. Given the vector field $\mathbf{E} = 4zy^2\cos 2x\mathbf{a}_x + 2zy\sin 2x\mathbf{a}_y + y^2\sin 2x\mathbf{a}_z$ for the region $|x|$, $|y|$, and $|z|$ less than 2, find:

a) the surfaces on which $E_y = 0$. With $E_y = 2zy\sin 2x = 0$, the surfaces are 1) the plane $\underline{z = 0}$, with $|x| < 2$, $|y| < 2$; 2) the plane $\underline{y = 0}$, with $|x| < 2$, $|z| < 2$; 3) the plane $\underline{x = 0}$, with $|y| < 2$, $|z| < 2$; 4) the plane $\underline{x = \pi/2}$, with $|y| < 2$, $|z| < 2$.

b) the region in which $E_y = E_z$: This occurs when $2zy\sin 2x = y^2\sin 2x$, or on the plane $\underline{2z = y}$, with $|x| < 2$, $|y| < 2$, $|z| < 1$.

c) the region in which $\mathbf{E} = 0$: We would have $E_x = E_y = E_z = 0$, or $zy^2\cos 2x = zy\sin 2x = y^2\sin 2x = 0$. This condition is met on the plane $\underline{y = 0}$, with $|x| < 2$, $|z| < 2$.

1.8. Demonstrate the ambiguity that results when the cross product is used to find the angle between two vectors by finding the angle between $\mathbf{A} = 3\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z$. Does this ambiguity exist when the dot product is used?

We use the relation $\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta \mathbf{n}$. With the given vectors we find

$$\mathbf{A} \times \mathbf{B} = 14\mathbf{a}_y + 7\mathbf{a}_z = 7\sqrt{5} \underbrace{\left[\frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}} \right]}_{\pm\mathbf{n}} = \sqrt{9+4+16}\sqrt{4+1+4} \, \sin\theta \, \mathbf{n}$$

where $\mathbf{n}$ is identified as shown; we see that $\mathbf{n}$ can be positive or negative, as $\sin\theta$ can be positive or negative. This apparent sign ambiguity is not the real problem, however, as we really want the magnitude of the angle anyway. Choosing the positive sign, we are left with $\sin\theta = 7\sqrt{5}/(\sqrt{29}\sqrt{9}) = 0.969$. *Two* values of θ ($75.7°$ and $104.3°$) satisfy this equation, and hence the real ambiguity.

In using the dot product, we find $\mathbf{A} \cdot \mathbf{B} = 6 - 2 - 8 = -4 = |\mathbf{A}||\mathbf{B}| \cos\theta = 3\sqrt{29} \cos\theta$, or $\cos\theta = -4/(3\sqrt{29}) = -0.248 \Rightarrow \theta = -75.7°$. Again, the minus sign is not important, as we care only about the angle magnitude. The main point is that *only one* θ value results when using the dot product, so no ambiguity.

1.9. A field is given as

$$\mathbf{G} = \frac{25}{(x^2 + y^2)}(x\mathbf{a}_x + y\mathbf{a}_y)$$

Find:

a) a unit vector in the direction of $\mathbf{G}$ at $P(3, 4, -2)$: Have $\mathbf{G}_p = 25/(9+16) \times (3, 4, 0) = 3\mathbf{a}_x + 4\mathbf{a}_y$, and $|\mathbf{G}_p| = 5$. Thus $\mathbf{a}_G = \underline{(0.6, 0.8, 0)}$.

b) the angle between $\mathbf{G}$ and $\mathbf{a}_x$ at P: The angle is found through $\mathbf{a}_G \cdot \mathbf{a}_x = \cos\theta$. So $\cos\theta = (0.6, 0.8, 0) \cdot (1, 0, 0) = 0.6$. Thus $\underline{\theta = 53°}$.

c) the value of the following double integral on the plane $y = 7$:

$$\int_0^4 \int_0^2 \mathbf{G} \cdot \mathbf{a}_y dz dx$$

$$\int_0^4 \int_0^2 \frac{25}{x^2 + y^2}(x\mathbf{a}_x + y\mathbf{a}_y) \cdot \mathbf{a}_y dz dx = \int_0^4 \int_0^2 \frac{25}{x^2 + 49} \times 7 \, dz dx = \int_0^4 \frac{350}{x^2 + 49} dx$$

$$= 350 \times \frac{1}{7} \left[\tan^{-1}\left(\frac{4}{7}\right) - 0 \right] = \underline{26}$$

1.10. By expressing diagonals as vectors and using the definition of the dot product, find the smaller angle between any two diagonals of a cube, where each diagonal connects diametrically opposite corners, and passes through the center of the cube:

Assuming a side length, b, two diagonal vectors would be $\mathbf{A} = b(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$ and $\mathbf{B} = b(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z)$. Now use $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$, or $b^2(1 - 1 + 1) = (\sqrt{3}b)(\sqrt{3}b) \cos\theta \Rightarrow \cos\theta = 1/3 \Rightarrow \theta = \underline{70.53°}$. This result (in magnitude) is the same for *any* two diagonal vectors.

1.11. Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find:

a) the vector $\mathbf{R}_{MN}$: $\mathbf{R}_{MN} = (-0.2, 0.1, 0.3) - (0.1, -0.2, -0.1) = \underline{(-0.3, 0.3, 0.4)}$.

b) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$: $\mathbf{R}_{MP} = (0.4, 0, 0.1) - (0.1, -0.2, -0.1) = (0.3, 0.2, 0.2)$. $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP} = (-0.3, 0.3, 0.4) \cdot (0.3, 0.2, 0.2) = -0.09 + 0.06 + 0.08 = \underline{0.05}$.

c) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$:

$$\mathbf{R}_{MN} \cdot \mathbf{a}_{RMP} = (-0.3, 0.3, 0.4) \cdot \frac{(0.3, 0.2, 0.2)}{\sqrt{0.09 + 0.04 + 0.04}} = \frac{0.05}{\sqrt{0.17}} = \underline{0.12}$$

d) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$:

$$\theta_M = \cos^{-1}\left(\frac{\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}}{|\mathbf{R}_{MN}||\mathbf{R}_{MP}|}\right) = \cos^{-1}\left(\frac{0.05}{\sqrt{0.34}\sqrt{0.17}}\right) = \underline{78^\circ}$$

1.12. Write an expression in rectangular components for the vector that extends from (x_1, y_1, z_1) to (x_2, y_2, z_2) and determine the magnitude of this vector.

The two points can be written as vectors from the origin:

$$\mathbf{A}_1 = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z \quad \text{and} \quad \mathbf{A}_2 = x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z$$

The desired vector will now be the difference:

$$\mathbf{A}_{12} = \mathbf{A}_2 - \mathbf{A}_1 = (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z$$

whose magnitude is

$$|\mathbf{A}_{12}| = \sqrt{\mathbf{A}_{12} \cdot \mathbf{A}_{12}} = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

1.13. a) Find the vector component of $\mathbf{F} = (10, -6, 5)$ that is parallel to $\mathbf{G} = (0.1, 0.2, 0.3)$:

$$\mathbf{F}_{||G} = \frac{\mathbf{F} \cdot \mathbf{G}}{|\mathbf{G}|^2}\mathbf{G} = \frac{(10, -6, 5) \cdot (0.1, 0.2, 0.3)}{0.01 + 0.04 + 0.09}(0.1, 0.2, 0.3) = \underline{(0.93, 1.86, 2.79)}$$

b) Find the vector component of $\mathbf{F}$ that is perpendicular to $\mathbf{G}$:

$$\mathbf{F}_{pG} = \mathbf{F} - \mathbf{F}_{||G} = (10, -6, 5) - (0.93, 1.86, 2.79) = \underline{(9.07, -7.86, 2.21)}$$

c) Find the vector component of $\mathbf{G}$ that is perpendicular to $\mathbf{F}$:

$$\mathbf{G}_{pF} = \mathbf{G} - \mathbf{G}_{||F} = \mathbf{G} - \frac{\mathbf{G} \cdot \mathbf{F}}{|\mathbf{F}|^2}\mathbf{F} = (0.1, 0.2, 0.3) - \frac{1.3}{100 + 36 + 25}(10, -6, 5) = \underline{(0.02, 0.25, 0.26)}$$

1.14. Given that $\mathbf{A} + \mathbf{B} + \mathbf{C} = 0$, where the three vectors represent line segments and extend from a common origin,

a) must the three vectors be coplanar?

In terms of the components, the vector sum will be

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = (A_x + B_x + C_x)\mathbf{a}_x + (A_y + B_y + C_y)\mathbf{a}_y + (A_z + B_z + C_z)\mathbf{a}_z$$

which we require to be zero. Suppose the coordinate system is configured so that vectors **A** and **B** lie in the x-y plane; in this case $A_z = B_z = 0$. Then C_z has to be zero in order for the three vectors to sum to zero. Therefore, the three vectors must be coplanar.

b) If $\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = 0$, are the four vectors coplanar?

The vector sum is now

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} = (A_x + B_x + C_x + D_x)\mathbf{a}_x + (A_y + B_y + C_y + D_y)\mathbf{a}_y + (A_z + B_z + C_z + D_z)\mathbf{a}_z$$

Now, for example, if **A** and **B** lie in the x-y plane, **C** and **D** need not, as long as $C_z + D_z = 0$. So the four vectors need not be coplanar to have a zero sum.

1.15. Three vectors extending from the origin are given as $\mathbf{r}_1 = (7, 3, -2)$, $\mathbf{r}_2 = (-2, 7, -3)$, and $\mathbf{r}_3 = (0, 2, 3)$. Find:

a) a unit vector perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{a}_{p12} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|} = \frac{(5, 25, 55)}{60.6} = \underline{(0.08, 0.41, 0.91)}$$

b) a unit vector perpendicular to the vectors $\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_2 - \mathbf{r}_3$: $\mathbf{r}_1 - \mathbf{r}_2 = (9, -4, 1)$ and $\mathbf{r}_2 - \mathbf{r}_3 = (-2, 5, -6)$. So $\mathbf{r}_1 - \mathbf{r}_2 \times \mathbf{r}_2 - \mathbf{r}_3 = (19, 52, 32)$. Then

$$\mathbf{a}_p = \frac{(19, 52, 32)}{|(19, 52, 32)|} = \frac{(19, 52, 32)}{63.95} = \underline{(0.30, 0.81, 0.50)}$$

c) the area of the triangle defined by $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\text{Area} = \frac{1}{2}|\mathbf{r}_1 \times \mathbf{r}_2| = \underline{30.3}$$

d) the area of the triangle defined by the heads of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$:

$$\text{Area} = \frac{1}{2}|(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_2 - \mathbf{r}_3)| = \frac{1}{2}|(-9, 4, -1) \times (-2, 5, -6)| = \underline{32.0}$$

1.16. If $\mathbf{A}$ represents a vector one unit in length directed due east, $\mathbf{B}$ represents a vector three units in length directed due north, and $\mathbf{A}+\mathbf{B}=2\mathbf{C}-\mathbf{D}$ and $2\mathbf{A}-\mathbf{B}=\mathbf{C}+2\mathbf{D}$, determine the length and direction of $\mathbf{C}$. (difficulty 1)

Take north as the positive y direction, and then east as the positive x direction. Then we may write

$$\mathbf{A}+\mathbf{B}=\mathbf{a}_x+3\mathbf{a}_y=2\mathbf{C}-\mathbf{D}$$

and

$$2\mathbf{A}-\mathbf{B}=2\mathbf{a}_x-3\mathbf{a}_y=\mathbf{C}+2\mathbf{D}$$

Multiplying the first equation by 2, and then adding the result to the second equation eliminates $\mathbf{D}$, and we get

$$4\mathbf{a}_x+3\mathbf{a}_y=5\mathbf{C} \quad\Rightarrow\quad \mathbf{C}=\frac{4}{5}\mathbf{a}_x+\frac{3}{5}\mathbf{a}_y$$

The length of $\mathbf{C}$ is $|\mathbf{C}|=\left[(4/5)^2+(3/5)^2\right]^{1/2}=\underline{1}$

$\mathbf{C}$ lies in the x-y plane at angle from due north (the y axis) given by $\alpha=\tan^{-1}(4/3)=53.1^\circ$ (or 36.9° from the x axis). For those having nautical leanings, this is very close to the compass point NE$\frac{3}{4}$E (not required).

1.17. Point $A(-4,2,5)$ and the two vectors, $\mathbf{R}_{AM}=(20,18,-10)$ and $\mathbf{R}_{AN}=(-10,8,15)$, define a triangle.

a) Find a unit vector perpendicular to the triangle: Use

$$\mathbf{a}_p=\frac{\mathbf{R}_{AM}\times\mathbf{R}_{AN}}{|\mathbf{R}_{AM}\times\mathbf{R}_{AN}|}=\frac{(350,-200,340)}{527.35}=\underline{(0.664,-0.379,0.645)}$$

The vector in the opposite direction to this one is also a valid answer.

b) Find a unit vector in the plane of the triangle and perpendicular to $\mathbf{R}_{AN}$:

$$\mathbf{a}_{AN}=\frac{(-10,8,15)}{\sqrt{389}}=(-0.507,0.406,0.761)$$

Then

$$\mathbf{a}_{pAN}=\mathbf{a}_p\times\mathbf{a}_{AN}=(0.664,-0.379,0.645)\times(-0.507,0.406,0.761)=\underline{(-0.550,-0.832,0.077)}$$

The vector in the opposite direction to this one is also a valid answer.

c) Find a unit vector in the plane of the triangle that bisects the interior angle at A: A non-unit vector in the required direction is $(1/2)(\mathbf{a}_{AM}+\mathbf{a}_{AN})$, where

$$\mathbf{a}_{AM}=\frac{(20,18,-10)}{|(20,18,-10)|}=(0.697,0.627,-0.348)$$

Now

$$\frac{1}{2}(\mathbf{a}_{AM}+\mathbf{a}_{AN})=\frac{1}{2}[(0.697,0.627,-0.348)+(-0.507,0.406,0.761)]=(0.095,0.516,0.207)$$

Finally,

$$\mathbf{a}_{bis}=\frac{(0.095,0.516,0.207)}{|(0.095,0.516,0.207)|}=\underline{(0.168,0.915,0.367)}$$

1.18. A certain vector field is given as $\mathbf{G} = (y+1)\mathbf{a}_x + x\mathbf{a}_y$. a) Determine $\mathbf{G}$ at the point (3,-2,4): $\mathbf{G}(3,-2,4) = \underline{-\mathbf{a}_x + 3\mathbf{a}_y}$.

b) obtain a unit vector defining the direction of $\mathbf{G}$ at (3,-2,4).
$|\mathbf{G}(3,-2,4)| = [1+3^2]^{1/2} = \sqrt{10}$. So the unit vector is

$$\mathbf{a}_G(3,-2,4) = \frac{-\mathbf{a}_x + 3\mathbf{a}_y}{\sqrt{10}}$$

1.19. a) Express the field $\mathbf{D} = (x^2+y^2)^{-1}(x\mathbf{a}_x + y\mathbf{a}_y)$ in cylindrical components and cylindrical variables: Have $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $x^2+y^2 = \rho^2$. Therefore

$$\mathbf{D} = \frac{1}{\rho}(\cos\phi\mathbf{a}_x + \sin\phi\mathbf{a}_y)$$

Then

$$D_\rho = \mathbf{D}\cdot\mathbf{a}_\rho = \frac{1}{\rho}\left[\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\rho) + \sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\rho)\right] = \frac{1}{\rho}\left[\cos^2\phi + \sin^2\phi\right] = \frac{1}{\rho}$$

and

$$D_\phi = \mathbf{D}\cdot\mathbf{a}_\phi = \frac{1}{\rho}\left[\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\phi) + \sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\phi)\right] = \frac{1}{\rho}\left[\cos\phi(-\sin\phi) + \sin\phi\cos\phi\right] = 0$$

Therefore

$$\underline{\mathbf{D} = \frac{1}{\rho}\mathbf{a}_\rho}$$

b) Evaluate $\mathbf{D}$ at the point where $\rho = 2$, $\phi = 0.2\pi$, and $z = 5$, expressing the result in cylindrical and cartesian coordinates: At the given point, and in cylindrical coordinates, $\underline{\mathbf{D} = 0.5\mathbf{a}_\rho}$. To express this in cartesian, we use

$$\mathbf{D} = 0.5(\mathbf{a}_\rho\cdot\mathbf{a}_x)\mathbf{a}_x + 0.5(\mathbf{a}_\rho\cdot\mathbf{a}_y)\mathbf{a}_y = 0.5\cos 36^\circ\mathbf{a}_x + 0.5\sin 36^\circ\mathbf{a}_y = \underline{0.41\mathbf{a}_x + 0.29\mathbf{a}_y}$$

1.20. If the three sides of a triangle are represented by the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, all directed counter-clockwise, show that $|\mathbf{C}|^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})$ and expand the product to obtain the law of cosines.

With the vectors drawn as described above, we find that $\mathbf{C} = -(\mathbf{A}+\mathbf{B})$ and so $|\mathbf{C}|^2 = C^2 = \mathbf{C}\cdot\mathbf{C} = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})$ So far so good. Now if we expand the product, obtain

$$(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) = A^2 + B^2 + 2\mathbf{A}\cdot\mathbf{B}$$

where $\mathbf{A}\cdot\mathbf{B} = AB\cos(180^\circ - \alpha) = -AB\cos\alpha$ where α is the interior angle at the junction of $\mathbf{A}$ and $\mathbf{B}$. Using this, we have $C^2 = A^2 + B^2 - 2AB\cos\alpha$, which is the law of cosines.

1.21. Express in cylindrical components:

a) the vector from $C(3, 2, -7)$ to $D(-1, -4, 2)$:
$C(3, 2, -7) \rightarrow C(\rho = 3.61, \phi = 33.7^\circ, z = -7)$ and
$D(-1, -4, 2) \rightarrow D(\rho = 4.12, \phi = -104.0^\circ, z = 2)$.
Now $\mathbf{R}_{CD} = (-4, -6, 9)$ and $R_\rho = \mathbf{R}_{CD} \cdot \mathbf{a}_\rho = -4\cos(33.7) - 6\sin(33.7) = -6.66$. Then $R_\phi = \mathbf{R}_{CD} \cdot \mathbf{a}_\phi = 4\sin(33.7) - 6\cos(33.7) = -2.77$. So $\mathbf{R}_{CD} = \underline{-6.66\mathbf{a}_\rho - 2.77\mathbf{a}_\phi + 9\mathbf{a}_z}$

b) a unit vector at D directed toward C:
$\mathbf{R}_{CD} = (4, 6, -9)$ and $R_\rho = \mathbf{R}_{DC} \cdot \mathbf{a}_\rho = 4\cos(-104.0) + 6\sin(-104.0) = -6.79$. Then $R_\phi = \mathbf{R}_{DC} \cdot \mathbf{a}_\phi = 4[-\sin(-104.0)] + 6\cos(-104.0) = 2.43$. So $\mathbf{R}_{DC} = -6.79\mathbf{a}_\rho + 2.43\mathbf{a}_\phi - 9\mathbf{a}_z$
Thus $\mathbf{a}_{DC} = \underline{-0.59\mathbf{a}_\rho + 0.21\mathbf{a}_\phi - 0.78\mathbf{a}_z}$

c) a unit vector at D directed toward the origin: Start with $\mathbf{r}_D = (-1, -4, 2)$, and so the vector toward the origin will be $-\mathbf{r}_D = (1, 4, -2)$. Thus in cartesian the unit vector is $\mathbf{a} = (0.22, 0.87, -0.44)$. Convert to cylindrical:
$a_\rho = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\rho = 0.22\cos(-104.0) + 0.87\sin(-104.0) = -0.90$, and
$a_\phi = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\phi = 0.22[-\sin(-104.0)] + 0.87\cos(-104.0) = 0$, so that finally, $\mathbf{a} = \underline{-0.90\mathbf{a}_\rho - 0.44\mathbf{a}_z}$.

1.22. A sphere of radius a, centered at the origin, rotates about the z axis at angular velocity Ω rad/s. The rotation direction is clockwise when one is looking in the positive z direction.

a) Using spherical components, write an expression for the velocity field, $\mathbf{v}$, which gives the tangential velocity at any point within the sphere:

As in problem 1.20, we find the tangential velocity as the product of the angular velocity and the perperdicular distance from the rotation axis. With clockwise rotation, we obtain

$$\mathbf{v}(r, \theta) = \underline{\Omega r \sin\theta \, \mathbf{a}_\phi \ (r < a)}$$

b) Convert to rectangular components:

From here, the problem is the same as part c in Problem 1.20, except the rotation direction is reversed. The answer is $\mathbf{v}(x, y) = \underline{\Omega\,[-y\,\mathbf{a}_x + x\,\mathbf{a}_y]}$, where $(x^2 + y^2 + z^2)^{1/2} < a$.

1.23. The surfaces $\rho = 3$, $\rho = 5$, $\phi = 100^\circ$, $\phi = 130^\circ$, $z = 3$, and $z = 4.5$ define a closed surface.

a) Find the enclosed volume:

$$\text{Vol} = \int_3^{4.5} \int_{100^\circ}^{130^\circ} \int_3^5 \rho \, d\rho \, d\phi \, dz = \underline{6.28}$$

NOTE: The limits on the ϕ integration must be converted to radians (as was done here, but not shown).

b) Find the total area of the enclosing surface:

$$\text{Area} = 2\int_{100^\circ}^{130^\circ} \int_3^5 \rho \, d\rho \, d\phi \ + \int_3^{4.5} \int_{100^\circ}^{130^\circ} 3 \, d\phi \, dz$$
$$+ \int_3^{4.5} \int_{100^\circ}^{130^\circ} 5 \, d\phi \, dz \ + \ 2\int_3^{4.5} \int_3^5 d\rho \, dz = \underline{20.7}$$

1.23c) Find the total length of the twelve edges of the surfaces:

$$\text{Length} = 4 \times 1.5 \; + \; 4 \times 2 \; + \; 2 \times \left[\frac{30^\circ}{360^\circ} \times 2\pi \times 3 \; + \; \frac{30^\circ}{360^\circ} \times 2\pi \times 5 \right] = \underline{22.4}$$

d) Find the length of the longest straight line that lies entirely within the volume: This will be between the points A($\rho = 3$, $\phi = 100^\circ$, $z = 3$) and B($\rho = 5$, $\phi = 130^\circ$, $z = 4.5$). Performing point transformations to cartesian coordinates, these become A($x = -0.52$, $y = 2.95$, $z = 3$) and B($x = -3.21$, $y = 3.83$, $z = 4.5$). Taking A and B as vectors directed from the origin, the requested length is

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = |(-2.69, 0.88, 1.5)| = \underline{3.21}$$

1.24. Two unit vectors, $\mathbf{a}_1$ and $\mathbf{a}_2$ lie in the xy plane and pass through the origin. They make angles ϕ_1 and ϕ_2 with the x axis respectively.

a) Express each vector in rectangular components; Have $\mathbf{a}_1 = A_{x1}\mathbf{a}_x + A_{y1}\mathbf{a}_y$, so that $A_{x1} = \mathbf{a}_1 \cdot \mathbf{a}_x = \cos\phi_1$. Then, $A_{y1} = \mathbf{a}_1 \cdot \mathbf{a}_y = \cos(90 - \phi_1) = \sin\phi_1$. Therefore,

$$\mathbf{a}_1 = \cos\phi_1\,\mathbf{a}_x + \sin\phi_1\,\mathbf{a}_y \quad \text{and similarly,} \quad \mathbf{a}_2 = \cos\phi_2\,\mathbf{a}_x + \sin\phi_2\,\mathbf{a}_y$$

b) take the dot product and verify the trigonometric identity, $\cos(\phi_1 - \phi_2) = \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2$: From the definition of the dot product,

$$\begin{aligned}\mathbf{a}_1 \cdot \mathbf{a}_2 &= (1)(1)\cos(\phi_1 - \phi_2) \\ &= (\cos\phi_1\,\mathbf{a}_x + \sin\phi_1\,\mathbf{a}_y) \cdot (\cos\phi_2\,\mathbf{a}_x + \sin\phi_2\,\mathbf{a}_y) = \cos\phi_1\,\cos\phi_2 + \sin\phi_1\,\sin\phi_2\end{aligned}$$

c) take the cross product and verify the trigonometric identity $\sin(\phi_2 - \phi_1) = \sin\phi_2\cos\phi_1 - \cos\phi_2\sin\phi_1$: From the definition of the cross product, and since $\mathbf{a}_1$ and $\mathbf{a}_2$ both lie in the x-y plane,

$$\begin{aligned}\mathbf{a}_1 \times \mathbf{a}_2 = (1)(1)\sin(\phi_1 - \phi_2)\,\mathbf{a}_z &= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \cos\phi_1 & \sin\phi_1 & 0 \\ \cos\phi_2 & \sin\phi_2 & 0 \end{vmatrix} \\ &= [\sin\phi_2\cos\phi_1 - \cos\phi_2\sin\phi_1]\;\mathbf{a}_z\end{aligned}$$

thus verified.

1.25. Given point $P(r = 0.8, \theta = 30^\circ, \phi = 45^\circ)$, and

$$\mathbf{E} = \frac{1}{r^2}\left(\cos\phi\,\mathbf{a}_r + \frac{\sin\phi}{\sin\theta}\,\mathbf{a}_\phi\right)$$

a) Find $\mathbf{E}$ at P: $\mathbf{E} = \underline{1.10\mathbf{a}_\rho + 2.21\mathbf{a}_\phi}$.

b) Find $|\mathbf{E}|$ at P: $|\mathbf{E}| = \sqrt{1.10^2 + 2.21^2} = \underline{2.47}$.

c) Find a unit vector in the direction of $\mathbf{E}$ at P:

$$\mathbf{a}_E = \frac{\mathbf{E}}{|\mathbf{E}|} = \underline{0.45\mathbf{a}_r + 0.89\mathbf{a}_\phi}$$

1.26. Express the uniform vector field, $\mathbf{F} = 5\,\mathbf{a}_x$ in

a) cylindrical components: $F_\rho = 5\,\mathbf{a}_x \cdot \mathbf{a}_\rho = 5\cos\phi$, and $F_\phi = 5\,\mathbf{a}_x \cdot \mathbf{a}_\phi = -5\sin\phi$. Combining, we obtain $\underline{\mathbf{F}(\rho,\phi) = 5(\cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi)}$.

b) spherical components: $F_r = 5\,\mathbf{a}_x \cdot \mathbf{a}_r = 5\sin\theta\cos\phi$; $F_\theta = 5\,\mathbf{a}_x \cdot \mathbf{a}_\theta = 5\cos\theta\cos\phi$; $F_\phi = 5\,\mathbf{a}_x \cdot \mathbf{a}_\phi = -5\sin\phi$. Combining, we obtain $\underline{\mathbf{F}(r,\theta,\phi) = 5\left[\sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi\right]}$.

1.27. The surfaces $r = 2$ and 4, $\theta = 30^\circ$ and 50°, and $\phi = 20^\circ$ and 60° identify a closed surface.

a) Find the enclosed volume: This will be

$$\text{Vol} = \int_{20^\circ}^{60^\circ}\int_{30^\circ}^{50^\circ}\int_2^4 r^2\sin\theta\,dr\,d\theta\,d\phi = \underline{2.91}$$

where degrees have been converted to radians.

b) Find the total area of the enclosing surface:

$$\text{Area} = \int_{20^\circ}^{60^\circ}\int_{30^\circ}^{50^\circ}(4^2+2^2)\sin\theta\,d\theta\,d\phi + \int_2^4\int_{20^\circ}^{60^\circ} r(\sin 30^\circ + \sin 50^\circ)\,dr\,d\phi$$
$$+\,2\int_{30^\circ}^{50^\circ}\int_2^4 r\,dr\,d\theta = \underline{12.61}$$

c) Find the total length of the twelve edges of the surface:

$$\text{Length} = 4\int_2^4 dr \;+\; 2\int_{30^\circ}^{50^\circ}(4+2)\,d\theta + \int_{20^\circ}^{60^\circ}(4\sin 50^\circ + 4\sin 30^\circ + 2\sin 50^\circ + 2\sin 30^\circ)\,d\phi$$
$$= \underline{17.49}$$

d) Find the length of the longest straight line that lies entirely within the surface: This will be from $A(r = 2, \theta = 50^\circ, \phi = 20^\circ)$ to $B(r = 4, \theta = 30^\circ, \phi = 60^\circ)$ or

$$A(x = 2\sin 50^\circ\cos 20^\circ, y = 2\sin 50^\circ\sin 20^\circ, z = 2\cos 50^\circ)$$

to

$$B(x = 4\sin 30^\circ\cos 60^\circ, y = 4\sin 30^\circ\sin 60^\circ, z = 4\cos 30^\circ)$$

or finally $A(1.44, 0.52, 1.29)$ to $B(1.00, 1.73, 3.46)$. Thus $\mathbf{B} - \mathbf{A} = (-0.44, 1.21, 2.18)$ and

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = \underline{2.53}$$

1.28. State whether or not $\mathbf{A} = \mathbf{B}$ and, if not, what conditions are imposed on $\mathbf{A}$ and $\mathbf{B}$ when

a) $\mathbf{A} \cdot \mathbf{a}_x = \mathbf{B} \cdot \mathbf{a}_x$: For this to be true, both $\mathbf{A}$ and $\mathbf{B}$ must be oriented at the same angle, θ, from the x axis. But this would allow either vector to lie anywhere along a conical surface of angle θ about the x axis. Therefore, $\mathbf{A}$ *can* be equal to $\mathbf{B}$, but not necessarily.

b) $\mathbf{A} \times \mathbf{a}_x = \mathbf{B} \times \mathbf{a}_x$: This is a more restrictive condition because the cross product gives a vector. For both cross products to lie in the same direction, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{a}_x$ must be coplanar. But if $\mathbf{A}$ lies at angle θ to the x axis, $\mathbf{B}$ could lie at θ *or* at $180^\circ - \theta$ to give the same cross product. So again, $\mathbf{A}$ *can* be equal to $\mathbf{B}$, but not necessarily.

1.28c) $\mathbf{A}\cdot\mathbf{a}_x = \mathbf{B}\cdot\mathbf{a}_x$ *and* $\mathbf{A}\times\mathbf{a}_x = \mathbf{B}\times\mathbf{a}_x$: In this case, we need to satisfy both requirements in parts *a* and *b* – that is, **A**, **B**, and $\mathbf{a}_x$ must be coplanar, *and* **A** and **B** must lie at the same angle, θ, to $\mathbf{a}_x$. With coplanar vectors, this latter condition might imply that both $+\theta$ and $-\theta$ would therefore work. But the negative angle reverses the direction of the cross product direction. Therefore both vectors must lie in the same plane *and* lie at the same angle to x; i.e., **A** *must* be equal to **B**.

d) $\mathbf{A}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{C}$ *and* $\mathbf{A}\times\mathbf{C} = \mathbf{B}\times\mathbf{C}$ where **C** is any vector except $\mathbf{C} = 0$: This is just the general case of part *c*. Since we can orient our coordinate system in any manner we choose, we can arrange it so that the x axis coincides with the direction of vector **C**. Thus all the arguments of part *c* apply, and again we conclude that **A** *must* be equal to **B**.

1.29. Express the unit vector $\mathbf{a}_x$ in spherical components at the point:

a) $r = 2$, $\theta = 1$ rad, $\phi = 0.8$ rad: Use

$$\mathbf{a}_x = (\mathbf{a}_x\cdot\mathbf{a}_r)\mathbf{a}_r + (\mathbf{a}_x\cdot\mathbf{a}_\theta)\mathbf{a}_\theta + (\mathbf{a}_x\cdot\mathbf{a}_\phi)\mathbf{a}_\phi =$$
$$\sin(1)\cos(0.8)\mathbf{a}_r + \cos(1)\cos(0.8)\mathbf{a}_\theta + (-\sin(0.8))\mathbf{a}_\phi = \underline{0.59\mathbf{a}_r + 0.38\mathbf{a}_\theta - 0.72\mathbf{a}_\phi}$$

b) $x = 3$, $y = 2$, $z = -1$: First, transform the point to spherical coordinates. Have $r = \sqrt{14}$, $\theta = \cos^{-1}(-1/\sqrt{14}) = 105.5°$, and $\phi = \tan^{-1}(2/3) = 33.7°$. Then

$$\mathbf{a}_x = \sin(105.5°)\cos(33.7°)\mathbf{a}_r + \cos(105.5°)\cos(33.7°)\mathbf{a}_\theta + (-\sin(33.7°))\mathbf{a}_\phi$$
$$= \underline{0.80\mathbf{a}_r - 0.22\mathbf{a}_\theta - 0.55\mathbf{a}_\phi}$$

c) $\rho = 2.5$, $\phi = 0.7$ rad, $z = 1.5$: Again, convert the point to spherical coordinates. $r = \sqrt{\rho^2 + z^2} = \sqrt{8.5}$, $\theta = \cos^{-1}(z/r) = \cos^{-1}(1.5/\sqrt{8.5}) = 59.0°$, and $\phi = 0.7\,\text{rad} = 40.1°$. Now

$$\mathbf{a}_x = \sin(59°)\cos(40.1°)\mathbf{a}_r + \cos(59°)\cos(40.1°)\mathbf{a}_\theta + (-\sin(40.1°))\mathbf{a}_\phi$$
$$= \underline{0.66\mathbf{a}_r + 0.39\mathbf{a}_\theta - 0.64\mathbf{a}_\phi}$$

1.30. Consider a problem analogous to the varying wind velocities encountered by transcontinental aircraft. We assume a constant altitude, a plane earth, a flight along the x axis from 0 to 10 units, no vertical velocity component, and no change in wind velocity with time. Assume $\mathbf{a}_x$ to be directed to the east and $\mathbf{a}_y$ to the north. The wind velocity at the operating altitude is assumed to be:

$$\mathbf{v}(x,y) = \frac{(0.01x^2 - 0.08x + 0.66)\mathbf{a}_x - (0.05x - 0.4)\mathbf{a}_y}{1 + 0.5y^2}$$

a) Determine the location and magnitude of the maximum tailwind encountered: Tailwind would be x-directed, and so we look at the x component only. Over the flight range, this function maximizes at a value of $0.86/(1 + 0.5y^2)$ at $x = 10$ (at the end of the trip). It reaches a local minimum of $0.50/(1 + 0.5y^2)$ at $x = 4$, and has another local maximum of $0.66/(1 + 0.5y^2)$ at the trip start, $x = 0$.

b) Repeat for headwind: The x component is always positive, and so therefore no headwind exists over the travel range.

c) Repeat for crosswind: Crosswind will be found from the y component, which is seen to maximize over the flight range at a value of $0.4/(1 + 0.5y^2)$ at the trip start ($x = 0$).

d) Would more favorable tailwinds be available at some other latitude? If so, where? Minimizing the denominator accomplishes this; in particular, the lattitude associated with $y = 0$ gives the strongest tailwind.

CHAPTER 2

2.1. Three point charges are positioned in the x-y plane as follows: 5nC at $y = 5$ cm, -10 nC at $y = -5$ cm, 15 nC at $x = -5$ cm. Find the required x-y coordinates of a 20-nC fourth charge that will produce a zero electric field at the origin.

With the charges thus configured, the electric field at the origin will be the superposition of the individual charge fields:

$$\mathbf{E}_0 = \frac{1}{4\pi\epsilon_0}\left[\frac{15}{(5)^2}\,\mathbf{a}_x - \frac{5}{(5)^2}\,\mathbf{a}_y - \frac{10}{(5)^2}\,\mathbf{a}_y\right] = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\right)[\mathbf{a}_x - \mathbf{a}_y] \quad \text{nC/m}$$

The field, $\mathbf{E}_{20}$, associated with the 20-nC charge (evaluated at the origin) must exactly cancel this field, so we write:

$$\mathbf{E}_{20} = \frac{-1}{4\pi\epsilon_0}\left(\frac{3}{5}\right)[\mathbf{a}_x - \mathbf{a}_y] = \frac{-20}{4\pi\epsilon_0\rho^2}\left(\frac{1}{\sqrt{2}}\right)[\mathbf{a}_x - \mathbf{a}_y]$$

From this, we identify the distance from the origin: $\rho = \sqrt{100/(3\sqrt{2})} = 4.85$. The x and y coordinates of the 20-nC charge will both be equal in magnitude to $4.85/\sqrt{2} = 3.43$. The coodinates of the 20-nC charge are then $\underline{(3.43, -3.43)}$.

2.2. Point charges of 1nC and -2nC are located at (0,0,0) and (1,1,1), respectively, in free space. Determine the vector force acting on each charge.

First, the electric field intensity associated with the 1nC charge, evalutated at the -2nC charge location is:

$$\mathbf{E}_{12} = \frac{1}{4\pi\epsilon_0(3)}\left(\frac{1}{\sqrt{3}}\right)(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \quad \text{nC/m}$$

in which the distance between charges is $\sqrt{3}$ m. The force on the -2nC charge is then

$$\mathbf{F}_{12} = q_2\mathbf{E}_{12} = \frac{-2}{12\sqrt{3}\,\pi\epsilon_0}(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) = \underline{\frac{-1}{10.4\,\pi\epsilon_0}(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \quad \text{nN}}$$

The force on the 1nC charge at the origin is just the opposite of this result, or

$$\mathbf{F}_{21} = \underline{\frac{+1}{10.4\,\pi\epsilon_0}(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \quad \text{nN}}$$

2.3. Point charges of 50nC each are located at $A(1,0,0)$, $B(-1,0,0)$, $C(0,1,0)$, and $D(0,-1,0)$ in free space. Find the total force on the charge at A.

The force will be:

$$\mathbf{F} = \frac{(50\times 10^{-9})^2}{4\pi\epsilon_0}\left[\frac{\mathbf{R}_{CA}}{|\mathbf{R}_{CA}|^3} + \frac{\mathbf{R}_{DA}}{|\mathbf{R}_{DA}|^3} + \frac{\mathbf{R}_{BA}}{|\mathbf{R}_{BA}|^3}\right]$$

where $\mathbf{R}_{CA} = \mathbf{a}_x - \mathbf{a}_y$, $\mathbf{R}_{DA} = \mathbf{a}_x + \mathbf{a}_y$, and $\mathbf{R}_{BA} = 2\mathbf{a}_x$. The magnitudes are $|\mathbf{R}_{CA}| = |\mathbf{R}_{DA}| = \sqrt{2}$, and $|\mathbf{R}_{BA}| = 2$. Substituting these leads to

$$\mathbf{F} = \frac{(50\times 10^{-9})^2}{4\pi\epsilon_0}\left[\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{2}{8}\right]\mathbf{a}_x = \underline{21.5\mathbf{a}_x\ \mu\text{N}}$$

where distances are in meters.

2.5. Let a point charge $Q_1 = 25$ nC be located at $P_1(4,-2,7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3,4,-2)$.

a) If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at $P_3(1,2,3)$: This field will be

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25\mathbf{R}_{13}}{|\mathbf{R}_{13}|^3} + \frac{60\mathbf{R}_{23}}{|\mathbf{R}_{23}|^3}\right]$$

where $\mathbf{R}_{13} = -3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{R}_{23} = 4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{41}$ and $|\mathbf{R}_{23}| = \sqrt{45}$. So

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25 \times (-3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z)}{(41)^{1.5}} + \frac{60 \times (4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z)}{(45)^{1.5}}\right]$$
$$= \underline{4.58\mathbf{a}_x - 0.15\mathbf{a}_y + 5.51\mathbf{a}_z}$$

b) At what point on the y axis is $E_x = 0$? P_3 is now at $(0, y, 0)$, so $\mathbf{R}_{13} = -4\mathbf{a}_x + (y+2)\mathbf{a}_y - 7\mathbf{a}_z$ and $\mathbf{R}_{23} = 3\mathbf{a}_x + (y-4)\mathbf{a}_y + 2\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{65 + (y+2)^2}$ and $|\mathbf{R}_{23}| = \sqrt{13 + (y-4)^2}$. Now the x component of $\mathbf{E}$ at the new P_3 will be:

$$E_x = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25 \times (-4)}{[65 + (y+2)^2]^{1.5}} + \frac{60 \times 3}{[13 + (y-4)^2]^{1.5}}\right]$$

2.5b (continued) To obtain $E_x = 0$, we require the expression in the large brackets to be zero. This expression simplifies to the following quadratic:

$$0.48y^2 + 13.92y + 73.10 = 0$$

which yields the two values: $y = \underline{-6.89, -22.11}$

2.6. Two point charges of equal magnitude q are positioned at $z = \pm d/2$.

a) find the electric field everywhere on the z axis: For a point charge at any location, we have

$$\mathbf{E} = \frac{q(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}$$

In the case of two charges, we would therefore have

$$\mathbf{E}_T = \frac{q_1(\mathbf{r} - \mathbf{r}_1')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1'|^3} + \frac{q_2(\mathbf{r} - \mathbf{r}_2')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2'|^3} \qquad (1)$$

In the present case, we assign $q_1 = q_2 = q$, the observation point position vector as $\mathbf{r} = z\mathbf{a}_z$, and the charge position vectors as $\mathbf{r}_1' = (d/2)\mathbf{a}_z$, and $\mathbf{r}_2' = -(d/2)\mathbf{a}_z$ Therefore

$$\mathbf{r} - \mathbf{r}_1' = [z - (d/2)]\mathbf{a}_z, \quad \mathbf{r} - \mathbf{r}_2' = [z + (d/2)]\mathbf{a}_z,$$

then

$$|\mathbf{r} - \mathbf{r}_1|^3 = [z - (d/2)]^3 \quad \text{and} \quad |\mathbf{r} - \mathbf{r}_2|^3 = [z + (d/2)]^3$$

Substitute these results into (1) to obtain:

$$\mathbf{E}_T(z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{[z - (d/2)]^2} + \frac{1}{[z + (d/2)]^2}\right]\mathbf{a}_z \quad \text{V/m} \qquad (2)$$

b) find the electric field everywhere on the x axis: We proceed as in part a, except that now $\mathbf{r} = x\mathbf{a}_x$. Eq. (1) becomes

$$\mathbf{E}_T(x) = \frac{q}{4\pi\epsilon_0}\left[\frac{x\mathbf{a}_x - (d/2)\mathbf{a}_z}{|x\mathbf{a}_x - (d/2)\mathbf{a}_z|^3} + \frac{x\mathbf{a}_x + (d/2)\mathbf{a}_z}{|x\mathbf{a}_x + (d/2)\mathbf{a}_z|^3}\right] \qquad (3)$$

where

$$|x\mathbf{a}_x - (d/2)\mathbf{a}_z| = |x\mathbf{a}_x + (d/2)\mathbf{a}_z| = \left[x^2 + (d/2)^2\right]^{1/2}$$

Therefore (3) becomes

$$\mathbf{E}_T(x) = \frac{2qx\,\mathbf{a}_x}{4\pi\epsilon_0\,[x^2 + (d/2)^2]^{3/2}}$$

c) repeat parts a and b if the charge at $z = -d/2$ is $-q$ instead of $+q$: The field along the z axis is quickly found by changing the sign of the second term in (2):

$$\mathbf{E}_T(z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{[z - (d/2)]^2} - \frac{1}{[z + (d/2)]^2}\right]\mathbf{a}_z \quad \text{V/m}$$

In like manner, the field along the x axis is found from (3) by again changing the sign of the second term. The result is

$$-2qd\,\mathbf{a}_z$$

2.7. A 2 μC point charge is located at $A(4,3,5)$ in free space. Find E_ρ, E_ϕ, and E_z at $P(8,12,2)$. Have

$$\mathbf{E}_P = \frac{2\times 10^{-6}}{4\pi\epsilon_0}\frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3} = \frac{2\times 10^{-6}}{4\pi\epsilon_0}\left[\frac{4\mathbf{a}_x + 9\mathbf{a}_y - 3\mathbf{a}_z}{(106)^{1.5}}\right] = 65.9\mathbf{a}_x + 148.3\mathbf{a}_y - 49.4\mathbf{a}_z$$

Then, at point P, $\rho = \sqrt{8^2+12^2} = 14.4$, $\phi = \tan^{-1}(12/8) = 56.3°$, and $z = z$. Now,

$$E_\rho = \mathbf{E}_p \cdot \mathbf{a}_\rho = 65.9(\mathbf{a}_x\cdot\mathbf{a}_\rho) + 148.3(\mathbf{a}_y\cdot\mathbf{a}_\rho) = 65.9\cos(56.3°) + 148.3\sin(56.3°) = \underline{159.7}$$

and

$$E_\phi = \mathbf{E}_p \cdot \mathbf{a}_\phi = 65.9(\mathbf{a}_x\cdot\mathbf{a}_\phi) + 148.3(\mathbf{a}_y\cdot\mathbf{a}_\phi) = -65.9\sin(56.3°) + 148.3\cos(56.3°) = \underline{27.4}$$

Finally, $E_z = \underline{-49.4 \text{ V/m}}$

2.8. A crude device for measuring charge consists of two small insulating spheres of radius a, one of which is fixed in position. The other is movable along the x axis, and is subject to a restraining force kx, where k is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of Q coulombs:

a) Obtain the expression by which Q may be found as a function of x: The spheres will attract, and so the movable sphere at $x = 0$ will move toward the other until the spring and Coulomb forces balance. This will occur at location x for the movable sphere. With equal and opposite forces, we have

$$\frac{Q^2}{4\pi\epsilon_0(d-x)^2} = kx$$

from which $\underline{Q = 2(d-x)\sqrt{\pi\epsilon_0 kx}}$.

b) Determine the maximum charge that can be measured in terms of ϵ_0, k, and d, and state the separation of the spheres then: With increasing charge, the spheres move toward each other until they just touch at $\underline{x_{max} = d - 2a}$. Using the part a result, we find the maximum measurable charge: $Q_{max} = \underline{4a\sqrt{\pi\epsilon_0 k(d-2a)}}$. Presumably some form of stop mechanism is placed at $x = x_{max}^-$ to prevent the spheres from actually touching.

c) What happens if a larger charge is applied? No further motion is possible, so nothing happens.

2.9. A 100 nC point charge is located at $A(-1,1,3)$ in free space.

a) Find the locus of all points $P(x,y,z)$ at which $E_x = 500$ V/m: The total field at P will be:

$$\mathbf{E}_P = \frac{100\times 10^{-9}}{4\pi\epsilon_0}\frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3}$$

where $\mathbf{R}_{AP} = (x+1)\mathbf{a}_x + (y-1)\mathbf{a}_y + (z-3)\mathbf{a}_z$, and where $|\mathbf{R}_{AP}| = [(x+1)^2+(y-1)^2+(z-3)^2]^{1/2}$. The x component of the field will be

$$E_x = \frac{100\times 10^{-9}}{4\pi\epsilon_0}\left[\frac{(x+1)}{[(x+1)^2+(y-1)^2+(z-3)^2]^{1.5}}\right] = 500 \text{ V/m}$$

And so our condition becomes:

$$\underline{(x+1) = 0.56\,[(x+1)^2+(y-1)^2+(z-3)^2]^{1.5}}$$

2.9b) Find y_1 if $P(-2, y_1, 3)$ lies on that locus: At point P, the condition of part a becomes

$$3.19 = \left[1 + (y_1 - 1)^2\right]^3$$

from which $(y_1 - 1)^2 = 0.47$, or $y_1 = \underline{1.69 \text{ or } 0.31}$

2.11. A charge Q_0 located at the origin in free space produces a field for which $E_z = 1$ kV/m at point $P(-2, 1, -1)$.

a) Find Q_0: The field at P will be

$$\mathbf{E}_P = \frac{Q_0}{4\pi\epsilon_0}\left[\frac{-2\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z}{6^{1.5}}\right]$$

Since the z component is of value 1 kV/m, we find $Q_0 = -4\pi\epsilon_0 6^{1.5} \times 10^3 = \underline{-1.63\ \mu\text{C}}$.

b) Find $\mathbf{E}$ at $M(1, 6, 5)$ in cartesian coordinates: This field will be:

$$\mathbf{E}_M = \frac{-1.63 \times 10^{-6}}{4\pi\epsilon_0}\left[\frac{\mathbf{a}_x + 6\mathbf{a}_y + 5\mathbf{a}_z}{[1 + 36 + 25]^{1.5}}\right]$$

or $\mathbf{E}_M = \underline{-30.11\mathbf{a}_x - 180.63\mathbf{a}_y - 150.53\mathbf{a}_z}$.

c) Find $\mathbf{E}$ at $M(1, 6, 5)$ in cylindrical coordinates: At M, $\rho = \sqrt{1 + 36} = 6.08$, $\phi = \tan^{-1}(6/1) = 80.54°$, and $z = 5$. Now

$$E_\rho = \mathbf{E}_M \cdot \mathbf{a}_\rho = -30.11\cos\phi - 180.63\sin\phi = -183.12$$

$$E_\phi = \mathbf{E}_M \cdot \mathbf{a}_\phi = -30.11(-\sin\phi) - 180.63\cos\phi = 0 \quad \text{(as expected)}$$

so that $\mathbf{E}_M = \underline{-183.12\mathbf{a}_\rho - 150.53\mathbf{a}_z}$.

d) Find $\mathbf{E}$ at $M(1, 6, 5)$ in spherical coordinates: At M, $r = \sqrt{1 + 36 + 25} = 7.87$, $\phi = 80.54°$ (as before), and $\theta = \cos^{-1}(5/7.87) = 50.58°$. Now, since the charge is at the origin, we expect to obtain only a radial component of $\mathbf{E}_M$. This will be:

$$E_r = \mathbf{E}_M \cdot \mathbf{a}_r = -30.11\sin\theta\cos\phi - 180.63\sin\theta\sin\phi - 150.53\cos\theta = \underline{-237.1}$$

2.12. Electrons are in random motion in a fixed region in space. During any 1μs interval, the probability of finding an electron in a subregion of volume 10^{-15} m^2 is 0.27. What volume charge density, appropriate for such time durations, should be assigned to that subregion?

The finite probabilty effectively reduces the net charge quantity by the probability fraction. With $e = -1.602 \times 10^{-19}$ C, the density becomes

$$\rho_v = -\frac{0.27 \times 1.602 \times 10^{-19}}{10^{-15}} = \underline{-43.3\ \mu\text{C/m}^3}$$

2.13. A uniform volume charge density of 0.2 μC/m^3 is present throughout the spherical shell extending from $r = 3$ cm to $r = 5$ cm. If $\rho_v = 0$ elsewhere:

a) find the total charge present throughout the shell: This will be

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_{.03}^{.05} 0.2\, r^2 \sin\theta\ dr\, d\theta\, d\phi = \left[4\pi(0.2)\frac{r^3}{3}\right]_{.03}^{.05} = 8.21 \times 10^{-5}\ \mu\text{C} = \underline{82.1\ \text{pC}}$$

b) find r_1 if half the total charge is located in the region $3\,\text{cm} < r < r_1$: If the integral over r in part a is taken to r_1, we would obtain

$$\left[4\pi(0.2)\frac{r^3}{3}\right]_{.03}^{r_1} = 4.105 \times 10^{-5}$$

Thus

$$r_1 = \left[\frac{3 \times 4.105 \times 10^{-5}}{0.2 \times 4\pi} + (.03)^3\right]^{1/3} = \underline{4.24\ \text{cm}}$$

2.14. The electron beam in a certain cathode ray tube possesses cylindrical symmetry, and the charge density is represented by $\rho_v = -0.1/(\rho^2 + 10^{-8})$ pC/m^3 for $0 < \rho < 3 \times 10^{-4}$ m, and $\rho_v = 0$ for $\rho > 3 \times 10^{-4}$ m.

a) Find the total charge per meter along the length of the beam: We integrate the charge density over the cylindrical volume having radius 3×10^{-4} m, and length 1m.

$$q = \int_0^1 \int_0^{2\pi} \int_0^{3\times10^{-4}} \frac{-0.1}{(\rho^2 + 10^{-8})}\, \rho\, d\rho\, d\phi\, dz$$

From integral tables, this evaluates as

$$q = -0.2\pi \left(\frac{1}{2}\right) \ln\left(\rho^2 + 10^{-8}\right)\Big|_0^{3\times10^{-4}} = 0.1\pi \ln(10) = \underline{-0.23\pi\ \text{pC/m}}$$

b) if the electron velocity is 5×10^7 m/s, and with one ampere defined as 1C/s, find the beam current:

Current $=$ charge/m $\times\, v = -0.23\pi\,[\text{pC/m}] \times 5 \times 10^7\,[\text{m/s}] = -11.5\pi \times 10^6\,[\text{pC/s}] = \underline{-11.5\pi\,\mu\text{A}}$

2.15. A spherical volume having a 2 μm radius contains a uniform volume charge density of 10^{15} C/m^3.

a) What total charge is enclosed in the spherical volume?
This will be $Q = (4/3)\pi(2 \times 10^{-6})^3 \times 10^{15} = \underline{3.35 \times 10^{-2}\ \text{C}}$.

b) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3mm on a side, and that there is no charge between spheres. What is the average volume charge density throughout this large region? Each cube will contain the equivalent of one little sphere. Neglecting the little sphere volume, the average density becomes

$$\rho_{v,avg} = \frac{3.35 \times 10^{-2}}{(0.003)^3} = \underline{1.24 \times 10^6\ \text{C/m}^3}$$

2.17. A uniform line charge of 16 nC/m is located along the line defined by $y = -2$, $z = 5$. If $\epsilon = \epsilon_0$:

a) Find $\mathbf{E}$ at $P(1, 2, 3)$: This will be

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0} \frac{\mathbf{R}_P}{|\mathbf{R}_P|^2}$$

where $\mathbf{R}_P = (1, 2, 3) - (1, -2, 5) = (0, 4, -2)$, and $|\mathbf{R}_P|^2 = 20$. So

$$\mathbf{E}_P = \frac{16 \times 10^{-9}}{2\pi\epsilon_0} \left[\frac{4\mathbf{a}_y - 2\mathbf{a}_z}{20}\right] = \underline{57.5\mathbf{a}_y - 28.8\mathbf{a}_z\ \text{V/m}}$$

b) Find $\mathbf{E}$ at that point in the $z = 0$ plane where the direction of $\mathbf{E}$ is given by $(1/3)\mathbf{a}_y - (2/3)\mathbf{a}_z$:
With $z = 0$, the general field will be

$$\mathbf{E}_{z=0} = \frac{\rho_l}{2\pi\epsilon_0} \left[\frac{(y+2)\mathbf{a}_y - 5\mathbf{a}_z}{(y+2)^2 + 25}\right]$$

We require $|E_z| = -|2E_y|$, so $2(y + 2) = 5$. Thus $y = 1/2$, and the field becomes:

$$\mathbf{E}_{z=0} = \frac{\rho_l}{2\pi\epsilon_0} \left[\frac{2.5\mathbf{a}_y - 5\mathbf{a}_z}{(2.5)^2 + 25}\right] = \underline{23\mathbf{a}_y - 46\mathbf{a}_z}$$

2.18. a) Find **E** in the plane $z = 0$ that is produced by a uniform line charge, ρ_L, extending along the z axis over the range $-L < z < L$ in a cylindrical coordinate system: We find **E** through

$$\mathbf{E} = \int_{-L}^{L} \frac{\rho_L dz(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where the observation point position vector is $\mathbf{r} = \rho\mathbf{a}_\rho$ (anywhere in the x-y plane), and where the position vector that locates any differential charge element on the z axis is $\mathbf{r}' = z\mathbf{a}_z$. So $\mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z\mathbf{a}_z$, and $|\mathbf{r} - \mathbf{r}'| = (\rho^2 + z^2)^{1/2}$. These relations are substituted into the integral to yield:

$$\mathbf{E} = \int_{-L}^{L} \frac{\rho_L dz(\rho\mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 + z^2)^{3/2}} = \frac{\rho_L\,\rho\,\mathbf{a}_\rho}{4\pi\epsilon_0} \int_{-L}^{L} \frac{dz}{(\rho^2 + z^2)^{3/2}} = E_\rho\,\mathbf{a}_\rho$$

Note that the second term in the left-hand integral (involving $z\mathbf{a}_z$) has effectively vanished because it produces equal and opposite sign contributions when the integral is taken over symmetric limits (odd parity). Evaluating the integral results in

$$E_\rho = \frac{\rho_L\,\rho}{4\pi\epsilon_0} \frac{z}{\rho^2\sqrt{\rho^2 + z^2}}\Big|_{-L}^{L} = \frac{\rho_L}{2\pi\epsilon_0\rho} \frac{L}{\sqrt{\rho^2 + L^2}} = \frac{\rho_L}{2\pi\epsilon_0\rho} \frac{1}{\sqrt{1 + (\rho/L)^2}}$$

Note that as $L \to \infty$, the expression reduces to the expected field of the infinite line charge in free space, $\rho_L/(2\pi\epsilon_0\rho)$.

b) if the finite line charge is approximated by an infinite line charge ($L \to \infty$), by what percentage is E_ρ in error if $\rho = 0.5L$? The percent error in this situation will be

$$\%\,\text{error} = \left[1 - \frac{1}{\sqrt{1 + (\rho/L)^2}}\right] \times 100$$

For $\rho = 0.5L$, this becomes $\%\,\text{error} = \underline{10.6\,\%}$

c) repeat b with $\rho = 0.1L$. For this value, obtain $\%\,\text{error} = \underline{0.496\,\%}$.

2.19. A uniform line charge of 2 μC/m is located on the z axis. Find **E** in rectangular coordinates at $P(1, 2, 3)$ if the charge extends from

a) $-\infty < z < \infty$: With the infinite line, we know that the field will have only a radial component in cylindrical coordinates (or x and y components in cartesian). The field from an infinite line on the z axis is generally $\mathbf{E} = [\rho_l/(2\pi\epsilon_0\rho)]\mathbf{a}_\rho$. Therefore, at point P:

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0} \frac{\mathbf{R}_{zP}}{|\mathbf{R}_{zP}|^2} = \frac{(2 \times 10^{-6})}{2\pi\epsilon_0} \frac{\mathbf{a}_x + 2\mathbf{a}_y}{5} = \underline{7.2\mathbf{a}_x + 14.4\mathbf{a}_y \ \ \text{kV/m}}$$

where $\mathbf{R}_{zP}$ is the vector that extends from the line charge to point P, and is perpendicular to the z axis; i.e., $\mathbf{R}_{zP} = (1, 2, 3) - (0, 0, 3) = (1, 2, 0)$.

b) $-4 \le z \le 4$: Here we use the general relation

$$\mathbf{E}_P = \int \frac{\rho_l dz}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

2.19b (continued) where $\mathbf{r} = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{r}' = z\mathbf{a}_z$. So the integral becomes

$$\mathbf{E}_P = \frac{(2 \times 10^{-6})}{4\pi\epsilon_0} \int_{-4}^{4} \frac{\mathbf{a}_x + 2\mathbf{a}_y + (3-z)\mathbf{a}_z}{[5 + (3-z)^2]^{1.5}}\, dz$$

Using integral tables, we obtain:

$$\mathbf{E}_P = 3597 \left[\frac{(\mathbf{a}_x + 2\mathbf{a}_y)(z-3) + 5\mathbf{a}_z}{(z^2 - 6z + 14)} \right]_{-4}^{4} \text{ V/m} = \underline{4.9\mathbf{a}_x + 9.8\mathbf{a}_y + 4.9\mathbf{a}_z \text{ kV/m}}$$

The student is invited to verify that when evaluating the above expression over the limits $-\infty < z < \infty$, the z component vanishes and the x and y components become those found in part a.

2.20. A line charge of uniform charge density ρ_0 C/m and of length ℓ, is oriented along the z axis at $-\ell/2 < z < \ell/2$.

a) Find the electric field strength, $\mathbf{E}$, in magnitude and direction at any position along the x axis: This follows the method in Problem 2.18. We find $\mathbf{E}$ through

$$\mathbf{E} = \int_{-\ell/2}^{\ell/2} \frac{\rho_0\, dz(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where the observation point position vector is $\mathbf{r} = x\mathbf{a}_x$ (anywhere on the x axis), and where the position vector that locates any differential charge element on the z axis is $\mathbf{r}' = z\mathbf{a}_z$. So $\mathbf{r} - \mathbf{r}' = x\mathbf{a}_x - z\mathbf{a}_z$, and $|\mathbf{r} - \mathbf{r}'| = (x^2 + z^2)^{1/2}$. These relations are substituted into the integral to yield:

$$\mathbf{E} = \int_{-\ell/2}^{\ell/2} \frac{\rho_0 dz(x\mathbf{a}_x - z\mathbf{a}_z)}{4\pi\epsilon_0 (x^2 + z^2)^{3/2}} = \frac{\rho_0\, x\, \mathbf{a}_x}{4\pi\epsilon_0} \int_{-\ell/2}^{\ell/2} \frac{dz}{(x^2 + z^2)^{3/2}} = E_x\, \mathbf{a}_x$$

Note that the second term in the left-hand integral (involving $z\mathbf{a}_z$) has effectively vanished because it produces equal and opposite sign contributions when the integral is taken over symmetric limits (odd parity). Evaluating the integral results in

$$E_x = \frac{\rho_0\, x}{4\pi\epsilon_0} \frac{z}{x^2\sqrt{x^2 + z^2}} \Bigg|_{-\ell/2}^{\ell/2} = \frac{\rho_0}{2\pi\epsilon_0 x} \frac{\ell/2}{\sqrt{x^2 + (\ell/2)^2}} = \frac{\rho_0}{2\pi\epsilon_0 x} \frac{1}{\sqrt{1 + (2x/\ell)^2}}$$

b) with the given line charge in position, find the force acting on an identical line charge that is oriented along the x axis at $\ell/2 < x < 3\ell/2$: The differential force on an element of the x-directed line charge will be $d\mathbf{F} = dq\mathbf{E} = (\rho_0\, dx)\mathbf{E}$, where $\mathbf{E}$ is the field as determined in part a. The net force is then the integral of the differential force over the length of the horizontal line charge, or

$$\mathbf{F} = \int_{\ell/2}^{3\ell/2} \frac{\rho_0^2}{2\pi\epsilon_0 x} \frac{1}{\sqrt{1 + (2x/\ell)^2}}\, dx\ \mathbf{a}_x$$

This can be re-written and then evaluated using integral tables as

$$\mathbf{F} = \frac{\rho_0^2\, \ell\, \mathbf{a}_x}{4\pi\epsilon_0} \int_{\ell/2}^{3\ell/2} \frac{dx}{x\sqrt{x^2 + (\ell/2)^2}} = \frac{-\rho_0^2\, \ell\, \mathbf{a}_x}{4\pi\epsilon_0} \left(\frac{1}{(\ell/2)} \ln \left[\frac{\ell/2 + \sqrt{x^2 + (\ell/2)^2}}{x} \right]_{\ell/2}^{3\ell/2} \right)$$

$$= \frac{-\rho_0^2\, \mathbf{a}_x}{2\pi\epsilon_0} \ln \left[\frac{(\ell/2)\left(1 + \sqrt{10}\right)}{3(\ell/2)\left(1 + \sqrt{2}\right)} \right] = \frac{\rho_0^2\, \mathbf{a}_x}{2\pi\epsilon_0} \ln \left[\frac{3(1 + \sqrt{2})}{1 + \sqrt{10}} \right] = \frac{0.55\rho_0^2}{2\pi\epsilon_0}\, \mathbf{a}_x \text{ N}$$

2.21. Two identical uniform line charges with $\rho_l = 75$ nC/m are located in free space at $x = 0$, $y = \pm 0.4$ m. What force per unit length does each line charge exert on the other? The charges are parallel to the z axis and are separated by 0.8 m. Thus the field from the charge at $y = -0.4$ evaluated at the location of the charge at $y = +0.4$ will be $\mathbf{E} = [\rho_l/(2\pi\epsilon_0(0.8))]\mathbf{a}_y$. The force on a differential length of the line at the positive y location is $d\mathbf{F} = dq\mathbf{E} = \rho_l dz \mathbf{E}$. Thus the force per unit length acting on the line at postive y arising from the charge at negative y is

$$\mathbf{F} = \int_0^1 \frac{\rho_l^2\, dz}{2\pi\epsilon_0(0.8)}\,\mathbf{a}_y = 1.26 \times 10^{-4}\,\mathbf{a}_y \text{ N/m} = \underline{126\,\mathbf{a}_y\ \mu\text{N/m}}$$

The force on the line at negative y is of course the same, but with $-\mathbf{a}_y$.

2.22. Two identical uniform sheet charges with $\rho_s = 100$ nC/m^2 are located in free space at $z = \pm 2.0$ cm. What force per unit area does each sheet exert on the other?

The field from the top sheet is $\mathbf{E} = -\rho_s/(2\epsilon_0)\,\mathbf{a}_z$ V/m. The differential force produced by this field on the bottom sheet is the charge density on the bottom sheet times the differential area there, multiplied by the electric field from the top sheet: $d\mathbf{F} = \rho_s da\mathbf{E}$. The force per unit area is then just $\mathbf{F} = \rho_s\mathbf{E} = (100 \times 10^{-9})(-100 \times 10^{-9})/(2\epsilon_0)\,\mathbf{a}_z = \underline{-5.6 \times 10^{-4}\,\mathbf{a}_z \text{ N/m}^2}$.

2.23. Given the surface charge density, $\rho_s = 2\,\mu$C/m^2, in the region $\rho < 0.2$ m, $z = 0$. Find $\mathbf{E}$ at:

a) $P_A(\rho = 0, z = 0.5)$: First, we recognize from symmetry that only a z component of $\mathbf{E}$ will be present. Considering a general point z on the z axis, we have $\mathbf{r} = z\mathbf{a}_z$. Then, with $\mathbf{r}' = \rho\mathbf{a}_\rho$, we obtain $\mathbf{r} - \mathbf{r}' = z\mathbf{a}_z - \rho\mathbf{a}_\rho$. The superposition integral for the z component of $\mathbf{E}$ will be:

$$E_{z,P_A} = \frac{\rho_s}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{0.2}\frac{z\,\rho\,d\rho\,d\phi}{(\rho^2+z^2)^{1.5}} = -\frac{2\pi\rho_s}{4\pi\epsilon_0}z\left[\frac{1}{\sqrt{z^2+\rho^2}}\right]_0^{0.2}$$
$$= \frac{\rho_s}{2\epsilon_0}z\left[\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{z^2+0.04}}\right]$$

With $z = 0.5$ m, the above evaluates as $E_{z,P_A} = \underline{8.1 \text{ kV/m}}$.

b) $P_B(\rho = 0, z = -0.5)$. With z at -0.5 m, we evaluate the expression for E_z to obtain $E_{z,P_B} = \underline{-8.1 \text{ kV/m}}$.

c) Show that the field along the z axis reduces to that of an infinite sheet charge at small values of z: In general, the field can be expressed as

$$E_z = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+0.04}}\right]$$

At small z, this reduces to $E_z \doteq \rho_s/2\epsilon_0$, which is the infinite sheet charge field.

d) Show that the z axis field reduces to that of a point charge at large values of z: The development is as follows:

$$E_z = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+0.04}}\right] = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{z}{z\sqrt{1+0.04/z^2}}\right] \doteq \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{1}{1+(1/2)(0.04)/z^2}\right]$$

where the last approximation is valid if $z >> .04$. Continuing:

$$E_z \doteq \frac{\rho_s}{2\epsilon_0}\left[1 - [1 - (1/2)(0.04)/z^2]\right] = \frac{0.04\rho_s}{4\epsilon_0 z^2} = \frac{\pi(0.2)^2\rho_s}{4\pi\epsilon_0 z^2}$$

This the point charge field, where we identify $q = \pi(0.2)^2\rho_s$ as the total charge on the disk (which now looks like a point).

2.24. a) Find the electric field on the z axis produced by an annular ring of uniform surface charge density ρ_s in free space. The ring occupies the region $z = 0$, $a \le \rho \le b$, $0 \le \phi \le 2\pi$ in cylindrical coordinates: We find the field through

$$\mathbf{E} = \int\int \frac{\rho_s da(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where the integral is taken over the surface of the annular ring, and where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. The integral then becomes

$$\mathbf{E} = \int_0^{2\pi} \int_a^b \frac{\rho_s\, \rho\, d\rho\, d\phi\, (z\mathbf{a}_z - \rho\mathbf{a}_\rho)}{4\pi\epsilon_0 (z^2 + \rho^2)^{3/2}}$$

In evaluating this integral, we first note that the term involving $\rho\mathbf{a}_\rho$ integrates to zero over the ϕ integration range of 0 to 2π. This is because we need to introduce the ϕ dependence in $\mathbf{a}_\rho$ by writing it as $\mathbf{a}_\rho = \cos\phi\, \mathbf{a}_x + \sin\phi\, \mathbf{a}_y$, where $\mathbf{a}_x$ and $\mathbf{a}_y$ are invariant in their orientation as ϕ varies. So the integral now simplifies to

$$\mathbf{E} = \frac{2\pi\rho_s\, z\, \mathbf{a}_z}{4\pi\epsilon_0} \int_a^b \frac{\rho\, d\rho}{(z^2 + \rho^2)^{3/2}} = \frac{\rho_s\, z\, \mathbf{a}_z}{2\epsilon_0} \left[\frac{-1}{\sqrt{z^2 + \rho^2}} \right]_a^b$$

$$= \frac{\rho_s}{2\epsilon_0} \left[\frac{1}{\sqrt{1 + (a/z)^2}} - \frac{1}{\sqrt{1 + (b/z)^2}} \right] \mathbf{a}_z$$

b) from your part a result, obtain the field of an infinite uniform sheet charge by taking appropriate limits. The infinite sheet is obtained by letting $a \to 0$ and $b \to \infty$, in which case $\mathbf{E} \to \rho_s/(2\epsilon_0)\, \mathbf{a}_z$ as expected.

2.25. Find $\mathbf{E}$ at the origin if the following charge distributions are present in free space: point charge, 12 nC at $P(2, 0, 6)$; uniform line charge density, 3nC/m at $x = -2$, $y = 3$; uniform surface charge density, $0.2\,\text{nC/m}^2$ at $x = 2$. The sum of the fields at the origin from each charge in order is:

$$\mathbf{E} = \left[\frac{(12 \times 10^{-9})}{4\pi\epsilon_0} \frac{(-2\mathbf{a}_x - 6\mathbf{a}_z)}{(4 + 36)^{1.5}} \right] + \left[\frac{(3 \times 10^{-9})}{2\pi\epsilon_0} \frac{(2\mathbf{a}_x - 3\mathbf{a}_y)}{(4 + 9)} \right] - \left[\frac{(0.2 \times 10^{-9})\mathbf{a}_x}{2\epsilon_0} \right]$$

$$= \underline{-3.9\mathbf{a}_x - 12.4\mathbf{a}_y - 2.5\mathbf{a}_z \text{ V/m}}$$

2.27. Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find:

a) the equation of the streamline that passes through the point $P(2, 3, -4)$: We write

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-(2x + 4y)}{(4x - 2y)}$$

Thus

$$2(x\,dy + y\,dx) = y\,dy - x\,dx$$

or

$$2\,d(xy) = \frac{1}{2}\,d(y^2) - \frac{1}{2}\,d(x^2)$$

So

$$C_1 + 2xy = \frac{1}{2}y^2 - \frac{1}{2}x^2$$

or

$$y^2 - x^2 = 4xy + C_2$$

Evaluating at $P(2, 3, -4)$, obtain:

$$9 - 4 = 24 + C_2, \text{ or } C_2 = -19$$

Finally, at P, the requested equation is

$$\underline{y^2 - x^2 = 4xy - 19}$$

b) a unit vector specifying the direction of $\mathbf{E}$ at $Q(3, -2, 5)$: Have $\mathbf{E}_Q = [4(3) + 2(2)]\mathbf{a}_x - [2(3) - 4(2)]\mathbf{a}_y = 16\mathbf{a}_x + 2\mathbf{a}_y$. Then $|\mathbf{E}| = \sqrt{16^2 + 4} = 16.12$ So

$$\mathbf{a}_Q = \frac{16\mathbf{a}_x + 2\mathbf{a}_y}{16.12} = \underline{0.99\mathbf{a}_x + 0.12\mathbf{a}_y}$$

2.28 An electric dipole (discussed in detail in Sec. 4.7) consists of two point charges of equal and opposite magnitude $\pm Q$ spaced by distance d. With the charges along the z axis at positions $z = \pm d/2$ (with the positive charge at the positive z location), the electric field in spherical coordinates is given by $\mathbf{E}(r,\theta) = \left[Qd/(4\pi\epsilon_0 r^3)\right][2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta]$, where $r >> d$. Using rectangular coordinates, determine expressions for the vector force on a point charge of magnitude q:

a) at $(0,0,z)$: Here, $\theta = 0$, $\mathbf{a}_r = \mathbf{a}_z$, and $r = z$. Therefore

$$\mathbf{F}(0,0,z) = \frac{qQd\,\mathbf{a}_z}{4\pi\epsilon_0 z^3} \text{ N}$$

b) at $(0,y,0)$: Here, $\theta = 90°$, $\mathbf{a}_\theta = -\mathbf{a}_z$, and $r = y$. The force is

$$\mathbf{F}(0,y,0) = \frac{-qQd\,\mathbf{a}_z}{4\pi\epsilon_0 y^3} \text{ N}$$

2.29. If $\mathbf{E} = 20e^{-5y}(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y)$, find:

a) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$: Substituting this point, we obtain $\mathbf{E}_P = -10.6\mathbf{a}_x - 6.1\mathbf{a}_y$, and so $|\mathbf{E}_P| = \underline{12.2}$.

b) a unit vector in the direction of $\mathbf{E}_P$: The unit vector associated with $\mathbf{E}$ is $(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y)$, which evaluated at P becomes $\mathbf{a}_E = \underline{-0.87\mathbf{a}_x - 0.50\mathbf{a}_y}$.

c) the equation of the direction line passing through P: Use

$$\frac{dy}{dx} = \frac{-\sin 5x}{\cos 5x} = -\tan 5x \quad \Rightarrow \quad dy = -\tan 5x\,dx$$

Thus $y = \frac{1}{5}\ln\cos 5x + C$. Evaluating at P, we find $C = 0.13$, and so

$$\underline{y = \frac{1}{5}\ln\cos 5x + 0.13}$$

2.30. For fields that do not vary with z in cylindrical coordinates, the equations of the streamlines are obtained by solving the differential equation $E_\rho/E_\phi = d\rho(\rho d\phi)$. Find the equation of the line passing through the point $(2, 30°, 0)$ for the field $\mathbf{E} = \rho\cos 2\phi\,\mathbf{a}_\rho - \rho\sin 2\phi\,\mathbf{a}_\phi$:

$$\frac{E_\rho}{E_\phi} = \frac{d\rho}{\rho d\phi} = \frac{-\rho\cos 2\phi}{\rho\sin 2\phi} = -\cot 2\phi \;\Rightarrow\; \frac{d\rho}{\rho} = -\cot 2\phi\,d\phi$$

Integrate to obtain

$$2\ln\rho = \ln\sin 2\phi + \ln C = \ln\left[\frac{C}{\sin 2\phi}\right] \;\Rightarrow\; \rho^2 = \frac{C}{\sin 2\phi}$$

At the given point, we have $4 = C/\sin(60°) \;\Rightarrow\; C = 4\sin 60° = 2\sqrt{3}$. Finally, the equation for the streamline is $\underline{\rho^2 = 2\sqrt{3}/\sin 2\phi}$.

CHAPTER 3

3.1. Suppose that the Faraday concentric sphere experiment is performed in free space using a central charge at the origin, Q_1, and with hemispheres of radius a. A second charge Q_2 (this time a point charge) is located at distance R from Q_1, where $R >> a$.

a) What is the force on the point charge before the hemispheres are assembled around Q_1? This will be simply the force beween two point charges, or

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\,\mathbf{a}_r$$

b) What is the force on the point charge after the hemispheres are assembled but before they are discharged? The answer will be the <u>same as in part *a*</u> because induced charge Q_1 now resides as a surface charge layer on the sphere exterior. This produces the same electric field at the Q_2 location as before, and so the force acting on Q_2 is the same.

c) What is the force on the point charge after the hemispheres are assembled and after they are discharged? Discharging the hemispheres (connecting them to ground) neutralizes the positive outside surface charge layer, thus zeroing the net field outside the sphere. The force on Q_2 is now <u>zero</u>.

d) Qualitatively, describe what happens as Q_2 is moved toward the sphere assembly to the extent that the condition $R >> a$ is no longer valid. Q_2 itself begins to induce negative surface charge on the sphere. An attractive force thus begins to strengthen as the charge moves closer. The point charge field approximation used in parts *a* through *c* is no longer valid.

3.2. An electric field in free space is $\mathbf{E} = (5z^2/\epsilon_0)\,\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a cube, centered at the origin, of 4-m side length, in which all sides are parallel to coordinate axes (and therefore each side intersects an axis at ± 2.

The flux density is $\mathbf{D} = \epsilon_0\mathbf{E} = 5z^2\,\mathbf{a}_z$. As $\mathbf{D}$ is z-directed only, it will intersect only the top and bottom surfaces (both parallel to the x-y plane). From Gauss' law, the charge in the cube is equal to the net outward flux of $\mathbf{D}$, which in this case is

$$Q_{encl} = \oint \mathbf{D}\cdot\mathbf{n}\,da = \int_{-2}^{2}\int_{-2}^{2} 5(2)^2\mathbf{a}_z\cdot\mathbf{a}_z\,dx\,dy + \int_{-2}^{2}\int_{-2}^{2} 5(-2)^2\mathbf{a}_z\cdot(-\mathbf{a}_z)\,dx\,dy = \underline{0}$$

where the first and second integrals on the far right are over the top and bottom surfaces respectively.

3.3. The cylindrical surface $\rho = 8$ cm contains the surface charge density, $\rho_s = 5e^{-20|z|}$ nC/m^2.

a) What is the total amount of charge present? We integrate over the surface to find:

$$Q = 2\int_0^\infty \int_0^{2\pi} 5e^{-20z}(.08)d\phi\, dz \text{ nC} = 20\pi(.08)\left(\frac{-1}{20}\right)e^{-20z}\Big|_0^\infty = \underline{0.25\,\text{nC}}$$

b) How much flux leaves the surface $\rho = 8\,\text{cm}$, $1\,\text{cm} < z < 5\text{cm}$, $30° < \phi < 90°$? We just integrate the charge density on that surface to find the flux that leaves it.

$$\Phi = Q' = \int_{.01}^{.05}\int_{30°}^{90°} 5e^{-20z}(.08)\,d\phi\,dz \text{ nC} = \left(\frac{90-30}{360}\right)2\pi(5)(.08)\left(\frac{-1}{20}\right)e^{-20z}\Big|_{.01}^{.05}$$
$$= 9.45 \times 10^{-3}\,\text{nC} = \underline{9.45\,\text{pC}}$$

3.4. An electric field in free space is $\mathbf{E} = (5z^3/\epsilon_0)\,\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a sphere of 3-m radius, centered at the origin. Using Gauss' law, we set up the integral in free space over the sphere surface, whose outward unit normal is $\mathbf{a}_r$:

$$Q = \oint \epsilon_0 \mathbf{E}\cdot\mathbf{n}\,da = \int_0^{2\pi}\int_0^{\pi} 5z^3\,\mathbf{a}_z\cdot\mathbf{a}_r\,(3)^2\sin\theta\,d\theta\,d\phi$$

where in this case $z = 3\cos\theta$ and (in all cases) $\mathbf{a}_z\cdot\mathbf{a}_r = \cos\theta$. These are substituted to yield

$$Q = 2\pi\int_0^{\pi} 5(3)^5\cos^4\theta\sin\theta d\theta = -2\pi(5)(3)^5\left(\frac{1}{5}\right)\cos^5\theta\Big|_0^{2\pi} = \underline{972\pi}$$

3.5. Let $\mathbf{D} = 4xy\mathbf{a}_x + 2(x^2+z^2)\mathbf{a}_y + 4yz\mathbf{a}_z$ C/m^2 and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped $0 < x < 2$, $0 < y < 3$, $0 < z < 5$ m: Of the 6 surfaces to consider, only 2 will contribute to the net outward flux. Why? First consider the planes at $y = 0$ and 3. The y component of $\mathbf{D}$ will penetrate those surfaces, but will be inward at $y = 0$ and outward at $y = 3$, while having the same magnitude in both cases. These fluxes will thus cancel. At the $x = 0$ plane, $D_x = 0$ and at the $z = 0$ plane, $D_z = 0$, so there will be no flux contributions from these surfaces. This leaves the 2 remaining surfaces at $x = 2$ and $z = 5$. The net outward flux becomes:

$$\Phi = \int_0^5\int_0^3 \mathbf{D}\Big|_{x=2}\cdot\mathbf{a}_x\,dy\,dz + \int_0^3\int_0^2 \mathbf{D}\Big|_{z=5}\cdot\mathbf{a}_z\,dx\,dy$$
$$= 5\int_0^3 4(2)y\,dy \;+\; 2\int_0^3 4(5)y\,dy = \underline{360\text{ C}}$$

3.6. In free space, volume charge of constant density $\rho_v = \rho_0$ exists within the region $-\infty < x < \infty$, $-\infty < y < \infty$, and $-d/2 < z < d/2$. Find **D** and **E** everywhere.

From the symmetry of the configuration, we surmise that the field will be everywhere z-directed, and will be uniform with x and y at fixed z. For finding the field inside the charge, an appropriate Gaussian surface will be that which encloses a rectangular region defined by $-1 < x < 1$, $-1 < y < 1$, and $|z| < d/2$. The outward flux from this surface will be limited to that through the two parallel surfaces at $\pm z$:

$$\Phi_{in} = \oint \mathbf{D} \cdot d\mathbf{S} = 2\int_{-1}^{1}\int_{-1}^{1} D_z\,dxdy = Q_{encl} = \int_{-z}^{z}\int_{-1}^{1}\int_{-1}^{1} \rho_0\,dxdydz'$$

where the factor of 2 in the second integral account for the equal fluxes through the two surfaces. The above readily simplifies, as both D_z and ρ_0 are constants, leading to $\underline{\mathbf{D}_{in} = \rho_0 z\,\mathbf{a}_z \text{ C/m}^2 \quad (|z| < d/2)}$, and therefore $\underline{\mathbf{E}_{in} = (\rho_0 z/\epsilon_0)\,\mathbf{a}_z \text{ V/m} \quad (|z| < d/2)}$.

Outside the charge, the Gaussian surface is the same, except that the parallel boundaries at $\pm z$ occur at $|z| > d/2$. As a result, the calculation is nearly the same as before, with the only change being the limits on the total charge integral:

$$\Phi_{out} = \oint \mathbf{D} \cdot d\mathbf{S} = 2\int_{-1}^{1}\int_{-1}^{1} D_z\,dxdy = Q_{encl} = \int_{-d/2}^{d/2}\int_{-1}^{1}\int_{-1}^{1} \rho_0\,dxdydz'$$

Solve for D_z to find the constant values:

$$\mathbf{D}_{out} = \begin{cases} (\rho_0 d/2)\,\mathbf{a}_z & (z > d/2) \\ -(\rho_0 d/2)\,\mathbf{a}_z & (z < d/2) \end{cases} \text{ C/m}^2 \quad \text{and} \quad \mathbf{E}_{out} = \begin{cases} (\rho_0 d/2\epsilon_0)\,\mathbf{a}_z & (z > d/2) \\ -(\rho_0 d/2\epsilon_0)\,\mathbf{a}_z & (z < d/2) \end{cases} \text{ V/m}$$

3.7. Volume charge density is located in free space as $\rho_v = 2e^{-1000r}$ nC/m^3 for $0 < r < 1$ mm, and $\rho_v = 0$ elsewhere.

a) Find the total charge enclosed by the spherical surface $r = 1$ mm: To find the charge we integrate:

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{.001} 2e^{-1000r} r^2 \sin\theta\,dr\,d\theta\,d\phi$$

Integration over the angles gives a factor of 4π. The radial integration we evaluate using tables; we obtain

$$Q = 8\pi\left[\frac{-r^2 e^{-1000r}}{1000}\bigg|_0^{.001} + \frac{2}{1000}\frac{e^{-1000r}}{(1000)^2}(-1000r - 1)\bigg|_0^{.001}\right] = \underline{4.0 \times 10^{-9}\,\text{nC}}$$

b) By using Gauss's law, calculate the value of D_r on the surface $r = 1$ mm: The gaussian surface is a spherical shell of radius 1 mm. The enclosed charge is the result of part a. We thus write $4\pi r^2 D_r = Q$, or

$$D_r = \frac{Q}{4\pi r^2} = \frac{4.0 \times 10^{-9}}{4\pi(.001)^2} = \underline{3.2 \times 10^{-4} \text{ nC/m}^2}$$

3.8. Use Gauss's law in integral form to show that an inverse distance field in spherical coordinates, $\mathbf{D} = A\mathbf{a}_r/r$, where A is a constant, requires every spherical shell of 1 m thickness to contain $4\pi A$ coulombs of charge. Does this indicate a continuous charge distribution? If so, find the charge density variation with r.

The net outward flux of this field through a spherical surface of radius r is

$$\Phi = \oint \mathbf{D} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} \frac{A}{r}\mathbf{a}_r \cdot \mathbf{a}_r \, r^2 \sin\theta \, d\theta \, d\phi = 4\pi A r = Q_{encl}$$

We see from this that with every increase in r by one m, the enclosed charge increases by $4\pi A$ (done). It is evident that the charge density is continuous, and we can find the density indirectly by constructing the integral for the enclosed charge, in which we already found the latter from Gauss's law:

$$Q_{encl} = 4\pi A r = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \rho(r') \, (r')^2 \sin\theta \, dr' \, d\theta \, d\phi = 4\pi \int_0^{r} \rho(r') \, (r')^2 \, dr'$$

To obtain the correct enclosed charge, the integrand must be $\rho(r) = \underline{A/r^2}$.

3.9. A uniform volume charge density of $80\,\mu\text{C/m}^3$ is present throughout the region $8\,\text{mm} < r < 10\,\text{mm}$. Let $\rho_v = 0$ for $0 < r < 8\,\text{mm}$.

a) Find the total charge inside the spherical surface $r = 10$ mm: This will be

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_{.008}^{.010} (80 \times 10^{-6}) r^2 \sin\theta \, dr \, d\theta \, d\phi = 4\pi \times (80 \times 10^{-6}) \frac{r^3}{3}\Big|_{.008}^{.010}$$
$$= 1.64 \times 10^{-10}\,\text{C} = \underline{164\,\text{pC}}$$

b) Find D_r at $r = 10$ mm: Using a spherical gaussian surface at $r = 10$, Gauss' law is written as $4\pi r^2 D_r = Q = 164 \times 10^{-12}$, or

$$D_r(10\,\text{mm}) = \frac{164 \times 10^{-12}}{4\pi(.01)^2} = 1.30 \times 10^{-7}\,\text{C/m}^2 = \underline{130\,\text{nC/m}^2}$$

c) If there is no charge for $r > 10$ mm, find D_r at $r = 20$ mm: This will be the same computation as in part b, except the gaussian surface now lies at 20 mm. Thus

$$D_r(20\,\text{mm}) = \frac{164 \times 10^{-12}}{4\pi(.02)^2} = 3.25 \times 10^{-8}\,\text{C/m}^2 = \underline{32.5\,\text{nC/m}^2}$$

3.10. An infinitely long cylindrical dielectric of radius b contains charge within its volume of density $\rho_v = a\rho^2$, where a is a constant. Find the electric field strength, $\mathbf{E}$, both inside and outside the cylinder.

Inside, we note from symmetry that $\mathbf{D}$ will be radially-directed, in the manner of a line charge field. So we apply Gauss' law to a cylindrical surface of radius ρ, concentric with the charge distribution, having unit length in z, and where $\rho < b$. The outward normal to the surface is $\mathbf{a}_\rho$.

$$\oint \mathbf{D}\cdot\mathbf{n}\,da = \int_0^1\int_0^{2\pi} D_\rho\,\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,\rho\,d\phi\,dz = Q_{encl} = \int_0^1\int_0^{2\pi}\int_0^{\rho} a(\rho')^2\rho'\,d\rho'\,d\phi\,dz$$

in which the dummy variable ρ' must be used in the far-right integral because the upper radial limit is ρ. D_ρ is constant over the surface and can be factored outside the integral. Evaluating both integrals leads to

$$2\pi(1)\rho D_\rho = 2\pi a\left(\frac{1}{4}\right)\rho^4 \;\Rightarrow\; D_\rho = \frac{a\rho^3}{4} \quad \text{or} \quad \underline{\mathbf{E}_{in} = \frac{a\rho^3}{4\epsilon_0}\,\mathbf{a}_\rho\ (\rho < b)}$$

To find the field outside the cylinder, we apply Gauss' law to a cylinder of radius $\rho > b$. The setup now changes only by the upper radius limit for the charge integral, which is now the charge radius, b:

$$\oint \mathbf{D}\cdot\mathbf{n}\,da = \int_0^1\int_0^{2\pi} D_\rho\,\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,\rho\,d\phi\,dz = Q_{encl} = \int_0^1\int_0^{2\pi}\int_0^{b} a\rho^2\rho\,d\rho\,d\phi\,dz$$

where the dummy variable is no longer needed. Evaluating as before, the result is

$$D_\rho = \frac{ab^4}{4\rho} \quad \text{or} \quad \underline{\mathbf{E}_{out} = \frac{ab^4}{4\epsilon_0\rho}\,\mathbf{a}_\rho\ (\rho > b)}$$

3.11. In cylindrical coordinates, let $\rho_v = 0$ for $\rho < 1$ mm, $\rho_v = 2\sin(2000\pi\rho)$ nC/m^3 for $1\,\text{mm} < \rho < 1.5\,\text{mm}$, and $\rho_v = 0$ for $\rho > 1.5\,\text{mm}$. Find $\mathbf{D}$ everywhere: Since the charge varies only with radius, and is in the form of a cylinder, symmetry tells us that the flux density will be radially-directed and will be constant over a cylindrical surface of a fixed radius. Gauss' law applied to such a surface of unit length in z gives:

a) for $\rho < 1$ mm, $\underline{D_\rho = 0}$, since no charge is enclosed by a cylindrical surface whose radius lies within this range.

b) for $1\,\text{mm} < \rho < 1.5\,\text{mm}$, we have

$$2\pi\rho D_\rho = 2\pi\int_{.001}^{\rho} 2\times10^{-9}\sin(2000\pi\rho')\rho'\,d\rho'$$
$$= 4\pi\times10^{-9}\left[\frac{1}{(2000\pi)^2}\sin(2000\pi\rho) - \frac{\rho}{2000\pi}\cos(2000\pi\rho)\right]_{.001}^{\rho}$$

or finally,

$$D_\rho = \frac{10^{-15}}{2\pi^2\rho}\Big[\sin(2000\pi\rho) + 2\pi\left[1 - 10^3\rho\cos(2000\pi\rho)\right]\Big]\ \text{C/m}^2 \quad (1\,\text{mm} < \rho < 1.5\,\text{mm})$$

3.11c) for $\rho > 1.5\,\text{mm}$, the gaussian cylinder now lies at radius ρ *outside* the charge distribution, so the integral that evaluates the enclosed charge now includes the entire charge distribution. To accomplish this, we change the upper limit of the integral of part b from ρ to 1.5 mm, finally obtaining:

$$D_\rho = \frac{2.5 \times 10^{-15}}{\pi\rho} \text{ C/m}^2 \quad (\rho > 1.5\,\text{mm})$$

3.12. The sun radiates a total power of about 3.86×10^{26} watts (W). If we imagine the sun's surface to be marked off in latitude and longitude and assume uniform radiation,

a) What power is radiated by the region lying between latitude 50° N and 60° N and longitude 12° W and 27° W?

50° N lattitude and 60° N lattitude correspond respectively to $\theta = 40°$ and $\theta = 30°$. 12° and 27° correspond directly to the limits on ϕ. Since the sun for our purposes is spherically-symmetric, the flux density emitted by it is $\mathbf{I} = 3.86 \times 10^{26}/(4\pi r^2)\,\mathbf{a}_r$ W/m^2. The required power is now found through

$$P_1 = \int_{12°}^{27°}\int_{30°}^{40°} \frac{3.86 \times 10^{26}}{4\pi r^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, r^2 \sin\theta\, d\theta\, d\phi$$
$$= \frac{3.86 \times 10^{26}}{4\pi}\,[\cos(30°) - \cos(40°)]\,(27° - 12°)\left(\frac{2\pi}{360}\right) = \underline{8.1 \times 10^{23} \text{ W}}$$

b) What is the power density on a spherical surface 93,000,000 miles from the sun in W/m^2?

First, 93,000,000 miles = 155,000,000 km = 1.55×10^{11} m. Use this distance in the flux density expression above to obtain

$$\mathbf{I} = \frac{3.86 \times 10^{26}}{4\pi(1.55 \times 10^{11})^2}\,\mathbf{a}_r = \underline{1200\,\mathbf{a}_r \text{ W/m}^2}$$

3.13. Spherical surfaces at $r = 2$, 4, and 6 m carry uniform surface charge densities of 20 nC/m^2, -4 nC/m^2, and ρ_{s0}, respectively.

a) Find $\mathbf{D}$ at $r = 1, 3$ and 5 m: Noting that the charges are spherically-symmetric, we ascertain that $\mathbf{D}$ will be radially-directed and will vary only with radius. Thus, we apply Gauss' law to spherical shells in the following regions: $r < 2$: Here, no charge is enclosed, and so $\underline{D_r = 0}$.

$$2 < r < 4: \quad 4\pi r^2 D_r = 4\pi(2)^2(20 \times 10^{-9}) \quad \Rightarrow \quad D_r = \frac{80 \times 10^{-9}}{r^2} \text{ C/m}^2$$

So $D_r(r = 3) = \underline{8.9 \times 10^{-9} \text{ C/m}^2}$.

$$4 < r < 6: \quad 4\pi r^2 D_r = 4\pi(2)^2(20 \times 10^{-9}) + 4\pi(4)^2(-4 \times 10^{-9}) \quad \Rightarrow \quad D_r = \frac{16 \times 10^{-9}}{r^2}$$

So $D_r(r = 5) = \underline{6.4 \times 10^{-10} \text{ C/m}^2}$.

b) Determine ρ_{s0} such that $\mathbf{D} = 0$ at $r = 7$ m. Since fields will decrease as $1/r^2$, the question could be re-phrased to ask for ρ_{s0} such that $\mathbf{D} = 0$ at *all* points where $r > 6$ m. In this region, the total field will be

$$D_r(r > 6) = \frac{16 \times 10^{-9}}{r^2} + \frac{\rho_{s0}(6)^2}{r^2}$$

Requiring this to be zero, we find $\rho_{s0} = \underline{-(4/9) \times 10^{-9} \text{ C/m}^2}$.

3.14. A certain light-emitting diode (LED) is centered at the origin with its surface in the xy plane. At far distances, the LED appears as a point, but the glowing surface geometry produces a far-field radiation pattern that follows a raised cosine law: That is, the optical power (flux) density in Watts/m^2 is given in spherical coordinates by

$$\mathbf{P}_d = P_0 \frac{\cos^2\theta}{2\pi r^2}\,\mathbf{a}_r \qquad \text{Watts/m}^2$$

where θ is the angle measured with respect to the normal to the LED surface (in this case, the z axis), and r is the radial distance from the origin at which the power is detected.

a) Find, in terms of P_0, the total power in Watts emitted in the upper half-space by the LED: We evaluate the surface integral of the power density over a hemispherical surface of radius r:

$$P_t = \int_0^{2\pi}\int_0^{\pi/2} P_0 \frac{\cos^2\theta}{2\pi r^2}\,\mathbf{a}_r\cdot\mathbf{a}_r\, r^2\sin\theta\, d\theta\, d\phi = -\frac{P_0}{3}\cos^3\theta\Big|_0^{\pi/2} = \underline{\frac{P_0}{3}}$$

b) Find the cone angle, θ_1, within which half the total power is radiated; i.e., within the range $0 < \theta < \theta_1$: We perform the same integral as in part a except the upper limit for θ is now θ_1. The result must be one-half that of part a, so we write:

$$\frac{P_t}{2} = \frac{P_0}{6} = -\frac{P_0}{3}\cos^3\theta\Big|_0^{\theta_1} = \frac{P_0}{3}\left(1-\cos^3\theta_1\right) \;\Rightarrow\; \theta_1 = \cos^{-1}\left(\frac{1}{2^{1/3}}\right) = \underline{37.5^\circ}$$

c) An optical detector, having a 1 mm^2 cross-sectional area, is positioned at $r = 1$ m and at $\theta = 45^\circ$, such that it faces the LED. If one nanowatt (stated in error as 1mW) is measured by the detector, what (to a very good estimate) is the value of P_0? Start with

$$\mathbf{P}_d(45^\circ) = P_0\frac{\cos^2(45^\circ)}{2\pi r^2}\mathbf{a}_r = \frac{P_0}{4\pi r^2}\mathbf{a}_r$$

Then the detected power in a 1-mm^2 area at $r = 1$ m approximates as

$$P[W] \doteq \frac{P_0}{4\pi}\times 10^{-6} = 10^{-9} \;\Rightarrow\; P_0 \doteq \underline{4\pi\times 10^{-3}\ \text{W}}$$

If the originally stated 1mW value is used for the detected power, the answer would have been 4π kW (!).

3.15. Volume charge density is located as follows: $\rho_v = 0$ for $\rho < 1$ mm and for $\rho > 2$ mm, $\rho_v = 4\rho\ \mu\text{C/m}^3$ for $1 < \rho < 2$ mm.

a) Calculate the total charge in the region $0 < \rho < \rho_1$, $0 < z < L$, where $1 < \rho_1 < 2$ mm: We find,

$$Q = \int_0^L\int_0^{2\pi}\int_{.001}^{\rho_1} 4\rho\,\rho\, d\rho\, d\phi\, dz = \underline{\frac{8\pi L}{3}[\rho_1^3 - 10^{-9}]\ \mu\text{C}}$$

b) Use Gauss' law to determine D_ρ at $\rho = \rho_1$: Gauss' law states that $2\pi\rho_1 L D_\rho = Q$, where Q is the result of part a. So, with ρ_1 in meters,

$$D_\rho(\rho_1) = \underline{\frac{4(\rho_1^3 - 10^{-9})}{3\rho_1}\ \mu\text{C/m}^2}$$

3.15c) Evaluate D_ρ at $\rho = 0.8\,\text{mm},\ 1.6\,\text{mm},$ and $2.4\,\text{mm}$: At $\rho = 0.8\,\text{mm}$, no charge is enclosed by a cylindrical gaussian surface of that radius, so $D_\rho(0.8\text{mm}) = \underline{0}$. At $\rho = 1.6\,\text{mm}$, we evaluate the part b result at $\rho_1 = 1.6$ to obtain:

$$D_\rho(1.6\text{mm}) = \frac{4[(.0016)^3 - (.0010)^3]}{3(.0016)} = \underline{3.6 \times 10^{-6}\ \mu\text{C/m}^2}$$

At $\rho = 2.4$, we evaluate the charge integral of part a from .001 to .002, and Gauss' law is written as

$$2\pi\rho L D_\rho = \frac{8\pi L}{3}[(.002)^2 - (.001)^2]\ \mu\text{C}$$

from which $D_\rho(2.4\text{mm}) = \underline{3.9 \times 10^{-6}\ \mu\text{C/m}^2}$.

3.16. An electric flux density is given by $\mathbf{D} = D_0\,\mathbf{a}_\rho$, where D_0 is a given constant.

a) What charge density generates this field? Charge density is found by taking the divergence: With radial $\mathbf{D}$ only, we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{d}{d\rho}(\rho D_0) = \frac{D_0}{\rho}\ \ \text{C/m}^3$$

b) For the specified field, what total charge is contained within a cylinder of radius a and height b, where the cylinder axis is the z axis? We can either integrate the charge density over the specified volume, or integrate $\mathbf{D}$ over the surface that contains the specified volume:

$$Q = \int_0^b \int_0^{2\pi} \int_0^a \frac{D_0}{\rho}\,\rho\,d\rho\,d\phi\,dz = \int_0^b \int_0^{2\pi} D_0\,\mathbf{a}_\rho \cdot \mathbf{a}_\rho\,a\,d\phi\,dz = \underline{2\pi a b D_0}\ \text{C}$$

3.17. A cube is defined by $1 < x, y, z < 1.2$. If $\mathbf{D} = 2x^2 y\mathbf{a}_x + 3x^2 y^2 \mathbf{a}_y$ C/m^2:

a) apply Gauss' law to find the total flux leaving the closed surface of the cube. We call the surfaces at $x = 1.2$ and $x = 1$ the front and back surfaces respectively, those at $y = 1.2$ and $y = 1$ the right and left surfaces, and those at $z = 1.2$ and $z = 1$ the top and bottom surfaces. To evaluate the total charge, we integrate $\mathbf{D} \cdot \mathbf{n}$ over all six surfaces and sum the results. We note that there is no z component of $\mathbf{D}$, so there will be no outward flux contributions from the top and bottom surfaces. The fluxes through the remaining four are

$$\Phi = Q = \oint \mathbf{D} \cdot \mathbf{n}\,da = \underbrace{\int_1^{1.2}\int_1^{1.2} 2(1.2)^2 y\,dy\,dz}_{\text{front}} + \underbrace{\int_1^{1.2}\int_1^{1.2} -2(1)^2 y\,dy\,dz}_{\text{back}}$$

$$+ \underbrace{\int_1^{1.2}\int_1^{1.2} -3x^2(1)^2\,dx\,dz}_{\text{left}} + \underbrace{\int_1^{1.2}\int_1^{1.2} 3x^2(1.2)^2\,dx\,dz}_{\text{right}} = \underline{0.1028\,\text{C}}$$

b) evaluate $\nabla \cdot \mathbf{D}$ at the center of the cube: This is

$$\nabla \cdot \mathbf{D} = \left[4xy + 6x^2 y\right]_{(1.1,1.1)} = 4(1.1)^2 + 6(1.1)^3 = \underline{12.83}$$

c) Estimate the total charge enclosed within the cube by using Eq. (8): This is

$$Q \doteq \nabla \cdot \mathbf{D}\Big|_{\text{center}} \times \Delta v = 12.83 \times (0.2)^3 = \underline{0.1026}\ \ \text{Close!}$$

3.18. State whether the divergence of the following vector fields is positive, negative, or zero:

a) the thermal energy flow in $\mathrm{J/(m^2 - s)}$ at any point in a freezing ice cube: One way to visualize this is to consider that heat is escaping through the surface of the ice cube as it freezes. Therefore the net outward flux of thermal energy through the surface is positive. Calling the thermal flux density $\mathbf{F}$, the divergence theorem says

$$\oint_s \mathbf{F} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{F}\, dv$$

and so if we identify the left integral as positive, the right integral (and its integrand) must also be positive. Answer: positive.

b) the current density in $\mathrm{A/m^2}$ in a bus bar carrying direct current: In this case, we have no accumulation or dissipation of charge within any small volume, since current is dc; this also means that the net outward current flux through the surface that surrounds any small volume is zero. Therefore the divergence must be zero.

c) the mass flow rate in $\mathrm{kg/(m^2 - s)}$ below the surface of water in a basin, in which the water is circulating clockwise as viewed from above: Here again, taking any small volume in the water, the net outward flow through the surface that surrounds the small volume is zero; i.e., there is no accumulation or dissipation of mass that would result in a change in density at any point. Divergence is therefore zero.

3.19. A spherical surface of radius 3 mm is centered at $P(4,1,5)$ in free space. Let $\mathbf{D} = x\mathbf{a}_x$ C/m^2. Use the results of Sec. 3.4 to estimate the net electric flux leaving the spherical surface: We use $\Phi \doteq \nabla \cdot \mathbf{D}\Delta v$, where in this case $\nabla \cdot \mathbf{D} = (\partial/\partial x)x = 1$ C/m^3. Thus

$$\Phi \doteq \frac{4}{3}\pi(.003)^3(1) = 1.13 \times 10^{-7}\,\mathrm{C} = \underline{113\,\mathrm{nC}}$$

3.20. A radial electric field distribution in free space is given in spherical coordinates as:

$$\mathbf{E}_1 = \frac{r\rho_0}{3\epsilon_0}\,\mathbf{a}_r \qquad (r \le a)$$

$$\mathbf{E}_2 = \frac{(2a^3 - r^3)\rho_0}{3\epsilon_0\, r^2}\,\mathbf{a}_r \qquad (a \le r \le b)$$

$$\mathbf{E}_3 = \frac{(2a^3 - b^3)\rho_0}{3\epsilon_0\, r^2}\,\mathbf{a}_r \qquad (r \ge b)$$

where ρ_0, a, and b are constants.

a) Determine the volume charge density in the entire region $(0 \le r \le \infty)$ by appropriate use of $\nabla \cdot \mathbf{D} = \rho_v$. We find ρ_v by taking the divergence of $\mathbf{D}$ in all three regions, where $\mathbf{D} = \epsilon_0 \mathbf{E}$. As $\mathbf{D}$ has only a radial component, the divergences become:

$$\rho_{v1} = \nabla \cdot \mathbf{D}_1 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 D_1\right) = \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^3\rho_0}{3}\right) = \underline{\rho_0} \qquad (r \le a)$$

$$\rho_{v2} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 D_2\right) = \frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{3}(2a^3 - r^3)\rho_0\right) = \underline{-\rho_0} \qquad (a \le r \le b)$$

$$\rho_{v3} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 D_3\right) = \frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{3}(2a^3 - b^3)\rho_0\right) = \underline{0} \qquad (r \ge b)$$

3.20b) Find, in terms of given parameters, the total charge, Q, within a sphere of radius r where $r > b$. We integrate the charge densities (piecewise) over the spherical volume of radius b:

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho_0 \, r^2 \sin\theta \, dr \, d\theta \, d\phi - \int_0^{2\pi} \int_0^{\pi} \int_a^{b} \rho_0 \, r^2 \sin\theta \, dr \, d\theta \, d\phi = \underline{\frac{4}{3}\pi \left(2a^3 - b^3\right) \rho_0}$$

3.21. Calculate the divergence of $\mathbf{D}$ at the point specified if

a) $\mathbf{D} = (1/z^2)\left[10xyz\,\mathbf{a}_x + 5x^2 z\,\mathbf{a}_y + (2z^3 - 5x^2 y)\,\mathbf{a}_z\right]$ at $P(-2,3,5)$: We find

$$\nabla \cdot \mathbf{D} = \left[\frac{10y}{z} + 0 + 2 + \frac{10x^2 y}{z^3}\right]_{(-2,3,5)} = \underline{8.96}$$

b) $\mathbf{D} = 5z^2 \mathbf{a}_\rho + 10\rho z\,\mathbf{a}_z$ at $P(3, -45°, 5)$: In cylindrical coordinates, we have

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} = \left[\frac{5z^2}{\rho} + 10\rho\right]_{(3,-45°,5)} = \underline{71.67}$$

c) $\mathbf{D} = 2r\sin\theta\sin\phi\,\mathbf{a}_r + r\cos\theta\sin\phi\,\mathbf{a}_\theta + r\cos\phi\,\mathbf{a}_\phi$ at $P(3, 45°, -45°)$: In spherical coordinates, we have

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

$$= \left[6\sin\theta\sin\phi + \frac{\cos 2\theta \sin\phi}{\sin\theta} - \frac{\sin\phi}{\sin\theta}\right]_{(3,45°,-45°)} = \underline{-2}$$

3.22. (a) A flux density field is given as $\mathbf{F}_1 = 5\mathbf{a}_z$. Evaluate the outward flux of $\mathbf{F}_1$ through the hemispherical surface, $r = a$, $0 < \theta < \pi/2$, $0 < \phi < 2\pi$.

The flux integral is

$$\Phi_1 = \int_{hem.} \mathbf{F}_1 \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi/2} 5 \underbrace{\mathbf{a}_z \cdot \mathbf{a}_r}_{\cos\theta} \, a^2 \sin\theta \, d\theta \, d\phi = -2\pi(5)a^2 \frac{\cos^2\theta}{2}\Big|_0^{\pi/2} = \underline{5\pi a^2}$$

b) What simple observation would have saved a lot of work in part a? The field is constant, and so the inward flux through the base of the hemisphere (of area πa^2) would be equal in magnitude to the outward flux through the upper surface (the flux through the base is a much easier calculation).

c) Now suppose the field is given by $\mathbf{F}_2 = 5z\mathbf{a}_z$. Using the appropriate surface integrals, evaluate the net outward flux of $\mathbf{F}_2$ through the closed surface consisting of the hemisphere of part a and its circular base in the xy plane:

Note that the integral over the base is zero, since $\mathbf{F}_2 = 0$ there. The remaining flux integral is that over the hemisphere:

$$\Phi_2 = \int_0^{2\pi} \int_0^{\pi/2} 5z\,\mathbf{a}_z \cdot \mathbf{a}_r \, a^2 \sin\theta \, d\theta \, d\phi = \int_0^{2\pi} \int_0^{\pi/2} 5(a\cos\theta)\cos\theta \, a^2 \sin\theta \, d\theta \, d\phi$$

$$= 10\pi a^3 \int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta \, d\phi = -\frac{10}{3}\pi a^3 \cos^3\theta \Big|_0^{\pi/2} = \underline{\frac{10}{3}\pi a^3}$$

d) Repeat part c by using the divergence theorem and an appropriate volume integral:

The divergence of $\mathbf{F}_2$ is just $dF_2/dz = 5$. We then integrate this over the hemisphere volume, which in this case involves just multiplying 5 by $(2/3)\pi a^3$, giving the same answer as in part c.

3.23. a) A point charge Q lies at the origin. Show that div $\mathbf{D}$ is zero everywhere except at the origin. For a point charge at the origin we know that $\mathbf{D} = Q/(4\pi r^2)\,\mathbf{a}_r$. Using the formula for divergence in spherical coordinates (see problem 3.21 solution), we find in this case that

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{Q}{4\pi r^2}\right) = 0$$

The above is true provided $r > 0$. When $r = 0$, we have a singularity in $\mathbf{D}$, so its divergence is not defined.

b) Replace the point charge with a uniform volume charge density ρ_{v0} for $0 < r < a$. Relate ρ_{v0} to Q and a so that the total charge is the same. Find div $\mathbf{D}$ everywhere: To achieve the same net charge, we require that $(4/3)\pi a^3 \rho_{v0} = Q$, so $\underline{\rho_{v0} = 3Q/(4\pi a^3)\ \text{C/m}^3}$. Gauss' law tells us that inside the charged sphere

$$4\pi r^2 D_r = \frac{4}{3}\pi r^3 \rho_{v0} = \frac{Qr^3}{a^3}$$

Thus

$$D_r = \frac{Qr}{4\pi a^3}\ \text{C/m}^2 \ \text{and}\ \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(\frac{Qr^3}{4\pi a^3}\right) = \frac{3Q}{4\pi a^3}$$

as expected. Outside the charged sphere, $\mathbf{D} = Q/(4\pi r^2)\,\mathbf{a}_r$ as before, and the divergence is zero.

3.24. In a region in free space, electric flux density is found to be:

$$\mathbf{D} = \begin{cases} \rho_0(z+2d)\,\mathbf{a}_z \ \ \text{C/m}^2 & (-2d \le z \le 0) \\ -\rho_0(z-2d)\,\mathbf{a}_z \ \ \text{C/m}^2 & (0 \le z \le 2d) \end{cases}$$

Everywhere else, $\mathbf{D} = 0$.

a) Using $\nabla \cdot \mathbf{D} = \rho_v$, find the volume charge density as a function of position everywhere: Use

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{dD_z}{dz} = \begin{cases} \rho_0 & (-2d \le z \le 0) \\ -\rho_0 & (0 \le z \le 2d) \end{cases}$$

b) determine the electric flux that passes through the surface defined by $z = 0$, $-a \le x \le a$, $-b \le y \le b$: In the x-y plane, $\mathbf{D}$ evaluates as the constant $\mathbf{D}(0) = 2d\rho_0\,\mathbf{a}_z$. Therefore the flux passing through the given area will be

$$\Phi = \int_{-a}^{a}\int_{-b}^{b} 2d\rho_0\, dx dy = \underline{8abd\,\rho_0}\ \text{C}$$

c) determine the total charge contained within the region $-a \le x \le a$, $-b \le y \le b$, $-d \le z \le d$: From part a, we have equal and opposite charge densities above and below the x-y plane. This means that within a region having equal volumes above and below the plane, the net charge is <u>zero</u>.

d) determine the total charge contained within the region $-a \le x \le a$, $-b \le y \le b$, $0 \le z \le 2d$. In this case,

$$Q = -\rho_0\,(2a)\,(2b)\,(2d) = \underline{-8abd\,\rho_0}\ \text{C}$$

This is equivalent to the net *inward* flux of $\mathbf{D}$ into the volume, as was found in part b.

3.25. Within the spherical shell, $3 < r < 4$ m, the electric flux density is given as

$$\mathbf{D} = 5(r-3)^3\,\mathbf{a}_r \text{ C/m}^2$$

a) What is the volume charge density at $r = 4$? In this case we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{5}{r}(r-3)^2(5r-6) \text{ C/m}^3$$

which we evaluate at $r = 4$ to find $\rho_v(r=4) = \underline{17.50 \text{ C/m}^3}$.

b) What is the electric flux density at $r = 4$? Substitute $r = 4$ into the given expression to find $\mathbf{D}(4) = \underline{5\,\mathbf{a}_r \text{ C/m}^2}$

c) How much electric flux leaves the sphere $r = 4$? Using the result of part *b*, this will be $\Phi = 4\pi(4)^2(5) = \underline{320\pi \text{ C}}$

d) How much charge is contained within the sphere, $r = 4$? From Gauss' law, this will be the same as the outward flux, or again, $Q = \underline{320\pi \text{ C}}$.

3.26. If we have a perfect gas of mass density ρ_m kg/m^3, and assign a velocity $\mathbf{U}$ m/s to each differential element, then the mass flow rate is $\rho_m\mathbf{U}$ kg/(m^2 − s). Physical reasoning then leads to the *continuity equation*, $\nabla \cdot (\rho_m \mathbf{U}) = -\partial\rho_m/\partial t$.

a) Explain in words the physical interpretation of this equation: The quantity $\rho_m\mathbf{U}$ is the flow (or flux) density of mass. Then the divergence of $\rho_m\mathbf{U}$ is the outward mass flux per unit volume at a point. This must be equivalent to the rate of depletion of mass per unit volume at the same point, as the continuity equation states.

b) Show that $\oint_s \rho_m \mathbf{U} \cdot d\mathbf{S} = -dM/dt$, where M is the total mass of the gas within the constant closed surface, S, and explain the physical significance of the equation.

Applying the divergence theorem, we have

$$\oint_s \rho_m \mathbf{U} \cdot d\mathbf{S} = \int_v \nabla \cdot (\rho_m \mathbf{U})\, dv = \int_v -\frac{\partial \rho_m}{\partial t}\, dv = -\frac{d}{dt}\int_v \rho_m\, dv = -\frac{dM}{dt}$$

This states in large-scale form what was already stated in part *a*. That is – the net outward mass flow (in kg/s) through a closed surface is equal to the negative time rate of change in total mass within the enclosed volume.

3.27. Let $\mathbf{D} = 5.00r^2\mathbf{a}_r$ mC/m^2 for $r \leq 0.08$ m and $\mathbf{D} = 0.205\,\mathbf{a}_r/r^2$ μC/m^2 for $r \geq 0.08$ m (note error in problem statement).

a) Find ρ_v for $r = 0.06$ m: This radius lies within the first region, and so

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{1}{r^2}\frac{d}{dr}(5.00r^4) = 20r \text{ mC/m}^3$$

which when evaluated at $r = 0.06$ yields $\rho_v(r = .06) = \underline{1.20 \text{ mC/m}^3}$.

b) Find ρ_v for $r = 0.1$ m: This is in the region where the second field expression is valid. The $1/r^2$ dependence of this field yields a zero divergence (shown in Problem 3.23), and so the volume charge density is <u>zero</u> at 0.1 m.

3.27c) What surface charge density could be located at $r = 0.08$ m to cause $\mathbf{D} = 0$ for $r > 0.08$ m?

The total surface charge should be equal and opposite to the total volume charge. The latter is

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{.08} 20r(\text{mC/m}^3)\, r^2 \sin\theta\, dr\, d\theta\, d\phi = 2.57 \times 10^{-3} \text{ mC} = 2.57\,\mu\text{C}$$

So now

$$\rho_s = -\left[\frac{2.57}{4\pi(.08)^2}\right] = \underline{-32\,\mu\text{C/m}^2}$$

3.28. Repeat Problem 3.8, but use $\nabla \cdot \mathbf{D} = \rho_v$ and take an appropriate volume integral.

We begin by finding the charge density directly through

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{A}{r}\right) = \underline{\frac{A}{r^2}}$$

Then, within each spherical shell of unit thickness, the contained charge is

$$Q(1) = 4\pi \int_r^{r+1} \frac{A}{(r')^2}(r')^2\, dr' = 4\pi A(r + 1 - r) = \underline{4\pi A}$$

3.29. In the region of free space that includes the volume $2 < x, y, z < 3$,

$$\mathbf{D} = \frac{2}{z^2}(yz\,\mathbf{a}_x + xz\,\mathbf{a}_y - 2xy\,\mathbf{a}_z) \text{ C/m}^2$$

a) Evaluate the volume integral side of the divergence theorem for the volume defined above: In cartesian, we find $\nabla \cdot \mathbf{D} = 8xy/z^3$. The volume integral side is now

$$\int_{vol} \nabla \cdot \mathbf{D}\, dv = \int_2^3 \int_2^3 \int_2^3 \frac{8xy}{z^3}\, dxdydz = (9-4)(9-4)\left(\frac{1}{4} - \frac{1}{9}\right) = \underline{3.47 \text{ C}}$$

b. Evaluate the surface integral side for the corresponding closed surface: We call the surfaces at $x = 3$ and $x = 2$ the front and back surfaces respectively, those at $y = 3$ and $y = 2$ the right and left surfaces, and those at $z = 3$ and $z = 2$ the top and bottom surfaces. To evaluate the surface integral side, we integrate $\mathbf{D} \cdot \mathbf{n}$ over all six surfaces and sum the results. Note that since the x component of $\mathbf{D}$ does not vary with x, the outward fluxes from the front and back surfaces will cancel each other. The same is true for the left and right surfaces, since D_y does not vary with y. This leaves only the top and bottom surfaces, where the fluxes are:

$$\oint \mathbf{D} \cdot d\mathbf{S} = \underbrace{\int_2^3 \int_2^3 \frac{-4xy}{3^2}\, dxdy}_{\text{top}} - \underbrace{\int_2^3 \int_2^3 \frac{-4xy}{2^2}\, dxdy}_{\text{bottom}} = (9-4)(9-4)\left(\frac{1}{4} - \frac{1}{9}\right) = \underline{3.47 \text{ C}}$$

3.30 a) Use Maxwell's first equation, $\nabla \cdot \mathbf{D} = \rho_v$, to describe the variation of the electric field intensity with x in a region in which no charge density exists and in which a non-homogeneous dielectric has a permittivity that increases exponentially with x. The field has an x component only: The permittivity can be written as $\epsilon(x) = \epsilon_1 \exp(\alpha_1 x)$, where ϵ_1 and α_1 are constants. Then

$$\nabla \cdot \mathbf{D} = \nabla \cdot [\epsilon(x)\mathbf{E}(x)] = \frac{d}{dx}\left[\epsilon_1 e^{\alpha_1 x} E_x(x)\right] = \epsilon_1 \left[\alpha_1 e^{\alpha_1 x} E_x + e^{\alpha_1 x}\frac{dE_x}{dx}\right] = 0$$

This reduces to

$$\frac{dE_x}{dx} + \alpha_1 E_x = 0 \;\Rightarrow\; E_x(x) = \underline{E_0\, e^{-\alpha_1 x}}$$

where E_0 is a constant.

b) Repeat part a, but with a radially-directed electric field (spherical coordinates), in which again $\rho_v = 0$, but in which the permittivity *decreases* exponentially with r. In this case, the permittivity can be written as $\epsilon(r) = \epsilon_2 \exp(-\alpha_2 r)$, where ϵ_2 and α_2 are constants. Then

$$\nabla \cdot \mathbf{D} = \nabla \cdot [\epsilon(r)\mathbf{E}(r)] = \frac{1}{r^2}\frac{d}{dr}\left[r^2 \epsilon_2\, e^{-\alpha_2 r}\, E_r\right] = \frac{\epsilon_2}{r^2}\left[2rE_r - \alpha_2 r^2 E_r + r^2\frac{dE_r}{dr}\right] e^{-\alpha_2 r} = 0$$

This reduces to

$$\frac{dE_r}{dr} + \left(\frac{2}{r} - \alpha_2\right) E_r = 0$$

whose solution is

$$E_r(r) = E_0 \exp\left[-\int \left(\frac{2}{r} - \alpha_2\right) dr\right] = E_0 \exp\left[-2\ln r + \alpha_2 r\right] = \underline{\frac{E_0}{r^2} e^{\alpha_2 r}}$$

where E_0 is a constant.

3.31. Given the flux density

$$\mathbf{D} = \frac{16}{r}\cos(2\theta)\,\mathbf{a}_\theta \ \text{C/m}^2,$$

use two different methods to find the total charge within the region $1 < r < 2$ m, $1 < \theta < 2$ rad, $1 < \phi < 2$ rad: We use the divergence theorem and first evaluate the surface integral side. We are evaluating the net outward flux through a curvilinear "cube", whose boundaries are defined by the specified ranges. The flux contributions will be only through the surfaces of constant θ, however, since $\mathbf{D}$ has only a θ component. On a constant-theta surface, the differential area is $da = r\sin\theta dr d\phi$, where θ is fixed at the surface location. Our flux integral becomes

$$\oint \mathbf{D} \cdot d\mathbf{S} = \underbrace{-\int_1^2\int_1^2 \frac{16}{r}\cos(2)\, r\sin(1)\, dr d\phi}_{\theta=1} + \underbrace{\int_1^2\int_1^2 \frac{16}{r}\cos(4)\, r\sin(2)\, dr d\phi}_{\theta=2}$$
$$= -16\left[\cos(2)\sin(1) - \cos(4)\sin(2)\right] = \underline{-3.91 \text{ C}}$$

We next evaluate the volume integral side of the divergence theorem, where in this case,

$$\nabla \cdot \mathbf{D} = \frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta\, D_\theta) = \frac{1}{r\sin\theta}\frac{d}{d\theta}\left[\frac{16}{r}\cos 2\theta \sin\theta\right] = \frac{16}{r^2}\left[\frac{\cos 2\theta\cos\theta}{\sin\theta} - 2\sin 2\theta\right]$$

3.31 (continued) We now evaluate:

$$\int_{vol} \nabla \cdot \mathbf{D}\, dv = \int_1^2 \int_1^2 \int_1^2 \frac{16}{r^2} \left[\frac{\cos 2\theta \cos\theta}{\sin\theta} - 2\sin 2\theta \right] r^2 \sin\theta \, dr d\theta d\phi$$

The integral simplifies to

$$\int_1^2 \int_1^2 \int_1^2 16[\cos 2\theta \cos\theta - 2\sin 2\theta \sin\theta]\, dr d\theta d\phi = 8 \int_1^2 [3\cos 3\theta - \cos\theta]\, d\theta = \underline{-3.91 \text{ C}}$$

CHAPTER 4

4.1. The value of $\mathbf{E}$ at $P(\rho = 2,\ \phi = 40°,\ z = 3)$ is given as $\mathbf{E} = 100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z$ V/m. Determine the incremental work required to move a 20 μC charge a distance of 6 μm:

a) in the direction of $\mathbf{a}_\rho$: The incremental work is given by $dW = -q\,\mathbf{E}\cdot d\mathbf{L}$, where in this case, $d\mathbf{L} = d\rho\,\mathbf{a}_\rho = 6\times 10^{-6}\,\mathbf{a}_\rho$. Thus

$$dW = -(20\times 10^{-6}\,\mathrm{C})(100\,\mathrm{V/m})(6\times 10^{-6}\,\mathrm{m}) = -12\times 10^{-9}\ \mathrm{J} = \underline{-12\,\mathrm{nJ}}$$

b) in the direction of $\mathbf{a}_\phi$: In this case $d\mathbf{L} = 2\,d\phi\,\mathbf{a}_\phi = 6\times 10^{-6}\,\mathbf{a}_\phi$, and so

$$dW = -(20\times 10^{-6})(-200)(6\times 10^{-6}) = 2.4\times 10^{-8}\,\mathrm{J} = \underline{24\ \mathrm{nJ}}$$

c) in the direction of $\mathbf{a}_z$: Here, $d\mathbf{L} = dz\,\mathbf{a}_z = 6\times 10^{-6}\,\mathbf{a}_z$, and so

$$dW = -(20\times 10^{-6})(300)(6\times 10^{-6}) = -3.6\times 10^{-8}\,\mathrm{J} = \underline{-36\ \mathrm{nJ}}$$

d) in the direction of $\mathbf{E}$: Here, $d\mathbf{L} = 6\times 10^{-6}\,\mathbf{a}_E$, where

$$\mathbf{a}_E = \frac{100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z}{[100^2 + 200^2 + 300^2]^{1/2}} = 0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z$$

Thus

$$\begin{aligned} dW &= -(20\times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z](6\times 10^{-6}) \\ &= \underline{-44.9\,\mathrm{nJ}} \end{aligned}$$

e) In the direction of $\mathbf{G} = 2\,\mathbf{a}_x - 3\,\mathbf{a}_y + 4\,\mathbf{a}_z$: In this case, $d\mathbf{L} = 6\times 10^{-6}\,\mathbf{a}_G$, where

$$\mathbf{a}_G = \frac{2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z}{[2^2 + 3^2 + 4^2]^{1/2}} = 0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z$$

So now

$$\begin{aligned} dW &= -(20\times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z](6\times 10^{-6}) \\ &= -(20\times 10^{-6})\,[37.1(\mathbf{a}_\rho\cdot\mathbf{a}_x) - 55.7(\mathbf{a}_\rho\cdot\mathbf{a}_y) - 74.2(\mathbf{a}_\phi\cdot\mathbf{a}_x) + 111.4(\mathbf{a}_\phi\cdot\mathbf{a}_y) \\ &\quad + 222.9]\,(6\times 10^{-6}) \end{aligned}$$

where, at P, $(\mathbf{a}_\rho\cdot\mathbf{a}_x) = (\mathbf{a}_\phi\cdot\mathbf{a}_y) = \cos(40°) = 0.766$, $(\mathbf{a}_\rho\cdot\mathbf{a}_y) = \sin(40°) = 0.643$, and $(\mathbf{a}_\phi\cdot\mathbf{a}_x) = -\sin(40°) = -0.643$. Substituting these results in

$$dW = -(20\times 10^{-6})[28.4 - 35.8 + 47.7 + 85.3 + 222.9](6\times 10^{-6}) = \underline{-41.8\,\mathrm{nJ}}$$

4.2. A positive point charge of magnitude q_1 lies at the origin. Derive an expression for the incremental work done in moving a second point charge q_2 through a distance dx from the starting position (x, y, z), in the direction of $-\mathbf{a}_x$: The incremental work is given by

$$dW = -q_2 \mathbf{E}_{12} \cdot d\mathbf{L}$$

where $\mathbf{E}_{12}$ is the electric field arising from q_1 evaluated at the location of q_2, and where $d\mathbf{L} = -dx\, \mathbf{a}_x$. Taking the location of q_2 at spherical coordinates (r, θ, ϕ), we write:

$$dW = \frac{-q_2 q_1}{4\pi\epsilon_0 r^2} \mathbf{a}_r \cdot (-dx)\mathbf{a}_x$$

where $r^2 = x^2 + y^2 + z^2$, and where $\mathbf{a}_r \cdot \mathbf{a}_x = \sin\theta \cos\phi$. So

$$dW = \frac{q_2 q_1}{4\pi\epsilon_0 (x^2+y^2+z^2)} \underbrace{\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}}_{\sin\theta} \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\cos\phi} dx = \underline{\frac{q_2 q_1\, x\, dx}{4\pi\epsilon_0 (x^2+y^2+z^2)^{3/2}}}$$

4.3. If $\mathbf{E} = 120\, \mathbf{a}_\rho$ V/m, find the incremental amount of work done in moving a $50\,\mu$m charge a distance of 2 mm from:

a) $P(1, 2, 3)$ toward $Q(2, 1, 4)$: The vector along this direction will be $Q - P = (1, -1, 1)$ from which $\mathbf{a}_{PQ} = [\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z]/\sqrt{3}$. We now write

$$dW = -q\mathbf{E} \cdot d\mathbf{L} = -(50 \times 10^{-6}) \left[120\mathbf{a}_\rho \cdot \frac{(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z}{\sqrt{3}} \right] (2 \times 10^{-3})$$
$$= -(50 \times 10^{-6})(120) \left[(\mathbf{a}_\rho \cdot \mathbf{a}_x) - (\mathbf{a}_\rho \cdot \mathbf{a}_y) \right] \frac{1}{\sqrt{3}} (2 \times 10^{-3})$$

At P, $\phi = \tan^{-1}(2/1) = 63.4°$. Thus $(\mathbf{a}_\rho \cdot \mathbf{a}_x) = \cos(63.4) = 0.447$ and $(\mathbf{a}_\rho \cdot \mathbf{a}_y) = \sin(63.4) = 0.894$. Substituting these, we obtain $dW = \underline{3.1\,\mu\text{J}}$.

b) $Q(2, 1, 4)$ toward $P(1, 2, 3)$: A little thought is in order here: Note that the field has only a radial component and does not depend on ϕ or z. Note also that P and Q are at the same radius ($\sqrt{5}$) from the z axis, but have different ϕ and z coordinates. We could just as well position the two points at the same z location and the problem would not change. If this were so, then moving along a straight line between P and Q would thus involve moving along a chord of a circle whose radius is $\sqrt{5}$. Halfway along this line is a point of symmetry in the field (make a sketch to see this). This means that when starting from either point, the initial force will be the same. Thus the answer is $dW = \underline{3.1\,\mu\text{J}}$ as in part a. This is also found by going through the same procedure as in part a, but with the direction (roles of P and Q) reversed.

4.4. An electric field in free space is given by $\mathbf{E} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ V/m. Find the work done in moving a 1μC charge through this field

a) from (1,1,1) to (0,0,0): The work will be

$$W = -q \int \mathbf{E} \cdot d\mathbf{L} \;=\; -10^{-6} \left[\int_1^0 x dx \;+\; \int_1^0 y dy \;+\; \int_1^0 z dz \right] \text{J} \;=\; \underline{1.5\,\mu\text{J}}$$

b) from ($\rho = 2$, $\phi = 0$) to ($\rho = 2$, $\phi = 90°$): The path involves changing ϕ with ρ and z fixed, and therefore $d\mathbf{L} = \rho\, d\phi\, \mathbf{a}_\phi$. We set up the integral for the work as

$$W = -10^{-6} \int_0^{\pi/2} (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z) \cdot \rho\, d\phi\, \mathbf{a}_\phi$$

where $\rho = 2$, $\mathbf{a}_x \cdot \mathbf{a}_\phi = -\sin\phi$, $\mathbf{a}_y \cdot \mathbf{a}_\phi = \cos\phi$, and $\mathbf{a}_z \cdot \mathbf{a}_\phi = 0$. Also, $x = 2\cos\phi$ and $y = 2\sin\phi$. Substitute all of these to get

$$W = -10^{-6} \int_0^{\pi/2} \left[-(2)^2 \cos\phi \sin\phi + (2)^2 \cos\phi \sin\phi \right] d\phi = \underline{0}$$

Given that the field is conservative (and so work is path-independent), can you see a much easier way to obtain this result?

c) from ($r = 10$, $\theta = \theta_0$) to ($r = 10$, $\theta = \theta_0 + 180°$): In this case, we are moving only in the $\mathbf{a}_\theta$ direction. The work is set up as

$$W = -10^{-6} \int_{\theta_0}^{\theta_0 + \pi} (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z) \cdot r\, d\theta\, \mathbf{a}_\theta$$

Now, substitute the following relations: $r = 10$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, $\mathbf{a}_x \cdot \mathbf{a}_\theta = \cos\theta\cos\phi$, $\mathbf{a}_y \cdot \mathbf{a}_\theta = \cos\theta\sin\phi$, and $\mathbf{a}_z \cdot \mathbf{a}_\theta = -\sin\theta$. Obtain

$$W = -10^{-6} \int_{\theta_0}^{\theta_0 + \pi} (10)^2 \left[\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \cos\theta\sin\theta \right] d\theta = \underline{0}$$

where we use $\cos^2\phi + \sin^2\phi = 1$.

4.5. Compute the value of $\int_A^P \mathbf{G} \cdot d\mathbf{L}$ for $\mathbf{G} = 2y\mathbf{a}_x$ with $A(1, -1, 2)$ and $P(2, 1, 2)$ using the path:

a) straight-line segments $A(1, -1, 2)$ to $B(1, 1, 2)$ to $P(2, 1, 2)$: In general we would have

$$\int_A^P \mathbf{G} \cdot d\mathbf{L} = \int_A^P 2y\, dx$$

The change in x occurs when moving between B and P, during which $y = 1$. Thus

$$\int_A^P \mathbf{G} \cdot d\mathbf{L} = \int_B^P 2y\, dx = \int_1^2 2(1) dx = \underline{2}$$

b) straight-line segments $A(1, -1, 2)$ to $C(2, -1, 2)$ to $P(2, 1, 2)$: In this case the change in x occurs when moving from A to C, during which $y = -1$. Thus

$$\int_A^P \mathbf{G} \cdot d\mathbf{L} = \int_A^C 2y\, dx = \int_1^2 2(-1) dx = \underline{-2}$$

4.6. An electric field in free space is given as $\mathbf{E} = x\,\hat{\mathbf{a}}_x + 4z\,\hat{\mathbf{a}}_y + 4y\,\hat{\mathbf{a}}_z$. Given $V(1,1,1) = 10$ V. Determine $V(3,3,3)$. The potential difference is expressed as

$$V(3,3,3) - V(1,1,1) = -\int_{1,1,1}^{3,3,3} (x\,\hat{\mathbf{a}}_x + 4z\,\hat{\mathbf{a}}_y + 4y\,\hat{\mathbf{a}}_z) \cdot (dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z)$$
$$= -\left[\int_1^3 x\,dx + \int_1^3 4z\,dy + \int_1^3 4y\,dz\right]$$

We choose the following path: 1) move along x from 1 to 3; 2) move along y from 1 to 3, holding x at 3 and z at 1; 3) move along z from 1 to 3, holding x at 3 and y at 3. The integrals become:

$$V(3,3,3) - V(1,1,1) = -\left[\int_1^3 x\,dx + \int_1^3 4(1)\,dy + \int_1^3 4(3)\,dz\right] = -36$$

So

$$V(3,3,3) = -36 + V(1,1,1) = -36 + 10 = \underline{-26}\text{ V}$$

4.7. Let $\mathbf{G} = 3xy^3\mathbf{a}_x + 2z\mathbf{a}_y$. Given an initial point $P(2,1,1)$ and a final point $Q(4,3,1)$, find $\int \mathbf{G} \cdot d\mathbf{L}$ using the path:

a) straight line: $y = x - 1$, $z = 1$: We obtain:

$$\int \mathbf{G} \cdot d\mathbf{L} = \int_2^4 3xy^2\,dx + \int_1^3 2z\,dy = \int_2^4 3x(x-1)^2\,dx + \int_1^3 2(1)\,dy = \underline{90}$$

b) parabola: $6y = x^2 + 2$, $z = 1$: We obtain:

$$\int \mathbf{G} \cdot d\mathbf{L} = \int_2^4 3xy^2\,dx + \int_1^3 2z\,dy = \int_2^4 \frac{1}{12}x(x^2+2)^2\,dx + \int_1^3 2(1)\,dy = \underline{82}$$

4.8. Given $\mathbf{E} = -x\mathbf{a}_x + y\mathbf{a}_y$, a) find the work involved in moving a unit positive charge on a circular arc, the circle centered at the origin, from $x = a$ to $x = y = a/\sqrt{2}$.

In moving along the arc, we start at $\phi = 0$ and move to $\phi = \pi/4$. The setup is

$$W = -q\int \mathbf{E} \cdot d\mathbf{L} = -\int_0^{\pi/4} \mathbf{E} \cdot a d\phi\,\mathbf{a}_\phi = -\int_0^{\pi/4} (-x\,\underbrace{\mathbf{a}_x \cdot \mathbf{a}_\phi}_{-\sin\phi} + y\,\underbrace{\mathbf{a}_y \cdot \mathbf{a}_\phi}_{\cos\phi})a\,d\phi$$
$$= -\int_0^{\pi/4} 2a^2 \sin\phi\cos\phi\,d\phi = -\int_0^{\pi/4} a^2\sin(2\phi)\,d\phi = \frac{a^2}{2}\cos(2\phi)\Big|_0^{\pi/4} = \underline{-\frac{a^2}{2}}$$

where $q = 1$, $x = a\cos\phi$, and $y = a\sin\phi$.

Note that the field is conservative, so we would get the same result by integrating along a two-segment path over x and y as shown:

$$W = -\int \mathbf{E} \cdot d\mathbf{L} = -\left[\int_a^{a/\sqrt{2}} (-x)\,dx + \int_0^{a/\sqrt{2}} y\,dy\right] = -a^2/2$$

4.8b) Verify that the work done in moving the charge around the full circle from $x = a$ is zero: In this case, the setup is the same, but the integration limits change:

$$W = -\int_0^{2\pi} a^2 \sin(2\phi)\, d\phi = \frac{a^2}{2} \cos(2\phi)\Big|_0^{2\pi} = \underline{0}$$

4.9. A uniform surface charge density of 20 nC/m^2 is present on the spherical surface $r = 0.6\,\text{cm}$ in free space.

a) Find the absolute potential at $P(r = 1\,\text{cm}, \theta = 25^\circ, \phi = 50^\circ)$: Since the charge density is uniform and is spherically-symmetric, the angular coordinates do not matter. The potential function for $r > 0.6$ cm will be that of a point charge of $Q = 4\pi a^2 \rho_s$, or

$$V(r) = \frac{4\pi(0.6 \times 10^{-2})^2 (20 \times 10^{-9})}{4\pi\epsilon_0 r} = \frac{0.081}{r} \text{ V} \quad \text{with } r \text{ in meters}$$

At $r = 1\,\text{cm}$, this becomes $V(r = 1\,\text{cm}) = \underline{8.14 \text{ V}}$

b) Find V_{AB} given points $A(r = 2\,\text{cm}, \theta = 30^\circ, \phi = 60^\circ)$ and $B(r = 3\,\text{cm}, \theta = 45^\circ, \phi = 90^\circ)$: Again, the angles do not matter because of the spherical symmetry. We use the part a result to obtain

$$V_{AB} = V_A - V_B = 0.081 \left[\frac{1}{0.02} - \frac{1}{0.03}\right] = \underline{1.36 \text{ V}}$$

4.10. A sphere of radius a carries a surface charge density of ρ_{s0} C/m^2.

a) Find the absolute potential at the sphere surface: The setup for this is

$$V_0 = -\int_\infty^a \mathbf{E} \cdot d\mathbf{L}$$

where, from Gauss' law:

$$\mathbf{E} = \frac{a^2 \rho_{s0}}{\epsilon_0 r^2} \mathbf{a}_r \quad \text{V/m}$$

So

$$V_0 = -\int_\infty^a \frac{a^2 \rho_{s0}}{\epsilon_0 r^2} \mathbf{a}_r \cdot \mathbf{a}_r \, dr = \frac{a^2 \rho_{s0}}{\epsilon_0 r}\Big|_\infty^a = \underline{\frac{a\rho_{s0}}{\epsilon_0}} \text{ V}$$

b) A grounded conducting shell of radius b where $b > a$ is now positioned around the charged sphere. What is the potential at the inner sphere surface in this case? With the outer sphere grounded, the field exists only between the surfaces, and is zero for $r > b$. The potential is then

$$V_0 = -\int_b^a \frac{a^2 \rho_{s0}}{\epsilon_0 r^2} \mathbf{a}_r \cdot \mathbf{a}_r \, dr = \frac{a^2 \rho_{s0}}{\epsilon_0 r}\Big|_b^a = \underline{\frac{a^2 \rho_{s0}}{\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]} \text{ V}$$

4.11. Let a uniform surface charge density of $5\,\text{nC/m}^2$ be present at the $z = 0$ plane, a uniform line charge density of $8\,\text{nC/m}$ be located at $x = 0$, $z = 4$, and a point charge of $2\,\mu\text{C}$ be present at $P(2, 0, 0)$. If $V = 0$ at $M(0, 0, 5)$, find V at $N(1, 2, 3)$: We need to find a potential function for the combined charges which is zero at M. That for the point charge we know to be

$$V_p(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Potential functions for the sheet and line charges can be found by taking indefinite integrals of the electric fields for those distributions. For the line charge, we have

$$V_l(\rho) = -\int \frac{\rho_l}{2\pi\epsilon_0\rho} d\rho + C_1 = -\frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) + C_1$$

For the sheet charge, we have

$$V_s(z) = -\int \frac{\rho_s}{2\epsilon_0} dz + C_2 = -\frac{\rho_s}{2\epsilon_0} z + C_2$$

The total potential function will be the sum of the three. Combining the integration constants, we obtain:

$$V = \frac{Q}{4\pi\epsilon_0 r} - \frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) - \frac{\rho_s}{2\epsilon_0} z + C$$

The terms in this expression are not referenced to a common origin, since the charges are at different positions. The parameters r, ρ, and z are *scalar distances* from the charges, and will be treated as such here. To evaluate the constant, C, we first look at point M, where $V_T = 0$. At M, $r = \sqrt{2^2 + 5^2} = \sqrt{29}$, $\rho = 1$, and $z = 5$. We thus have

$$0 = \frac{2 \times 10^{-6}}{4\pi\epsilon_0\sqrt{29}} - \frac{8 \times 10^{-9}}{2\pi\epsilon_0}\ln(1) - \frac{5 \times 10^{-9}}{2\epsilon_0} 5 + C \quad \Rightarrow \quad C = -1.93 \times 10^3 \text{ V}$$

At point N, $r = \sqrt{1 + 4 + 9} = \sqrt{14}$, $\rho = \sqrt{2}$, and $z = 3$. The potential at N is thus

$$V_N = \frac{2 \times 10^{-6}}{4\pi\epsilon_0\sqrt{14}} - \frac{8 \times 10^{-9}}{2\pi\epsilon_0}\ln(\sqrt{2}) - \frac{5 \times 10^{-9}}{2\epsilon_0}(3) - 1.93 \times 10^3 = 1.98 \times 10^3 \text{ V} = \underline{1.98\,\text{kV}}$$

4.12. In spherical coordinates, $\mathbf{E} = 2r/(r^2 + a^2)^2\,\mathbf{a}_r$ V/m. Find the potential at any point, using the reference

a) $V = 0$ at infinity: We write in general

$$V(r) = -\int \frac{2r\,dr}{(r^2+a^2)^2} + C = \frac{1}{r^2+a^2} + C$$

With a zero reference at $r \to \infty$, $C = 0$ and therefore $V(r) = \underline{1/(r^2 + a^2)}$.

b) $V = 0$ at $r = 0$: Using the general expression, we find

$$V(0) = \frac{1}{a^2} + C = 0 \quad \Rightarrow \quad C = -\frac{1}{a^2}$$

Therefore

$$V(r) = \frac{1}{r^2+a^2} - \frac{1}{a^2} = \frac{-r^2}{a^2(r^2+a^2)}$$

c) $V = 100$V at $r = a$: Here, we find

$$V(a) = \frac{1}{2a^2} + C = 100 \quad \Rightarrow \quad C = 100 - \frac{1}{2a^2}$$

Therefore

$$V(r) = \frac{1}{r^2+a^2} - \frac{1}{2a^2} + 100 = \frac{a^2-r^2}{2a^2(r^2+a^2)} + 100$$

4.13. Three identical point charges of 4 pC each are located at the corners of an equilateral triangle 0.5 mm on a side in free space. How much work must be done to move one charge to a point equidistant from the other two and on the line joining them? This will be the magnitude of the charge times the potential difference between the finishing and starting positions, or

$$W = \frac{(4\times10^{-12})^2}{2\pi\epsilon_0}\left[\frac{1}{2.5} - \frac{1}{5}\right] \times 10^4 = 5.76\times10^{-10}\,\text{J} = \underline{576\,\text{pJ}}$$

4.14. Given the electric field $\mathbf{E} = (y+1)\mathbf{a}_x + (x-1)\mathbf{a}_y + 2\mathbf{a}_z$, find the potential difference between the points

a) (2,-2,-1) and (0,0,0): We choose a path along which motion occurs in one coordinate direction at a time. Starting at the origin, first move along x from 0 to 2, where $y = 0$; then along y from 0 to -2, where x is 2; then along z from 0 to -1. The setup is

$$V_b - V_a = -\int_0^2 (y+1)\Big|_{y=0} dx \; - \int_0^{-2} (x-1)\Big|_{x=2} dy \; - \int_0^{-1} 2\,dz = \underline{2}$$

b) (3,2,-1) and (-2,-3,4): Following similar reasoning,

$$V_b - V_a = -\int_{-2}^3 (y+1)\Big|_{y=-3} dx \; - \int_{-3}^{2} (x-1)\Big|_{x=3} dy \; - \int_4^{-1} 2\,dz = \underline{10}$$

4.15. Two uniform line charges, 8 nC/m each, are located at $x = 1$, $z = 2$, and at $x = -1$, $y = 2$ in free space. If the potential at the origin is 100 V, find V at $P(4, 1, 3)$: The net potential function for the two charges would in general be:

$$V = -\frac{\rho_l}{2\pi\epsilon_0}\ln(R_1) - \frac{\rho_l}{2\pi\epsilon_0}\ln(R_2) + C$$

At the origin, $R_1 = R_2 = \sqrt{5}$, and $V = 100$ V. Thus, with $\rho_l = 8 \times 10^{-9}$,

$$100 = -2\frac{(8 \times 10^{-9})}{2\pi\epsilon_0}\ln(\sqrt{5}) + C \quad \Rightarrow \quad C = 331.6 \text{ V}$$

At $P(4, 1, 3)$, $R_1 = |(4, 1, 3) - (1, 1, 2)| = \sqrt{10}$ and $R_2 = |(4, 1, 3) - (-1, 2, 3)| = \sqrt{26}$. Therefore

$$V_P = -\frac{(8 \times 10^{-9})}{2\pi\epsilon_0}\left[\ln(\sqrt{10}) + \ln(\sqrt{26})\right] + 331.6 = \underline{-68.4 \text{ V}}$$

4.16. A spherically-symmetric charge distribution in free space (with $a < r < \infty$) – *note typo in problem statement*, which says $(0 < r < \infty)$ – is known to have a potential function $V(r) = V_0 a^2/r^2$, where V_0 and a are constants.

a) Find the electric field intensity: This is found through

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\,\mathbf{a}_r = \underline{2V_0\,\frac{a^2}{r^3}\,\mathbf{a}_r \text{ V/m}}$$

b) Find the volume charge density: Use Maxwell's first equation:

$$\rho_v = \nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_0 \mathbf{E}) = \frac{1}{r^2}\frac{d}{dr}\left[r^2\left(2\epsilon_0 V_0 \frac{a^2}{r^3}\right)\right] = \underline{-2\epsilon_0 V_0 \frac{a^2}{r^4} \text{ C/m}^3}$$

c) Find the charge contained inside radius a: Here, we do not know the charge density inside radius a, but we do know the flux density *at* that radius. We use Gauss' law to integrate $\mathbf{D}$ over the spherical surface at $r = a$ to find the charge enclosed:

$$Q_{encl} = \oint_{r=a} \mathbf{D} \cdot d\mathbf{S} = 4\pi a^2 D|_{r=a} = 4\pi a^2\left(2\epsilon_0 V_0 \frac{a^2}{a^3}\right) = \underline{8\pi\epsilon_0 a V_0} \text{ C}$$

d) Find the total energy stored in the charge (or equivalently, in its electric field) *in the region* $(a < r < \infty)$. We integrate the energy density in the field over the region:

$$W_e = \int_v \frac{1}{2}\mathbf{D} \cdot \mathbf{E}\,dv = \int_0^{2\pi}\int_0^{\pi}\int_a^{\infty} 2\epsilon_0 V_0^2 \frac{a^4}{r^6} r^2 \sin\theta\,dr\,d\theta\,d\phi$$

$$= -8\pi\, V_0^2\, \epsilon_0\, a^4 \frac{1}{3r^3}\Big|_a^{\infty} = \underline{8\pi\epsilon_0 a V_0^2/3} \text{ J}$$

4.17. Uniform surface charge densities of 6 and 2 nC/m^2 are present at $\rho = 2$ and 6 cm respectively, in free space. Assume $V = 0$ at $\rho = 4$ cm, and calculate V at:

a) $\rho = 5$ cm: Since $V = 0$ at 4 cm, the potential at 5 cm will be the potential difference between points 5 and 4:

$$V_5 = -\int_4^5 \mathbf{E} \cdot d\mathbf{L} = -\int_4^5 \frac{a\rho_{sa}}{\epsilon_0 \rho}\, d\rho = -\frac{(.02)(6 \times 10^{-9})}{\epsilon_0} \ln\left(\frac{5}{4}\right) = \underline{-3.026\,\text{V}}$$

b) $\rho = 7$ cm: Here we integrate piecewise from $\rho = 4$ to $\rho = 7$:

$$V_7 = -\int_4^6 \frac{a\rho_{sa}}{\epsilon_0 \rho} d\rho - \int_6^7 \frac{(a\rho_{sa} + b\rho_{sb})}{\epsilon_0 \rho} d\rho$$

With the given values, this becomes

$$V_7 = -\left[\frac{(.02)(6 \times 10^{-9})}{\epsilon_0}\right] \ln\left(\frac{6}{4}\right) - \left[\frac{(.02)(6 \times 10^{-9}) + (.06)(2 \times 10^{-9})}{\epsilon_0}\right] \ln\left(\frac{7}{6}\right)$$
$$= \underline{-9.678\,\text{V}}$$

4.18. Find the potential at the origin produced by a line charge $\rho_L = kx/(x^2 + a^2)$ extending along the x axis from $x = a$ to $+\infty$, where $a > 0$. Assume a zero reference at infinity.

Think of the line charge as an array of point charges, each of charge $dq = \rho_L dx$, and each having potential at the origin of $dV = \rho_L dx/(4\pi\epsilon_0 x)$. The total potential at the origin is then the sum of all these potentials, or

$$V = \int_a^\infty \frac{\rho_L\, dx}{4\pi\epsilon_0 x} = \int_a^\infty \frac{k\, dx}{4\pi\epsilon_0 (x^2 + a^2)} = \frac{k}{4\pi\epsilon_0 a} \tan^{-1}\left(\frac{x}{a}\right)_a^\infty = \frac{k}{4\pi\epsilon_0 a}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{k}{16\epsilon_0 a}$$

4.19. The annular surface, $1\,\text{cm} < \rho < 3\,\text{cm}$, $z = 0$, carries the nonuniform surface charge density $\rho_s = 5\rho\,\text{nC/m}^2$. Find V at $P(0, 0, 2\,\text{cm})$ if $V = 0$ at infinity: We use the superposition integral form:

$$V_P = \int\int \frac{\rho_s\, da}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}$$

where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. We integrate over the surface of the annular region, with $da = \rho\, d\rho\, d\phi$. Substituting the given values, we find

$$V_P = \int_0^{2\pi} \int_{.01}^{.03} \frac{(5 \times 10^{-9})\rho^2\, d\rho\, d\phi}{4\pi\epsilon_0 \sqrt{\rho^2 + z^2}}$$

Substituting $z = .02$, and using tables, the integral evaluates as

$$V_P = \left[\frac{(5 \times 10^{-9})}{2\epsilon_0}\right]\left[\frac{\rho}{2}\sqrt{\rho^2 + (.02)^2} - \frac{(.02)^2}{2} \ln(\rho + \sqrt{\rho^2 + (.02)^2})\right]_{.01}^{.03} = \underline{.081\,\text{V}}$$

4.20. In a certain medium, the electric potential is given by

$$V(x) = \frac{\rho_0}{a\epsilon_0}\left(1 - e^{-ax}\right)$$

where ρ_0 and a are constants.

a) Find the electric field intensity, **E**:

$$\mathbf{E} == -\nabla V = -\frac{d}{dx}\left[\frac{\rho_0}{a\epsilon_0}\left(1 - e^{-ax}\right)\right]\mathbf{a}_x = \underline{-\frac{\rho_0}{\epsilon_0}\, e^{-ax}\,\mathbf{a}_x} \text{ V/m}$$

b) find the potential difference between the points $x = d$ and $x = 0$:

$$V_{d0} = V(d) - V(0) = \underline{\frac{\rho_0}{a\epsilon_0}\left(1 - e^{-ad}\right)} \text{ V}$$

c) if the medium permittivity is given by $\epsilon(x) = \epsilon_0 e^{ax}$, find the electric flux density, **D**, and the volume charge density, ρ_v, in the region:

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0 e^{ax}\left(-\frac{\rho_0}{\epsilon_0}\, e^{-ax}\,\mathbf{a}_x\right) = \underline{-\rho_0\,\mathbf{a}_x} \text{ C/m}^2$$

Then $\rho_v = \nabla \cdot \mathbf{D} = \underline{0}$.

d) Find the stored energy in the region $(0 < x < d)$, $(0 < y < 1)$, $(0 < z < 1)$:

$$W_e = \int_v \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dv = \int_0^1\int_0^1\int_0^d \frac{\rho_0^2}{2\epsilon_0} e^{-ax}\,dx\,dy\,dz = \frac{-\rho_0^2}{2\epsilon_0 a}\,e^{-ax}\Big|_0^d = \underline{\frac{\rho_0^2}{2\epsilon_0 a}\left(1 - e^{-ad}\right)} \text{ J}$$

4.21. Let $V = 2xy^2z^3 + 3\ln(x^2 + 2y^2 + 3z^2)$ V in free space. Evaluate each of the following quantities at $P(3, 2, -1)$:

a) V: Substitute P directly to obtain: $V = \underline{-15.0\,\text{V}}$
b) $|V|$. This will be just $\underline{15.0\,\text{V}}$.
c) **E**: We have

$$\mathbf{E}\Big|_P = -\nabla V\Big|_P = -\left[\left(2y^2z^3 + \frac{6x}{x^2+2y^2+3z^2}\right)\mathbf{a}_x + \left(4xyz^3 + \frac{12y}{x^2+2y^2+3z^2}\right)\mathbf{a}_y\right.$$
$$\left. + \left(6xy^2z^2 + \frac{18z}{x^2+2y^2+3z^2}\right)\mathbf{a}_z\right]_P = \underline{7.1\mathbf{a}_x + 22.8\mathbf{a}_y - 71.1\mathbf{a}_z \text{ V/m}}$$

d) $|\mathbf{E}|_P$: taking the magnitude of the part c result, we find $|\mathbf{E}|_P = \underline{75.0\,\text{V/m}}$.

e) $\mathbf{a}_N$: By definition, this will be

$$\mathbf{a}_N\Big|_P = -\frac{\mathbf{E}}{|\mathbf{E}|} = \underline{-0.095\,\mathbf{a}_x - 0.304\,\mathbf{a}_y + 0.948\,\mathbf{a}_z}$$

f) **D**: This is $\mathbf{D}\Big|_P = \epsilon_0\mathbf{E}\Big|_P = \underline{62.8\,\mathbf{a}_x + 202\,\mathbf{a}_y - 629\,\mathbf{a}_z \text{ pC/m}^2}$.

4.22. A line charge of infinite length lies along the z axis, and carries a uniform linear charge density of ρ_ℓ C/m. A perfectly-conducting cylindrical shell, whose axis is the z axis, surrounds the line charge. The cylinder (of radius b), is at ground potential. Under these conditions, the potential function inside the cylinder ($\rho < b$) is given by

$$V(\rho) = k - \frac{\rho_\ell}{2\pi\epsilon_0} \ln(\rho)$$

where k is a constant.

a) Find k in terms of given or known parameters: At radius b,

$$V(b) = k - \frac{\rho_\ell}{2\pi\epsilon_0} \ln(b) = 0 \;\Rightarrow\; k = \underline{\frac{\rho_\ell}{2\pi\epsilon_0} \ln(b)}$$

b) find the electric field strength, $\mathbf{E}$, for $\rho < b$:

$$\mathbf{E}_{in} = -\nabla V = -\frac{d}{d\rho}\left[\frac{\rho_\ell}{2\pi\epsilon_0} \ln(b) - \frac{\rho_\ell}{2\pi\epsilon_0} \ln(\rho)\right] \mathbf{a}_\rho = \underline{\frac{\rho_\ell}{2\pi\epsilon_0 \rho} \mathbf{a}_\rho \text{ V/m}}$$

c) find the electric field strength, $\mathbf{E}$, for $\rho > b$: $\mathbf{E}_{out} = \underline{0}$ because the cylinder is at ground potential.

d) Find the stored energy in the electric field *per unit length* in the z direction within the volume defined by $\rho > a$, where $a < b$:

$$W_e = \int_v \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\, dv = \int_0^1 \int_0^{2\pi} \int_a^b \frac{\rho_\ell^2}{8\pi^2\epsilon_0\rho^2}\, \rho\, d\rho\, d\phi\, dz = \underline{\frac{\rho_\ell^2}{4\pi\epsilon_0} \ln\left(\frac{b}{a}\right) \text{ J}}$$

4.23. It is known that the potential is given as $V = 80\rho^{.6}$ V. Assuming free space conditions, find:

a) $\mathbf{E}$: We find this through

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\mathbf{a}_\rho = \underline{-48\rho^{-.4} \text{ V/m}}$$

b) the volume charge density at $\rho = .5$ m: Using $\mathbf{D} = \epsilon_0\mathbf{E}$, we find the charge density through

$$\rho_v\Big|_{.5} = [\nabla\cdot\mathbf{D}]_{.5} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}(\rho D_\rho)\Big|_{.5} = -28.8\epsilon_0\rho^{-1.4}\Big|_{.5} = \underline{-673 \text{ pC/m}^3}$$

c) the total charge lying within the closed surface $\rho = .6$, $0 < z < 1$: The easiest way to do this calculation is to evaluate D_ρ at $\rho = .6$ (noting that it is constant), and then multiply by the cylinder area: Using part a, we have $D_\rho\Big|_{.6} = -48\epsilon_0(.6)^{-.4} = -521$ pC/m^2. Thus $Q = -2\pi(.6)(1)521 \times 10^{-12}$ C $= \underline{-1.96 \text{ nC}}$.

4.24. A certain spherically-symmetric charge configuration in free space produces an electric field given in spherical coordinates by:

$$\mathbf{E}(r) = \begin{cases} (\rho_0 r^2)/(100\epsilon_0)\,\mathbf{a}_r \text{ V/m} & (r \le 10) \\ (100\rho_0)/(\epsilon_0 r^2)\,\mathbf{a}_r \text{ V/m} & (r \ge 10) \end{cases}$$

where ρ_0 is a constant.

a) Find the charge density as a function of position:

$$\rho_v(r \le 10) = \nabla \cdot (\epsilon_0 \mathbf{E}_1) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{\rho_0 r^2}{100}\right) = \underline{\frac{\rho_0 r}{25}} \text{ C/m}^3$$

$$\rho_v(r \ge 10) = \nabla \cdot (\epsilon_0 \mathbf{E}_2) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{100\rho_0}{r^2}\right) = \underline{0}$$

b) find the absolute potential as a function of position in the two regions, $r \le 10$ and $r \ge 10$:

$$V(r \le 10) = -\int_\infty^{10} \frac{100\rho_0}{\epsilon_0 r^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, dr \;-\; \int_{10}^{r} \frac{\rho_0 (r')^2}{100\epsilon_0}\,\mathbf{a}_r \cdot \mathbf{a}_r\, dr'$$
$$= \frac{100\rho_0}{\epsilon_0 r}\Big|_\infty^{10} - \frac{\rho_0 (r')^3}{300\epsilon_0}\Big|_{10}^{r} = \underline{\frac{10\rho_0}{3\epsilon_0}\left[4 - (10^{-3}r^3)\right] \text{ V}}$$

$$V(r \ge 10) = -\int_\infty^{r} \frac{100\rho_0}{\epsilon_0 (r')^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, dr' = \frac{100\rho_0}{\epsilon_0 r'}\Big|_\infty^{r} = \underline{\frac{100\rho_0}{\epsilon_0 r}} \text{ V}$$

c) check your result of part b by using the gradient:

$$\mathbf{E}_1 = -\nabla V(r \le 10) = -\frac{d}{dr}\left[\frac{10\rho_0}{3\epsilon_0}\left[4 - (10^{-3}r^3)\right]\right]\mathbf{a}_r = \frac{10\rho_0}{3\epsilon_0}(3r^2)(10^{-3})\,\mathbf{a}_r = \frac{\rho_0 r^2}{100\epsilon_0}\,\mathbf{a}_r$$

$$\mathbf{E}_2 = -\nabla V(r \ge 10) = -\frac{d}{dr}\left[\frac{100\rho_0}{\epsilon_0 r}\right]\mathbf{a}_r = \frac{100\rho_0}{\epsilon_0 r^2}\,\mathbf{a}_r$$

d) find the stored energy in the charge by an integral of the form of Eq. (42) (not Eq. (43)):

$$W_e = \frac{1}{2}\int_v \rho_v V dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{10} \frac{1}{2}\frac{\rho_0 r}{25}\left[\frac{10\rho_0}{3\epsilon_0}\left[4 - (10^{-3}r^3)\right]\right] r^2 \sin\theta\, dr\, d\theta\, d\phi$$
$$= \frac{4\pi\rho_0^2}{150\epsilon_0}\int_0^{10}\left[40r^3 - \frac{r^6}{100}\right] dr = \frac{4\pi\rho_0^2}{150\epsilon_0}\left[10r^4 - \frac{r^7}{700}\right]_0^{10} = \underline{7.18 \times 10^3\,\frac{\rho_0^2}{\epsilon_0}}$$

e) Find the stored energy in the field by an integral of the form of Eq. (44) (not Eq. (45)).

$$W_e = \int_{(r\le 10)} \frac{1}{2}\mathbf{D}_1 \cdot \mathbf{E}_1\, dv \;+\; \int_{(r \ge 10)} \frac{1}{2}\mathbf{D}_2 \cdot \mathbf{E}_2\, dv$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{10} \frac{\rho_0^2 r^4}{(2\times 10^4)\epsilon_0} r^2 \sin\theta\, dr\, d\theta\, d\phi + \int_0^{2\pi}\int_0^{\pi}\int_{10}^{\infty} \frac{10^4\rho_0^2}{2\epsilon_0 r^4} r^2 \sin\theta\, dr\, d\theta\, d\phi$$
$$= \frac{2\pi\rho_0^2}{\epsilon_0}\left[10^{-4}\int_0^{10} r^6\, dr \;+\; 10^4\int_{10}^{\infty}\frac{dr}{r^2}\right] = \frac{2\pi\rho_0^2}{\epsilon_0}\left[\frac{1}{7}(10^3) \;+\; 10^3\right] = \underline{7.18 \times 10^3\,\frac{\rho_0^2}{\epsilon_0}}$$

4.25. Within the cylinder $\rho = 2$, $0 < z < 1$, the potential is given by $V = 100 + 50\rho + 150\rho \sin\phi$ V.

a) Find V, $\mathbf{E}$, $\mathbf{D}$, and ρ_v at $P(1, 60^\circ, 0.5)$ in free space: First, substituting the given point, we find $V_P = \underline{279.9\,\text{V}}$. Then,

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi = -\left[50 + 150\sin\phi\right]\mathbf{a}_\rho - \left[150\cos\phi\right]\mathbf{a}_\phi$$

Evaluate the above at P to find $\mathbf{E}_P = \underline{-179.9\mathbf{a}_\rho - 75.0\mathbf{a}_\phi \text{ V/m}}$

Now $\mathbf{D} = \epsilon_0 \mathbf{E}$, so $\mathbf{D}_P = \underline{-1.59\mathbf{a}_\rho - .664\mathbf{a}_\phi \text{ nC/m}^2}$. Then

$$\rho_v = \nabla \cdot \mathbf{D} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} = \left[-\frac{1}{\rho}(50 + 150\sin\phi) + \frac{1}{\rho}150\sin\phi\right]\epsilon_0 = -\frac{50}{\rho}\epsilon_0 \text{ C}$$

At P, this is $\rho_{vP} = \underline{-443 \text{ pC/m}^3}$.

b) How much charge lies within the cylinder? We will integrate ρ_v over the volume to obtain:

$$Q = \int_0^1 \int_0^{2\pi} \int_0^2 -\frac{50\epsilon_0}{\rho}\rho\, d\rho\, d\phi\, dz = -2\pi(50)\epsilon_0(2) = \underline{-5.56\,\text{nC}}$$

4.26. Let us assume that we have a very thin, square, imperfectly conducting plate 2m on a side, located in the plane $z = 0$ with one corner at the origin such that it lies entirely within the first quadrant. The potential at any point in the plate is given as $V = -e^{-x}\sin y$.

a) An electron enters the plate at $x = 0$, $y = \pi/3$ with zero initial velocity; in what direction is its initial movement? We first find the electric field associated with the given potential:

$$\mathbf{E} = -\nabla V = -e^{-x}[\sin y\, \mathbf{a}_x - \cos y\, \mathbf{a}_y]$$

Since we have an electron, its motion is opposite that of the field, so the direction on entry is that of $-\mathbf{E}$ at $(0, \pi/3)$, or $\underline{\sqrt{3}/2\,\mathbf{a}_x - 1/2\,\mathbf{a}_y}$.

b) Because of collisions with the particles in the plate, the electron achieves a relatively low velocity and little acceleration (the work that the field does on it is converted largely into heat). The electron therefore moves approximately along a streamline. Where does it leave the plate and in what direction is it moving at the time? Considering the result of part *a*, we would expect the exit to occur along the bottom edge of the plate. The equation of the streamline is found through

$$\frac{E_y}{E_x} = \frac{dy}{dx} = -\frac{\cos y}{\sin y} \quad \Rightarrow \quad x = -\int \tan y\, dy + C = \ln(\cos y) + C$$

At the entry point $(0, \pi/3)$, we have $0 = \ln[\cos(\pi/3)] + C$, from which $C = 0.69$. Now, along the bottom edge ($y = 0$), we find $x = 0.69$, and so the exit point is $\underline{(0.69, 0)}$. From the field expression evaluated at the exit point, we find the direction on exit to be $\underline{-\mathbf{a}_y}$.

4.27. Two point charges, 1 nC at $(0, 0, 0.1)$ and -1 nC at $(0, 0, -0.1)$, are in free space.

a) Calculate V at $P(0.3, 0, 0.4)$: Use

$$V_P = \frac{q}{4\pi\epsilon_0 |\mathbf{R}^+|} - \frac{q}{4\pi\epsilon_0 |\mathbf{R}^-|}$$

where $\mathbf{R}^+ = (.3, 0, .3)$ and $\mathbf{R}^- = (.3, 0, .5)$, so that $|\mathbf{R}^+| = 0.424$ and $|\mathbf{R}^-| = 0.583$. Thus

$$V_P = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{1}{.424} - \frac{1}{.583}\right] = \underline{5.78\,\mathrm{V}}$$

b) Calculate $|\mathbf{E}|$ at P: Use

$$\mathbf{E}_P = \frac{q(.3\mathbf{a}_x + .3\mathbf{a}_z)}{4\pi\epsilon_0 (.424)^3} - \frac{q(.3\mathbf{a}_x + .5\mathbf{a}_z)}{4\pi\epsilon_0 (.583)^3} = \frac{10^{-9}}{4\pi\epsilon_0}\left[2.42\mathbf{a}_x + 1.41\mathbf{a}_z\right]\ \mathrm{V/m}$$

Taking the magnitude of the above, we find $|\mathbf{E}_P| = \underline{25.2\,\mathrm{V/m}}$.

c) Now treat the two charges as a dipole at the origin and find V at P: In spherical coordinates, P is located at $r = \sqrt{.3^2 + .4^2} = .5$ and $\theta = \sin^{-1}(.3/.5) = 36.9°$. Assuming a dipole in far-field, we have

$$V_P = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{10^{-9}(.2)\cos(36.9°)}{4\pi\epsilon_0 (.5)^2} = \underline{5.76\,\mathrm{V}}$$

4.28. Use the electric field intensity of the dipole (Sec. 4.7, Eq. (36)) to find the difference in potential between points at θ_a and θ_b, each point having the same r and ϕ coordinates. Under what conditions does the answer agree with Eq. (34), for the potential at θ_a?

We perform a line integral of Eq. (36) along an arc of constant r and ϕ:

$$V_{ab} = -\int_{\theta_b}^{\theta_a} \frac{qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right]\cdot\mathbf{a}_\theta\, r\,d\theta = -\int_{\theta_b}^{\theta_a} \frac{qd}{4\pi\epsilon_0 r^2}\sin\theta\,d\theta$$
$$= \frac{qd}{4\pi\epsilon_0 r^2}\left[\cos\theta_a - \cos\theta_b\right]$$

This result agrees with Eq. (34) if θ_a (the ending point in the path) is $90°$ (the xy plane). Under this condition, we note that if $\theta_b > 90°$, positive work is done when moving (against the field) to the xy plane; if $\theta_b < 90°$, negative work is done since we move with the field.

4.29. A dipole having a moment $\mathbf{p} = 3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z$ nC $\cdot$ m is located at $Q(1, 2, -4)$ in free space. Find V at $P(2, 3, 4)$: We use the general expression for the potential in the far field:

$$V = \frac{\mathbf{p}\cdot(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = P - Q = (1, 1, 8)$. So

$$V_P = \frac{(3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z)\cdot(\mathbf{a}_x + \mathbf{a}_y + 8\mathbf{a}_z)\times 10^{-9}}{4\pi\epsilon_0 [1^2 + 1^2 + 8^2]^{1.5}} = \underline{1.31\ \mathrm{V}}$$

4.30. A dipole for which $\mathbf{p} = 10\epsilon_0\,\mathbf{a}_z$ C · m is located at the origin. What is the equation of the surface on which $E_z = 0$ but $\mathbf{E} \neq 0$?

First we find the z component:

$$E_z = \mathbf{E}\cdot\mathbf{a}_z = \frac{10}{4\pi r^3}\left[2\cos\theta\,(\mathbf{a}_r\cdot\mathbf{a}_z) + \sin\theta\,(\mathbf{a}_\theta\cdot\mathbf{a}_z)\right] = \frac{5}{2\pi r^3}\left[2\cos^2\theta - \sin^2\theta\right]$$

This will be zero when $\left[2\cos^2\theta - \sin^2\theta\right] = 0$. Using identities, we write

$$2\cos^2\theta - \sin^2\theta = \frac{1}{2}[1 + 3\cos(2\theta)]$$

The above becomes zero on the cone surfaces, $\underline{\theta = 54.7°}$ and $\underline{\theta = 125.3°}$.

4.31. A potential field in free space is expressed as $V = 20/(xyz)$ V.

a) Find the total energy stored within the cube $1 < x, y, z < 2$. We integrate the energy density over the cube volume, where $w_E = (1/2)\epsilon_0\mathbf{E}\cdot\mathbf{E}$, and where

$$\mathbf{E} = -\nabla V = 20\left[\frac{1}{x^2yz}\mathbf{a}_x + \frac{1}{xy^2z}\mathbf{a}_y + \frac{1}{xyz^2}\mathbf{a}_z\right]\ \text{V/m}$$

The energy is now

$$W_E = 200\epsilon_0\int_1^2\int_1^2\int_1^2\left[\frac{1}{x^4y^2z^2} + \frac{1}{x^2y^4z^2} + \frac{1}{x^2y^2z^4}\right]dx\,dy\,dz$$

The integral evaluates as follows:

$$\begin{aligned}
W_E &= 200\epsilon_0\int_1^2\int_1^2\left[-\left(\frac{1}{3}\right)\frac{1}{x^3y^2z^2} - \frac{1}{xy^4z^2} - \frac{1}{xy^2z^4}\right]_1^2 dy\,dz \\
&= 200\epsilon_0\int_1^2\int_1^2\left[\left(\frac{7}{24}\right)\frac{1}{y^2z^2} + \left(\frac{1}{2}\right)\frac{1}{y^4z^2} + \left(\frac{1}{2}\right)\frac{1}{y^2z^4}\right]dy\,dz \\
&= 200\epsilon_0\int_1^2\left[-\left(\frac{7}{24}\right)\frac{1}{yz^2} - \left(\frac{1}{6}\right)\frac{1}{y^3z^2} - \left(\frac{1}{2}\right)\frac{1}{yz^4}\right]_1^2 dz \\
&= 200\epsilon_0\int_1^2\left[\left(\frac{7}{48}\right)\frac{1}{z^2} + \left(\frac{7}{48}\right)\frac{1}{z^2} + \left(\frac{1}{4}\right)\frac{1}{z^4}\right]dz \\
&= 200\epsilon_0(3)\left[\frac{7}{96}\right] = \underline{387\,\text{pJ}}
\end{aligned}$$

b) What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube? At $C(1.5, 1.5, 1.5)$ the energy density is

$$w_E = 200\epsilon_0(3)\left[\frac{1}{(1.5)^4(1.5)^2(1.5)^2}\right] = 2.07\times10^{-10}\ \text{J/m}^3$$

This, multiplied by a cube volume of 1, produces an energy value of $\underline{207\,\text{pJ}}$.

4.32. Using Eq. (36), a) find the energy stored in the dipole field in the region $r > a$:

We start with

$$\mathbf{E}(r,\theta) = \frac{qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right]$$

Then the energy will be

$$W_e = \int_{vol}\frac{1}{2}\epsilon_0\mathbf{E}\cdot\mathbf{E}\,dv = \int_0^{2\pi}\int_0^{\pi}\int_a^{\infty}\frac{(qd)^2}{32\pi^2\epsilon_0 r^6}\underbrace{\left[4\cos^2\theta + \sin^2\theta\right]}_{3\cos^2\theta+1} r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{-2\pi(qd)^2}{32\pi^2\epsilon_0}\,\frac{1}{3r^3}\Big|_a^{\infty}\int_0^{\pi}\left[3\cos^2\theta + 1\right]\sin\theta\,d\theta = \frac{(qd)^2}{48\pi^2\epsilon_0 a^3}\underbrace{\left[-\cos^3\theta - \cos\theta\right]_0^{\pi}}_{4}$$

$$= \frac{(qd)^2}{12\pi\epsilon_0 a^3}\ \text{J}$$

b) Why can we not let a approach zero as a limit? From the above result, a singularity in the energy occurs as $a \to 0$. More importantly, a cannot be too small, or the original far-field assumption used to derive Eq. (36) ($a >> d$) will not hold, and so the field expression will not be valid.

4.33. A copper sphere of radius 4 cm carries a uniformly-distributed total charge of $5\,\mu$C in free space.

a) Use Gauss' law to find **D** external to the sphere: with a spherical Gaussian surface at radius r, D will be the total charge divided by the area of this sphere, and will be $\mathbf{a}_r$-directed. Thus

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r = \frac{5\times 10^{-6}}{4\pi r^2}\mathbf{a}_r\ \text{C/m}^2$$

b) Calculate the total energy stored in the electrostatic field: Use

$$W_E = \int_{vol}\frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dv = \int_0^{2\pi}\int_0^{\pi}\int_{.04}^{\infty}\frac{1}{2}\,\frac{(5\times 10^{-6})^2}{16\pi^2\epsilon_0 r^4}\,r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

$$= (4\pi)\left(\frac{1}{2}\right)\frac{(5\times 10^{-6})^2}{16\pi^2\epsilon_0}\int_{.04}^{\infty}\frac{dr}{r^2} = \frac{25\times 10^{-12}}{8\pi\epsilon_0}\,\frac{1}{.04} = \underline{2.81\,\text{J}}$$

c) Use $W_E = Q^2/(2C)$ to calculate the capacitance of the isolated sphere: We have

$$C = \frac{Q^2}{2W_E} = \frac{(5\times 10^{-6})^2}{2(2.81)} = 4.45\times 10^{-12}\,\text{F} = \underline{4.45\,\text{pF}}$$

4.34. A sphere of radius a contains volume charge of uniform density ρ_0 C/m^3. Find the total stored energy by applying

a) Eq. (43): We first need the potential everywhere inside the sphere. The electric field inside and outside is readily found from Gauss's law:

$$\mathbf{E}_1 = \frac{\rho_0 r}{3\epsilon_0}\,\mathbf{a}_r \quad r \le a \quad \text{and} \quad \mathbf{E}_2 = \frac{\rho_0 a^3}{3\epsilon_0 r^2}\,\mathbf{a}_r \quad r \ge a$$

The potential at position r inside the sphere is now the work done in moving a unit positive point charge from infinity to position r:

$$V(r) = -\int_\infty^a \mathbf{E}_2 \cdot \mathbf{a}_r\,dr - \int_a^r \mathbf{E}_1 \cdot \mathbf{a}_r\,dr' = -\int_\infty^a \frac{\rho_0 a^3}{3\epsilon_0 r^2}\,dr - \int_a^r \frac{\rho_0 r'}{3\epsilon_0}\,dr' = \frac{\rho_0}{6\epsilon_0}\left(3a^2 - r^2\right)$$

Now, using this result in (43) leads to the energy associated with the charge in the sphere:

$$W_e = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^a \frac{\rho_0^2}{6\epsilon_0}\left(3a^2 - r^2\right) r^2 \sin\theta\,dr\,d\theta\,d\phi = \frac{\pi\rho_0}{3\epsilon_0}\int_0^a \left(3a^2 r^2 - r^4\right)\,dr = \frac{4\pi a^5 \rho_0^2}{15\epsilon_0}$$

b) Eq. (45): Using the given fields we find the energy densities

$$w_{e1} = \frac{1}{2}\epsilon_0 \mathbf{E}_1 \cdot \mathbf{E}_1 = \frac{\rho_0^2 r^2}{18\epsilon_0} \quad r \le a \quad \text{and} \quad w_{e2} = \frac{1}{2}\epsilon_0 \mathbf{E}_2 \cdot \mathbf{E}_2 = \frac{\rho_0^2 a^6}{18\epsilon_0 r^4} \quad r \ge a$$

We now integrate these over their respective volumes to find the total energy:

$$W_e = \int_0^{2\pi}\int_0^{\pi}\int_0^a \frac{\rho_0^2 r^2}{18\epsilon_0} r^2 \sin\theta\,dr\,d\theta\,d\phi + \int_0^{2\pi}\int_0^{\pi}\int_a^\infty \frac{\rho_0^2 a^6}{18\epsilon_0 r^4} r^2 \sin\theta\,dr\,d\theta\,d\phi = \frac{4\pi a^5 \rho_0^2}{15\epsilon_0}$$

4.35. Four 0.8 nC point charges are located in free space at the corners of a square 4 cm on a side.

a) Find the total potential energy stored: This will be given by

$$W_E = \frac{1}{2}\sum_{n=1}^{4} q_n V_n$$

where V_n in this case is the potential at the location of any one of the point charges that arises from the other three. This will be (for charge 1)

$$V_1 = V_{21} + V_{31} + V_{41} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{.04} + \frac{1}{.04} + \frac{1}{.04\sqrt{2}}\right]$$

Taking the summation produces a factor of 4, since the situation is the same at all four points. Consequently,

$$W_E = \frac{1}{2}(4)q_1 V_1 = \frac{(.8\times 10^{-9})^2}{2\pi\epsilon_0(.04)}\left[2 + \frac{1}{\sqrt{2}}\right] = 7.79\times 10^{-7}\,\text{J} = \underline{0.779\,\mu\text{J}}$$

b) A fifth 0.8 nC charge is installed at the center of the square. Again find the total stored energy: This will be the energy found in part *a* plus the amount of work done in moving the fifth charge into position from infinity. The latter is just the potential at the square center arising from the original four charges, times the new charge value, or

$$\Delta W_E = \frac{4(.8\times 10^{-9})^2}{4\pi\epsilon_0(.04\sqrt{2}/2)} = .813\,\mu\text{J}$$

The total energy is now

$$W_{E\,net} = W_E(\text{part a}) + \Delta W_E = .779 + .813 = \underline{1.59\,\mu\text{J}}$$

4.36 Surface charge of uniform density ρ_s lies on a spherical shell of radius b, centered at the origin in free space.

a) Find the absolute potential everywhere, with zero reference at infinity: First, the electric field, found from Gauss' law, is

$$\mathbf{E} = \frac{b^2\rho_s}{\epsilon_0 r^2}\,\mathbf{a}_r\ \text{V/m}$$

Then

$$V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{L} = -\int_\infty^r \frac{b^2\rho_s}{\epsilon_0(r')^2}\,dr' = \underline{\frac{b^2\rho_s}{\epsilon_0 r}}\ \text{V}$$

b) find the stored energy in the sphere by considering the charge density and the potential in a two-dimensional version of Eq. (42):

$$W_e = \frac{1}{2}\int_S \rho_s V(b)\,da = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi} \rho_s\frac{b^2\rho_s}{\epsilon_0 b}\,b^2\sin\theta\,d\theta\,d\phi = \underline{\frac{2\pi\rho_s^2 b^3}{\epsilon_0}}$$

c) find the stored energy in the electric field and show that the results of parts *b* and *c* are identical.

$$W_e = \int_v \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dv = \int_0^{2\pi}\int_0^{\pi}\int_b^{\infty} \frac{1}{2}\frac{b^4\rho_s^2}{\epsilon_0 r^4}\,r^2\sin\theta\,dr\,d\theta\,d\phi = \underline{\frac{2\pi\rho_s^2 b^3}{\epsilon_0}}$$

CHAPTER 5

5.1. Given the current density $\mathbf{J} = -10^4[\sin(2x)e^{-2y}\mathbf{a}_x + \cos(2x)e^{-2y}\mathbf{a}_y]\,\text{kA/m}^2$:

a) Find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1$, $0 < z < 2$: This is found through

$$I = \int\int_S \mathbf{J}\cdot\mathbf{n}\Big|_S da = \int_0^2\int_0^1 \mathbf{J}\cdot\mathbf{a}_y\Big|_{y=1} dx\,dz = \int_0^2\int_0^1 -10^4\cos(2x)e^{-2}\,dx\,dz$$
$$= -10^4(2)\frac{1}{2}\sin(2x)\Big|_0^1 e^{-2} = \underline{-1.23\,\text{MA}}$$

b) Find the total current leaving the region $0 < x, x < 1$, $2 < z < 3$ by integrating $\mathbf{J}\cdot\mathbf{dS}$ over the surface of the cube: Note first that current through the top and bottom surfaces will not exist, since $\mathbf{J}$ has no z component. Also note that there will be no current through the $x = 0$ plane, since $J_x = 0$ there. Current will pass through the three remaining surfaces, and will be found through

$$I = \int_2^3\int_0^1 \mathbf{J}\cdot(-\mathbf{a}_y)\Big|_{y=0} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_y)\Big|_{y=1} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_x)\Big|_{x=1} dy\,dz$$
$$= 10^4\int_2^3\int_0^1 \left[\cos(2x)e^{-0} - \cos(2x)e^{-2}\right] dx\,dz - 10^4\int_2^3\int_0^1 \sin(2)e^{-2y}\,dy\,dz$$
$$= 10^4\left(\frac{1}{2}\right)\sin(2x)\Big|_0^1(3-2)\left[1-e^{-2}\right] + 10^4\left(\frac{1}{2}\right)\sin(2)e^{-2y}\Big|_0^1(3-2) = \underline{0}$$

c) Repeat part b, but use the divergence theorem: We find the net outward current through the surface of the cube by integrating the divergence of $\mathbf{J}$ over the cube volume. We have

$$\nabla\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = -10^{-4}\left[2\cos(2x)e^{-2y} - 2\cos(2x)e^{-2y}\right] = \underline{0} \text{ as expected}$$

5.2. Given $\mathbf{J} = -10^{-4}(y\mathbf{a}_x + x\mathbf{a}_y)\,\text{A/m}^2$, find the current crossing the $y = 0$ plane in the $-\mathbf{a}_y$ direction between $z = 0$ and 1, and $x = 0$ and 2.

At $y = 0$, $\mathbf{J}(x,0) = -10^4 x\mathbf{a}_y$, so that the current through the plane becomes

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_0^1\int_0^2 -10^4 x\,\mathbf{a}_y\cdot(-\mathbf{a}_y)\,dx\,dz = \underline{2\times 10^{-4}}\text{ A}$$

5.3. Let

$$\mathbf{J} = \frac{400 \sin\theta}{r^2 + 4}\, \mathbf{a}_r \text{ A/m}^2$$

a) Find the total current flowing through that portion of the spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi$, $0 < \phi < 2\pi$: This will be

$$I = \int\int \mathbf{J} \cdot \mathbf{n}\Big|_S da = \int_0^{2\pi} \int_{.1\pi}^{.3\pi} \frac{400 \sin\theta}{(.8)^2 + 4} (.8)^2 \sin\theta \, d\theta \, d\phi = \frac{400(.8)^2 2\pi}{4.64} \int_{.1\pi}^{.3\pi} \sin^2 d\theta$$

$$= 346.5 \int_{.1\pi}^{.3\pi} \frac{1}{2}[1 - \cos(2\theta)] \, d\theta = \underline{77.4 \text{ A}}$$

b) Find the average value of $\mathbf{J}$ over the defined area. The area is

$$\text{Area} = \int_0^{2\pi} \int_{.1\pi}^{.3\pi} (.8)^2 \sin\theta \, d\theta \, d\phi = 1.46 \text{ m}^2$$

The average current density is thus $\mathbf{J}_{avg} = (77.4/1.46)\, \mathbf{a}_r = \underline{53.0\, \mathbf{a}_r \text{ A/m}^2}$.

5.4. If volume charge density is given as $\rho_v = (\cos\omega t)/r^2$ C/m^3 in spherical coordinates, find $\mathbf{J}$. It is reasonable to assume that $\mathbf{J}$ is not a function of θ or ϕ.

We use the continuity equation (5), along with the assumption of no angular variation to write

$$\nabla \cdot \mathbf{J} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 J_r\right) = -\frac{\partial \rho_v}{\partial t} = -\frac{\partial}{\partial t} \left(\frac{\cos\omega t}{r^2}\right) = \frac{\omega \sin\omega t}{r^2}$$

So we may now solve

$$\frac{\partial}{\partial r} \left(r^2 J_r\right) = \omega \sin\omega t$$

by direct integration to obtain:

$$\mathbf{J} = J_r\, \mathbf{a}_r = \underline{\frac{\omega \sin\omega t}{r}\, \mathbf{a}_r} \text{ A/m}^2$$

where the integration constant is set to zero because a steady current will not be created by a time-varying charge density.

5.5. Let

$$\mathbf{J} = \frac{25}{\rho}\,\mathbf{a}_\rho - \frac{20}{\rho^2 + 0.01}\,\mathbf{a}_z \text{ A/m}^2$$

a) Find the total current crossing the plane $z = 0.2$ in the $\mathbf{a}_z$ direction for $\rho < 0.4$: Use

$$I = \int\int_S \mathbf{J}\cdot\mathbf{n}\Big|_{z=.2} da = \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2 + .01}\rho\,d\rho\,d\phi$$
$$= -\left(\frac{1}{2}\right) 20\ln(.01 + \rho^2)\Big|_0^{.4}(2\pi) = -20\pi\ln(17) = \underline{-178.0\,\text{A}}$$

b) Calculate $\partial\rho_v/\partial t$: This is found using the equation of continuity:

$$\frac{\partial\rho_v}{\partial t} = -\nabla\cdot\mathbf{J} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho J_\rho) + \frac{\partial J_z}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(25) + \frac{\partial}{\partial z}\left(\frac{-20}{\rho^2 + .01}\right) = \underline{0}$$

c) Find the outward current crossing the closed surface defined by $\rho = 0.01$, $\rho = 0.4$, $z = 0$, and $z = 0.2$: This will be

$$I = \int_0^{.2}\int_0^{2\pi} \frac{25}{.01}\mathbf{a}_\rho\cdot(-\mathbf{a}_\rho)(.01)\,d\phi\,dz + \int_0^{.2}\int_0^{2\pi} \frac{25}{.4}\mathbf{a}_\rho\cdot(\mathbf{a}_\rho)(.4)\,d\phi\,dz$$
$$+ \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2 + .01}\mathbf{a}_z\cdot(-\mathbf{a}_z)\,\rho\,d\rho\,d\phi + \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2 + .01}\mathbf{a}_z\cdot(\mathbf{a}_z)\,\rho\,d\rho\,d\phi = \underline{0}$$

since the integrals will cancel each other.

d) Show that the divergence theorem is satisfied for $\mathbf{J}$ and the surface specified in part *b*. In part *c*, the net outward flux was found to be zero, and in part *b*, the divergence of $\mathbf{J}$ was found to be zero (as will be its volume integral). Therefore, the divergence theorem is satisfied.

5.6. In spherical coordinates, a current density $\mathbf{J} = -k/(r\sin\theta)\,\mathbf{a}_\theta$ A/m^2 exists in a conducting medium, where k is a constant. Determine the total current in the $\mathbf{a}_z$ direction that crosses a circular disk of radius R, centered on the z axis and located at a) $z = 0$; b) $z = h$.

Integration over a disk means that we use cylindrical coordinates. The general flux integral assumes the form:

$$I = \int_s \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^R \frac{-k}{r\sin\theta}\underbrace{\mathbf{a}_\theta\cdot\mathbf{a}_z}_{-\sin\theta}\,\rho\,d\rho\,d\phi$$

Then, using $r = \sqrt{\rho^2 + z^2}$, this becomes

$$I = \int_0^{2\pi}\int_0^R \frac{k\rho}{\sqrt{\rho^2 + z^2}} = 2\pi k\sqrt{\rho^2 + z^2}\Big|_0^R = 2\pi k\left[\sqrt{R^2 + z^2} - z\right]$$

At $z = 0$ (part *a*), we have $I(0) = \underline{2\pi kR}$, and at $z = h$ (part *b*): $I(h) = \underline{2\pi k\left[\sqrt{R^2 + h^2} - h\right]}$.

5.7. Assuming that there is no transformation of mass to energy or vice-versa, it is possible to write a continuity equation for mass.

a) If we use the continuity equation for charge as our model, what quantities correspond to **J** and ρ_v? These would be, respectively, mass flux density in $(\text{kg/m}^2 - \text{s})$ and mass density in (kg/m^3).

b) Given a cube 1 cm on a side, experimental data show that the rates at which mass is leaving each of the six faces are 10.25, -9.85, 1.75, -2.00, -4.05, and 4.45 mg/s. If we assume that the cube is an incremental volume element, determine an approximate value for the time rate of change of density at its center. We may write the continuity equation for mass as follows, also invoking the divergence theorem:

$$\int_v \frac{\partial \rho_m}{\partial t} dv = -\int_v \nabla \cdot \mathbf{J}_m \, dv = -\oint_s \mathbf{J}_m \cdot \mathbf{dS}$$

where

$$\oint_s \mathbf{J}_m \cdot \mathbf{dS} = 10.25 - 9.85 + 1.75 - 2.00 - 4.05 + 4.45 = 0.550 \text{ mg/s}$$

Treating our 1 cm^3 volume as differential, we find

$$\frac{\partial \rho_m}{\partial t} \doteq -\frac{0.550 \times 10^{-3} \,\text{g/s}}{10^{-6} \ \text{m}^3} = \underline{-550 \,\text{g/m}^3 - \text{s}}$$

5.8. A truncated cone has a height of 16 cm. The circular faces on the top and bottom have radii of 2mm and 0.1mm, respectively. If the material from which this solid cone is constructed has a conductivity of 2×10^6 S/m, use some good approximations to determine the resistance between the two circular faces.

Consider the cone upside down and centered on the positive z axis. The 1-mm radius end is at distance $z = \ell$ from the x-y plane; the wide end (2-mm radius) lies at $z = \ell + 16\,\text{cm}$. ℓ is chosen such that if the cone were not truncated, its vertex would occur at the origin. The cone surface subtends angle θ_c from the z axis (in spherical coordinates). Therefore, we may write

$$\ell = \frac{0.1\text{mm}}{\tan\theta_c} \quad \text{and} \quad \tan\theta_c = \frac{2\,\text{mm}}{160 + \ell}$$

Solving these, we find $\ell = 8.4$ mm, $\tan\theta_c = 1.19 \times 10^{-2}$, and so $\theta_c = 0.68°$, which gives us a very thin cone! With this understanding, we can assume that the current density is uniform with θ and ϕ and will vary only with spherical radius, r. So the current density will be constant over a spherical cap (of constant r) anywhere within the cone. As the cone is thin, we can also assume constant current density over any *flat* surface within the cone at a specifed z. That is, any spherical cap looks flat if the cap radius, r, is large compared to its radius as measured from the z axis (ρ). This is our primary assumption.

Now, assuming constant current density at constant r, and net current, I, we may write

$$I = \int_0^{2\pi} \int_0^{\theta_c} J(r)\, \mathbf{a}_r \cdot \mathbf{a}_r \, r^2 \sin\theta \, d\theta \, d\phi = 2\pi r^2 J(r)(1 - \cos\theta_c)$$

or

$$\mathbf{J}(r) = \frac{I}{2\pi r^2 (1 - \cos\theta_c)} \, \mathbf{a}_r = \frac{(1.42 \times 10^4) I}{2\pi r^2} \, \mathbf{a}_r$$

5.8 (continued) The electric field is now

$$\mathbf{E}(r) = \frac{\mathbf{J}(r)}{\sigma} = \frac{(1.42 \times 10^4)I}{2\pi r^2(2 \times 10^6)}\,\mathbf{a}_r = (7.1 \times 10^{-3})\frac{I}{2\pi r^2}\,\mathbf{a}_r \text{ V/m}$$

The voltage between the ends is now

$$V_0 = -\int_{r_{out}}^{r_{in}} \mathbf{E} \cdot \mathbf{a}_r\, dr$$

where $r_{in} = \ell/\cos\theta_c \doteq \ell$ and where $r_{out} = (160 + \ell)/\cos\theta_c \doteq 160 + \ell$. The voltage is

$$V_0 = -\int_{(160+\ell)\times 10^{-3}}^{\ell \times 10^{-3}} (7.1 \times 10^{-3})\frac{I}{2\pi r^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, dr = (7.1 \times 10^{-3})\frac{I}{2\pi}\left[\frac{1}{.0084} - \frac{1}{.1684}\right]$$
$$= \frac{0.40}{\pi} I$$

from which we identify the resistance as $R = 0.40/\pi = \underline{0.128}$ ohms.

A second method uses the idea that we can construct the cone from a stack of thin circular plates of linearly-increasing radius, ρ. Assuming each plate is of differential thickness, dz, the differential resistance of a plate will be

$$dR = \frac{dz}{\sigma \pi \rho^2}$$

where $\rho = z \tan\theta_c = z(1.19 \times 10^{-2})$. The cone resistance will be the resistance of the stack of plates (in series), found through

$$R = \int dR = \int_{\ell \times 10^{-3}}^{(160+\ell)\times 10^{-3}} \frac{dz}{\sigma \pi \rho^2} = \int_{\ell \times 10^{-3}}^{(160+\ell)\times 10^{-3}} \frac{dz}{(2 \times 10^6)\pi z^2 (1.19 \times 10^{-2})^2}$$
$$= \frac{3.53 \times 10^{-3}}{\pi}\left[\frac{1}{.0084} - \frac{1}{.1684}\right] = \underline{0.127} \text{ ohms}$$

5.9. a) Using data tabulated in Appendix C, calculate the required diameter for a 2-m long nichrome wire that will dissipate an average power of 450 W when 120 V rms at 60 Hz is applied to it:
The required resistance will be

$$R = \frac{V^2}{P} = \frac{l}{\sigma(\pi a^2)}$$

Thus the diameter will be

$$d = 2a = 2\sqrt{\frac{lP}{\sigma \pi V^2}} = 2\sqrt{\frac{2(450)}{(10^6)\pi(120)^2}} = 2.8 \times 10^{-4} \text{ m} = \underline{0.28\,\text{mm}}$$

b) Calculate the rms current density in the wire: The rms current will be $I = 450/120 = 3.75$ A. Thus

$$J = \frac{3.75}{\pi\left(2.8 \times 10^{-4}/2\right)^2} = \underline{6.0 \times 10^7 \text{ A/m}^2}$$

5.10. A large brass washer has a 2-cm inside diameter, a 5-cm outside diameter, and is 0.5 cm thick. Its conductivity is $\sigma = 1.5 \times 10^7$ S/m. The washer is cut in half along a diameter, and a voltage is applied between the two rectangular faces of one part. The resultant electric field in the interior of the half-washer is $\mathbf{E} = (0.5/\rho)\,\mathbf{a}_\phi$ V/m in cylindrical coordinates, where the z axis is the axis of the washer.

a) What potential difference exists between the two rectangular faces? First, we orient the washer in the x-y plane with the cut faces aligned with the x axis. To find the voltage, we integrate $\mathbf{E}$ over a circular path of radius ρ inside the washer, between the two cut faces:

$$V_0 = -\int \mathbf{E} \cdot d\mathbf{L} = -\int_{\pi}^{0} \frac{0.5\pi}{\rho}\,\mathbf{a}_\phi \cdot \mathbf{a}_\phi\,\rho\,d\phi = \underline{0.5\pi}\text{ V}$$

b) What total current is flowing? First, the current density is $\mathbf{J} = \sigma\mathbf{E}$, so

$$\mathbf{J} = \frac{1.5 \times 10^7(0.5)}{\rho}\,\mathbf{a}_\phi = \frac{7.5 \times 10^6}{\rho}\,\mathbf{a}_\phi \text{ A/m}^2$$

Current is then found by integrating $\mathbf{J}$ over any transverse plane in the washer (the rectangular cross-section):

$$I = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^{0.5\times10^{-2}} \int_{10^{-2}}^{2.5\times10^{-2}} \frac{7.5 \times 10^6}{\rho}\,\mathbf{a}_\phi \cdot \mathbf{a}_\phi\,d\rho\,dz$$
$$= 0.5 \times 10^{-2}(7.5 \times 10^6)\ln\left(\frac{2.5}{1}\right) = \underline{3.4 \times 10^4}\text{ A}$$

c) What is the resistance between the two faces?

$$R = \frac{V_0}{I} = \frac{0.5\pi}{3.4 \times 10^4} = \underline{4.6 \times 10^{-5}}\text{ ohms}$$

5.11. Two perfectly-conducting cylindrical surfaces of length l are located at $\rho = 3$ and $\rho = 5$ cm. The total current passing radially outward through the medium between the cylinders is 3 A dc.

a) Find the voltage and resistance between the cylinders, and $\mathbf{E}$ in the region between the cylinders, if a conducting material having $\sigma = 0.05$ S/m is present for $3 < \rho < 5$ cm: Given the current, and knowing that it is radially-directed, we find the current density by dividing it by the area of a cylinder of radius ρ and length l:

$$\mathbf{J} = \frac{3}{2\pi\rho l}\,\mathbf{a}_\rho \text{ A/m}^2$$

5.11a) (continued)

Then the electric field is found by dividing this result by σ:

$$\mathbf{E} = \frac{3}{2\pi\sigma\rho l}\,\mathbf{a}_\rho = \underline{\frac{9.55}{\rho l}\mathbf{a}_\rho \text{ V/m}}$$

The voltage between cylinders is now:

$$V = -\int_5^3 \mathbf{E}\cdot\mathbf{dL} = \int_3^5 \frac{9.55}{\rho l}\mathbf{a}_\rho\cdot\mathbf{a}_\rho d\rho = \frac{9.55}{l}\ln\left(\frac{5}{3}\right) = \underline{\frac{4.88}{l}\text{ V}}$$

Now, the resistance will be

$$R = \frac{V}{I} = \frac{4.88}{3l} = \underline{\frac{1.63}{l}\ \Omega}$$

b) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power: We calculate

$$P = \int_v \mathbf{E}\cdot\mathbf{J}\,dv = \int_0^l\int_0^{2\pi}\int_{.03}^{.05}\frac{3^2}{(2\pi)^2\rho^2(.05)l^2}\,\rho\,d\rho\,d\phi\,dz = \frac{3^2}{2\pi(.05)l}\ln\left(\frac{5}{3}\right) = \frac{14.64}{l}\text{ W}$$

We also find the power by taking the product of voltage and current:

$$P = VI = \frac{4.88}{l}(3) = \frac{14.64}{l}\text{ W}$$

which is in agreement with the power density integration.

5.12. Two identical conducting plates, each having area A, are located at $z = 0$ and $z = d$. The region between plates is filled with a material having z-dependent conductivity, $\sigma(z) = \sigma_0 e^{-z/d}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters:

a) the resistance of the material: We start with the differential resistance of a thin slab of the material of thickness dz, which is

$$dR = \frac{dz}{\sigma A} = \frac{e^{z/d}dz}{\sigma_0 A} \quad\text{so that}\quad R = \int dR = \int_0^d \frac{e^{z/d}dz}{\sigma_0 A} = \frac{d}{\sigma_0 A}(e-1) = \frac{1.72d}{\sigma_0 A}\ \Omega$$

b) the total current flowing between plates: We use

$$I = \frac{V_0}{R} = \frac{\sigma_0 A V_0}{1.72\,d}$$

c) the electric field intensity $\mathbf{E}$ within the material: First the current density is

$$\mathbf{J} = -\frac{I}{A}\,\mathbf{a}_z = \frac{-\sigma_0 V_0}{1.72\,d}\,\mathbf{a}_z \quad\text{so that}\quad \mathbf{E} = \frac{\mathbf{J}}{\sigma(z)} = \frac{-V_0 e^{z/d}}{1.72\,d}\,\mathbf{a}_z \text{ V/m}$$

5.13. A hollow cylindrical tube with a rectangular cross-section has external dimensions of 0.5 in by 1 in and a wall thickness of 0.05 in. Assume that the material is brass, for which $\sigma = 1.5 \times 10^7$ S/m. A current of 200 A dc is flowing down the tube.

a) What voltage drop is present across a 1m length of the tube? Converting all measurements to meters, the tube resistance over a 1 m length will be:

$$R_1 = \frac{1}{(1.5 \times 10^7)\left[(2.54)(2.54/2) \times 10^{-4} - 2.54(1-.1)(2.54/2)(1-.2) \times 10^{-4}\right]}$$
$$= 7.38 \times 10^{-4}\ \Omega$$

The voltage drop is now $V = IR_1 = 200(7.38 \times 10^{-4} = \underline{0.147\text{ V}}$.

b) Find the voltage drop if the interior of the tube is filled with a conducting material for which $\sigma = 1.5 \times 10^5$ S/m: The resistance of the filling will be:

$$R_2 = \frac{1}{(1.5 \times 10^5)(1/2)(2.54)^2 \times 10^{-4}(.9)(.8)} = 2.87 \times 10^{-2}\ \Omega$$

The total resistance is now the parallel combination of R_1 and R_2:
$R_T = R_1R_2/(R_1 + R_2) = 7.19 \times 10^{-4}\ \Omega$, and the voltage drop is now $V = 200R_T = \underline{.144\,\text{V}}$.

5.14. A rectangular conducting plate lies in the xy plane, occupying the region $0 < x < a$, $0 < y < b$. An identical conducting plate is positioned directly above and parallel to the first, at $z = d$. The region between plates is filled with material having conductivity $\sigma(x) = \sigma_0 e^{-x/a}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters:

a) the electric field intensity $\mathbf{E}$ within the material: We know that $\mathbf{E}$ will be z-directed, but the conductivity varies with x. We therefore expect no z variation in $\mathbf{E}$, and also note that the line integral of $\mathbf{E}$ between the bottom and top plates must always give V_0. Therefore $\mathbf{E} = \underline{-V_0/d\,\mathbf{a}_z \text{ V/m}}$.

b) the total current flowing between plates: We have

$$\mathbf{J} = \sigma(x)\mathbf{E} = \frac{-\sigma_0 e^{-x/a} V_0}{d}\,\mathbf{a}_z$$

Using this, we find

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_0^b \int_0^a \frac{-\sigma_0 e^{-x/a} V_0}{d}\,\mathbf{a}_z \cdot (-\mathbf{a}_z)\,dx\,dy = \frac{\sigma_0 ab V_0}{d}(1 - e^{-1}) = \frac{0.63 ab\sigma_0 V_0}{d}\ \text{A}$$

c) the resistance of the material: We use

$$R = \frac{V_0}{I} = \frac{d}{0.63\,ab\,\sigma_0}\ \Omega$$

5.15. Let $V = 10(\rho+1)z^2\cos\phi\,\mathrm{V}$ in free space.

a) Let the equipotential surface $V = 20\,\mathrm{V}$ define a conductor surface. Find the equation of the conductor surface: Set the given potential function equal to 20, to find:

$$\underline{(\rho+1)z^2\cos\phi = 2}$$

b) Find ρ and $\mathbf{E}$ at that point on the conductor surface where $\phi = 0.2\pi$ and $z = 1.5$: At the given values of ϕ and z, we solve the equation of the surface found in part a for ρ, obtaining $\rho = \underline{.10}$. Then

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi - \frac{\partial V}{\partial z}\mathbf{a}_z$$
$$= -10z^2\cos\phi\,\mathbf{a}_\rho + 10\,\frac{\rho+1}{\rho}z^2\sin\phi\,\mathbf{a}_\phi - 20(\rho+1)z\cos\phi\,\mathbf{a}_z$$

Then

$$\mathbf{E}(.10, .2\pi, 1.5) = \underline{-18.2\,\mathbf{a}_\rho + 145\,\mathbf{a}_\phi - 26.7\,\mathbf{a}_z\ \mathrm{V/m}}$$

c) Find $|\rho_s|$ at that point: Since $\mathbf{E}$ is at the perfectly-conducting surface, it will be normal to the surface, so we may write:

$$\rho_s = \epsilon_0 \mathbf{E}\cdot\mathbf{n}\Big|_{\text{surface}} = \epsilon_0\frac{\mathbf{E}\cdot\mathbf{E}}{|\mathbf{E}|} = \epsilon_0\sqrt{\mathbf{E}\cdot\mathbf{E}} = \epsilon_0\sqrt{(18.2)^2+(145)^2+(26.7)^2} = \underline{1.32\ \mathrm{nC/m^2}}$$

5.16. A coaxial transmission line has inner and outer conductor radii a and b. Between conductors $(a < \rho < b)$ lies a conductive medium whose conductivity is $\sigma(\rho) = \sigma_0/\rho$, where σ_0 is a constant. The inner conductor is charged to potential V_0, and the outer conductor is grounded.

a) Assuming dc radial current I per unit length in z, determine the radial current density field $\mathbf{J}$ in $\mathrm{A/m^2}$: This will be the current divided by the cross-sectional area that is normal to the current direction:

$$\mathbf{J} = \frac{I}{2\pi\rho(1)}\,\mathbf{a}_\rho\ \mathrm{A/m^2}$$

b) Determine the electric field intensity $\mathbf{E}$ in terms of I and other parameters, given or known:

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{I\rho}{2\pi\sigma_0\rho}\,\mathbf{a}_\rho = \frac{I}{2\pi\sigma_0}\,\mathbf{a}_\rho\ \mathrm{V/m}$$

c) by taking an appropriate line integral of $\mathbf{E}$ as found in part b, find an expression that relates V_0 to I:

$$V_0 = -\int_b^a \mathbf{E}\cdot d\mathbf{L} = -\int_b^a \frac{I}{2\pi\sigma_0}\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,d\rho = \frac{I(b-a)}{2\pi\sigma_0}\ \mathrm{V}$$

d) find an expression for the conductance of the line per unit length, G:

$$G = \frac{I}{V_0} = \frac{2\pi\sigma_0}{(b-a)}\ \mathrm{S/m}$$

5.17. Given the potential field $V = 100xz/(x^2+4)$ V. in free space:

a) Find $\mathbf{D}$ at the surface $z = 0$: Use

$$\mathbf{E} = -\nabla V = -100z\frac{\partial}{\partial x}\left(\frac{x}{x^2+4}\right)\mathbf{a}_x - 0\,\mathbf{a}_y - \frac{100x}{x^2+4}\mathbf{a}_z \text{ V/m}$$

At $z = 0$, we use this to find $\mathbf{D}(z=0) = \epsilon_0\mathbf{E}(z=0) = \underline{-100\epsilon_0 x/(x^2+4)\,\mathbf{a}_z \text{ C/m}^2}$.

b) Show that the $z = 0$ surface is an equipotential surface: There are two reasons for this: 1) $\mathbf{E}$ at $z = 0$ is everywhere z-directed, and so moving a charge around on the surface involves doing no work; 2) When evaluating the given potential function at $z = 0$, the result is 0 for all x and y.

c) Assume that the $z = 0$ surface is a conductor and find the total charge on that portion of the conductor defined by $0 < x < 2$, $-3 < y < 0$: We have

$$\rho_s = \mathbf{D}\cdot\mathbf{a}_z\Big|_{z=0} = -\frac{100\epsilon_0 x}{x^2+4} \text{ C/m}^2$$

So

$$Q = \int_{-3}^{0}\int_{0}^{2} -\frac{100\epsilon_0 x}{x^2+4}\,dx\,dy = -(3)(100)\epsilon_0\left(\frac{1}{2}\right)\ln(x^2+4)\Big|_0^2 = -150\epsilon_0\ln 2 = \underline{-0.92\,\text{nC}}$$

5.18. Two parallel circular plates of radius a are located at $z = 0$ and $z = d$. The top plate ($z = d$) is raised to potential V_0; the bottom plate is grounded. Between the plates is a conducting material having radial-dependent conductivity, $\sigma(\rho) = \sigma_0\rho$, where σ_0 is a constant.

a) Find the *ρ-independent* electric field strength, $\mathbf{E}$, between plates: The integral of $\mathbf{E}$ between plates must give V_0, independent of position on the plates. Therefore, it must be true that

$$\mathbf{E} = -\frac{V_0}{d}\mathbf{a}_z \text{ V/m} \quad (0 < \rho < a)$$

b) Find the current density, $\mathbf{J}$ between plates:

$$\mathbf{J} = \sigma\mathbf{E} = -\frac{\sigma_0 V_0 \rho}{d}\mathbf{a}_z \text{ A/m}^2$$

c) Find the total current, I, in the structure:

$$I = \int_0^{2\pi}\int_0^a -\frac{\sigma_0 V_0\rho}{d}\mathbf{a}_z\cdot(-\mathbf{a}_z)\,\rho\,d\rho\,d\phi = \frac{2\pi a^3\sigma_0 V_0}{3d} \text{ A}$$

d) Find the resistance between plates:

$$R = \frac{V_0}{I} = \frac{3d}{2\pi a^3\sigma_0} \text{ ohms}$$

5.19. Let $V = 20x^2yz - 10z^2$ V in free space.

a) Determine the equations of the equipotential surfaces on which $V = 0$ and 60 V: Setting the given potential function equal to 0 and 60 and simplifying results in:

$$\text{At } 0\,\text{V}: \quad 2x^2y - z = 0$$

$$\text{At } 60\,\text{V}: \quad 2x^2y - z = \frac{6}{z}$$

b) Assume these are conducting surfaces and find the surface charge density at that point on the $V = 60$ V surface where $x = 2$ and $z = 1$. It is known that $0 \leq V \leq 60$ V is the field-containing region: First, on the 60 V surface, we have

$$2x^2y - z - \frac{6}{z} = 0 \;\Rightarrow\; 2(2)^2y(1) - 1 - 6 = 0 \;\Rightarrow\; y = \frac{7}{8}$$

Now

$$\mathbf{E} = -\nabla V = -40xyz\,\mathbf{a}_x - 20x^2z\,\mathbf{a}_y - [20xy - 20z]\,\mathbf{a}_z$$

Then, at the given point, we have

$$\mathbf{D}(2, 7/8, 1) = \epsilon_0\mathbf{E}(2, 7/8, 1) = -\epsilon_0[70\,\mathbf{a}_x + 80\,\mathbf{a}_y + 50\,\mathbf{a}_z]\,\text{C/m}^2$$

We know that since this is the higher potential surface, **D** must be directed away from it, and so the charge density would be positive. Thus

$$\rho_s = \sqrt{\mathbf{D}\cdot\mathbf{D}} = 10\epsilon_0\sqrt{7^2 + 8^2 + 5^2} = \underline{1.04\,\text{nC/m}^2}$$

c) Give the unit vector at this point that is normal to the conducting surface and directed toward the $V = 0$ surface: This will be in the direction of **E** and **D** as found in part *b*, or

$$\mathbf{a}_n = -\left[\frac{7\mathbf{a}_x + 8\mathbf{a}_y + 5\mathbf{a}_z}{\sqrt{7^2 + 8^2 + 5^2}}\right] = \underline{-[0.60\mathbf{a}_x + 0.68\mathbf{a}_y + 0.43\mathbf{a}_z]}$$

5.20. Two point charges of -100π μC are located at (2,-1,0) and (2,1,0). The surface $x = 0$ is a conducting plane.

a) Determine the surface charge density at the origin. I will solve the general case first, in which we find the charge density anywhere on the y axis. With the conducting plane in the yz plane, we will have two image charges, each of $+100\pi$ μC, located at (-2, -1, 0) and (-2, 1, 0). The electric flux density on the y axis from these four charges will be

$$\mathbf{D}(y) = \frac{-100\pi}{4\pi}\Bigg[\underbrace{\frac{[(y-1)\,\mathbf{a}_y - 2\,\mathbf{a}_x]}{[(y-1)^2+4]^{3/2}} + \frac{[(y+1)\,\mathbf{a}_y - 2\,\mathbf{a}_x]}{[(y+1)^2+4]^{3/2}}}_{\text{given charges}}$$

$$- \underbrace{\frac{[(y-1)\,\mathbf{a}_y + 2\,\mathbf{a}_x]}{[(y-1)^2+4]^{3/2}} - \frac{[(y+1)\,\mathbf{a}_y + 2\,\mathbf{a}_x]}{[(y+1)^2+4]^{3/2}}}_{\text{image charges}}\Bigg]\;\mu\text{C/m}^2$$

5.20 a) (continued)

In the expression, all y components cancel, and we are left with

$$\mathbf{D}(y) = 100\left[\frac{1}{[(y-1)^2+4]^{3/2}} + \frac{1}{[(y+1)^2+4]^{3/2}}\right] \mathbf{a}_x \ \mu\text{C/m}^2$$

We now find the charge density at the origin:

$$\rho_s(0,0,0) = \mathbf{D}\cdot\mathbf{a}_x\Big|_{y=0} = \underline{17.9\ \mu\text{C/m}^2}$$

b) Determine ρ_S at $P(0,h,0)$. This will be

$$\rho_s(0,h,0) = \mathbf{D}\cdot\mathbf{a}_x\Big|_{y=h} = 100\left[\frac{1}{[(h-1)^2+4]^{3/2}} + \frac{1}{[(h+1)^2+4]^{3/2}}\right] \ \mu\text{C/m}^2$$

5.21. Let the surface $y = 0$ be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at $x = 0$, $y = 1$, and $x = 0$, $y = 2$.

a) Let $V = 0$ at the plane $y = 0$, and find V at $P(1,2,0)$: The line charges will image across the plane, producing image line charges of -30 nC/m each at $x = 0$, $y = -1$, and $x = 0$, $y = -2$. We find the potential at P by evaluating the work done in moving a unit positive charge from the $y = 0$ plane (we choose the origin) to P: For each line charge, this will be:

$$V_P - V_{0,0,0} = -\frac{\rho_l}{2\pi\epsilon_0}\ln\left[\frac{\text{final distance from charge}}{\text{initial distance from charge}}\right]$$

where $V_{0,0,0} = 0$. Considering the four charges, we thus have

$$\begin{aligned}V_P &= -\frac{\rho_l}{2\pi\epsilon_0}\left[\ln\left(\frac{1}{2}\right) + \ln\left(\frac{\sqrt{2}}{1}\right) - \ln\left(\frac{\sqrt{10}}{1}\right) - \ln\left(\frac{\sqrt{17}}{2}\right)\right] \\ &= \frac{\rho_l}{2\pi\epsilon_0}\left[\ln(2) + \ln\left(\frac{1}{\sqrt{2}}\right) + \ln\left(\sqrt{10}\right) + \ln\left(\frac{\sqrt{17}}{2}\right)\right] = \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\left[\frac{\sqrt{10}\sqrt{17}}{\sqrt{2}}\right] \\ &= \underline{1.20\,\text{kV}}\end{aligned}$$

b) Find $\mathbf{E}$ at P: Use

$$\begin{aligned}\mathbf{E}_P &= \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(1,2,0)-(0,1,0)}{|(1,1,0)|^2} + \frac{(1,2,0)-(0,2,0)}{|(1,0,0)|^2}\right. \\ &\quad \left. - \frac{(1,2,0)-(0,-1,0)}{|(1,3,0)|^2} - \frac{(1,2,0)-(0,-2,0)}{|(1,4,0)|^2}\right] \\ &= \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(1,1,0)}{2} + \frac{(1,0,0)}{1} - \frac{(1,3,0)}{10} - \frac{(1,4,0)}{17}\right] = \underline{723\,\mathbf{a}_x - 18.9\,\mathbf{a}_y \ \text{V/m}}\end{aligned}$$

5.22. The line segment $x = 0$, $-1 \le y \le 1$, $z = 1$, carries a linear charge density $\rho_L = \pi|y|$ μC/m. Let $z = 0$ be a conducting plane and determine the surface charge density at: (a) (0,0,0); (b) (0,1,0).

We consider the line charge to be made up of a string of differential segments of length, dy', and of charge $dq = \rho_L\, dy'$. A given segment at location $(0, y', 1)$ will have a corresponding image charge segment at location $(0, y', -1)$. The differential flux density on the y axis that is associated with the segment-image pair will be

$$d\mathbf{D} = \frac{\rho_L\, dy'[(y-y')\,\mathbf{a}_y - \mathbf{a}_z]}{4\pi[(y-y')^2+1]^{3/2}} - \frac{\rho_L\, dy'[(y-y')\,\mathbf{a}_y + \mathbf{a}_z]}{4\pi[(y-y')^2+1]^{3/2}} = \frac{-\rho_L\, dy'\,\mathbf{a}_z}{2\pi[(y-y')^2+1]^{3/2}}$$

In other words, each charge segment and its image produce a net field in which the y components have cancelled. The total flux density from the line charge and its image is now

$$\begin{aligned}
\mathbf{D}(y) &= \int d\mathbf{D} = \int_{-1}^{1} \frac{-\pi|y'|\,\mathbf{a}_z\, dy'}{2\pi[(y-y')^2+1]^{3/2}} \\
&= -\frac{\mathbf{a}_z}{2}\int_0^1 \left[\frac{y'}{[(y-y')^2+1]^{3/2}} + \frac{y'}{[(y+y')^2+1]^{3/2}}\right] dy' \\
&= \frac{\mathbf{a}_z}{2}\left[\frac{y(y-y')+1}{[(y-y')^2+1]^{1/2}} + \frac{y(y+y')+1}{[(y+y')^2+1]^{1/2}}\right]_0^1 \\
&= \frac{\mathbf{a}_z}{2}\left[\frac{y(y-1)+1}{[(y-1)^2+1]^{1/2}} + \frac{y(y+1)+1}{[(y+1)^2+1]^{1/2}} - 2(y^2+1)^{1/2}\right]
\end{aligned}$$

Now, at the origin (part a), we find the charge density through

$$\rho_s(0,0,0) = \mathbf{D}\cdot\mathbf{a}_z\Big|_{y=0} = \frac{\mathbf{a}_z}{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 2\right] = \underline{-0.29\ \mu\text{C/m}^2}$$

Then, at (0,1,0) (part b), the charge density is

$$\rho_s(0,1,0) = \mathbf{D}\cdot\mathbf{a}_z\Big|_{y=1} = \frac{\mathbf{a}_z}{2}\left[1 + \frac{3}{\sqrt{5}} - 2\right] = \underline{-0.24\ \mu\text{C/m}^2}$$

5.23. A dipole with $\mathbf{p} = 0.1\mathbf{a}_z$ μC $\cdot$ m is located at $A(1,0,0)$ in free space, and the $x = 0$ plane is perfectly-conducting.

a) Find V at $P(2,0,1)$. We use the far-field potential for a z-directed dipole:

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{p}{4\pi\epsilon_0}\frac{z}{[x^2+y^2+z^2]^{1.5}}$$

The dipole at $x = 1$ will image in the plane to produce a second dipole of the opposite orientation at $x = -1$. The potential at any point is now:

$$V = \frac{p}{4\pi\epsilon_0}\left[\frac{z}{[(x-1)^2+y^2+z^2]^{1.5}} - \frac{z}{[(x+1)^2+y^2+z^2]^{1.5}}\right]$$

Substituting $P(2,0,1)$, we find

$$V = \frac{.1\times10^6}{4\pi\epsilon_0}\left[\frac{1}{2\sqrt{2}} - \frac{1}{10\sqrt{10}}\right] = \underline{289.5\ \text{V}}$$

5.23 b) Find the equation of the 200-V equipotential surface in cartesian coordinates: We just set the potential exression of part a equal to 200 V to obtain:

$$\left[\frac{z}{[(x-1)^2+y^2+z^2]^{1.5}} - \frac{z}{[(x+1)^2+y^2+z^2]^{1.5}}\right] = 0.222$$

5.24. At a certain temperature, the electron and hole mobilities in intrinsic germanium are given as 0.43 and 0.21 $\mathrm{m^2/V \cdot s}$, respectively. If the electron and hole concentrations are both 2.3×10^{19} $\mathrm{m^{-3}}$, find the conductivity at this temperature.

With the electron and hole charge magnitude of 1.6×10^{-19} C, the conductivity in this case can be written:

$$\sigma = |\rho_e|\mu_e + \rho_h\mu_h = (1.6 \times 10^{-19})(2.3 \times 10^{19})(0.43 + 0.21) = \underline{2.36 \text{ S/m}}$$

5.25. Electron and hole concentrations increase with temperature. For pure silicon, suitable expressions are $\rho_h = -\rho_e = 6200T^{1.5}e^{-7000/T}$ $\mathrm{C/m^3}$. The functional dependence of the mobilities on temperature is given by $\mu_h = 2.3 \times 10^5 T^{-2.7}$ $\mathrm{m^2/V \cdot s}$ and $\mu_e = 2.1 \times 10^5 T^{-2.5}$ $\mathrm{m^2/V \cdot s}$, where the temperature, T, is in degrees Kelvin. The conductivity will thus be

$$\begin{aligned}\sigma &= -\rho_e\mu_e + \rho_h\mu_h = 6200T^{1.5}e^{-7000/T}\left[2.1 \times 10^5 T^{-2.5} + 2.3 \times 10^5 T^{-2.7}\right] \\ &= \frac{1.30 \times 10^9}{T} e^{-7000/T}\left[1 + 1.095T^{-.2}\right] \text{ S/m}\end{aligned}$$

Find σ at:

a) 0° C: With $T = 273$°K, the expression evaluates as $\sigma(0) = \underline{4.7 \times 10^{-5} \text{ S/m}}$.

b) 40° C: With $T = 273 + 40 = 313$, we obtain $\sigma(40) = \underline{1.1 \times 10^{-3} \text{ S/m}}$.

c) 80° C: With $T = 273 + 80 = 353$, we obtain $\sigma(80) = \underline{1.2 \times 10^{-2} \text{ S/m}}$.

5.26. A semiconductor sample has a rectangular cross-section 1.5 by 2.0 mm, and a length of 11.0 mm. The material has electron and hole densities of 1.8×10^{18} and 3.0×10^{15} $\mathrm{m^{-3}}$, respectively. If $\mu_e = 0.082$ $\mathrm{m^2/V \cdot s}$ and $\mu_h = 0.0021$ $\mathrm{m^2/V \cdot s}$, find the resistance offered between the end faces of the sample.

Using the given values along with the electron charge, the conductivity is

$$\sigma = (1.6 \times 10^{-19})\left[(1.8 \times 10^{18})(0.082) + (3.0 \times 10^{15})(0.0021)\right] = 0.0236 \text{ S/m}$$

The resistance is then

$$R = \frac{\ell}{\sigma A} = \frac{0.011}{(0.0236)(0.002)(0.0015)} = \underline{155 \text{ k}\Omega}$$

5.27. Atomic hydrogen contains 5.5×10^{25} atoms/m^3 at a certain temperature and pressure. When an electric field of 4 kV/m is applied, each dipole formed by the electron and positive nucleus has an effective length of 7.1×10^{-19} m.

a) Find P: With all identical dipoles, we have

$$P = Nqd = (5.5 \times 10^{25})(1.602 \times 10^{-19})(7.1 \times 10^{-19}) = 6.26 \times 10^{-12}\ \text{C/m}^2 = \underline{6.26\ \text{pC/m}^2}$$

b) Find ϵ_r: We use $P = \epsilon_0 \chi_e E$, and so

$$\chi_e = \frac{P}{\epsilon_0 E} = \frac{6.26 \times 10^{-12}}{(8.85 \times 10^{-12})(4 \times 10^3)} = 1.76 \times 10^{-4}$$

Then $\epsilon_r = 1 + \chi_e = \underline{1.000176}$.

5.28. Find the dielectric constant of a material in which the electric flux density is four times the polarization.

First we use $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + (1/4)\mathbf{D}$. Therefore $\mathbf{D} = (4/3)\epsilon_0 \mathbf{E}$, so we identify $\epsilon_r = \underline{4/3}$.

5.29. A coaxial conductor has radii $a = 0.8$ mm and $b = 3$ mm and a polystyrene dielectric for which $\epsilon_r = 2.56$. If $\mathbf{P} = (2/\rho)\mathbf{a}_\rho$ nC/m^2 in the dielectric, find:

a) $\mathbf{D}$ and $\mathbf{E}$ as functions of ρ: Use

$$\mathbf{E} = \frac{\mathbf{P}}{\epsilon_0(\epsilon_r - 1)} = \frac{(2/\rho) \times 10^{-9}\mathbf{a}_\rho}{(8.85 \times 10^{-12})(1.56)} = \underline{\frac{144.9}{\rho}\mathbf{a}_\rho\ \text{V/m}}$$

Then

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \frac{2 \times 10^{-9}\mathbf{a}_\rho}{\rho}\left[\frac{1}{1.56} + 1\right] = \frac{3.28 \times 10^{-9}\mathbf{a}_\rho}{\rho}\ \text{C/m}^2 = \underline{\frac{3.28\mathbf{a}_\rho}{\rho}\ \text{nC/m}^2}$$

b) Find V_{ab} and χ_e: Use

$$V_{ab} = -\int_3^{0.8} \frac{144.9}{\rho}\, d\rho = 144.9 \ln\left(\frac{3}{0.8}\right) = \underline{192\,\text{V}}$$

$\chi_e = \epsilon_r - 1 = \underline{1.56}$, as found in part a.

c) If there are 4×10^{19} molecules per cubic meter in the dielectric, find $\mathbf{p}(\rho)$: Use

$$\mathbf{p} = \frac{\mathbf{P}}{N} = \frac{(2 \times 10^{-9}/\rho)}{4 \times 10^{19}}\,\mathbf{a}_\rho = \underline{\frac{5.0 \times 10^{-29}}{\rho}\,\mathbf{a}_\rho\ \text{C} \cdot \text{m}}$$

5.30. Consider a composite material made up of two species, having number densities N_1 and N_2 molecules/m^3 respectively. The two materials are uniformly mixed, yielding a total number density of $N = N_1 + N_2$. The presence of an electric field $\mathbf{E}$, induces molecular dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$ within the individual species, whether mixed or not. Show that the dielectric constant of the composite material is given by $\epsilon_r = f\epsilon_{r1} + (1-f)\epsilon_{r2}$, where f is the number fraction of species 1 dipoles in the composite, and where ϵ_{r1} and ϵ_{r2} are the dielectric constants that the unmixed species would have if each had number density N.

We may write the total polarization vector as

$$\mathbf{P}_{tot} = N_1\mathbf{p}_1 + N_2\mathbf{p}_2 = N\left(\frac{N_1}{N}\mathbf{p}_1 + \frac{N_2}{N}\mathbf{p}_2\right) = N\left[f\mathbf{p}_1 + (1-f)\mathbf{p}_2\right] = f\mathbf{P}_1 + (1-f)\mathbf{P}_2$$

In terms of the susceptibilities, this becomes $\mathbf{P}_{tot} = \epsilon_0\left[f\chi_{e1} + (1-f)\chi_{e2}\right]\mathbf{E}$, where χ_{e1} and χ_{e2} are evaluated at the composite number density, N. Now

$$\mathbf{D} = \epsilon_r\epsilon_0\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}_{tot} = \epsilon_0\underbrace{\left[1 + f\chi_{e1} + (1-f)\chi_{e2}\right]}_{\epsilon_r}\mathbf{E}$$

Identifying ϵ_r as shown, we may rewrite it by adding and subracting f:

$$\begin{aligned}\epsilon_r &= \left[1 + f - f + f\chi_{e1} + (1-f)\chi_{e2}\right] = \left[f(1+\chi_{e1}) + (1-f)(1+\chi_{e2})\right] \\ &= \left[f\epsilon_{r1} + (1-f)\epsilon_{r2}\right] \quad \text{Q.E.D.}\end{aligned}$$

5.31. The surface $x = 0$ separates two perfect dielectrics. For $x > 0$, let $\epsilon_r = \epsilon_{r1} = 3$, while $\epsilon_{r2} = 5$ where $x < 0$. If $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ V/m, find:

a) E_{N1}: This will be $\mathbf{E}_1 \cdot \mathbf{a}_x = \underline{80\,\text{V/m}}$.

b) $\mathbf{E}_{T1}$. This has components of $\mathbf{E}_1$ *not* normal to the surface, or $\mathbf{E}_{T1} = \underline{-60\mathbf{a}_y - 30\mathbf{a}_z\ \text{V/m}}$.

c) $E_{T1} = \sqrt{(60)^2 + (30)^2} = \underline{67.1\,\text{V/m}}$.

d) $E_1 = \sqrt{(80)^2 + (60)^2 + (30)^2} = \underline{104.4\,\text{V/m}}$.

e) The angle θ_1 between $\mathbf{E}_1$ and a normal to the surface: Use

$$\cos\theta_1 = \frac{\mathbf{E}_1 \cdot \mathbf{a}_x}{E_1} = \frac{80}{104.4} \;\Rightarrow\; \theta_1 = \underline{40.0^\circ}$$

f) $D_{N2} = D_{N1} = \epsilon_{r1}\epsilon_0 E_{N1} = 3(8.85 \times 10^{-12})(80) = \underline{2.12\,\text{nC/m}^2}$.

g) $D_{T2} = \epsilon_{r2}\epsilon_0 E_{T1} = 5(8.85 \times 10^{-12})(67.1) = \underline{2.97\,\text{nC/m}^2}$.

h) $\mathbf{D}_2 = \epsilon_{r1}\epsilon_0 E_{N1}\mathbf{a}_x + \epsilon_{r2}\epsilon_0\mathbf{E}_{T1} = \underline{2.12\mathbf{a}_x - 2.66\mathbf{a}_y - 1.33\mathbf{a}_z\ \text{nC/m}^2}$.

i) $\mathbf{P}_2 = \mathbf{D}_2 - \epsilon_0\mathbf{E}_2 = \mathbf{D}_2\left[1 - (1/\epsilon_{r2})\right] = (4/5)\mathbf{D}_2 = \underline{1.70\mathbf{a}_x - 2.13\mathbf{a}_y - 1.06\mathbf{a}_z\ \text{nC/m}^2}$.

j) the angle θ_2 between $\mathbf{E}_2$ and a normal to the surface: Use

$$\cos\theta_2 = \frac{\mathbf{E}_2 \cdot \mathbf{a}_x}{E_2} = \frac{\mathbf{D}_2 \cdot \mathbf{a}_x}{D_2} = \frac{2.12}{\sqrt{(2.12)^2 = (2.66)^2 + (1.33)^2}} = .581$$

Thus $\theta_2 = \cos^{-1}(.581) = \underline{54.5^\circ}$.

5.32. Two equal but opposite-sign point charges of 3μC are held x meters apart by a spring that provides a repulsive force given by $F_{sp} = 12(0.5 - x)$ N. Without any force of attraction, the spring would be fully-extended to 0.5m.

a) Determine the charge separation: The Coulomb and spring forces must be equal in magnitude. We set up

$$\frac{(3 \times 10^{-6})^2}{4\pi\epsilon_0 x^2} = \frac{9 \times 10^{-12}}{4\pi(8.85 \times 10^{-12})x^2} = 12(0.5 - x)$$

which leads to the cubic equation:

$$x^3 - 0.5x^2 + 6.74 \times 10^{-3}$$

whose solution, found using a calculator, is $x \doteq \underline{0.136}$ m.

b) what is the dipole moment?

Dipole moment magnitude will be $p = qd = (3 \times 10^{-6})(0.136) = \underline{4.08 \times 10^{-7}}$ C-m.

5.33. Two perfect dielectrics have relative permittivities $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 8$. The planar interface between them is the surface $x - y + 2z = 5$. The origin lies in region 1. If $\mathbf{E}_1 = 100\mathbf{a}_x + 200\mathbf{a}_y - 50\mathbf{a}_z$ V/m, find $\mathbf{E}_2$: We need to find the components of $\mathbf{E}_1$ that are normal and tangent to the boundary, and then apply the appropriate boundary conditions. The normal component will be $E_{N1} = \mathbf{E}_1 \cdot \mathbf{n}$. Taking $f = x - y + 2z$, the unit vector that is normal to the surface is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}\left[\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right]$$

This normal will point in the direction of increasing f, which will be away from the origin, or into region 2 (you can visualize a portion of the surface as a triangle whose vertices are on the three coordinate axes at $x = 5$, $y = -5$, and $z = 2.5$). So $E_{N1} = (1/\sqrt{6})[100 - 200 - 100] = -81.7\,$V/m. Since the magnitude is negative, the normal component points into region 1 from the surface. Then

$$\mathbf{E}_{N1} = -81.65\left(\frac{1}{\sqrt{6}}\right)\left[\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right] = -33.33\mathbf{a}_x + 33.33\mathbf{a}_y - 66.67\mathbf{a}_z \text{ V/m}$$

Now, the tangential component will be $\mathbf{E}_{T1} = \mathbf{E}_1 - \mathbf{E}_{N1} = 133.3\mathbf{a}_x + 166.7\mathbf{a}_y + 16.67\mathbf{a}_z$. Our boundary conditions state that $\mathbf{E}_{T2} = \mathbf{E}_{T1}$ and $\mathbf{E}_{N2} = (\epsilon_{r1}/\epsilon_{r2})\mathbf{E}_{N1} = (1/4)\mathbf{E}_{N1}$. Thus

$$\mathbf{E}_2 = \mathbf{E}_{T2} + \mathbf{E}_{N2} = \mathbf{E}_{T1} + \frac{1}{4}\mathbf{E}_{N1} = 133.3\mathbf{a}_x + 166.7\mathbf{a}_y + 16.67\mathbf{a}_z - 8.3\mathbf{a}_x + 8.3\mathbf{a}_y - 16.67\mathbf{a}_z$$
$$= \underline{125\mathbf{a}_x + 175\mathbf{a}_y \text{ V/m}}$$

5.34. Region 1 ($x \geq 0$) is a dielectric with $\epsilon_{r1} = 2$, while region 2 ($x < 0$) has $\epsilon_{r2} = 5$. Let $\mathbf{E}_1 = 20\mathbf{a}_x - 10\mathbf{a}_y + 50\mathbf{a}_z$ V/m.

a) Find $\mathbf{D}_2$: One approach is to first find $\mathbf{E}_2$. This will have the same y and z (tangential) components as $\mathbf{E}_1$, but the normal component, E_x, will differ by the ratio $\epsilon_{r1}/\epsilon_{r2}$; this arises from $D_{x1} = D_{x2}$ (normal component of $\mathbf{D}$ is continuous across a non-charged interface). Therefore $\mathbf{E}_2 = 20(\epsilon_{r1}/\epsilon_{r2})\,\mathbf{a}_x - 10\,\mathbf{a}_y + 50\,\mathbf{a}_z = 8\,\mathbf{a}_x - 10\,\mathbf{a}_y + 50\,\mathbf{a}_z$. The flux density is then

$$\mathbf{D}_2 = \epsilon_{r2}\epsilon_0\mathbf{E}_2 = 40\epsilon_0\,\mathbf{a}_x - 50\epsilon_0\,\mathbf{a}_y + 250\epsilon_0\,\mathbf{a}_z = \underline{0.35\,\mathbf{a}_x - 0.44\,\mathbf{a}_y + 2.21\,\mathbf{a}_z \text{ nC/m}^2}$$

b) Find the energy density in both regions: These will be

$$w_{e1} = \frac{1}{2}\epsilon_{r1}\epsilon_0\mathbf{E}_1 \cdot \mathbf{E}_1 = \frac{1}{2}(2)\epsilon_0\left[(20)^2 + (10)^2 + (50)^2\right] = 3000\epsilon_0 = \underline{26.6 \text{ nJ/m}^3}$$

$$w_{e2} = \frac{1}{2}\epsilon_{r2}\epsilon_0\mathbf{E}_2 \cdot \mathbf{E}_2 = \frac{1}{2}(5)\epsilon_0\left[(8)^2 + (10)^2 + (50)^2\right] = 6660\epsilon_0 = \underline{59.0 \text{ nJ/m}^3}$$

5.35. Let the cylindrical surfaces $\rho = 4$ cm and $\rho = 9$ cm enclose two wedges of perfect dielectrics, $\epsilon_{r1} = 2$ for $0 < \phi < \pi/2$, and $\epsilon_{r2} = 5$ for $\pi/2 < \phi < 2\pi$. If $\mathbf{E}_1 = (2000/\rho)\mathbf{a}_\rho$ V/m, find:

a) $\mathbf{E}_2$: The interfaces between the two media will lie on planes of constant ϕ, to which $\mathbf{E}_1$ is parallel. Thus the field is the same on either side of the boundaries, and so $\underline{\mathbf{E}_2 = \mathbf{E}_1}$.

b) the total electrostatic energy stored in a 1m length of each region: In general we have $w_E = (1/2)\epsilon_r\epsilon_0 E^2$. So in region 1:

$$W_{E1} = \int_0^1 \int_0^{\pi/2} \int_4^9 \frac{1}{2}(2)\epsilon_0 \frac{(2000)^2}{\rho_2}\,\rho\,d\rho\,d\phi\,dz = \frac{\pi}{2}\epsilon_0(2000)^2 \ln\left(\frac{9}{4}\right) = \underline{45.1\,\mu\text{J}}$$

In region 2, we have

$$W_{E2} = \int_0^1 \int_{\pi/2}^{2\pi} \int_4^9 \frac{1}{2}(5)\epsilon_0 \frac{(2000)^2}{\rho_2}\,\rho\,d\rho\,d\phi\,dz = \frac{15\pi}{4}\epsilon_0(2000)^2 \ln\left(\frac{9}{4}\right) = \underline{338\,\mu\text{J}}$$

CHAPTER 6.

6.1. Consider a coaxial capacitor having inner radius a, outer radius b, unit length, and filled with a material with dielectric constant, ϵ_r. Compare this to a parallel-plate capacitor having plate width, w, plate separation d, filled with the same dielectric, and having unit length. Express the ratio b/a in terms of the ratio d/w, such that the two structures will store the same energy for a given applied voltage.

Storing the same energy for a given applied voltage means that the capacitances will be equal. With both structures having unit length and containing the same dielectric (permittivity ϵ), we equate the two capacitances:

$$\frac{2\pi\epsilon}{\ln(b/a)} = \frac{\epsilon w}{d} \Rightarrow \underline{\frac{b}{a} = \exp\left(2\pi\frac{d}{w}\right)}$$

6.2. Let $S = 100\,\text{mm}^2$, $d = 3$ mm, and $\epsilon_r = 12$ for a parallel-plate capacitor.

a) Calculate the capacitance:

$$C = \frac{\epsilon_r\epsilon_0 A}{d} = \frac{12\epsilon_0(100\times10^{-6})}{3\times10^{-3}} = 0.4\epsilon_0 = \underline{3.54\text{ pf}}$$

b) After connecting a 6 V battery across the capacitor, calculate E, D, Q, and the total stored electrostatic energy: First,

$$E = V_0/d = 6/(3\times10^{-3}) = \underline{2000\text{ V/m}}, \quad \text{then} \quad D = \epsilon_r\epsilon_0 E = 2.4\times10^4\epsilon_0 = \underline{0.21\ \mu\text{C/m}^2}$$

The charge in this case is

$$Q = \mathbf{D}\cdot\mathbf{n}|_s = DA = 0.21\times(100\times10^{-6}) = 0.21\times10^{-4}\ \mu\text{C} = \underline{21\text{ pC}}$$

Finally, $W_e = (1/2)QV_0 = 0.5(21)(6) = \underline{63\text{ pJ}}$.

c) With the source still connected, the dielectric is carefully withdrawn from between the plates. With the dielectric gone, re-calculate E, D, Q, and the energy stored in the capacitor.

$$E = V_0/d = 6/(3\times10^{-3}) = \underline{2000\text{ V/m}}, \quad \text{as before.} \quad D = \epsilon_0 E = 2000\epsilon_0 = \underline{17.7\text{ nC/m}^2}$$

The charge is now $Q = DA = 17.7\times(100\times10^{-6})\text{ nC} = \underline{1.8\text{ pC}}$.

Finally, $W_e = (1/2)QV_0 = 0.5(1.8)(6) = \underline{5.4\text{ pJ}}$.

d) If the charge and energy found in (c) are less than that found in (b) (which you should have discovered), what became of the missing charge and energy? In the absence of friction in removing the dielectric, the charge and energy have returned to the battery that gave it.

6.3. Capacitors tend to be more expensive as their capacitance and maximum voltage, V_{max}, increase. The voltage V_{max} is limited by the field strength at which the dielectric breaks down, E_{BD}. Which of these dielectrics will give the largest CV_{max} product for equal plate areas: (a) air: $\epsilon_r = 1$, $E_{BD} = 3$ MV/m; (b) barium titanate: $\epsilon_r = 1200$, $E_{BD} = 3$ MV/m; (c) silicon dioxide: $\epsilon_r = 3.78$, $E_{BD} = 16$ MV/m; (d) polyethylene: $\epsilon_r = 2.26$, $E_{BD} = 4.7$ MV/m? Note that $V_{max} = EBDd$, where d is the plate separation. Also, $C = \epsilon_r \epsilon_0 A/d$, and so $V_{max}C = \epsilon_r \epsilon_0 A E_{BD}$, where A is the plate area. The maximum CV_{max} product is found through the maximum $\epsilon_r E_{BD}$ product. Trying this with the given materials yields the winner, which is barium titanate.

6.4. An air-filled parallel-plate capacitor with plate separation d and plate area A is connected to a battery which applies a voltage V_0 between plates. With the battery left connected, the plates are moved apart to a distance of $10d$. Determine by what factor each of the following quantities changes:

a) V_0: Remains the same, since the battery is left connected.

b) C: As $C = \epsilon_0 A/d$, increasing d by a factor of ten decreases C by a factor of 0.1.

c) E: We require $E \times d = V_0$, where V_0 has not changed. Therefore, E has decreased by a factor of 0.1.

d) D: As $D = \epsilon_0 E$, and since E has decreased by 0.1, D decreases by 0.1.

e) Q: Since $Q = CV_0$, and as C is down by 0.1, Q also decreases by 0.1.

f) ρ_S: As Q is reduced by 0.1, ρ_S reduces by 0.1. This is also consistent with D having been reduced by 0.1.

g) W_e: Use $W_e = 1/2\, CV_0^2$, to observe its reduction by 0.1, since C is reduced by that factor.

6.5. A parallel plate capacitor is filled with a nonuniform dielectric characterized by $\epsilon_r = 2 + 2 \times 10^6 x^2$, where x is the distance from one plate. If $S = 0.02\,\text{m}^2$, and $d = 1$ mm, find C: Start by assuming charge density ρ_s on the top plate. **D** will, as usual, be x-directed, originating at the top plate and terminating on the bottom plate. The key here is that **D** *will be constant over the distance between plates*. This can be understood by considering the x-varying dielectric as constructed of many thin layers, each having constant permittivity. The permittivity changes from layer to layer to approximate the given function of x. The approximation becomes exact as the layer thicknesses approach zero. We know that **D**, which is normal to the layers, will be continuous across each boundary, and so **D** is constant over the plate separation distance, and will be given in magnitude by ρ_s. The electric field magnitude is now

$$E = \frac{D}{\epsilon_0 \epsilon_r} = \frac{\rho_s}{\epsilon_0 (2 + 2 \times 10^6 x^2)}$$

The voltage beween plates is then

$$V_0 = \int_0^{10^{-3}} \frac{\rho_s\, dx}{\epsilon_0 (2 + 2 \times 10^6 x^2)} = \frac{\rho_s}{\epsilon_0} \frac{1}{\sqrt{4 \times 10^6}} \tan^{-1}\left(\frac{x\sqrt{4 \times 10^6}}{2} \right)\Bigg|_0^{10^{-3}} = \frac{\rho_s}{\epsilon_0} \frac{1}{2 \times 10^3} \left(\frac{\pi}{4} \right)$$

Now $Q = \rho_s(.02)$, and so

$$C = \frac{Q}{V_0} = \frac{\rho_s(.02)\epsilon_0 (2 \times 10^3)(4)}{\rho_s \pi} = 4.51 \times 10^{-10}\,\text{F} = \underline{451\,\text{pF}}$$

6.6. Repeat Problem 6.4 assuming the battery is disconnected before the plate separation is increased: The ordering of parameters is changed over that in Problem 6.4, as the progression of thought on the matter is different.

a) Q: Remains the <u>same</u>, since with the battery disconnected, the charge has nowhere to go.

b) ρ_S: As Q is unchanged, ρ_S is also <u>unchanged</u>, since the plate area is the same.

c) D: As $D = \rho_S$, it will remain the <u>same</u> also.

d) E: Since $E = D/\epsilon_0$, and as D is not changed, E will also remain the <u>same</u>.

e) V_0: We require $E \times d = V_0$, where E has not changed. Therefore, V_0 has increased by a factor of <u>10</u>.

f) C: As $C = \epsilon_0 A/d$, increasing d by a factor of ten decreases C by a factor of <u>0.1</u>. The same result occurs because $C = Q/V_0$, where V_0 is increased by 10, whereas Q has not changed.

g) W_e: Use $W_e = 1/2\, CV_0^2 = 1/2\, QV_0$, to observe its increase by a factor of <u>10</u>.

6.7. Let $\epsilon_{r1} = 2.5$ for $0 < y < 1$ mm, $\epsilon_{r2} = 4$ for $1 < y < 3$ mm, and ϵ_{r3} for $3 < y < 5$ mm. Conducting surfaces are present at $y = 0$ and $y = 5$ mm. Calculate the capacitance per square meter of surface area if: a) ϵ_{r3} is that of air; b) $\epsilon_{r3} = \epsilon_{r1}$; c) $\epsilon_{r3} = \epsilon_{r2}$; d) region 3 is silver: The combination will be three capacitors in series, for which

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{d_1}{\epsilon_{r1}\epsilon_0(1)} + \frac{d_2}{\epsilon_{r2}\epsilon_0(1)} + \frac{d_3}{\epsilon_{r3}\epsilon_0(1)} = \frac{10^{-3}}{\epsilon_0}\left[\frac{1}{2.5} + \frac{2}{4} + \frac{2}{\epsilon_{r3}}\right]$$

So that

$$C = \frac{(5 \times 10^{-3})\epsilon_0\epsilon_{r3}}{10 + 4.5\epsilon_{r3}}$$

Evaluating this for the four cases, we find a) $C = \underline{3.05\,\text{nF}}$ for $\epsilon_{r3} = 1$, b) $C = \underline{5.21\,\text{nF}}$ for $\epsilon_{r3} = 2.5$, c) $C = \underline{6.32\,\text{nF}}$ for $\epsilon_{r3} = 4$, and d) $C = \underline{9.83\,\text{nF}}$ if silver (taken as a perfect conductor) forms region 3; this has the effect of removing the term involving ϵ_{r3} from the original formula (first equation line), or equivalently, allowing ϵ_{r3} to approach infinity.

6.8. A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially-dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho^2/a^2)$. Find:

a) **E**: Since ϵ does not vary in the z direction, and since we must always obtain V_0 when integrating **E** between plates, it must follow that $\mathbf{E} = \underline{-V_0/d\,\mathbf{a}_z \text{ V/m}}$.

b) **D**: $\mathbf{D} = \epsilon\mathbf{E} = \underline{-[\epsilon_0(1 + \rho^2/a^2)V_0/d]\ \mathbf{a}_z \text{ C/m}^2}$.

c) Q: Here we find the integral of the surface charge density over the top plate:

$$Q = \int_S \mathbf{D} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^a \frac{-\epsilon_0(1 + \rho^2/a^2)V_0}{d} \mathbf{a}_z \cdot (-\mathbf{a}_z)\, \rho\, d\rho\, d\phi$$

$$= \frac{2\pi\epsilon_0 V_0}{d} \int_0^a (\rho + \rho^3/a^2)\, d\rho = \underline{\frac{3\pi\epsilon_0 a^2}{2d} V_0}$$

d) C: We use $C = Q/V_0$ and our previous result to find $C = \underline{3\epsilon_0(\pi a^2)/(2d) \text{ F}}$.

6.9. Two coaxial conducting cylinders of radius 2 cm and 4 cm have a length of 1m. The region between the cylinders contains a layer of dielectric from $\rho = c$ to $\rho = d$ with $\epsilon_r = 4$. Find the capacitance if

a) $c = 2$ cm, $d = 3$ cm: This is two capacitors in series, and so

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\pi\epsilon_0}\left[\frac{1}{4}\ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right)\right] \Rightarrow C = \underline{143\,\text{pF}}$$

b) $d = 4$ cm, and the volume of the dielectric is the same as in part a: Having equal volumes requires that $3^2 - 2^2 = 4^2 - c^2$, from which $c = 3.32$ cm. Now

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\pi\epsilon_0}\left[\ln\left(\frac{3.32}{2}\right) + \frac{1}{4}\ln\left(\frac{4}{3.32}\right)\right] \Rightarrow C = \underline{101\,\text{pF}}$$

6.10. A coaxial cable has conductor dimensions of $a = 1.0$ mm and $b = 2.7$ mm. The inner conductor is supported by dielectric spacers ($\epsilon_r = 5$) in the form of washers with a hole radius of 1 mm and an outer radius of 2.7 mm, and with a thickness of 3.0 mm. The spacers are located every 2 cm down the cable.

a) By what factor do the spacers increase the capacitance per unit length? The net capacitance can be constructed as a composite quantity, composed of weighted contributions from the air-filled and dielectric-filled regions:

$$C_{net} = \frac{2\pi\epsilon_0}{\ln(b/a)}\, f_1 + \frac{2\pi\epsilon_r\epsilon_0}{\ln(b/a)}\, f_2$$

where $f_1 = (2 - 0.3)/2$ and $f_2 = 0.3/2$ are the filling factors for air and dielectric. Substituting these gives

$$C_{net} = \frac{2\pi\epsilon_0}{\ln(b/a)} \underbrace{\left[\frac{1.7}{2} + 0.15\epsilon_r\right]}_{f}$$

where the bracketed term, f, is the capactance increase factor that we seek. Substituting $\epsilon_r = 5$ gives $\underline{f = 1.6}$.

b) If 100V is maintained across the cable, find **E** at all points:

Method 1: We recall the expression for electric field in a coaxial line from Gauss' Law:

$$\mathbf{E} = \frac{a\rho_s}{\epsilon\rho}\, \mathbf{a}_\rho$$

where ρ_s is the surface charge density on the inner conductor. We also note that electric field will be the same both inside and outside the dielectric rings because the integral of **E** between conductors must always give 100 V. Another way to justify this is through the dielectric boundary condition that requires tangential electric field to be continuous across an interface between two dielectrics. This again means that **E** in our case will be the same inside and outside the rings. We can therefore set up the integral for the voltage:

$$V_0 = -\int_b^a \mathbf{E}\cdot d\mathbf{L} = -\int_b^a \frac{a\rho_s}{\epsilon\rho}\, d\rho = \frac{a\rho_s}{\epsilon}\ln(b/a) = 100$$

Then

$$\frac{a\rho_s}{\epsilon} = \frac{100}{\ln(2.7)} = 101 \;\Rightarrow\; \underline{\mathbf{E} = \frac{101}{\rho}\mathbf{a}_\rho} \text{ V/m}$$

Method 2: Solve Laplace's equation. This was done for the cylindrical geometry in Example 6.3, which gave the potential distribution between conductors, Eq. (35):

$$V(\rho) = V_0 \frac{\ln(b/\rho)}{\ln(b/a)}$$

from which

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\,\mathbf{a}_\rho = \frac{V_0}{\rho\ln(b/a)}\,\mathbf{a}_\rho = \frac{100}{\rho\ln(2.7)}\,\mathbf{a}_\rho = \underline{\frac{101}{\rho}\mathbf{a}_\rho} \text{ V/m}$$

as before.

6.11. Two conducting spherical shells have radii $a = 3$ cm and $b = 6$ cm. The interior is a perfect dielectric for which $\epsilon_r = 8$.

a) Find C: For a spherical capacitor, we know that:

$$C = \frac{4\pi\epsilon_r\epsilon_0}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi(8)\epsilon_0}{\left(\frac{1}{3} - \frac{1}{6}\right)(100)} = 1.92\pi\epsilon_0 = \underline{53.3\,\text{pF}}$$

b) A portion of the dielectric is now removed so that $\epsilon_r = 1.0$, $0 < \phi < \pi/2$, and $\epsilon_r = 8$, $\pi/2 < \phi < 2\pi$. Again, find C: We recognize here that removing that portion leaves us with two capacitors in parallel (whose C's will add). We use the fact that with the dielectric *completely* removed, the capacitance would be $C(\epsilon_r = 1) = 53.3/8 = 6.67\,\text{pF}$. With one-fourth the dielectric removed, the total capacitance will be

$$C = \frac{1}{4}(6.67) + \frac{3}{4}(53.4) = \underline{41.7\,\text{pF}}$$

6.12. a) Determine the capacitance of an isolated conducting sphere of radius a in free space (consider an outer conductor existing at $r \to \infty$). If we assume charge Q on the sphere, the electric field will be

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\mathbf{a}_r \text{ V/m} \quad (a < r < \infty)$$

The potential on the sphere surface will then be

$$V_0 = -\int_{\infty}^{a} \mathbf{E}\cdot d\mathbf{L} = -\int_{\infty}^{a} \frac{Q\,dr}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon_0 a}$$

Capacitance is then

$$C = \frac{Q}{V_0} = \underline{4\pi\epsilon_0 a} \text{ F}$$

b) The sphere is to be covered with a dielectric layer of thickness d and dielectric contant ϵ_r. If $\epsilon_r = 3$, find d in terms of a such that the capacitance is twice that of part a: Let the dielectric radius be b, where $b = d + a$. The potential at the conductor surface is then

$$V_0 = -\int_{\infty}^{b} \frac{Q}{4\pi\epsilon_0 r^2}\,dr - \int_{b}^{a} \frac{Q}{4\pi\epsilon_r\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{b} + \frac{1}{\epsilon_r}\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$

Then, substituting capacitance, $C = Q/V_0$, and solving for b, we find, after a little algebra:

$$b = \frac{a(\epsilon_r - 1)}{a\epsilon_r\,(4\pi\epsilon_0/C) - 1}$$

Then, substitute $\epsilon_r = 3$ and $C = 8\pi\epsilon_0 a$ (twice the part a result) to obtain:

$$b = 4a \;\Rightarrow\; d = b - a = \underline{3a}$$

6.13. With reference to Fig. 6.5, let $b = 6$ m, $h = 15$ m, and the conductor potential be 250 V. Take $\epsilon = \epsilon_0$. Find values for K_1, ρ_L, a, and C: We have

$$K_1 = \left[\frac{h + \sqrt{h^2 + b^2}}{b}\right]^2 = \left[\frac{15 + \sqrt{(15)^2 + (6)^2}}{6}\right]^2 = \underline{23.0}$$

We then have

$$\rho_L = \frac{4\pi\epsilon_0 V_0}{\ln K_1} = \frac{4\pi\epsilon_0(250)}{\ln(23)} = \underline{8.87\,\text{nC/m}}$$

Next, $a = \sqrt{h^2 - b^2} = \sqrt{(15)^2 - (6)^2} = \underline{13.8\,\text{m}}$. Finally,

$$C = \frac{2\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{2\pi\epsilon_0}{\cosh^{-1}(15/6)} = \underline{35.5\,\text{pF}}$$

6.14. Two #16 copper conductors (1.29-mm diameter) are parallel with a separation d between axes. Determine d so that the capacitance between wires in air is 30 pF/m.

We use

$$\frac{C}{L} = 60\text{ pF/m} = \frac{2\pi\epsilon_0}{\cosh^{-1}(h/b)}$$

The above expression evaluates the capacitance of one of the wires suspended over a plane at mid-span, $h = d/2$. Therefore the capacitance of that structure is doubled over that required (from 30 to 60 pF/m). Using this,

$$\frac{h}{b} = \cosh\left(\frac{2\pi\epsilon_0}{C/L}\right) = \cosh\left(\frac{2\pi \times 8.854}{60}\right) = 1.46$$

Therefore, $d = 2h = 2b(1.46) = 2(1.29/2)(1.46) = \underline{1.88\text{ mm}}$.

6.15. A 2 cm diameter conductor is suspended in air with its axis 5 cm from a conducting plane. Let the potential of the cylinder be 100 V and that of the plane be 0 V. Find the surface charge density on the:

a) cylinder at a point nearest the plane: The cylinder will image across the plane, producing an equivalent two-cylinder problem, with the second one at location 5 cm below the plane. We will take the plane as the zy plane, with the cylinder positions at $x = \pm 5$. Now $b = 1$ cm, $h = 5$ cm, and $V_0 = 100$ V. Thus $a = \sqrt{h^2 - b^2} = 4.90$ cm. Then $K_1 = [(h + a)/b]^2 = 98.0$, and $\rho_L = (4\pi\epsilon_0 V_0)/\ln K_1 = 2.43\,\text{nC/m}$. Now

$$\mathbf{D} = \epsilon_0\mathbf{E} = -\frac{\rho_L}{2\pi}\left[\frac{(x+a)\mathbf{a}_x + y\mathbf{a}_y}{(x+a)^2 + y^2} - \frac{(x-a)\mathbf{a}_x + y\mathbf{a}_y}{(x-a)^2 + y^2}\right]$$

and

$$\rho_{s,\,max} = \mathbf{D}\cdot(-\mathbf{a}_x)\Big|_{x=h-b,y=0} = \frac{\rho_L}{2\pi}\left[\frac{h-b+a}{(h-b+a)^2} - \frac{h-b-a}{(h-b-a)^2}\right] = \underline{473\,\text{nC/m}^2}$$

6.15b) plane at a point nearest the cylinder: At $x = y = 0$,

$$\mathbf{D}(0,0) = -\frac{\rho_L}{2\pi}\left[\frac{a\mathbf{a}_x}{a^2} - \frac{-a\mathbf{a}_x}{a^2}\right] = -\frac{\rho_L}{2\pi}\frac{2}{a}\mathbf{a}_x$$

from which

$$\rho_s = \mathbf{D}(0,0)\cdot\mathbf{a}_x = -\frac{\rho_L}{\pi a} = \underline{-15.8\,\mathrm{nC/m^2}}$$

c) Find the capacitance per unit length. This will be $C = \rho_L/V_0 = 2.43$ [nC/m]/100 = <u>24.3 pF/m</u>.

6.16. Consider an arrangement of two isolated conducting surfaces of any shape that form a capacitor. Use the definitions of capacitance (Eq. (2) in this chapter) and resistance (Eq. (14) in Chapter 5) to show that when the region between the conductors is filled with either conductive material (conductivity σ) or with a perfect dielectric (permittivity ϵ), the resulting resistance and capacitance of the structures are related through the simple formula $RC = \epsilon/\sigma$. What basic properties must be true about both the dielectric and the conducting medium for this condition to hold for certain?

Considering the two surfaces, with location a one surface, and location b on the other, the definitions are written as:

$$C = \frac{\oint_s \epsilon\mathbf{E}\cdot d\mathbf{S}}{-\int_b^a \mathbf{E}\cdot d\mathbf{L}} \quad \text{and} \quad R = \frac{-\int_b^a \mathbf{E}\cdot d\mathbf{L}}{\oint_s \sigma\mathbf{E}\cdot d\mathbf{S}}$$

Note that the integration surfaces in the two definitions are (in this case) closed, because each must completely surround one of the two conductors. The surface integrals would thus yield the total charge on – or the total current flowing out of – the conductor inside.

The two formulas are muliplied together to give

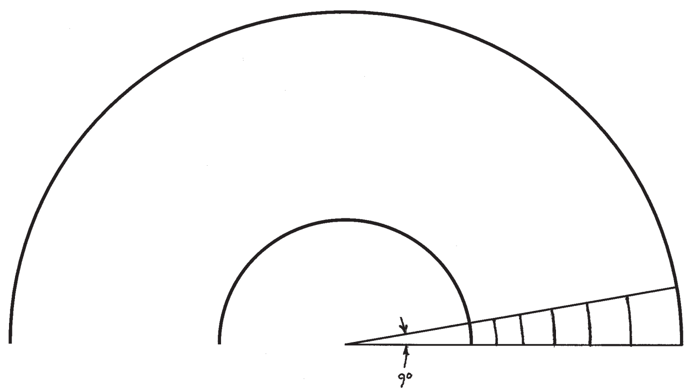
$$RC = \frac{\oint_s \epsilon\mathbf{E}\cdot d\mathbf{S}}{\oint_s \sigma\mathbf{E}\cdot d\mathbf{S}} = \frac{\epsilon}{\sigma}$$

The far-right result is valid provided that ϵ and σ are constant-valued over the integration surfaces. Since in prinicple *any* surface can be chosen over which to integrate, the safest (and correct) conclusion is that ϵ and σ must be constants over the capacitor (or resistor) volume; i.e., the medium must be homogeneous. A little more subtle points are that ϵ and σ generally cannot vary with field orientation (isotropic medium), and cannot vary with field intensity (linear medium), for the simple relation $RC = \epsilon/\sigma$ to work.

6.17 Construct a curvilinear square map for a coaxial capacitor of 3-cm inner radius and 8-cm outer radius. These dimensions are suitable for the drawing.

a) Use your sketch to calculate the capacitance per meter length, assuming $\epsilon_R = 1$: The sketch is shown below. Note that only a 9° sector was drawn, since this would then be duplicated 40 times around the circumference to complete the drawing. The capacitance is thus

$$C \doteq \epsilon_0 \frac{N_Q}{N_V} = \epsilon_0 \frac{40}{6} = \underline{59 \text{ pF/m}}$$



b) Calculate an exact value for the capacitance per unit length: This will be

$$C = \frac{2\pi\epsilon_0}{\ln(8/3)} = \underline{57 \text{ pF/m}}$$

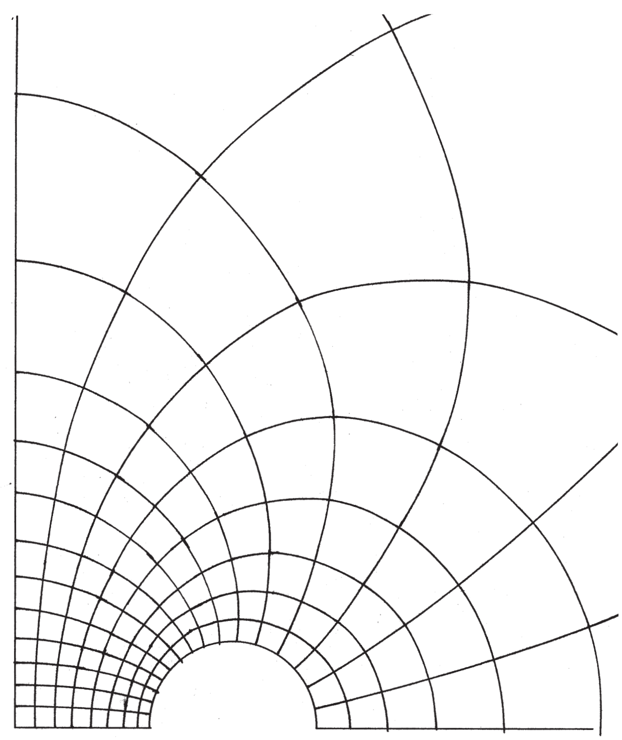
6.18 Construct a curvilinear-square map of the potential field about two parallel circular cylinders, each of 2.5 cm radius, separated by a center-to-center distance of 13cm. These dimensions are suitable for the actual sketch if symmetry is considered. As a check, compute the capacitance per meter both from your sketch and from the exact formula. Assume $\epsilon_R = 1$.

Symmetry allows us to plot the field lines and equipotentials over just the first quadrant, as is done in the sketch below (shown to one-half scale). The capacitance is found from the formula $C = (N_Q/N_V)\epsilon_0$, where N_Q is twice the number of squares around the perimeter of the half-circle and N_V is twice the number of squares between the half-circle and the left vertical plane. The result is

$$C = \frac{N_Q}{N_V}\epsilon_0 = \frac{32}{16}\epsilon_0 = 2\epsilon_0 = \underline{17.7 \text{ pF/m}}$$

We check this result with that using the exact formula:

$$C = \frac{\pi\epsilon_0}{\cosh^{-1}(d/2a)} = \frac{\pi\epsilon_0}{\cosh^{-1}(13/5)} = 1.95\epsilon_0 = \underline{17.3 \text{ pF/m}}$$



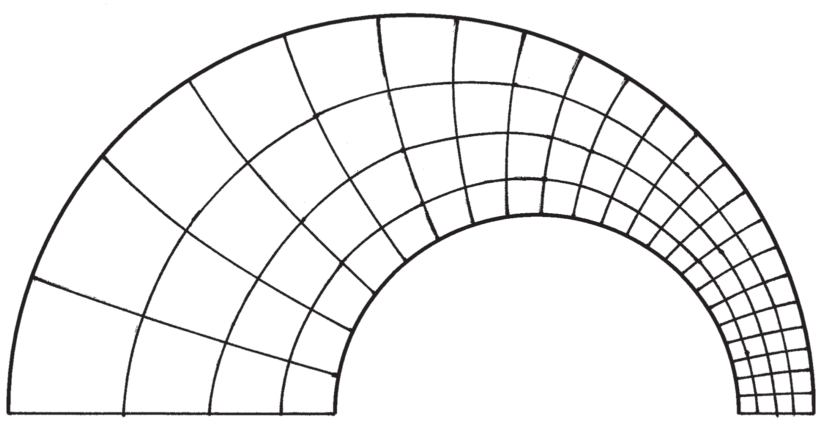
6.19. Construct a curvilinear square map of the potential field between two parallel circular cylinders, one of 4-cm radius inside one of 8-cm radius. The two axes are displaced by 2.5 cm. These dimensions are suitable for the drawing. As a check on the accuracy, compute the capacitance per meter from the sketch and from the exact expression:

$$C = \frac{2\pi\epsilon}{\cosh^{-1}\left[(a^2 + b^2 - D^2)/(2ab)\right]}$$

where a and b are the conductor radii and D is the axis separation.

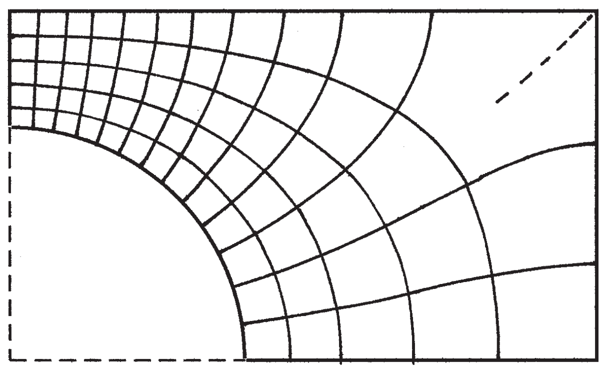
The drawing is shown below. Use of the exact expression above yields a capacitance value of $C = \underline{11.5\epsilon_0 \text{ F/m}}$. Use of the drawing produces:

$$C \doteq \frac{22 \times 2}{4}\epsilon_0 = \underline{11\epsilon_0 \text{ F/m}}$$



6.20. A solid conducting cylinder of 4-cm radius is centered within a rectangular conducting cylinder with a 12-cm by 20-cm cross-section.

a) Make a full-size sketch of one quadrant of this configuration and construct a curvilinear-square map for its interior: The result below could still be improved a little, but is nevertheless sufficient for a reasonable capacitance estimate. Note that the five-sided region in the upper right corner has been partially subdivided (dashed line) in anticipation of how it would look when the next-level subdivision is done (doubling the number of field lines and equipotentials).

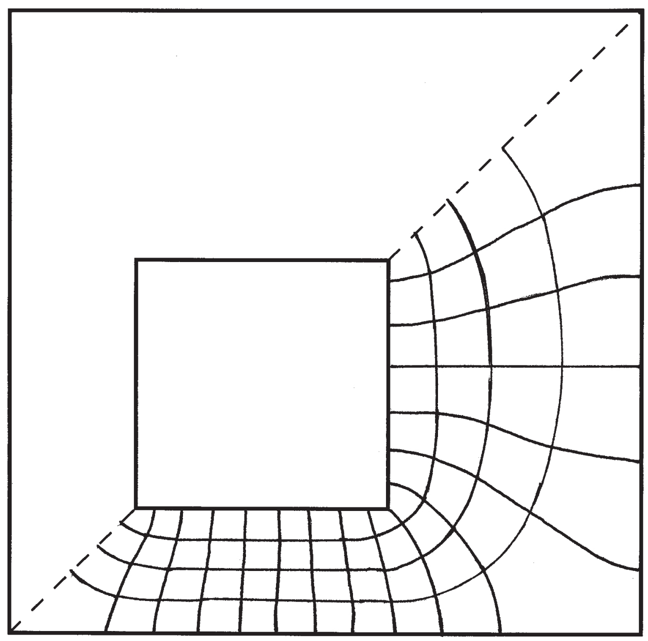


b) Assume $\epsilon = \epsilon_0$ and estimate C per meter length: In this case N_Q is the number of squares around the full perimeter of the circular conductor, or four times the number of squares shown in the drawing. N_V is the number of squares between the circle and the rectangle, or 5. The capacitance is estimated to be

$$C = \frac{N_Q}{N_V}\epsilon_0 = \frac{4 \times 13}{5}\epsilon_0 = 10.4\epsilon_0 \doteq \underline{90\ \text{pF/m}}$$

6.21. The inner conductor of the transmission line shown in Fig. 6.14 has a square cross-section $2a \times 2a$, while the outer square is $5a \times 5a$. The axes are displaced as shown. (a) Construct a good-sized drawing of the transmission line, say with $a = 2.5$ cm, and then prepare a curvilinear-square plot of the electrostatic field between the conductors. (b) Use the map to calculate the capacitance per meter length if $\epsilon = 1.6\epsilon_0$. (c) How would your result to part b change if $a = 0.6$ cm?

a) The plot is shown below. Some improvement is possible, depending on how much time one wishes to spend.



b) From the plot, the capacitance is found to be

$$C \doteq \frac{16 \times 2}{4}(1.6)\epsilon_0 = 12.8\epsilon_0 \doteq \underline{110 \text{ pF/m}}$$

c) If a is changed, the result of part b would not change, since all dimensions retain the same relative scale.

6.22. Two conducting plates, each 3 by 6 cm, and three slabs of dielectric, each 1 by 3 by 6 cm, and having dielectric constants of 1, 2, and 3 are assembled into a capacitor with $d = 3$ cm. Determine the two values of capacitance obtained by the two possible methods of assembling the capactor.

The two possible configurations are 1) all slabs positioned vertically, side-by-side; 2) all slabs positioned horizontally, stacked on top of one another. For vertical positioning, the 1x3 surfaces of each slab are in contact with the plates, and we have three capacitors in parallel. The individual capacitances will thus add to give:

$$C_{vert} = C_1 + C_2 + C_3 = \frac{\epsilon_0(1 \times 3)}{3}(1+2+3) = \underline{6\epsilon_0}$$

With the slabs positioned horizontally, the configuration becomes three capacitors in series, with the total capacitance found through:

$$\frac{1}{C_{horiz}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{3}{\epsilon_0(3)}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{6\epsilon_0} \;\Rightarrow\; C_{horiz} = \underline{\frac{6\epsilon_0}{11}}$$

6.23. A two-wire transmission line consists of two parallel perfectly-conducting cylinders, each having a radius of 0.2 mm, separated by center-to-center distance of 2 mm. The medium surrounding the wires has $\epsilon_r = 3$ and $\sigma = 1.5$ mS/m. A 100-V battery is connected between the wires. Calculate:

a) the magnitude of the charge per meter length on each wire: Use

$$C = \frac{\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{\pi \times 3 \times 8.85 \times 10^{-12}}{\cosh^{-1}(1/0.2)} = 3.64 \times 10^{-9}\ \text{C/m}$$

Then the charge per unit length will be

$$Q = CV_0 = (3.64 \times 10^{-11})(100) = 3.64 \times 10^{-9}\ \text{C/m} = \underline{3.64\,\text{nC/m}}$$

b) the battery current: Use

$$RC = \frac{\epsilon}{\sigma} \quad\Rightarrow\quad R = \frac{3 \times 8.85 \times 10^{-12}}{(1.5 \times 10^{-3})(3.64 \times 10^{-11})} = 486\,\Omega$$

Then

$$I = \frac{V_0}{R} = \frac{100}{486} = 0.206\,\text{A} = \underline{206\,\text{mA}}$$

6.24. A potential field in free space is given in spherical coordinates as:

$$V(r) = \begin{cases} [\rho_0/(6\epsilon_0)]\,[3a^2 - r^2] & (r \le a) \\ (a^3\rho_0)/(3\epsilon_0 r) & (r \ge a) \end{cases}$$

where ρ_0 and a are constants.

a) Use Poisson's equation to find the volume charge density everywhere: Inside $r = a$, we apply Poisson's equation to the potential there:

$$\nabla^2 V_1 = -\frac{\rho_v}{\epsilon_0} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV_1}{dr}\right) = \frac{\rho_0}{6\epsilon_0}\frac{1}{r^2}\frac{d}{dr}\left(r^2(-2r)\right) = -\frac{\rho_0}{\epsilon_0}$$

from which we identify $\rho_v = \underline{\rho_0} \quad (r \le a)$.

Outside $r = a$, we use

$$\nabla^2 V_2 = -\frac{\rho_v}{\epsilon_0} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV_2}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\left(\frac{-a^3\rho_0}{3\epsilon_0 r^2}\right)\right) = 0$$

from which $\rho_v = \underline{0} \quad (r \ge a)$.

b) find the total charge present: We have a constant charge density confined within a spherical volume of radius a. The total charge is therefore $Q = \underline{(4/3)\pi a^3 \rho_0}$ C.

6.25. Let $V = 2xy^2z^3$ and $\epsilon = \epsilon_0$. Given point $P(1, 2, -1)$, find:

a) V at P: Substituting the coordinates into V, find $V_P = \underline{-8\,\text{V}}$.

b) $\mathbf{E}$ at P: We use $\mathbf{E} = -\nabla V = -2y^2z^3\mathbf{a}_x - 4xyz^3\mathbf{a}_y - 6xy^2z^2\mathbf{a}_z$, which, when evaluated at P, becomes $\mathbf{E}_P = \underline{8\mathbf{a}_x + 8\mathbf{a}_y - 24\mathbf{a}_z \text{ V/m}}$

c) ρ_v at P: This is $\rho_v = \nabla \cdot \mathbf{D} = -\epsilon_0 \nabla^2 V = \underline{-4xz(z^2 + 3y^2) \text{ C/m}^3}$

d) the equation of the equipotential surface passing through P: At P, we know $V = -8$ V, so the equation will be $\underline{xy^2z^3 = -4}$.

e) the equation of the streamline passing through P: First,

$$\frac{E_y}{E_x} = \frac{dy}{dx} = \frac{4xyz^3}{2y^2z^3} = \frac{2x}{y}$$

Thus

$$ydy = 2xdx, \text{ and so } \frac{1}{2}y^2 = x^2 + C_1$$

Evaluating at P, we find $C_1 = 1$. Next,

$$\frac{E_z}{E_x} = \frac{dz}{dx} = \frac{6xy^2z^2}{2y^2z^3} = \frac{3x}{z}$$

Thus

$$3xdx = zdz, \text{ and so } \frac{3}{2}x^2 = \frac{1}{2}z^2 + C_2$$

Evaluating at P, we find $C_2 = 1$. The streamline is now specified by the equations: $\underline{y^2 - 2x^2 = 2}$ and $\underline{3x^2 - z^2 = 2}$.

f) Does V satisfy Laplace's equation? No, since the charge density is not zero.

6.26. Given the spherically-symmetric potential field in free space, $V = V_0 e^{-r/a}$, find:

a) ρ_v at $r = a$; Use Poisson's equation, $\nabla^2 V = -\rho_v/\epsilon$, which in this case becomes

$$-\frac{\rho_v}{\epsilon_0} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = \frac{-V_0}{ar^2}\frac{d}{dr}\left(r^2 e^{-r/a}\right) = \frac{-V_0}{ar}\left(2 - \frac{r}{a}\right)e^{-r/a}$$

from which

$$\rho_v(r) = \frac{\epsilon_0 V_0}{ar}\left(2 - \frac{r}{a}\right)e^{-r/a} \quad \Rightarrow \quad \rho_v(a) = \frac{\epsilon_0 V_0}{a^2}e^{-1}\ \text{C/m}^3$$

b) the electric field at $r = a$; this we find through the negative gradient:

$$\mathbf{E}(r) = -\nabla V = -\frac{dV}{dr}\,\mathbf{a}_r = \frac{V_0}{a}e^{-r/a}\,\mathbf{a}_r \quad \Rightarrow \quad \mathbf{E}(a) = \frac{V_0}{a}e^{-1}\mathbf{a}_r\ \text{V/m}$$

c) the total charge: The easiest way is to first find the electric flux density, which from part b is $\mathbf{D} = \epsilon_0 \mathbf{E} = (\epsilon_0 V_0/a)e^{-r/a}\,\mathbf{a}_r$. Then the net outward flux of $\mathbf{D}$ through a sphere of radius r would be

$$\Phi(r) = Q_{encl}(r) = 4\pi r^2 D = 4\pi\epsilon_0 V_0 r^2 e^{-r/a}\ \text{C}$$

As $r \to \infty$, this result approaches zero, so the total charge is therefore $Q_{net} = \underline{0}$.

6.27. Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$: Since $\rho_v = 0$, we know that $\nabla^2 V = 0$, and so

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 16e^{2x} + \frac{d^2 f}{dx^2} - 6 = 0$$

Therefore

$$\frac{d^2 f}{dx^2} = -16e^{2x} + 6 \quad \Rightarrow \quad \frac{df}{dx} = -8e^{2x} + 6x + C_1$$

Now

$$E_x = \frac{\partial V}{\partial x} = 8e^{2x} + \frac{df}{dx}$$

and at the origin, this becomes

$$E_x(0) = 8 + \left.\frac{df}{dx}\right|_{x=0} = 0(\text{as given})$$

Thus $df/dx\,|_{x=0} = -8$, and so it follows that $C_1 = 0$. Integrating again, we find

$$f(x, y) = -4e^{2x} + 3x^2 + C_2$$

which at the origin becomes $f(0, 0) = -4 + C_2$. However, $V(0, 0) = 0 = 4 + f(0, 0)$. So $f(0, 0) = -4$ and $C_2 = 0$. Finally, $f(x, y) = \underline{-4e^{2x} + 3x^2}$, and $V(x, y) = 4e^{2x} - 4e^{2x} + 3x^2 - 3y^2 = \underline{3(x^2 - y^2)}$.

6.28. Show that in a homogeneous medium of conductivity σ, the potential field V satisfies Laplace's equation if any volume charge density present does not vary with time: We begin with the continuity equation, Eq. (5), Chapter 5:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

where $\mathbf{J} = \sigma \mathbf{E}$, and where, in our homogeneous medium, σ is constant with position. Now write

$$\nabla \cdot \mathbf{J} = \nabla \cdot (\sigma \mathbf{E}) = \sigma \nabla \cdot (-\nabla V) = -\sigma \nabla^2 V = -\frac{\partial \rho_v}{\partial t}$$

This becomes

$$\nabla^2 V = 0 \text{ (Laplace's equation)}$$

when ρ_v is time-independent. Q.E.D.

6.29. Given the potential field $V = (A\rho^4 + B\rho^{-4}) \sin 4\phi$:

a) Show that $\nabla^2 V = 0$: In cylindrical coordinates,

$$\begin{aligned}
\nabla^2 V &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} \\
&= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho(4A\rho^3 - 4B\rho^{-5})\right) \sin 4\phi - \frac{1}{\rho^2} 16(A\rho^4 + B\rho^{-4}) \sin 4\phi \\
&= \frac{16}{\rho}(A\rho^3 + B\rho^{-5}) \sin 4\phi - \frac{16}{\rho^2}(A\rho^4 + B\rho^{-4}) \sin 4\phi = 0
\end{aligned}$$

b) Select A and B so that $V = 100$ V and $|\mathbf{E}| = 500$ V/m at $P(\rho = 1, \phi = 22.5^\circ, z = 2)$: First,

$$\begin{aligned}
\mathbf{E} = -\nabla V &= -\frac{\partial V}{\partial \rho}\,\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\,\mathbf{a}_\phi \\
&= -4\left[(A\rho^3 - B\rho^{-5}) \sin 4\phi\, \mathbf{a}_\rho + (A\rho^3 + B\rho^{-5}) \cos 4\phi\, \mathbf{a}_\phi\right]
\end{aligned}$$

and at P, $\mathbf{E}_P = -4(A - B)\,\mathbf{a}_\rho$. Thus $|\mathbf{E}_P| = \pm 4(A - B)$. Also, $V_P = A + B$. Our two equations are:

$$4(A - B) = \pm 500$$

and

$$A + B = 100$$

We thus have two pairs of values for A and B:

$$\underline{A = 112.5,\ B = -12.5} \quad \text{or} \quad \underline{A = -12.5,\ B = 112.5}$$

6.30. A parallel-plate capacitor has plates located at $z = 0$ and $z = d$. The region between plates is filled with a material containing volume charge of uniform density ρ_0 C/m^3, and which has permittivity ϵ. Both plates are held at ground potential.

a) Determine the potential field between plates: We solve Poisson's equation, under the assumption that V varies only with z:

$$\nabla^2 V = \frac{d^2V}{dz^2} = -\frac{\rho_0}{\epsilon} \;\Rightarrow\; V = \frac{-\rho_0 z^2}{2\epsilon} + C_1 z + C_2$$

At $z = 0$, $V = 0$, and so $C_2 = 0$. Then, at $z = d$, $V = 0$ as well, so we find $C_1 = \rho_0 d/2\epsilon$. Finally, $V(z) = \underline{(\rho_0 z/2\epsilon)[d - z]}$.

b) Determine the electric field intensity, **E** between plates: Taking the answer to part *a*, we find **E** through

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\,\mathbf{a}_z = -\frac{d}{dz}\left[\frac{\rho_0 z}{2\epsilon}(d - z)\right] = \frac{\rho_0}{2\epsilon}(2z - d)\,\mathbf{a}_z \text{ V/m}$$

c) Repeat *a* and *b* for the case of the plate at $z = d$ raised to potential V_0, with the $z = 0$ plate grounded: Begin with

$$V(z) = \frac{-\rho_0 z^2}{2\epsilon} + C_1 z + C_2$$

with $C_2 = 0$ as before, since $V(z = 0) = 0$. Then

$$V(z = d) = V_0 = \frac{-\rho_0 d^2}{2\epsilon} + C_1 d \;\Rightarrow\; C_1 = \frac{V_0}{d} + \frac{\rho_0 d}{2\epsilon}$$

So that

$$V(z) = \frac{V_0}{d} z + \frac{\rho_0 z}{2\epsilon}(d - z)$$

We recognize this as the simple superposition of the voltage as found in part *a* and the voltage of a capacitor carrying voltage V_0, but without the charged dielectric. The electric field is now

$$\mathbf{E} = -\frac{dV}{dz}\,\mathbf{a}_z = \frac{-V_0}{d}\,\mathbf{a}_z + \frac{\rho_0}{2\epsilon}(2z - d)\,\mathbf{a}_z \text{ V/m}$$

6.31. Let $V = (\cos 2\phi)/\rho$ in free space.

a) Find the volume charge density at point $A(0.5, 60^\circ, 1)$: Use Poisson's equation:

$$\rho_v = -\epsilon_0 \nabla^2 V = -\epsilon_0 \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}\right)$$
$$= -\epsilon_0 \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{-\cos 2\phi}{\rho}\right) - \frac{4}{\rho^2}\frac{\cos 2\phi}{\rho}\right) = \frac{3\epsilon_0 \cos 2\phi}{\rho^3}$$

So at A we find:

$$\rho_{vA} = \frac{3\epsilon_0 \cos(120^\circ)}{0.5^3} = -12\epsilon_0 = \underline{-106\,\text{pC/m}^3}$$

b) Find the surface charge density on a conductor surface passing through $B(2, 30^\circ, 1)$: First, we find $\mathbf{E}$:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial\rho}\,\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial\phi}\,\mathbf{a}_\phi$$
$$= \frac{\cos 2\phi}{\rho^2}\,\mathbf{a}_\rho + \frac{2\sin 2\phi}{\rho^2}\,\mathbf{a}_\phi$$

At point B the field becomes

$$\mathbf{E}_B = \frac{\cos 60^\circ}{4}\,\mathbf{a}_\rho + \frac{2\sin 60^\circ}{4}\,\mathbf{a}_\phi = 0.125\,\mathbf{a}_\rho + 0.433\,\mathbf{a}_\phi$$

The surface charge density will now be

$$\rho_{sB} = \pm|\mathbf{D}_B| = \pm\epsilon_0|\mathbf{E}_B| = \pm 0.451\epsilon_0 = \underline{\pm 0.399\,\text{pC/m}^2}$$

The charge is positive or negative depending on which side of the surface we are considering. The problem did not provide information necessary to determine this.

6.32. A uniform volume charge has constant density $\rho_v = \rho_0$ C/m^3, and fills the region $r < a$, in which permittivity ϵ as assumed. A conducting spherical shell is located at $r = a$, and is held at ground potential. Find:

a) the potential everywhere: Inside the sphere, we solve Poisson's equation, assuming radial variation only:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = \frac{-\rho_0}{\epsilon} \quad \Rightarrow \quad V(r) = \frac{-\rho_0 r^2}{6\epsilon_0} + \frac{C_1}{r} + C_2$$

We require that V is finite at the orgin (or as $r \to 0$), and so therefore $C_1 = 0$. Next, $V = 0$ at $r = a$, which gives $C_2 = \rho_0 a^2/6\epsilon$. Outside, $r > a$, we know the potential must be zero, since the sphere is grounded. To show this, solve Laplace's equation:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \quad \Rightarrow \quad V(r) = \frac{C_1}{r} + C_2$$

Requiring $V = 0$ at both $r = a$ and at infinity leads to $C_1 = C_2 = 0$. To summarize

$$V(r) = \begin{cases} \frac{\rho_0}{6\epsilon}(a^2 - r^2) & r < a \\ 0 & r > a \end{cases}$$

b) the electric field intensity, $\mathbf{E}$, everywhere: Use

$$\mathbf{E} = -\nabla V = \frac{-dV}{dr}\,\mathbf{a}_r = \frac{\rho_0 r}{3\epsilon}\,\mathbf{a}_r \quad r < a$$

Outside $(r > a)$, the potential is zero, and so $\mathbf{E} = 0$ there as well.

6.33. The functions $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$ both satisfy Laplace's equation in the region $a < \rho < b$, $0 \leq \phi < 2\pi$, $-L < z < L$; each is zero on the surfaces $\rho = b$ for $-L < z < L$; $z = -L$ for $a < \rho < b$; and $z = L$ for $a < \rho < b$; and each is 100 V on the surface $\rho = a$ for $-L < z < L$.

a) In the region specified above, is Laplace's equation satisfied by the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? Yes for the first three, since Laplace's equation is linear. No for $V_1 V_2$.

b) On the boundary surfaces specified, are the potential values given above obtained from the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? At the 100 V surface $(\rho = a)$, No for all. At the 0 V surfaces, yes, except for $V_1 + 3$.

c) Are the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$ identical with V_1? Only V_2 is, since it is given as satisfying all the boundary conditions that V_1 does. Therefore, by the uniqueness theorem, $V_2 = V_1$. The others, not satisfying the boundary conditions, are not the same as V_1.

6.34. Consider the parallel-plate capacitor of Problem 6.30, but this time the charged dielectric exists only between $z = 0$ and $z = b$, where $b < d$. Free space fills the region $b < z < d$. Both plates are at ground potential. No surface charge exists at $z = b$, so that both V and $\mathbf{D}$ are continuous there. By solving Laplace's *and* Poisson's equations, find:

a) $V(z)$ for $0 < z < d$: In Region 1 ($z < b$), we solve Poisson's equation, assuming z variation only:

$$\frac{d^2V_1}{dz^2} = \frac{-\rho_0}{\epsilon} \quad \Rightarrow \quad \frac{dV_1}{dz} = \frac{-\rho_0 z}{\epsilon} + C_1 \quad (z < b)$$

In Region 2 ($z > b$), we solve Laplace's equation, assuming z variation only:

$$\frac{d^2V_2}{dz^2} = 0 \quad \Rightarrow \quad \frac{dV_2}{dz} = C_1' \quad (z > b)$$

At this stage we apply the first boundary condition, which is continuity of $\mathbf{D}$ across the interface at $z = b$. Knowing that the electric field magnitude is given by dV/dz, we write

$$\epsilon \frac{dV_1}{dz}\bigg|_{z=b} = \epsilon_0 \frac{dV_2}{dz}\bigg|_{z=b} \quad \Rightarrow \quad -\rho_0 b + \epsilon C_1 = \epsilon_0 C_1' \quad \Rightarrow \quad C_1' = \frac{-\rho_0 b}{\epsilon_0} + \frac{\epsilon}{\epsilon_0} C_1$$

Substituting the above expression for C_1', and performing a second integration on the Poisson and Laplace equations, we find

$$V_1(z) = -\frac{\rho_0 z^2}{2\epsilon} + C_1 z + C_2 \quad (z < b)$$

and

$$V_2(z) = -\frac{\rho_0 b z}{2\epsilon_0} + \frac{\epsilon}{\epsilon_0} C_1 z + C_2' \quad (z > b)$$

Next, requiring $V_1 = 0$ at $z = 0$ leads to $C_2 = 0$. Then, the requirement that $V_2 = 0$ at $z = d$ leads to

$$0 = -\frac{\rho_0 b d}{\epsilon_0} + \frac{\epsilon}{\epsilon_0} C_1 d + C_2' \quad \Rightarrow \quad C_2' = \frac{\rho_0 b d}{\epsilon_0} - \frac{\epsilon}{\epsilon_0} C_1 d$$

With C_2 and C_2' known, the voltages now become

$$V_1(z) = -\frac{\rho_0 z^2}{2\epsilon} + C_1 z \quad \text{and} \quad V_2(z) = \frac{\rho_0 b}{\epsilon_0}(d - z) - \frac{\epsilon}{\epsilon_0} C_1 (d - z)$$

Finally, to evaluate C_1, we equate the two voltage expressions at $z = b$:

$$V_1|_{z=b} = V_2|_{z=b} \quad \Rightarrow \quad C_1 = \frac{\rho_0 b}{2\epsilon}\left[\frac{b + 2\epsilon_r(d - b)}{b + \epsilon_r(d - b)}\right]$$

where $\epsilon_r = \epsilon/\epsilon_0$. Substituting C_1 as found above into V_1 and V_2 leads to the final expressions for the voltages:

$$V_1(z) = \frac{\rho_0 b z}{2\epsilon}\left[\left(\frac{b + 2\epsilon_r(d - b)}{b + \epsilon_r(d - b)}\right) - \frac{z}{b}\right] \quad (z < b)$$

$$V_2(z) = \frac{\rho_0 b^2}{2\epsilon_0}\left[\frac{d - z}{b + \epsilon_r(d - b)}\right] \quad (z > b)$$

6.34b) the electric field intensity for $0 < z < d$: This involves taking the negative gradient of the final voltage expressions of part a. We find

$$\mathbf{E}_1 = -\frac{dV_1}{dz}\,\mathbf{a}_z = \frac{\rho_0}{\epsilon}\left[z - \frac{b}{2}\left(\frac{b+2\epsilon_r(d-b)}{b+\epsilon_r(d-b)}\right)\right]\mathbf{a}_z \ \ \text{V/m} \quad (z < b)$$

$$\mathbf{E}_2 = -\frac{dV_2}{dz}\,\mathbf{a}_z = \frac{\rho_0 b^2}{2\epsilon_0}\left[\frac{1}{b+\epsilon_r(d-b)}\right]\mathbf{a}_z \ \ \text{V/m} \quad (z > b)$$

6.35. The conducting planes $2x + 3y = 12$ and $2x + 3y = 18$ are at potentials of 100 V and 0, respectively. Let $\epsilon = \epsilon_0$ and find:

a) V at $P(5, 2, 6)$: The planes are parallel, and so we expect variation in potential in the direction normal to them. Using the two boundary condtions, our general potential function can be written:

$$V(x, y) = A(2x + 3y - 12) + 100 = A(2x + 3y - 18) + 0$$

and so $A = -100/6$. We then write

$$V(x, y) = -\frac{100}{6}(2x + 3y - 18) = -\frac{100}{3}x - 50y + 300$$

and $V_P = -\frac{100}{3}(5) - 100 + 300 = \underline{33.33\ \text{V}}$.

b) Find $\mathbf{E}$ at P: Use

$$\mathbf{E} = -\nabla V = \underline{\frac{100}{3}\,\mathbf{a}_x + 50\,\mathbf{a}_y\ \text{V/m}}$$

6.36. The derivation of Laplace's and Poisson's equations assumed constant permittivity, but there are cases of spatially-varying permittivity in which the equations will still apply. Consider the vector identity, $\nabla \cdot (\psi \mathbf{G}) = \mathbf{G} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{G}$, where ψ and $\mathbf{G}$ are scalar and vector functions, respectively. Determine a general rule on the allowed *directions* in which ϵ may vary with respect to the electric field.

In the original derivation of Poisson's equation, we started with $\nabla \cdot \mathbf{D} = \rho_v$, where $\mathbf{D} = \epsilon\mathbf{E}$. Therefore

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon\mathbf{E}) = -\nabla \cdot (\epsilon\nabla V) = -\nabla V \cdot \nabla\epsilon - \epsilon\nabla^2 V = \rho_v$$

We see from this that Poisson's equation, $\nabla^2 V = -\rho_v/\epsilon$, results when $\nabla V \cdot \nabla\epsilon = 0$. In words, ϵ is allowed to vary, provided it does so in directions that are normal to the local electric field.

6.37. Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100V and the outer at 0V, find:

a) the location of the 20V equipotential surface: From Eq. (35) we have

$$V(\rho) = 100\frac{\ln(.012/\rho)}{\ln(.012/.005)} \text{ V}$$

We seek ρ at which $V = 20$ V, and thus we need to solve:

$$20 = 100\frac{\ln(.012/\rho)}{\ln(2.4)} \quad \Rightarrow \quad \rho = \frac{.012}{(2.4)^{0.2}} = \underline{1.01 \text{ cm}}$$

b) $E_{\rho\, max}$: We have

$$E_\rho = -\frac{\partial V}{\partial \rho} = -\frac{dV}{d\rho} = \frac{100}{\rho \ln(2.4)}$$

whose maximum value will occur at the inner cylinder, or at $\rho = .5$ cm:

$$E_{\rho\, max} = \frac{100}{.005\ln(2.4)} = 2.28 \times 10^4 \text{ V/m} = \underline{22.8 \text{ kV/m}}$$

c) ϵ_r if the charge per meter length on the inner cylinder is 20 nC/m: The capacitance per meter length is

$$C = \frac{2\pi\epsilon_0\epsilon_r}{\ln(2.4)} = \frac{Q}{V_0}$$

We solve for ϵ_r:

$$\epsilon_r = \frac{(20 \times 10^{-9})\ln(2.4)}{2\pi\epsilon_0(100)} = \underline{3.15}$$

6.38. Repeat Problem 6.37, but with the dielectric only partially filling the volume, within $0 < \phi < \pi$, and with free space in the remaining volume.

We note that the dielectric changes with ϕ, and not with ρ. Also, since $\mathbf{E}$ is radially-directed and varies only with radius, Laplace's equation for this case is valid (see Problem 6.36) and is the same as that which led to the potential and field in Problem 6.37. Therefore, the solutions to parts *a* and *b* are unchanged from Problem 6.37. Part *c*, however, is different. We write the charge per unit length as the sum of the charges along each half of the center conductor (of radius *a*)

$$Q = \epsilon_r\epsilon_0 E_{\rho,max}(\pi a) + \epsilon_0 E_{\rho,max}(\pi a) = \epsilon_0 E_{\rho,max}(\pi a)(1+\epsilon_r) \text{ C/m}$$

Using the numbers given or found in Problem 6.37, we obtain

$$1 + \epsilon_r = \frac{20 \times 10^{-9} \text{ C/m}}{(8.852 \times 10^{-12})(22.8 \times 10^3 \text{ V/m})(0.5 \times 10^{-2} \text{ m})\pi} = 6.31 \quad \Rightarrow \quad \epsilon_r = \underline{5.31}$$

We may also note that the *average* dielectric constant in this problem, $(\epsilon_r + 1)/2$, is the same as that of the uniform dielectric constant found in Problem 6.37.

6.39. The two conducting planes illustrated in Fig. 6.14 are defined by $0.001 < \rho < 0.120$ m, $0 < z < 0.1$ m, $\phi = 0.179$ and 0.188 rad. The medium surrounding the planes is air. For region 1, $0.179 < \phi < 0.188$, neglect fringing and find:

a) $V(\phi)$: The general solution to Laplace's equation will be $V = C_1\phi + C_2$, and so

$$20 = C_1(.188) + C_2 \quad \text{and} \quad 200 = C_1(.179) + C_2$$

Subtracting one equation from the other, we find

$$-180 = C_1(.188 - .179) \quad \Rightarrow \quad C_1 = -2.00 \times 10^4$$

Then

$$20 = -2.00 \times 10^4(.188) + C_2 \quad \Rightarrow \quad C_2 = 3.78 \times 10^3$$

Finally, $V(\phi) = \underline{(-2.00 \times 10^4)\phi + 3.78 \times 10^3 \text{ V}}$.

b) $\mathbf{E}(\rho)$: Use

$$\mathbf{E}(\rho) = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi} = \underline{\frac{2.00 \times 10^4}{\rho}\mathbf{a}_\phi \text{ V/m}}$$

c) $\mathbf{D}(\rho) = \epsilon_0\mathbf{E}(\rho) = \underline{(2.00 \times 10^4\epsilon_0/\rho)\,\mathbf{a}_\phi \text{ C/m}^2}$.

d) ρ_s on the upper surface of the lower plane: We use

$$\rho_s = \mathbf{D}\cdot\mathbf{n}\Big|_{surface} = \frac{2.00 \times 10^4}{\rho}\mathbf{a}_\phi \cdot \mathbf{a}_\phi = \underline{\frac{2.00 \times 10^4}{\rho} \text{ C/m}^2}$$

e) Q on the upper surface of the lower plane: This will be

$$Q_t = \int_0^{.1}\int_{.001}^{.120} \frac{2.00 \times 10^4\epsilon_0}{\rho}\,d\rho\,dz = 2.00 \times 10^4\epsilon_0(.1)\ln(120) = 8.47 \times 10^{-8} \text{ C} = \underline{84.7 \text{ nC}}$$

f) Repeat *a*) to *c*) for region 2 by letting the location of the upper plane be $\phi = .188 - 2\pi$, and then find ρ_s and Q on the lower surface of the lower plane. Back to the beginning, we use

$$20 = C_1'(.188 - 2\pi) + C_2' \quad \text{and} \quad 200 = C_1'(.179) + C_2'$$

Subtracting one from the other, we find

$$-180 = C_1'(.009 - 2\pi) \quad \Rightarrow \quad C_1' = 28.7$$

Then $200 = 28.7(.179) + C_2' \;\Rightarrow\; C_2' = 194.9$. Thus $V(\phi) = \underline{28.7\phi + 194.9}$ in region 2. Then

$$\underline{\mathbf{E} = -\frac{28.7}{\rho}\mathbf{a}_\phi \text{ V/m}} \quad \text{and} \quad \underline{\mathbf{D} = -\frac{28.7\epsilon_0}{\rho}\mathbf{a}_\phi \text{ C/m}^2}$$

ρ_s on the lower surface of the lower plane will now be

$$\rho_s = -\frac{28.7\epsilon_0}{\rho}\mathbf{a}_\phi \cdot (-\mathbf{a}_\phi) = \underline{\frac{28.7\epsilon_0}{\rho} \text{ C/m}^2}$$

The charge on that surface will then be $Q_b = 28.7\epsilon_0(.1)\ln(120) = \underline{122 \text{ pC}}$.

6.39g) Find the total charge on the lower plane and the capacitance between the planes: Total charge will be $Q_{net} = Q_t + Q_b = 84.7 \text{ nC} + 0.122 \text{ nC} = \underline{84.8 \text{ nC}}$. The capacitance will be

$$C = \frac{Q_{net}}{\Delta V} = \frac{84.8}{200 - 20} = 0.471 \text{ nF} = \underline{471 \text{ pF}}$$

6.40. A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially-dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho^2/a^2)$. Find:

a) $V(z)$: Since ϵ varies in the direction normal to **E**, Laplace's equation applies, and we write

$$\nabla^2 V = \frac{d^2V}{dz^2} = 0 \quad \Rightarrow \quad V(z) = C_1 z + C_2$$

With the given boundary conditions, $C_2 = 0$, and $C_1 = V_0/d$. Therefore $\underline{V(z) = V_0 z/d \text{ V}}$.

b) **E**: This will be $\mathbf{E} = -\nabla V = -dV/dz\, \mathbf{a}_z = \underline{-(V_0/d)\, \mathbf{a}_z \text{ V/m}}$.

c) Q: First we find the electric flux density: $\mathbf{D} = \epsilon \mathbf{E} = -\epsilon_0(1 + \rho^2/a^2)(V_0/d)\, \mathbf{a}_z$ C/m^2. The charge density on the top plate is then $\rho_s = \mathbf{D} \cdot -\mathbf{a}_z = \epsilon_0(1 + \rho^2/a^2)(V_0/d)$ C/m^2. From this we find the charge on the top plate:

$$Q = \int_0^{2\pi} \int_0^a \epsilon_0(1 + \rho^2/a^2)(V_0/d)\, \rho\, d\rho\, d\phi = \frac{3\pi a^2 \epsilon_0 V_0}{2d} \text{ C}$$

d) C. The capacitance is $C = Q/V_0 = \underline{3\pi a^2 \epsilon_0/(2d) \text{ F}}$.

6.41. Concentric conducting spheres are located at $r = 5$ mm and $r = 20$ mm. The region between the spheres is filled with a perfect dielectric. If the inner sphere is at 100 V and the outer sphere at 0 V:

a) Find the location of the 20 V equipotential surface: Solving Laplace's equation gives us

$$V(r) = V_0 \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$$

where $V_0 = 100$, $a = 5$ and $b = 20$. Setting $V(r) = 20$, and solving for r produces $r = \underline{12.5\,\text{mm}}$.

b) Find $E_{r,max}$: Use

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = \frac{V_0\, \mathbf{a}_r}{r^2 \left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$E_{r,max} = E(r = a) = \frac{V_0}{a(1 - (a/b))} = \frac{100}{5(1 - (5/20))} = 26.7 \text{ V/mm} = \underline{26.7 \text{ kV/m}}$$

c) Find ϵ_r if the surface charge density on the inner sphere is $1.0\,\mu$C/m^2: ρ_s will be equal in magnitude to the electric flux density at $r = a$. So $\rho_s = (2.67 \times 10^4 \text{ V/m})\epsilon_r \epsilon_0 = 10^{-6}$ C/m^2. Thus $\epsilon_r = \underline{4.23}$. Note, in the first printing, the given charge density was $100\,\mu$C/m^2, leading to a ridiculous answer of $\epsilon_r = 423$.

6.42. The hemisphere $0 < r < a$, $0 < \theta < \pi/2$, is composed of homogeneous conducting material of conductivity σ. The flat side of the hemisphere rests on a perfectly-conducting plane. Now, the material within the conical region $0 < \theta < \alpha$, $0 < r < a$, is drilled out, and replaced with material that is perfectly-conducting. An air gap is maintained between the $r = 0$ tip of this new material and the plane. What resistance is measured between the two perfect conductors? Neglect fringing fields.

With no fringing fields, we have θ-variation only in the potential. Laplace's equation is therefore:

$$\nabla^2 V = \frac{1}{r^2 \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0$$

This reduces to

$$\frac{dV}{d\theta} = \frac{C_1}{\sin\theta} \quad \Rightarrow \quad V(\theta) = C_1 \ln\tan(\theta/2) + C_2$$

We assume zero potential on the plane (at $\theta = \pi/2$), which means that $C_2 = 0$. On the cone (at $\theta = \alpha$), we assume potential V_0, and so $V_0 = C_1 \ln\tan(\alpha/2)$ $\Rightarrow$ $C_1 = V_0/\ln\tan(\alpha/2)$ The potential function is now

$$V(\theta) = V_0 \frac{\ln\tan(\theta/2)}{\ln\tan(\alpha/2)} \quad \alpha < \theta < \pi/2$$

The electric field is then

$$\mathbf{E} = -\nabla V = -\frac{1}{r}\frac{dV}{d\theta}\,\mathbf{a}_\theta = -\frac{V_0}{r\sin\theta\ln\tan(\alpha/2)}\,\mathbf{a}_\theta \ \ \text{V/m}$$

The total current can now be found by integrating the current density, $\mathbf{J} = \sigma\mathbf{E}$, over any cross-section. Choosing the lower plane at $\theta = \pi/2$, this becomes

$$I = \int_0^{2\pi}\int_0^a -\frac{\sigma V_0}{r\sin(\pi/2)\ln\tan(\alpha/2)}\,\mathbf{a}_\theta \cdot \mathbf{a}_\theta\, r\,dr\,d\phi = -\frac{2\pi a\sigma V_0}{\ln\tan(\alpha/2)} \ \ \text{A}$$

The resistance is finally

$$R = \frac{V_0}{I} = -\frac{\ln\tan(\alpha/2)}{2\pi a\sigma} \ \text{ohms}$$

Note that R is in fact positive (despite the minus sign) since $\ln\tan(\alpha/2)$ is negative when $\alpha < \pi/2$ (which it must be).

6.43. Two coaxial conducting cones have their vertices at the origin and the z axis as their axis. Cone A has the point $A(1,0,2)$ on its surface, while cone B has the point $B(0,3,2)$ on its surface. Let $V_A = 100$ V and $V_B = 20$ V. Find:

a) α for each cone: Have $\alpha_A = \tan^{-1}(1/2) = \underline{26.57^\circ}$ and $\alpha_B = \tan^{-1}(3/2) = \underline{56.31^\circ}$.

b) V at $P(1,1,1)$: The potential function between cones can be written as

$$V(\theta) = C_1 \ln \tan(\theta/2) + C_2$$

Then

$$20 = C_1 \ln \tan(56.31/2) + C_2 \quad \text{and} \quad 100 = C_1 \ln \tan(26.57/2) + C_2$$

Solving these two equations, we find $C_1 = -97.7$ and $C_2 = -41.1$. Now at P, $\theta = \tan^{-1}(\sqrt{2}) = 54.7^\circ$. Thus

$$V_P = -97.7 \ln \tan(54.7/2) - 41.1 = \underline{23.3\,\text{V}}$$

6.44. A potential field in free space is given as $V = 100 \ln \tan(\theta/2) + 50$ V.

a) Find the maximum value of $|\mathbf{E}_\theta|$ on the surface $\theta = 40^\circ$ for $0.1 < r < 0.8$ m, $60^\circ < \phi < 90^\circ$. First

$$\mathbf{E} = -\frac{1}{r}\frac{dV}{d\theta}\mathbf{a}_\theta = -\frac{100}{2r\tan(\theta/2)\cos^2(\theta/2)}\mathbf{a}_\theta = -\frac{100}{2r\sin(\theta/2)\cos(\theta/2)}\mathbf{a}_\theta = -\frac{100}{r\sin\theta}\mathbf{a}_\theta$$

This will maximize at the smallest value of r, or 0.1:

$$\mathbf{E}_{max}(\theta = 40^\circ) = \mathbf{E}(r = 0.1, \theta = 40^\circ) = -\frac{100}{0.1\sin(40)}\mathbf{a}_\theta = \underline{1.56\,\mathbf{a}_\theta\ \text{kV/m}}$$

b) Describe the surface $V = 80$ V: Set $100 \ln \tan\theta/2 + 50 = 80$ and solve for θ: Obtain $\ln \tan\theta/2 = 0.3 \quad \Rightarrow \quad \tan\theta/2 = e^{.3} = 1.35 \quad \Rightarrow \quad \theta = \underline{107^\circ}$ (the cone surface at $\theta = 107$ degrees).

6.45. In free space, let $\rho_v = 200\epsilon_0/r^{2.4}$.

a) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2 E_r \to 0$ when $r \to 0$, and also that $V \to 0$ as $r \to \infty$: With r variation only, we have

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -\frac{\rho_v}{\epsilon} = -200r^{-2.4}$$

or

$$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = -200r^{-.4}$$

Integrate once:

$$\left(r^2 \frac{dV}{dr}\right) = -\frac{200}{.6}r^{.6} + C_1 = -333.3r^{.6} + C_1$$

or

$$\frac{dV}{dr} = -333.3r^{-1.4} + \frac{C_1}{r^2} = \nabla V \text{ (in this case)} = -E_r$$

Our first boundary condition states that $r^2 E_r \to 0$ when $r \to 0$ Therefore $C_1 = 0$. Integrate again to find:

$$V(r) = \frac{333.3}{.4}r^{-.4} + C_2$$

From our second boundary condition, $V \to 0$ as $r \to \infty$, we see that $C_2 = 0$. Finally,

$$V(r) = \underline{833.3r^{-.4} \text{ V}}$$

b) Now find $V(r)$ by using Gauss' Law and a line integral: Gauss' law applied to a spherical surface of radius r gives:

$$4\pi r^2 D_r = 4\pi \int_0^r \frac{200\epsilon_0}{(r')^{2.4}}(r')^2 dr = 800\pi\epsilon_0 \frac{r^{.6}}{.6}$$

Thus

$$E_r = \frac{D_r}{\epsilon_0} = \frac{800\pi\epsilon_0 r^{.6}}{.6(4\pi)\epsilon_0 r^2} = 333.3r^{-1.4} \text{ V/m}$$

Now

$$V(r) = -\int_\infty^r 333.3(r')^{-1.4} dr' = \underline{833.3r^{-.4} \text{ V}}$$

6.46. By appropriate solution of Laplace's *and* Poisson's equations, determine the absolute potential at the center of a sphere of radius a, containing uniform volume charge of density ρ_0. Assume permittivity ϵ_0 everywhere. HINT: What must be true about the potential and the electric field at $r = 0$ and at $r = a$?

With radial dependence only, Poisson's equation (applicable to $r \leq a$) becomes

$$\nabla^2 V_1 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV_1}{dr}\right) = -\frac{\rho_0}{\epsilon_0} \quad \Rightarrow \quad V_1(r) = -\frac{\rho_0 r^2}{6\epsilon_0} + \frac{C_1}{r} + C_2 \quad (r \leq a)$$

For region 2 ($r \geq a$) there is no charge and so Laplace's equation becomes

$$\nabla^2 V_2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV_2}{dr}\right) = 0 \quad \Rightarrow \quad V_2(r) = \frac{C_3}{r} + C_4 \quad (r \geq a)$$

Now, as $r \to \infty$, $V_2 \to 0$, so therefore $C_4 = 0$. Also, as $r \to 0$, V_1 must be finite, so therefore $C_1 = 0$. Then, V must be continuous across the boundary, $r = a$:

$$V_1\Big|_{r=a} = V_2\Big|_{r=a} \quad \Rightarrow \quad -\frac{\rho_0 a^2}{6\epsilon_0} + C_2 = \frac{C_3}{a} \quad \Rightarrow \quad C_2 = \frac{C_3}{a} + \frac{\rho_0 a^2}{6\epsilon_0}$$

So now

$$V_1(r) = \frac{\rho_0}{6\epsilon_0}(a^2 - r^2) + \frac{C_3}{a} \quad \text{and} \quad V_2(r) = \frac{C_3}{r}$$

Finally, since the permittivity is ϵ_0 everywhere, the electric field will be continuous at $r = a$. This is equivalent to the continuity of the voltage derivatives:

$$\frac{dV_1}{dr}\Big|_{r=a} = \frac{dV_2}{dr}\Big|_{r=a} \quad \Rightarrow \quad -\frac{\rho_0 a}{3\epsilon_0} = -\frac{C_3}{a^2} \quad \Rightarrow \quad C_3 = \frac{\rho_0 a^3}{3\epsilon_0}$$

So the potentials in their final forms are

$$V_1(r) = \frac{\rho_0}{6\epsilon_0}(3a^2 - r^2) \quad \text{and} \quad V_2(r) = \frac{\rho_0 a^3}{3\epsilon_0 r}$$

The requested absolute potential at the origin is now $V_1(r = 0) = \underline{\rho_0 a^2/(2\epsilon_0)\ \text{V}}$.

CHAPTER 7

7.1a. Find $\mathbf{H}$ in cartesian components at $P(2,3,4)$ if there is a current filament on the z axis carrying 8 mA in the $\mathbf{a}_z$ direction:
Applying the Biot-Savart Law, we obtain

$$\mathbf{H}_a = \int_{-\infty}^{\infty} \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [2\mathbf{a}_x + 3\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2 - 8z + 29)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[2\mathbf{a}_y - 3\mathbf{a}_x]}{4\pi(z^2 - 8z + 29)^{3/2}}$$

Using integral tables, this evaluates as

$$\mathbf{H}_a = \frac{I}{4\pi}\left[\frac{2(2z-8)(2\mathbf{a}_y - 3\mathbf{a}_x)}{52(z^2 - 8z + 29)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{26\pi}(2\mathbf{a}_y - 3\mathbf{a}_x)$$

Then with $I = 8$ mA, we finally obtain $\mathbf{H}_a = \underline{-294\mathbf{a}_x + 196\mathbf{a}_y\ \mu\text{A/m}}$

b. Repeat if the filament is located at $x = -1$, $y = 2$: In this case the Biot-Savart integral becomes

$$\mathbf{H}_b = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [(2+1)\mathbf{a}_x + (3-2)\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2 - 8z + 26)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[3\mathbf{a}_y - \mathbf{a}_x]}{4\pi(z^2 - 8z + 26)^{3/2}}$$

Evaluating as before, we obtain with $I = 8$ mA:

$$\mathbf{H}_b = \frac{I}{4\pi}\left[\frac{2(2z-8)(3\mathbf{a}_y - \mathbf{a}_x)}{40(z^2 - 8z + 26)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{20\pi}(3\mathbf{a}_y - \mathbf{a}_x) = \underline{-127\mathbf{a}_x + 382\mathbf{a}_y\ \mu\text{A/m}}$$

c. Find $\mathbf{H}$ if both filaments are present: This will be just the sum of the results of parts *a* and *b*, or

$$\mathbf{H}_T = \mathbf{H}_a + \mathbf{H}_b = \underline{-421\mathbf{a}_x + 578\mathbf{a}_y\ \mu\text{A/m}}$$

This problem can also be done (somewhat more simply) by using the known result for $\mathbf{H}$ from an infinitely-long wire in cylindrical components, and transforming to cartesian components. The Biot-Savart method was used here for the sake of illustration.

7.2. A filamentary conductor is formed into an equilateral triangle with sides of length ℓ carrying current I. Find the magnetic field intensity at the center of the triangle.

I will work this one from scratch, using the Biot-Savart law. Consider one side of the triangle, oriented along the z axis, with its end points at $z = \pm\ell/2$. Then consider a point, x_0, on the x axis, which would correspond to the center of the triangle, and at which we want to find $\mathbf{H}$ associated with the wire segment. We thus have $Id\mathbf{L} = Idz\,\mathbf{a}_z$, $R = \sqrt{z^2 + x_0^2}$, and $\mathbf{a}_R = [x_0\,\mathbf{a}_x - z\,\mathbf{a}_z]/R$. The differential magnetic field at x_0 is now

$$d\mathbf{H} = \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{Idz\,\mathbf{a}_z \times (x_0\mathbf{a}_x - z\,\mathbf{a}_z)}{4\pi(x_0^2 + z^2)^{3/2}} = \frac{I\,dz\,x_0\,\mathbf{a}_y}{4\pi(x_0^2 + z^2)^{3/2}}$$

where $\mathbf{a}_y$ would be normal to the plane of the triangle. The magnetic field at x_0 is then

$$\mathbf{H} = \int_{-\ell/2}^{\ell/2} \frac{I\,dz\,x_0\,\mathbf{a}_y}{4\pi(x_0^2 + z^2)^{3/2}} = \frac{I\,z\,\mathbf{a}_y}{4\pi x_0\sqrt{x_0^2 + z^2}}\Bigg|_{-\ell/2}^{\ell/2} = \frac{I\ell\,\mathbf{a}_y}{2\pi x_0\sqrt{\ell^2 + 4x_0^2}}$$

7.2. (continued). Now, x_0 lies at the center of the equilateral triangle, and from the geometry of the triangle, we find that $x_0 = (\ell/2)\tan(30°) = \ell/(2\sqrt{3})$. Substituting this result into the just-found expression for $\mathbf{H}$ leads to $\mathbf{H} = 3I/(2\pi\ell)\,\mathbf{a}_y$. The contributions from the other two sides of the triangle effectively multiply the above result by three. The final answer is therefore $\underline{\mathbf{H}_{net} = 9I/(2\pi\ell)\,\mathbf{a}_y \text{ A/m}}$. It is also possible to work this problem (somewhat more easily) by using Eq. (9), applied to the triangle geometry.

7.3. Two semi-infinite filaments on the z axis lie in the regions $-\infty < z < -a$ (note typographical error in problem statement) and $a < z < \infty$. Each carries a current I in the $\mathbf{a}_z$ direction.

a) Calculate $\mathbf{H}$ as a function of ρ and ϕ at $z = 0$: One way to do this is to use the field from an infinite line and subtract from it that portion of the field that would arise from the current segment at $-a < z < a$, found from the Biot-Savart law. Thus,

$$\mathbf{H} = \frac{I}{2\pi\rho}\,\mathbf{a}_\phi - \int_{-a}^{a} \frac{I\,dz\,\mathbf{a}_z \times [\rho\,\mathbf{a}_\rho - z\,\mathbf{a}_z]}{4\pi[\rho^2 + z^2]^{3/2}}$$

The integral part simplifies and is evaluated:

$$\int_{-a}^{a} \frac{I\,dz\,\rho\,\mathbf{a}_\phi}{4\pi[\rho^2 + z^2]^{3/2}} = \frac{I\rho}{4\pi}\,\mathbf{a}_\phi \left.\frac{z}{\rho^2\sqrt{\rho^2 + z^2}}\right|_{-a}^{a} = \frac{Ia}{2\pi\rho\sqrt{\rho^2 + a^2}}\,\mathbf{a}_\phi$$

Finally,

$$\mathbf{H} = \frac{I}{2\pi\rho}\left[1 - \frac{a}{\sqrt{\rho^2 + a^2}}\right]\mathbf{a}_\phi \text{ A/m}$$

b) What value of a will cause the magnitude of $\mathbf{H}$ at $\rho = 1$, $z = 0$, to be one-half the value obtained for an infinite filament? We require

$$\left[1 - \frac{a}{\sqrt{\rho^2 + a^2}}\right]_{\rho=1} = \frac{1}{2} \quad \Rightarrow \quad \frac{a}{\sqrt{1 + a^2}} = \frac{1}{2} \quad \Rightarrow \quad a = \underline{1/\sqrt{3}}$$

7.4. Two circular current loops are centered on the z axis at $z = \pm h$. Each loop has radius a and carries current I in the $\mathbf{a}_\phi$ direction.

a) Find $\mathbf{H}$ on the z axis over the range $-h < z < h$: As a first step, we find the magnetic field on the z axis arising from a current loop of radius a, centered at the origin in the plane $z = 0$. It carries a current I in the $\mathbf{a}_\phi$ direction. Using the Biot-Savart law, we have $Id\mathbf{L} = Iad\phi\,\mathbf{a}_\phi$, $R = \sqrt{a^2 + z^2}$, and $\mathbf{a}_R = (z\mathbf{a}_z - a\mathbf{a}_\rho)/\sqrt{a^2 + z^2}$. The field on the z axis is then

$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,\mathbf{a}_\phi \times (z\mathbf{a}_z - a\mathbf{a}_\rho)}{4\pi(a^2 + z^2)^{3/2}} = \int_0^{2\pi} \frac{Ia^2 d\phi\,\mathbf{a}_z}{4\pi(a^2 + z^2)^{3/2}} = \frac{a^2 I}{2(a^2 + z^2)^{3/2}}\,\mathbf{a}_z \text{ A/m}$$

In obtaining this result, the term involving $\mathbf{a}_\phi \times z\mathbf{a}_z = z\mathbf{a}_\rho$ has integrated to zero, when taken over the range $0 < \phi < 2\pi$. Substitute $\mathbf{a}_\rho = \mathbf{a}_x \cos\phi + \mathbf{a}_y \sin\phi$ to show this.

We now have two loops, displaced from the x-y plane to $z = \pm h$. The field is now the superposition of the two loop fields, which we can construct using displaced versions of the $\mathbf{H}$ field we just found:

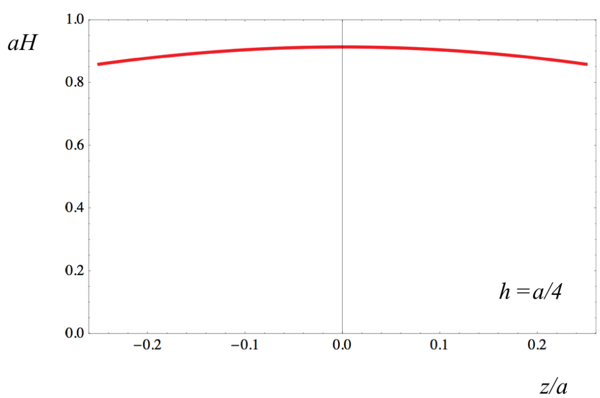
$$\mathbf{H} = \frac{a^2 I}{2}\left[\frac{1}{[(z-h)^2 + a^2]^{3/2}} + \frac{1}{[(z+h)^2 + a^2]^{3/2}}\right]\mathbf{a}_z \text{ A/m}$$

7.4 (continued) We can rewrite this in terms of normalized distances, z/a, and h/a:

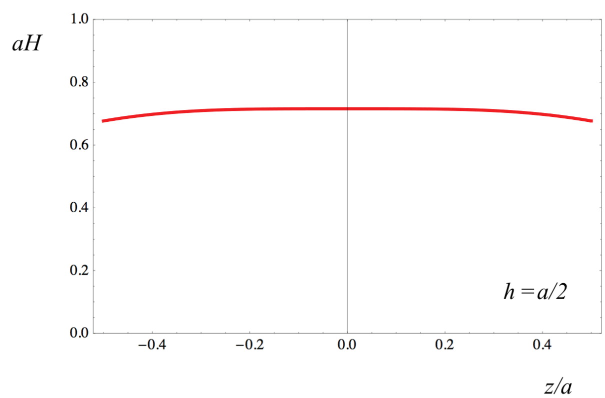
$$a\mathbf{H} = \frac{I}{2}\left[\left(\left(\frac{z}{a}-\frac{h}{a}\right)^2+1\right)^{-3/2}+\left(\left(\frac{z}{a}+\frac{h}{a}\right)^2+1\right)^{-3/2}\right]\mathbf{a}_z \ \text{A}$$

Take $I = 1$ A and plot $|\mathbf{H}|$ as a function of z/a if:

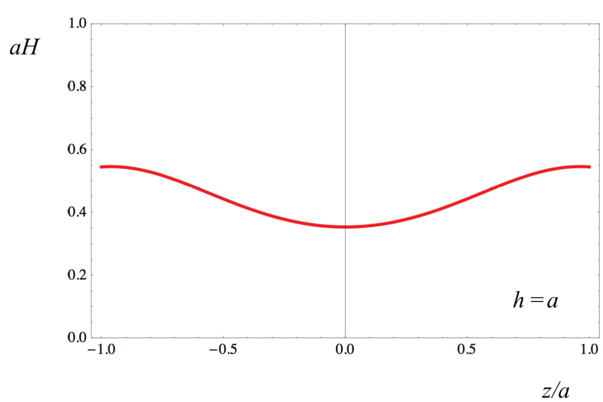
b) $h = a/4$,



c) $h = a/2$:



d) $h = a$.



Which choice for h gives the most uniform field? From the results, $h = a/2$ is evidently the best. This is the Helmholtz coil configuration – in which the spacing is equal to the coil radius.

7.5. The parallel filamentary conductors shown in Fig. 8.21 lie in free space. Plot $|\mathbf{H}|$ versus y, $-4 < y < 4$, along the line $x = 0$, $z = 2$: We need an expression for $\mathbf{H}$ in cartesian coordinates. We can start with the known $\mathbf{H}$ in cylindrical for an infinite filament along the z axis: $\mathbf{H} = I/(2\pi\rho)\,\mathbf{a}_\phi$, which we transform to cartesian to obtain:

$$\mathbf{H} = \frac{-Iy}{2\pi(x^2+y^2)}\,\mathbf{a}_x + \frac{Ix}{2\pi(x^2+y^2)}\,\mathbf{a}_y$$

If we now rotate the filament so that it lies along the x axis, with current flowing in positive x, we obtain the field from the above expression by replacing x with y and y with z:

$$\mathbf{H} = \frac{-Iz}{2\pi(y^2+z^2)}\,\mathbf{a}_y + \frac{Iy}{2\pi(y^2+z^2)}\,\mathbf{a}_z$$

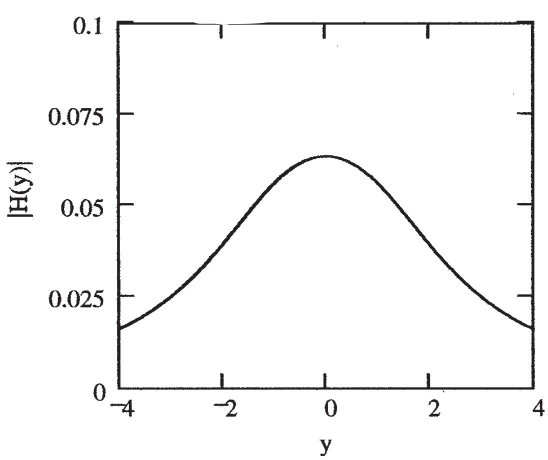
Now, with two filaments, displaced from the x axis to lie at $y = \pm 1$, and with the current directions as shown in the figure, we use the previous expression to write

$$\mathbf{H} = \left[\frac{Iz}{2\pi[(y+1)^2+z^2]} - \frac{Iz}{2\pi[(y-1)^2+z^2]}\right]\mathbf{a}_y + \left[\frac{I(y-1)}{2\pi[(y-1)^2+z^2]} - \frac{I(y+1)}{2\pi[(y+1)^2+z^2]}\right]\mathbf{a}_z$$

We now evaluate this at $z = 2$, and find the magnitude ($\sqrt{\mathbf{H}\cdot\mathbf{H}}$), resulting in

$$|\mathbf{H}| = \frac{I}{2\pi}\left[\left(\frac{2}{y^2+2y+5} - \frac{2}{y^2-2y+5}\right)^2 + \left(\frac{(y-1)}{y^2-2y+5} - \frac{(y+1)}{y^2+2y+5}\right)^2\right]^{1/2}$$

This function is plotted below



7.6. A disk of radius a lies in the xy plane, with the z axis through its center. Surface charge of uniform density ρ_s lies on the disk, which rotates about the z axis at angular velocity Ω rad/s. Find $\mathbf{H}$ at any point on the z axis.

We use the Biot-Savart law in the form of Eq. (6), with the following parameters: $\mathbf{K} = \rho_s\mathbf{v} = \rho_s\rho\Omega\,\mathbf{a}_\phi$, $R = \sqrt{z^2+\rho^2}$, and $\mathbf{a}_R = (z\,\mathbf{a}_z - \rho\,\mathbf{a}_\rho)/R$. The differential field at point z is

$$d\mathbf{H} = \frac{\mathbf{K}da \times \mathbf{a}_R}{4\pi R^2} = \frac{\rho_s\rho\,\Omega\,\mathbf{a}_\phi \times (z\,\mathbf{a}_z - \rho\,\mathbf{a}_\rho)}{4\pi(z^2+\rho^2)^{3/2}}\,\rho\,d\rho\,d\phi = \frac{\rho_s\rho\,\Omega\,(z\,\mathbf{a}_\rho + \rho\,\mathbf{a}_z)}{4\pi(z^2+\rho^2)^{3/2}}\,\rho\,d\rho\,d\phi$$

7.6. (continued). On integrating the above over ϕ around a complete circle, the $\mathbf{a}_\rho$ components cancel from symmetry, leaving us with

$$\mathbf{H}(z) = \int_0^{2\pi}\int_0^a \frac{\rho_s \rho\,\Omega\,\rho\,\mathbf{a}_z}{4\pi(z^2+\rho^2)^{3/2}}\,\rho\,d\rho\,d\phi = \int_0^a \frac{\rho_s\,\Omega\,\rho^3\,\mathbf{a}_z}{2(z^2+\rho^2)^{3/2}}\,d\rho$$

$$= \frac{\rho_s\Omega}{2}\left[\sqrt{z^2+\rho^2}+\frac{z^2}{\sqrt{z^2+\rho^2}}\right]_0^a \mathbf{a}_z = \frac{\rho_s\Omega}{2z}\left[\frac{a^2+2z^2\left(1-\sqrt{1+a^2/z^2}\right)}{\sqrt{1+a^2/z^2}}\right]\mathbf{a}_z \text{ A/m}$$

7.7. A filamentary conductor carrying current I in the $\mathbf{a}_z$ direction extends along the entire negative z axis. At $z=0$ it connects to a copper sheet that fills the $x>0$, $y>0$ quadrant of the xy plane.

a) Set up the Biot-Savart law and find $\mathbf{H}$ everywhere on the z axis (Hint: express $\mathbf{a}_\phi$ in terms of $\mathbf{a}_x$ and $\mathbf{a}_y$ and angle ϕ in the integral): First, the contribution to the field at z from the current on the negative z axis will be zero, because the cross product, $Id\mathbf{L}\times\mathbf{a}_R = 0$ for all current elements on the z axis. This leaves the contribution of the current sheet in the first quadrant. On exiting the origin, current fans out over the first quadrant in the $\mathbf{a}_\rho$ direction and is uniform at a given radius. The surface current density can therefore be written as $\mathbf{K}(\rho) = 2I/(\pi\rho)\,\mathbf{a}_\rho$ A/m^2 over the region $(0<\phi<\pi/2)$. The Biot-Savart law applicable to surface current is written as

$$\mathbf{H} = \int_s \frac{\mathbf{K}\times\mathbf{a}_R}{4\pi R^2}\,dA$$

where $R=\sqrt{z^2+\rho^2}$ and $\mathbf{a}_R = (z\mathbf{a}_z - \rho\mathbf{a}_\rho)/\sqrt{z^2+\rho^2}$ Substituting these and integrating over the first quadrant yields the setup:

$$\mathbf{H} = \int_0^{\pi/2}\int_0^\infty \frac{2I\,\mathbf{a}_\rho\times(z\mathbf{a}_z-\rho\mathbf{a}_\rho)}{4\pi^2\rho(z^2+\rho^2)^{3/2}}\,\rho\,d\rho\,d\phi = \int_0^{\pi/2}\int_0^\infty \frac{-Iz\,\mathbf{a}_\phi}{2\pi^2(z^2+\rho^2)^{3/2}}\,d\rho\,d\phi$$

Following the hint, we now substitute $\mathbf{a}_\phi = \mathbf{a}_y\cos\phi - \mathbf{a}_x\sin\phi$, and write:

$$\mathbf{H} = \frac{-Iz}{2\pi^2}\int_0^{\pi/2}\int_0^\infty \frac{(\mathbf{a}_y\cos\phi-\mathbf{a}_x\sin\phi)}{(z^2+\rho^2)^{3/2}}\,d\rho\,d\phi = \frac{Iz}{2\pi^2}(\mathbf{a}_x-\mathbf{a}_y)\int_0^\infty \frac{d\rho}{(z^2+\rho^2)^{3/2}}$$

$$= \frac{Iz}{2\pi^2}(\mathbf{a}_x-\mathbf{a}_y)\left.\frac{\rho}{z^2\sqrt{z^2+\rho^2}}\right|_0^\infty = \underline{\frac{I}{2\pi^2 z}(\mathbf{a}_x-\mathbf{a}_y)} \text{ A/m}$$

b) repeat part a, but with the copper sheet occupying the *entire* xy plane. In this case, the ϕ limits are $(0<\phi<2\pi)$. The $\cos\phi$ and $\sin\phi$ terms would then integrate to zero, so the answer is just that: $\mathbf{H} = \underline{0}$.

7.8. For the finite-length current element on the z axis, as shown in Fig. 8.5, use the Biot-Savart law to derive Eq. (9) of Sec. 8.1: The Biot-Savart law reads:

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_{\rho\tan\alpha_1}^{\rho\tan\alpha_2} \frac{Idz\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi(\rho^2+z^2)^{3/2}} = \int_{\rho\tan\alpha_1}^{\rho\tan\alpha_2} \frac{I\rho\mathbf{a}_\phi\,dz}{4\pi(\rho^2+z^2)^{3/2}}$$

The integral is evaluated (using tables) and gives the desired result:

$$\mathbf{H} = \frac{Iz\mathbf{a}_\phi}{4\pi\rho\sqrt{\rho^2+z^2}}\Big|_{\rho\tan\alpha_1}^{\rho\tan\alpha_2} = \frac{I}{4\pi\rho}\left[\frac{\tan\alpha_2}{\sqrt{1+\tan^2\alpha_2}} - \frac{\tan\alpha_1}{\sqrt{1+\tan^2\alpha_1}}\right]\mathbf{a}_\phi$$
$$= \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$$

7.9. A current sheet $\mathbf{K} = 8\mathbf{a}_x$ A/m flows in the region $-2 < y < 2$ in the plane $z = 0$. Calculate H at $P(0,0,3)$: Using the Biot-Savart law, we write

$$\mathbf{H}_P = \int\int \frac{\mathbf{K}\times\mathbf{a}_R\,dx\,dy}{4\pi R^2} = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8\mathbf{a}_x \times (-x\mathbf{a}_x - y\mathbf{a}_y + 3\mathbf{a}_z)}{4\pi(x^2+y^2+9)^{3/2}}\,dx\,dy$$

Taking the cross product gives:

$$\mathbf{H}_P = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8(-y\mathbf{a}_z - 3\mathbf{a}_y)\,dx\,dy}{4\pi(x^2+y^2+9)^{3/2}}$$

We note that the z component is anti-symmetric in y about the origin (odd parity). Since the limits are symmetric, the integral of the z component over y is zero. We are left with

$$\mathbf{H}_P = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{-24\,\mathbf{a}_y\,dx\,dy}{4\pi(x^2+y^2+9)^{3/2}} = -\frac{6}{\pi}\mathbf{a}_y \int_{-2}^{2} \frac{x}{(y^2+9)\sqrt{x^2+y^2+9}}\Big|_{-\infty}^{\infty} dy$$
$$= -\frac{6}{\pi}\mathbf{a}_y\int_{-2}^{2}\frac{2}{y^2+9}\,dy = -\frac{12}{\pi}\mathbf{a}_y\,\frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right)\Big|_{-2}^{2} = -\frac{4}{\pi}(2)(0.59)\,\mathbf{a}_y = \underline{-1.50\,\mathbf{a}_y \text{ A/m}}$$

7.10. A hollow spherical conducting shell of radius a has filamentary connections made at the top ($r = a$, $\theta = 0$) and bottom ($r = a$, $\theta = \pi$). A direct current I flows down the upper filament, down the spherical surface, and out the lower filament. Find **H** in spherical coordinates (a) inside and (b) outside the sphere.

Applying Ampere's circuital law, we use a circular contour, centered on the z axis, and find that within the sphere, no current is enclosed, and so $\underline{\mathbf{H} = 0 \text{ when } r < a}$. The same contour drawn outside the sphere at any z position will always enclose I amps, flowing in the negative z direction, and so

$$\mathbf{H} = -\frac{I}{2\pi\rho}\,\mathbf{a}_\phi = -\frac{I}{2\pi r\sin\theta}\,\mathbf{a}_\phi \text{ A/m} \quad (r > a)$$

7.11. An infinite filament on the z axis carries 20π mA in the $\mathbf{a}_z$ direction. Three uniform cylindrical current sheets are also present: 400 mA/m at $\rho = 1$ cm, -250 mA/m at $\rho = 2$ cm, and -300 mA/m at $\rho = 3$ cm. Calculate H_ϕ at $\rho = 0.5$, 1.5, 2.5, and 3.5 cm: We find H_ϕ at each of the required radii by applying Ampere's circuital law to circular paths of those radii; the paths are centered on the z axis. So, at $\rho_1 = 0.5$ cm:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho_1 H_{\phi 1} = I_{encl} = 20\pi \times 10^{-3} \text{ A}$$

Thus

$$H_{\phi 1} = \frac{10 \times 10^{-3}}{\rho_1} = \frac{10 \times 10^{-3}}{0.5 \times 10^{-2}} = \underline{2.0 \text{ A/m}}$$

At $\rho = \rho_2 = 1.5$ cm, we enclose the first of the current cylinders at $\rho = 1$ cm. Ampere's law becomes:

$$2\pi\rho_2 H_{\phi 2} = 20\pi + 2\pi(10^{-2})(400) \text{ mA} \quad \Rightarrow \quad H_{\phi 2} = \frac{10 + 4.00}{1.5 \times 10^{-2}} = \underline{933 \text{ mA/m}}$$

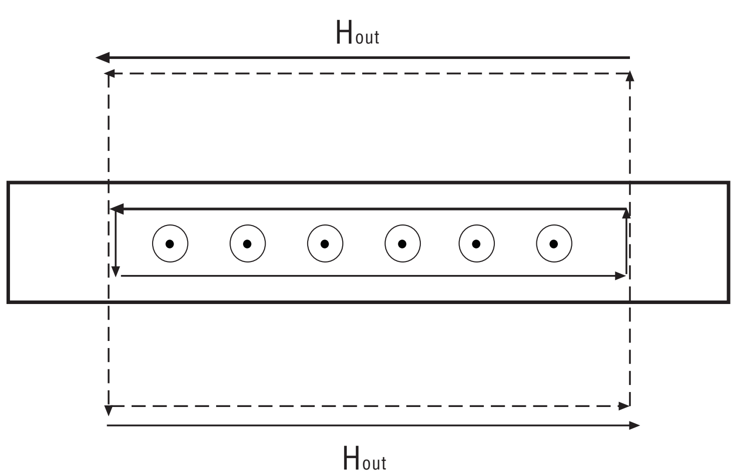
Following this method, at 2.5 cm:

$$H_{\phi 3} = \frac{10 + 4.00 - (2 \times 10^{-2})(250)}{2.5 \times 10^{-2}} = \underline{360 \text{ mA/m}}$$

and at 3.5 cm,

$$H_{\phi 4} = \frac{10 + 4.00 - 5.00 - (3 \times 10^{-2})(300)}{3.5 \times 10^{-2}} = \underline{0}$$

7.12. In Fig. 8.22, let the regions $0 < z < 0.3$ m and $0.7 < z < 1.0$ m be conducting slabs carrying uniform current densities of 10 A/m^2 in opposite directions as shown. The problem asks you to find $\mathbf{H}$ at various positions. Before continuing, we need to know how to find $\mathbf{H}$ for this type of current configuration. The sketch below shows one of the slabs (of thickness D) oriented with the current coming out of the page. The problem statement implies that both slabs are of infinite length and width. To find the magnetic field *inside* a slab, we apply Ampere's circuital law to the rectangular path of height d and width w, as shown, since by symmetry, $\mathbf{H}$ should be oriented horizontally. For example, if the sketch below shows the upper slab in Fig. 8.22, current will be in the positive y direction. Thus $\mathbf{H}$ will be in the positive x direction above the slab midpoint, and will be in the negative x direction below the midpoint.



7.12. (continued). In taking the line integral in Ampere's law, the two vertical path segments will cancel each other. Ampere's circuital law for the interior loop becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2H_{in} \times w = I_{encl} = J \times w \times d \quad \Rightarrow \quad H_{in} = \frac{Jd}{2}$$

The field outside the slab is found similarly, but with the enclosed current now bounded by the slab thickness, rather than the integration path height:

$$2H_{out} \times w = J \times w \times D \quad \Rightarrow \quad H_{out} = \frac{JD}{2}$$

where H_{out} is directed from right to left below the slab and from left to right above the slab (right hand rule). Reverse the current, and the fields, of course, reverse direction. We are now in a position to solve the problem. Find **H** at:

a) $z = -0.2$m: Here the fields from the top and bottom slabs (carrying opposite currents) will cancel, and so $\mathbf{H} = \underline{0}$.

b) $z = 0.2$m. This point lies within the lower slab above its midpoint. Thus the field will be oriented in the negative x direction. Referring to Fig. 8.22 and to the sketch on the previous page, we find that $d = 0.1$. The total field will be this field plus the contribution from the upper slab current:

$$\mathbf{H} = \underbrace{\frac{-10(0.1)}{2}\mathbf{a}_x}_{\text{lower slab}} - \underbrace{\frac{10(0.3)}{2}\mathbf{a}_x}_{\text{upper slab}} = \underline{-2\mathbf{a}_x \text{ A/m}}$$

c) $z = 0.4$m: Here the fields from both slabs will add constructively in the negative x direction:

$$\mathbf{H} = -2\frac{10(0.3)}{2}\mathbf{a}_x = \underline{-3\mathbf{a}_x \text{ A/m}}$$

d) $z = 0.75$m: This is in the interior of the upper slab, whose midpoint lies at $z = 0.85$. Therefore $d = 0.2$. Since 0.75 lies below the midpoint, magnetic field from the upper slab will lie in the negative x direction. The field from the lower slab will be negative x-directed as well, leading to:

$$\mathbf{H} = \underbrace{\frac{-10(0.2)}{2}\mathbf{a}_x}_{\text{upper slab}} - \underbrace{\frac{10(0.3)}{2}\mathbf{a}_x}_{\text{lower slab}} = \underline{-2.5\mathbf{a}_x \text{ A/m}}$$

e) $z = 1.2$m: This point lies above both slabs, where again fields cancel completely: Thus $\mathbf{H} = \underline{0}$.

7.13. A hollow cylindrical shell of radius a is centered on the z axis and carries a uniform surface current density of $K_a\mathbf{a}_\phi$.

a) Show that H is not a function of ϕ or z: Consider this situation as illustrated in Fig. 8.11. There (sec. 8.2) it was stated that the field will be entirely z-directed. We can see this by applying Ampere's circuital law to a closed loop path whose orientation we choose such that current is enclosed by the path. The only way to enclose current is to set up the loop (which we choose to be rectangular) such that it is oriented with two parallel opposing segments lying in the z direction; one of these lies inside the cylinder, the other outside. The other two parallel segments lie in the ρ direction. The loop is now cut by the current sheet, and if we assume a length of the loop in z of d, then the enclosed current will be given by Kd A. There will be no ϕ variation in the field because where we position the loop around the circumference of the cylinder does not affect the result of Ampere's law. If we assume an infinite cylinder length, there will be no z dependence in the field, since as we lengthen the loop in the z direction, the path length (over which the integral is taken) increases, but then so does the enclosed current – by the same factor. Thus H would not change with z. There would also be no change if the loop was simply moved along the z direction.

b) Show that H_ϕ and H_ρ are everywhere zero. First, if H_ϕ were to exist, then we should be able to find a closed loop path *that encloses current*, in which all or or portion of the path lies in the ϕ direction. This we cannot do, and so H_ϕ must be zero. Another argument is that when applying the Biot-Savart law, there is no current element that would produce a ϕ component. Again, using the Biot-Savart law, we note that radial field components will be produced by individual current elements, but such components will cancel from two elements that lie at symmetric distances in z on either side of the observation point.

c) Show that $H_z = 0$ for $\rho > a$: Suppose the rectangular loop was drawn such that the outside z-directed segment is moved further and further away from the cylinder. We would expect H_z outside to decrease (as the Biot-Savart law would imply) but the same amount of current is always enclosed no matter how far away the outer segment is. We therefore must conclude that the field outside is zero.

d) Show that $H_z = K_a$ for $\rho < a$: With our rectangular path set up as in part a, we have no path integral contributions from the two radial segments, and no contribution from the outside z-directed segment. Therefore, Ampere's circuital law would state that

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z d = I_{encl} = K_a d \quad \Rightarrow \quad H_z = K_a$$

where d is the length of the loop in the z direction.

e) A second shell, $\rho = b$, carries a current $K_b\mathbf{a}_\phi$. Find $\mathbf{H}$ everywhere: For $\rho < a$ we would have both cylinders contributing, or $H_z(\rho < a) = K_a + K_b$. Between the cylinders, we are outside the inner one, so its field will not contribute. Thus $H_z(a < \rho < b) = K_b$. Outside ($\rho > b$) the field will be zero.

7.14. A toroid having a cross section of rectangular shape is defined by the following surfaces: the cylinders $\rho = 2$ and $\rho = 3$ cm, and the planes $z = 1$ and $z = 2.5$ cm. The toroid carries a surface current density of $-50\mathbf{a}_z$ A/m on the surface $\rho = 3$ cm. Find $\mathbf{H}$ at the point $P(\rho, \phi, z)$: The construction is similar to that of the toroid of round cross section as done on p.239. Again, magnetic field exists only inside the toroid cross section, and is given by

$$\mathbf{H} = \frac{I_{encl}}{2\pi\rho}\mathbf{a}_\phi \quad (2 < \rho < 3) \text{ cm}, \quad (1 < z < 2.5) \text{ cm}$$

where I_{encl} is found from the given current density: On the outer radius, the current is

$$I_{outer} = -50(2\pi \times 3 \times 10^{-2}) = -3\pi \text{ A}$$

This current is directed along negative z, which means that the current on the *inner* radius ($\rho = 2$) is directed along *positive* z. Inner and outer currents have the same magnitude. It is the inner current that is enclosed by the circular integration path in $\mathbf{a}_\phi$ within the toroid that is used in Ampere's law. So $I_{encl} = +3\pi$ A. We can now proceed with what is requested:

a) $P_A(1.5\text{cm}, 0, 2\text{cm})$: The radius, $\rho = 1.5$ cm, lies outside the cross section, and so $\mathbf{H}_A = \underline{0}$.

b) $P_B(2.1\text{cm}, 0, 2\text{cm})$: This point does lie inside the cross section, and the ϕ and z values do not matter. We find

$$\mathbf{H}_B = \frac{I_{encl}}{2\pi\rho}\mathbf{a}_\phi = \frac{3\mathbf{a}_\phi}{2(2.1 \times 10^{-2})} = \underline{71.4\,\mathbf{a}_\phi \text{ A/m}}$$

c) $P_C(2.7\text{cm}, \pi/2, 2\text{cm})$: again, ϕ and z values make no difference, so

$$\mathbf{H}_C = \frac{3\mathbf{a}_\phi}{2(2.7 \times 10^{-2})} = \underline{55.6\,\mathbf{a}_\phi \text{ A/m}}$$

d) $P_D(3.5\text{cm}, \pi/2, 2\text{cm})$. This point lies outside the cross section, and so $\mathbf{H}_D = \underline{0}$.

7.15. Assume that there is a region with cylindrical symmetry in which the conductivity is given by $\sigma = 1.5e^{-150\rho}$ kS/m. An electric field of $30\,\mathbf{a}_z$ V/m is present.

a) Find $\mathbf{J}$: Use

$$\mathbf{J} = \sigma\mathbf{E} = \underline{45e^{-150\rho}\,\mathbf{a}_z \text{ kA/m}^2}$$

b) Find the total current crossing the surface $\rho < \rho_0$, $z = 0$, all ϕ:

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\rho_0} 45e^{-150\rho}\rho\, d\rho\, d\phi = \frac{2\pi(45)}{(150)^2}e^{-150\rho}\left[-150\rho - 1\right]\Big|_0^{\rho_0} \text{ kA}$$
$$= \underline{12.6\left[1 - (1 + 150\rho_0)e^{-150\rho_0}\right] \text{ A}}$$

c) Make use of Ampere's circuital law to find $\mathbf{H}$: Symmetry suggests that $\mathbf{H}$ will be ϕ-directed only, and so we consider a circular path of integration, centered on and perpendicular to the z axis. Ampere's law becomes: $2\pi\rho H_\phi = I_{encl}$, where I_{encl} is the current found in part *b*, except with ρ_0 replaced by the variable, ρ. We obtain

$$\underline{H_\phi = \frac{2.00}{\rho}\left[1 - (1 + 150\rho)e^{-150\rho}\right] \text{ A/m}}$$

7.16. A current filament carrying I in the $-\mathbf{a}_z$ direction lies along the entire positive z axis. At the origin, it connects to a conducting sheet that forms the xy plane.

a) Find $\mathbf{K}$ in the conducting sheet: The current fans outward radially with uniform surface current density at a fixed radius. The current density at radius ρ will be the total current, I, divided by the circumference at radius ρ:

$$\mathbf{K} = \frac{I}{2\pi\rho}\,\mathbf{a}_\rho \text{ A/m}$$

b) Use Ampere's circuital law to find $\mathbf{H}$ everywhere for $z > 0$: Circular lines of $\mathbf{H}$ are expected, centered on the z axis – in the $-\mathbf{a}_\phi$ direction. Ampere's law is set up by considering a circular path integral taken around the wire at fixed z. The enclosed current is that which passes through *any* surface that is bounded by the line integration path:

$$\oint \mathbf{H}\cdot d\mathbf{L} = 2\pi\rho H_\phi = I_{encl}$$

If the surface is that of the disk whose perimeter is the integration path, then the enclosed current is just I, and the magnetic field becomes

$$H_\phi = -\frac{I}{2\pi\rho} \quad \Rightarrow \quad \mathbf{H} = -\frac{I}{2\pi\rho}\,\mathbf{a}_\phi \text{ A/m}$$

But the disk surface can be "stretched" so that it forms a balloon shape. Suppose the "balloon" is a right circular cylinder, with its open top circumference at the path integral location. The cylinder extends downward, intersecting the surface current in the x-y plane, with the bottom of the cylinder below the x-y plane. Now, the path integral is unchanged from before, and the enclosed current is the radial current in the x-y plane that passes through the side of the cylinder. This current will be $I = 2\pi\rho[I/(2\pi\rho)] = I$, as before. So the answer given above for $\mathbf{H}$ applies to anywhere in the region $z > 0$.

c) Find $\mathbf{H}$ for $z < 0$: Consider the same cylinder as in part *b*, except take the path integral of $\mathbf{H}$ around the *bottom* circumference (below the x-y plane). The enclosed current now consists of the filament current that enters through the top, plus the radial current that exits though the side. The two currents are equal magnitude but opposite in sign. Therefore, the net enclosed current is *zero*, and thus $\underline{\mathbf{H} = 0}$ ($z < 0$).

7.17. A current filament on the z axis carries a current of 7 mA in the $\mathbf{a}_z$ direction, and current sheets of $0.5\,\mathbf{a}_z$ A/m and $-0.2\,\mathbf{a}_z$ A/m are located at $\rho = 1$ cm and $\rho = 0.5$ cm, respectively. Calculate $\mathbf{H}$ at:

a) $\rho = 0.5$ cm: Here, we are either just inside or just outside the first current sheet, so both we will calculate $\mathbf{H}$ for both cases. Just inside, applying Ampere's circuital law to a circular path centered on the z axis produces:

$$2\pi\rho H_\phi = 7 \times 10^{-3} \quad \Rightarrow \quad \mathbf{H}(\text{just inside}) = \frac{7 \times 10^{-3}}{2\pi(0.5 \times 10^{-2}}\mathbf{a}_\phi = \underline{2.2 \times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$$

Just outside the current sheet at .5 cm, Ampere's law becomes

$$2\pi\rho H_\phi = 7 \times 10^{-3} - 2\pi(0.5 \times 10^{-2})(0.2)$$

$$\Rightarrow \quad \mathbf{H}(\text{just outside}) = \frac{7.2 \times 10^{-4}}{2\pi(0.5 \times 10^{-2})}\mathbf{a}_\phi = \underline{2.3 \times 10^{-2}\mathbf{a}_\phi \text{ A/m}}$$

b) $\rho = 1.5$ cm: Here, all three currents are enclosed, so Ampere's law becomes

$$2\pi(1.5 \times 10^{-2})H_\phi = 7 \times 10^{-3} - 6.28 \times 10^{-3} + 2\pi(10^{-2})(0.5)$$

$$\Rightarrow \quad \mathbf{H}(\rho = 1.5) = \underline{3.4 \times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$$

c) $\rho = 4$ cm: Ampere's law as used in part b applies here, except we replace $\rho = 1.5$ cm with $\rho = 4$ cm on the left hand side. The result is $\mathbf{H}(\rho = 4) = \underline{1.3 \times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$.

d) What current sheet should be located at $\rho = 4$ cm so that $\mathbf{H} = 0$ for all $\rho > 4$ cm? We require that the total enclosed current be zero, and so the net current in the proposed cylinder at 4 cm must be negative the right hand side of the first equation in part b. This will be -3.2×10^{-2}, so that the surface current density at 4 cm must be

$$\mathbf{K} = \frac{-3.2 \times 10^{-2}}{2\pi(4 \times 10^{-2})}\mathbf{a}_z = \underline{-1.3 \times 10^{-1}\,\mathbf{a}_z \text{ A/m}}$$

7.18. A wire of 3-mm radius is made up of an inner material ($0 < \rho < 2$ mm) for which $\sigma = 10^7$ S/m, and an outer material (2mm $< \rho <$ 3mm) for which $\sigma = 4 \times 10^7$ S/m. If the wire carries a total current of 100 mA dc, determine $\mathbf{H}$ everywhere as a function of ρ.

Since the materials have different conductivities, the current densities within them will differ. Electric field, however is constant throughout. The current can be expressed as

$$I = \pi(.002)^2 J_1 + \pi[(.003)^2 - (.002)^2]J_2 = \pi\left[(.002)^2\sigma_1 + [(.003)^2 - (.002)^2]\sigma_2\right]E$$

Solve for E to obtain

$$E = \frac{0.1}{\pi[(4 \times 10^{-6})(10^7) + (9 \times 10^{-6} - 4 \times 10^{-6})(4 \times 10^7)]} = 1.33 \times 10^{-4} \text{ V/m}$$

We next apply Ampere's circuital law to a circular path of radius ρ, where $\rho <$ 2mm:

$$2\pi\rho H_{\phi 1} = \pi\rho^2 J_1 = \pi\rho^2\sigma_1 E \quad \Rightarrow \quad H_{\phi 1} = \frac{\sigma_1 E\rho}{2} = \underline{663 \text{ A/m}}$$

7.18 (continued) . Next, for the region 2mm $< \rho <$ 3mm, Ampere's law becomes

$$2\pi\rho H_{\phi 2} = \pi[(4\times 10^{-6})(10^7) + (\rho^2 - 4\times 10^{-6})(4\times 10^7)]E$$
$$\Rightarrow \quad H_{\phi 2} = 2.7\times 10^3 \rho \; - \; \frac{8.0\times 10^{-3}}{\rho} \text{ A/m}$$

Finally, for $\rho >$ 3mm, the field outside is that for a long wire:

$$H_{\phi 3} = \frac{I}{2\pi\rho} = \frac{0.1}{2\pi\rho} = \frac{1.6\times 10^{-2}}{\rho} \text{ A/m}$$

7.19. In spherical coordinates, the surface of a solid conducting cone is described by $\theta = \pi/4$ and a conducting plane by $\theta = \pi/2$. Each carries a total current I. The current flows as a surface current radially inward on the plane to the vertex of the cone, and then flows radially-outward throughout the cross-section of the conical conductor.

a) Express the surface current density as a function of r: This will be the total current divided by the circumference of a circle of radius r in the plane, directed toward the origin:

$$\mathbf{K}(r) = -\frac{I}{2\pi r}\mathbf{a}_r \text{ A/m}^2 \quad (\theta = \pi/2)$$

b) Express the volume current density inside the cone as a function of r: This will be the total current divided by the area of the spherical cap subtending angle $\theta = \pi/4$:

$$\mathbf{J}(r) = I\left[\int_0^{2\pi}\int_0^{\pi/4} r^2 \sin\theta'\, d\theta'\, d\phi\right]^{-1} \mathbf{a}_r \;=\; \underline{\frac{I\,\mathbf{a}_r}{2\pi r^2(1-1/\sqrt{2})}} \text{ A/m}^2 \quad (0<\theta<\pi/4)$$

c) Determine $\mathbf{H}$ as a function of r and θ in the region between the cone and the plane: From symmetry, we expect $\mathbf{H}$ to be ϕ-directed and uniform at constant r and θ. Ampere's circuital law can therefore be stated as:

$$\oint \mathbf{H}\cdot d\mathbf{L} = 2\pi r\sin\theta\, H_\phi = I \;\Rightarrow\; \mathbf{H} = \frac{I}{2\pi r\sin\theta}\mathbf{a}_\phi \text{ A/m} \quad (\pi/4<\theta<\pi/2)$$

d) Determine $\mathbf{H}$ as a function of r and θ inside the cone: Again, ϕ-directed $\mathbf{H}$ is anticipated, so we apply Ampere's law in the following way:

$$\oint \mathbf{H}\cdot d\mathbf{L} = 2\pi r\sin\theta\, H_\phi = \int_s \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\theta} \frac{I\,\mathbf{a}_r}{2\pi r^2(1-1/\sqrt{2})}\cdot \mathbf{a}_r\, r^2 \sin\theta'\, d\theta'\, d\phi$$

This becomes

$$2\pi r\sin\theta\, H_\phi = -2\pi\frac{I}{2\pi(1-1/\sqrt{2})}\cos\theta'\Big|_0^{\theta}$$

or

$$\underline{\mathbf{H} = \frac{I}{2\pi r(1-1/\sqrt{2})}\left[\frac{(1-\cos\theta)}{\sin\theta}\right]\mathbf{a}_\phi} \text{ A/m} \quad (0<\theta<\pi/4)$$

As a test of this, note that the inside and outside fields (results of parts *c* and *d*) are equal at the cone surface ($\theta = \pi/4$) as they must be.

7.20. A solid conductor of circular cross-section with a radius of 5 mm has a conductivity that varies with radius. The conductor is 20 m long and there is a potential difference of 0.1 V dc between its two ends. Within the conductor, $\mathbf{H} = 10^5 \rho^2 \mathbf{a}_\phi$ A/m.

a) Find σ as a function of ρ: Start by finding $\mathbf{J}$ from $\mathbf{H}$ by taking the curl. With $\mathbf{H}$ ϕ-directed, and varying with radius only, the curl becomes:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho H_\phi\right) \mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(10^5 \rho^3\right) \mathbf{a}_z = 3 \times 10^5 \rho \, \mathbf{a}_z \ \text{A/m}^2$$

Then $\mathbf{E} = 0.1/20 = 0.005 \, \mathbf{a}_z$ V/m, which we then use with $\mathbf{J} = \sigma \mathbf{E}$ to find

$$\sigma = \frac{J}{E} = \frac{3 \times 10^5 \rho}{0.005} = \underline{6 \times 10^7 \rho \ \ \text{S/m}}$$

b) What is the resistance between the two ends? The current in the wire is

$$I = \int_s \mathbf{J} \cdot d\mathbf{S} = 2\pi \int_0^a \left(3 \times 10^5 \rho\right) \rho \, d\rho = 6\pi \times 10^5 \left(\frac{1}{3} a^3\right) = 2\pi \times 10^5 (0.005)^3 = 0.079 \ \text{A}$$

Finally, $R = V_0 / I = 0.1/0.079 = \underline{1.3 \ \Omega}$

7.21. A cylindrical wire of radius a is oriented with the z axis down its center line. The wire carries a non-uniform current down its length of density $\mathbf{J} = b\rho\,\mathbf{a}_z$ A/m^2, where b is a constant.

a) What total current flows in the wire? We integrate the current density over the wire cross-section:

$$I_{tot} = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^a b\rho\,\mathbf{a}_z \cdot \mathbf{a}_z\,\rho\,d\rho\,d\phi = \underline{\frac{2\pi b a^3}{3}} \text{ A}$$

b) find $\mathbf{H}_{in}$ $(0 < \rho < a)$, as a function of ρ: From the symmetry, ϕ-directed $\mathbf{H}$ $(= H_\phi\,\mathbf{a}_\phi)$ is expected in the interior; this will be constant at a fixed radius, ρ. Apply Ampere's circuital law to a circular path of radius ρ inside:

$$\oint \mathbf{H}_{in} \cdot d\mathbf{L} = 2\pi\rho H_{\phi,in} = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\rho} b\rho'\,\mathbf{a}_z \cdot \mathbf{a}_z\,\rho'\,d\rho'\,d\phi = \frac{2\pi b\rho^3}{3}$$

So that

$$\mathbf{H}_{in} = \underline{\frac{b\rho^2}{3}\,\mathbf{a}_\phi} \text{ A/m} \quad (0 < \rho < a)$$

c) find $\mathbf{H}_{out}(\rho > a)$, as a function of ρ Same as part b, except the path integral is taken at a radius outside the wire:

$$\oint \mathbf{H}_{out} \cdot d\mathbf{L} = 2\pi\rho H_{\phi,out} = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{a} b\rho\,\mathbf{a}_z \cdot \mathbf{a}_z\,\rho\,d\rho\,d\phi = \frac{2\pi b a^3}{3}$$

So that

$$\mathbf{H}_{out} = \underline{\frac{ba^3}{3\rho}\,\mathbf{a}_\phi} \text{ A/m} \quad (\rho > a)$$

d) verify your results of parts b and c by using $\nabla \times \mathbf{H} = \mathbf{J}$: With a ϕ component of $\mathbf{H}$ only, varying only with ρ, the curl in cylindrical coordinates reduces to

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z$$

Apply this to the inside field to get

$$\mathbf{J}_{in} = \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{b\rho^3}{3}\right)\mathbf{a}_z = b\rho\,\mathbf{a}_z$$

For the outside field, we find

$$\mathbf{J}_{out} = \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{\rho b a^3}{3\rho}\right)\mathbf{a}_z = 0$$

as expected.

7.22. A solid cylinder of radius a and length L, where $L >> a$, contains volume charge of uniform density ρ_0 C/m^3. The cylinder rotates about its axis (the z axis) at angular velocity Ω rad/s.

a) Determine the current density $\mathbf{J}$, as a function of position within the rotating cylinder: Use $\mathbf{J} = \rho_0 \mathbf{v} = \underline{\rho_0 \rho \Omega\, \mathbf{a}_\phi \text{ A/m}^2}$.

b) Determine the magnetic field intensity $\mathbf{H}$ inside and outside: It helps initially to obtain the field on-axis. To do this, we use the result of Problem 8.6, but give the rotating charged disk in that problem a differential thickness, dz. We can then evaluate the on-axis field in the rotating cylinder as the superposition of fields from a stack of disks which exist between $\pm L/2$. Here, we make the problem easier by letting $L \to \infty$ (since $L >> a$) thereby specializing our evaluation to positions near the half-length. The on-axis field is therefore:

$$H_z(\rho = 0) = \int_{-\infty}^{\infty} \frac{\rho_0 \Omega}{2z} \left[\frac{a^2 + 2z^2 \left(1 - \sqrt{1 + a^2/z^2}\right)}{\sqrt{1 + a^2/z^2}} \right] dz$$

$$= 2 \int_0^{\infty} \frac{\rho_0 \Omega}{2} \left[\frac{a^2}{\sqrt{z^2 + a^2}} + \frac{2z^2}{\sqrt{z^2 + a^2}} - 2z \right] dz$$

$$= 2\rho_0 \Omega \left[\frac{a^2}{2} \ln(z + \sqrt{z^2 + a^2}) + \frac{z}{2}\sqrt{z^2 + a^2} - \frac{a^2}{2} \ln(z + \sqrt{z^2 + a^2}) - \frac{z^2}{2} \right]_0^{\infty}$$

$$= \rho_0 \Omega \left[z\sqrt{z^2 + a^2} - z^2 \right]_0^{\infty} = \rho_0 \Omega \left[z\sqrt{z^2 + a^2} - z^2 \right]_{z \to \infty}$$

Using the large z approximation in the radical, we obtain

$$H_z(\rho = 0) = \rho_0 \Omega \left[z^2 \left(1 + \frac{a^2}{2z^2}\right) - z^2 \right] = \frac{\rho_0 \Omega a^2}{2}$$

To find the field as a function of radius, we apply Ampere's circuital law to a rectangular loop, drawn in two locations described as follows: First, construct the rectangle with one side along the z axis, and with the opposite side lying at any radius *outside* the cylinder. In taking the line integral of $\mathbf{H}$ around the rectangle, we note that the two segments that are perpendicular to the cylinder axis will have their path integrals exactly cancel, since the two path segments are oppositely-directed, while from symmetry the field should not be different along each segment. This leaves only the path segment that coindides with the axis, and that lying parallel to the axis, but outside. Choosing the length of these segments to be ℓ, Ampere's circuital law becomes:

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z(\rho = 0)\ell + H_z(\rho > a)\ell = I_{encl} = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^{\ell} \int_0^a \rho_0 \rho \Omega\, \mathbf{a}_\phi \cdot \mathbf{a}_\phi \, d\rho \, dz$$

$$= \ell \frac{\rho_0 \Omega a^2}{2}$$

But we found earlier that $H_z(\rho = 0) = \rho_0 \Omega a^2/2$. Therefore, we identify the outside field, $H_z(\rho > a) = 0$. Next, change the rectangular path only by displacing the central path component off-axis by distance ρ, but still lying within the cylinder. The enclosed current is now somewhat less, and Ampere's law becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z(\rho)\ell + H_z(\rho > a)\ell = I_{encl} = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^{\ell} \int_\rho^a \rho_0 \rho' \Omega\, \mathbf{a}_\phi \cdot \mathbf{a}_\phi \, d\rho \, dz$$

$$= \ell \frac{\rho_0 \Omega}{2}(a^2 - \rho^2) \quad \Rightarrow \quad \mathbf{H}(\rho) = \frac{\rho_0 \Omega}{2}(a^2 - \rho^2)\, \mathbf{a}_z \text{ A/m}$$

7.22c) Check your result of part *b* by taking the curl of **H**. With **H** *z*-directed, and varying only with ρ, the curl in cylindrical coordinates becomes

$$\nabla \times \mathbf{H} = -\frac{dH_z}{d\rho}\,\mathbf{a}_\phi = \rho_0 \Omega \rho\,\mathbf{a}_\phi \text{ A/m}^2 = \mathbf{J}$$

as expected.

7.23. Given the field $\mathbf{H} = 20\rho^2\,\mathbf{a}_\phi$ A/m:

a) Determine the current density **J**: This is found through the curl of **H**, which simplifies to a single term, since **H** varies only with ρ and has only a ϕ component:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d(\rho H_\phi)}{d\rho}\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(20\rho^3\right)\,\mathbf{a}_z = \underline{60\rho\,\mathbf{a}_z \text{ A/m}^2}$$

b) Integrate **J** over the circular surface $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$, to determine the total current passing through that surface in the $\mathbf{a}_z$ direction: The integral is:

$$I = \int\!\!\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 60\rho\mathbf{a}_z \cdot \rho\, d\rho\, d\phi\mathbf{a}_z = \underline{40\pi \text{ A}}$$

c) Find the total current once more, this time by a line integral around the circular path $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$:

$$I = \oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} 20\rho^2\,\mathbf{a}_\phi\Big|_{\rho=1} \cdot (1)d\phi\mathbf{a}_\phi = \int_0^{2\pi} 20\, d\phi = \underline{40\pi \text{ A}}$$

7.24. Infinitely-long filamentary conductors are located in the $y = 0$ plane at $x = n$ meters where $n = 0, \pm 1, \pm 2, \ldots$ Each carries 1 A in the $\mathbf{a}_z$ direction.

a) Find $\mathbf{H}$ on the y axis. As a help,

$$\sum_{n=1}^{\infty} \frac{y}{y^2 + n^2} = \frac{\pi}{2} - \frac{1}{2y} + \frac{\pi}{e^{2\pi y} - 1}$$

We can begin by determining the field on the y axis arising from two wires only, located at $x = \pm n$. We start from basics, using the Biot-Savart law:

$$\mathbf{H} = \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

where $R = \left(n^2 + y^2 + z^2\right)^2$ for both wires, and where, for the wire at $x = +n$

$$\mathbf{a}_R^+ = \frac{-n\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z}{\left(n^2 + y^2 + z^2\right)^{1/2}} \qquad \text{and} \qquad \mathbf{a}_R^- = \frac{n\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z}{\left(n^2 + y^2 + z^2\right)^{1/2}}$$

for the wire located at $x = -n$. Then with $I d\mathbf{L} = dz\, \mathbf{a}_z$ $(I = 1)$ for both wires, the Biot-Savart construction becomes

$$\mathbf{H} = \int_{-\infty}^{\infty} \frac{dz\, \mathbf{a}_z \times [(-n\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z) + (n\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)]}{4\pi \left(n^2 + y^2 + z^2\right)^{3/2}} = \frac{-y\mathbf{a}_x}{2\pi} \int_{-\infty}^{\infty} \frac{dz}{\left(n^2 + y^2 + z^2\right)^{3/2}}$$

This evaluates as

$$\mathbf{H} = -\frac{1}{\pi} \left(\frac{y}{n^2 + y^2} \right) \mathbf{a}_x \text{ A/m}$$

Now if we include *all* wire pairs, the result is the superposition of an infinite number of fields of the above form. Specifically,

$$\mathbf{H}_{net} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{y}{n^2 + y^2} \right) \mathbf{a}_x \text{ A/m}$$

Using the given closed form of the series expansion, our final answer is

$$\underline{\mathbf{H}_{net} = -\left[\frac{1}{2} - \frac{1}{2\pi y} + \frac{1}{e^{2\pi y} - 1} \right]} \mathbf{a}_x \text{ A/m}$$

b) Compare your result of part *a* to that obtained if the filaments are replaced by a current sheet in the $y = 0$ plane that carries surface current density $\mathbf{K} = 1\mathbf{a}_z$ A/m: This is found through Eq. (11):

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N = \frac{1}{2}\mathbf{a}_z \times \mathbf{a}_y = -\frac{1}{2}\mathbf{a}_x$$

Our answer of part *a* approaches this value as $y \to \infty$, demonstrating that at large distances, the parallel wires act like a current sheet. Interestingly, at close-in locations, such that $2\pi y << 1$, we may expand $e^{2\pi y} \doteq 1 + 2\pi y$, leading to the cancellation of the last two terms in the part *a* result, and again, $\mathbf{H}_{net} \doteq -1/2\, \mathbf{a}_x$.

7.25. When x, y, and z are positive and less than 5, a certain magnetic field intensity may be expressed as $\mathbf{H} = [x^2yz/(y+1)]\mathbf{a}_x + 3x^2z^2\mathbf{a}_y - [xyz^2/(y+1)]\mathbf{a}_z$. Find the total current in the $\mathbf{a}_x$ direction that crosses the strip, $x = 2$, $1 \le y \le 4$, $3 \le z \le 4$, by a method utilizing:

a) a surface integral: We need to find the current density by taking the curl of the given $\mathbf{H}$. Actually, since the strip lies parallel to the yz plane, we need only find the x component of the current density, as only this component will contribute to the requested current. This is

$$J_x = (\nabla \times \mathbf{H})_x = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = -\left(\frac{xz^2}{(y+1)^2} + 6x^2z\right)\mathbf{a}_x$$

The current through the strip is then

$$I = \int_s \mathbf{J}\cdot\mathbf{a}_x\, da = -\int_3^4\int_1^4 \left(\frac{2z^2}{(y+1)^2} + 24z\right) dy\, dz = -\int_3^4 \left(\frac{-2z^2}{(y+1)} + 24zy\right)_1^4 dz$$

$$= -\int_3^4 \left(\frac{3}{5}z^2 + 72z\right) dz = -\left(\frac{1}{5}z^3 + 36z^2\right)_3^4 = \underline{-259}$$

b) a closed line integral: We integrate counter-clockwise around the strip boundary (using the right-hand convention), where the path normal is positive $\mathbf{a}_x$. The current is then

$$I = \oint \mathbf{H}\cdot d\mathbf{L} = \int_1^4 3(2)^2(3)^2\, dy + \int_3^4 -\frac{2(4)z^2}{(4+1)}\, dz + \int_4^1 3(2)^2(4)^2\, dy + \int_4^3 -\frac{2(1)z^2}{(1+1)}\, dz$$

$$= 108(3) - \frac{8}{15}(4^3 - 3^3) + 192(1-4) - \frac{1}{3}(3^3 - 4^3) = -259$$

7.26. Consider a sphere of radius $r = 4$ centered at (0,0,3). Let S_1 be that portion of the spherical surface that lies above the xy plane. Find $\int_{S_1}(\nabla \times \mathbf{H})\cdot d\mathbf{S}$ if $\mathbf{H} = 3\rho\,\mathbf{a}_\phi$ in cylindrical coordinates: First, the intersection of the sphere with the x-y plane is a disk in the plane of radius $\rho_d = \sqrt{4^2 - 3^2} = \sqrt{7}$. The curl of the given field (having a ϕ component that varies only with ρ) is

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(3\rho^2\right)\mathbf{a}_z = 6\,\mathbf{a}_z$$

Since this is a constant field, its flux through S_1 will simply be the flux through the disk, which in turn is simply the product of the field with the disk area:

$$\Phi = 6 \times \pi(\sqrt{7})^2 = \underline{42\pi}$$

Another way to solve the problem is to use Stokes' theorem and write the flux as

$$\Phi = \oint \mathbf{H}\cdot d\mathbf{L}$$

where the path integral is taken around the disk perimeter. Doing this gives:

$$\Phi = 2\pi\rho_d H_\phi|_{\rho_d} = 2\pi\sqrt{7} \times 3\sqrt{7} = \underline{42\pi}$$

7.27. The magnetic field intensity is given in a certain region of space as

$$\mathbf{H} = \frac{x+2y}{z^2}\,\mathbf{a}_y + \frac{2}{z}\,\mathbf{a}_z \text{ A/m}$$

a) Find $\nabla\times\mathbf{H}$: For this field, the general curl expression in rectangular coordinates simplifies to

$$\nabla\times\mathbf{H} = -\frac{\partial H_y}{\partial z}\,\mathbf{a}_x + \frac{\partial H_y}{\partial x}\,\mathbf{a}_z = \underline{\frac{2(x+2y)}{z^3}\,\mathbf{a}_x + \frac{1}{z^2}\mathbf{a}_z \text{ A/m}}$$

b) Find $\mathbf{J}$: This will be the answer of part a, since $\nabla\times\mathbf{H} = \mathbf{J}$.

c) Use $\mathbf{J}$ to find the total current passing through the surface $z = 4$, $1 < x < 2$, $3 < y < 5$, in the $\mathbf{a}_z$ direction: This will be

$$I = \int\int \mathbf{J}\big|_{z=4}\cdot\mathbf{a}_z\,dx\,dy = \int_3^5\int_1^2 \frac{1}{4^2}dx\,dy = \underline{1/8 \text{ A}}$$

d) Show that the same result is obtained using the other side of Stokes' theorem: We take $\oint\mathbf{H}\cdot d\mathbf{L}$ over the square path at $z = 4$ as defined in part c. This involves two integrals of the y component of $\mathbf{H}$ over the range $3 < y < 5$. Integrals over x, to complete the loop, do not exist since there is no x component of $\mathbf{H}$. We have

$$I = \oint \mathbf{H}\big|_{z=4}\cdot d\mathbf{L} = \int_3^5 \frac{2+2y}{16}\,dy + \int_5^3 \frac{1+2y}{16}\,dy = \frac{1}{8}(2) - \frac{1}{16}(2) = \underline{1/8 \text{ A}}$$

7.28. Given $\mathbf{H} = (3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\mathbf{a}_\phi$ A/m in free space:

a) find the total current in the $\mathbf{a}_\theta$ direction through the conical surface $\theta = 20^\circ, 0 \le \phi \le 2\pi$, $0 \le r \le 5$, by whatever side of Stokes' theorem you like best. I chose the line integral side, where the integration path is the circular path in ϕ around the top edge of the cone, at $r = 5$. The path direction is chosen to be *clockwise* looking down on the xy plane. This, by convention, leads to the normal from the cone surface that points in the positive $\mathbf{a}_\theta$ direction (right hand rule). We find

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int_0^{2\pi} \left[(3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\mathbf{a}_\phi\right]_{r=5,\theta=20} \cdot 5\sin(20^\circ)\,d\phi\,(-\mathbf{a}_\phi)$$
$$= -2\pi(54)(25)\cos(20^\circ)\sin(20^\circ) = \underline{-2.73\times 10^3\text{ A}}$$

This result means that there is a component of current that enters the cone surface in the $-\mathbf{a}_\theta$ direction, to which is associated a component of $\mathbf{H}$ in the positive $\mathbf{a}_\phi$ direction.

b) Check the result by using the other side of Stokes' theorem: We first find the current density through the curl of the magnetic field, where three of the six terms in the spherical coordinate formula survive:

$$\nabla\times\mathbf{H} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(54r\cos\theta\sin\theta\right)\,\mathbf{a}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(54r^2\cos\theta\right)\,\mathbf{a}_\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{3r^3}{\sin\theta}\right)\,\mathbf{a}_\phi = \mathbf{J}$$

Thus

$$\mathbf{J} = 54\cot\theta\,\mathbf{a}_r - 108\cos\theta\,\mathbf{a}_\theta + \frac{9r}{\sin\theta}\,\mathbf{a}_\phi$$

The calculation of the other side of Stokes' theorem now involves integrating $\mathbf{J}$ over the surface of the cone, where the outward normal is positive $\mathbf{a}_\theta$, as defined in part a:

$$\int_S(\nabla\times\mathbf{H})\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^5\left[54\cot\theta\,\mathbf{a}_r - 108\cos\theta\,\mathbf{a}_\theta + \frac{9r}{\sin\theta}\,\mathbf{a}_\phi\right]_{20^\circ}\cdot\mathbf{a}_\theta\,r\sin(20^\circ)\,dr\,d\phi$$
$$= -\int_0^{2\pi}\int_0^5 108\cos(20^\circ)\sin(20^\circ)r\,dr\,d\phi = -2\pi(54)(25)\cos(20^\circ)\sin(20^\circ)$$
$$= \underline{-2.73\times 10^3\text{ A}}$$

7.29. A long straight non-magnetic conductor of 0.2 mm radius carries a uniformly-distributed current of 2 A dc.

a) Find **J** within the conductor: Assuming the current is $+z$ directed,

$$\mathbf{J} = \frac{2}{\pi(0.2 \times 10^{-3})^2}\mathbf{a}_z = \underline{1.59 \times 10^7\,\mathbf{a}_z\ \text{A/m}^2}$$

b) Use Ampere's circuital law to find **H** and **B** within the conductor: Inside, at radius ρ, we have

$$2\pi\rho H_\phi = \pi\rho^2 J \quad \Rightarrow \quad \mathbf{H} = \frac{\rho J}{2}\mathbf{a}_\phi = \underline{7.96 \times 10^6 \rho\,\mathbf{a}_\phi\ \text{A/m}}$$

Then $\mathbf{B} = \mu_0\mathbf{H} = (4\pi \times 10^{-7})(7.96 \times 10^6)\rho\mathbf{a}_\phi = \underline{10\rho\,\mathbf{a}_\phi\ \text{Wb/m}^2}$.

c) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ within the conductor: Using the result of part b, we find,

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{1.59 \times 10^7 \rho^2}{2}\right)\mathbf{a}_z = \underline{1.59 \times 10^7\,\mathbf{a}_z\ \text{A/m}^2} = \mathbf{J}$$

d) Find **H** and **B** *outside* the conductor (note typo in book): Outside, the entire current is enclosed by a closed path at radius ρ, and so

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \underline{\frac{1}{\pi\rho}\,\mathbf{a}_\phi\ \text{A/m}}$$

Now $\mathbf{B} = \mu_0\mathbf{H} = \underline{\mu_0/(\pi\rho)\,\mathbf{a}_\phi\ \text{Wb/m}^2}$.

e) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ outside the conductor: Here we use **H** outside the conductor and write:

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{1}{\pi\rho}\right)\mathbf{a}_z = \underline{0} \quad \text{(as expected)}$$

7.30. (an inversion of Problem 8.20). A solid nonmagnetic conductor of circular cross-section has a radius of 2mm. The conductor is inhomogeneous, with $\sigma = 10^6(1 + 10^6\rho^2)$ S/m. If the conductor is 1m in length and has a voltage of 1mV between its ends, find:

a) **H** inside: With current along the cylinder length (along $\mathbf{a}_z$, and with ϕ symmetry, **H** will be ϕ-directed only. We find $\mathbf{E} = (V_0/d)\mathbf{a}_z = 10^{-3}\mathbf{a}_z$ V/m. Then $\mathbf{J} = \sigma\mathbf{E} = 10^3(1 + 10^6\rho^2)\mathbf{a}_z$ A/m^2. Next we apply Ampere's circuital law to a circular path of radius ρ, centered on the z axis and normal to the axis:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = \int\int_S \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\rho} 10^3(1 + 10^6(\rho')^2)\mathbf{a}_z \cdot \mathbf{a}_z \rho' d\rho' d\phi$$

Thus

$$H_\phi = \frac{10^3}{\rho}\int_0^{\rho} \rho' + 10^6(\rho')^3 d\rho' = \frac{10^3}{\rho}\left[\frac{\rho^2}{2} + \frac{10^6}{4}\rho^4\right]$$

Finally, $\mathbf{H} = \underline{500\rho(1 + 5 \times 10^5\rho^3)\mathbf{a}_\phi\ \text{A/m}\ (0 < \rho < 2\text{mm})}$.

b) the total magnetic flux inside the conductor: With field in the ϕ direction, a plane normal to **B** will be that in the region $0 < \rho < 2$ mm, $0 < z < 1$ m. The flux will be

$$\Phi = \int\int_S \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int_0^1 \int_0^{2\times10^{-3}} \left(500\rho + 2.5 \times 10^8\rho^3\right) d\rho dz = 8\pi \times 10^{-10}\,\text{Wb} = \underline{2.5\,\text{nWb}}$$

7.31. The cylindrical shell defined by 1 cm $< \rho <$ 1.4 cm consists of a non-magnetic conducting material and carries a total current of 50 A in the $\mathbf{a}_z$ direction. Find the total magnetic flux crossing the plane $\phi = 0$, $0 < z < 1$:

a) $0 < \rho < 1.2\,\text{cm}$: We first need to find **J**, **H**, and **B**: The current density will be:

$$\mathbf{J} = \frac{50}{\pi[(1.4 \times 10^{-2})^2 - (1.0 \times 10^{-2})^2]}\,\mathbf{a}_z = 1.66 \times 10^5\,\mathbf{a}_z \text{ A/m}^2$$

Next we find H_ϕ at radius ρ between 1.0 and 1.4 cm, by applying Ampere's circuital law, and noting that the current density is zero at radii less than 1 cm:

$$2\pi\rho H_\phi = I_{encl} = \int_0^{2\pi}\int_{10^{-2}}^{\rho} 1.66 \times 10^5 \rho'\,d\rho'\,d\phi$$

$$\Rightarrow \quad H_\phi = 8.30 \times 10^4 \frac{(\rho^2 - 10^{-4})}{\rho} \text{ A/m} \quad (10^{-2}\,\text{m} < \rho < 1.4 \times 10^{-2}\,\text{m})$$

Then $\mathbf{B} = \mu_0 \mathbf{H}$, or

$$\mathbf{B} = 0.104 \frac{(\rho^2 - 10^{-4})}{\rho}\,\mathbf{a}_\phi \text{ Wb/m}^2$$

Now,

$$\Phi_a = \int\int \mathbf{B} \cdot d\mathbf{S} = \int_0^1 \int_{10^{-2}}^{1.2\times10^{-2}} 0.104 \left[\rho - \frac{10^{-4}}{\rho}\right] d\rho\,dz$$
$$= 0.104 \left[\frac{(1.2 \times 10^{-2})^2 - 10^{-4}}{2} - 10^{-4} \ln\left(\frac{1.2}{1.0}\right)\right] = 3.92 \times 10^{-7}\,\text{Wb} = \underline{0.392\,\mu\text{Wb}}$$

b) $1.0\,\text{cm} < \rho < 1.4\,\text{cm}$ (note typo in book): This is part a over again, except we change the upper limit of the radial integration:

$$\Phi_b = \int\int \mathbf{B} \cdot d\mathbf{S} = \int_0^1 \int_{10^{-2}}^{1.4\times10^{-2}} 0.104 \left[\rho - \frac{10^{-4}}{\rho}\right] d\rho\,dz$$
$$= 0.104 \left[\frac{(1.4 \times 10^{-2})^2 - 10^{-4}}{2} - 10^{-4} \ln\left(\frac{1.4}{1.0}\right)\right] = 1.49 \times 10^{-6}\,\text{Wb} = \underline{1.49\,\mu\text{Wb}}$$

c) $1.4\,\text{cm} < \rho < 20\,\text{cm}$: This is entirely outside the current distribution, so we need **B** there: We modify the Ampere's circuital law result of part a to find:

$$\mathbf{B}_{out} = 0.104 \frac{[(1.4 \times 10^{-2})^2 - 10^{-4}]}{\rho}\,\mathbf{a}_\phi = \frac{10^{-5}}{\rho}\,\mathbf{a}_\phi \text{ Wb/m}^2$$

We now find

$$\Phi_c = \int_0^1 \int_{1.4\times10^{-2}}^{20\times10^{-2}} \frac{10^{-5}}{\rho}\,d\rho\,dz = 10^{-5} \ln\left(\frac{20}{1.4}\right) = 2.7 \times 10^{-5}\,\text{Wb} = \underline{27\,\mu\text{Wb}}$$

7.32. The free space region defined by $1 < z < 4$ cm and $2 < \rho < 3$ cm is a toroid of rectangular cross-section. Let the surface at $\rho = 3$ cm carry a surface current $\mathbf{K} = 2\mathbf{a}_z$ kA/m.

a) Specify the current densities on the surfaces at $\rho = 2$ cm, $z = 1$cm, and $z = 4$cm. All surfaces must carry equal currents. With this requirement, we find: $\mathbf{K}(\rho = 2) = \underline{-3\,\mathbf{a}_z \text{ kA/m}}$. Next, the current densities on the $z = 1$ and $z = 4$ surfaces must transistion between the current density values at $\rho = 2$ and $\rho = 3$. Knowing the the radial current density will vary as $1/\rho$, we find $\mathbf{K}(z = 1) = \underline{(60/\rho)\mathbf{a}_\rho \text{ A/m}}$ with ρ in meters. Similarly, $\mathbf{K}(z = 4) = \underline{-(60/\rho)\mathbf{a}_\rho \text{ A/m}}$.

b) Find $\mathbf{H}$ everywhere: Outside the toroid, $\mathbf{H} = 0$. Inside, we apply Ampere's circuital law in the manner of Problem 8.14:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = \int_0^{2\pi} \mathbf{K}(\rho = 2) \cdot \mathbf{a}_z \, (2 \times 10^{-2}) \, d\phi$$

$$\Rightarrow \quad \mathbf{H} = -\frac{2\pi(3000)(.02)}{\rho}\mathbf{a}_\phi = \underline{-60/\rho \, \mathbf{a}_\phi \text{ A/m (inside)}}$$

c) Calculate the total flux within the toriod: We have $\mathbf{B} = -(60\mu_0/\rho)\mathbf{a}_\phi$ Wb/m^2. Then

$$\Phi = \int_{.01}^{.04} \int_{.02}^{.03} \frac{-60\mu_0}{\rho} \, \mathbf{a}_\phi \cdot (-\mathbf{a}_\phi) \, d\rho \, dz = (.03)(60)\mu_0 \ln\left(\frac{3}{2}\right) = \underline{0.92 \, \mu\text{Wb}}$$

7.33. Use an expansion in rectangular coordinates to show that the curl of the gradient of any scalar field G is identically equal to zero. We begin with

$$\nabla G = \frac{\partial G}{\partial x}\,\mathbf{a}_x + \frac{\partial G}{\partial y}\,\mathbf{a}_y + \frac{\partial G}{\partial z}\,\mathbf{a}_z$$

and

$$\nabla \times \nabla G = \left[\frac{\partial}{\partial y}\left(\frac{\partial G}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial G}{\partial y}\right)\right]\mathbf{a}_x + \left[\frac{\partial}{\partial z}\left(\frac{\partial G}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial z}\right)\right]\mathbf{a}_y + \left[\frac{\partial}{\partial x}\left(\frac{\partial G}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x}\right)\right]\mathbf{a}_z = \underline{0} \text{ for any } G$$

7.34. A filamentary conductor on the z axis carries a current of 16A in the $\mathbf{a}_z$ direction, a conducting shell at $\rho = 6$ carries a total current of 12A in the $-\mathbf{a}_z$ direction, and another shell at $\rho = 10$ carries a total current of 4A in the $-\mathbf{a}_z$ direction.

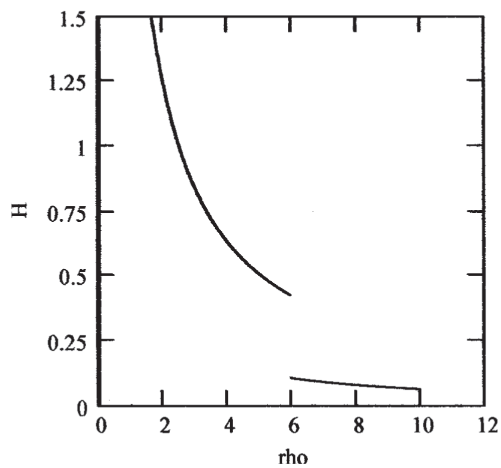
a) Find $\mathbf{H}$ for $0 < \rho < 12$: Ampere's circuital law states that $\oint \mathbf{H} \cdot d\mathbf{L} = I_{encl}$, where the line integral and current direction are related in the usual way through the right hand rule. Therefore, if I is in the positive z direction, $\mathbf{H}$ is in the $\mathbf{a}_\phi$ direction. We proceed as follows:

$$0 < \rho < 6: \quad 2\pi\rho H_\phi = 16 \quad \Rightarrow \quad \mathbf{H} = \underline{16/(2\pi\rho)\mathbf{a}_\phi}$$

$$6 < \rho < 10: \quad 2\pi\rho H_\phi = 16 - 12 \quad \Rightarrow \quad \mathbf{H} = \underline{4/(2\pi\rho)\mathbf{a}_\phi}$$

$$\rho > 10: \quad 2\pi\rho H_\phi = 16 - 12 - 4 = 0 \quad \Rightarrow \quad \mathbf{H} = \underline{0}$$

b) Plot H_ϕ vs. ρ:



c) Find the total flux Φ crossing the surface $1 < \rho < 7$, $0 < z < 1$: This will be

$$\Phi = \int_0^1 \int_1^6 \frac{16\mu_0}{2\pi\rho}\, d\rho\, dz + \int_0^1 \int_6^7 \frac{4\mu_0}{2\pi\rho}\, d\rho\, dz = \frac{2\mu_0}{\pi}\left[4\ln 6 + \ln(7/6)\right] = \underline{5.9\ \mu\text{Wb}}$$

7.35. A current sheet, $\mathbf{K} = 20\,\mathbf{a}_z$ A/m, is located at $\rho = 2$, and a second sheet, $\mathbf{K} = -10\,\mathbf{a}_z$ A/m is located at $\rho = 4$.

a.) Let $V_m = 0$ at $P(\rho = 3, \phi = 0, z = 5)$ and place a barrier at $\phi = \pi$. Find $V_m(\rho, \phi, z)$ for $-\pi < \phi < \pi$: Since the current is cylindrically-symmetric, we know that $\mathbf{H} = I/(2\pi\rho)\,\mathbf{a}_\phi$, where I is the current enclosed, equal in this case to $2\pi(2)K = 80\pi$ A. Thus, using the result of Section 8.6, we find

$$V_m = -\frac{I}{2\pi}\,\phi = -\frac{80\pi}{2\pi}\,\phi = \underline{-40\phi \text{ A}}$$

which is valid over the region $2 < \rho < 4$, $-\pi < \phi < \pi$, and $-\infty < z < \infty$. For $\rho > 4$, the outer current contributes, leading to a total enclosed current of

$$I_{net} = 2\pi(2)(20) - 2\pi(4)(10) = 0$$

With zero enclosed current, $H_\phi = 0$, and the magnetic potential is zero as well.

b) Let $\mathbf{A} = 0$ at P and find $\mathbf{A}(\rho, \phi, z)$ for $2 < \rho < 4$: Again, we know that $\mathbf{H} = H_\phi(\rho)$, since the current is cylindrically symmetric. With the current only in the z direction, and again using symmmetry, we expect only a z component of $\mathbf{A}$ which varies only with ρ. We can then write:

$$\nabla \times \mathbf{A} = -\frac{dA_z}{d\rho}\mathbf{a}_\phi = \mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

Thus

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi\rho} \quad \Rightarrow \quad A_z = -\frac{\mu_0 I}{2\pi}\ln(\rho) + C$$

We require that $A_z = 0$ at $\rho = 3$. Therefore $C = [(\mu_0 I)/(2\pi)]\ln(3)$, Then, with $I = 80\pi$, we finally obtain

$$\mathbf{A} = -\frac{\mu_0(80\pi)}{2\pi}\left[\ln(\rho) - \ln(3)\right]\mathbf{a}_z = \underline{40\mu_0 \ln\left(\frac{3}{\rho}\right)\mathbf{a}_z \text{ Wb/m}}$$

7.36. Let $\mathbf{A} = (3y - z)\mathbf{a}_x + 2xz\mathbf{a}_y$ Wb/m in a certain region of free space.

a) Show that $\nabla \cdot \mathbf{A} = 0$:

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(3y - z) + \frac{\partial}{\partial y}2xz = \underline{0}$$

b) At $P(2, -1, 3)$, find $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{J}$: First $\mathbf{A}_P = \underline{-6\mathbf{a}_x + 12\mathbf{a}_y}$. Then, using the curl formula in cartesian coordinates,

$$\mathbf{B} = \nabla \times \mathbf{A} = -2x\mathbf{a}_x - \mathbf{a}_y + (2z - 3)\mathbf{a}_z \quad \Rightarrow \quad \mathbf{B}_P = \underline{-4\mathbf{a}_x - \mathbf{a}_y + 3\mathbf{a}_z \text{ Wb/m}^2}$$

Now

$$\mathbf{H}_P = (1/\mu_0)\mathbf{B}_P = \underline{-3.2 \times 10^6\mathbf{a}_x - 8.0 \times 10^5\mathbf{a}_y + 2.4 \times 10^6\mathbf{a}_z \text{ A/m}}$$

Then $\mathbf{J} = \nabla \times \mathbf{H} = (1/\mu_0)\nabla \times \mathbf{B} = \underline{0}$, as the curl formula in cartesian coordinates shows.

7.37. Let $N = 1000$, $I = 0.8$ A, $\rho_0 = 2$ cm, and $a = 0.8$ cm for the toroid shown in Fig. 8.12b. Find V_m in the interior of the toroid if $V_m = 0$ at $\rho = 2.5$ cm, $\phi = 0.3\pi$. Keep ϕ within the range $0 < \phi < 2\pi$: Well-within the toroid, we have

$$\mathbf{H} = \frac{NI}{2\pi\rho}\mathbf{a}_\phi = -\nabla V_m = -\frac{1}{\rho}\frac{dV_m}{d\phi}\mathbf{a}_\phi$$

Thus

$$V_m = -\frac{NI\phi}{2\pi} + C$$

Then,

$$0 = -\frac{1000(0.8)(0.3\pi)}{2\pi} + C$$

or C = 120. Finally

$$V_m = \underline{\left[120 - \frac{400}{\pi}\phi\right] \text{ A} \quad (0 < \phi < 2\pi)}$$

7.38. A square filamentary differential current loop, dL on a side, is centered at the origin in the $z = 0$ plane in free space. The current I flows generally in the $\mathbf{a}_\phi$ direction.

a) Assuming that $r >> dL$, and following a method similar to that in Sec. 4.7, show that

$$d\mathbf{A} = \frac{\mu_0 I (dL)^2 \sin\theta}{4\pi r^2} \mathbf{a}_\phi$$

We begin with the expression for the differential vector potential, Eq. (48):

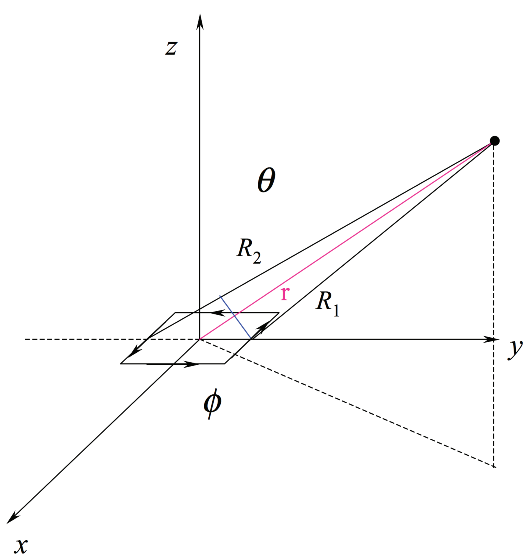
$$d\mathbf{A} = \frac{\mu_0 I d\mathbf{L}}{4\pi R}$$

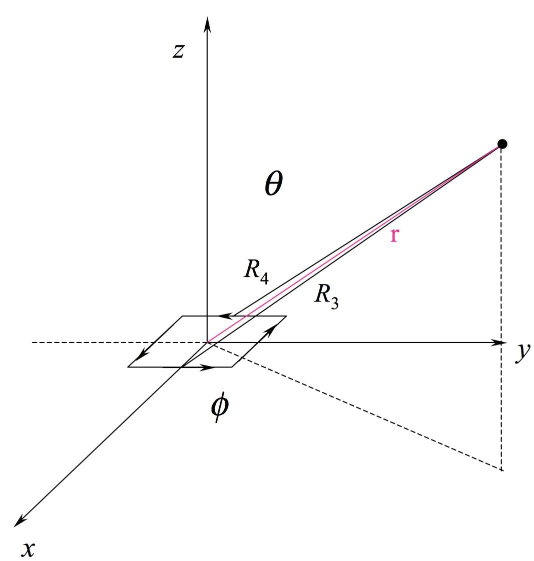
where in our case, we have four differential elements. The net vector potential at some distant point will consist of the vector sum of the four individual potentials. Referring to the figures below, the net potential at the indicated point is initially constructed as:

$$d\mathbf{A} = \frac{\mu_0 I dL}{4\pi} \left[\mathbf{a}_x \left(\frac{1}{R_2} - \frac{1}{R_1} \right) + \mathbf{a}_y \left(\frac{1}{R_3} - \frac{1}{R_4} \right) \right]$$

The challenge is to determine the four distances, R_1 through R_4, in terms of spherical coordinates, r, θ, and ϕ, thus referencing the four potentials to a common origin. This is where the treatment in Sec. 4.7 is useful, although it is more complicated here because the problem is three-dimensional.

The diagram for the y-displaced elements is shown in the left-hand figure. The three distance lines, r, R_1, and R_2, are approximately parallel because the observation point is in the far zone. Therefore, beyond the blue line segment that crosses the three lines, the lengths are essentially equal. The difference in lengths between r and R_1, for example, is the length of the line segment along r, from the origin to the blue line. This length will be the projection of the distance vector $dL/2\,\mathbf{a}_y$ along $\mathbf{a}_r$, or $dL/2\,\mathbf{a}_y \cdot \mathbf{a}_r$. As a look ahead, this principle is discussed with illustrations in Sec. 14.5, which handles antenna arrays of two elements.





7.38a) (continued). We now may write:

$$R_1 \doteq r - \left[\frac{dL}{2}\mathbf{a}_y \cdot \mathbf{a}_r\right] \qquad \text{and} \qquad R_2 \doteq r + \left[\frac{dL}{2}\mathbf{a}_y \cdot \mathbf{a}_r\right]$$

Referring to the right-hand figure and applying similar reasoning there leads to

$$R_3 \doteq r - \left[\frac{dL}{2}\mathbf{a}_x \cdot \mathbf{a}_r\right] \qquad \text{and} \qquad R_4 \doteq r + \left[\frac{dL}{2}\mathbf{a}_x \cdot \mathbf{a}_r\right]$$

where we know that $\mathbf{a}_x \cdot \mathbf{a}_r = \sin\theta\cos\phi$ and $\mathbf{a}_y \cdot \mathbf{a}_r = \sin\theta\sin\phi$ We now substitute all these relations into the original expression for $d\mathbf{A}$:

$$\begin{aligned} d\mathbf{A} = \frac{\mu_0 I dL}{4\pi}&\left[\left(\left(r + \frac{dL}{2}\sin\theta\sin\phi\right)^{-1} - \left(r - \frac{dL}{2}\sin\theta\sin\phi\right)^{-1}\right)\mathbf{a}_x\right. \\ &+ \left.\left(\left(r - \frac{dL}{2}\sin\theta\cos\phi\right)^{-1} - \left(r + \frac{dL}{2}\sin\theta\cos\phi\right)^{-1}\right)\mathbf{a}_y\right] \\ = \frac{\mu_0 I dL}{4\pi r}&\left[\left(\left(1 + \frac{dL}{2r}\sin\theta\sin\phi\right)^{-1} - \left(1 - \frac{dL}{2r}\sin\theta\sin\phi\right)^{-1}\right)\mathbf{a}_x\right. \\ &+ \left.\left(\left(1 - \frac{dL}{2r}\sin\theta\cos\phi\right)^{-1} - \left(1 + \frac{dL}{2r}\sin\theta\cos\phi\right)^{-1}\right)\mathbf{a}_y\right] \end{aligned}$$

Now, since $dL/r << 1$, this simplifies to

$$\begin{aligned} d\mathbf{A} \doteq \frac{\mu_0 I dL}{4\pi r}&\left[\left(\left(1 - \frac{dL}{2r}\sin\theta\sin\phi\right) - \left(1 + \frac{dL}{2r}\sin\theta\sin\phi\right)\right)\mathbf{a}_x\right. \\ &+ \left.\left(\left(1 + \frac{dL}{2r}\sin\theta\cos\phi\right) - \left(1 - \frac{dL}{2r}\sin\theta\cos\phi\right)\right)\mathbf{a}_y\right] \\ &= \frac{\mu_0 I (dL)^2\sin\theta}{4\pi r^2}\left[-\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y\right] \\ &= \frac{\mu_0 I (dL)^2\sin\theta}{4\pi r^2}\mathbf{a}_\phi \end{aligned}$$

b) Show that

$$d\mathbf{H} = \frac{I(dL)^2}{4\pi r^3}\left(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right)$$

Using the part a expression, we construct $d\mathbf{H} = (1/\mu_0)\nabla\times d\mathbf{A}$, where in this case, we have a ϕ-directed $d\mathbf{A}$ that varies with r and θ. The curl expression in spherical coordinates reduces to:

$$\nabla\times d\mathbf{A} = \frac{1}{r\sin\theta}\frac{\partial(dA\sin\theta)}{\partial\theta}\mathbf{a}_r - \frac{1}{r}\frac{\partial(rdA)}{\partial r}\mathbf{a}_\theta$$

or

$$\begin{aligned} \nabla\times d\mathbf{A} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{I(dL)^2\sin^2\theta}{4\pi r^2}\right)\mathbf{a}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{I(dL)^2\sin\theta}{4\pi r}\right)\mathbf{a}_\theta \\ &= \frac{I(dL)^2}{4\pi r^3}\left(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right) \end{aligned}$$

7.39. Planar current sheets of $\mathbf{K} = 30\mathbf{a}_z$ A/m and $-30\mathbf{a}_z$ A/m are located in free space at $x = 0.2$ and $x = -0.2$ respectively. For the region $-0.2 < x < 0.2$:

a) Find $\mathbf{H}$: Since we have parallel current sheets carrying equal and opposite currents, we use Eq. (12), $\mathbf{H} = \mathbf{K} \times \mathbf{a}_N$, where $\mathbf{a}_N$ is the unit normal directed into the region between currents, and where either one of the two currents are used. Choosing the sheet at $x = 0.2$, we find

$$\mathbf{H} = 30\mathbf{a}_z \times -\mathbf{a}_x = \underline{-30\mathbf{a}_y \text{ A/m}}$$

b) Obtain and expression for V_m if $V_m = 0$ at $P(0.1, 0.2, 0.3)$: Use

$$\mathbf{H} = -30\mathbf{a}_y = -\nabla V_m = -\frac{dV_m}{dy}\mathbf{a}_y$$

So

$$\frac{dV_m}{dy} = 30 \quad \Rightarrow \quad V_m = 30y + C_1$$

Then

$$0 = 30(0.2) + C_1 \quad \Rightarrow \quad C_1 = -6 \quad \Rightarrow \quad V_m = \underline{30y - 6 \text{ A}}$$

c) Find $\mathbf{B}$: $\mathbf{B} = \mu_0 \mathbf{H} = \underline{-30\mu_0 \mathbf{a}_y \text{ Wb/m}^2}$.

d) Obtain an expression for $\mathbf{A}$ if $\mathbf{A} = 0$ at P: We expect $\mathbf{A}$ to be z-directed (with the current), and so from $\nabla \times \mathbf{A} = \mathbf{B}$, where $\mathbf{B}$ is y-directed, we set up

$$-\frac{dA_z}{dx} = -30\mu_0 \quad \Rightarrow \quad A_z = 30\mu_0 x + C_2$$

Then $0 = 30\mu_0(0.1) + C_2 \quad \Rightarrow \quad C_2 = -3\mu_0$. So finally $\mathbf{A} = \underline{\mu_0(30x - 3)\mathbf{a}_z \text{ Wb/m}}$.

7.40. Show that the line integral of the vector potential $\mathbf{A}$ about any closed path is equal to the magnetic flux enclosed by the path, or $\oint \mathbf{A} \cdot d\mathbf{L} = \int \mathbf{B} \cdot d\mathbf{S}$.

We use the fact that $\mathbf{B} = \nabla \times \mathbf{A}$, and substitute this into the desired relation to find

$$\oint \mathbf{A} \cdot d\mathbf{L} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

This is just a statement of Stokes' theorem (already proved), so we are done.

7.41. Assume that $\mathbf{A} = 50\rho^2\mathbf{a}_z$ Wb/m in a certain region of free space.

a) Find **H** and **B**: Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = \underline{-100\rho\,\mathbf{a}_\phi \text{ Wb/m}^2}$$

Then $\mathbf{H} = \mathbf{B}/\mu_0 = \underline{-100\rho/\mu_0\,\mathbf{a}_\phi \text{ A/m}}$.

b) Find **J**: Use

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi)\mathbf{a}_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{-100\rho^2}{\mu_0}\right)\mathbf{a}_z = \underline{-\frac{200}{\mu_0}\mathbf{a}_z \text{ A/m}^2}$$

c) Use **J** to find the total current crossing the surface $0 \leq \rho \leq 1$, $0 \leq \phi < 2\pi$, $z = 0$: The current is

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \frac{-200}{\mu_0}\mathbf{a}_z \cdot \mathbf{a}_z\,\rho\,d\rho\,d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ MA}}$$

d) Use the value of H_ϕ at $\rho = 1$ to calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ for $\rho = 1$, $z = 0$: Have

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_0^{2\pi} \frac{-100}{\mu_0}\mathbf{a}_\phi \cdot \mathbf{a}_\phi\,(1)d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ MA}}$$

7.42. Show that $\nabla_2(1/R_{12}) = -\nabla_1(1/R_{12}) = \mathbf{R}_{21}/R_{12}^3$. First

$$\nabla_2\left(\frac{1}{R_{12}}\right) = \nabla_2\left[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\right]^{-1/2}$$

$$= -\frac{1}{2}\left[\frac{2(x_2-x_1)\mathbf{a}_x + 2(y_2-y_1)\mathbf{a}_y + 2(z_2-z_1)\mathbf{a}_z}{[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2]^{3/2}}\right] = \frac{-\mathbf{R}_{12}}{R_{12}^3} = \frac{\mathbf{R}_{21}}{R_{12}^3}$$

Also note that $\nabla_1(1/R_{12})$ would give the same result, but of opposite sign.

7.43. Compute the vector magnetic potential within the outer conductor for the coaxial line whose vector magnetic potential is shown in Fig. 8.20 if the outer radius of the outer conductor is $7a$. Select the proper zero reference and sketch the results on the figure: We do this by first finding **B** within the outer conductor and then "uncurling" the result to find **A**. With $-z$-directed current I in the outer conductor, the current density is

$$\mathbf{J}_{out} = -\frac{I}{\pi(7a)^2 - \pi(5a)^2}\mathbf{a}_z = -\frac{I}{24\pi a^2}\mathbf{a}_z$$

Since current I flows in both conductors, but in opposite directions, Ampere's circuital law inside the outer conductor gives:

$$2\pi\rho H_\phi = I - \int_0^{2\pi}\int_{5a}^{\rho} \frac{I}{24\pi a^2}\rho'\,d\rho'\,d\phi \quad \Rightarrow \quad H_\phi = \frac{I}{2\pi\rho}\left[\frac{49a^2 - \rho^2}{24a^2}\right]$$

Now, with $\mathbf{B} = \mu_0\mathbf{H}$, we note that $\nabla \times \mathbf{A}$ will have a ϕ component only, and from the direction and symmetry of the current, we expect **A** to be z-directed, and to vary only with ρ. Therefore

$$\nabla \times \mathbf{A} = -\frac{dA_z}{d\rho}\mathbf{a}_\phi = \mu_0\mathbf{H}$$

and so

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi\rho}\left[\frac{49a^2 - \rho^2}{24a^2}\right]$$

Then by direct integration,

$$A_z = \int \frac{-\mu_0 I(49)}{48\pi\rho}\,d\rho + \int \frac{\mu_0 I\rho}{48\pi a^2}\,d\rho + C = \frac{\mu_0 I}{96\pi}\left[\frac{\rho^2}{a^2} - 98\ln\rho\right] + C$$

As per Fig. 8.20, we establish a zero reference at $\rho = 5a$, enabling the evaluation of the integration constant:

$$C = -\frac{\mu_0 I}{96\pi}\left[25 - 98\ln(5a)\right]$$

Finally,

$$\underline{A_z = \frac{\mu_0 I}{96\pi}\left[\left(\frac{\rho^2}{a^2} - 25\right) + 98\ln\left(\frac{5a}{\rho}\right)\right] \text{ Wb/m}}$$

A plot of this continues the plot of Fig. 8.20, in which the curve goes negative at $\rho = 5a$, and then approaches a minimum of $-.09\mu_0 I/\pi$ at $\rho = 7a$, at which point the slope becomes zero.

7.44. By expanding Eq.(58), Sec. 8.7 in cartesian coordinates, show that (59) is correct. Eq. (58) can be rewritten as

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}$$

We begin with

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Then the x component of $\nabla(\nabla \cdot \mathbf{A})$ is

$$[\nabla(\nabla \cdot \mathbf{A})]_x = \underline{\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}}$$

Now

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z$$

and the x component of $\nabla \times \nabla \times \mathbf{A}$ is

$$[\nabla \times \nabla \times \mathbf{A}]_x = \underline{\frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial y}}$$

Then, using the underlined results

$$[\nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}]_x = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = \nabla^2 A_x$$

Similar results will be found for the other two components, leading to

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} = \nabla^2 A_x \mathbf{a}_x + \nabla^2 A_y \mathbf{a}_y + \nabla^2 A_z \mathbf{a}_z \equiv \nabla^2 \mathbf{A} \quad \text{QED}$$

CHAPTER 8

8.1. A point charge, $Q = -0.3\,\mu$C and $m = 3 \times 10^{-16}$ kg, is moving through the field $\mathbf{E} = 30\,\mathbf{a}_z$ V/m. Use Eq. (1) and Newton's laws to develop the appropriate differential equations and solve them, subject to the initial conditions at $t = 0$: $\mathbf{v} = 3 \times 10^5\,\mathbf{a}_x$ m/s at the origin. At $t = 3\,\mu$s, find:

a) the position $P(x, y, z)$ of the charge: The force on the charge is given by $\mathbf{F} = q\mathbf{E}$, and Newton's second law becomes:

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{z}}{dt^2} = q\mathbf{E} = (-0.3 \times 10^{-6})(30\,\mathbf{a}_z)$$

describing motion of the charge in the z direction. The initial velocity in x is constant, and so no force is applied in that direction. We integrate once:

$$\frac{dz}{dt} = v_z = \frac{qE}{m}t + C_1$$

The initial velocity along z, $v_z(0)$ is zero, and so $C_1 = 0$. Integrating a second time yields the z coordinate:

$$z = \frac{qE}{2m}t^2 + C_2$$

The charge lies at the origin at $t = 0$, and so $C_2 = 0$. Introducing the given values, we find

$$z = \frac{(-0.3 \times 10^{-6})(30)}{2 \times 3 \times 10^{-16}}t^2 = -1.5 \times 10^{10}t^2 \text{ m}$$

At $t = 3\ \mu$s, $z = -(1.5 \times 10^{10})(3 \times 10^{-6})^2 = -.135$ cm. Now, considering the initial constant velocity in x, the charge in 3 μs attains an x coordinate of $x = vt = (3\times10^5)(3\times10^{-6}) = .90$ m. In summary, at $t = 3\ \mu$s we have $P(x, y, z) = \underline{(.90, 0, -.135)}$.

b) the velocity, $\mathbf{v}$: After the first integration in part a, we find

$$v_z = \frac{qE}{m}t = -(3 \times 10^{10})(3 \times 10^{-6}) = -9 \times 10^4 \text{ m/s}$$

Including the intial x-directed velocity, we finally obtain $\mathbf{v} = \underline{3 \times 10^5\,\mathbf{a}_x - 9 \times 10^4\mathbf{a}_z \text{ m/s}}$.

c) the kinetic energy of the charge: Have

$$\text{K.E.} = \frac{1}{2}m|v|^2 = \frac{1}{2}(3 \times 10^{-16})(1.13 \times 10^5)^2 = \underline{1.5 \times 10^{-5}\text{ J}}$$

8.2. Compare the magnitudes of the electric and magnetic forces on an electron that has attained a velocity of 10^7 m/s. Assume an electric field intensity of 10^5 V/m, and a magnetic flux density associated with that of the Earth's magnetic field in temperate latitudes, 0.5 gauss. We use the Lorentz Law, $\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where $|\mathbf{B}| = 0.5$ G $= 5.0 \times 10^{-5}$ T. We find

$$|\mathbf{F}_e| = (1.6 \times 10^{-19}\,\text{C})(10^5\,\text{V/m}) = \underline{1.6 \times 10^{-14}\,\text{N}}$$

$$|\mathbf{F}_m| = (1.6 \times 10^{-19}\,\text{C})(10^7\,\text{m/s})(5.0 \times 10^{-5}\,\text{T}) = \underline{8.0 \times 10^{-17}\,\text{N}} = 0.005|\mathbf{F}_e|$$

8.3. A point charge for which $Q = 2 \times 10^{-16}$ C and $m = 5 \times 10^{-26}$ kg is moving in the combined fields $\mathbf{E} = 100\mathbf{a}_x - 200\mathbf{a}_y + 300\mathbf{a}_z$ V/m and $\mathbf{B} = -3\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z$ mT. If the charge velocity at $t = 0$ is $\mathbf{v}(0) = (2\mathbf{a}_x - 3\mathbf{a}_y - 4\mathbf{a}_z) \times 10^5$ m/s:

a) give the unit vector showing the direction in which the charge is accelerating at $t = 0$: Use $\mathbf{F}(t = 0) = q[\mathbf{E} + (\mathbf{v}(0) \times \mathbf{B})]$, where

$$\mathbf{v}(0) \times \mathbf{B} = (2\mathbf{a}_x - 3\mathbf{a}_y - 4\mathbf{a}_z)10^5 \times (-3\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z)10^{-3} = 1100\mathbf{a}_x + 1400\mathbf{a}_y - 500\mathbf{a}_z$$

So the force in newtons becomes

$$\mathbf{F}(0) = (2 \times 10^{-16})[(100 + 1100)\mathbf{a}_x + (1400 - 200)\mathbf{a}_y + (300 - 500)\mathbf{a}_z] = 4 \times 10^{-14}[6\mathbf{a}_x + 6\mathbf{a}_y - \mathbf{a}_z]$$

The unit vector that gives the acceleration direction is found from the force to be

$$\mathbf{a}_F = \frac{6\mathbf{a}_x + 6\mathbf{a}_y - \mathbf{a}_z}{\sqrt{73}} = \underline{.70\mathbf{a}_x + .70\mathbf{a}_y - .12\mathbf{a}_z}$$

b) find the kinetic energy of the charge at $t = 0$:

$$\text{K.E.} = \frac{1}{2}m|\mathbf{v}(0)|^2 = \frac{1}{2}(5 \times 10^{-26}\,\text{kg})(5.39 \times 10^5\,\text{m/s})^2 = 7.25 \times 10^{-15}\,\text{J} = \underline{7.25\,\text{fJ}}$$

8.4. Show that a charged particle in a uniform magnetic field describes a circular orbit with an orbital period that is independent of the radius. Find the relationship between the angular velocity and magnetic flux density for an electron (the *cyclotron frequency*).

A circular orbit can be established if the magnetic force on the particle is balanced by the centripital force associated with the circular path. We assume a circular path of radius R, in which $\mathbf{B} = B_0\,\mathbf{a}_z$ is normal to the plane of the path. Then, with particle angular velocity Ω, the velocity is $\mathbf{v} = R\Omega\,\mathbf{a}_\phi$. The magnetic force is then $\mathbf{F}_m = q\mathbf{v} \times \mathbf{B} = qR\Omega\,\mathbf{a}_\phi \times B_0\,\mathbf{a}_z = qR\Omega B_0\,\mathbf{a}_\rho$. This force will be negative (pulling the particle toward the center of the path) if the charge is positive and motion is in the $-\mathbf{a}_\phi$ direction, or if the charge is negative, and motion is in positive $\mathbf{a}_\phi$. In either case, the centripital force must counteract the magnetic force. Assuming particle mass m, the force balance equation is $qR\Omega B_0 = m\Omega^2 R$, from which $\Omega = qB_0/m$. The revolution period is $T = 2\pi/\Omega = 2\pi m/(qB_0)$, which is independent of R. For an electron, we have $q = 1.6 \times 10^{-9}$ C, and $m = 9.1 \times 10^{31}$ kg. The cyclotron frequency is therefore

$$\Omega_c = \frac{q}{m}B_0 = 1.76 \times 10^{11} B_0 \ \ \text{s}^{-1}$$

8.5. A rectangular loop of wire in free space joins points $A(1,0,1)$ to $B(3,0,1)$ to $C(3,0,4)$ to $D(1,0,4)$ to A. The wire carries a current of 6 mA, flowing in the $\mathbf{a}_z$ direction from B to C. A filamentary current of 15 A flows along the entire z axis in the $\mathbf{a}_z$ direction.

a) Find $\mathbf{F}$ on side BC:

$$\mathbf{F}_{BC} = \int_B^C I_{\text{loop}} d\mathbf{L} \times \mathbf{B}_{\text{from wire at BC}}$$

Thus

$$\mathbf{F}_{BC} = \int_1^4 (6\times10^{-3})\,dz\,\mathbf{a}_z \times \frac{15\mu_0}{2\pi(3)}\,\mathbf{a}_y = -1.8\times10^{-8}\mathbf{a}_x\ \text{N} = \underline{-18\mathbf{a}_x\ \text{nN}}$$

b) Find $\mathbf{F}$ on side AB: The field from the long wire now varies with position along the loop segment. We include that dependence and write

$$\mathbf{F}_{AB} = \int_1^3 (6\times10^{-3})\,dx\,\mathbf{a}_x \times \frac{15\mu_0}{2\pi x}\,\mathbf{a}_y = \frac{45\times10^{-3}}{\pi}\mu_0 \ln 3\ \mathbf{a}_z = \underline{19.8\mathbf{a}_z\ \text{nN}}$$

c) Find $\mathbf{F}_{\text{total}}$ on the loop: This will be the vector sum of the forces on the four sides. Note that by symmetry, the forces on sides AB and CD will be equal and opposite, and so will cancel. This leaves the sum of forces on sides BC (part a) and DA, where

$$\mathbf{F}_{DA} = \int_1^4 -(6\times10^{-3})\,dz\,\mathbf{a}_z \times \frac{15\mu_0}{2\pi(1)}\,\mathbf{a}_y = 54\mathbf{a}_x\ \text{nN}$$

The total force is then $\mathbf{F}_{\text{total}} = \mathbf{F}_{DA} + \mathbf{F}_{BC} = (54-18)\mathbf{a}_x = \underline{36\,\mathbf{a}_x\ \text{nN}}$

8.6. Show that the differential work in moving a current element $Id\mathbf{L}$ through a distance $d\mathbf{l}$ in a magnetic field $\mathbf{B}$ is the negative of that done in moving the element $Id\mathbf{l}$ through a distance $d\mathbf{L}$ in the same field: The two differential work quantities are written as:

$$dW = (Id\mathbf{L}\times\mathbf{B})\cdot d\mathbf{l} \quad \text{and} \quad dW' = (Id\mathbf{l}\times\mathbf{B})\cdot d\mathbf{L}$$

We now apply the vector identity, Eq.(A.6), Appendix A: $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = (\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}$, and write:

$$(Id\mathbf{L}\times\mathbf{B})\cdot d\mathbf{l} = (\mathbf{B}\times d\mathbf{l})\cdot Id\mathbf{L} = -(Id\mathbf{l}\times\mathbf{B})\cdot d\mathbf{L} \qquad \text{QED}$$

8.7. Uniform current sheets are located in free space as follows: $8\mathbf{a}_z$ A/m at $y=0$, $-4\mathbf{a}_z$ A/m at $y=1$, and $-4\mathbf{a}_z$ A/m at $y=-1$. Find the vector force per meter length exerted on a current filament carrying 7 mA in the $\mathbf{a}_L$ direction if the filament is located at:

a) $x=0$, $y=0.5$, and $\mathbf{a}_L = \mathbf{a}_z$: We first note that within the region $-1<y<1$, the magnetic fields from the two outer sheets (carrying $-4\mathbf{a}_z$ A/m) cancel, leaving only the field from the center sheet. Therefore, $\mathbf{H} = -4\mathbf{a}_x$ A/m $(0<y<1)$ and $\mathbf{H} = 4\mathbf{a}_x$ A/m $(-1<y<0)$. Outside ($y>1$ and $y<-1$) the fields from all three sheets cancel, leaving $\mathbf{H}=0$ ($y>1$, $y<-1$). So at $x=0$, $y=.5$, the force per meter length will be

$$\mathbf{F}/\text{m} = I\mathbf{a}_z\times\mathbf{B} = (7\times10^{-3})\mathbf{a}_z \times -4\mu_0\mathbf{a}_x = \underline{-35.2\mathbf{a}_y\ \text{nN/m}}$$

b.) $y=0.5$, $z=0$, and $\mathbf{a}_L = \mathbf{a}_x$: $\mathbf{F}/\text{m} = I\mathbf{a}_x \times -4\mu_0\mathbf{a}_x = \underline{0}$.

c) $x=0$, $y=1.5$, $\mathbf{a}_L=\mathbf{a}_z$: Since $y=1.5$, we are in the region in which $\mathbf{B}=0$, and so the force is <u>zero</u>.

8.8. Two conducting strips, having infinite length in the z direction, lie in the xz plane. One occupies the region $d/2 < x < b + d/2$ and carries surface current density $\mathbf{K} = K_0\mathbf{a}_z$; the other is situated at $-(b + d/2) < x < -d/2$ and carries surface current density $-K_0\mathbf{a}_z$.

a) Find the force per unit length in z that tends to separate the two strips:

We begin by evaluating the magnetic field arising from the left-hand strip (in the region $x < 0$) at any location on the x axis. Because the source strip is infinite in z, this field will not depend on z and will be valid at any location in the x-z plane. We use the Biot-Savart law and find the field at a fixed point x_0 on the x axis. The Biot-Savart law reads:

$$\mathbf{H}(x_0) = \int_s \frac{\mathbf{K} \times \mathbf{a}_R}{4\pi R^2}\, da$$

where the integral is taken over the left strip area, and where R is the distance from point (x, z) on the strip to the fixed observation point, x_0. Thus $R = \sqrt{(x - x_0)^2 + z^2}$, and

$$\mathbf{a}_R = \frac{-(x - x_0)\,\mathbf{a}_x - z\,\mathbf{a}_z}{\sqrt{(x - x_0)^2 + z^2}}$$

so that

$$\mathbf{H}(x_0) = \int_{-\infty}^{\infty}\int_{-(\frac{d}{2}+b)}^{-\frac{d}{2}} \frac{-K_0\,\mathbf{a}_z \times [-(x - x_0)\,\mathbf{a}_x - z\,\mathbf{a}_z]}{4\pi\left[(x - x_0)^2 + z^2\right]^{3/2}}\, dx\, dz$$

Taking the cross product leaves only a y component:

$$\mathbf{H}(x_0) = \int_{-\infty}^{\infty}\int_{-(\frac{d}{2}+b)}^{-\frac{d}{2}} \frac{K_0\,\mathbf{a}_y(x - x_0)}{4\pi\left[(x - x_0)^2 + z^2\right]^{3/2}}\, dx\, dz$$

It is easiest to evaluate the z integral first, leading to

$$\mathbf{H}_0 = \frac{K_0}{4\pi}\,\mathbf{a}_y \int_{-(\frac{d}{2}+b)}^{-\frac{d}{2}} \frac{z}{(x - x_0)\sqrt{(x - x_0)^2 + z^2}}\,\Big|_{-\infty}^{\infty} dx = \frac{K_0}{2\pi}\,\mathbf{a}_y \int_{-(\frac{d}{2}+b)}^{-\frac{d}{2}} \frac{dx}{(x - x_0)}$$

Evaluate the x integral to find:

$$\mathbf{H}_0 = \frac{K_0}{2\pi}\,\mathbf{a}_y \ln(x - x_0)\Big|_{-(\frac{d}{2}+b)}^{-\frac{d}{2}} = -\frac{K_0}{2\pi}\ln\left[\frac{\frac{d}{2} + b + x_0}{\frac{d}{2} + x_0}\right]\mathbf{a}_y \ \text{ A/m}$$

Now the force acting on the right-hand strip per unit length is

$$\mathbf{F} = \int_s \mathbf{K} \times \mathbf{B}\, da$$

where $\mathbf{K}$ is the surface current density in the right-hand strip, and $\mathbf{B}$ is the magnetic flux density ($\mu_0\mathbf{H}$) arising from the left strip, evaluated within the right strip area (over which the integral is taken). Over a unit lengh in z, the force integral is written:

$$\mathbf{F} = \int_0^1 \int_{\frac{d}{2}}^{(b+\frac{d}{2})} k_0\,\mathbf{a}_z \times \frac{-\mu_0 K_0}{2\pi}\ln\left[\frac{\frac{d}{2} + b + x_0}{\frac{d}{2} + x_0}\right]\mathbf{a}_y\ dx_0\, dz$$

8.8a (continued)

The z integration yields a factor of 1, the cross product gives an x-directed force, and we can rewrite the expression as:

$$\mathbf{F} = \frac{\mu_0 K_0^2}{2\pi}\,\mathbf{a}_x \int_{\frac{d}{2}}^{(b+\frac{d}{2})} \left[\ln\left(\frac{d}{2}+b+x_0\right) - \ln\left(\frac{d}{2}+x_0\right)\right] dx_0$$

$$= \frac{\mu_0 K_0^2}{2\pi}\,\mathbf{a}_x \left[\left(\frac{d}{2}+b+x_0\right)\ln\left(\frac{d}{2}+b+x_0\right) - x_0 - \left(\frac{d}{2}+x_0\right)\ln\left(\frac{d}{2}+x_0\right) + x_0\right]_{\frac{d}{2}}^{(b+\frac{d}{2})}$$

Evaluating this result over the integration limits and then simplifying results in the following expression, which is one of many ways of writing the result:

$$\mathbf{F} = \frac{\mu_0 d K_0^2}{2\pi}\,\mathbf{a}_x \left[\left(1+\frac{2b}{d}\right)\ln\left(\frac{1+\frac{2b}{d}}{1+\frac{b}{d}}\right) - \ln\left(1+\frac{b}{d}\right)\right]$$

b) let b approach zero while maintaining constant current, $I = K_0 b$, and show that the force per unit length approaches $\mu_0 I^2/(2\pi d)$ N/m.

As b gets small, so does the ratio b/d. We may then write:

$$\frac{1}{1+\frac{b}{d}} \doteq \left[1 - \frac{b}{d} + \left(\frac{b}{d}\right)^2\right]$$

The force expression now becomes:

$$\mathbf{F} = \frac{\mu_0 d K_0^2}{2\pi}\,\mathbf{a}_x \left[\left(1+\frac{2b}{d}\right)\ln\left[\left(1+\frac{2b}{d}\right)\left(1-\frac{b}{d}+\left(\frac{b}{d}\right)^2\right)\right] - \ln\left[1+\frac{b}{d}\right]\right]$$

The product in the natural log function is expanded:

$$\mathbf{F} = \frac{\mu_0 d K_0^2}{2\pi}\,\mathbf{a}_x \left[\left(1+\frac{2b}{d}\right)\ln\left[1-\frac{b}{d}+\frac{b^2}{d^2}+\frac{2b}{d}-\frac{2b^2}{d^2}+\frac{2b^2}{d^2}\right] - \ln\left[1+\frac{b}{d}\right]\right]$$

All terms in the natural log functions involve $1+f(b/d)$ where $f(b/d) << 1$. Therefore, we may expand the log functions in power series, keeping only the first terms: i.e., $\ln(1+f) \doteq f$, if $f << 1$. With this simplification, the force becomes:

$$\mathbf{F} \doteq \frac{\mu_0 d K_0^2}{2\pi}\,\mathbf{a}_x \left[\left(1+\frac{2b}{d}\right)\left(\frac{b}{d}-\frac{b^2}{d^2}+\frac{2b^3}{d^3}\right) - \frac{b}{d}\right]$$

Expanding and simplifying, the final result is

$$\mathbf{F} \doteq \frac{\mu_0 (bK_0)^2}{2\pi d}\,\mathbf{a}_x \text{ N/m} \qquad \left(\frac{b}{d} << 1\right)$$

where the term, $4b^4/d^4$, has been neglected.

8.9. A current of $-100\mathbf{a}_z$ A/m flows on the conducting cylinder $\rho = 5$ mm and $+500\mathbf{a}_z$ A/m is present on the conducting cylinder $\rho = 1$ mm. Find the magnitude of the total force acting to split the outer cylinder apart along its length: The differential force acting on the outer cylinder arising from the field of the inner cylinder is $d\mathbf{F} = \mathbf{K}_{\text{outer}} \times \mathbf{B}$, where $\mathbf{B}$ is the field from the inner cylinder, evaluated at the outer cylinder location:

$$\mathbf{B} = \frac{2\pi(1)(500)\mu_0}{2\pi(5)}\mathbf{a}_\phi = 100\mu_0\,\mathbf{a}_\phi \text{ T}$$

Thus $d\mathbf{F} = -100\mathbf{a}_z \times 100\mu_0\mathbf{a}_\phi = 10^4\mu_0\mathbf{a}_\rho$ N/m^2. We wish to find the force acting to split the outer cylinder, which means we need to evaluate the net force in one cartesian direction on one half of the cylinder. We choose the "upper" half $(0 < \phi < \pi)$, and integrate the y component of $d\mathbf{F}$ over this range, and over a unit length in the z direction:

$$F_y = \int_0^1 \int_0^\pi 10^4\mu_0\mathbf{a}_\rho \cdot \mathbf{a}_y(5\times 10^{-3})\,d\phi\,dz = \int_0^\pi 50\mu_0 \sin\phi\,d\phi = 100\mu_0 = \underline{4\pi \times 10^{-5} \text{ N/m}}$$

Note that we did not include the "self force" arising from the outer cylinder's $\mathbf{B}$ field on itself. Since the outer cylinder is a two-dimensional current sheet, its field exists only just outside the cylinder, and so no force exists. If this cylinder possessed a finite thickness, then we would need to include its self-force, since there would be an interior field and a volume current density that would spatially overlap.

8.10. A planar transmission line consists of two conducting planes of width b separated d m in air, carrying equal and opposite currents of I A. If $b >> d$, find the force of repulsion per meter of length between the two conductors.

Take the current in the top plate in the positive z direction, and so the bottom plate current is directed along negative z. Furthermore, the bottom plate is at $y = 0$, and the top plate is at $y = d$. The magnetic field stength at the bottom plate arising from the current in the top plate is $\mathbf{H} = K/2\,\mathbf{a}_x$ A/m, where the top plate surface current density is $\mathbf{K} = I/b\,\mathbf{a}_z$ A/m. Now the force per unit length on the bottom plate is

$$\mathbf{F} = \int_0^1 \int_0^b \mathbf{K}_b \times \mathbf{B}_b\,dS$$

where $\mathbf{K}_b$ is the surface current density on the bottom plate, and $\mathbf{B}_b$ is the magnetic flux density arising from the top plate current, evaluated at the bottom plate location. We obtain

$$\mathbf{F} = \int_0^1 \int_0^b -\frac{I}{b}\,\mathbf{a}_z \times \frac{\mu_0 I}{2b}\,\mathbf{a}_x\,dS = -\frac{\mu_0 I^2}{2b}\,\mathbf{a}_y \text{ N/m}$$

8.11. a) Use Eq. (14), Sec. 8.3, to show that the force of attraction per unit length between two filamentary conductors in free space with currents $I_1\mathbf{a}_z$ at $x = 0$, $y = d/2$, and $I_2\mathbf{a}_z$ at $x = 0$, $y = -d/2$, is $\mu_0 I_1 I_2/(2\pi d)$: The force on I_2 is given by

$$\mathbf{F}_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\oint \frac{\mathbf{a}_{R12} \times d\mathbf{L}_1}{R_{12}^2} \right] \times d\mathbf{L}_2$$

Let z_1 indicate the z coordinate along I_1, and z_2 indicate the z coordinate along I_2. We then have $R_{12} = \sqrt{(z_2 - z_1)^2 + d^2}$ and

$$\mathbf{a}_{R12} = \frac{(z_2 - z_1)\mathbf{a}_z - d\mathbf{a}_y}{\sqrt{(z_2 - z_1)^2 + d^2}}$$

Also, $d\mathbf{L}_1 = dz_1\mathbf{a}_z$ and $d\mathbf{L}_2 = dz_2\mathbf{a}_z$ The "inside" integral becomes:

$$\oint \frac{\mathbf{a}_{R12} \times d\mathbf{L}_1}{R_{12}^2} = \oint \frac{[(z_2 - z_1)\mathbf{a}_z - d\mathbf{a}_y] \times dz_1\mathbf{a}_z}{[(z_2 - z_1)^2 + d^2]^{1.5}} = \int_{-\infty}^{\infty} \frac{-d\, dz_1\, \mathbf{a}_x}{[(z_2 - z_1)^2 + d^2]^{1.5}}$$

The force expression now becomes

$$\mathbf{F}_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\int_{-\infty}^{\infty} \frac{-d\, dz_1\, \mathbf{a}_x}{[(z_2 - z_1)^2 + d^2]^{1.5}} \times dz_2\mathbf{a}_z \right] = \mu_0 \frac{I_1 I_2}{4\pi} \int_0^1 \int_{-\infty}^{\infty} \frac{d\, dz_1\, dz_2\, \mathbf{a}_y}{[(z_2 - z_1)^2 + d^2]^{1.5}}$$

Note that the "outside" integral is taken over a unit length of current I_2. Evaluating, obtain,

$$\mathbf{F}_2 = \mu_0 \frac{I_1 I_2 d\, \mathbf{a}_y}{4\pi d^2}(2) \int_0^1 dz_2 = \frac{\mu_0 I_1 I_2}{2\pi d}\mathbf{a}_y \text{ N/m}$$

as expected.

b) Show how a simpler method can be used to check your result: We use $d\mathbf{F}_2 = I_2 d\mathbf{L}_2 \times \mathbf{B}_{12}$, where the field from current 1 at the location of current 2 is

$$\mathbf{B}_{12} = \frac{\mu_0 I_1}{2\pi d}\mathbf{a}_x \text{ T}$$

so over a unit length of I_2, we obtain

$$\mathbf{F}_2 = I_2\mathbf{a}_z \times \frac{\mu_0 I_1}{2\pi d}\mathbf{a}_x = \mu_0 \frac{I_1 I_2}{2\pi d}\mathbf{a}_y \text{ N/m}$$

This second method is really just the first over again, since we recognize the inside integral of the first method as the Biot-Savart law, used to find the field from current 1 at the current 2 location.

8.12. Two circular wire rings are parallel to each other, share the same axis, are of radius a, and are separated by distance d, where $d << a$. Each ring carries current I. Find the approximate force of attraction and indicate the relative orientations of the currents.

With the loops very close to each other, the primary force on each "segment" of current in either loop can be assumed to arise from the local B field from the immediately adjacent segment in the other loop. So the problem becomes essentially that of straightening out both loops and considering them as two parallel wires of length $L = 2\pi a$. The magnetic induction at points *very* close to either current loop is approximately that of an infinite straight wire, or $B \doteq \mu_0 I/(2\pi d)$. The force is therefore found using Eq. (11) to be

$$F = ILB = I(2\pi a)\frac{\mu_0 I}{2\pi d} = \underline{\frac{\mu_0 a I^2}{d}}$$

If the force is attractive, then the currents must be in the same direction.

8.13. A current of 6A flows from $M(2,0,5)$ to $N(5,0,5)$ in a straight solid conductor in free space. An infinite current filament lies along the z axis and carries 50A in the $\mathbf{a}_z$ direction. Compute the vector torque on the wire segment using:

a) an origin at $(0,0,5)$: The $\mathbf{B}$ field from the long wire at the short wire is $\mathbf{B} = (\mu_0 I_z \mathbf{a_y})/(2\pi x)$ T. Then the force acting on a differential length of the wire segment is

$$d\mathbf{F} = I_w d\mathbf{L} \times \mathbf{B} = I_w dx\, \mathbf{a}_x \times \frac{\mu_0 I_z}{2\pi x}\mathbf{a}_y = \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z \text{ N}$$

Now the differential torque about $(0,0,5)$ will be

$$d\mathbf{T} = \mathbf{R}_T \times d\mathbf{F} = x\mathbf{a}_x \times \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z = -\frac{\mu_0 I_w I_z}{2\pi}\, dx\, \mathbf{a}_y$$

The net torque is now found by integrating the differential torque over the length of the wire segment:

$$\mathbf{T} = \int_2^5 -\frac{\mu_0 I_w I_z}{2\pi}\, dx\, \mathbf{a}_y = -\frac{3\mu_0(6)(50)}{2\pi}\, \mathbf{a}_y = \underline{-1.8 \times 10^{-4}\, \mathbf{a}_y \;\; \text{N} \cdot \text{m}}$$

b) an origin at $(0,0,0)$: Here, the only modification is in $\mathbf{R}_T$, which is now $\mathbf{R}_T = x\,\mathbf{a}_x + 5\,\mathbf{a}_z$ So now

$$d\mathbf{T} = \mathbf{R}_T \times d\mathbf{F} = [x\mathbf{a}_x + 5\mathbf{a}_z] \times \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z = -\frac{\mu_0 I_w I_z}{2\pi}\, dx\, \mathbf{a}_y$$

Everything from here is the same as in part a, so again, $\mathbf{T} = \underline{-1.8 \times 10^{-4}\, \mathbf{a}_y \;\; \text{N} \cdot \text{m}}$.

c) an origin at $(3,0,0)$: In this case, $\mathbf{R}_T = (x-3)\mathbf{a}_x + 5\mathbf{a}_z$, and the differential torque is

$$d\mathbf{T} = [(x-3)\mathbf{a}_x + 5\mathbf{a}_z] \times \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z = -\frac{\mu_0 I_w I_z (x-3)}{2\pi x}\, dx\, \mathbf{a}_y$$

Thus

$$\mathbf{T} = \int_2^5 -\frac{\mu_0 I_w I_z (x-3)}{2\pi x}\, dx\, \mathbf{a}_y = -6.0 \times 10^{-5} \left[3 - 3\ln\left(\frac{5}{2}\right)\right] \mathbf{a}_y = \underline{-1.5 \times 10^{-5}\, \mathbf{a}_y \;\; \text{N} \cdot \text{m}}$$

8.14. A solenoid is 25cm long, 3cm in diameter, and carries 4 A dc in its 400 turns. Its axis is perpendicular to a uniform magnetic field of 0.8 Wb/m^2 in air. Using an origin at the center of the solenoid, calculate the torque acting on it.

First, we consider the torque, referenced to the origin, of a single wire loop of radius a in the plane $z = z_0$. This is one loop of the solenoid. We will take the applied magnetic flux density as in the $\mathbf{a}_x$ direction and write it as $\mathbf{B} = B_0\mathbf{a}_x$. The differential torque associated with a differential current element on this loop, referenced to the origin, will be:

$$d\mathbf{T} = \mathbf{R} \times d\mathbf{F}$$

where $\mathbf{R}$ is the vector directed from the origin to the current element, and is given by $\mathbf{R} = z_0\mathbf{a}_z + a\mathbf{a}_\rho$, and where

$$d\mathbf{F} = Id\mathbf{L} \times \mathbf{B} = Ia\, d\phi\, \mathbf{a}_\phi \times B_0\, \mathbf{a}_x = -IaB_0 \cos\phi\, d\phi\, \mathbf{a}_z$$

So now

$$d\mathbf{T} = (z_0\mathbf{a}_z + a\mathbf{a}_\rho) \times (-IaB_0 \cos\phi\, d\phi\, \mathbf{a}_z) = a^2 IB_0 \cos\phi\, d\phi\, \mathbf{a}_\phi$$

Using $\mathbf{a}_\phi = \cos\phi\, \mathbf{a}_y - \sin\phi\, \mathbf{a}_x$, the differential torque becomes

$$d\mathbf{T} = a^2 IB_0 \cos\phi\, (\cos\phi\, \mathbf{a}_y - \sin\phi\, \mathbf{a}_x)\, d\phi$$

Note that there is no dependence on z_0. The net torque on the loop is now

$$\mathbf{T} = \int d\mathbf{T} = \int_0^{2\pi} a^2 IB_0 \left(\cos^2\phi\, \mathbf{a}_y - \cos\phi \sin\phi\, \mathbf{a}_x\right) d\phi = \pi a^2 IB_0\, \mathbf{a}_y$$

But we have 400 identical turns at different z locations (which don't matter), so the total torque will be just the above result times 400, or:

$$\mathbf{T} = 400\pi a^2 IB_0\, \mathbf{a}_y = 400\pi(1.5 \times 10^{-2})^2(4)(0.8)\mathbf{a}_y = \underline{0.91}\mathbf{a}_y \ \text{N} \cdot \text{m}$$

8.15. A solid conducting filament extends from $x = -b$ to $x = b$ along the line $y = 2$, $z = 0$. This filament carries a current of 3 A in the $\mathbf{a}_x$ direction. An infinite filament on the z axis carries 5 A in the $\mathbf{a}_z$ direction. Obtain an expression for the torque exerted on the finite conductor about an origin located at $(0, 2, 0)$: The differential force on the wire segment arising from the field from the infinite wire is

$$d\mathbf{F} = 3\, dx\, \mathbf{a}_x \times \frac{5\mu_0}{2\pi\rho}\, \mathbf{a}_\phi = -\frac{15\mu_0 \cos\phi\, dx}{2\pi\sqrt{x^2+4}}\, \mathbf{a}_z = -\frac{15\mu_0 x\, dx}{2\pi(x^2+4)}\, \mathbf{a}_z$$

So now the differential torque about the $(0, 2, 0)$ origin is

$$d\mathbf{T} = \mathbf{R}_T \times d\mathbf{F} = x\, \mathbf{a}_x \times -\frac{15\mu_0 x\, dx}{2\pi(x^2+4)}\, \mathbf{a}_z = \frac{15\mu_0 x^2\, dx}{2\pi(x^2+4)}\, \mathbf{a}_y$$

The torque is then

$$\mathbf{T} = \int_{-b}^{b} \frac{15\mu_0 x^2\, dx}{2\pi(x^2+4)}\, \mathbf{a}_y = \frac{15\mu_0}{2\pi}\, \mathbf{a}_y \left[x - 2\tan^{-1}\left(\frac{x}{2}\right)\right]_{-b}^{b}$$

$$= \underline{(6 \times 10^{-6}) \left[b - 2\tan^{-1}\left(\frac{b}{2}\right)\right] \mathbf{a}_y \ \ \text{N} \cdot \text{m}}$$

8.16. Assume that an electron is describing a circular orbit of radius a about a positively-charged nucleus.

a) By selecting an appropriate current and area, show that the equivalent orbital dipole moment is $ea^2\omega/2$, where ω is the electron's angular velocity: The current magnitude will be $I = \frac{e}{T}$, where e is the electron charge and T is the orbital period. The latter is $T = 2\pi/\omega$, and so $I = e\omega/(2\pi)$. Now the dipole moment magnitude will be $m = IA$, where A is the loop area. Thus

$$m = \frac{e\omega}{2\pi}\,\pi a^2 = \frac{1}{2}ea^2\omega \quad //$$

b) Show that the torque produced by a magnetic field parallel to the plane of the orbit is $ea^2\omega B/2$: With B assumed constant over the loop area, we would have $\mathbf{T} = \mathbf{m} \times \mathbf{B}$. With $\mathbf{B}$ parallel to the loop plane, $\mathbf{m}$ and $\mathbf{B}$ are orthogonal, and so $T = mB$. So, using part a, $T = ea^2\omega B/2$.

c) by equating the Coulomb and centrifugal forces, show that ω is $(4\pi\epsilon_0 m_e a^3/e^2)^{-1/2}$, where m_e is the electron mass: The force balance is written as

$$\frac{e^2}{4\pi\epsilon_0 a^2} = m^e\omega^2 a \quad \Rightarrow \quad \omega = \left(\frac{4\pi\epsilon_0 m_e a^3}{e^2}\right)^{-1/2} \qquad //$$

d) Find values for the angular velocity, torque, and the orbital magnetic moment for a hydrogen atom, where a is about 6×10^{-11} m; let $B = 0.5$ T: First

$$\omega = \left[\frac{(1.60 \times 10^{-19})^2}{4\pi(8.85 \times 10^{-12})(9.1 \times 10^{-31})(6 \times 10^{-11})^3}\right]^{1/2} = \underline{3.42 \times 10^{16} \text{ rad/s}}$$

$$T = \frac{1}{2}(3.42 \times 10^{16})(1.60 \times 10^{-19})(0.5)(6 \times 10^{-11})^2 = \underline{4.93 \times 10^{-24} \text{ N} \cdot \text{m}}$$

Finally,

$$m = \frac{T}{B} = \underline{9.86 \times 10^{-24} \text{ A} \cdot \text{m}^2}$$

8.17. The hydrogen atom described in Problem 16 is now subjected to a magnetic field having the same direction as that of the atom. Show that the forces caused by B result in a decrease of the angular velocity by $eB/(2m_e)$ and a decrease in the orbital moment by $e^2a^2B/(4m_e)$. What are these decreases for the hydrogen atom in parts per million for an external magnetic flux density of 0.5 T? We first write down all forces on the electron, in which we equate its coulomb force toward the nucleus to the sum of the centrifugal force and the force associated with the applied B field. With the field applied in the same direction as that of the atom, this would yield a Lorentz force that is radially outward – in the same direction as the centrifugal force.

$$F_e = F_{cent} + F_B \quad \Rightarrow \quad \frac{e^2}{4\pi\epsilon_0 a^2} = m_e\omega^2 a + \underbrace{e\omega aB}_{QvB}$$

With $B = 0$, we solve for ω to find:

$$\omega = \omega_0 = \sqrt{\frac{e^2}{4\pi\epsilon_0 m_e a^3}}$$

Then with B present, we find

$$\omega^2 = \frac{e^2}{4\pi\epsilon_0 m_e a^3} - \frac{e\omega B}{m_e} = \omega_0^2 - \frac{e\omega B}{m_e}$$

Therefore

$$\omega = \omega_0\sqrt{1 - \frac{e\omega B}{\omega_0^2 m_e}} \doteq \omega_0\left(1 - \frac{e\omega B}{2\omega_0^2 m_e}\right)$$

But $\omega \doteq \omega_0$, and so

$$\omega \doteq \omega_0\left(1 - \frac{eB}{2\omega_0 m_e}\right) = \omega_0 - \frac{eB}{2m_e} \quad //$$

As for the magnetic moment, we have

$$m = IS = \frac{e\omega}{2\pi}\pi a^2 = \frac{1}{2}\omega e a^2 \doteq \frac{1}{2}ea^2\left(\omega_0 - \frac{eB}{2m_e}\right) = \frac{1}{2}\omega_0 e a^2 - \frac{1}{4}\frac{e^2a^2B}{m_e} \quad //$$

Finally, for $a = 6 \times 10^{-11}$ m, $B = 0.5$ T, we have

$$\frac{\Delta\omega}{\omega} = \frac{eB}{2m_e}\frac{1}{\omega} \doteq \frac{eB}{2m_e}\frac{1}{\omega_0} = \frac{1.60 \times 10^{-19} \times 0.5}{2 \times 9.1 \times 10^{-31} \times 3.4 \times 10^{16}} = \underline{1.3 \times 10^{-6}}$$

where $\omega_0 = 3.4 \times 10^{16}$ sec^{-1} is found from Problem 16. Finally,

$$\frac{\Delta m}{m} = \frac{e^2a^2B}{4m_e} \times \frac{2}{\omega e a^2} \doteq \frac{eB}{2m_e\omega_0} = \underline{1.3 \times 10^{-6}}$$

8.18. Calculate the vector torque on the square loop shown in Fig. 8.16 about an origin at A in the field $\mathbf{B}$, given:

a) $A(0,0,0)$ and $\mathbf{B} = 100\mathbf{a}_y$ mT: The field is uniform and so does not produce any translation of the loop. Therefore, we may use $\mathbf{T} = I\mathbf{S} \times \mathbf{B}$ about any origin, where $I = 0.6$ A and $\mathbf{S} = 16\mathbf{a}_z$ m^2. We find $\mathbf{T} = 0.6(16)\mathbf{a}_z \times 0.100\mathbf{a}_y = \underline{-0.96\,\mathbf{a}_x \text{ N}-\text{m}}$.

b) $A(0,0,0)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y$ mT: Using the same reasoning as in part a, we find

$$\mathbf{T} = 0.6(16)\mathbf{a}_z \times (0.200\mathbf{a}_x + 0.100\mathbf{a}_y) = \underline{-0.96\mathbf{a}_x + 1.92\mathbf{a}_y \text{ N}-\text{m}}$$

c) $A(1,2,3)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y - 300\mathbf{a}_z$ mT: We observe two things here: 1) The field is again uniform and so again the torque is independent of the origin chosen, and 2) The field differs from that of part b only by the addition of a z component. With $\mathbf{S}$ in the z direction, this new component of $\mathbf{B}$ will produce no torque, so the answer is the *same as part b*, or $\mathbf{T} = \underline{-0.96\mathbf{a}_x + 1.92\mathbf{a}_y \text{ N}-\text{m}}$.

d) $A(1,2,3)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y - 300\mathbf{a}_z$ mT for $x \geq 2$ and $\mathbf{B} = 0$ elsewhere: Now, force is acting only on the y-directed segment at $x = +2$, so we need to be careful, since translation will occur. So we must use the given origin. The differential torque acting on the differential wire segment at location (2,y) is $d\mathbf{T} = \mathbf{R}(y) \times d\mathbf{F}$, where

$$d\mathbf{F} = Id\mathbf{L} \times \mathbf{B} = 0.6\,dy\,\mathbf{a}_y \times [0.2\mathbf{a}_x + 0.1\mathbf{a}_y - 0.3\mathbf{a}_z] = [-0.18\mathbf{a}_x - 0.12\mathbf{a}_z]\,dy$$

and $\mathbf{R}(y) = (2,y,0) - (1,2,3) = \mathbf{a}_x + (y-2)\mathbf{a}_y - 3\mathbf{a}_z$. We thus find

$$\begin{aligned} d\mathbf{T} &= \mathbf{R}(y) \times d\mathbf{F} = [\mathbf{a}_x + (y-2)\mathbf{a}_y - 3\mathbf{a}_z] \times [-0.18\mathbf{a}_x - 0.12\mathbf{a}_z]\,dy \\ &= [-0.12(y-2)\mathbf{a}_x + 0.66\mathbf{a}_y + 0.18(y-2)\mathbf{a}_z]\;dy \end{aligned}$$

The net torque is now

$$\mathbf{T} = \int_{-2}^{2} [-0.12(y-2)\mathbf{a}_x + 0.66\mathbf{a}_y + 0.18(y-2)\mathbf{a}_z]\;dy = \underline{0.96\mathbf{a}_x + 2.64\mathbf{a}_y - 1.44\mathbf{a}_z \text{ N}-\text{m}}$$

8.19. Given a material for which $\chi_m = 3.1$ and within which $\mathbf{B} = 0.4y\mathbf{a}_z$ T, find:

a) $\mathbf{H}$: We use $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H}$, or

$$\mathbf{H} = \frac{0.4y\mathbf{a}_y}{(1+3.1)\mu_0} = \underline{77.6y\mathbf{a}_z \text{ kA/m}}$$

b) $\mu = (1+3.1)\mu_0 = \underline{5.15 \times 10^{-6} \text{ H/m}}$.

c) $\mu_r = (1+3.1) = \underline{4.1}$.

d) $\mathbf{M} = \chi_m\mathbf{H} = (3.1)(77.6y\mathbf{a}_y) = \underline{241y\mathbf{a}_z \text{ kA/m}}$

e) $\mathbf{J} = \nabla \times \mathbf{H} = (dH_z)/(dy)\,\mathbf{a}_x = \underline{77.6\,\mathbf{a}_x \text{ kA/m}^2}$.

f) $\mathbf{J}_b = \nabla \times \mathbf{M} = (dM_z)/(dy)\,\mathbf{a}_x = \underline{241\,\mathbf{a}_x \text{ kA/m}^2}$.

g) $\mathbf{J}_T = \nabla \times \mathbf{B}/\mu_0 = \underline{318\mathbf{a}_x \text{ kA/m}^2}$.

8.20. Find **H** in a material where:

a) $\mu_r = 4.2$, there are 2.7×10^{29} atoms/m^3, and each atom has a dipole moment of $2.6 \times 10^{-30}\,\mathbf{a}_y$ A $\cdot$ m^2. Since all dipoles are identical, we may write $\mathbf{M} = N\mathbf{m} = (2.7 \times 10^{29})(2.6 \times 10^{-30}\mathbf{a}_y) = 0.70\mathbf{a}_y$ A/m. Then

$$\mathbf{H} = \frac{\mathbf{M}}{\mu_r - 1} = \frac{0.70\,\mathbf{a}_y}{4.2 - 1} = \underline{0.22\,\mathbf{a}_y \text{ A/m}}$$

b) $\mathbf{M} = 270\,\mathbf{a}_z$ A/m and $\mu = 2\ \mu$H/m: Have $\mu_r = \mu/\mu_0 = (2 \times 10^{-6})/(4\pi \times 10^{-7}) = 1.59$. Then $\mathbf{H} = 270\mathbf{a}_z/(1.59 - 1) = \underline{456\,\mathbf{a}_z \text{ A/m}}$.

c) $\chi_m = 0.7$ and $\mathbf{B} = 2\mathbf{a}_z$ T: Use

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0(1 + \chi_m)} = \frac{2\mathbf{a}_z}{(4\pi \times 10^{-7})(1.7)} = \underline{936\,\mathbf{a}_z \text{ kA/m}}$$

d) Find **M** in a material where bound surface current densities of $12\,\mathbf{a}_z$ A/m and $-9\,\mathbf{a}_z$ A/m exist at $\rho = 0.3$ m and $\rho = 0.4$ m, respectively: We use $\oint \mathbf{M} \cdot d\mathbf{L} = I_b$, where, since currents are in the z direction and are symmetric about the z axis, we chose the path integrals to be circular loops centered on and normal to z. From the symmetry, **M** will be ϕ-directed and will vary only with radius. Note first that for $\rho < 0.3$ m, no bound current will be enclosed by a path integral, so we conclude that $\mathbf{M} = \underline{0 \text{ for } \rho < 0.3\text{m}}$. At radii between the currents the path integral will enclose only the inner current so,

$$\oint \mathbf{M} \cdot d\mathbf{L} = 2\pi\rho M_\phi = 2\pi(0.3)12 \ \Rightarrow\ \mathbf{M} = \underline{\frac{3.6}{\rho}\,\mathbf{a}_\phi \text{ A/m} \ \ (0.3 < \rho < 0.4\text{m})}$$

Finally, for $\rho > 0.4$ m, the total enclosed bound current is $I_{b,tot} = 2\pi(0.3)(12) - 2\pi(0.4)(9) = 0$, so therefore $\mathbf{M} = \underline{0 \ (\rho > 0.4\text{m})}$.

8.21. Find the magnitude of the magnetization in a material for which:

a) the magnetic flux density is 0.02 Wb/m^2 *and the magnetic susceptibility is 0.003* (note that this latter quantity is missing in the original problem statement): From $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ and from $\mathbf{M} = \chi_m\mathbf{H}$, we write

$$M = \frac{B}{\mu_0}\left(\frac{1}{\chi_m} + 1\right)^{-1} = \frac{B}{\mu_0(334)} = \frac{0.02}{(4\pi \times 10^{-7})(334)} = \underline{47.7 \text{ A/m}}$$

b) the magnetic field intensity is 1200 A/m and the relative permeability is 1.005: From $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0\mu_r\mathbf{H}$, we write

$$M = (\mu_r - 1)H = (.005)(1200) = \underline{6.0 \text{ A/m}}$$

c) there are 7.2×10^{28} atoms per cubic meter, each having a dipole moment of 4×10^{-30} A $\cdot$ m^2 in the same direction, and the magnetic susceptibility is 0.0003: With all dipoles identical the dipole moment density becomes

$$M = n\,m = (7.2 \times 10^{28})(4 \times 10^{-30}) = \underline{0.288 \text{ A/m}}$$

8.22. Under some conditions, it is possible to approximate the effects of ferromagnetic materials by assuming linearity in the relationship of **B** and **H**. Let $\mu_r = 1000$ for a certain material of which a cylindrical wire of radius 1mm is made. If $I = 1$ A and the current distribution is uniform, find

a) **B**: We apply Ampere's circuital law to a circular path of radius ρ around the wire axis, and where $\rho < a$:

$$2\pi\rho H = \frac{\pi\rho^2}{\pi a^2} I \quad \Rightarrow \quad H = \frac{I\rho}{2\pi a^2} \quad \Rightarrow \mathbf{B} = \frac{1000\mu_0 I\rho}{2\pi a^2}\,\mathbf{a}_\phi = \frac{(10^3)4\pi \times 10^{-7}(1)\rho}{2\pi \times 10^{-6}}\,\mathbf{a}_\phi$$
$$= 200\rho\,\mathbf{a}_\phi \text{ Wb/m}^2$$

b) **H**: Using part a, $\mathbf{H} = \mathbf{B}/\mu_r\mu_0 = \underline{\rho/(2\pi) \times 10^6\,\mathbf{a}_\phi \text{ A/m}}$.

c) **M**:

$$\mathbf{M} = \mathbf{B}/\mu_0 - \mathbf{H} = \frac{(2000 - 2)\rho}{4\pi} \times 10^6\,\mathbf{a}_\phi = \underline{1.59 \times 10^8 \rho\,\mathbf{a}_\phi \text{ A/m}}$$

d) **J**:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d(\rho H_\phi)}{d\rho}\,\mathbf{a}_z = \underline{3.18 \times 10^5\,\mathbf{a}_z \text{ A/m}}$$

e) $\mathbf{J}_b$ within the wire:

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \frac{1}{\rho}\frac{d(\rho M_\phi)}{d\rho}\,\mathbf{a}_z = \underline{3.18 \times 10^8\,\mathbf{a}_z \text{ A/m}^2}$$

8.23. Calculate values for H_ϕ, B_ϕ, and M_ϕ at $\rho = c$ for a coaxial cable with $a = 2.5$ mm and $b = 6$ mm if it carries current $I = 12$ A in the center conductor, and $\mu = 3\ \mu$H/m for $2.5 < \rho < 3.5$ mm, $\mu = 5\ \mu$H/m for $3.5 < \rho < 4.5$ mm, and $\mu = 10\ \mu$H/m for $4.5 < \rho < 6$ mm. Compute for:

a) $c = 3$ mm: Have

$$H_\phi = \frac{I}{2\pi\rho} = \frac{12}{2\pi(3 \times 10^{-3})} = \underline{637 \text{ A/m}}$$

Then $B_\phi = \mu H_\phi = (3 \times 10^{-6})(637) = \underline{1.91 \times 10^{-3} \text{ Wb/m}^2}$.
Finally, $M_\phi = (1/\mu_0)B_\phi - H_\phi = \underline{884 \text{ A/m}}$.

b. $c = 4$ mm: Have

$$H_\phi = \frac{I}{2\pi\rho} = \frac{12}{2\pi(4 \times 10^{-3})} = \underline{478 \text{ A/m}}$$

Then $B_\phi = \mu H_\phi = (5 \times 10^{-6})(478) = \underline{2.39 \times 10^{-3} \text{ Wb/m}^2}$.

Finally, $M_\phi = (1/\mu_0)B_\phi - H_\phi = \underline{1.42 \times 10^3 \text{ A/m}}$.

c) $c = 5$ mm: Have

$$H_\phi = \frac{I}{2\pi\rho} = \frac{12}{2\pi(5 \times 10^{-3})} = \underline{382 \text{ A/m}}$$

Then $B_\phi = \mu H_\phi = (10 \times 10^{-6})(382) = \underline{3.82 \times 10^{-3} \text{ Wb/m}^2}$.
Finally, $M_\phi = (1/\mu_0)B_\phi - H_\phi = \underline{2.66 \times 10^3 \text{ A/m}}$.

8.24. Two current sheets, $K_0\mathbf{a}_y$ A/m at $z = 0$ and $-K_0\mathbf{a}_y$ A/m at $z = d$, are separated by an inhomogeneous material for which $\mu_r = az + 1$, where a is a constant.

a) Find expressions for **H** and **B** in the material: The z variation in the permeability leaves the **H** field unaffected, and so we may find this using Ampere's circuital law. This is done in Chapter 7, culminating in Eq. (12) there. Applying this to the conductor in the $z = 0$ plane, we find

$$\mathbf{H} = \mathbf{K} \times \mathbf{a}_n = K_0\,\mathbf{a}_y \times \mathbf{a}_z = \underline{K_0\,\mathbf{a}_x} \text{ A/m}$$

b) find the total flux that crosses a 1m^2 area on the yz plane: Because the permeability varies with z, the flux will depend on the location and dimensions of the 1m^2 area. Choose a rectangle located in the range $0 < y < y_1$, and $z_1 < z < z_2$, where we require that $(z_2 - z_1)y_1 = 1$. Therefore, $y_1 = 1/(z_2 - z_1)$. The flux through this area is now

$$\Phi_m = \int_s \mu\mathbf{H} \cdot d\mathbf{S} = \int_{z_1}^{z_2} \int_0^{1/(z_2-z_1)} \mu_0 K_0 (az+1)\,\mathbf{a}_x \cdot \mathbf{a}_x\,dy\,dz = \frac{\mu_0 K_0}{(z_2 - z_1)} \int_{z_1}^{z_2} (az+1)\,dz$$
$$= \frac{\mu_0 K_0}{(z_2 - z_1)} \left[\frac{a}{2}(z_2^2 - z_1^2) + (z_2 - z_1)\right] = \underline{\mu_0 K_0 \left[\frac{a}{2}(z_2 + z_1) + 1\right]} \text{ Wb/m}^2$$

8.25. A conducting filament at $z = 0$ carries 12 A in the $\mathbf{a}_z$ direction. Let $\mu_r = 1$ for $\rho < 1$ cm, $\mu_r = 6$ for $1 < \rho < 2$ cm, and $\mu_r = 1$ for $\rho > 2$ cm. Find

a) **H** everywhere: This result will depend on the current and not the materials, and is:

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \underline{\frac{1.91}{\rho} \text{ A/m} \quad (0 < \rho < \infty)}$$

b) **B** everywhere: We use $\mathbf{B} = \mu_r\mu_0\mathbf{H}$ to find:

$\mathbf{B}(\rho < 1\,\text{cm}) = (1)\mu_0(1.91/\rho) = \underline{(2.4 \times 10^{-6}/\rho)\mathbf{a}_\phi \text{ T}}$
$\mathbf{B}(1 < \rho < 2\,\text{cm}) = (6)\mu_0(1.91/\rho) = \underline{(1.4 \times 10^{-5}/\rho)\mathbf{a}_\phi \text{ T}}$
$\mathbf{B}(\rho > 2\,\text{cm}) = (1)\mu_0(1.91/\rho) = \underline{(2.4 \times 10^{-6}/\rho)\mathbf{a}_\phi \text{ T}}$ where ρ is in meters.

8.26. A long solenoid has a radius of 3cm, 5,000 turns/m, and carries current $I = 0.25$ A. The region $0 < \rho < a$ within the solenoid has $\mu_r = 5$, while $\mu_r = 1$ for $a < \rho < 3$ cm. Determine a so that

a) a total flux of 10 μWb is present: First, the magnetic flux density in the coil is written in general as $\mathbf{B} = \mu n I\,\mathbf{a}_z$ Wb/m^2. Using $b = 0.03$m as the outer radius, the total flux in the coil becomes

$$\Phi_m = \int_s \mathbf{B} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^a 5\mu_0 n I \rho\,d\rho\,d\phi + \int_0^{2\pi} \int_a^b \mu_0 n I\,\rho\,d\rho\,d\phi$$
$$= \mu_0 n I \left[5a^2 + (b^2 - a^2)\right] = \mu_0 n I \left[4a^2 - b^2\right]$$

Substituting the given numbers, we have

$$\Phi_m = (4\pi \times 10^{-7})(5000)(0.25)\left[4a^2 - 0.03^2\right] = 10^{-5} \text{ Wb (as required)}$$

Solve for a to find $a = \underline{2.7}$ cm.

b) Find a so that the flux is equally-divided between the regions $0 < \rho < a$ and $a < \rho < 3$ cm: Using the expression for the flux in part a, we set

$$5a^2 = b^2 - a^2 \;\Rightarrow\; a = \frac{b}{\sqrt{6}} = \frac{3}{\sqrt{6}} = \underline{1.22} \text{ cm}$$

8.27. Let $\mu_{r1} = 2$ in region 1, defined by $2x+3y-4z > 1$, while $\mu_{r2} = 5$ in region 2 where $2x+3y-4z < 1$. In region 1, $\mathbf{H}_1 = 50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z$ A/m. Find:

a) $\mathbf{H}_{N1}$ (normal component of $\mathbf{H}_1$ at the boundary): We first need a unit vector normal to the surface, found through

$$\mathbf{a}_N = \frac{\nabla(2x+3y-4z)}{|\nabla(2x+3y-4z)|} = \frac{2\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z}{\sqrt{29}} = .37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z$$

Since this vector is found through the gradient, it will point in the direction of increasing values of $2x+3y-4z$, and so will be directed into region 1. Thus we write $\mathbf{a}_N = \mathbf{a}_{N21}$. The normal component of $\mathbf{H}_1$ will now be:

$$\begin{aligned}\mathbf{H}_{N1} &= (\mathbf{H}_1 \cdot \mathbf{a}_{N21})\mathbf{a}_{N21} \\ &= [(50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z)\cdot(.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z)](.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z) \\ &= \underline{-4.83\mathbf{a}_x - 7.24\mathbf{a}_y + 9.66\mathbf{a}_z \text{ A/m}}\end{aligned}$$

b) $\mathbf{H}_{T1}$ (tangential component of $\mathbf{H}_1$ at the boundary):

$$\begin{aligned}\mathbf{H}_{T1} &= \mathbf{H}_1 - \mathbf{H}_{N1} \\ &= (50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z) - (-4.83\mathbf{a}_x - 7.24\mathbf{a}_y + 9.66\mathbf{a}_z) \\ &= \underline{54.83\mathbf{a}_x - 22.76\mathbf{a}_y + 10.34\mathbf{a}_z \text{ A/m}}\end{aligned}$$

c) $\mathbf{H}_{T2}$ (tangential component of $\mathbf{H}_2$ at the boundary): Since tangential components of $\mathbf{H}$ are continuous across a boundary between two media of different permeabilities, we have

$$\mathbf{H}_{T2} = \mathbf{H}_{T1} = \underline{54.83\mathbf{a}_x - 22.76\mathbf{a}_y + 10.34\mathbf{a}_z \text{ A/m}}$$

d) $\mathbf{H}_{N2}$ (normal component of $\mathbf{H}_2$ at the boundary): Since normal components of $\mathbf{B}$ are continuous across a boundary between media of different permeabilities, we write $\mu_1\mathbf{H}_{N1} = \mu_2\mathbf{H}_{N2}$ or

$$\mathbf{H}_{N2} = \frac{\mu_{r1}}{\mu_R 2}\mathbf{H}_{N1} = \frac{2}{5}(-4.83\mathbf{a}_x - 7.24\mathbf{a}_y + 9.66\mathbf{a}_z) = \underline{-1.93\mathbf{a}_x - 2.90\mathbf{a}_y + 3.86\mathbf{a}_z \text{ A/m}}$$

e) θ_1, the angle between $\mathbf{H}_1$ and $\mathbf{a}_{N21}$: This will be

$$\cos\theta_1 = \frac{\mathbf{H}_1}{|\mathbf{H}_1|}\cdot\mathbf{a}_{N21} = \left[\frac{50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z}{(50^2+30^2+20^2)^{1/2}}\right]\cdot(.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z) = -0.21$$

Therefore $\theta_1 = \cos^{-1}(-.21) = \underline{102°}$.

f) θ_2, the angle between $\mathbf{H}_2$ and $\mathbf{a}_{N21}$: First,

$$\begin{aligned}\mathbf{H}_2 = \mathbf{H}_{T2} + \mathbf{H}_{N2} &= (54.83\mathbf{a}_x - 22.76\mathbf{a}_y + 10.34\mathbf{a}_z) + (-1.93\mathbf{a}_x - 2.90\mathbf{a}_y + 3.86\mathbf{a}_z) \\ &= 52.90\mathbf{a}_x - 25.66\mathbf{a}_y + 14.20\mathbf{a}_z \text{ A/m}\end{aligned}$$

$$\cos\theta_2 = \frac{\mathbf{H}_2}{|\mathbf{H}_2|}\cdot\mathbf{a}_{N21} = \left[\frac{52.90\mathbf{a}_x - 25.66\mathbf{a}_y + 14.20\mathbf{a}_z}{60.49}\right]\cdot(.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z) = -0.09$$

Therefore $\theta_2 = \cos^{-1}(-.09) = \underline{95°}$.

8.28. For values of B below the knee on the magnetization curve for silicon steel, approximate the curve by a straight line with $\mu = 5$ mH/m. The core shown in Fig. 8.17 has areas of 1.6 cm^2 and lengths of 10 cm in each outer leg, and an area of 2.5 cm^2 and a length of 3 cm in the central leg. A coil of 1200 turns carrying 12 mA is placed around the central leg. Find B in the:

a) center leg: We use $mmf = \Phi R$, where, in the central leg,

$$R_c = \frac{L_{in}}{\mu A_{in}} = \frac{3 \times 10^{-2}}{(5 \times 10^{-3})(2.5 \times 10^{-4})} = 2.4 \times 10^4 \text{ H}$$

In each outer leg, the reluctance is

$$R_o = \frac{L_{out}}{\mu A_{out}} = \frac{10 \times 10^{-2}}{(5 \times 10^{-3})(1.6 \times 10^{-4})} = 1.25 \times 10^5 \text{ H}$$

The magnetic circuit is formed by the center leg in series with the parallel combination of the two outer legs. The total reluctance seen at the coil location is $R_T = R_c + (1/2)R_o = 8.65 \times 10^4$ H. We now have

$$\Phi = \frac{mmf}{R_T} = \frac{14.4}{8.65 \times 10^4} = 1.66 \times 10^{-4} \text{ Wb}$$

The flux density in the center leg is now

$$B = \frac{\Phi}{A} = \frac{1.66 \times 10^{-4}}{2.5 \times 10^{-4}} = \underline{0.666 \text{ T}}$$

b) center leg, if a 0.3-mm air gap is present in the center leg: The air gap reluctance adds to the total reluctance already calculated, where

$$R_{air} = \frac{0.3 \times 10^{-3}}{(4\pi \times 10^{-7})(2.5 \times 10^{-4})} = 9.55 \times 10^5 \text{ H}$$

Now the total reluctance is $R_{net} = R_T + R_{air} = 8.56 \times 10^4 + 9.55 \times 10^5 = 1.04 \times 10^6$. The flux in the center leg is now

$$\Phi = \frac{14.4}{1.04 \times 10^6} = 1.38 \times 10^{-5} \text{ Wb}$$

and

$$B = \frac{1.38 \times 10^{-5}}{2.5 \times 10^{-4}} = \underline{55.3 \text{ mT}}$$

8.29. In Problem 8.28, the linear approximation suggested in the statement of the problem leads to a flux density of 0.666 T in the center leg. Using this value of B and the magnetization curve for silicon steel, what current is required in the 1200-turn coil? With $B = 0.666$ T, we read $H_{in} \doteq 120$ A $\cdot$ t/m in Fig. 8.11. The flux in the center leg is $\Phi = 0.666(2.5 \times 10^{-4}) = 1.66 \times 10^{-4}$ Wb. This divides equally in the two outer legs, so that the flux density in each outer leg is

$$B_{out} = \left(\frac{1}{2}\right) \frac{1.66 \times 10^{-4}}{1.6 \times 10^{-4}} = 0.52 \text{ Wb/m}^2$$

Using Fig. 8.11 with this result, we find $H_{out} \doteq 90$ A $\cdot$ t/m We now use

$$\oint \mathbf{H} \cdot d\mathbf{L} = NI$$

to find

$$I = \frac{1}{N}\left(H_{in}L_{in} + H_{out}L_{out}\right) = \frac{(120)(3 \times 10^{-2}) + (90)(10 \times 10^{-2})}{1200} = \underline{10.5 \text{ mA}}$$

8.30. A rectangular core has fixed permeability $\mu_r >> 1$, a square cross-section of dimensions $a \times a$, and has centerline dimensions around its perimeter of b and d. Coils 1 and 2, having turn numbers N_1 and N_2, are wound on the core. Consider a selected core cross-sectional plane as lying within the xy plane, such that the surface is defined by $0 < x < a$, $0 < y < a$.

a) With current I_1 in coil 1, use Ampere's circuital law to find the magnetic flux density as a function of position over the core cross-section: Along the midline of the core (at which $x = d/2$), the path integral for $\mathbf{H}$ in Ampere's law becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = (2b + 2d)H$$

At all other points in the core interior, but off the midline, the path integral becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = [2(d + a - 2x) + 2(b + a - 2x)]\, H = I_{encl} = N_1 I_1$$

The flux density magnitudes are therefore

$$B_{11} = B_{12} = \mu H = \frac{\mu_r \mu_0 N_1 I_1}{2(d + b + 2a - 4x)}$$

in which we are assuming no y variation.

b) Integrate your result of part a to determine the total magnetic flux within the core: This will be the integral of B over the core cross-section:

$$\Phi_m = \int_s \mathbf{B} \cdot d\mathbf{S} = \int_0^a \int_0^a \frac{\mu_r \mu_0 N_1 I_1}{2(d + b + 2a - 4x)}\, dx\, dy = -\frac{1}{8} \mu_r \mu_0 N_1 I_1 a \ln\left[d + b + 2a - 4x\right] \Big|_0^a$$
$$= \frac{1}{8} \mu_r \mu_0 N_1 I_1 a \ln\left[\frac{d + b + 2a}{d + b - 2a}\right] \text{ Wb}$$

c) Find the self-inductance of coil 1:

$$L_{11} = \frac{N_1 B_{11}}{I_1} = \frac{1}{8} \mu_r \mu_0 N_1^2 a \ln\left[\frac{d + b + 2a}{d + b - 2a}\right] \text{ H}$$

d) find the mutual inductance between coils 1 and 2.

$$M_{12} = M = \frac{N_2 B_{12}}{I_1} = \frac{1}{8} \mu_r \mu_0 N_1 N_2 a \ln\left[\frac{d + b + 2a}{d + b - 2a}\right] \text{ H}$$

8.31. A toroid is constructed of a magnetic material having a cross-sectional area of 2.5 cm^2 and an effective length of 8 cm. There is also a short air gap 0.25 mm length and an effective area of 2.8 cm^2. An mmf of 200 A $\cdot$ t is applied to the magnetic circuit. Calculate the total flux in the toroid if:

a) the magnetic material is assumed to have infinite permeability: In this case the core reluctance, $R_c = l/(\mu A)$, is zero, leaving only the gap reluctance. This is

$$R_g = \frac{d}{\mu_0 A_g} = \frac{0.25 \times 10^{-3}}{(4\pi \times 10^{-7})(2.5 \times 10^{-4})} = 7.1 \times 10^5 \text{ H}$$

Now

$$\Phi = \frac{mmf}{\mathcal{R}_g} = \frac{200}{7.1 \times 10^5} = \underline{2.8 \times 10^{-4} \text{ Wb}}$$

b) the magnetic material is assumed to be linear with $\mu_r = 1000$: Now the core reluctance is no longer zero, but

$$R_c = \frac{8 \times 10^{-2}}{(1000)(4\pi \times 10^{-7})(2.5 \times 10^{-4})} = 2.6 \times 10^5 \text{ H}$$

The flux is then

$$\Phi = \frac{mmf}{R_c + R_g} = \frac{200}{9.7 \times 10^5} = \underline{2.1 \times 10^{-4} \text{ Wb}}$$

c) the magnetic material is silicon steel: In this case we use the magnetization curve, Fig. 8.11, and employ an iterative process to arrive at the final answer. We can begin with the value of Φ found in part a, assuming infinite permeability: $\Phi^{(1)} = 2.8 \times 10^{-4}$ Wb. The flux density in the core is then $B_c^{(1)} = (2.8 \times 10^{-4})/(2.5 \times 10^{-4}) = 1.1$ Wb/m^2. From Fig. 8.11, this corresponds to magnetic field strength $H_c^{(1)} \doteq 270$ A/m. We check this by applying Ampere's circuital law to the magnetic circuit:

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_c^{(1)} L_c + H_g^{(1)} d$$

where $H_c^{(1)} L_c = (270)(8 \times 10^{-2}) = 22$, and where $H_g^{(1)} d = \Phi^{(1)} \mathcal{R}_g = (2.8 \times 10^{-4})(7.1 \times 10^5) = 199$. But we require that

$$\oint \mathbf{H} \cdot d\mathbf{L} = 200 \text{ A} \cdot \text{t}$$

whereas the actual result in this first calculation is $199 + 22 = 221$, which is too high. So, for a second trial, we reduce B to $B_c^{(2)} = 1$ Wb/m^2. This yields $H_c^{(2)} = 200$ A/m from Fig. 8.11, and thus $\Phi^{(2)} = 2.5 \times 10^{-4}$ Wb. Now

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_c^{(2)} L_c + \Phi^{(2)} R_g = 200(8 \times 10^{-2}) + (2.5 \times 10^{-4})(7.1 \times 10^5) = 194$$

This is less than 200, meaning that the actual flux is $\underline{\text{slightly higher than } 2.5 \times 10^{-4} \text{ Wb}}$. I will leave the answer at that, considering the lack of fine resolution in Fig. 8.11.

8.32. a) Find an expression for the magnetic energy stored per unit length in a coaxial transmission line consisting of conducting sleeves of negligible thickness, having radii a and b. A medium of relative permeability μ_r fills the region between conductors. Assume current I flows in both conductors, in opposite directions.

Within the coax, the magnetic field is $\mathbf{H} = I/(2\pi\rho)\,\mathbf{a}_\phi$. The energy density is then

$$w_m = \frac{1}{2}\mathbf{B}\cdot\mathbf{H} = \frac{\mu_r\mu_0 I^2}{8\pi^2\rho^2} \text{ J/m}^3$$

The energy per unit length in z is therefore

$$W_m = \int_v w_m\,dv = \int_0^1\int_0^{2\pi}\int_a^b \frac{\mu_r\mu_0 I^2}{8\pi^2\rho^2}\,\rho\,d\rho\,d\phi\,dz = \frac{\mu_r\mu_0 I^2}{4\pi}\ln\left(\frac{b}{a}\right) \text{ J/m}$$

b) Obtain the inductance, L, per unit length of line by equating the energy to $(1/2)LI^2$.

$$L = \frac{2W_m}{I^2} = \frac{\mu_r\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \text{ H/m}$$

8.33. A toroidal core has a square cross section, 2.5 cm $< \rho <$ 3.5 cm, -0.5 cm $< z <$ 0.5 cm. The upper half of the toroid, $0 < z < 0.5$ cm, is constructed of a linear material for which $\mu_r = 10$, while the lower half, -0.5 cm $< z < 0$, has $\mu_r = 20$. An mmf of 150 A $\cdot$ t establishes a flux in the $\mathbf{a}_\phi$ direction. For $z > 0$, find:

a) $H_\phi(\rho)$: Ampere's circuital law gives:

$$2\pi\rho H_\phi = NI = 150 \quad\Rightarrow\quad H_\phi = \frac{150}{2\pi\rho} = \underline{23.9/\rho \text{ A/m}}$$

b) $B_\phi(\rho)$: We use $B_\phi = \mu_r\mu_0 H_\phi = (10)(4\pi\times10^{-7})(23.9/\rho) = \underline{3.0\times10^{-4}/\rho \text{ Wb/m}^2}$.

c) $\Phi_{z>0}$: This will be

$$\Phi_{z>0} = \int\int \mathbf{B}\cdot d\mathbf{S} = \int_0^{.005}\int_{.025}^{.035} \frac{3.0\times10^{-4}}{\rho}d\rho dz = (.005)(3.0\times10^{-4})\ln\left(\frac{.035}{.025}\right)$$
$$= \underline{5.0\times10^{-7} \text{ Wb}}$$

d) Repeat for $z < 0$: First, the magnetic field strength will be the same as in part a, since the calculation is material-independent. Thus $H_\phi = \underline{23.9/\rho \text{ A/m}}$. Next, B_ϕ is modified only by the new permeability, which is twice the value used in part a: Thus $B_\phi = \underline{6.0\times10^{-4}/\rho \text{ Wb/m}^2}$. Finally, since B_ϕ is twice that of part a, the flux will be increased by the same factor, since the area of integration for $z < 0$ is the same. Thus $\Phi_{z<0} = \underline{1.0\times10^{-6} \text{ Wb}}$.

e) Find Φ_{total}: This will be the sum of the values found for $z < 0$ and $z > 0$, or $\Phi_{\text{total}} = \underline{1.5\times10^{-6} \text{ Wb}}$.

8.34. Determine the energy stored per unit length in the internal magnetic field of an infinitely-long straight wire of radius a, carrying uniform current I.

We begin with $\mathbf{H} = I\rho/(2\pi a^2)\,\mathbf{a}_\phi$, and find the integral of the energy density over the unit length in z:

$$W_e = \int_{vol} \frac{1}{2}\mu_0 H^2\, dv = \int_0^1 \int_0^{2\pi} \int_0^a \frac{\mu_0 \rho^2 I^2}{8\pi^2 a^4} \rho\, d\rho\, d\phi\, dz = \frac{\mu_0 I^2}{16\pi} \text{ J/m}$$

8.35. The cones $\theta = 21°$ and $\theta = 159°$ are conducting surfaces and carry total currents of 40 A, as shown in Fig. 8.18. The currents return on a spherical conducting surface of 0.25 m radius.

a) Find $\mathbf{H}$ in the region $0 < r < 0.25$, $21° < \theta < 159°$, $0 < \phi < 2\pi$: We can apply Ampere's circuital law and take advantage of symmetry. We expect to see $\mathbf{H}$ in the $\mathbf{a}_\phi$ direction and it would be constant at a given distance from the z axis. We thus perform the line integral of $\mathbf{H}$ over a circle, centered on the z axis, and parallel to the xy plane:

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \mathbf{a}_\phi \cdot r\sin\theta \mathbf{a}_\phi\, d\phi = I_{encl.} = 40 \text{ A}$$

Assuming that H_ϕ is constant over the integration path, we take it outside the integral and solve:

$$H_\phi = \frac{40}{2\pi r\sin\theta} \quad\Rightarrow\quad \mathbf{H} = \underline{\frac{20}{\pi r\sin\theta}\mathbf{a}_\phi \text{ A/m}}$$

b) How much energy is stored in this region? This will be

$$W_H = \int_v \frac{1}{2}\mu_0 H_\phi^2 = \int_0^{2\pi}\int_{21°}^{159°}\int_0^{.25} \frac{200\mu_0}{\pi^2 r^2 \sin^2\theta} r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{100\mu_0}{\pi}\int_{21°}^{159°} \frac{d\theta}{\sin\theta}$$
$$= \frac{100\mu_0}{\pi} \ln\left[\frac{\tan(159/2)}{\tan(21/2)}\right] = \underline{1.35\times 10^{-4} \text{ J}}$$

8.36. The dimensions of the outer conductor of a coaxial cable are b and c, where $c > b$. Assuming $\mu = \mu_0$, find the magnetic energy stored per unit length in the region $b < \rho < c$ for a uniformly-distributed total current I flowing in opposite directions in the inner and outer conductors.

We first need to find the magnetic field inside the outer conductor volume. Ampere's circuital law is applied to a circular path of radius ρ, where $b < \rho < c$. This encloses the entire center conductor current (assumed in the positive z direction), plus that part of the $-z$-directed outer conductor current that lies inside ρ. We obtain:

$$2\pi\rho H = I - I\left[\frac{\rho^2 - b^2}{c^2 - b^2}\right] = I\left[\frac{c^2-\rho^2}{c^2-b^2}\right]$$

So that

$$\mathbf{H} = \frac{I}{2\pi\rho}\left[\frac{c^2-\rho^2}{c^2-b^2}\right]\mathbf{a}_\phi \text{ A/m} \quad (b<\rho<c)$$

The energy within the outer conductor is now

$$W_m = \int_{vol}\frac{1}{2}\mu_0 H^2\, dv = \int_0^1\int_0^{2\pi}\int_b^c \frac{\mu_0 I^2}{8\pi^2(c^2-b^2)^2}\left[\frac{c^2}{\rho^2} - 2c^2 + \rho^2\right]\rho\, d\rho\, d\phi, dz$$
$$= \frac{\mu_0 I^2}{4\pi(1-b^2/c^2)^2}\left[\ln(c/b) - (1-b^2/c^2) + \frac{1}{4}(1-b^4/c^4)\right] \text{ J}$$

8.37. Find the inductance of the cone-sphere configuration described in Problem 8.35 and Fig. 8.18. The inductance is that offered at the origin between the vertices of the cone: From Problem 8.35, the magnetic flux density is $B_\phi = 20\mu_0/(\pi r \sin\theta)$. We integrate this over the crossectional area defined by $0 < r < 0.25$ and $21^\circ < \theta < 159^\circ$, to find the total flux:

$$\Phi = \int_{21^\circ}^{159^\circ} \int_0^{0.25} \frac{20\mu_0}{\pi r \sin\theta} r\, dr\, d\theta = \frac{5\mu_0}{\pi} \ln\left[\frac{\tan(159/2)}{\tan(21/2)}\right] = \frac{5\mu_0}{\pi}(3.37) = 6.74 \times 10^{-6} \text{ Wb}$$

Now $L = \Phi/I = 6.74 \times 10^{-6}/40 = \underline{0.17\ \mu\text{H}}$.
Second method: Use the energy computation of Problem 8.35, and write

$$L = \frac{2W_H}{I^2} = \frac{2(1.35 \times 10^{-4})}{(40)^2} = \underline{0.17\ \mu\text{H}}$$

8.38. A toroidal core has a rectangular cross section defined by the surfaces $\rho = 2$ cm, $\rho = 3$ cm, $z = 4$ cm, and $z = 4.5$ cm. The core material has a relative permeability of 80. If the core is wound with a coil containing 8000 turns of wire, find its inductance: First we apply Ampere's circuital law to a circular loop of radius ρ in the interior of the toroid, and in the $\mathbf{a}_\phi$ direction.

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = NI \quad \Rightarrow \quad H_\phi = \frac{NI}{2\pi\rho}$$

The flux in the toroid is then the integral over the cross section of $\mathbf{B}$:

$$\Phi = \int\int \mathbf{B} \cdot d\mathbf{L} = \int_{.04}^{.045} \int_{.02}^{.03} \frac{\mu_r \mu_0 NI}{2\pi\rho}\, d\rho\, dz = (.005)\frac{\mu_r \mu_0 NI}{2\pi} \ln\left(\frac{.03}{.02}\right)$$

The flux linkage is then given by $N\Phi$, and the inductance is

$$L = \frac{N\Phi}{I} = \frac{(.005)(80)(4\pi \times 10^{-7})(8000)^2}{2\pi} \ln(1.5) = \underline{2.08\ \text{H}}$$

8.39. Conducting planes in air at $z = 0$ and $z = d$ carry surface currents of $\pm K_0 \mathbf{a}_x$ A/m.

a) Find the energy stored in the magnetic field per unit length ($0 < x < 1$) in a width w ($0 < y < w$): First, assuming current flows in the $+\mathbf{a}_x$ direction in the sheet at $z = d$, and in $-\mathbf{a}_x$ in the sheet at $z = 0$, we find that both currents together yield $\mathbf{H} = K_0 \mathbf{a}_y$ for $0 < z < d$ and zero elsewhere. The stored energy within the specified volume will be:

$$W_H = \int_v \frac{1}{2}\mu_0 H^2 dv = \int_0^d \int_0^w \int_0^1 \frac{1}{2}\mu_0 K_0^2 \, dx \, dy \, dz = \underline{\frac{1}{2} w d \mu_0 K_0^2 \text{ J/m}}$$

b) Calculate the inductance per unit length of this transmission line from $W_H = (1/2)LI^2$, where I is the total current in a width w in either conductor: We have $I = wK_0$, and so

$$L = \frac{2}{I^2}\frac{wd}{2}\mu_0 K_0^2 = \frac{2}{w^2 K_0^2}\frac{dw}{2}\mu_0 K_0^2 = \underline{\frac{\mu_0 d}{w} \text{ H/m}}$$

c) Calculate the total flux passing through the rectangle $0 < x < 1$, $0 < z < d$, in the plane $y = 0$, and from this result again find the inductance per unit length:

$$\Phi = \int_0^d \int_0^1 \mu_0 H \mathbf{a}_y \cdot \mathbf{a}_y \, dx \, dz = \int_0^d \int_0^1 \mu_0 K_0 dx \, dy = \mu_0 d K_0$$

Then

$$L = \frac{\Phi}{I} = \frac{\mu_0 d K_0}{w K_0} = \underline{\frac{\mu_0 d}{w} \text{ H/m}}$$

8.40. A coaxial cable has conductor radii a and b, where $a < b$. Material of permeability $\mu_r \neq 1$ exists in the region $a < \rho < c$, while the region $c < \rho < b$ is air-filled. Find an expression for the inductance per unit length.

In both regions, the magnetic field will be $\mathbf{H} = I/(2\pi\rho)\,\mathbf{a}_\phi$ A/m. So the flux per unit length between conductors will be the sum of the fluxes in both regions. We integrate over a plane surface of constant ϕ, unit length in z, and between radii a and b:

$$\Phi_m = \int_s \mathbf{B}\cdot d\mathbf{S} = \int_0^1 \int_a^c \frac{\mu_r \mu_0 I}{2\pi\rho} \, d\rho \, dz + \int_0^1 \int_c^b \frac{\mu_0 I}{2\pi\rho} \, d\rho \, dz = \frac{\mu_0 I}{2\pi}\left[\mu_r \ln\left(\frac{c}{a}\right) + \ln\left(\frac{b}{c}\right)\right] \text{ Wb/m}$$

The inductance per unit length is then $L = \Phi_m / I$ (with one turn), or

$$L = \frac{\mu_0}{2\pi}\left[\mu_r \ln\left(\frac{c}{a}\right) + \ln\left(\frac{b}{c}\right)\right] \text{ H/m}$$

8.41. A rectangular coil is composed of 150 turns of a filamentary conductor. Find the mutual inductance in free space between this coil and an infinite straight filament on the z axis if the four corners of the coil are located at

a) (0,1,0), (0,3,0), (0,3,1), and (0,1,1): In this case the coil lies in the yz plane. If we assume that the filament current is in the $+\mathbf{a}_z$ direction, then the $\mathbf{B}$ field from the filament penetrates the coil in the $-\mathbf{a}_x$ direction (normal to the loop plane). The flux through the loop will thus be

$$\Phi = \int_0^1 \int_1^3 \frac{-\mu_0 I}{2\pi y} \mathbf{a}_x \cdot (-\mathbf{a}_x)\, dy\, dz = \frac{\mu_0 I}{2\pi} \ln 3$$

The mutual inductance is then

$$M = \frac{N\Phi}{I} = \frac{150\mu_0}{2\pi} \ln 3 = \underline{33\ \mu\text{H}}$$

b) (1,1,0), (1,3,0), (1,3,1), and (1,1,1): Now the coil lies in the $x = 1$ plane, and the field from the filament penetrates in a direction that is not normal to the plane of the coil. We write the $\mathbf{B}$ field from the filament at the coil location as

$$\mathbf{B} = \frac{\mu_0 I \mathbf{a}_\phi}{2\pi\sqrt{y^2+1}}$$

The flux through the coil is now

$$\Phi = \int_0^1 \int_1^3 \frac{\mu_0 I \mathbf{a}_\phi}{2\pi\sqrt{y^2+1}} \cdot (-\mathbf{a}_x)\, dy\, dz = \int_0^1 \int_1^3 \frac{\mu_0 I \sin\phi}{2\pi\sqrt{y^2+1}}\, dy\, dz$$
$$= \int_0^1 \int_1^3 \frac{\mu_0 I y}{2\pi(y^2+1)}\, dy\, dz = \frac{\mu_0 I}{2\pi} \ln(y^2+1)\Big|_1^3 = (1.6 \times 10^{-7})I$$

The mutual inductance is then

$$M = \frac{N\Phi}{I} = (150)(1.6 \times 10^{-7}) = \underline{24\ \mu\text{H}}$$

8.42. Find the mutual inductance between two filaments forming circular rings of radii a and Δa, where $\Delta a << a$. The field should be determined by approximate methods. The rings are coplanar and concentric.

We use the result of Problem 8.4, which asks for the magnetic field at the origin, arising from a circular current loop of radius a. That solution is reproduced below: Using the Biot-Savart law, we have $Id\mathbf{L} = Iad\pi\, \mathbf{a}_\phi$, $R = a$, and $\mathbf{a}_R = -\mathbf{a}_\rho$. The field at the center of the circle is then

$$\mathbf{H}_{circ} = \int_0^{2\pi} \frac{Iad\phi\, \mathbf{a}_\phi \times (-\mathbf{a}_\rho)}{4\pi a^2} = \int_0^{2\pi} \frac{Id\phi\, \mathbf{a}_z}{4\pi a} = \frac{I}{2a} \mathbf{a}_z \text{ A/m}$$

We now approximate that field as constant over a circular area of radius Δa, and write the flux linkage (for the single turn) as

$$\Phi_m \doteq \pi(\Delta a)^2 B_{outer} = \frac{\mu_0 I \pi(\Delta a)^2}{2a} \quad \Rightarrow \quad M = \frac{\Phi_m}{I} = \frac{\mu_0 \pi(\Delta a)^2}{2a}$$

8.43. a) Use energy relationships to show that the internal inductance of a nonmagnetic cylindrical wire of radius a carrying a uniformly-distributed current I is $\mu_0/(8\pi)$ H/m. We first find the magnetic field inside the conductor, then calculate the energy stored there. From Ampere's circuital law:

$$2\pi\rho H_\phi = \frac{\pi\rho^2}{\pi a^2} I \quad \Rightarrow \quad H_\phi = \frac{I\rho}{2\pi a^2} \text{ A/m}$$

Now

$$W_H = \int_v \frac{1}{2}\mu_0 H_\phi^2 \, dv = \int_0^1 \int_0^{2\pi} \int_0^a \frac{\mu_0 I^2 \rho^2}{8\pi^2 a^4} \rho \, d\rho \, d\phi \, dz = \frac{\mu_0 I^2}{16\pi} \text{ J/m}$$

Now, with $W_H = (1/2)LI^2$, we find $L_{int} = \mu_0/(8\pi)$ as expected.

b) Find the internal inductance if the portion of the conductor for which $\rho < c < a$ is removed: The hollowed-out conductor still carries current I, so Ampere's circuital law now reads:

$$2\pi\rho H_\phi = \frac{\pi(\rho^2 - c^2)}{\pi(a^2 - c^2)} \quad \Rightarrow \quad H_\phi = \frac{I}{2\pi\rho}\left[\frac{\rho^2 - c^2}{a^2 - c^2}\right] \text{ A/m}$$

and the energy is now

$$\begin{aligned} W_H &= \int_0^1 \int_0^{2\pi} \int_c^a \frac{\mu_0 I^2 (\rho^2 - c^2)^2}{8\pi^2 \rho^2 (a^2 - c^2)^2} \rho \, d\rho \, d\phi \, dz = \frac{\mu_0 I^2}{4\pi(a^2 - c^2)^2} \int_c^a \left[\rho^3 - 2c^2\rho + \frac{C^4}{\rho}\right] d\rho \\ &= \frac{\mu_0 I^2}{4\pi(a^2 - c^2)^2}\left[\frac{1}{4}(a^4 - c^4) - c^2(a^2 - c^2) + c^4 \ln\left(\frac{a}{c}\right)\right] \text{ J/m} \end{aligned}$$

The internal inductance is then

$$L_{int} = \frac{2W_H}{I^2} = \underline{\frac{\mu_0}{8\pi}\left[\frac{a^4 - 4a^2c^2 + 3c^4 + 4c^4 \ln(a/c)}{(a^2 - c^2)^2}\right] \text{ H/m}}$$

8.44. Show that the external inductance per unit length of a two-wire transmission line carrying equal and opposite currents is approximately $(\mu/\pi)\ln(d/a)$ H/m, where a is the radius of each wire and d is the center-to-center wire spacing. On what basis is the approximation valid?

Suppose that one line is positioned along the z axis, with the other in the x-z plane at $x = d$. With equal and opposite currents, I, and with the z axis wire current in the positive z direction, the magnetic flux density in the x-z plane (arising from both currents) will be

$$\mathbf{B}(x) = \frac{\mu I}{\pi x} \mathbf{a}_y \text{ Wb/m}^2 \quad (a < x < d - a)$$

The flux per unit length in z between conductors is now:

$$\Phi_m = \int_s \mathbf{B} \cdot d\mathbf{S} = \int_0^1 \int_a^{(d-a)} \frac{\mu I}{\pi x} \mathbf{a}_y \cdot \mathbf{a}_y \, dx \, dz = \frac{\mu I}{\pi} \ln\left(\frac{d-a}{a}\right)$$

Now, the external inductance will be $L = \Phi_m/I$, which becomes

$$L \doteq \frac{\mu}{\pi} \ln\left(\frac{d}{a}\right) \text{ H/m}$$

under the assumption that $a << d$.

CHAPTER 9

9.1. In Fig. 9.4, let $B = 0.2\cos 120\pi t$ T, and assume that the conductor joining the two ends of the resistor is perfect. It may be assumed that the magnetic field produced by $I(t)$ is negligible. Find:

a) $V_{ab}(t)$: Since B is constant over the loop area, the flux is $\Phi = \pi(0.15)^2 B = 1.41 \times 10^{-2}\cos 120\pi t$ Wb. Now, $emf = V_{ba}(t) = -d\Phi/dt = (120\pi)(1.41 \times 10^{-2})\sin 120\pi t$. Then $V_{ab}(t) = -V_{ba}(t) = \underline{-5.33\sin 120\pi t \text{ V}}$.

b) $I(t) = V_{ba}(t)/R = 5.33\sin(120\pi t)/250 = \underline{21.3\sin(120\pi t) \text{ mA}}$

9.2. In the example described by Fig. 9.1, replace the constant magnetic flux density by the time-varying quantity $\mathbf{B} = B_0 \sin\omega t\ \mathbf{a}_z$. Assume that $\mathbf{v}$ is constant and that the displacement y of the bar is zero at $t = 0$. Find the emf at any time, t.

The magnetic flux through the loop area is

$$\Phi_m = \int_s \mathbf{B}\cdot d\mathbf{S} = \int_0^{vt}\int_0^d B_0 \sin\omega t\ (\mathbf{a}_z \cdot \mathbf{a}_z)\, dx\, dy = B_0\, v\, t\, d \sin\omega t$$

Then the emf is

$$emf = \oint \mathbf{E}\cdot d\mathbf{L} = -\frac{d\Phi_m}{dt} = \underline{-B_0\, d\, v\ [\sin\omega t\ +\ \omega t\cos\omega t]}\ \text{V}$$

9.3. Given $\mathbf{H} = 300\,\mathbf{a}_z\ \cos(3\times 10^8 t - y)$ A/m in free space, find the emf developed in the general $\mathbf{a}_\phi$ direction about the closed path having corners at

a) (0,0,0), (1,0,0), (1,1,0), and (0,1,0): The magnetic flux will be:

$$\begin{aligned}\Phi &= \int_0^1\int_0^1 300\mu_0 \cos(3\times 10^8 t - y)\, dx\, dy = 300\mu_0 \sin(3\times 10^8 t - y)\big|_0^1 \\ &= 300\mu_0 \left[\sin(3\times 10^8 t - 1) - \sin(3\times 10^8 t)\right]\ \text{Wb}\end{aligned}$$

Then

$$\begin{aligned}\text{emf} &= -\frac{d\Phi}{dt} = -300(3\times 10^8)(4\pi\times 10^{-7})\left[\cos(3\times 10^8 t - 1) - \cos(3\times 10^8 t)\right] \\ &= \underline{-1.13\times 10^5\left[\cos(3\times 10^8 t - 1) - \cos(3\times 10^8 t)\right]\ \text{V}}\end{aligned}$$

b) corners at (0,0,0), (2π,0,0), (2π,2π,0), (0,2π,0): In this case, the flux is

$$\Phi = 2\pi \times 300\mu_0 \sin(3\times 10^8 t - y)\big|_0^{2\pi} = 0$$

The emf is therefore $\underline{0}$.

9.4. A rectangular loop of wire containing a high-resistance voltmeter has corners initially at $(a/2, b/2, 0)$, $(-a/2, b/2, 0)$, $(-a/2, -b/2, 0)$, and $(a/2, -b/2, 0)$. The loop begins to rotate about the x axis at constant angular velocity ω, with the first-named corner moving in the $\mathbf{a}_z$ direction at $t = 0$. Assume a uniform magnetic flux density $\mathbf{B} = B_0\mathbf{a}_z$. Determine the induced emf in the rotating loop and specify the direction of the current.

The magnetic flux though the loop is found (as usual) through

$$\Phi_m = \int_s \mathbf{B} \cdot d\mathbf{S}, \text{ where } \mathbf{S} = \mathbf{n}\, da$$

Because the loop is rotating, the direction of the normal, $\mathbf{n}$, changing, and is in this case given by

$$\mathbf{n} = \cos\omega t\, \mathbf{a}_z - \sin\omega t\, \mathbf{a}_y$$

Therefore,

$$\Phi_m = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} B_0\mathbf{a}_z \cdot (\cos\omega t\, \mathbf{a}_z - \sin\omega t\, \mathbf{a}_y)\; dx\, dy = abB_0\cos\omega t$$

The integral is taken over the entire loop area (regardless of its immediate orientation). The important result is that the component of $\mathbf{B}$ that is normal to the loop area is varying sinusoidally, and so it is fine to think of the $\mathbf{B}$ field itself rotating about the x axis in the opposite direction while the loop is stationary. Now the emf is

$$emf = \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d\Phi_m}{dt} = \underline{ab\,\omega B_0 \sin\omega t} \text{ V}$$

The direction of the current is the same as the direction of $\mathbf{E}$ in the emf expression. It is easiest to picture this by considering the $\mathbf{B}$ field rotating and the loop fixed. By convention, $d\mathbf{L}$ will be counter-clockwise when looking down on the loop from the upper half-space (in the opposite direction of the normal vector to the plane). The current will be counter-clockwise whenever the emf is positive, and will be clockwise whenever the emf is negative.

9.5. The location of the sliding bar in Fig. 9.5 is given by $x = 5t + 2t^3$, and the separation of the two rails is 20 cm. Let $\mathbf{B} = 0.8x^2\mathbf{a}_z$ T. Find the voltmeter reading at:

a) $t = 0.4$ s: The flux through the loop will be

$$\Phi = \int_0^{0.2}\int_0^{x} 0.8(x')^2\, dx'\, dy = \frac{0.16}{3}x^3 = \frac{0.16}{3}(5t + 2t^3)^3 \text{ Wb}$$

Then

$$\text{emf} = -\frac{d\Phi}{dt} = \frac{0.16}{3}(3)(5t+2t^3)^2(5+6t^2) = -(0.16)[5(.4)+2(.4)^3]^2[5+6(.4)^2] = \underline{-4.32 \text{ V}}$$

b) $x = 0.6$ m: Have $0.6 = 5t + 2t^3$, from which we find $t = 0.1193$. Thus

$$\text{emf} = -(0.16)[5(.1193) + 2(.1193)^3]^2[5 + 6(.1193)^2] = \underline{-.293 \text{ V}}$$

9.6. Let the wire loop of Problem 9.4 be stationary in its $t = 0$ position and find the induced emf that results from a magnetic flux density given by $\mathbf{B}(y,t) = B_0 \cos(\omega t - \beta y)\,\mathbf{a}_z$, where ω and β are constants.

We begin by finding the net magnetic flux through the loop:

$$\Phi_m = \int_s \mathbf{B} \cdot d\mathbf{S} = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} B_0 \cos(\omega t - \beta y)\,\mathbf{a}_z \cdot \mathbf{a}_z\, dx\, dy$$
$$= \frac{B_0 a}{\beta}\left[\sin(\omega t + \beta b/2) - \sin(\omega t - \beta b/2)\right]$$

Now the emf is

$$emf = \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d\Phi_m}{dt} = -\frac{B_0 a \omega}{\beta}\left[\cos(\omega t + \beta b/2) - \cos(\omega t - \beta b/2)\right]$$

Using the trig identity, $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$, we may write the above result as

$$\underline{emf = +2B_0\, a\, \frac{\omega}{\beta} \sin(\omega t)\sin(\beta b/2)\ \text{V}}$$

9.7. The rails in Fig. 9.7 each have a resistance of 2.2 Ω/m. The bar moves to the right at a constant speed of 9 m/s in a uniform magnetic field of 0.8 T. Find $I(t)$, $0 < t < 1$ s, if the bar is at $x = 2$ m at $t = 0$ and

a) a 0.3 Ω resistor is present across the left end with the right end open-circuited: The flux in the left-hand closed loop is

$$\Phi_l = B \times \text{area} = (0.8)(0.2)(2 + 9t)$$

Then, $\text{emf}_l = -d\Phi_l/dt = -(0.16)(9) = -1.44$ V. With the bar in motion, the loop resistance is increasing with time, and is given by $R_l(t) = 0.3 + 2[2.2(2 + 9t)]$. The current is now

$$I_l(t) = \frac{\text{emf}_l}{R_l(t)} = \underline{\frac{-1.44}{9.1 + 39.6t}\ \text{A}}$$

Note that the sign of the current indicates that it is flowing in the direction opposite that shown in the figure.

b) Repeat part *a*, but with a resistor of 0.3 Ω across each end: In this case, there will be a contribution to the current from the right loop, which is now closed. The flux in the right loop, whose area decreases with time, is

$$\Phi_r = (0.8)(0.2)[(16 - 2) - 9t]$$

and $\text{emf}_r = -d\Phi_r/dt = (0.16)(9) = 1.44$ V. The resistance of the right loop is $R_r(t) = 0.3 + 2[2.2(14 - 9t)]$, and so the contribution to the current from the right loop will be

$$I_r(t) = \frac{-1.44}{61.9 - 39.6t}\ \text{A}$$

The minus sign has been inserted because again the current must flow in the opposite direction as that indicated in the figure, with the flux decreasing with time. The total current is found by adding the part *a* result, or

$$\underline{I_T(t) = -1.44\left[\frac{1}{61.9 - 39.6t} + \frac{1}{9.1 + 39.6t}\right]\ \text{A}}$$

9.8. A perfectly-conducting filament is formed into a circular ring of radius a. At one point a resistance R is inserted into the circuit, and at another a battery of voltage V_0 is inserted. Assume that the loop current itself produces negligible magnetic field.

a) Apply Faraday's law, Eq. (4), evaluating each side of the equation carefully and independently to show the equality: With no $\mathbf{B}$ field present, and no time variation, the right-hand side of Faraday's law is zero, and so therefore

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

This is just a statement of Kirchoff's voltage law around the loop, stating that the battery voltage is equal and opposite to the resistor voltage.

b) Repeat part a, assuming the battery removed, the ring closed again, and a linearly-increasing $\mathbf{B}$ field applied in a direction normal to the loop surface: The situation now becomes the same as that shown in Fig. 9.4, except the loop radius is now a, and the resistor value is not specified. Consider the loop as in the x-y plane with the positive z axis directed out of the page. The $\mathbf{a}_\phi$ direction is thus counter-clockwise around the loop. The $\mathbf{B}$ field (out of the page as shown) can be written as $\mathbf{B}(t) = B_0 t\, \mathbf{a}_z$. With the normal to the loop specified as $\mathbf{a}_z$, the direction of $d\mathbf{L}$ is, by the right hand convention, $\mathbf{a}_\phi$. Since the wire is perfectly-conducting, the only voltage appears across the resistor, and is given as V_R. Faraday's law becomes

$$\oint \mathbf{E} \cdot d\mathbf{L} = V_R = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\int_s B_0 t\, \mathbf{a}_z \cdot \mathbf{a}_z \, da = -\pi a^2 B_0$$

This indicates that the resistor voltage, $V_R = \pi a^2 B_0$, has polarity such that the positive terminal is at point a in the figure, while the negative terminal is at point b. Current flows in the clockwise direction, and is given in magnitude by $I = \pi a^2 B_0 / R$.

9.9. A square filamentary loop of wire is 25 cm on a side and has a resistance of 125 Ω per meter length. The loop lies in the $z = 0$ plane with its corners at $(0,0,0)$, $(0.25,0,0)$, $(0.25,0.25,0)$, and $(0,0.25,0)$ at $t = 0$. The loop is moving with velocity $v_y = 50$ m/s in the field $B_z = 8\cos(1.5 \times 10^8 t - 0.5x)$ μT. Develop a function of time which expresses the ohmic power being delivered to the loop: First, since the field does not vary with y, the loop motion in the y direction does not produce any time-varying flux, and so this motion is immaterial. We can evaluate the flux at the original loop position to obtain:

$$\Phi(t) = \int_0^{.25}\int_0^{.25} 8 \times 10^{-6} \cos(1.5 \times 10^8 t - 0.5x)\, dx\, dy$$
$$= -(4 \times 10^{-6})\left[\sin(1.5 \times 10^8 t - 0.13) - \sin(1.5 \times 10^8 t)\right] \text{ Wb}$$

Now, $emf = V(t) = -d\Phi/dt = 6.0 \times 10^2 \left[\cos(1.5 \times 10^8 t - 0.13) - \cos(1.5 \times 10^8 t)\right]$, The total loop resistance is $R = 125(0.25 + 0.25 + 0.25 + 0.25) = 125\,\Omega$. Then the ohmic power is

$$P(t) = \frac{V^2(t)}{R} = \underline{2.9 \times 10^3 \left[\cos(1.5 \times 10^8 t - 0.13) - \cos(1.5 \times 10^8 t)\right]^2 \text{ Watts}}$$

9.10 a) Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\sigma/\omega\epsilon$ for the applied field $E = E_m \cos\omega t$. Assume $\mu = \mu_0$. First, $D = \epsilon E = \epsilon E_m \cos\omega t$. Then the displacement current density is $\partial D/\partial t = -\omega\epsilon E_m \sin\omega t$. Second, $J_c = \sigma E = \sigma E_m \cos\omega t$. Using these results we find $|J_c|/|J_d| = \sigma/\omega\epsilon$.

b) What is the amplitude ratio if the applied field is $E = E_m e^{-t/\tau}$, where τ is real? As before, find $D = \epsilon E = \epsilon E_m e^{-t/\tau}$, and so $J_d = \partial D/\partial t = -(\epsilon/\tau)E_m e^{-t/\tau}$. Also, $J_c = \sigma E_m e^{-t/\tau}$. Finally, $|J_c|/|J_d| = \underline{\sigma\tau/\epsilon}$.

9.11. Let the internal dimension of a coaxial capacitor be $a = 1.2$ cm, $b = 4$ cm, and $l = 40$ cm. The homogeneous material inside the capacitor has the parameters $\epsilon = 10^{-11}$ F/m, $\mu = 10^{-5}$ H/m, and $\sigma = 10^{-5}$ S/m. If the electric field intensity is $\mathbf{E} = (10^6/\rho)\cos(10^5 t)\mathbf{a}_\rho$ V/m, find:

a) $\mathbf{J}$: Use

$$\mathbf{J} = \sigma\mathbf{E} = \underline{\left(\frac{10}{\rho}\right)\cos(10^5 t)\mathbf{a}_\rho \ \text{A/m}^2}$$

b) the total conduction current, I_c, through the capacitor: Have

$$I_c = \int\!\!\int \mathbf{J}\cdot d\mathbf{S} = 2\pi\rho l J = 20\pi l\cos(10^5 t) = \underline{8\pi\cos(10^5 t) \ \text{A}}$$

c) the total displacement current, I_d, through the capacitor: First find

$$\mathbf{J}_d = \frac{\partial\mathbf{D}}{\partial t} = \frac{\partial}{\partial t}(\epsilon\mathbf{E}) = -\frac{(10^5)(10^{-11})(10^6)}{\rho}\sin(10^5 t)\mathbf{a}_\rho = -\frac{1}{\rho}\sin(10^5 t) \ \text{A/m}$$

Now

$$I_d = 2\pi\rho l J_d = -2\pi l\sin(10^5 t) = \underline{-0.8\pi\sin(10^5 t) \ \text{A}}$$

d) the ratio of the amplitude of I_d to that of I_c, the quality factor of the capacitor: This will be

$$\frac{|I_d|}{|I_c|} = \frac{0.8}{8} = \underline{0.1}$$

9.12. Find the displacement current density associated with the magnetic field (assume zero conduction current):

$$\mathbf{H} = A_1 \sin(4x)\cos(\omega t - \beta z)\,\mathbf{a}_x + A_2 \cos(4x)\sin(\omega t - \beta z)\,\mathbf{a}_z$$

The displacement current density is given by

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} = \underline{(4A_2 + \beta A_1)\sin(4x)\cos(\omega t - \beta z)\,\mathbf{a}_y}\ \text{A/m}^2$$

9.13. Consider the region defined by $|x|$, $|y|$, and $|z| < 1$. Let $\epsilon_r = 5$, $\mu_r = 4$, and $\sigma = 0$. If $\mathbf{J}_d = 20\cos(1.5 \times 10^8 t - bx)\mathbf{a}_y\ \mu\text{A/m}^2$;

a) find **D** and **E**: Since $\mathbf{J}_d = \partial \mathbf{D}/\partial t$, we write

$$\mathbf{D} = \int \mathbf{J}_d dt + C = \frac{20 \times 10^{-6}}{1.5 \times 10^8}\sin(1.5 \times 10^8 - bx)\mathbf{a}_y$$
$$= \underline{1.33 \times 10^{-13}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_y\ \text{C/m}^2}$$

where the integration constant is set to zero (assuming no dc fields are present). Then

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{1.33 \times 10^{-13}}{(5 \times 8.85 \times 10^{-12})}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_y$$
$$= \underline{3.0 \times 10^{-3}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_y\ \text{V/m}}$$

b) use the point form of Faraday's law and an integration with respect to time to find **B** and **H**: In this case,

$$\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial x}\mathbf{a}_z = -b(3.0 \times 10^{-3})\cos(1.5 \times 10^8 t - bx)\mathbf{a}_z = -\frac{\partial \mathbf{B}}{\partial t}$$

Solve for **B** by integrating over time:

$$\mathbf{B} = \frac{b(3.0 \times 10^{-3})}{1.5 \times 10^8}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_z = \underline{(2.0)b \times 10^{-11}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_z\ \text{T}}$$

Now

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{(2.0)b \times 10^{-11}}{4 \times 4\pi \times 10^{-7}}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_z$$
$$= \underline{(4.0 \times 10^{-6})b\sin(1.5 \times 10^8 t - bx)\mathbf{a}_z\ \text{A/m}}$$

c) use $\nabla \times \mathbf{H} = \mathbf{J}_d + \mathbf{J}$ to find $\mathbf{J}_d$: Since $\sigma = 0$, there is no conduction current, so in this case

$$\nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial x}\mathbf{a}_y = \underline{4.0 \times 10^{-6} b^2 \cos(1.5 \times 10^8 t - bx)\mathbf{a}_y\ \text{A/m}^2} = \mathbf{J}_d$$

d) What is the numerical value of b? We set the given expression for $\mathbf{J}_d$ equal to the result of part c to obtain:

$$20 \times 10^{-6} = 4.0 \times 10^{-6} b^2 \quad \Rightarrow \quad b = \underline{\sqrt{5.0}\ \text{m}^{-1}}$$

9.14. A voltage source, $V_0 \sin\omega t$, is connected between two concentric conducting spheres, $r = a$ and $r = b$, $b > a$, where the region between them is a material for which $\epsilon = \epsilon_r \epsilon_0$, $\mu = \mu_0$, and $\sigma = 0$. Find the total displacement current through the dielectric and compare it with the source current as determined from the capacitance (Sec. 6.3) and circuit analysis methods: First, solving Laplace's equation, we find the voltage between spheres (see Eq. 39, Chapter 6):

$$V(t) = \frac{(1/r) - (1/b)}{(1/a) - (1/b)} V_0 \sin\omega t$$

Then

$$\mathbf{E} = -\nabla V = \frac{V_0 \sin\omega t}{r^2(1/a - 1/b)} \mathbf{a}_r \quad \Rightarrow \quad \mathbf{D} = \frac{\epsilon_r \epsilon_0 V_0 \sin\omega t}{r^2(1/a - 1/b)} \mathbf{a}_r$$

Now

$$\mathbf{J_d} = \frac{\partial \mathbf{D}}{\partial t} = \frac{\epsilon_r \epsilon_0 \omega V_0 \cos\omega t}{r^2(1/a - 1/b)} \mathbf{a}_r$$

The displacement current is then

$$I_d = 4\pi r^2 J_d = \frac{4\pi \epsilon_r \epsilon_0 \omega V_0 \cos\omega t}{(1/a - 1/b)} = C \frac{dV}{dt}$$

where, from Eq. 6, Chapter 6,

$$C = \frac{4\pi \epsilon_r \epsilon_0}{(1/a - 1/b)}$$

9.15. Let $\mu = 3 \times 10^{-5}$ H/m, $\epsilon = 1.2 \times 10^{-10}$ F/m, and $\sigma = 0$ everywhere. If $\mathbf{H} = 2\cos(10^{10}t - \beta x)\mathbf{a}_z$ A/m, use Maxwell's equations to obtain expressions for **B**, **D**, **E**, and β: First, $\mathbf{B} = \mu \mathbf{H} = \underline{6 \times 10^{-5} \cos(10^{10}t - \beta x)\mathbf{a}_z \text{ T}}$. Next we use

$$\nabla \times \mathbf{H} = -\frac{\partial \mathbf{H}}{\partial x}\mathbf{a}_y = 2\beta \sin(10^{10}t - \beta x)\mathbf{a}_y = \frac{\partial \mathbf{D}}{\partial t}$$

from which

$$\mathbf{D} = \int 2\beta \sin(10^{10}t - \beta x)\, dt + C = \underline{-\frac{2\beta}{10^{10}} \cos(10^{10}t - \beta x)\mathbf{a}_y \text{ C/m}^2}$$

where the integration constant is set to zero, since no dc fields are presumed to exist. Next,

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = -\frac{2\beta}{(1.2 \times 10^{-10})(10^{10})} \cos(10^{10}t - \beta x)\mathbf{a}_y = \underline{-1.67\beta \cos(10^{10}t - \beta x)\mathbf{a}_y \text{ V/m}}$$

Now

$$\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial x}\mathbf{a}_z = 1.67\beta^2 \sin(10^{10}t - \beta x)\mathbf{a}_z = -\frac{\partial \mathbf{B}}{\partial t}$$

So

$$\mathbf{B} = -\int 1.67\beta^2 \sin(10^{10}t - \beta x)\mathbf{a}_z dt = (1.67 \times 10^{-10})\beta^2 \cos(10^{10}t - \beta x)\mathbf{a}_z$$

We require this result to be consistent with the expression for **B** originally found. So

$$(1.67 \times 10^{-10})\beta^2 = 6 \times 10^{-5} \quad \Rightarrow \quad \beta = \underline{\pm 600 \text{ rad/m}}$$

9.16. Derive the continuity equation from Maxwell's equations: First, take the divergence of both sides of Ampere's circuital law:

$$\underbrace{\nabla\cdot\nabla\times\mathbf{H}}_{0} = \nabla\cdot\mathbf{J} + \frac{\partial}{\partial t}\nabla\cdot\mathbf{D} = \underline{\nabla\cdot\mathbf{J} + \frac{\partial\rho_v}{\partial t} = 0}$$

where we have used $\nabla\cdot\mathbf{D} = \rho_v$, another Maxwell equation.

9.17. The electric field intensity in the region $0 < x < 5$, $0 < y < \pi/12$, $0 < z < 0.06$ m in free space is given by $\mathbf{E} = C\sin(12y)\sin(az)\cos(2\times10^{10}t)\,\mathbf{a}_x$ V/m. Beginning with the $\nabla\times\mathbf{E}$ relationship, use Maxwell's equations to find a numerical value for a, if it is known that a is greater than zero: In this case we find

$$\nabla\times\mathbf{E} = \frac{\partial E_x}{\partial z}\,\mathbf{a}_y - \frac{\partial E_z}{\partial y}\,\mathbf{a}_z$$
$$= C\left[a\sin(12y)\cos(az)\mathbf{a}_y - 12\cos(12y)\sin(az)\mathbf{a}_z\right]\cos(2\times10^{10}t) = -\frac{\partial\mathbf{B}}{\partial t}$$

Then

$$\mathbf{H} = -\frac{1}{\mu_0}\int\nabla\times\mathbf{E}\;dt + C_1$$
$$= -\frac{C}{\mu_0(2\times10^{10}}\left[a\sin(12y)\cos(az)\mathbf{a}_y - 12\cos(12y)\sin(az)\mathbf{a}_z\right]\sin(2\times10^{10}t)\ \text{A/m}$$

where the integration constant, $C_1 = 0$, since there are no initial conditions. Using this result, we now find

$$\nabla\times\mathbf{H} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\mathbf{a}_x = -\frac{C(144+a^2)}{\mu_0(2\times10^{10})}\sin(12y)\sin(az)\sin(2\times10^{10}t)\,\mathbf{a}_x = \frac{\partial\mathbf{D}}{\partial t}$$

Now

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \int\frac{1}{\epsilon_0}\nabla\times\mathbf{H}\;dt + C_2 = \frac{C(144+a^2)}{\mu_0\epsilon_0(2\times10^{10})^2}\sin(12y)\sin(az)\cos(2\times10^{10}t)\,\mathbf{a}_x$$

where $C_2 = 0$. This field must be the same as the original field as stated, and so we require that

$$\frac{C(144+a^2)}{\mu_0\epsilon_0(2\times10^{10})^2} = 1$$

Using $\mu_0\epsilon_0 = (3\times10^8)^{-2}$, we find

$$a = \left[\frac{(2\times10^{10})^2}{(3\times10^8)^2} - 144\right]^{1/2} = \underline{66\ \text{m}^{-1}}$$

9.18. The parallel plate transmission line shown in Fig. 9.7 has dimensions $b = 4$ cm and $d = 8$ mm, while the medium between plates is characterized by $\mu_r = 1$, $\epsilon_r = 20$, and $\sigma = 0$. Neglect fields outside the dielectric. Given the field $\mathbf{H} = 5\cos(10^9 t - \beta z)\mathbf{a}_y$ A/m, use Maxwell's equations to help find:

a) β, if $\beta > 0$: Take

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = -5\beta \sin(10^9 t - \beta z)\mathbf{a}_x = 20\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

So

$$\mathbf{E} = \int \frac{-5\beta}{20\epsilon_0} \sin(10^9 t - \beta z)\mathbf{a}_x \, dt = \frac{\beta}{(4 \times 10^9)\epsilon_0} \cos(10^9 t - \beta z)\mathbf{a}_x$$

Then

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = \frac{\beta^2}{(4 \times 10^9)\epsilon_0} \sin(10^9 t - \beta z)\mathbf{a}_y = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

So that

$$\mathbf{H} = \int \frac{-\beta^2}{(4 \times 10^9)\mu_0\epsilon_0} \sin(10^9 t - \beta z)\mathbf{a}_x \, dt = \frac{\beta^2}{(4 \times 10^{18})\mu_0\epsilon_0} \cos(10^9 t - \beta z)$$
$$= 5\cos(10^9 t - \beta z)\mathbf{a}_y$$

where the last equality is required to maintain consistency. Therefore

$$\frac{\beta^2}{(4 \times 10^{18})\mu_0\epsilon_0} = 5 \quad \Rightarrow \quad \beta = \underline{14.9 \text{ m}^{-1}}$$

b) the displacement current density at $z = 0$: Since $\sigma = 0$, we have

$$\nabla \times \mathbf{H} = \mathbf{J}_d = -5\beta \sin(10^9 t - \beta z) = -74.5 \sin(10^9 t - 14.9z)\mathbf{a}_x$$
$$= \underline{-74.5 \sin(10^9 t)\mathbf{a}_x \text{ A/m}} \text{ at z} = 0$$

c) the total displacement current crossing the surface $x = 0.5d$, $0 < y < b$, and $0 < z < 0.1$ m in the $\mathbf{a}_x$ direction. We evaluate the flux integral of $\mathbf{J}_d$ over the given cross section:

$$I_d = -74.5b \int_0^{0.1} \sin(10^9 t - 14.9z)\, \mathbf{a}_x \cdot \mathbf{a}_x \, dz = \underline{0.20 \left[\cos(10^9 t - 1.49) - \cos(10^9 t)\right] \text{ A}}$$

9.19. In the first section of this chapter, Faraday's law was used to show that the field $\mathbf{E} = -\frac{1}{2}kB_0\rho e^{kt}\mathbf{a}_\phi$ results from the changing magnetic field $\mathbf{B} = B_0 e^{kt}\mathbf{a}_z$.

a) Show that these fields do not satisfy Maxwell's other curl equation: Note that $\mathbf{B}$ as stated is constant with position, and so will have zero curl. The electric field, however, varies with time, and so $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$ would have a zero left-hand side and a non-zero right-hand side. The equation is thus not valid with these fields.

b) If we let $B_0 = 1$ T and $k = 10^6$ s^{-1}, we are establishing a fairly large magnetic flux density in 1 μs. Use the $\nabla \times \mathbf{H}$ equation to show that the rate at which B_z should (but does not) change with ρ is only about 5×10^{-6} T/m in free space at $t = 0$: Assuming that $\mathbf{B}$ varies with ρ, we write

$$\nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial \rho}\mathbf{a}_\phi = -\frac{1}{\mu_0}\frac{dB_0}{d\rho}e^{kt} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{2}\epsilon_0 k^2 B_0 \rho e^{kt}$$

Thus

$$\frac{dB_0}{d\rho} = \frac{1}{2}\mu_0\epsilon_0 k^2 \rho B_0 = \frac{10^{12}(1)\rho}{2(3 \times 10^8)^2} = 5.6 \times 10^{-6}\rho$$

which is near the stated value if ρ is on the order of 1m.

9.20. Given Maxwell's equations in point form, assume that all fields vary as e^{st} and write the equations without explicitly involving time: Write all fields in the general form $\mathbf{A}(\mathbf{r},t) = \mathbf{A}_0(\mathbf{r})e^{st}$, where $\mathbf{r}$ is a position vector in any coordinate system. Maxwell's equations become:

$$\nabla \times \mathbf{E}_0(\mathbf{r})\, e^{st} = -\frac{\partial}{\partial t}\left(\mathbf{B}_0(\mathbf{r})\, e^{st}\right) = -s\mathbf{B}_0(\mathbf{r})\, e^{st}$$

$$\nabla \times \mathbf{H}_0(\mathbf{r})\, e^{st} = \mathbf{J}_0(\mathbf{r})e^{st} + \frac{\partial}{\partial t}\left(\mathbf{D}_0(\mathbf{r})\, e^{st}\right) = \mathbf{J}_0(\mathbf{r})e^{st} + s\mathbf{D}_0(\mathbf{r})\, e^{st}$$

$$\nabla \cdot \mathbf{D}_0(\mathbf{r})\, e^{st} = \rho_0(\mathbf{r})\, e^{st}$$

$$\nabla \cdot \mathbf{B}_0(\mathbf{r})\, e^{st} = 0$$

In all cases, the e^{st} terms divide out, leaving:

$$\nabla \times \mathbf{E}_0(\mathbf{r}) = -s\mathbf{B}_0(\mathbf{r})$$

$$\nabla \times \mathbf{H}_0(\mathbf{r}) = \mathbf{J}_0(\mathbf{r}) + s\mathbf{D}_0(\mathbf{r})$$

$$\nabla \cdot \mathbf{D}_0(\mathbf{r}) = \rho_0(\mathbf{r})$$

$$\nabla \cdot \mathbf{B}_0(\mathbf{r}) = 0$$

9.21. a) Show that under static field conditions, Eq. (55) reduces to Ampere's circuital law. First use the definition of the vector Laplacian:

$$\nabla^2 \mathbf{A} = -\nabla \times \nabla \times \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = -\mu \mathbf{J}$$

which is Eq. (55) with the time derivative set to zero. We also note that $\nabla \cdot \mathbf{A} = 0$ in steady state (from Eq. (54)). Now, since $\mathbf{B} = \nabla \times \mathbf{A}$, (55) becomes

$$-\nabla \times \mathbf{B} = -\mu \mathbf{J} \quad \Rightarrow \quad \nabla \times \mathbf{H} = \mathbf{J}$$

b) Show that Eq. (51) becomes Faraday's law when taking the curl: Doing this gives

$$\nabla \times \mathbf{E} = -\nabla \times \nabla V - \frac{\partial}{\partial t}\nabla \times \mathbf{A}$$

The curl of the gradient is identially zero, and $\nabla \times \mathbf{A} = \mathbf{B}$. We are left with

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$$

9.22. In a sourceless medium, in which $\mathbf{J} = 0$ and $\rho_v = 0$, assume a rectangular coordinate system in which $\mathbf{E}$ and $\mathbf{H}$ are functions only of z and t. The medium has permittivity ϵ and permeability μ.

a) If $\mathbf{E} = E_x \mathbf{a}_x$ and $\mathbf{H} = H_y \mathbf{a}_y$, begin with Maxwell's equations and determine the second order partial differential equation that E_x must satisfy.

First use

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \quad \frac{\partial E_x}{\partial z}\, \mathbf{a}_y = -\mu \frac{\partial H_y}{\partial t}\, \mathbf{a}_y$$

in which case, the curl has dictated the direction that $\mathbf{H}$ must lie in. Similarly, use the other Maxwell curl equation to find

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad \Rightarrow \quad -\frac{\partial H_y}{\partial z}\, \mathbf{a}_x = \epsilon \frac{\partial E_x}{\partial t}\, \mathbf{a}_x$$

Now, differentiate the first equation with respect to z, and the second equation with respect to t:

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu \frac{\partial^2 H_y}{\partial t \partial z} \quad \text{and} \quad \frac{\partial^2 H_y}{\partial z \partial t} = -\epsilon \frac{\partial^2 E_x}{\partial t^2}$$

Combining these two, we find

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2}$$

b) Show that $E_x = E_0 \cos(\omega t - \beta z)$ is a solution of that equation for a particular value of β: Substituting, we find

$$\frac{\partial^2 E_x}{\partial z^2} = -\beta^2 E_0 \cos(\omega t - \beta z) \quad \text{and} \quad \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} = -\omega^2 \mu\epsilon E_0 \cos(\omega t - \beta z)$$

These two will be equal provided the constant multipliers of $\cos(\omega t - \beta z)$ are equal.

c) Find β as a function of given parameters. Equating the two constants in part *b*, we find $\underline{\beta = \omega\sqrt{\mu\epsilon}}$.

9.23. In region 1, $z < 0$, $\epsilon_1 = 2 \times 10^{-11}$ F/m, $\mu_1 = 2 \times 10^{-6}$ H/m, and $\sigma_1 = 4 \times 10^{-3}$ S/m; in region 2, $z > 0$, $\epsilon_2 = \epsilon_1/2$, $\mu_2 = 2\mu_1$, and $\sigma_2 = \sigma_1/4$. It is known that $\mathbf{E}_1 = (30\mathbf{a}_x + 20\mathbf{a}_y + 10\mathbf{a}_z)\cos(10^9 t)$ V/m at $P_1(0, 0, 0^-)$.

a) Find $\mathbf{E}_{N1}$, $\mathbf{E}_{t1}$, $\mathbf{D}_{N1}$, and $\mathbf{D}_{t1}$: These will be

$$\mathbf{E}_{N1} = \underline{10\cos(10^9 t)\mathbf{a}_z \text{ V/m}} \quad \mathbf{E}_{t1} = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ V/m}}$$

$$\mathbf{D}_{N1} = \epsilon_1 \mathbf{E}_{N1} = (2 \times 10^{-11})(10)\cos(10^9 t)\mathbf{a}_z \text{ C/m}^2 = \underline{200\cos(10^9 t)\mathbf{a}_z \text{ pC/m}^2}$$

$$\mathbf{D}_{t1} = \epsilon_1 \mathbf{E}_{t1} = (2 \times 10^{-11})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(600\mathbf{a}_x + 400\mathbf{a}_y)\cos(10^9 t) \text{ pC/m}^2}$$

b) Find $\mathbf{J}_{N1}$ and $\mathbf{J}_{t1}$ at P_1:

$$\mathbf{J}_{N1} = \sigma_1 \mathbf{E}_{N1} = (4 \times 10^{-3})(10\cos(10^9 t))\mathbf{a}_z = \underline{40\cos(10^9 t)\mathbf{a}_z \text{ mA/m}^2}$$

$$\mathbf{J}_{t1} = \sigma_1 \mathbf{E}_{t1} = (4 \times 10^{-3})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(120\mathbf{a}_x + 80\mathbf{a}_y)\cos(10^9 t) \text{ mA/m}^2}$$

c) Find $\mathbf{E}_{t2}$, $\mathbf{D}_{t2}$, and $\mathbf{J}_{t2}$ at P_1: By continuity of tangential $\mathbf{E}$,

$$\mathbf{E}_{t2} = \mathbf{E}_{t1} = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ V/m}}$$

Then

$$\mathbf{D}_{t2} = \epsilon_2 \mathbf{E}_{t2} = (10^{-11})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(300\mathbf{a}_x + 200\mathbf{a}_y)\cos(10^9 t) \text{ pC/m}^2}$$

$$\mathbf{J}_{t2} = \sigma_2 \mathbf{E}_{t2} = (10^{-3})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ mA/m}^2}$$

d) (Harder) Use the continuity equation to help show that $J_{N1} - J_{N2} = \partial D_{N2}/\partial t - \partial D_{N1}/\partial t$ and then determine $\mathbf{E}_{N2}$, $\mathbf{D}_{N2}$, and $\mathbf{J}_{N2}$: We assume the existence of a surface charge layer at the boundary having density ρ_s C/m^2. If we draw a cylindrical "pillbox" whose top and bottom surfaces (each of area Δa) are on either side of the interface, we may use the continuity condition to write

$$(J_{N2} - J_{N1})\Delta a = -\frac{\partial \rho_s}{\partial t}\Delta a$$

where $\rho_s = D_{N2} - D_{N1}$. Therefore,

$$J_{N1} - J_{N2} = \frac{\partial}{\partial t}(D_{N2} - D_{N1})$$

In terms of the normal electric field components, this becomes

$$\sigma_1 E_{N1} - \sigma_2 E_{N2} = \frac{\partial}{\partial t}(\epsilon_2 E_{N2} - \epsilon_1 E_{N1})$$

Now let $E_{N2} = A\cos(10^9 t) + B\sin(10^9 t)$, while from before, $E_{N1} = 10\cos(10^9 t)$.

9.23d (continued)

These, along with the permittivities and conductivities, are substituted to obtain

$$
\begin{aligned}
&(4\times 10^{-3})(10)\cos(10^9 t) - 10^{-3}[A\cos(10^9 t) + B\sin(10^9 t)] \\
&= \frac{\partial}{\partial t}\left[10^{-11}[A\cos(10^9 t) + B\sin(10^9 t)] - (2\times 10^{-11})(10)\cos(10^9 t)\right] \\
&= -(10^{-2}A\sin(10^9 t) + 10^{-2}B\cos(10^9 t) + (2\times 10^{-1})\sin(10^9 t)
\end{aligned}
$$

We now equate coefficients of the sin and cos terms to obtain two equations:

$$4\times 10^{-2} - 10^{-3}A = 10^{-2}B$$

$$-10^{-3}B = -10^{-2}A + 2\times 10^{-1}$$

These are solved together to find $A = 20.2$ and $B = 2.0$. Thus

$$\mathbf{E}_{N2} = \left[20.2\cos(10^9 t) + 2.0\sin(10^9 t)\right]\mathbf{a}_z = \underline{20.3\cos(10^9 t + 5.6^\circ)\mathbf{a}_z \text{ V/m}}$$

Then

$$\mathbf{D}_{N2} = \epsilon_2 \mathbf{E}_{N2} = \underline{203\cos(10^9 t + 5.6^\circ)\mathbf{a}_z \text{ pC/m}^2}$$

and

$$\mathbf{J}_{N2} = \sigma_2 \mathbf{E}_{N2} = \underline{20.3\cos(10^9 t + 5.6^\circ)\mathbf{a}_z \text{ mA/m}^2}$$

9.24. A vector potential is given as $\mathbf{A} = A_0 \cos(\omega t - kz)\,\mathbf{a}_y$. a) Assuming as many components as possible are zero, find $\mathbf{H}$, $\mathbf{E}$, and V;

With $\mathbf{A}$ y-directed only, and varying spatially only with z, we find

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} = -\frac{1}{\mu}\frac{\partial A_y}{\partial z}\,\mathbf{a}_x = \underline{-\frac{kA_0}{\mu}\sin(\omega t - kz)\,\mathbf{a}_x \text{ A/m}}$$

Now, in a lossless medium we will have zero conductivity, so that the point form of Ampere's circuital law involves only the displacement current term:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

Using the magnetic field as found above, we find

$$\nabla \times \mathbf{H} = \frac{\partial H_x}{\partial z}\,\mathbf{a}_y = \frac{k^2 A_0}{\mu}\cos(\omega t - kz)\,\mathbf{a}_y = \epsilon\frac{\partial \mathbf{E}}{\partial t} \;\Rightarrow\; \mathbf{E} = \underline{\frac{k^2 A_0}{\omega\mu\epsilon}\sin(\omega t - kz)\,\mathbf{a}_y \text{ V/m}}$$

Now,

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \;\Rightarrow\; \nabla V = -\left[\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E}\right]$$

or

$$\nabla V = A_0\omega\left[1 - \frac{k^2}{\omega^2\mu\epsilon}\right]\sin(\omega t - kz)\,\mathbf{a}_y = \frac{\partial V}{\partial y}\,\mathbf{a}_y$$

Integrating over y we find

$$V = A_0\,\omega\,y\left[1 - \frac{k^2}{\omega^2\mu\epsilon}\right]\sin(\omega t - kz) + C$$

where C, the integration constant, can be taken as zero. In part b, it will be shown that $k = \omega\sqrt{\mu\epsilon}$, which means that $\underline{V = 0}$.

b) Specify k in terms of A_0, ω, and the constants of the lossless medium, ϵ and μ. Use the other Maxwell curl equation:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu\frac{\partial \mathbf{H}}{\partial t}$$

so that

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu}\,\nabla \times \mathbf{E} = \frac{1}{\mu}\frac{\partial E_y}{\partial z}\,\mathbf{a}_x = -\frac{k^3 A_0}{\omega\mu^2\epsilon}\cos(\omega t - kz)\,\mathbf{a}_x$$

Integrate over t (and set the integration constant to zero) and require the result to be consistant with part a:

$$\mathbf{H} = -\frac{k^3 A_0}{\omega^2\mu^2\epsilon}\sin(\omega t - kz)\,\mathbf{a}_x = \underbrace{-\frac{kA_0}{\mu}\sin(\omega t - kz)\,\mathbf{a}_x}_{\text{from part a}}$$

We identify

$$\underline{k = \omega\sqrt{\mu\epsilon}}$$

9.25. In a region where $\mu_r = \epsilon_r = 1$ and $\sigma = 0$, the retarded potentials are given by $V = x(z - ct)$ V and $\mathbf{A} = x[(z/c) - t]\mathbf{a}_z$ Wb/m, where $c = 1/\sqrt{\mu_0 \epsilon_0}$.

a) Show that $\nabla \cdot \mathbf{A} = -\mu\epsilon(\partial V/\partial t)$:

First,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = \frac{x}{c} = x\sqrt{\mu_0 \epsilon_0}$$

Second,

$$\frac{\partial V}{\partial t} = -cx = -\frac{x}{\sqrt{\mu_0 \epsilon_0}}$$

so we observe that $\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0(\partial V/\partial t)$ in free space, implying that the given statement would hold true in general media.

b) Find **B**, **H**, **E**, and **D**:

Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_x}{\partial x}\mathbf{a}_y = \underline{\left(t - \frac{z}{c}\right)\mathbf{a}_y \text{ T}}$$

Then

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \underline{\frac{1}{\mu_0}\left(t - \frac{z}{c}\right)\mathbf{a}_y \text{ A/m}}$$

Now,

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -(z - ct)\mathbf{a}_x - x\mathbf{a}_z + x\mathbf{a}_z = \underline{(ct - z)\mathbf{a}_x \text{ V/m}}$$

Then

$$\mathbf{D} = \epsilon_0 \mathbf{E} = \underline{\epsilon_0(ct - z)\mathbf{a}_x \text{ C/m}^2}$$

c) Show that these results satisfy Maxwell's equations if **J** and ρ_v are zero:

i. $\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon_0(ct - z)\mathbf{a}_x = 0$

ii. $\nabla \cdot \mathbf{B} = \nabla \cdot (t - z/c)\mathbf{a}_y = 0$

iii.

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = \frac{1}{\mu_0 c}\mathbf{a}_x = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{a}_x$$

which we require to equal $\partial \mathbf{D}/\partial t$:

$$\frac{\partial \mathbf{D}}{\partial t} = \epsilon_0 c\mathbf{a}_x = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{a}_x$$

iv.

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = -\mathbf{a}_y$$

which we require to equal $-\partial \mathbf{B}/\partial t$:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{a}_y$$

So all four Maxwell equations are satisfied.

9.26. Write Maxwell's equations in point form in terms of **E** and **H** as they apply to a sourceless medium, where **J** and ρ_v are both zero. Replace ϵ by μ, μ by ϵ, **E** by **H**, and **H** by $-$**E**, and show that the equations are unchanged. This is a more general expression of the *duality principle* in circuit theory.

Maxwell's equations in sourceless media can be written as:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{2}$$

$$\nabla \cdot \epsilon \mathbf{E} = 0 \tag{3}$$

$$\nabla \cdot \mu \mathbf{H} = 0 \tag{4}$$

In making the above substitutions, we find that (1) converts to (2), (2) converts to (1), and (3) and (4) convert to each other.

CHAPTER 10

10.1. The parameters of a certain transmission line operating at $\omega = 6 \times 10^8$ rad/s are $L = 0.35\ \mu$H/m, $C = 40$ pF/m, $G = 75\ \mu$S/m, and $R = 17\ \Omega$/m. Find γ, α, β, λ, and Z_0: We use

$$\begin{aligned}\gamma &= \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{[17 + j(6 \times 10^8)(0.35 \times 10^{-6})][75 \times 10^{-6} + j(6 \times 10^8)(40 \times 10^{-12})]} \\ &= \underline{0.094 + j2.25\ \text{m}^{-1}} = \alpha + j\beta\end{aligned}$$

Therefore, $\alpha = \underline{0.094\ \text{Np/m}}$, $\beta = \underline{2.25\ \text{rad/m}}$, and $\lambda = 2\pi/\beta = \underline{2.79\ \text{m}}$. Finally,

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{17 + j2.1 \times 10^2}{75 \times 10^{-6} + j2.4 \times 10^{-2}}} = \underline{93.6 - j3.64\ \Omega}$$

10.2. A sinusoidal wave on a transmission line is specified by voltage and current in phasor form:

$$V_s(z) = V_0\, e^{\alpha z}\, e^{j\beta z} \quad \text{and} \quad I_s(z) = I_0\, e^{\alpha z}\, e^{j\beta z}\, e^{j\phi}$$

where V_0 and I_0 are both real.

a) In which direction does this wave propagate and why? Propagation is in the *backward z* direction, because of the factor $e^{+j\beta z}$.

b) It is found that $\alpha = 0$, $Z_0 = 50\,\Omega$, and the wave velocity is $v_p = 2.5 \times 10^8$ m/s, with $\omega = 10^8\,\text{s}^{-1}$. Evaluate R, G, L, C, λ, and ϕ: First, the fact that $\alpha = 0$ means that the line is lossless, from which we immediately conclude that $\underline{R = G = 0}$. As this is true it follows that $Z_0 = \sqrt{L/C}$ and $v_p = 1/\sqrt{LC}$, from which

$$C = \frac{1}{Z_0 v_p} = \frac{1}{50(2.5 \times 10^8)} = 8.0 \times 10^{-11}\ \text{F} = \underline{80\ \text{pF}}$$

Then

$$L = CZ_0^2 = (8.0 \times 10^{-11})(50)^2 = 2.0 \times 10^{-7} = \underline{0.20\ \mu\text{H}}$$

Now,

$$\lambda = \frac{v_p}{f} = \frac{2\pi v_p}{\omega} = \frac{2\pi(2.5 \times 10^8)}{10^8} = \underline{15.7\ \text{m}}$$

Finally, the current phase is found through

$$I_0 e^{j\phi} = \frac{V_0}{Z_0}$$

Since V_0, I_0, and Z_0 are all real, it follows that $\underline{\phi = 0}$.

10.3. The characteristic impedance of a certain lossless transmission line is 72 Ω. If $L = 0.5\,\mu$H/m, find:

a) C: Use $Z_0 = \sqrt{L/C}$, or

$$C = \frac{L}{Z_0^2} = \frac{5 \times 10^{-7}}{(72)^2} = 9.6 \times 10^{-11}\,\text{F/m} = \underline{96\,\text{pF/m}}$$

v_p:

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(5 \times 10^{-7})(9.6 \times 10^{-11})}} = \underline{1.44 \times 10^8\ \text{m/s}}$$

c) β if $f = 80$ MHz:

$$\beta = \omega\sqrt{LC} = \frac{2\pi \times 80 \times 10^6}{1.44 \times 10^8} = \underline{3.5\ \text{rad/m}}$$

d) The line is terminated with a load of 60 Ω. Find Γ and s:

$$\Gamma = \frac{60 - 72}{60 + 72} = \underline{-0.09} \qquad\qquad s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + .09}{1 - .09} = \underline{1.2}$$

10.4. A sinusoidal voltage wave of amplitude V_0, frequency ω, and phase constant, β, propagates in the forward z direction toward the open load end in a lossless transmission line of characteristic impedance Z_0. At the end, the wave totally reflects with zero phase shift, and the reflected wave now interferes with the incident wave to yield a standing wave pattern over the line length (as per Example 10.1). Determine the standing wave pattern for the *current* in the line. Express the result in real instantaneous form and simplify.

In phasor form, the forward and backward waves are:

$$V_{sT}(z) = V_0 e^{-j\beta z} + V_0 e^{j\beta z}$$

The current is found from the voltage by dividing by Z_0 (while incorporating the proper sign for forward and backward waves):

$$I_{sT}(z) = \frac{V_0}{Z_0} e^{-j\beta z} - \frac{V_0}{Z_0} e^{j\beta z} = -\frac{V_0}{Z_0}\left(e^{j\beta z} - e^{-j\beta z}\right) = -j\frac{2V_0}{Z_0}\sin(\beta z)$$

The real instantaneous current is now

$$\mathcal{I}(z,t) = \mathcal{R}e\left\{I_{sT}(z)e^{j\omega t}\right\} = \mathcal{R}e\left\{-j\frac{2V_0}{Z_0}\sin(\beta z)[\underbrace{\cos(\omega t) + j\sin(\omega t)}_{e^{j\omega t}}]\right\}$$

$$= \frac{2V_0}{Z_0}\sin(\beta z)\sin(\omega t)$$

10.5. Two characteristics of a certain lossless transmission line are $Z_0 = 50\ \Omega$ and $\gamma = 0 + j0.2\pi\ \text{m}^{-1}$ at $f = 60$ MHz.

a) Find L and C for the line: We have $\beta = 0.2\pi = \omega\sqrt{LC}$ and $Z_0 = 50 = \sqrt{L/C}$. Thus

$$\frac{\beta}{Z_0} = \omega C \quad \Rightarrow \quad C = \frac{\beta}{\omega Z_0} = \frac{0.2\pi}{(2\pi \times 60 \times 10^6)(50)} = \frac{1}{3} \times 10^{10} = \underline{33.3\ \text{pF/m}}$$

Then $L = CZ_0^2 = (33.3 \times 10^{-12})(50)^2 = 8.33 \times 10^{-8}$ H/m $= \underline{83.3\ \text{nH/m}}$.

b) A load, $Z_L = 60 + j80\ \Omega$ is located at $z = 0$. What is the shortest distance from the load to a point at which $Z_{in} = R_{in} + j0$? I will do this using two different methods:

The Hard Way: We use the general expression

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \right]$$

We can then normalize the impedances with respect to Z_0 and write

$$z_{in} = \frac{Z_{in}}{Z_0} = \left[\frac{(Z_L/Z_0) + j\tan(\beta l)}{1 + j(Z_L/Z_0)\tan(\beta l)} \right] = \left[\frac{z_L + j\tan(\beta l)}{1 + jz_L \tan(\beta l)} \right]$$

where $z_L = (60 + j80)/50 = 1.2 + j1.6$. Using this, and defining $x = \tan(\beta l)$, we find

$$z_{in} = \left[\frac{1.2 + j(1.6 + x)}{(1 - 1.6x) + j1.2x} \right] \left[\frac{(1 - 1.6x) - j1.2x}{(1 - 1.6x) - j1.2x} \right]$$

The second bracketed term is a factor of one, composed of the complex conjugate of the denominator of the first term, divided by itself. Carrying out this product, we find

$$z_{in} = \left[\frac{1.2(1 - 1.6x) + 1.2x(1.6 + x) - j[(1.2)^2 x - (1.6 + x)(1 - 1.6x)]}{(1 - 1.6x)^2 + (1.2)^2 x^2} \right]$$

We require the imaginary part to be zero. Thus

$$(1.2)^2 x - (1.6 + x)(1 - 1.6x) = 0 \quad \Rightarrow \quad 1.6x^2 + 3x - 1.6 = 0$$

$$\text{So} \;\; x = \tan(\beta l) = \frac{-3 \pm \sqrt{9 + 4(1.6)^2}}{2(1.6)} = (.433, -2.31)$$

We take the positive root, and find

$$\beta l = \tan^{-1}(.433) = 0.409 \quad \Rightarrow \quad l = \frac{0.409}{0.2\pi} = 0.65\ \text{m} = \underline{65\ \text{cm}}$$

The Easy Way: We find

$$\Gamma = \frac{60 + j80 - 50}{60 + j80 + 50} = 0.405 + j0.432 = 0.59\angle 0.818$$

Thus $\phi = 0.818$ rad, and we use the fact that the input impedance will be purely real at the location of a voltage minimum or maximum. The first voltage maximum will occur at a distance in front of the load given by

$$z_{max} = \frac{\phi}{2\beta} = \frac{0.818}{2(0.2\pi)} = 0.65\ \text{m}$$

10.6. A 50-ohm load is attached to a 50m section of the transmission line of Problem 10.1, and a 100-W signal is fed to the input end of the line.

a) Evaluate the distributed line loss in dB/m: From Problem 10.1 (or from the answer in Appendix F) we have $\alpha = 0.094$ Np/m. Then

$$\text{Loss[dB/m]} = 8.69\alpha = 8.69(0.094) = \underline{0.82 \text{ dB/m}}$$

b) Evaluate the reflection coefficient at the load: We need the characteristic impedance of the line. Again, in solving Problem 10.1 (or looking up the answer in the appendix), we have $Z_0 = 93.6 - j3.64$ ohms. The reflection coefficient is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - (93.6 - j3.64)}{50 + (93.6 - j3.64)} = \underline{-0.304 + j0.0176 \;\; = \;\; 0.305\angle 177^\circ}$$

c) Evaluate the power that is dissipated by the load resistor: This will be

$$P_d = 100\text{W} \times e^{-2\alpha L} \times (1 - |\Gamma_L|^2) = 100\, e^{-2(0.094)(50)} \left[1 - (0.305)^2\right] = \underline{7.5 \text{ mW}}$$

d) What power drop in dB does the dissipated power in the load represent when compared to the original input power? This we find as a positive number through

$$P_d[\text{dB}] = 10\log_{10}\left[\frac{P_{in}}{P_d}\right] = 10\log_{10}\left[\frac{100}{0.0075}\right] = \underline{41.2 \text{ dB}}$$

e) on partial reflection from the load, how much power returns to the input and what dB drop does this represent when compared to the original 100-W input power? After one round trip plus a reflection at the load, the power returning to the input is expressed as

$$P_{out} = P_{in} \times e^{-2\alpha(2L)} \times |\Gamma_L|^2 = 100\, e^{200(0.094)} (0.305)^2 = 6.37 \times 10^{-10}\,\text{W} = \underline{637 \text{ pW}}$$

As a decibel reduction from the original input power, this becomes

$$P_{out}[\text{dB}] = 10\log_{10}\left[\frac{P_{in}}{P_{out}}\right] = 10\log_{10}\left[\frac{100}{6.37 \times 10^{-10}}\right] = \underline{112 \text{ dB}}$$

10.7. A transmitter and receiver are connected using a cascaded pair of transmission lines. At the operating frequency, Line 1 has a measured loss of 0.1 dB/m, and Line 2 is rated at 0.2 dB/m. The link is composed of 40m of Line 1, joined to 25m of Line 2. At the joint, a splice loss of 2 dB is measured. If the transmitted power is 100mW, what is the received power?

The total loss in the link in dB is $40(0.1) + 25(0.2) + 2 = 11$ dB. Then the received power is $P_r = 100\text{mW} \times 10^{-0.1(11)} = \underline{7.9 \text{ mW}}$.

10.8. An absolute measure of power is the dBm scale, in which power is specified in decibels relative to one milliwatt. Specifically, $P(\text{dBm}) = 10\log_{10}[P(\text{mW})/1\,\text{mW}]$. Suppose that a receiver is rated as having a *sensitivity* of -20 dBm, indicating the *mimimum* power that it must receive in order to adequately interpret the transmitted electronic data. Suppose this receiver is at the load end of a 50-ohm transmission line having 100-m length and loss rating of 0.09 dB/m. The receiver impedance is 75 ohms, and so is not matched to the line. What is the minimum required input power to the line in a) dBm, b) mW?

Method 1 – using decibels: The total loss in dB will be the sum of the transit loss in the line and the loss arising from partial transmission into the load. The latter will be

$$\text{Loss}_{load}\,[\text{dB}] = 10\log_{10}\left(\frac{1}{1-|\Gamma_L|^2}\right)$$

where $\Gamma_L = (75-50)/(75+50) = 0.20$. So

$$\text{Loss}_{load} = 10\log_{10}\left(\frac{1}{1-(0.20)^2}\right) = 0.18\text{ dB}$$

The transit loss will be

$$\text{Loss}_{trans} = 10\log_{10}\left(\frac{1}{e^{-2\alpha L}}\right) = (0.09\,\text{dB/m})(100\,\text{m}) = 9.0\,\text{dB}$$

The total loss in dB is then $\text{Loss}_{tot} = 9.0 + 0.18 = 9.2$ dB. The minimum required input power is now

$$P_{in}\,[\text{dBm}] = -20\,\text{dBm} + 9.2\,\text{dB} = \underline{-10.8\text{ dBm}}$$

In milliwatts, this is

$$P_{in}\,[\text{mW}] = 10^{-1.08} = \underline{8.3\times10^{-2}\text{ mW} = 83\ \mu\text{W}}$$

Method 2 – using loss factors: The 0.09 dB/m line loss corresponds to an exponential voltage attenuation coefficient of $\alpha = 0.09/8.69 = 1.04\times10^{-2}$ Np/m. Now, the power dropped at the load will be

$$P_{load} = P_{in}e^{-2\alpha L}(1-|\Gamma_L|^2) = P_{in}\exp\left[-2(1.04\times10^{-2})(100)\right]\left[1-(0.2)^2\right] = 0.12P_{in}$$

Since the minimum power at the load of -20 dBm in mW is 10^{-2}, the minimum input power will be

$$P_{in}[\text{mW}] = \frac{10^{-2}}{0.12} = 8.3\times10^{-2}\,\text{mW} = 83\ \mu\text{W}\quad\text{as before}$$

In dBm this is $P_{in} = 10\log_{10}\left(8.3\times10^{-2}\right) = -10.8$ dBm.

10.9. A sinusoidal voltage source drives the series combination of an impedance, $Z_g = 50 - j50\ \Omega$, and a lossless transmission line of length L, shorted at the load end. The line characteristic impedance is 50 Ω, and wavelength λ is measured on the line.

a) Determine, in terms of wavelength, the shortest line length that will result in the voltage source driving a total impedance of 50 Ω: Using Eq. (98), with $Z_L = 0$, we find the input impedance, $Z_{in} = jZ_0 \tan(\beta L)$, where $Z_0 = 50$ ohms. This input inpedance is in series with the generator impedance, giving a total of $Z_{tot} = 50 - j50 + j50\tan(\beta L)$. For this impedance to equal 50 ohms, the imaginary parts must cancel. Therefore, $\tan(\beta L) = 1$, or $\beta L = \pi/4$, at minimum. So $L = \pi/(4\beta) = \pi/(4 \times 2\pi/\lambda) = \underline{\lambda/8}$.

b) Will other line lengths meet the requirements of part *a*? If so what are they? Yes, the requirement being $\beta L = \pi/4 + m\pi$, where m is an integer. Therefore

$$L = \frac{\pi/4 + m\pi}{\beta} = \frac{\pi(1+4m)}{4 \times 2\pi/\lambda} = \frac{\lambda}{8} + m\frac{\lambda}{2}$$

10.10. Two lossless transmission lines having different characteristic impedances are to be joined end-to-end. The impedances are $Z_{01} = 100$ ohms and $Z_{03} = 25$ ohms. The operating frequency is 1 GHz.

a) Find the required characteristic impedance, Z_{02}, of a quarter-wave section to be inserted between the two, which will impedance-match the joint, thus allowing total power transmission through the three lines: The required inpedance will be $Z_{02} = \sqrt{Z_{01}Z_{03}} = \sqrt{(100)(25)} = \underline{50\text{ ohms}}$.

b) The capacitance per unit length of the intermediate line is found to be 100 pF/m. Find the shortest length in meters of this line that is needed to satisfy the impedance-matching condition: For the lossless intermediate line,

$$Z_{02} = \sqrt{\frac{L_2}{C_2}} \;\Rightarrow\; L_2 = C_2 Z_{02}^2 \quad \text{Then} \quad \beta_2 = \omega\sqrt{L_2 C_2} = 2\pi f\, C_2\, Z_{02}$$

The line length at $\lambda/4$ (the shortest length that will work) is then

$$\ell_2 = \frac{\lambda_2}{4} = \frac{1}{4}\left(\frac{2\pi}{\beta_2}\right) = \frac{1}{4fC_2Z_{02}} = \frac{1}{(4\times 10^9)(10^{-10})(50)} = \underline{0.05\text{ m}}$$

c) With the three-segment setup as found in parts *a* and *b*, the frequency is now doubled to 2 GHz. Find the input impedance at the Line 1-to-Line 2 junction, seen by waves incident from Line 1: With the frequency doubled, the wavelength is cut in half, which means that the intermediate section is now a half-wavelength long. In that case, the input impedance is just the impedance of the far line, or $Z_{in} = Z_{03} = \underline{25\text{ ohms}}$.

d) Under the conditions of part *c*, and with power incident from Line 1, evaluate the standing wave ratio that will be measured in Line 1, and the fraction of the incident power from Line 1 that is reflected and propagates back to the Line 1 input. The reflection coefficient at the junction is $\Gamma_{in} = (25-100)/(25+100) = -3/5$. So the VSWR $= (1+3/5)/(1-3/5) = \underline{4}$. The fraction of the power reflected at the junction is $|\Gamma|^2 = (3/5)^2 = \underline{0.36}$, or 36%.

10.11. A transmission line having primary constants L, C, R, and G, has length ℓ and is terminated by a load having complex impedance $R_L + jX_L$. At the input end of the line, a DC voltage source, V_0, is connected. Assuming all parameters are known at zero frequency, find the steady state power dissipated by the load if

a) $R = G = 0$: Here, the line just acts as a pair of lossless leads to the impedance. At zero frequency, the dissipated power is just $P_d = \underline{V_0^2/R_L}$.

b) $R \neq 0$, $G = 0$: In this case, the load is effectively in series with a resistance of value $R\ell$. The voltage at the load is therefore $V_L = V_0 R_L/(R\ell + R_L)$, and the dissipated power is $P_d = V_L^2/R_L = \underline{V_0^2 R_L/(R\ell + R_L)^2}$.

c) $R = 0$, $G \neq 0$: Now, the load is in parallel with a resistance, $1/(G\ell)$, but the voltage at the load is still V_0. Dissipated power by the load is $P_d = \underline{V_0^2/R_L}$.

d) $R \neq 0$, $G \neq 0$: One way to approach this problem is to think of the power at the load as arising from an incident voltage wave of vanishingly small frequency, and to assume that losses in the line are sufficient to allow steady state conditions to be reached after a single reflection from the load. The "forward-traveling" voltage as a function of z is given by $V(z) = V_0 \exp(-\gamma z)$, where $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \rightarrow \sqrt{RG}$ as frequency approaches zero. Considering a single reflection only, the voltage at the load is then $V_L = (1 + \Gamma)V_0 \exp\left(-\sqrt{RG}\,\ell\right)$. The reflection coefficient requires the line characteristic impedance, given by $Z_0 = [(R + j\omega L)/(G + j\omega C)]^{1/2} \rightarrow \sqrt{R/G}$ as $\omega \rightarrow 0$. The reflection coefficient is then $\Gamma = (R_L - \sqrt{R/G})/(R_L + \sqrt{R/G})$, and so the load voltage becomes:

$$V_L = \frac{2R_L}{R_L + \sqrt{R/G}} \exp\left(-\sqrt{RG}\,\ell\right)$$

The dissipated power is then

$$P_d = \frac{V_L^2}{R_L} = \frac{4R_L V_0^2}{\left(R_L + \sqrt{R/G}\right)^2} \exp\left(-2\sqrt{RG}\,\ell\right) \text{ W}$$

10.12. In a circuit in which a sinusoidal voltage source drives its internal impedance in series with a load impedance, it is known that maximum power transfer to the load occurs when the source and load impedances form a complex conjugate pair. Suppose the source (with its internal impedance) now drives a complex load of impedance $Z_L = R_L + jX_L$ that has been moved to the end of a lossless transmission line of length ℓ having characteristic impedance Z_0. If the source impedance is $Z_g = R_g + jX_g$, write an equation that can be solved for the required line length, ℓ, such that the displaced load will receive the maximum power.

The condition of maximum power transfer will be met if the *input impedance* to the line is the conjugate of the internal impedance. Using Eq. (98), we write

$$Z_{in} = Z_0 \left[\frac{(R_L + jX_L)\cos(\beta\ell) + jZ_0 \sin(\beta\ell)}{Z_0 \cos(\beta\ell) + j(R_L + jX_L)\sin(\beta\ell)} \right] = R_g - jX_g$$

This is the equation that we have to solve for ℓ – assuming that such a solution exists. To find out, we need to work with the equation a little. Multiplying both sides by the denominator of the left side gives

$$Z_0(R_L + jX_L)\cos(\beta\ell) + jZ_0^2 \sin(\beta\ell) = (R_g - jX_g)[Z_0 \cos(\beta\ell) + j(R_L + jX_L)\sin(\beta\ell)]$$

We next separate the equation by equating the real parts of both sides and the imaginary parts of both sides, giving

$$(R_L - R_g)\cos(\beta\ell) = \frac{X_L X_g}{Z_0} \sin(\beta\ell) \quad \text{(real parts)}$$

and

$$(X_L + X_g)\cos(\beta\ell) = \frac{R_g R_L - Z_0^2}{Z_0} \sin(\beta\ell) \quad \text{(imaginary parts)}$$

Using the two equations, we find two conditions on the tangent of $\beta\ell$:

$$\tan(\beta\ell) = \frac{Z_0(R_L - R_g)}{X_g X_L} = \frac{Z_0(X_L + X_g)}{R_g R_L - Z_0^2}$$

For a viable solution to exist for ℓ, both equalities must be satisfied, thus limiting the possible choices of the two impedances.

10.13. The incident voltage wave on a certain lossless transmission line for which $Z_0 = 50\,\Omega$ and $v_p = 2 \times 10^8$ m/s is $V^+(z,t) = 200\cos(\omega t - \pi z)$ V.

a) Find ω: We know $\beta = \pi = \omega/v_p$, so $\omega = \pi(2 \times 10^8) = \underline{6.28 \times 10^8 \text{ rad/s}}$.

b) Find $I^+(z,t)$: Since Z_0 is real, we may write

$$I^+(z,t) = \frac{V^+(z,t)}{Z_0} = \underline{4\cos(\omega t - \pi z)\ \text{A}}$$

The section of line for which $z > 0$ is replaced by a load $Z_L = 50 + j30\ \Omega$ at $z = 0$. Find

c) Γ_L: This will be

$$\Gamma_L = \frac{50 + j30 - 50}{50 + j30 + 50} = .0825 + j0.275 = \underline{0.287\angle 1.28 \text{ rad}}$$

d) $V_s^-(z) = \Gamma_L V_s^+(z) e^{j2\beta z} = 0.287(200)e^{j\pi z}e^{j1.28} = \underline{57.5e^{j(\pi z + 1.28)}}$

e) V_s at $z = -2.2$ m:

$$\begin{aligned} V_s(-2.2) &= V_s^+(-2.2) + V_s^-(-2.2) = 200e^{j2.2\pi} + 57.5e^{-j(2.2\pi - 1.28)} = 257.5e^{j0.63} \\ &= \underline{257.5\angle 36^\circ} \end{aligned}$$

10.14. A lossless transmission line having characteristic impedance $Z_0 = 50$ ohms is driven by a source at the input end that consists of the series combination of a 10-V sinusoidal generator and a 50-ohm resistor. The line is one-quarter wavelength long. At the other end of the line, a load impedance, $Z_L = 50 - j50$ ohms is attached.

a) Evaluate the input impedance to the line seen by the voltage source-resistor combination: For a quarter-wave section,

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{(50)^2}{50 - j50} = \underline{25 + j25 \text{ ohms}}$$

b) Evaluate the power that is dissipated by the load: This will be the same as the power dissipated by Z_{in}, assuming we replace the line-load section by a lumped element of impedance Z_{in}. The voltage across Z_{in} will be

$$V_{in} = V_{s0}\frac{Z_{in}}{Z_g + Z_{in}} = 10\left[\frac{25 + j25}{50 + 25 + j25}\right] = 10 + j5$$

The power will be

$$P_{in} = P_L = \frac{1}{2}\mathcal{R}e\left\{\frac{V_{in}V_{in}^*}{Z_{in}^*}\right\} = \frac{1}{2}\mathcal{R}e\left\{\frac{(10 + j5)(10 - j5)}{25 - j25}\right\} = \underline{1.25 \text{ W}}$$

c) Evaluate the voltage amplitude that appears across the load: The phasor voltage at any point in the line is given by the sum of forward and backward waves:

$$V_s(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

where $V_0^- = \Gamma_L V_0^+$, and where

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j50 - 50}{50 - j50 + 50} = 0.2 - j0.4$$

By our convention, the load is located at $z = 0$. The voltage at the line input, V_{in}, is therefore given by the above voltage expression evaluated at $z = -\ell$, where $\ell = -\lambda/4$. Thus $\beta\ell = \pi/2$, and

$$V_{in} = V_s(-\ell) = V_0^+\left[e^{j\beta\ell} + \Gamma_L\, e^{-j\beta\ell}\right] = V_0^+\left[j + (0.2 - j0.4)(-j)\right] = V_0^+(-0.4 + j0.8)$$

Using V_{in} from part b, we have

$$V_0^+ = \frac{(10 + j5)}{(-0.4 + j0.8)}$$

Now, the voltage at the load will be

$$V_L = V_0^+(1 + \Gamma_L) = \frac{(10 + j5)}{(-0.4 + j0.8)}(1 + 0.2 - j0.4) = \underline{-5 + j15 \text{ V}}$$

As a check,

$$P_L = \frac{1}{2}\mathcal{R}e\left\{\frac{V_L V_L^*}{Z_L^*}\right\} = \frac{1}{2}\mathcal{R}e\left\{\frac{(-5 + j15)(-5 - j15)}{50 + j50}\right\} = 1.25 \text{ W}$$

which is in agreement with part b.

10.15. For the transmission line represented in Fig. 10.29, find $V_{s,out}$ if $f =$:

a) 60 Hz: At this frequency,

$$\beta = \frac{\omega}{v_p} = \frac{2\pi \times 60}{(2/3)(3 \times 10^8)} = 1.9\times10^{-6} \text{ rad/m} \;\; \text{So } \beta l = (1.9\times10^{-6})(80) = 1.5\times10^{-4} << 1$$

The line is thus essentially a lumped circuit, where $Z_{in} \doteq Z_L = 80\ \Omega$. Therefore

$$V_{s,out} = 120\left[\frac{80}{12+80}\right] = \underline{104 \text{ V}}$$

b) 500 kHz: In this case

$$\beta = \frac{2\pi \times 5 \times 10^5}{2 \times 10^8} = 1.57 \times 10^{-2} \text{ rad/s} \;\; \text{So } \beta l = 1.57 \times 10^{-2}(80) = 1.26 \text{ rad}$$

Now

$$Z_{in} = 50\left[\frac{80\cos(1.26) + j50\sin(1.26)}{50\cos(1.26) + j80\sin(1.26)}\right] = 33.17 - j9.57 = 34.5\angle - .28$$

The equivalent circuit is now the voltage source driving the series combination of Z_{in} and the 12 ohm resistor. The voltage across Z_{in} is thus

$$V_{in} = 120\left[\frac{Z_{in}}{12 + Z_{in}}\right] = 120\left[\frac{33.17 - j9.57}{12 + 33.17 - j9.57}\right] = 89.5 - j6.46 = 89.7\angle - .071$$

The voltage at the line input is now the sum of the forward and backward-propagating waves just to the right of the input. We reference the load at $z = 0$, and so the input is located at $z = -80$ m. In general we write $V_{in} = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$, where

$$V_0^- = \Gamma_L V_0^+ = \frac{80-50}{80+50} V_0^+ = \frac{3}{13} V_0^+$$

At $z = -80$ m we thus have

$$V_{in} = V_0^+\left[e^{j1.26} + \frac{3}{13}e^{-j1.26}\right] \;\Rightarrow\; V_0^+ = \frac{89.5 - j6.46}{e^{j1.26} + (3/13)e^{-j1.26}} = 42.7 - j100 \text{ V}$$

Now

$$V_{s,out} = V_0^+(1 + \Gamma_L) = (42.7 - j100)(1 + 3/(13)) = 134\angle - 1.17\,\text{rad} = \underline{52.6 - j123 \text{ V}}$$

As a check, we can evaluate the average power reaching the load:

$$P_{avg,L} = \frac{1}{2}\frac{|V_{s,out}|^2}{R_L} = \frac{1}{2}\frac{(134)^2}{80} = 112 \text{ W}$$

This must be the same power that occurs at the input impedance:

$$P_{avg,in} = \frac{1}{2}\text{Re}\left\{V_{in}I_{in}^*\right\} = \frac{1}{2}\text{Re}\left\{(89.5 - j6.46)(2.54 + j0.54)\right\} = 112 \text{ W}$$

where $I_{in} = V_{in}/Z_{in} = (89.5 - j6.46)/(33.17 - j9.57) = 2.54 + j0.54$.

10.16. A 100-Ω lossless transmission line is connected to a second line of 40-Ω impedance, whose length is $\lambda/4$. The other end of the short line is terminated by a 25-Ω resistor. A sinusoidal wave (of frequency f) having 50 W average power is incident from the 100-Ω line.

a) Evaluate the input impedance to the quarter-wave line: For the quarter-wave section,

$$Z_{in} = \frac{Z_{02}^2}{Z_L} = \frac{(40)^2}{25} = \underline{64 \text{ ohms}}$$

b) Determine the steady state power that is dissipated by the resistor: This will be the same as the power dropped across a lumped element of impedance Z_{in} at the junction, which replaces the termimated 40-ohm line. The reflection coefficient at the junction is

$$\Gamma_{in} = \frac{Z_{in} - Z_{01}}{Z_{in} + Z_{01}} = \frac{64 - 100}{64 + 100} = -\frac{9}{41}$$

The dissipated power there is then

$$P_{in} = P_L = 50\left(1 - |\Gamma_{in}|^2\right) = 50\left(1 - \left(\frac{9}{41}\right)^2\right) = \underline{47.6 \text{ W}}$$

c) Now suppose the operating frequency is lowered to one-half its original value. Determine the new input impedance, Z'_{in}, for this case: Halving the frequency doubles the wavelength, so that now the 40-ohm section is of length $\ell = \lambda/8$. $\beta\ell$ is now $\pi/4$, and the input impedance, from Eq. (98) is:

$$Z'_{in} = 40\left[\frac{25\cos(\pi/4) + j40\sin(\pi/4)}{40\cos(\pi/4) + j25\sin(\pi/4)}\right] = \underline{36.0 + j17.5 \text{ ohms}}$$

d) For the new frequency, calculate the power in watts that returns to the input end of the line after reflection: The new reflection coefficient is

$$\Gamma'_{in} = \frac{Z'_{in} - Z_{01}}{Z'_{in} + Z_{01}} = \frac{36.0 + j17.5 - 100}{36.0 + j17.5 + 100} = -0.447 + j0.186$$

The reflected power (all of which returns to the input) is

$$P_{ref} = 50|\Gamma'_{in}|^2 = 50(0.234) = \underline{11.7 \text{ W}}$$

10.17. Determine the average power absorbed by each resistor in Fig. 10.30: The problem is made easier by first converting the current source/100 ohm resistor combination to its Thevenin equivalent. This is a $50\angle 0$ V voltage source in series with the 100 ohm resistor. The next step is to determine the input impedance of the 2.6λ length line, terminated by the 25 ohm resistor: We use $\beta l = (2\pi/\lambda)(2.6\lambda) = 16.33$ rad. This value, modulo 2π is (by subtracting 2π twice) 3.77 rad. Now

$$Z_{in} = 50\left[\frac{25\cos(3.77) + j50\sin(3.77)}{50\cos(3.77) + j25\sin(3.77)}\right] = 33.7 + j24.0$$

The equivalent circuit now consists of the series combination of 50 V source, 100 ohm resistor, and Z_{in}, as calculated above. The current in this circuit will be

$$I = \frac{50}{100 + 33.7 + j24.0} = 0.368\angle - .178$$

The power dissipated by the 25 ohm resistor is the same as the power dissipated by the real part of Z_{in}, or

$$P_{25} = P_{33.7} = \frac{1}{2}|I|^2 R = \frac{1}{2}(.368)^2(33.7) = \underline{2.28 \text{ W}}$$

To find the power dissipated by the 100 ohm resistor, we need to return to the Norton configuration, with the original current source in parallel with the 100 ohm resistor, and in parallel with Z_{in}. The voltage across the 100 ohm resistor will be the same as that across Z_{in}, or $V = IZ_{in} = (.368\angle - .178)(33.7 + j24.0) = 15.2\angle 0.44$. The power dissipated by the 100 ohm resistor is now

$$P_{100} = \frac{1}{2}\frac{|V|^2}{R} = \frac{1}{2}\frac{(15.2)^2}{100} = \underline{1.16 \text{ W}}$$

10.18 The line shown in Fig. 10.31 is lossless. Find s on both sections 1 and 2: For section 2, we consider the propagation of one forward and one backward wave, comprising the superposition of all reflected waves from both ends of the section. The ratio of the backward to the forward wave amplitude is given by the reflection coefficient at the load, which is

$$\Gamma_L = \frac{50 - j100 - 50}{50 - j100 + 50} = \frac{-j}{1-j} = \frac{1}{2}(1-j)$$

Then $|\Gamma_L| = (1/2)\sqrt{(1-j)(1+j)} = 1/\sqrt{2}$. Finally

$$s_2 = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+1/\sqrt{2}}{1-1/\sqrt{2}} = \underline{5.83}$$

For section 1, we need the reflection coefficient at the junction (location of the 100 Ω resistor) seen by waves incident from section 1: We first need the input impedance of the $.2\lambda$ length of section 2:

$$Z_{in2} = 50\left[\frac{(50-j100)\cos(\beta_2 l) + j50\sin(\beta_2 l)}{50\cos(\beta_2 l) + j(50-j100)\sin(\beta_2 l)}\right] = 50\left[\frac{(1-j2)(0.309) + j0.951}{0.309 + j(1-j2)(0.951)}\right]$$
$$= 8.63 + j3.82 = 9.44\angle 0.42\,\text{rad}$$

Now, this impedance is in parallel with the 100Ω resistor, leading to a net junction impedance found by

$$\frac{1}{Z_{inT}} = \frac{1}{100} + \frac{1}{8.63 + j3.82} \quad \Rightarrow \quad Z_{inT} = 8.06 + j3.23 = 8.69\angle 0.38\,\text{rad}$$

The reflection coefficient will be

$$\Gamma_j = \frac{Z_{inT} - 50}{Z_{inT} + 50} = -0.717 + j0.096 = 0.723\angle 3.0\,\text{rad}$$

and the standing wave ratio is $s_1 = (1 + 0.723)/(1 - 0.723) = \underline{6.22}$.

10.19. A lossless transmission line is 50 cm in length and operating at a frequency of 100 MHz. The line parameters are $L = 0.2\ \mu$H/m and $C = 80$ pF/m. The line is terminated by a short circuit at $z = 0$, and there is a load, $Z_L = 50 + j20$ ohms across the line at location $z = -20$ cm. What average power is delivered to Z_L if the input voltage is $100\angle 0$ V? With the given capacitance and inductance, we find

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{2 \times 10^{-7}}{8 \times 10^{-11}}} = 50\ \Omega$$

and

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(2 \times 10^{-7})(9 \times 10^{-11})}} = 2.5 \times 10^8 \text{ m/s}$$

Now $\beta = \omega/v_p = (2\pi \times 10^8)/(2.5 \times 10^8) = 2.5$ rad/s. We then find the input impedance to the shorted line section of length 20 cm (putting this impedance at the location of Z_L, so we can combine them): We have $\beta l = (2.5)(0.2) = 0.50$, and so, using the input impedance formula with a zero load impedance, we find $Z_{in1} = j50\tan(0.50) = j27.4$ ohms. Now, at the location of Z_L, the net impedance there is the parallel combination of Z_L and Z_{in1}: $Z_{net} = (50 + j20)||(j27.4) = 7.93 + j19.9$. We now transform this impedance to the line input, 30 cm to the left, obtaining (with $\beta l = (2.5)(.3) = 0.75$):

$$Z_{in2} = 50\left[\frac{(7.93 + j19.9)\cos(.75) + j50\sin(.75)}{50\cos(.75) + j(7.93 + j19.9)\sin(.75)}\right] = 35.9 + j98.0 = 104.3\angle 1.22$$

The power delivered to Z_L is the same as the power delivered to Z_{in2}: The current magnitude is $|I| = (100)/(104.3) = 0.96$ A. So finally,

$$P = \frac{1}{2}|I|^2 R = \frac{1}{2}(0.96)^2(35.9) = \underline{16.5 \text{ W}}$$

10.20 a) Determine s on the transmission line of Fig. 10.32. Note that the dielectric is air: The reflection coefficient at the load is

$$\Gamma_L = \frac{40 + j30 - 50}{40 + j30 + 50} = j0.333 = 0.333\angle 1.57\,\text{rad} \qquad \text{Then} \;\; s = \frac{1 + .333}{1 - .333} = \underline{2.0}$$

b) Find the input impedance: With the length of the line at 2.7λ, we have $\beta l = (2\pi)(2.7) = 16.96$ rad. The input impedance is then

$$Z_{in} = 50 \left[\frac{(40 + j30)\cos(16.96) + j50\sin(16.96)}{50\cos(16.96) + j(40 + j30)\sin(16.96)} \right] = 50 \left[\frac{-1.236 - j5.682}{1.308 - j3.804} \right] = \underline{61.8 - j37.5\ \Omega}$$

c) If $\omega L = 10\,\Omega$, find I_s: The source drives a total impedance given by $Z_{net} = 20 + j\omega L + Z_{in} = 20 + j10 + 61.8 - j37.5 = 81.8 - j27.5$. The current is now $I_s = 100/(81.8 - j27.5) = \underline{1.10 + j0.37\,\text{A}}$.

d) What value of L will produce a maximum value for $|I_s|$ at $\omega = 1$ Grad/s? To achieve this, the imaginary part of the total impedance of part c must be reduced to zero (so we need an inductor). The inductor impedance must be equal to negative the imaginary part of the line input impedance, or $\omega L = 37.5$, so that $L = 37.5/\omega = \underline{37.5\,\text{nH}}$. Continuing, for this value of L, calculate the average power:

e) supplied by the source: $P_s = (1/2)\text{Re}\{V_s I_s^*\} = (1/2)(100)(1.10) = \underline{55.0\,\text{W}}$.

f) delivered to $Z_L = 40 + j30\,\Omega$: The power delivered to the load will be the same as the power delivered to the input impedance. We write

$$P_L = \frac{1}{2}\text{Re}\{Z_{in}\}|I_s|^2 = \frac{1}{2}(61.8)[(1.10 + j.37)(1.10 - j.37)] = \underline{41.6\,\text{W}}$$

10.21. A lossless line having an air dielectric has a characteristic impedance of 400 Ω. The line is operating at 200 MHz and $Z_{in} = 200 - j200\ \Omega$. Use analytic methods or the Smith chart (or both) to find: (a) s; (b) Z_L if the line is 1 m long; (c) the distance from the load to the nearest voltage maximum: I will first use the analytic approach. Using normalized impedances, Eq. (13) becomes

$$z_{in} = \frac{Z_{in}}{Z_0} = \left[\frac{z_L\cos(\beta L) + j\sin(\beta L)}{\cos(\beta L) + jz_L\sin(\beta L)} \right] = \left[\frac{z_L + j\tan(\beta L)}{1 + jz_L\tan(\beta L)} \right]$$

Solve for z_L:

$$z_L = \left[\frac{z_{in} - j\tan(\beta L)}{1 - jz_{in}\tan(\beta L)} \right]$$

where, with $\lambda = c/f = 3 \times 10^8/2 \times 10^8 = 1.50$ m, we find $\beta L = (2\pi)(1)/(1.50) = 4.19$, and so $\tan(\beta L) = 1.73$. Also, $z_{in} = (200 - j200)/400 = 0.5 - j0.5$. So

$$z_L = \frac{0.5 - j0.5 - j1.73}{1 - j(0.5 - j0.5)(1.73)} = 2.61 + j0.174$$

Finally, $Z_L = z_L(400) = \underline{1.04 \times 10^3 + j69.8\ \Omega}$. Next

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{6.42 \times 10^2 + j69.8}{1.44 \times 10^3 + j69.8} = .446 + j2.68 \times 10^{-2} = .447\angle 6.0 \times 10^{-2}\ \text{rad}$$

10.21. (continued) Now

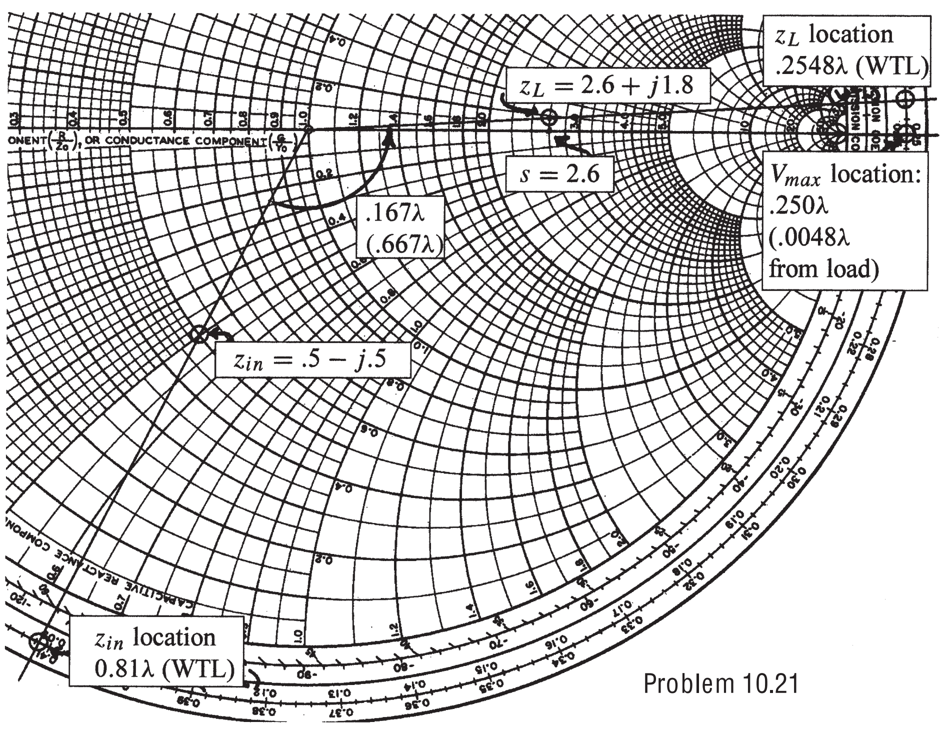
$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+.447}{1-.447} = \underline{2.62}$$

Finally

$$z_{max} = -\frac{\phi}{2\beta} = -\frac{\lambda\phi}{4\pi} = -\frac{(6.0\times10^{-2})(1.50)}{4\pi} = -7.2\times10^{-3}\text{ m} = \underline{-7.2\text{ mm}}$$

We next solve the problem using the Smith chart. Referring to the figure below, we first locate and mark the normalized input impedance, $z_{in} = 0.5 - j0.5$. A line drawn from the origin through this point intersects the outer chart boundary at the position $0.0881\,\lambda$ on the wavelengths toward load (WTL) scale. With a wavelength of 1.5 m, the 1 meter line is 0.6667 wavelengths long. On the WTL scale, we add 0.6667λ, or equivalently, 0.1667λ (since 0.5λ is once around the chart), obtaining $(0.0881+0.1667)\lambda) = 0.2548\lambda$, which is the position of the load. A straight line is now drawn from the origin though the 0.2548λ position. A compass is then used to measure the distance between the origin and z_{in}. With this distance set, the compass is then used to scribe off the same distance from the origin to the load impedance, along the line between the origin and the 0.2548λ position. That point is the normalized load impedance, which is read to be $z_L = 2.6 + j0.18$. Thus $Z_L = z_L(400) = 1040 + j72$. This is in reasonable agreement with the analytic result of $1040 + j69.8$. The difference in imaginary parts arises from uncertainty in reading the chart in that region.

In transforming from the input to the load positions, we cross the $r > 1$ real axis of the chart at r=2.6. This is close to the value of the VSWR, as we found earlier. We also see that the $r > 1$ real axis (at which the first V_{max} occurs) is a distance of 0.0048λ (marked as $.005\lambda$ on the chart) in front of the load. The actual distance is $z_{max} = -0.0048(1.5)\,\text{m} = -0.0072\,\text{m} = -7.2\,\text{mm}$.



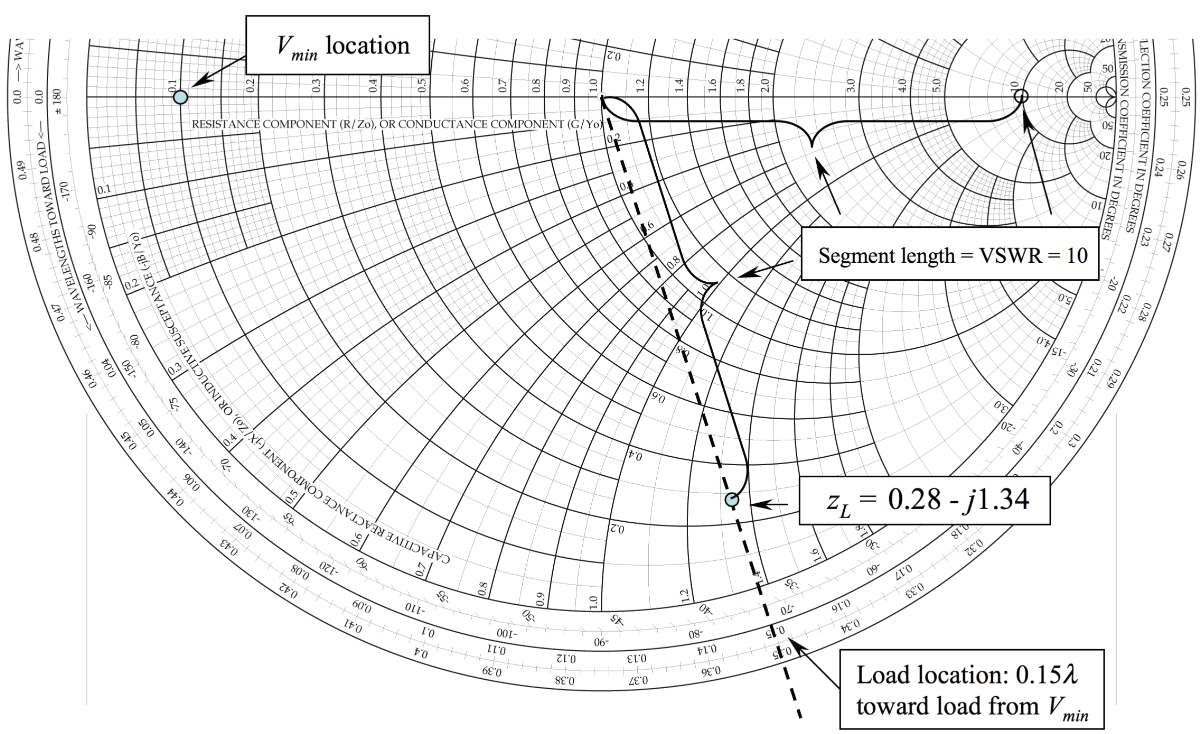
Problem 10.21

10.22. A lossless 75-ohm line is terminated by an unknown load impedance. A VSWR of 10 is measured, and the first voltage minimum occurs at a 0.15 wavelengths in front of the load. Using the Smith chart, find

a) The load impedance: Referring to the Smith chart section below, first mark the VSWR on the positive real axis and set the compass to that length. The voltage minimum will be located on the negative real axis and will have normalized impedance of the reciprocal of the VSWR, or 0.1. This value is marked and is labled as the V_{min} location. Now, move toward the load by a distance of 0.15 wavelenths (using the wavelenths toward load scale). The dashed line is drawn from the origin through the 0.15λ mark on the scale. Use the compass (set to the VSWR length) to scribe the point on the dashed line that is labled z_L. We identify that as the normalized load impedance, $z_L = 0.28 - j1.34$. The load impedance is then $Z_L = 75z_L = \underline{21.0 - j100}$ ohms

b) The magnitude and phase of the reflection coefficient: The magnitude of Γ_L can be found by measuring the compass span on the linear "Ref. coeff. E or I" scale on the bottom of the chart. Set the compass point at the center position, and then scribe on the scale to the left to find $|\Gamma_L| = 0.82$. The phase is the angle of the dashed line from the positive real axis, which is read from the "angle of reflection coefficient" scale as $\phi = -72°$. In summary, $\Gamma_L = \underline{0.82\angle - 72°}$

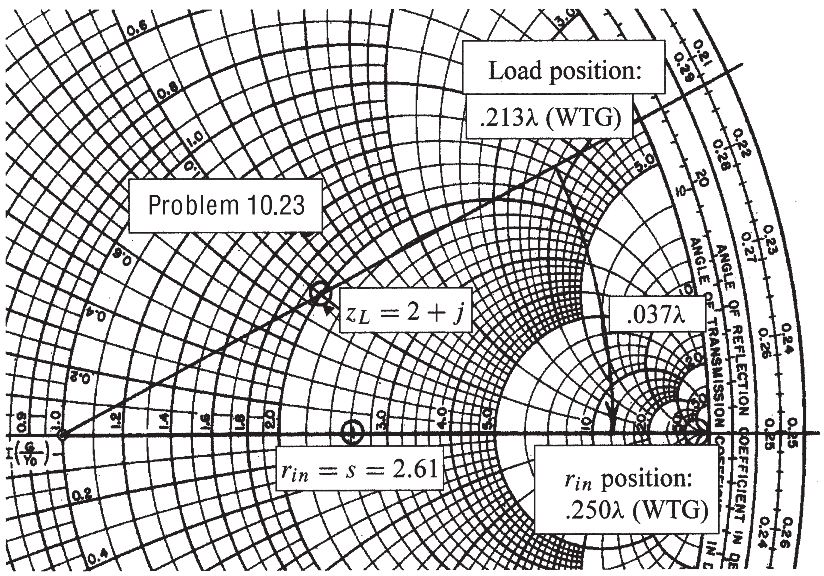
c) The shortest length of line necessary to achieve an entirely resistive input impedance: In moving toward the generator from the load, we look for the first real axis crossing. This occurs simply at the V_{min} location, and so we identify the shortest length as just $\underline{0.15\lambda}$.



10.23. The normalized load on a lossless transmission line is $z_L = 2 + j1$. Let $\lambda = 20$ m Make use of the Smith chart to find:

a) the shortest distance from the load to the point at which $z_{in} = r_{in} + j0$, where $r_{in} > 1$ (not greater than 0 as stated): Referring to the figure below, we start by marking the given z_L on the chart and drawing a line from the origin through this point to the outer boundary. On the WTG scale, we read the z_L location as 0.213λ. Moving from here toward the generator, we cross the positive Γ_R axis (at which the impedance is purely real and greater than 1) at 0.250λ. The distance is then $(0.250 - 0.213)\lambda = \underline{0.037\lambda}$ from the load. With $\lambda = 20$ m, the actual distance is $20(0.037) = 0.74$ m.

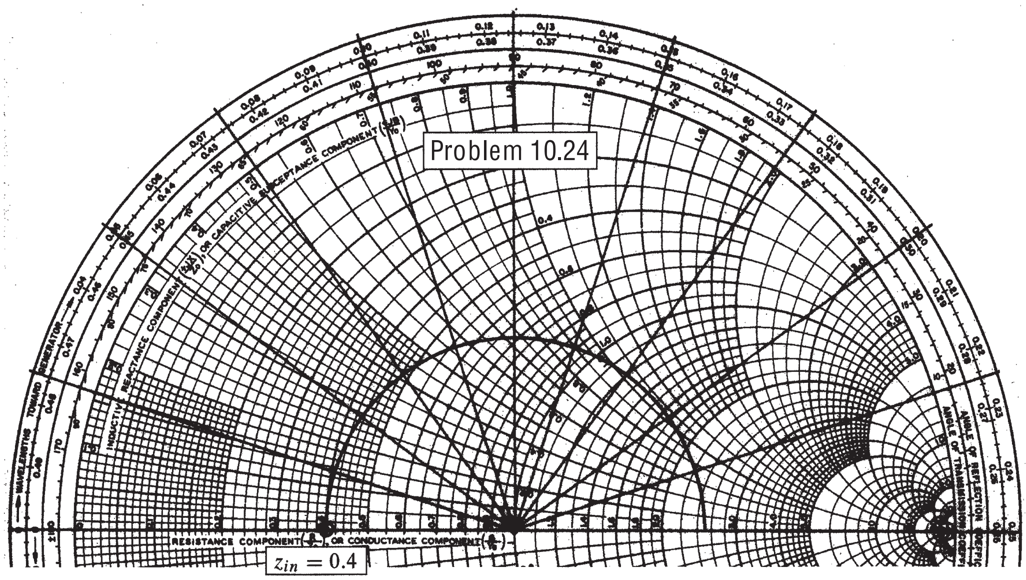
b) Find z_{in} at the point found in part a: Using a compass, we set its radius at the distance between the origin and z_L. We then scribe this distance along the real axis to find $z_{in} = r_{in} = \underline{2.61}$.



c) The line is cut at this point and the portion containing z_L is thrown away. A resistor $r = r_{in}$ of part a is connected across the line. What is s on the remainder of the line? This will be just s for the line as it was before. As we know, s will be the positive real axis value of the normalized impedance, or $s = \underline{2.61}$.

d) What is the shortest distance from this resistor to a point at which $z_{in} = 2 + j1$? This would return us to the original point, requiring a complete circle around the chart (one-half wavelength distance). The distance from the resistor will therefore be: $d = 0.500\,\lambda - 0.037\,\lambda = \underline{0.463\,\lambda}$. With $\lambda = 20$ m, the actual distance would be $20(0.463) = 9.26$ m.

10.24. With the aid of the Smith chart, plot a curve of $|Z_{in}|$ vs. l for the transmission line shown in Fig. 10.33. Cover the range $0 < l/\lambda < 0.25$. The required input impedance is that at the actual line input (to the left of the two 20Ω resistors. The input to the line section occurs just to the right of the 20Ω resistors, and the input impedance there we first find with the Smith chart. This impedance is in series with the two 20Ω resistors, so we add 40Ω to the calculated impedance from the Smith chart to find the net line input impedance. To begin, the 20Ω load resistor represents a normalized impedance of $z_l = 0.4$, which we mark on the chart (see below). Then, using a compass, draw a circle beginning at z_L and progressing clockwise to the positive real axis. The circle traces the locus of z_{in} values for line lengths over the range $0 < l < \lambda/4$.



On the chart, radial lines are drawn at positions corresponding to .025λ increments on the WTG scale. The intersections of the lines and the circle give a total of 11 z_{in} values. To these we add normalized impedance of $40/50 = 0.8$ to add the effect of the 40Ω resistors and obtain the normalized impedance at the line input. The magnitudes of these values are then found, and the results are multiplied by 50Ω. The table below summarizes the results.

l/λ	z_{inl} (to right of 40Ω)	$z_{in} = z_{inl} + 0.8$	$\|Z_{in}\| = 50\|z_{in}\|$
0	0.40	1.20	60
.025	0.41 + j.13	1.21 + j.13	61
.050	0.43 + j.27	1.23 + j.27	63
.075	0.48 + j.41	1.28 + j.41	67
.100	0.56 + j.57	1.36 + j.57	74
.125	0.68 + j.73	1.48 + j.73	83
.150	0.90 + j.90	1.70 + j.90	96
.175	1.20 + j1.05	2.00 + j1.05	113
.200	1.65 + j1.05	2.45 + j1.05	134
.225	2.2 + j.7	3.0 + j.7	154
.250	2.5	3.3	165

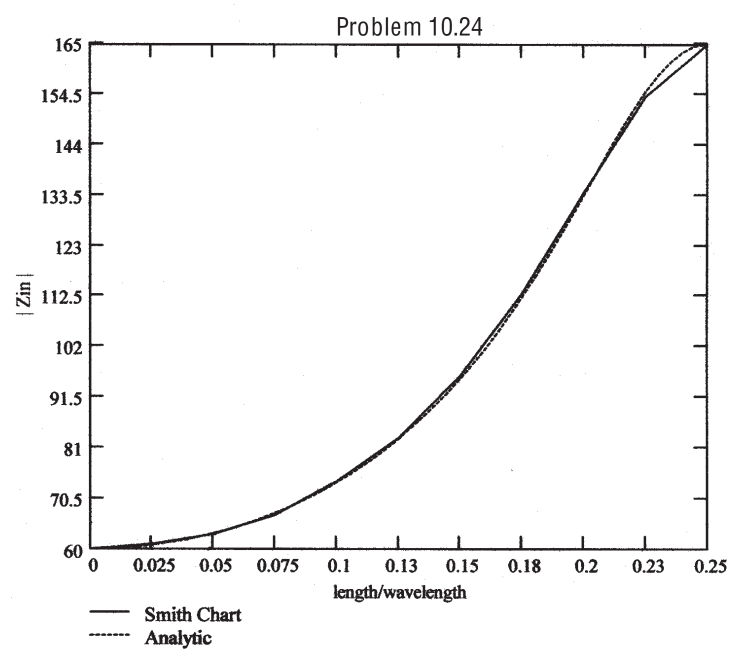
10.24. (continued) As a check, the line input input impedance can be found analytically through

$$Z_{in} = 40 + 50 \left[\frac{20\cos(2\pi l/\lambda) + j50\sin(2\pi l/\lambda)}{50\cos(2\pi l/\lambda) + j20\sin(2\pi l/\lambda)} \right] = 50 \left[\frac{60\cos(2\pi l/\lambda) + j66\sin(2\pi l/\lambda)}{50\cos(2\pi l/\lambda) + j20\sin(2\pi l/\lambda)} \right]$$

from which

$$|Z_{in}| = 50 \left[\frac{36\cos^2(2\pi l/\lambda) + 43.6\sin^2(2\pi l/\lambda)}{25\cos^2(2\pi l/\lambda) + 4\sin^2(2\pi l/\lambda)} \right]^{1/2}$$

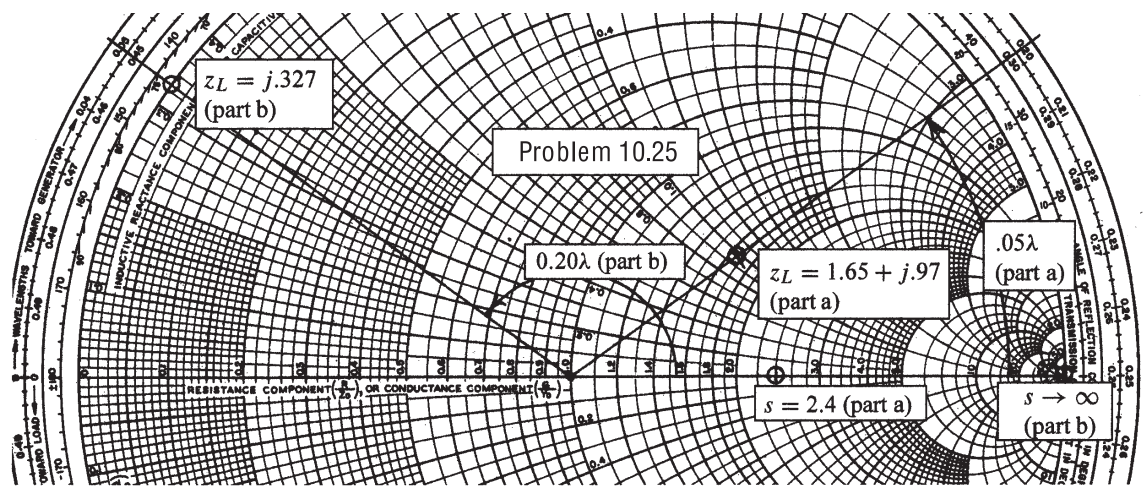
This function is plotted below along with the results obtained from the Smith chart. A fairly good comparison is obtained.



10.25. A 300-ohm transmission line is short-circuited at $z = 0$. A voltage maximum, $|V|_{max} = 10$ V, is found at $z = -25$ cm, and the minimum voltage, $|V|_{min} = 0$, is found at $z = -50$ cm. Use the Smith chart to find Z_L (with the short circuit replaced by the load) if the voltage readings are:

a) $|V|_{max} = 12$ V at $z = -5$ cm, and $|V|_{min} = 5$ V: First, we know that the maximum and minimum voltages are spaced by $\lambda/4$. Since this distance is given as 25 cm, we see that $\lambda = 100$ cm $= 1$ m. Thus the maximum voltage location is $5/100 = 0.05\lambda$ in front of the load. The standing wave ratio is $s = |V|_{max}/|V|_{min} = 12/5 = 2.4$. We mark this on the positive real axis of the chart (see next page). The load position is now 0.05 wavelengths toward the load from the $|V|_{max}$ position, or at 0.30 λ on the WTL scale. A line is drawn from the origin through this point on the chart, as shown. We next set the compass to the distance between the origin and the $z = r = 2.4$ point on the real axis. We then scribe this same distance along the line drawn through the .30 λ position. The intersection is the value of z_L, which we read as $z_L = 1.65 + j.97$. The actual load impedance is then $Z_L = 300 z_L = \underline{495 + j290\ \Omega}$.

b) $|V|_{max} = 17$ V at $z = -20$ cm, and $|V|_{min} = 0$. In this case the standing wave ratio is infinite, which puts the starting point on the $r \to \infty$ point on the chart. The distance of 20 cm corresponds to $20/100 = 0.20\,\lambda$, placing the load position at $0.45\,\lambda$ on the WTL scale. A line is drawn from the origin through this location on the chart. An infinite standing wave ratio places us on the outer boundary of the chart, so we read $z_L = j0.327$ at the $0.45\,\lambda$ WTL position. Thus $Z_L = j300(0.327) \doteq \underline{j98\,\Omega}$.



10.26. A 50-ohm lossless line is of length 1.1λ. It is terminated by an unknown load impedance. The input end of the 50-ohm line is attached to the load end of a lossless 75-ohm line. A VSWR of 4 is measured on the 75-ohm line, on which the first voltage maximum occurs at a distance of 0.2λ in front of the junction between the two lines. Use the Smith chart to find the unknown load impedance.

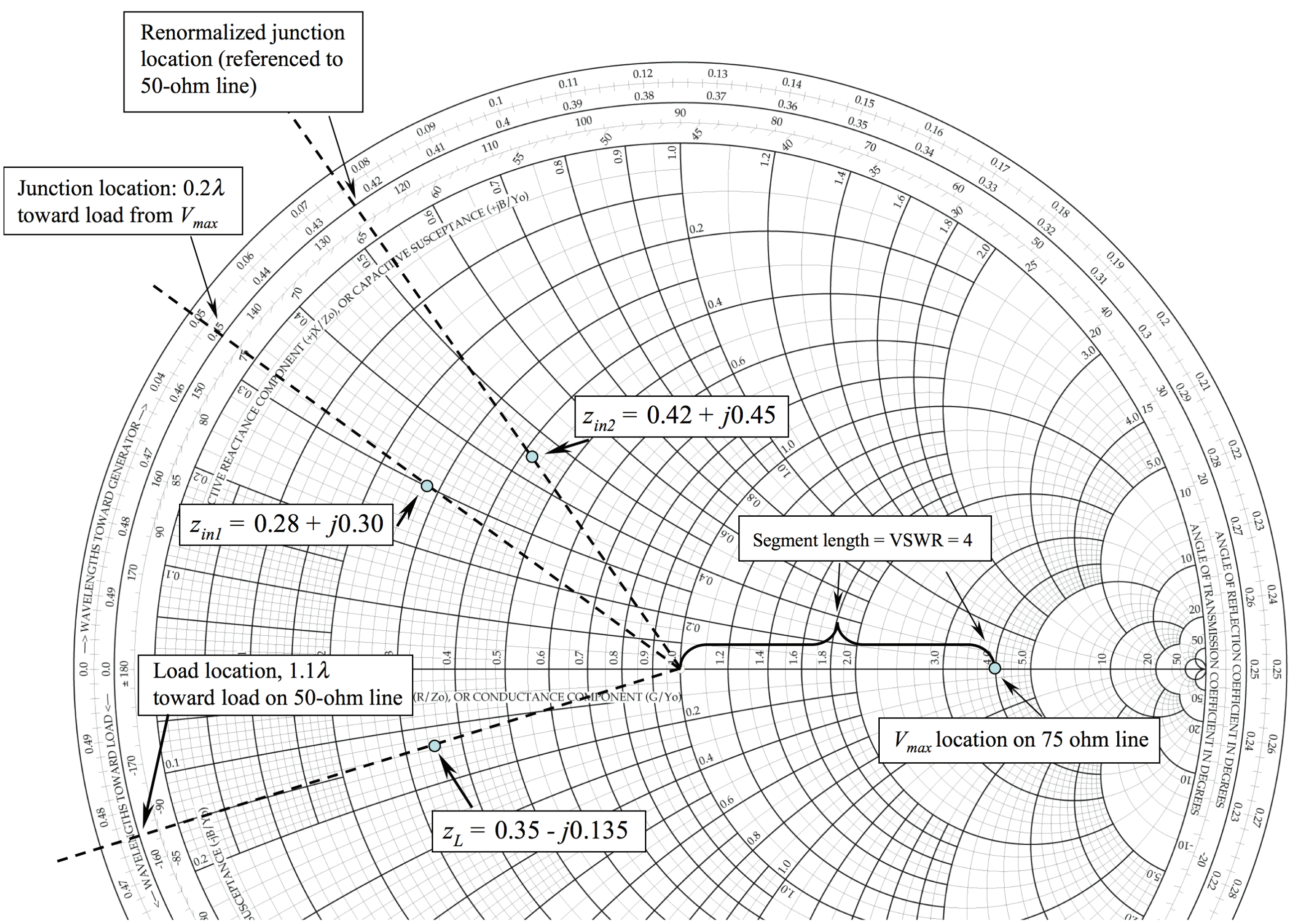
First, mark the VSWR on the positive real axis, which gives the magnitude of Γ as determined on the 75-ohm line. The starting point is thus $r = 4, x = 0$, which is the location of the first voltage maximum. From there, move toward the load by 0.2 wavelengths, and note the normalized impedance there, marked as $z_{in1} = 0.28 + j0.30$. This is the normalized load impedance at the junction, as seen by the 75-ohm line.

The next step is to re-normalize z_{in1} to the 50-ohm line to find z_{in2}. This will be

$$z_{in2} = z_{in1}\,\frac{75}{50} = 0.42 + j0.45$$

which is marked on the chart as shown. Now, this point is translated toward the load by 1.1λ (equivalent to 0.1λ) to obtain the normlized load impedance, $z_L = 0.35 - j0.135$, marked on the chart. The load impedance is thus

$$Z_L = 50(0.35 - j0.135) = \underline{17.5 - j6.8} \text{ ohms}$$

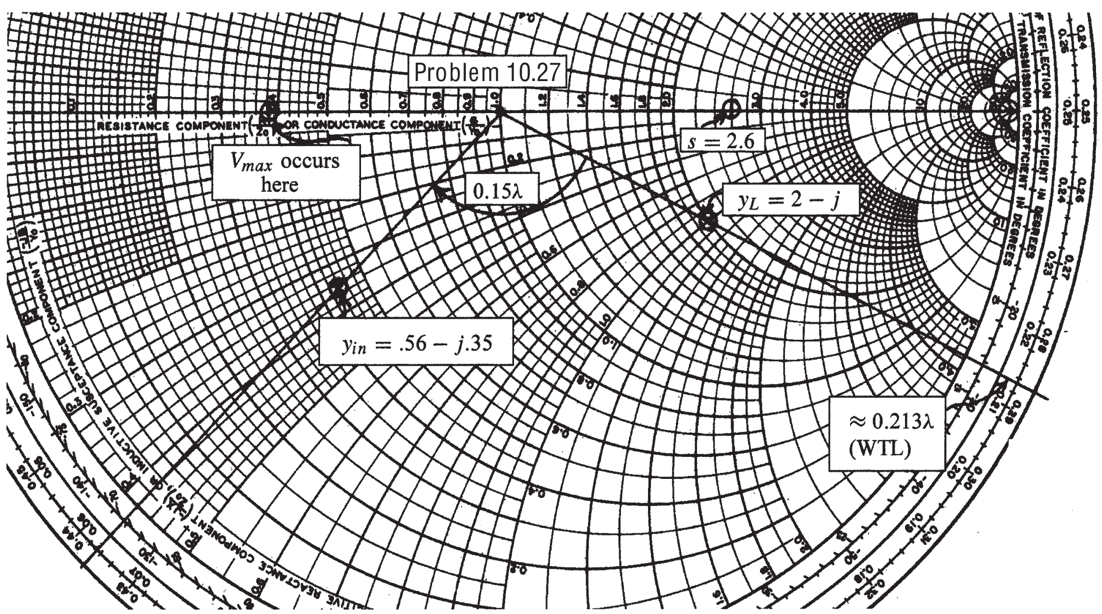


10.27. The characteristic admittance ($Y_0 = 1/Z_0$) of a lossless transmission line is 20 mS. The line is terminated in a load $Y_L = 40 - j20$ mS. Make use of the Smith chart to find:

a) s: We first find the normalized load admittance, which is $y_L = Y_L/Y_0 = 2 - j1$. This is plotted on the Smith chart below. We then set on the compass the distance between y_L and the origin. The same distance is then scribed along the positive real axis, and the value of s is read as 2.6.

b) Y_{in} if $l = 0.15\,\lambda$: First we draw a line from the origin through z_L and note its intersection with the WTG scale on the chart outer boundary. We note a reading on that scale of about 0.287 λ. To this we add 0.15 λ, obtaining about 0.437 λ, which we then mark on the chart (0.287 λ is not the precise value, but I have added 0.15 λ to that mark to obtain the point shown on the chart that is near to 0.437 λ. This "eyeballing" method increases the accuracy a little). A line drawn from the 0.437 λ position on the WTG scale to the origin passes through the input admittance. Using the compass, we scribe the distance found in part a across this line to find $y_{in} = 0.56 - j0.35$, or $Y_{in} = 20y_{in} = 11 - j7.0$ mS.

c) the distance in wavelengths from Y_L to the nearest voltage maximum: On the admittance chart, the V_{max} position is on the negative Γ_r axis. This is at the zero position on the WTL scale. The load is at the approximate $0.213\,\lambda$ point on the WTL scale, so this distance is the one we want.

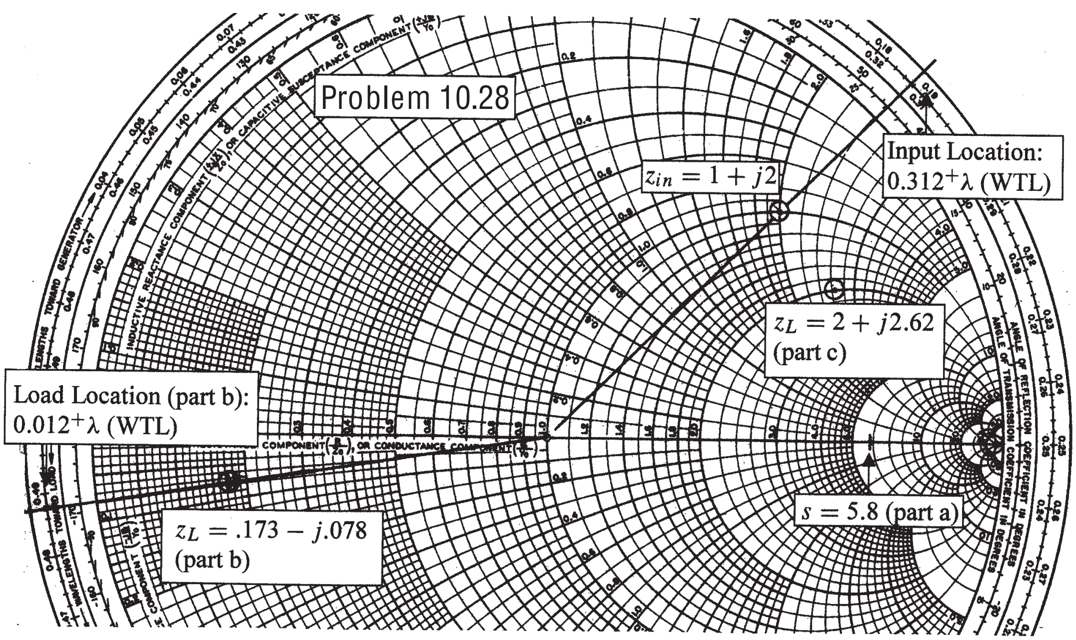


10.28. The wavelength on a certain lossless line is 10cm. If the normalized input impedance is $z_{in} = 1 + j2$, use the Smith chart to determine:

a) s: We begin by marking z_{in} on the chart (see below), and setting the compass at its distance from the origin. We then use the compass at that setting to scribe a mark on the positive real axis, noting the value there of $s = \underline{5.8}$.

b) z_L, if the length of the line is 12 cm: First, use a straight edge to draw a line from the origin through z_{in}, and through the outer scale. We read the input location as slightly more than 0.312λ on the WTL scale (this additional distance beyond the .312 mark is not measured, but is instead used to add a similar distance when the impedance is transformed). The line length of 12cm corresponds to 1.2 wavelengths. Thus, to transform to the load, we go counter-clockwise twice around the chart, plus 0.2λ, finally arriving at (again) slightly more than 0.012λ on the WTL scale. A line is drawn to the origin from that position, and the compass (with its previous setting) is scribed through the line. The intersection is the normalized load impedance, which we read as $z_L = \underline{0.173 - j0.078}$.

c) x_L, if $z_L = 2 + jx_L$, where $x_L > 0$. For this, use the compass at its original setting to scribe through the $r = 2$ circle in the upper half plane. At that point we read $x_L = \underline{2.62}$.



10.29. A standing wave ratio of 2.5 exists on a lossless 60 Ω line. Probe measurements locate a voltage minimum on the line whose location is marked by a small scratch on the line. When the load is replaced by a short circuit, the minima are 25 cm apart, and one minimum is located at a point 7 cm toward the source from the scratch. Find Z_L: We note first that the 25 cm separation between minima imply a wavelength of twice that, or $\lambda = 50$ cm. Suppose that the scratch locates the first voltage minimum. With the short in place, the first minimum occurs at the load, and the second at 25 cm in front of the load. The effect of replacing the short with the load is to move the minimum at 25 cm to a new location 7 cm toward the load, or at 18 cm. This is a possible location for the scratch, which would otherwise occur at multiples of a half-wavelength farther away from that point, toward the generator. Our assumed scratch position will be 18 cm or $18/50 = 0.36$ wavelengths from the load. Using the Smith chart (see below) we first draw a line from the origin through the 0.36λ point on the wavelengths toward load scale. We set the compass to the length corresponding to the $s = r = 2.5$ point on the chart, and then scribe this distance through the straight line. We read $z_L = 0.79 + j0.825$, from which $Z_L = 47.4 + j49.5\ \Omega$. As a check, I will do the problem analytically. First, we use

$$z_{min} = -18\,\text{cm} = -\frac{1}{2\beta}(\phi + \pi) \quad \Rightarrow \quad \phi = \left[\frac{4(18)}{50} - 1\right]\pi = 1.382\,\text{rad} = 79.2^\circ$$

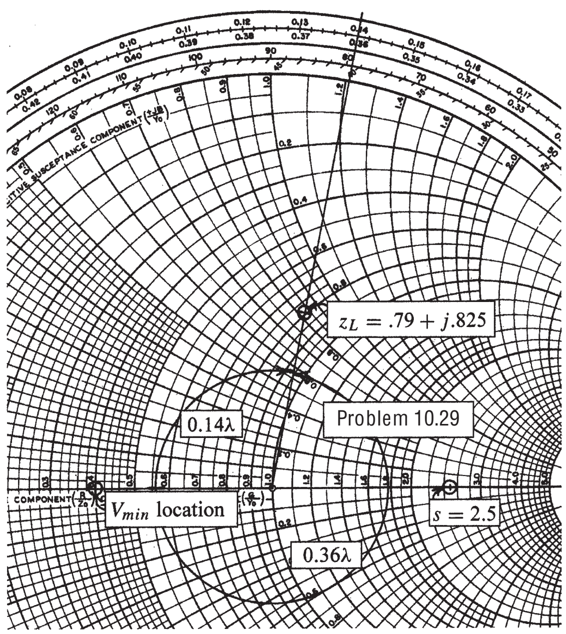
Now

$$|\Gamma_L| = \frac{s-1}{s+1} = \frac{2.5-1}{2.5+1} = 0.4286$$

and so $\Gamma_L = 0.4286\angle 1.382$. Using this, we find

$$z_L = \frac{1+\Gamma_L}{1-\Gamma_L} = 0.798 + j0.823$$

and thus $Z_L = z_L(60) = \underline{47.8 + j49.3\ \Omega}$.

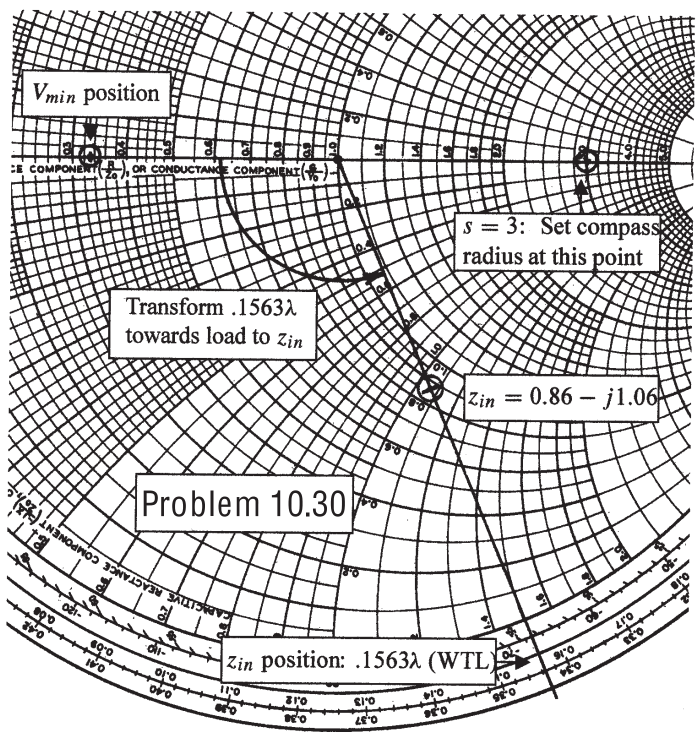


10.30. A 2-wire line, constructed of lossless wire of circular cross-section is gradually flared into a coupling loop that looks like an egg beater. At the point X, indicated by the arrow in Fig. 10.34, a short circuit is placed across the line. A probe is moved along the line and indicates that the first voltage minimum to the left of X is 16cm from X. With the short circuit removed, a voltage minimum is found 5cm to the left of X, and a voltage maximum is located that is 3 times voltage of the minimum. Use the Smith chart to determine:

a) f: No Smith chart is needed to find f, since we know that the first voltage minimum in front of a short circuit is one-half wavelength away. Therefore, $\lambda = 2(16) = 32$cm, and (assuming an air-filled line), $f = c/\lambda = 3 \times 10^8/0.32 = \underline{0.938\,\text{GHz}}$.

b) s: Again, no Smith chart is needed, since s is the ratio of the maximum to the minimum voltage amplitudes. Since we are given that $V_{max} = 3V_{min}$, we find $s = \underline{3}$.

c) the normalized input impedance of the egg beater as seen looking the right at point X: Now we need the chart. From the figure below, $s = 3$ is marked on the positive real axis, which determines the compass radius setting. This point is then transformed, using the compass, to the negative real axis, which corresponds to the location of a voltage minimum. Since the first V_{min} is 5cm in front of X, this corresponds to $(5/32)\lambda = 0.1563\lambda$ to the left of X. On the chart, we now move this distance from the V_{min} location toward the load, using the WTL scale. A line is drawn from the origin through the 0.1563λ mark on the WTL scale, and the compass is used to scribe the original radius through this line. The intersection is the normalized input impedance, which is read as $z_{in} = \underline{0.86 - j1.06}$.



10.31. In order to compare the relative sharpness of the maxima and minima of a standing wave, assume a load $z_L = 4 + j0$ is located at $z = 0$. Let $|V|_{min} = 1$ and $\lambda = 1$ m. Determine the width of the

a) minimum, where $|V| < 1.1$: We begin with the general phasor voltage in the line:

$$V(z) = V^+(e^{-j\beta z} + \Gamma e^{j\beta z})$$

With $z_L = 4 + j0$, we recognize the real part as the standing wave ratio. Since the load impedance is real, the reflection coefficient is also real, and so we write

$$\Gamma = |\Gamma| = \frac{s-1}{s+1} = \frac{4-1}{4+1} = 0.6$$

The voltage magnitude is then

$$|V(z)| = \sqrt{V(z)V^*(z)} = V^+ \left[(e^{-j\beta z} + \Gamma e^{j\beta z})(e^{j\beta z} + \Gamma e^{-j\beta z})\right]^{1/2}$$
$$= V^+ \left[1 + 2\Gamma \cos(2\beta z) + \Gamma^2\right]^{1/2}$$

Note that with $\cos(2\beta z) = \pm 1$, we obtain $|V| = V^+(1 \pm \Gamma)$ as expected. With $s = 4$ and with $|V|_{min} = 1$, we find $|V|_{max} = 4$. Then with $\Gamma = 0.6$, it follows that $V^+ = 2.5$. The net expression for $|V(z)|$ is then

$$V(z) = 2.5\sqrt{1.36 + 1.2\cos(2\beta z)}$$

To find the width in z of the voltage minimum, defined as $|V| < 1.1$, we set $|V(z)| = 1.1$ and solve for z: We find

$$\left(\frac{1.1}{2.5}\right)^2 = 1.36 + 1.2\cos(2\beta z) \quad \Rightarrow \quad 2\beta z = \cos^{-1}(-0.9726)$$

Thus $2\beta z = 2.904$. At this stage, we note the the $|V|_{min}$ point will occur at $2\beta z = \pi$. We therefore compute the range, Δz, over which $|V| < 1.1$ through the equation:

$$2\beta(\Delta z) = 2(\pi - 2.904) \quad \Rightarrow \quad \Delta z = \frac{\pi - 2.904}{2\pi/1} = 0.0378\,\text{m} = \underline{3.8\,\text{cm}}$$

where $\lambda = 1$ m has been used.

b) Determine the width of the maximum, where $|V| > 4/1.1$: We use the same equation for $|V(z)|$, which in this case reads:

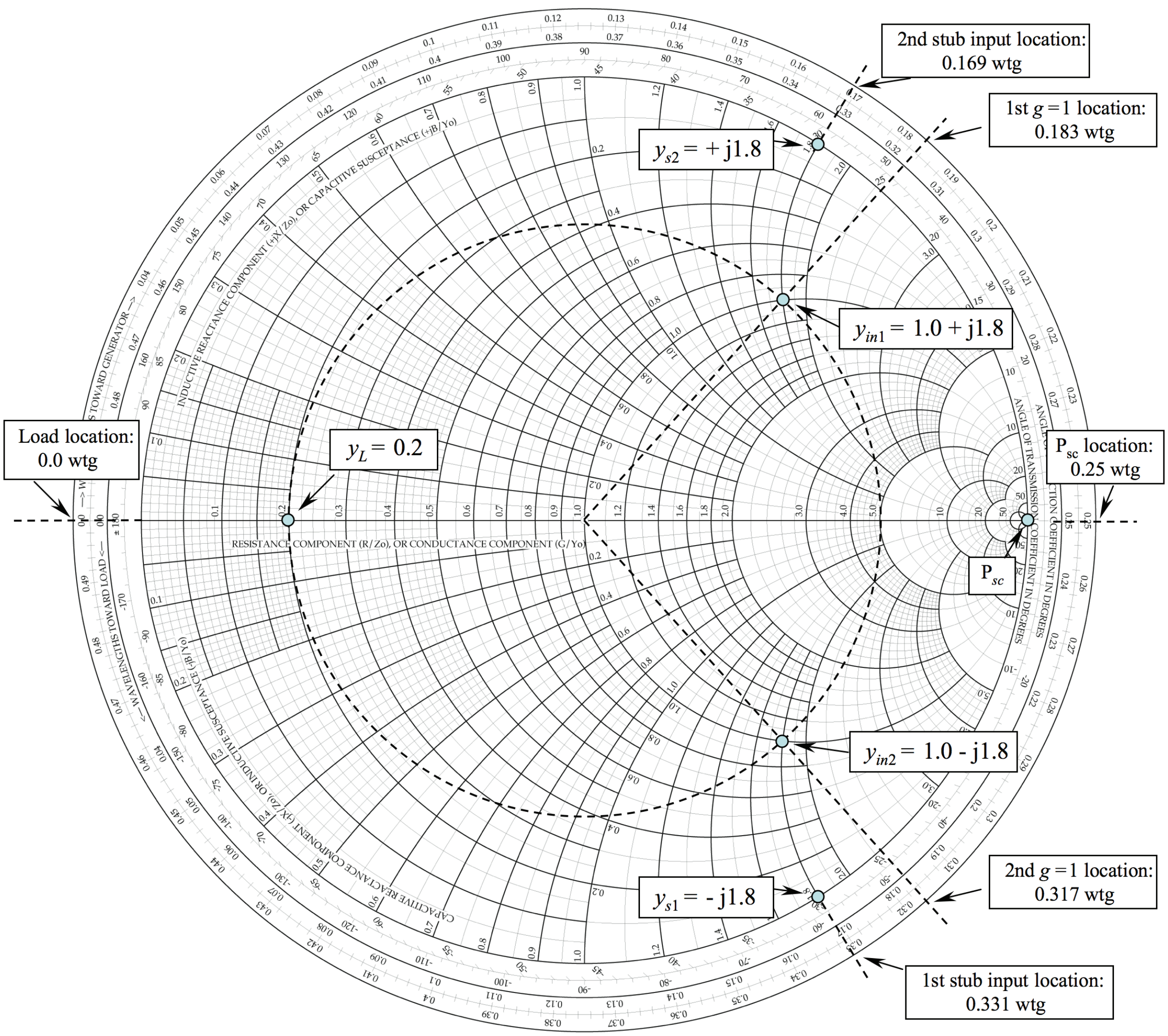
$$4/1.1 = 2.5\sqrt{1.36 + 1.2\cos(2\beta z)} \quad \Rightarrow \quad \cos(2\beta z) = 0.6298$$

Since the maximum corresponds to $2\beta z = 0$, we find the range through

$$2\beta\Delta z = 2\cos^{-1}(0.6298) \quad \Rightarrow \quad \Delta z = \frac{0.8896}{2\pi/1} = 0.142\,\text{m} = \underline{14.2\,\text{cm}}$$

10.32. In Fig. 10.7, let $Z_L = 250$ ohms, $Z_0 = 50$ ohms, find the shortest attachment distance d and the shortest length, d_1 of a short-circuited stub line that will provide a perfect match on the main line to the left of the stub. Express all answers in wavelengths.

The first step is to mark the normalized load admittance on the chart. This will be $y_L = 1/z_L = 50/250 = 0.20$. Its location is noted as 0.0 on the wavelengths toward generator (WTG) scale. Next, from the load, move clockwise (toward generator) until the admittance real part is unity. The first instance of this is at the point $y_{in1} = 1 + j1.8$, as shown. Moving farther, the second instance is at the point $y_{in2} = 1.0 - j1.8$. The distance in wavelenghs between y_L and y_{in1} is noted on the WTG scale, or $d_a = 0.183\lambda$. The distance in wavelenghs between y_L and y_{in2} is again noted on the WTG scale, or $d_b = 0.317\lambda$. These are the two possible attachment points for the shorted stub. The shortest of these is $\underline{d_a = 0.183\lambda}$. The corresponding stub length is found by transforming from the short circuit (load) position on the stub, P_{sc} toward generator until a normalized admittance of $y_s = b_s = -j1.8$ occurs. This is marked on the chart as the point y_{s1}, located at 0.331λ (WTG). The (shortest) stub length is thus $d_{1a} = (0.331 - 0.250)\lambda = \underline{0.81\lambda}$.



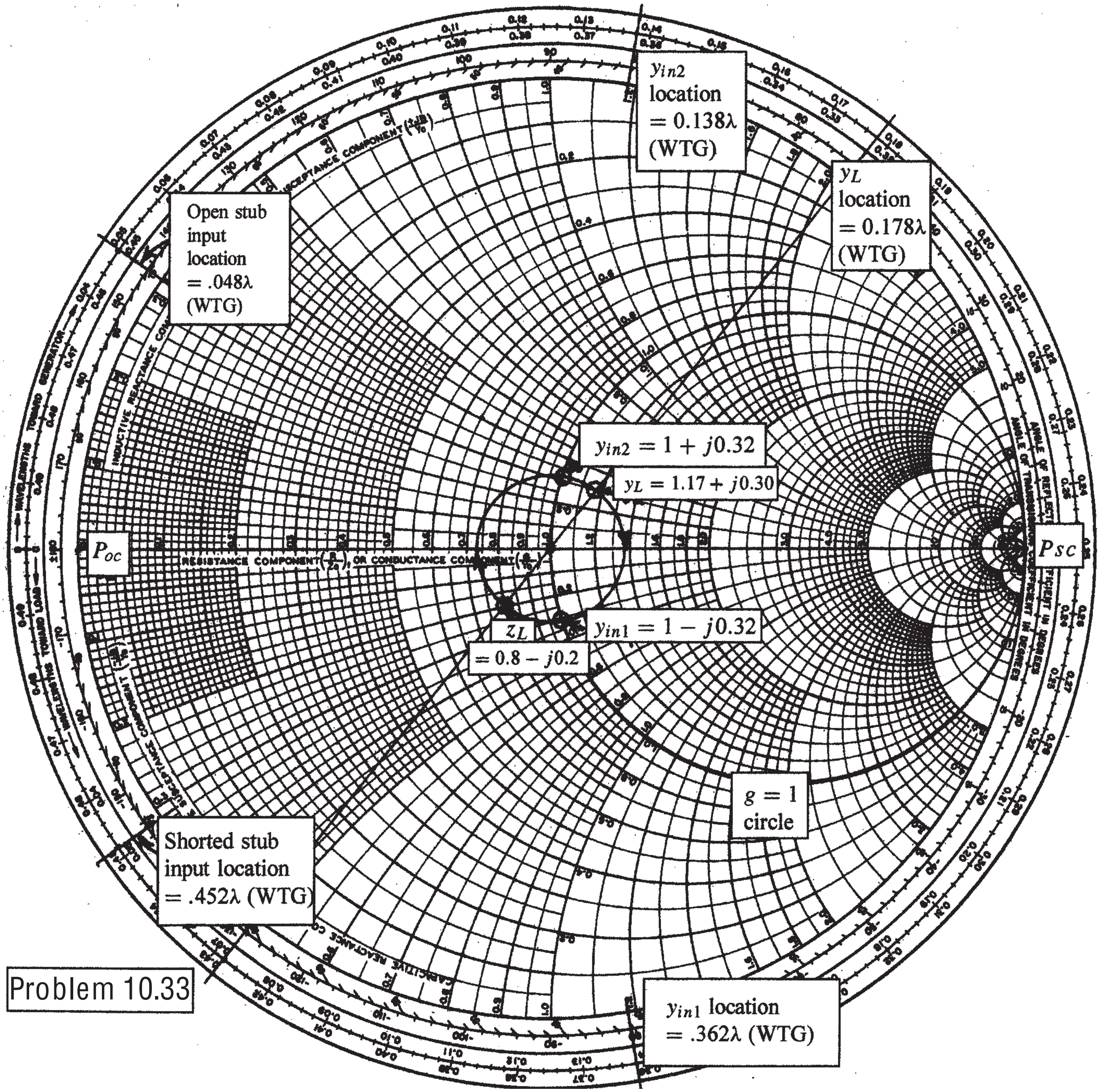
10.33. In Fig. 10.17, let $Z_L = 40 - j10\ \Omega$, $Z_0 = 50\ \Omega$, $f = 800$ MHz, and $v = c$.

a) Find the shortest length, d_1, of a short-circuited stub, and the shortest distance d that it may be located from the load to provide a perfect match on the main line to the left of the stub:

The Smith chart construction is shown on the next page. First we find $z_L = (40 - j10)/50 = 0.8 - j0.2$ and plot it on the chart. Next, we find $y_L = 1/z_L$ by transforming this point halfway around the chart, where we read $y_L = 1.17 + j0.30$. This point is to be transformed to a location at which the real part of the normalized admittance is unity. The $g = 1$ circle is highlighted on the chart; y_L transforms to two locations on it: $y_{in1} = 1 - j0.32$ and $y_{in2} = 1 + j0.32$. The stub is connected at either of these two points. The stub input admittance must cancel the imaginary part of the line admittance at that point. If y_{in2} is chosen, the stub must have input admittance of $-j0.32$. This point is marked on the outer circle and occurs at $0.452\,\lambda$ on the WTG scale. The length of the stub is found by computing the distance between its input, found above, and the short-circuit position (stub load end), marked as P_{sc}. This distance is $d_1 = (0.452 - 0.250)\lambda = 0.202\,\lambda$. With $f = 800$ MHz and $v = c$, the wavelength is $\lambda = (3 \times 10^8)/(8 \times 10^8) = 0.375$ m. The distance is thus $d_1 = (0.202)(0.375) = 0.758\,\text{m} = \underline{7.6\,\text{cm}}$. This is the shortest of the two possible stub lengths, since if we had used y_{in1}, we would have needed a stub input admittance of $+j0.32$, which would have required a longer stub length to realize. The length of the main line between its load and the stub attachment point is found on the chart by measuring the distance between y_L and y_{in2}, in moving clockwise (toward generator). This distance will be $d = [0.500 - (0.178 - 0.138)]\,\lambda = 0.46\,\lambda$. The actual length is then $d = (0.46)(0.375) = 0.173\text{m} = \underline{17.3\,\text{cm}}$.

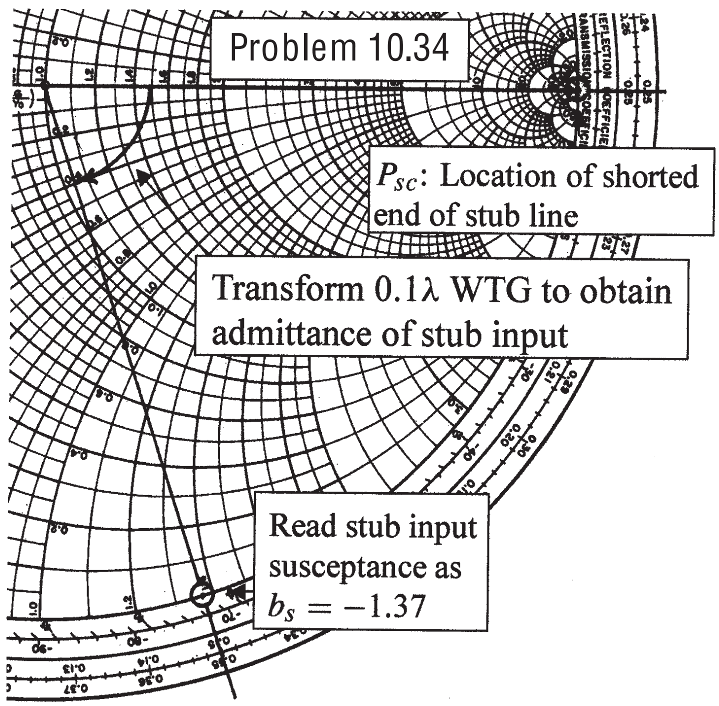
b) Repeat for an open-circuited stub:

In this case, everything is the same, except for the load-end position of the stub, which now occurs at the P_{oc} point on the chart. To use the shortest possible stub, we need to use $y_{in1} = 1 - j0.32$, requiring $y_s = +j0.32$. We find the stub length by moving from P_{oc} to the point at which the admittance is $j0.32$. This occurs at $0.048\,\lambda$ on the WTG scale, which thus determines the required stub length. Now $d_1 = (0.048)(0.375) = 0.18\,\text{m} = \underline{1.8\,\text{cm}}$. The attachment point is found by transforming y_L to y_{in1}, where the former point is located at $0.178\,\lambda$ on the WTG scale, and the latter is at $0.362\,\lambda$ on the same scale. The distance is then $d = (0.362 - 0.178)\lambda = 0.184\lambda$. The actual length is $d = (0.184)(0.375) = 0.069\,\text{m} = \underline{6.9\,\text{cm}}$.



Problem 10.33

10.34. The lossless line shown in Fig. 10.35 is operating with $\lambda = 100$cm. If $d_1 = 10$cm, $d = 25$cm, and the line is matched to the left of the stub, what is Z_L? For the line to be matched, it is required that the sum of the normalized input admittances of the shorted stub and the main line at the point where the stub is connected be unity. So the input susceptances of the two lines must cancel. To find the stub input susceptance, use the Smith chart to transform the short circuit point 0.1λ toward the generator, and read the input value as $b_s = -1.37$ (note that the stub length is one-tenth of a wavelength). The main line input admittance must now be $y_{in} = 1 + j1.37$. This line is one-quarter wavelength long, so the normalized load impedance is equal to the normalized input admittance. Thus $z_L = 1 + j1.37$, so that $Z_L = 300 z_L = \underline{300 + j411\ \Omega}$.

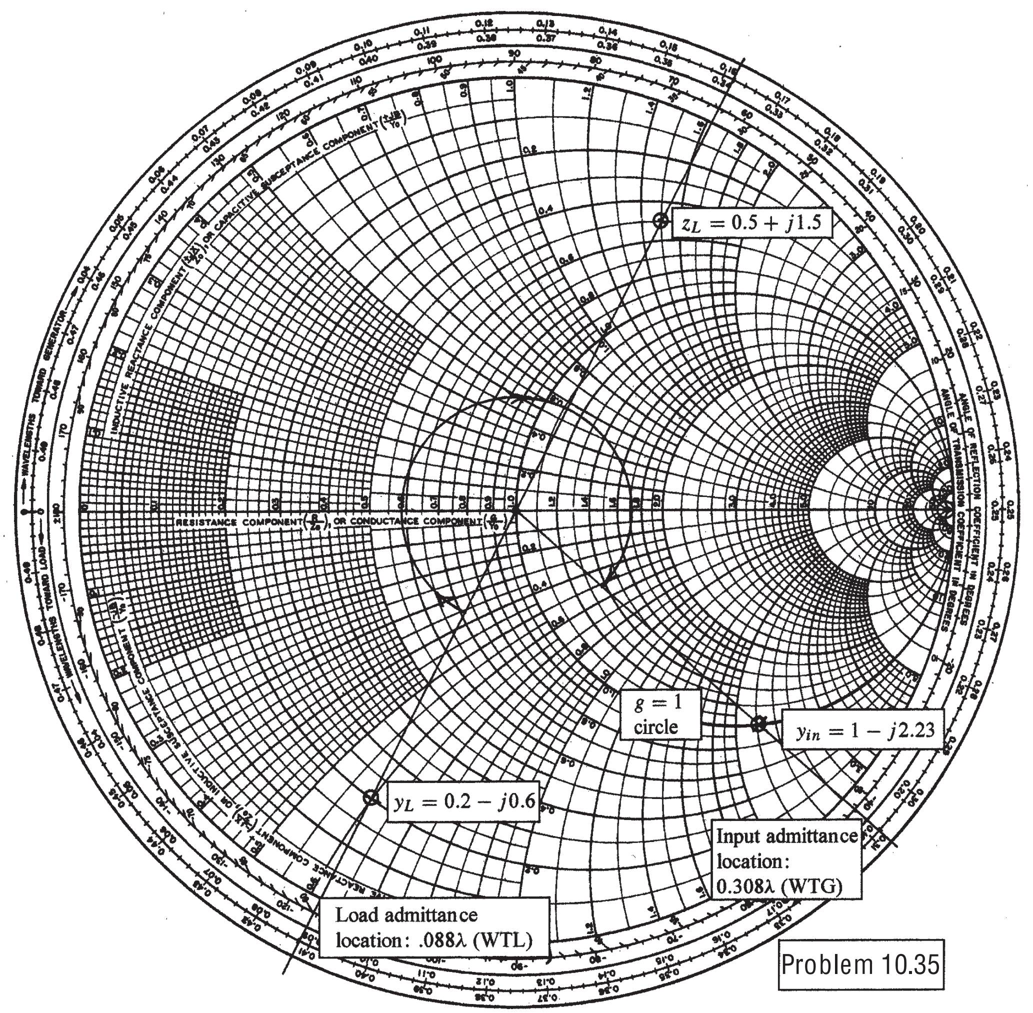


10.35. A load, $Z_L = 25 + j75\ \Omega$, is located at $z = 0$ on a lossless two-wire line for which $Z_0 = 50\ \Omega$ and $v = c$.

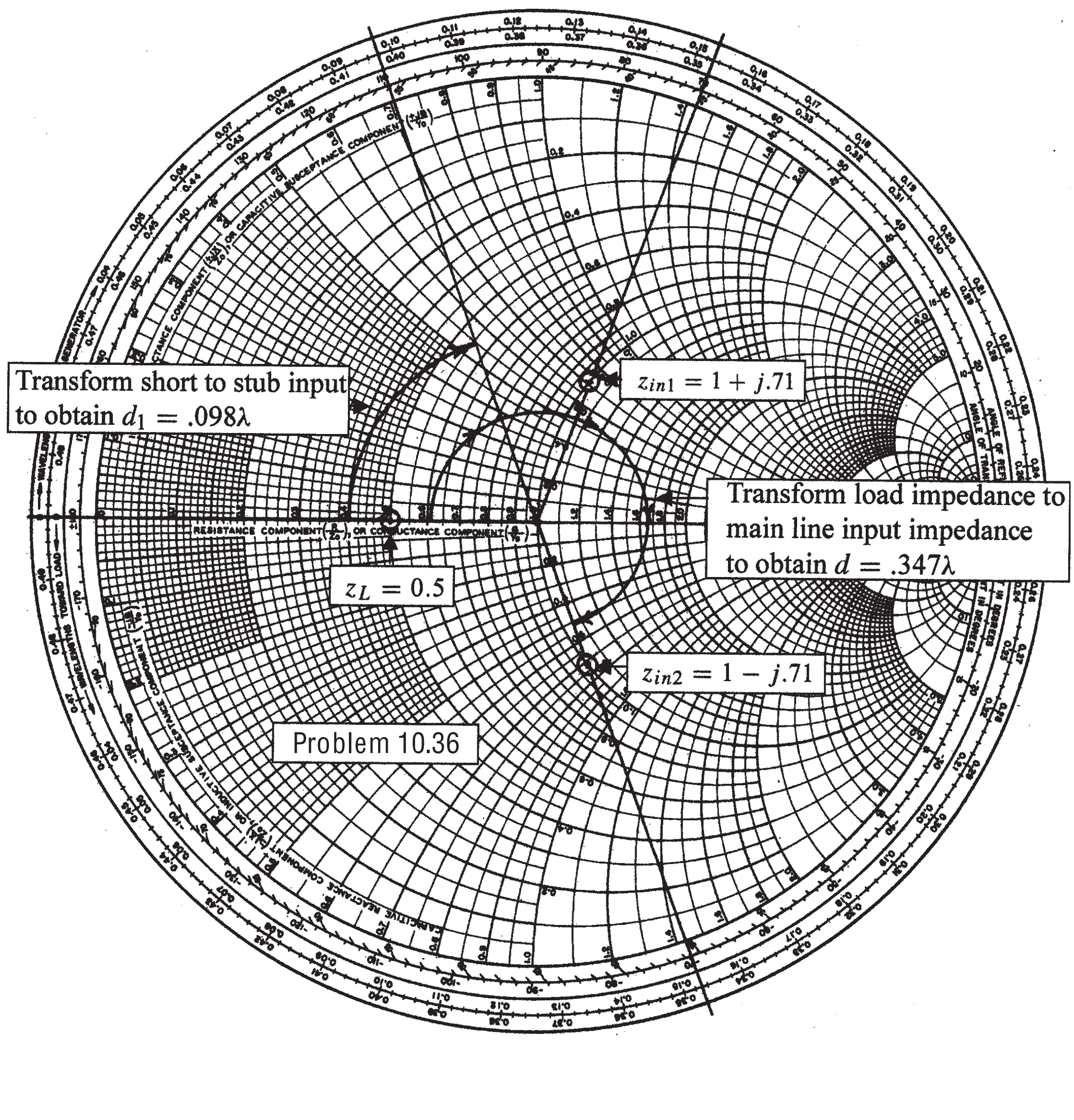
a) If $f = 300$ MHz, find the shortest distance d ($z = -d$) at which the input impedance has a real part equal to $1/Z_0$ and a negative imaginary part: The Smith chart construction is shown below. We begin by calculating $z_L = (25 + j75)/50 = 0.5 + j1.5$, which we then locate on the chart. Next, this point is transformed by rotation halfway around the chart to find $y_L = 1/z_L = 0.20 - j0.60$, which is located at 0.088 λ on the WTL scale. This point is then transformed toward the generator until it intersects the $g = 1$ circle (shown highlighted) with a negative imaginary part. This occurs at point $y_{in} = 1.0 - j2.23$, located at 0.308 λ on the WTG scale. The total distance between load and input is then $d = (0.088 + 0.308)\lambda = 0.396\lambda$. At 300 MHz, and with $v = c$, the wavelength is $\lambda = 1$ m. Thus the distance is $d = 0.396\,\text{m} = \underline{39.6\,\text{cm}}$.

b) What value of capacitance C should be connected across the line at that point to provide unity standing wave ratio on the remaining portion of the line? To cancel the input normalized susceptance of -2.23, we need a capacitive normalized susceptance of +2.23. We therefore write

$$\omega C = \frac{2.23}{Z_0} \quad \Rightarrow \quad C = \frac{2.23}{(50)(2\pi \times 3 \times 10^8)} = 2.4 \times 10^{-11}\ \text{F} = \underline{24\ \text{pF}}$$



10.36. The two-wire lines shown in Fig. 10.36 are all lossless and have $Z_0 = 200\,\Omega$. Find d and the shortest possible value for d_1 to provide a matched load if $\lambda = 100$cm. In this case, we have a series combination of the loaded line section and the shorted stub, so we use impedances and the Smith chart as an impedance diagram. The requirement for matching is that the total normalized impedance at the junction (consisting of the sum of the input impedances to the stub and main loaded section) is unity. First, we find $z_L = 100/200 = 0.5$ and mark this on the chart (see below). We then transform this point toward the generator until we reach the $r = 1$ circle. This happens at two possible points, indicated as $z_{in1} = 1 + j.71$ and $z_{in2} = 1 - j.71$. The stub input impedance must cancel the imaginary part of the loaded section input impedance, or $z_{ins} = \pm j.71$. The shortest stub length that accomplishes this is found by transforming the short circuit point on the chart to the point $z_{ins} = +j0.71$, which yields a stub length of $d_1 = .098\lambda = \underline{9.8\,\text{cm}}$. The length of the loaded section is then found by transforming $z_L = 0.5$ to the point $z_{in2} = 1 - j.71$, so that $z_{ins} + z_{in2} = 1$, as required. This transformation distance is $d = 0.347\lambda = \underline{37.7\,\text{cm}}$.

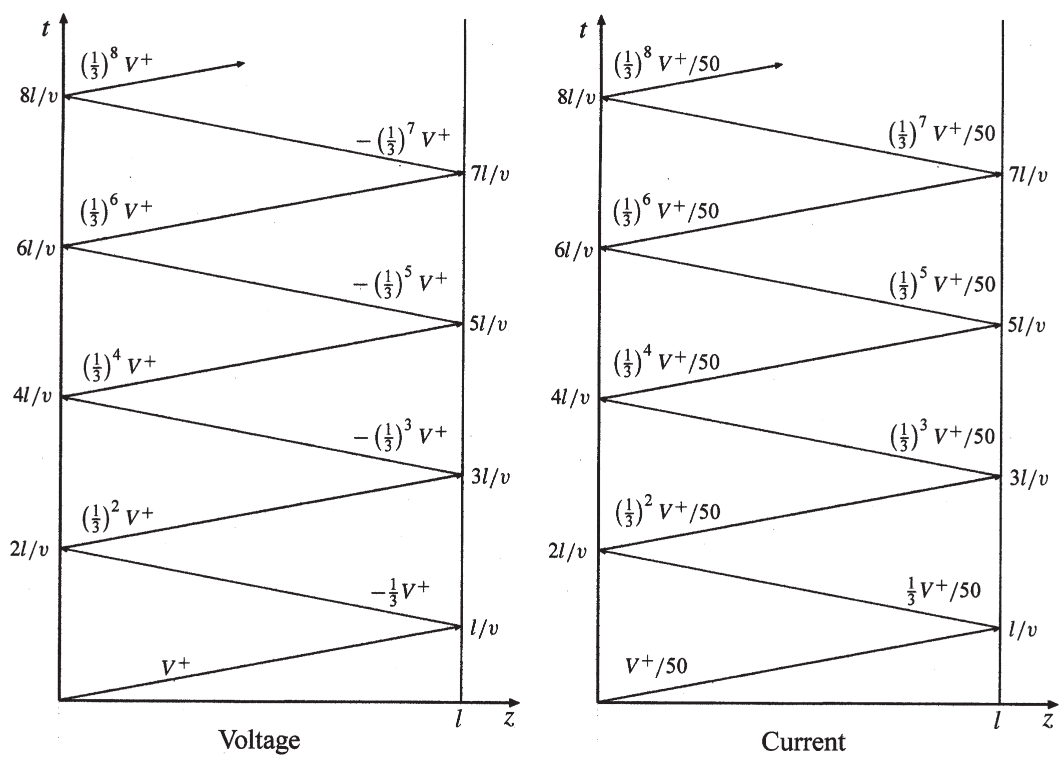


10.37. In the transmission line of Fig. 10.20, $R_g = Z_0 = 50\ \Omega$, and $R_L = 25\ \Omega$. Determine and plot the voltage at the load resistor and the current in the battery as functions of time by constructing appropriate voltage and current reflection diagrams: Referring to the figure, closing the switch launches a voltage wave whose value is given by Eq. (119):

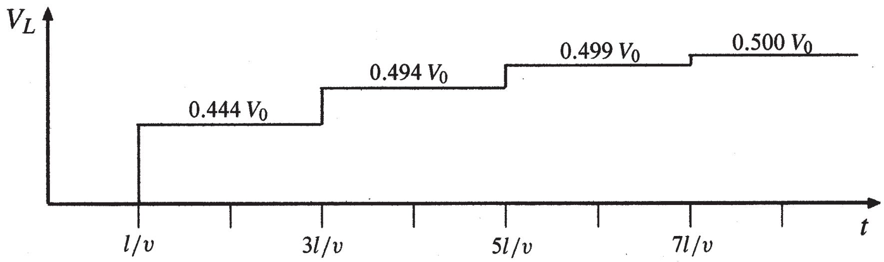
$$V_1^+ = \frac{V_0 Z_0}{R_g + Z_0} = \frac{50}{100} V_0 = \frac{1}{2} V_0$$

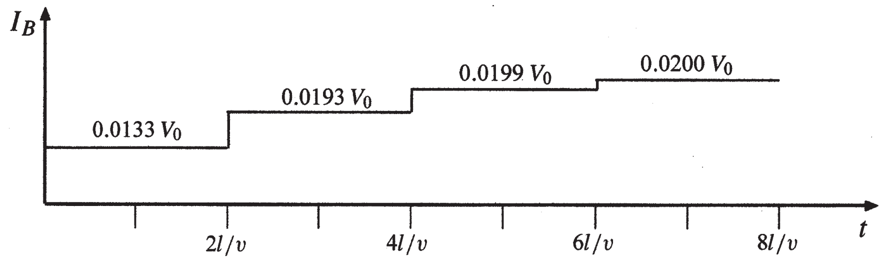
Now, $\Gamma_L = (25 - 50)/(25 + 50) = -1/3$. So on reflection from the load, the reflected wave is of value $V_1^- = -V_0/6$. On returning to the input end, the reflection coefficient there is zero, and so all is still. The voltage reflection diagram would be that shown in Fig. 10.21a, except that no waves are present after time $t = 2l/v$. Likewise, the current reflection diagram is that of Fig. 10.22a, except, again, no waves exist after $t = l/v$. The voltage at the load will be $V_L = V_1^+(1 + \Gamma_L) = V_0/3$ for times beyond l/v. The current through the battery is initially $I_B = V_1^+/Z_0 = V_0/100$ for times $(0 < t < 2l/v)$. When the reflected wave from the load returns to the input end (at time $t = 2l/v$), the reflected wave current, $I_1^- = V_0/300$, adds to the original current to give $I_B = V_0/75$ A for $(t > 2l/v)$.

10.38. Repeat Problem 37, with $Z_0 = 50\Omega$, and $R_L = R_g = 25\Omega$. Carry out the analysis for the time period $0 < t < 8l/v$. At the generator end, we have $\Gamma_g = -1/3$. At the load end, we have $\Gamma_L = -1/3$ as before. The initial wave is of magnitude $V^+ = (2/3)V_0$. Using these values, voltage and current reflection diagrams are constructed, and are shown below:



10.38. (continued) From the diagrams, voltage and current plots are constructed. First, the load voltage is found by adding voltages along the right side of the voltage diagram at the indicated times. Second, the current through the battery is found by adding currents along the left side of the current reflection diagram. Both plots are shown below, where currents and voltages are expressed to three significant figures. The steady state values, $V_L = 0.5$V and $I_B = 0.02$A, are expected as $t \to \infty$.





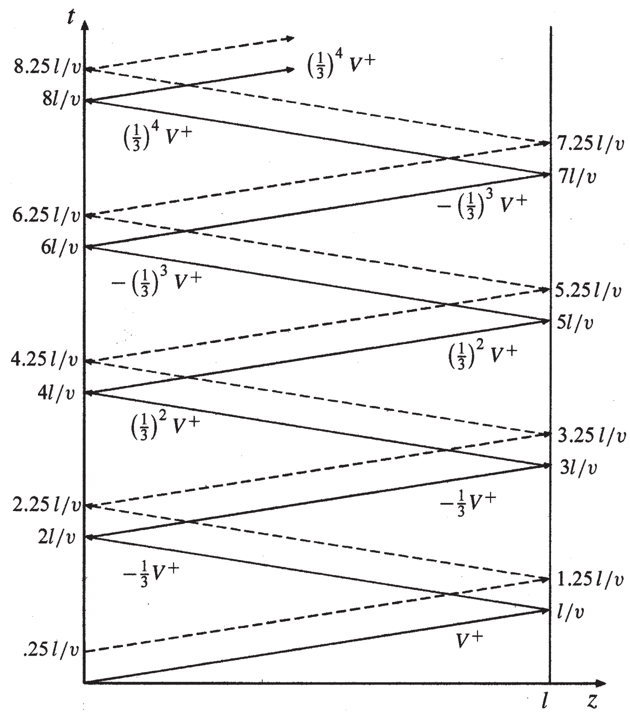
10.39. In the transmission line of Fig. 10.20, $Z_0 = 50\,\Omega$ and $R_L = R_g = 25\,\Omega$. The switch is closed at $t = 0$ and *is opened again* at time $t = l/4v$, thus creating a rectangular voltage *pulse* in the line. Construct an appropriate voltage reflection diagram for this case and use it to make a plot of the voltage at the load resistor as a function of time for $0 < t < 8l/v$ (note that the effect of opening the switch is to initiate a second voltage wave, whose value is such that it leaves a net current of zero in its wake): The value of the initial voltage wave, formed by closing the switch, will be

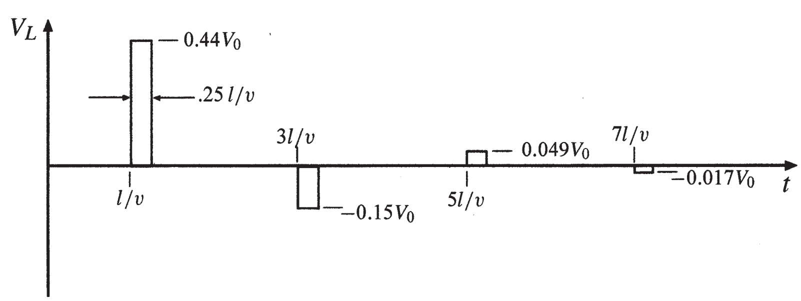
$$V^+ = \frac{Z_0}{R_g + Z_0} V_0 = \frac{50}{25 + 50} V_0 = \frac{2}{3} V_0$$

On opening the switch, a second wave, $V^{+\prime}$, is generated which leaves a net current behind it of zero. This means that $V^{+\prime} = -V^+ = -(2/3)V_0$. Note also that when the switch is opened, the reflection coefficient at the generator end of the line becomes unity. The reflection coefficient at the load end is $\Gamma_L = (25 - 50)/(25 + 50) = -(1/3)$. The reflection diagram is now constructed in the usual manner, and is shown on the next page. The path of the second wave as it reflects from either end is shown in dashed lines, and is a replica of the first wave path, displaced later in time by $l/(4v)$.a All values for the second wave after each reflection are equal but of opposite sign to the immediately preceding first wave values. The load voltage as a function of time is found by accumulating voltage values as they are read moving up along the right hand boundary of the chart. The resulting function, plotted just below the reflection diagram, is found to be a sequence of pulses that alternate signs. The pulse amplitudes are calculated as follows:

10.39. (continued)

$$\frac{l}{v} < t < \frac{5l}{4v}: \quad V_1 = \left(1 - \frac{1}{3}\right) V^+ = 0.44\, V_0$$
$$\frac{3l}{v} < t < \frac{13l}{4v}: \quad V_2 = -\frac{1}{3}\left(1 - \frac{1}{3}\right) V^+ = -0.15\, V_0$$
$$\frac{5l}{v} < t < \frac{21l}{4v}: \quad V_3 = \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{3}\right) V^+ = 0.049\, V_0$$
$$\frac{7l}{v} < t < \frac{29l}{4v}: \quad V_4 = -\left(\frac{1}{3}\right)^3 \left(1 - \frac{1}{3}\right) V^+ = -0.017\, V_0$$

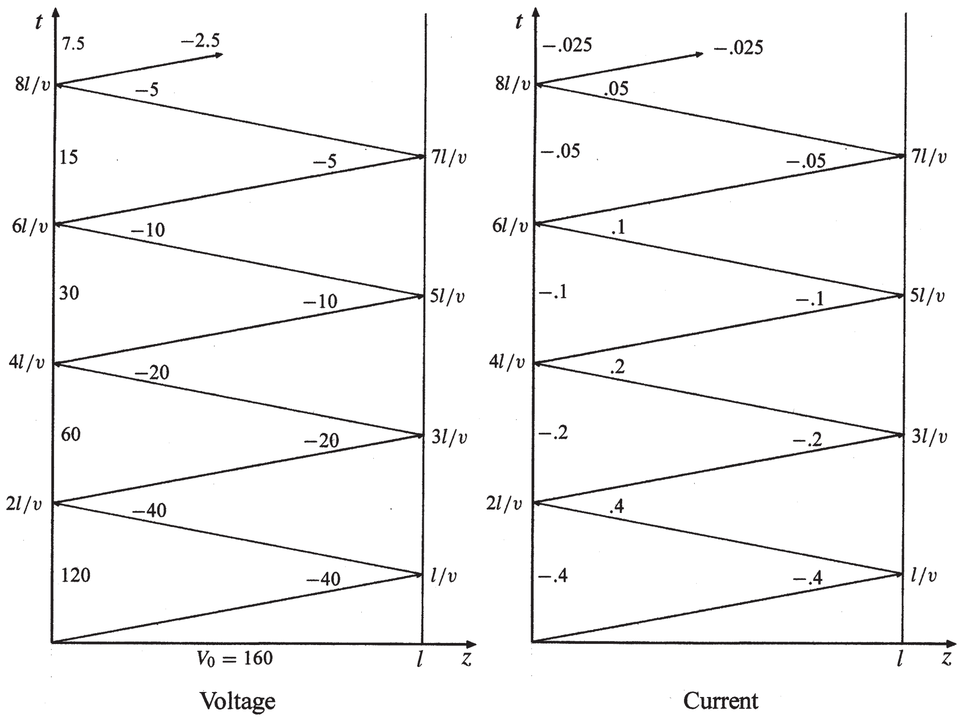


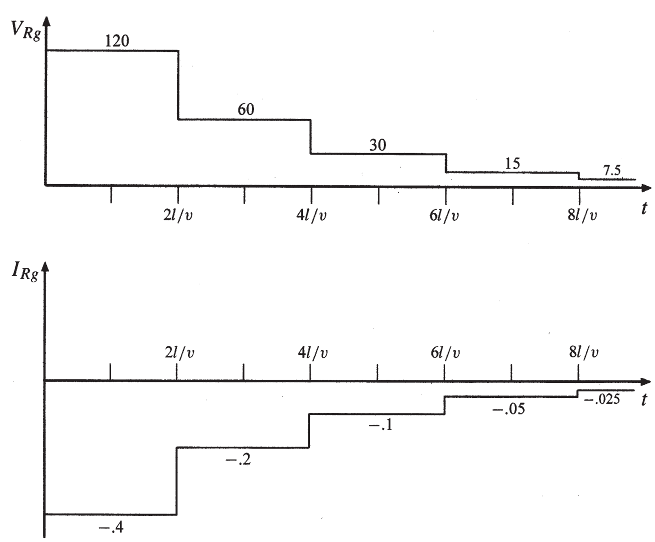


10.40. In the charged line of Fig. 10.25, the characteristic impedance is $Z_0 = 100\Omega$, and $R_g = 300\Omega$. The line is charged to initial voltage $V_0 = 160$ V, and the switch is closed at $t = 0$. Determine and plot the voltage and current through the resistor for time $0 < t < 8l/v$ (four round trips). This problem accompanies Example 11.12 as the other special case of the basic charged line problem, in which now $R_g > Z_0$. On closing the switch, the initial voltage wave is

$$V^+ = -V_0 \frac{Z_0}{R_g + Z_0} = -160 \frac{100}{400} = -40\,\text{V}$$

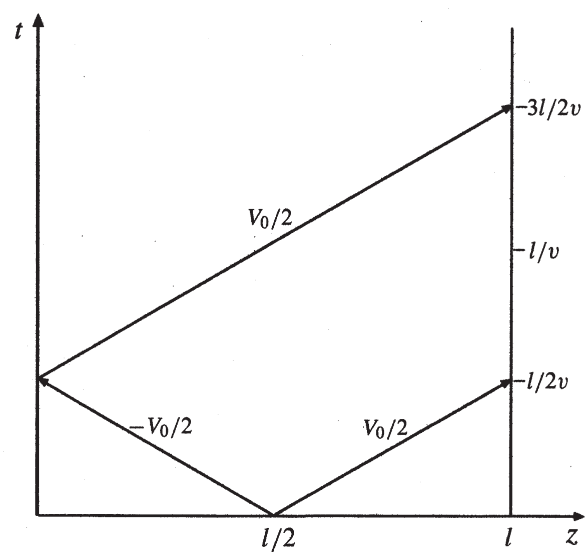
Now, with $\Gamma_g = 1/2$ and $\Gamma_L = 1$, the voltage and current reflection diagrams are constructed as shown below. Plots of the voltage and current at the resistor are then found by accumulating values from the left sides of the two charts, producing the plots as shown.

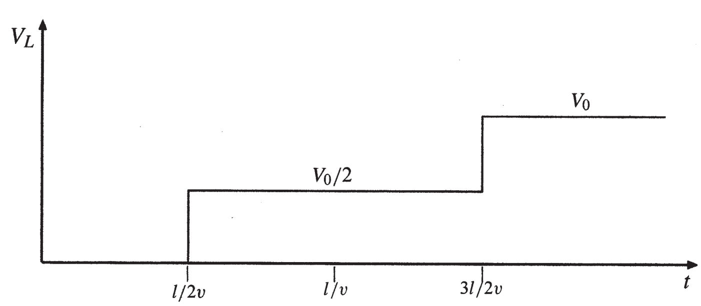




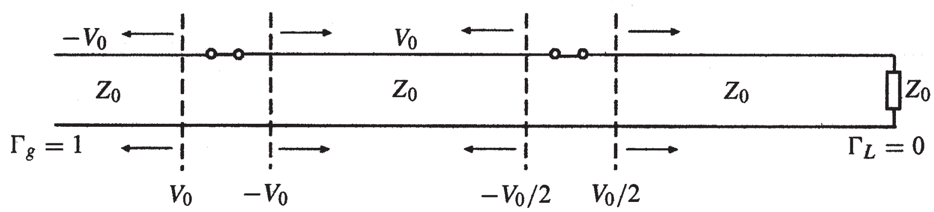
10.41. In the transmission line of Fig. 10.37, the switch is located *midway* down the line, and is closed at $t = 0$. Construct a voltage reflection diagram for this case, where $R_L = Z_0$. Plot the load resistor voltage as a function of time: With the left half of the line charged to V_0, closing the switch initiates (at the switch location) *two* voltage waves: The first is of value $-V_0/2$ and propagates toward the left; the second is of value $V_0/2$ and propagates toward the right. The backward wave reflects at the battery with $\Gamma_g = -1$. No reflection occurs at the load end, since the load is matched to the line. The reflection diagram and load voltage plot are shown below. The results are summarized as follows:

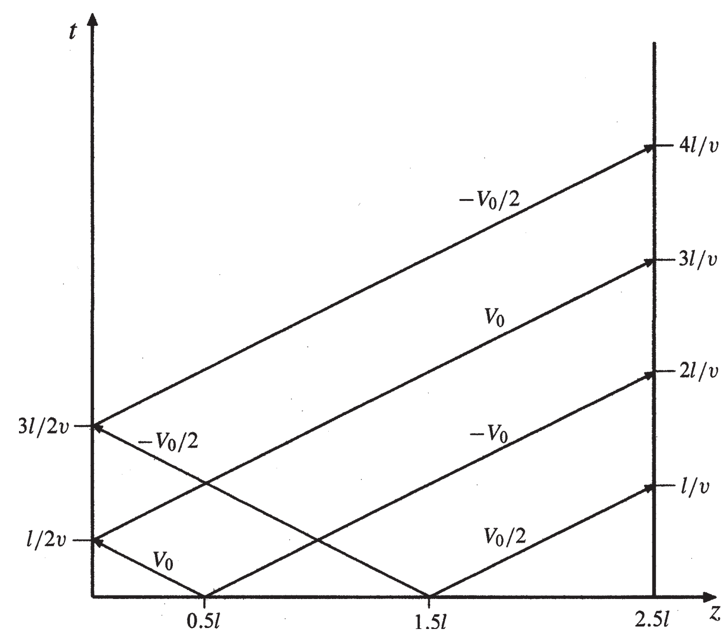
$$0 < t < \frac{l}{2v} : \quad V_L = 0$$
$$\frac{l}{2v} < t < \frac{3l}{2v} : \quad V_L = \frac{V_0}{2}$$
$$t > \frac{3l}{2v} : \quad V_L = V_0$$

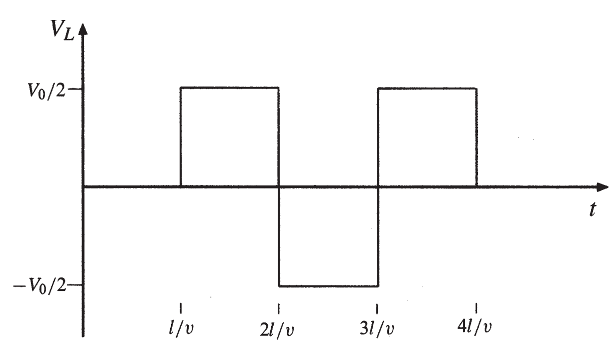




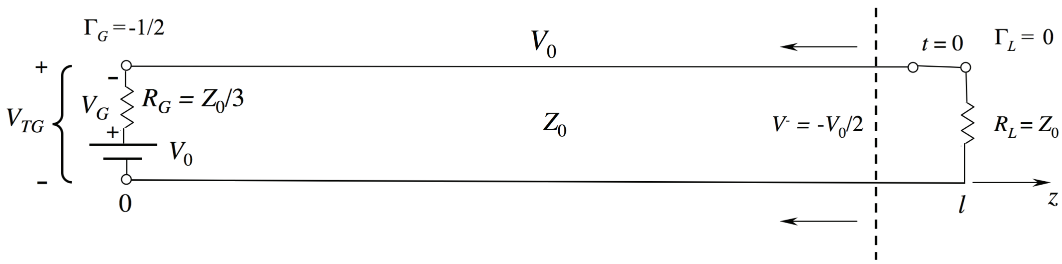
10.42. A simple *frozen wave generator* is shown in Fig. 10.38. Both switches are closed simultaneously at $t = 0$. Construct an appropriate voltage reflection diagram for the case in which $R_L = Z_0$. Determine and plot the load voltage as a function of time: Closing the switches sets up a total of four voltage waves as shown in the diagram below. Note that the first and second waves from the left are of magnitude V_0, since in fact we are superimposing voltage waves from the $-V_0$ and $+V_0$ charged sections acting alone. The reflection diagram is drawn and is used to construct the load voltage with time by accumulating voltages up the right hand vertical axis.



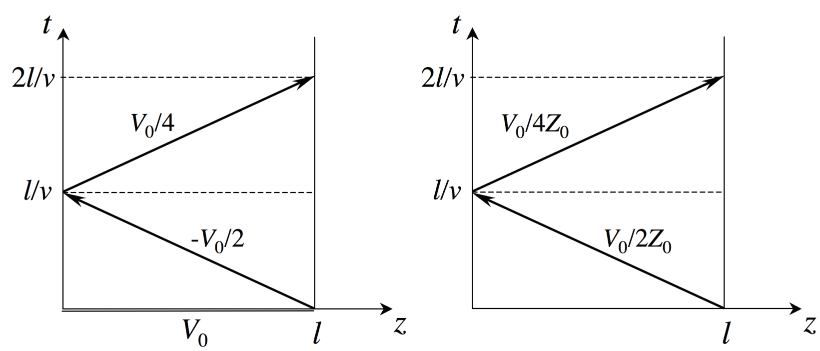




10.43. In Fig. 10.39, $R_L = Z_0$ and $R_g = Z_0/3$. The switch is closed at $t = 0$. Determine and plot as functions of time: a) the voltage across R_L; b) the voltage across R_g; c) the current through the battery.

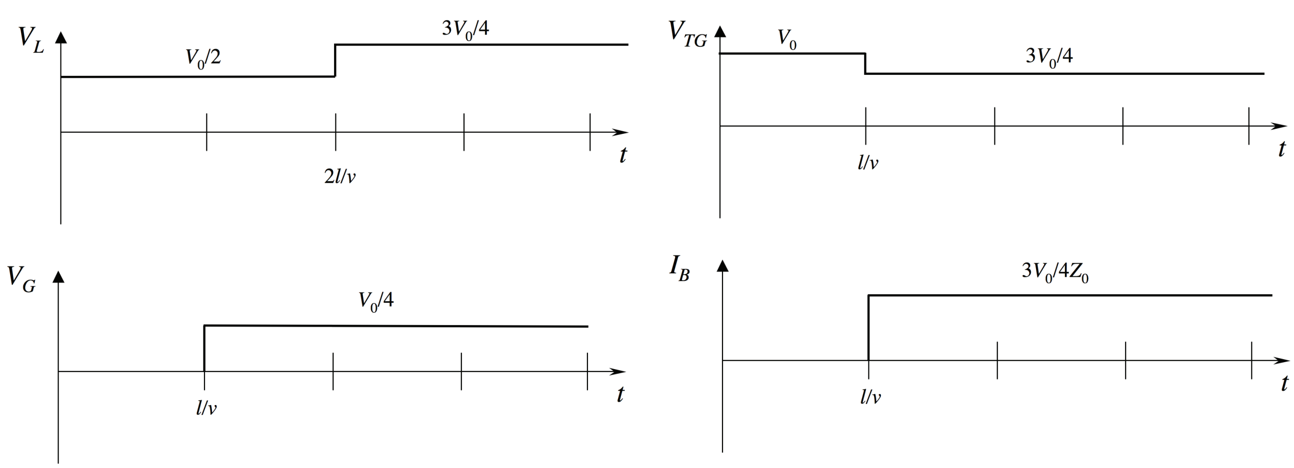


With the switch at the opposite end from the battery, the entire line is initially charged to V_0. So, on closing the switch, the initial wave propagates backward, originating at the switch, and is of value $V^- = -V_0 Z_0/(R_L + Z_0) = -V_0/2$. On reaching the left end, the wave reflects with reflection coefficient $\Gamma_G = (R_G - Z_0)/(R_G + Z_0) = -1/2$. The reflected wave returns to the switch end, sees a matched load there, and there is no further reflection. The resulting voltage and current reflection diagrams are shown below.



voltage current

The load voltage is read from the right side of the voltage diagram, and is plotted as V_L below. The voltage across the entire left end is read from the left side of the voltage diagram, and is plotted as V_{TG} below. The voltage across the resistor, R_G, will be $V_G = V_{TG} - V_0$ (choosing the resistor voltage polarity as positive), which leads to the V_G plot below. Finally, the battery current is read from the left side of the current diagram, and is plotted as I_B below.



CHAPTER 11

11.1. Show that $E_{xs} = Ae^{jk_0 z+\phi}$ is a solution to the vector Helmholtz equation, Sec. 11.1, Eq. (30), for $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and any ϕ and A: We take

$$\frac{d^2}{dz^2} Ae^{jk_0 z+\phi} = (jk_0)^2 Ae^{jk_0 z+\phi} = -k_0^2 E_{xs}$$

11.2. A 100-MHz uniform plane wave propagates in a lossless medium for which $\epsilon_r = 5$ and $\mu_r = 1$. Find:

a) v_p: $v_p = c/\sqrt{\epsilon_r} = 3 \times 10^8/\sqrt{5} = \underline{1.34 \times 10^8 \text{ m/s}}$.

b) β: $\beta = \omega/v_p = (2\pi \times 10^8)/(1.34 \times 10^8) = \underline{4.69 \text{ m}^{-1}}$.

c) λ: $\lambda = 2\pi/\beta = \underline{1.34 \text{ m}}$.

d) $\mathbf{E}_s$: Assume real amplitude E_0, forward z travel, and x polarization, and write $\mathbf{E}_s = E_0 \exp(-j\beta z)\mathbf{a}_x = \underline{E_0 \exp(-j4.69z)\, \mathbf{a}_x \text{ V/m}}$.

e) $\mathbf{H}_s$: First, the intrinsic impedance of the medium is $\eta = \eta_0/\sqrt{\epsilon_r} = 377/\sqrt{5} = 169\ \Omega$. Then $\mathbf{H}_s = (E_0/\eta)\exp(-j\beta z)\,\mathbf{a}_y = \underline{(E_0/169)\exp(-j4.69z)\,\mathbf{a}_y \text{ A/m}}$.

f) $< \mathbf{S} >= (1/2)\mathcal{R}e\,\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \underline{(E_0^2/337)\,\mathbf{a}_z \text{ W/m}^2}$

12.3. An $\mathbf{H}$ field in free space is given as $\mathcal{H}(x,t) = 10\cos(10^8 t - \beta x)\mathbf{a}_y$ A/m. Find

a) β: Since we have a uniform plane wave, $\beta = \omega/c$, where we identify $\omega = 10^8$ sec^{-1}. Thus $\beta = 10^8/(3 \times 10^8) = \underline{0.33 \text{ rad/m}}$.

b) λ: We know $\lambda = 2\pi/\beta = \underline{18.9 \text{ m}}$.

c) $\mathcal{E}(x,t)$ at $P(0.1, 0.2, 0.3)$ at $t = 1$ ns: Use $E(x,t) = -\eta_0 H(x,t) = -(377)(10)\cos(10^8 t - \beta x) = -3.77 \times 10^3 \cos(10^8 t - \beta x)$. The vector direction of $\mathbf{E}$ will be $-\mathbf{a}_z$, since we require that $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, where $\mathbf{S}$ is x-directed. At the given point, the relevant coordinate is $x = 0.1$. Using this, along with $t = 10^{-9}$ sec, we finally obtain

$$\mathbf{E}(x,t) = -3.77 \times 10^3 \cos[(10^8)(10^{-9}) - (0.33)(0.1)]\mathbf{a}_z = -3.77 \times 10^3 \cos(6.7 \times 10^{-2})\mathbf{a}_z$$
$$= \underline{-3.76 \times 10^3 \mathbf{a}_z \text{ V/m}}$$

11.4. Small antennas have low efficiencies (as will be seen in Chapter 14) and the efficiency increases with size up to the point at which a critical dimension of the antenna is an appreciable fraction of a wavelength, say $\lambda/8$.

a) An antenna is that is 12cm long is operated in air at 1 MHz. What fraction of a wavelength long is it? The free space wavelength will be

$$\lambda_{air} = \frac{c}{f} = \frac{3.0 \times 10^8 \text{ m/s}}{10^6 \text{ s}^{-1}} = 300 \text{ m, so that the fraction} = \frac{1.2}{300} = \underline{4.0 \times 10^{-3}}$$

b) The same antenna is embedded in a ferrite material for which $\epsilon_r = 20$ and $\mu_r = 2{,}000$. What fraction of a wavelength is it now?

$$\lambda_{ferrite} = \frac{\lambda_{air}}{\sqrt{\mu_r \epsilon_r}} = \frac{300}{\sqrt{(20)(2000)}} = 1.5\text{m} \;\Rightarrow\; \text{fraction} = \frac{1.2}{1.5} = \underline{0.8}$$

11.5. A 150-MHz uniform plane wave in free space is described by $\mathbf{H}_s = (4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$ A/m.

a) Find numerical values for ω, λ, and β: First, $\omega = 2\pi \times 150 \times 10^6 = \underline{3\pi \times 10^8 \text{ sec}^{-1}}$. Second, for a uniform plane wave in free space, $\lambda = 2\pi c/\omega = c/f = (3 \times 10^8)/(1.5 \times 10^8) = \underline{2\,\text{m}}$. Third, $\beta = 2\pi/\lambda = \underline{\pi\,\text{rad/m}}$.

b) Find $\mathcal{H}(z,t)$ at $t = 1.5$ ns, $z = 20$ cm: Use

$$\mathbf{H}(z,t) = \text{Re}\{\mathbf{H}_s e^{j\omega t}\} = \text{Re}\{(4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)(\cos(\omega t - \beta z) + j\sin(\omega t - \beta z)\}$$
$$= [8\cos(\omega t - \beta z) - 20\sin(\omega t - \beta z)]\,\mathbf{a}_x - [10\cos(\omega t - \beta z) + 4\sin(\omega t - \beta z)]\,\mathbf{a}_y$$

. Now at the given position and time, $\omega t - \beta z = (3\pi \times 10^8)(1.5 \times 10^{-9}) - \pi(0.20) = \pi/4$. And $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$. So finally,

$$\mathbf{H}(z = 20\text{cm}, t = 1.5\text{ns}) = -\frac{1}{\sqrt{2}}(12\mathbf{a}_x + 14\mathbf{a}_y) = \underline{-8.5\mathbf{a}_x - 9.9\mathbf{a}_y \text{ A/m}}$$

c) What is $|E|_{max}$? Have $|E|_{max} = \eta_0 |H|_{max}$, where

$$|H|_{max} = \sqrt{\mathbf{H}_s \cdot \mathbf{H}_s^*} = [4(4 + j10)(4 - j10) + (j)(-j)(4 + j10)(4 - j10)]^{1/2} = 24.1\,\text{A/m}$$

Then $|E|_{max} = 377(24.1) = \underline{9.08 \text{ kV/m}}$.

11.6. A uniform plane wave has electric field $\mathbf{E}_s = (E_{y0}\,\mathbf{a}_y - E_{z0}\,\mathbf{a}_z)\,e^{-\alpha x}\,e^{-j\beta x}$ V/m. The intrinsic impedance of the medium is given as $\eta = |\eta|\,e^{j\phi}$, where ϕ is a constant phase.

a) Describe the wave polarization and state the direction of propagation: The wave is linearly polarized in the y-z plane, and propagates in the forward x direction (from the $e^{-j\beta x}$ factor).

b) Find $\mathbf{H}_s$: Each component of $\mathbf{E}_s$, when crossed into its companion component of $\mathbf{H}_s^*$, must give a vector in the positive-x direction of travel. Using this rule, we find

$$\mathbf{H}_s = \left[\frac{E_y}{\eta}\,\mathbf{a}_z + \frac{E_z}{\eta}\,\mathbf{a}_y\right] = \left[\frac{E_{y0}}{|\eta|}\,\mathbf{a}_z + \frac{E_{z0}}{|\eta|}\,\mathbf{a}_y\right] e^{-\alpha x} e^{-j\phi} e^{-j\beta x} \text{ A/m}$$

c) Find $\mathcal{E}(x,t)$ and $\mathcal{H}(x,t)$: $\mathcal{E}(x,t) = \mathcal{R}e\left\{\mathbf{E}_s e^{j\omega t}\right\} = [E_{y0}\,\mathbf{a}_y - E_{z0}\,\mathbf{a}_z]\,e^{-\alpha x}\cos(\omega t - \beta x)$

$$\mathcal{H}(x,t) = \mathcal{R}e\left\{\mathbf{H}_s e^{j\omega t}\right\} = [E_{y0}\,\mathbf{a}_z + E_{z0}\,\mathbf{a}_y]\,e^{-\alpha x}\cos(\omega t - \beta x - \phi)$$

where all amplitudes are assumed real.

d) Find $<\mathbf{S}>$ in W/m^2:

$$<\mathbf{S}> = \frac{1}{2}\mathcal{R}e\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\left(E_{y0}^2 + E_{z0}^2\right) e^{-2\alpha x}\cos\phi\,\mathbf{a}_x \text{ W/m}^2$$

e) Find the time-average power in watts that is intercepted by an antenna of rectangular cross-section, having width w and height h, suspended parallel to the yz plane, and at a distance d from the wave source. This will be

$$P = \int\int_{plate} <\mathbf{S}> \cdot d\mathbf{S} = |<\mathbf{S}>|_{x=d} \times area = \frac{1}{2}(wh)\left(E_{y0}^2 + E_{z0}^2\right) e^{-2\alpha d}\cos\phi \text{ W}$$

11.7. The phasor magnetic field intensity for a 400-MHz uniform plane wave propagating in a certain lossless material is $(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}$ A/m. Knowing that the maximum amplitude of $\mathbf{E}$ is 1500 V/m, find β, η, λ, v_p, ϵ_r, μ_r, and $\mathcal{H}(x, y, z, t)$: First, from the phasor expression, we identify $\beta = \underline{25\ \text{m}^{-1}}$ from the argument of the exponential function. Next, we evaluate $H_0 = |\mathbf{H}| = \sqrt{\mathbf{H}\cdot\mathbf{H}^*} = \sqrt{2^2 + 5^2} = \sqrt{29}$. Then $\eta = E_0/H_0 = 1500/\sqrt{29} = \underline{278.5\ \Omega}$. Then $\lambda = 2\pi/\beta = 2\pi/25 = .25\ \text{m} = \underline{25\ \text{cm}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 400 \times 10^6}{25} = \underline{1.01 \times 10^8\ \text{m/s}}$$

Now we note that

$$\eta = 278.5 = 377\sqrt{\frac{\mu_r}{\epsilon_r}} \quad \Rightarrow \quad \frac{\mu_r}{\epsilon_r} = 0.546$$

And

$$v_p = 1.01 \times 10^8 = \frac{c}{\sqrt{\mu_r \epsilon_r}} \quad \Rightarrow \quad \mu_r \epsilon_r = 8.79$$

We solve the above two equations simultaneously to find $\epsilon_r = \underline{4.01}$ and $\mu_r = \underline{2.19}$. Finally,

$$\begin{aligned}\mathbf{H}(x, y, z, t) &= Re\left\{(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}e^{j\omega t}\right\}\\ &= 2\cos(2\pi \times 400 \times 10^6 t - 25x)\mathbf{a}_y + 5\sin(2\pi \times 400 \times 10^6 t - 25x)\mathbf{a}_z\\ &= \underline{2\cos(8\pi \times 10^8 t - 25x)\mathbf{a}_y + 5\sin(8\pi \times 10^8 t - 25x)\mathbf{a}_z\ \text{A/m}}\end{aligned}$$

11.8. An electric field in free space is given in spherical coordinates as $\mathbf{E}_s(r) = E_0(r)e^{-jkr}\,\mathbf{a}_\theta$ V/m.

a) find $\mathbf{H}_s(r)$ assuming uniform plane wave behavior: Knowing that the cross product of $\mathbf{E}_s$ with the complex conjugate of the phasor $\mathbf{H}_s$ field must give a vector in the direction of propagation, we obtain,

$$\mathbf{H}_s(r) = \frac{E_0(r)}{\eta_0}e^{-jkr}\,\mathbf{a}_\phi\ \text{A/m}$$

b) Find $<\mathbf{S}>$: This will be

$$<\mathbf{S}> = \frac{1}{2}\mathcal{R}e\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{E_0^2(r)}{2\eta_0}\,\mathbf{a}_r\ \text{W/m}^2$$

c) Express the average outward power in watts through a closed spherical shell of radius r, centered at the origin: The power will be (in this case) just the product of the power density magnitude in part b with the sphere area, or

$$P = 4\pi r^2\,\frac{E_0^2(r)}{2\eta_0}\ \text{W}$$

where $E_0(r)$ is assumed real.

d) Establish the required functional form of $E_0(r)$ that will enable the power flow in part c to be independent of radius: Evidently this condition is met when $\underline{E_0(r) \propto 1/r}$

11.9. A certain lossless material has $\mu_r = 4$ and $\epsilon_r = 9$. A 10-MHz uniform plane wave is propagating in the $\mathbf{a}_y$ direction with $E_{x0} = 400$ V/m and $E_{y0} = E_{z0} = 0$ at $P(0.6, 0.6, 0.6)$ at $t = 60$ ns.

a) Find β, λ, v_p, and η: For a uniform plane wave,

$$\beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}\sqrt{\mu_r\epsilon_r} = \frac{2\pi \times 10^7}{3 \times 10^8}\sqrt{(4)(9)} = \underline{0.4\pi \text{ rad/m}}$$

Then $\lambda = (2\pi)/\beta = (2\pi)/(0.4\pi) = \underline{5\,\text{m}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^7}{4\pi \times 10^{-1}} = \underline{5 \times 10^7 \text{ m/s}}$$

Finally,

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \eta_0\sqrt{\frac{\mu_r}{\epsilon_r}} = 377\sqrt{\frac{4}{9}} = \underline{251\ \Omega}$$

b) Find $E(t)$ (at P): We are given the amplitude at $t = 60$ ns and at $y = 0.6$ m. Let the maximum amplitude be E_{max}, so that in general, $E_x = E_{max}\cos(\omega t - \beta y)$. At the given position and time,

$$E_x = 400 = E_{max}\cos[(2\pi \times 10^7)(60 \times 10^{-9}) - (4\pi \times 10^{-1})(0.6)] = E_{max}\cos(0.96\pi)$$
$$= -0.99E_{max}$$

So $E_{max} = (400)/(-0.99) = -403\,\text{V/m}$. Thus at P, $E(t) = \underline{-403\cos(2\pi \times 10^7 t) \text{ V/m}}$.

c) Find $H(t)$: First, we note that if E at a given instant points in the negative x direction, while the wave propagates in the forward y direction, then H at that same position and time must point in the positive z direction. Since we have a lossless homogeneous medium, η is real, and we are allowed to write $H(t) = E(t)/\eta$, where η is treated as negative and real. Thus

$$H(t) = H_z(t) = \frac{E_x(t)}{\eta} = \frac{-403}{-251}\cos(2\pi \times 10^{-7}t) = \underline{1.61\cos(2\pi \times 10^{-7}t) \text{ A/m}}$$

11.10. In a medium characterized by intrinsic impedance $\eta = |\eta|e^{j\phi}$, a linearly-polarized plane wave propagates, with magnetic field given as $\mathbf{H}_s = (H_{0y}\mathbf{a}_y + H_{0z}\mathbf{a}_z)\,e^{-\alpha x}e^{-j\beta x}$. Find:

a) $\mathbf{E}_s$: Requiring orthogonal components of $\mathbf{E}_s$ for each component of $\mathbf{H}_s$, we find

$$\mathbf{E}_s = |\eta|\,[H_{0z}\,\mathbf{a}_y - H_{0y}\,\mathbf{a}_z]\; e^{-\alpha x}\, e^{-j\beta x}\, e^{j\phi}$$

b) $\mathcal{E}(x,t) = \mathcal{R}e\,\{\mathbf{E}_s e^{j\omega t}\} = |\eta|\,[H_{0z}\,\mathbf{a}_y - H_{0y}\,\mathbf{a}_z]\,e^{-\alpha x}\,\cos(\omega t - \beta x + \phi)$.

c) $\mathcal{H}(x,t) = \mathcal{R}e\,\{\mathbf{H}_s e^{j\omega t}\} = [H_{0y}\,\mathbf{a}_y + H_{0z}\,\mathbf{a}_z]\,e^{-\alpha x}\,\cos(\omega t - \beta x)$.

d) $$< \mathbf{S} >= \frac{1}{2}\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{1}{2}|\eta|\left[H_{0y}^2 + H_{0z}^2\right] e^{-2\alpha x}\,\cos\phi\,\mathbf{a}_x \text{ W/m}^2$$

11.11. A 2-GHz uniform plane wave has an amplitude of $E_{y0} = 1.4$ kV/m at $(0, 0, 0, t = 0)$ and is propagating in the $\mathbf{a}_z$ direction in a medium where $\epsilon'' = 1.6 \times 10^{-11}$ F/m, $\epsilon' = 3.0 \times 10^{-11}$ F/m, and $\mu = 2.5\,\mu$H/m. Find:

a) E_y at $P(0, 0, 1.8\text{cm})$ at 0.2 ns: To begin, we have the ratio, $\epsilon''/\epsilon' = 1.6/3.0 = 0.533$. So

$$\alpha = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]^{1/2}$$

$$= (2\pi \times 2 \times 10^9)\sqrt{\frac{(2.5 \times 10^{-6})(3.0 \times 10^{-11})}{2}}\left[\sqrt{1 + (.533)^2} - 1\right]^{1/2} = 28.1\,\text{Np/m}$$

Then

$$\beta = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]^{1/2} = 112\,\text{rad/m}$$

Thus in general,

$$E_y(z, t) = 1.4e^{-28.1z}\cos(4\pi \times 10^9 t - 112z)\ \text{kV/m}$$

Evaluating this at $t = 0.2$ ns and $z = 1.8$ cm, find

$$E_y(1.8\,\text{cm}, 0.2\,\text{ns}) = \underline{0.74\ \text{kV/m}}$$

b) H_x at P at 0.2 ns: We use the phasor relation, $H_{xs} = -E_{ys}/\eta$ where

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} = \sqrt{\frac{2.5 \times 10^{-6}}{3.0 \times 10^{-11}}}\frac{1}{\sqrt{1 - j(.533)}} = 263 + j65.7 = 271\angle\,14^\circ\ \Omega$$

So now

$$H_{xs} = -\frac{E_{ys}}{\eta} = -\frac{(1.4 \times 10^3)e^{-28.1z}e^{-j112z}}{271e^{j14^\circ}} = -5.16e^{-28.1z}e^{-j112z}e^{-j14^\circ}\ \text{A/m}$$

Then

$$H_x(z, t) = -5.16e^{-28.1z}\cos(4\pi \times 10^{-9}t - 112z - 14^\circ)$$

This, when evaluated at $t = 0.2$ ns and $z = 1.8$ cm, yields

$$H_x(1.8\,\text{cm}, 0.2\,\text{ns}) = \underline{-3.0\,\text{A/m}}$$

11.12. Describe how the attenuation coefficient of a liquid medium, assumed to be a good conductor, could be determined through measurement of wavelength in the liquid at a known frequency. What restrictions apply? Could this method be used to find the conductivity as well? In a good conductor, we may use the approximation:

$$\alpha \doteq \beta \doteq \sqrt{\frac{\omega\mu\sigma}{2}} \text{ where } \beta = \frac{2\pi}{\lambda}$$

Therefore, in the good conductor approximation, $\alpha \doteq 2\pi/\lambda$. From the above formula, we could also find

$$\sigma \doteq \frac{4\pi}{\lambda^2 f \mu}$$

which would work provided that again, we are certain that we have a good conductor, and that the permeability is known.

11.13. Let $jk = 0.2 + j1.5\,\mathrm{m}^{-1}$ and $\eta = 450 + j60\ \Omega$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction. If $\omega = 300$ Mrad/s, find μ, ϵ', and ϵ'': We begin with

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\,\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} = 450 + j60$$

and

$$jk = j\omega\sqrt{\mu\epsilon'}\,\sqrt{1 - j(\epsilon''/\epsilon')} = 0.2 + j1.5$$

Then

$$\eta\eta^* = \frac{\mu}{\epsilon'}\,\frac{1}{\sqrt{1 + (\epsilon''/\epsilon')^2}} = (450 + j60)(450 - j60) = 2.06 \times 10^5 \qquad (1)$$

and

$$(jk)(jk)^* = \omega^2\mu\epsilon'\,\sqrt{1 + (\epsilon''/\epsilon')^2} = (0.2 + j1.5)(0.2 - j1.5) = 2.29 \qquad (2)$$

Taking the ratio of (2) to (1),

$$\frac{(jk)(jk)^*}{\eta\eta^*} = \omega^2(\epsilon')^2\left(1 + (\epsilon''/\epsilon')^2\right) = \frac{2.29}{2.06 \times 10^5} = 1.11 \times 10^{-5}$$

Then with $\omega = 3 \times 10^8$,

$$(\epsilon')^2 = \frac{1.11 \times 10^{-5}}{(3 \times 10^8)^2\,(1 + (\epsilon''/\epsilon')^2)} = \frac{1.23 \times 10^{-22}}{(1 + (\epsilon''/\epsilon')^2)} \qquad (3)$$

Now, we use Eqs. (35) and (36). Squaring these and taking their ratio gives

$$\frac{\alpha^2}{\beta^2} = \frac{\sqrt{1 + (\epsilon''/\epsilon')^2}}{\sqrt{1 + (\epsilon''/\epsilon')^2}} = \frac{(0.2)^2}{(1.5)^2}$$

We solve this to find $\epsilon''/\epsilon' = 0.271$. Substituting this result into (3) gives $\epsilon' = 1.07 \times 10^{-11}$ F/m. Since $\epsilon''/\epsilon' = 0.271$, we then find $\epsilon'' = 2.90 \times 10^{-12}$ F/m. Finally, using these results in either (1) or (2) we find $\mu = 2.28 \times 10^{-6}$ H/m. Summary: $\mu = \underline{2.28 \times 10^{-6}\ \mathrm{H/m}}$, $\epsilon' = \underline{1.07 \times 10^{-11}\ \mathrm{F/m}}$, and $\epsilon'' = \underline{2.90 \times 10^{-12}\ \mathrm{F/m}}$.

11.14. A certain nonmagnetic material has the material constants $\epsilon'_r = 2$ and $\epsilon''/\epsilon' = 4 \times 10^{-4}$ at $\omega = 1.5$ Grad/s. Find the distance a uniform plane wave can propagate through the material before:

a) it is attenuated by 1 Np: First, $\epsilon'' = (4 \times 10^4)(2)(8.854 \times 10^{-12}) = 7.1 \times 10^{-15}$ F/m. Then, since $\epsilon''/\epsilon' << 1$, we use the approximate form for α, given by Eq. (51) (written in terms of ϵ''):

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{(1.5 \times 10^9)(7.1 \times 10^{-15})}{2}\frac{377}{\sqrt{2}} = 1.42 \times 10^{-3} \text{ Np/m}$$

The required distance is now $z_1 = (1.42 \times 10^{-3})^{-1} = \underline{706 \text{ m}}$

b) the power level is reduced by one-half: The governing relation is $e^{-2\alpha z_{1/2}} = 1/2$, or $z_{1/2} = \ln 2/2\alpha = \ln 2/2(1.42 \times 10{-}3) = \underline{244 \text{ m}}$.

c) the phase shifts 360°: This distance is defined as one wavelength, where $\lambda = 2\pi/\beta = (2\pi c)/(\omega\sqrt{\epsilon'_r}) = [2\pi(3 \times 10^8)]/[(1.5 \times 10^9)\sqrt{2}] = \underline{0.89 \text{ m}}$.

11.15. A 10 GHz radar signal may be represented as a uniform plane wave in a sufficiently small region. Calculate the wavelength in centimeters and the attenuation in nepers per meter if the wave is propagating in a non-magnetic material for which

a) $\epsilon'_r = 1$ and $\epsilon''_r = 0$: In a non-magnetic material, we would have:

$$\alpha = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon'_r}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''_r}{\epsilon'_r}\right)^2} - 1\right]^{1/2}$$

and

$$\beta = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon'_r}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''_r}{\epsilon'_r}\right)^2} + 1\right]^{1/2}$$

With the given values of ϵ'_r and ϵ''_r, it is clear that $\beta = \omega\sqrt{\mu_0\epsilon_0} = \omega/c$, and so

$\lambda = 2\pi/\beta = 2\pi c/\omega = 3 \times 10^{10}/10^{10} = \underline{3 \text{ cm}}$. It is also clear that $\alpha = \underline{0}$.

b) $\epsilon'_r = 1.04$ and $\epsilon''_r = 9.00 \times 10^{-4}$: In this case $\epsilon''_r/\epsilon'_r << 1$, and so $\beta \doteq \omega\sqrt{\epsilon'_r}/c = 2.13 \text{ cm}^{-1}$. Thus $\lambda = 2\pi/\beta = \underline{2.95 \text{ cm}}$. Then

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\omega\epsilon''_r}{2}\frac{\sqrt{\mu_0\epsilon_0}}{\sqrt{\epsilon'_r}} = \frac{\omega}{2c}\frac{\epsilon''_r}{\sqrt{\epsilon'_r}} = \frac{2\pi \times 10^{10}}{2 \times 3 \times 10^8}\frac{(9.00 \times 10^{-4})}{\sqrt{1.04}}$$
$$= \underline{9.24 \times 10^{-2} \text{ Np/m}}$$

c) $\epsilon'_r = 2.5$ and $\epsilon''_r = 7.2$: Using the above formulas, we obtain

$$\beta = \frac{2\pi \times 10^{10}\sqrt{2.5}}{(3 \times 10^{10})\sqrt{2}}\left[\sqrt{1 + \left(\frac{7.2}{2.5}\right)^2} + 1\right]^{1/2} = 4.71 \text{ cm}^{-1}$$

and so $\lambda = 2\pi/\beta = \underline{1.33 \text{ cm}}$. Then

$$\alpha = \frac{2\pi \times 10^{10}\sqrt{2.5}}{(3 \times 10^8)\sqrt{2}}\left[\sqrt{1 + \left(\frac{7.2}{2.5}\right)^2} - 1\right]^{1/2} = \underline{335 \text{ Np/m}}$$

11.16. Consider the power dissipation term, $\int \mathbf{E} \cdot \mathbf{J} dv$ in Poynting's theorem (Eq. (70)). This gives the power lost to heat within a volume into which electromagnetic waves enter. The term $p_d = \mathbf{E} \cdot \mathbf{J}$ is thus the power dissipation per unit volume in W/m^3. Following the same reasoning that resulted in Eq. (77), the time-average power dissipation per volume will be $< p_d >= (1/2)\mathcal{R}e\{\mathbf{E}_s \cdot \mathbf{J}_s^*\}$.

a) Show that in a conducting medium, through which a uniform plane wave of amplitude E_0 propagates in the forward z direction, $< p_d >= (\sigma/2)|E_0|^2 e^{-2\alpha z}$: Begin with the phasor expression for the electric field, assuming complex amplitude E_0, and x-polarization:

$$\mathbf{E}_s = E_0 e^{-\alpha z} e^{-j\beta z}\, \mathbf{a}_x \text{ V/m}^2$$

Then

$$\mathbf{J}_s = \sigma \mathbf{E}_s = \sigma E_0 e^{-\alpha z} e^{-j\beta z}\, \mathbf{a}_x \text{ A/m}^2$$

So that

$$< p_d >= (1/2)\mathcal{R}e\left\{E_0 e^{-\alpha z} e^{-j\beta z}\, \mathbf{a}_x \cdot \sigma E_0^* e^{-\alpha z} e^{+j\beta z}\, \mathbf{a}_x\right\} = (\sigma/2)|E_0|^2 e^{-2\alpha z}$$

b) Confirm this result for the special case of a good conductor by using the left hand side of Eq. (70), and consider a very small volume. In a good conductor, the intrinsic impedance is, from Eq. (85), $\eta_c = (1+j)/(\sigma\delta)$, where the skin depth, $\delta = 1/\alpha$. The magnetic field phasor is then

$$\mathbf{H}_s = \frac{E_s}{\eta_c}\, \mathbf{a}_y = \frac{\sigma}{(1+j)\alpha} E_0 e^{-\alpha z} e^{-j\beta z}\, \mathbf{a}_y \text{ A/m}$$

The time-average Poynting vector is then

$$< \mathbf{S} >= \frac{1}{2}\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{\alpha}{4\sigma}|E_0|^2 e^{-2\alpha z}\, \mathbf{a}_z \text{ W/m}^2$$

Now, consider a rectangular volume of side lengths, Δx, Δy, and Δz, all of which are very small. As the wave passes through this volume in the forward z direction, the power dissipated will be the difference between the power at entry (at $z = 0$), and the power that exits the volume (at $z = \Delta z$). With small z, we may approximate $e^{-2\alpha z} \doteq 1 - 2\alpha z$, and the dissipated power in the volume becomes

$$P_d = P_{in} - P_{out} = \left[\frac{\alpha}{4\sigma}|E_0|^2\right] \Delta x \Delta y - \left[\frac{\alpha}{4\sigma}|E_0|^2 (1 - 2\alpha\Delta z)\right] \Delta x \Delta y = \frac{\alpha}{2\sigma}|E_0|^2 (\Delta x \Delta y \Delta z)$$

This is just the result of part a, evaluated at $z = 0$ and multiplied by the volume. The relation becomes exact as $\Delta z \to 0$, in which case $< p_d > \to (\sigma/2)|E_0|^2$.

It is also possible to show the relation by using Eq. (69) (which involves taking the divergence of $< \mathbf{S} >$), or by removing the restriction of a small volume and evaluating the integrals in Eq. (70) without approximations. Either method is straightforward.

11.17. Let $\eta = 250 + j30\,\Omega$ and $jk = 0.2 + j2\,\mathrm{m}^{-1}$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction in a dielectric having some finite conductivity. If $|E_s| = 400$ V/m at $z = 0$, find:

a) $<\mathbf{S}>$ at $z = 0$ and $z = 60$ cm: Assume x-polarization for the electric field. Then

$$
\begin{aligned}
<\mathbf{S}> &= \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\mathrm{Re}\left\{400e^{-\alpha z}e^{-j\beta z}\mathbf{a}_x \times \frac{400}{\eta^*}e^{-\alpha z}e^{j\beta z}\mathbf{a}_y\right\} \\
&= \frac{1}{2}(400)^2 e^{-2\alpha z}\mathrm{Re}\left\{\frac{1}{\eta^*}\right\}\mathbf{a}_z = 8.0 \times 10^4 e^{-2(0.2)z}\mathrm{Re}\left\{\frac{1}{250 - j30}\right\}\mathbf{a}_z \\
&= 315\, e^{-2(0.2)z}\,\mathbf{a}_z \ \mathrm{W/m^2}
\end{aligned}
$$

Evaluating at $z = 0$, obtain $<\mathbf{S}>(z = 0) = \underline{315\,\mathbf{a}_z \ \mathrm{W/m^2}}$,
and at $z = 60$ cm, $\mathbf{P}_{z,av}(z = 0.6) = 315e^{-2(0.2)(0.6)}\mathbf{a}_z = \underline{248\,\mathbf{a}_z \ \mathrm{W/m^2}}$.

b) the average ohmic power dissipation in watts per cubic meter at $z = 60$ cm: At this point a flaw becomes evident in the problem statement, since solving this part in two different ways gives results that are not the same. I will demonstrate: In the first method, we use Poynting's theorem in point form Eq.(69), which we modify for the case of time-independent fields to read: $-\nabla\cdot <\mathbf{S}> = <\mathbf{J}\cdot\mathbf{E}>$, where the right hand side is the average power dissipation per volume. Note that the additional right-hand-side terms in Poynting's theorem that describe changes in energy stored in the fields will both be zero in steady state. We apply our equation to the result of part a:

$$
<\mathbf{J}\cdot\mathbf{E}> = -\nabla\cdot <\mathbf{S}> = -\frac{d}{dz}315\,e^{-2(0.2)z} = (0.4)(315)e^{-2(0.2)z} = 126e^{-0.4z} \ \mathrm{W/m^3}
$$

At $z = 60$ cm, this becomes $<\mathbf{J}\cdot\mathbf{E}> = 99.1$ W/m^3. In the second method, we solve for the conductivity and evaluate $<\mathbf{J}\cdot\mathbf{E}> = \sigma < E^2 >$. We use $jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j(\epsilon''/\epsilon')}$ and

$$
\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}}
$$

We take the ratio,

$$
\frac{jk}{\eta} = j\omega\epsilon'\left[1 - j\left(\frac{\epsilon''}{\epsilon'}\right)\right] = j\omega\epsilon' + \omega\epsilon''
$$

Identifying $\sigma = \omega\epsilon''$, we find

$$
\sigma = \mathrm{Re}\left\{\frac{jk}{\eta}\right\} = \mathrm{Re}\left\{\frac{0.2 + j2}{250 + j30}\right\} = 1.74 \times 10^{-3} \ \mathrm{S/m}
$$

Now we find the dissipated power per volume:

$$
\sigma < E^2 > = 1.74 \times 10^{-3}\left(\frac{1}{2}\right)\left(400e^{-0.2z}\right)^2
$$

At $z = 60$ cm, this evaluates as 109 W/m^3. One can show that consistency between the two methods requires that

$$
\mathrm{Re}\left\{\frac{1}{\eta^*}\right\} = \frac{\sigma}{2\alpha}
$$

This relation does not hold using the numbers as given in the problem statement and the value of σ found above. Note that in Problem 11.13, where all values are worked out, the relation does hold and consistent results are obtained using both methods.

11.18. Given, a 100MHz uniform plane wave in a medium known to be a good dielectric. The phasor electric field is $\mathbf{E}_s = 4e^{-0.5z}e^{-j20z}\mathbf{a}_x$ V/m. Not stated in the problem is the permeability, which we take to be μ_0. Determine:

a) ϵ': As a first step, it is useful to see just how much of a good dielectric we have. We use the good dielectric approximations, Eqs. (60a) and (60b), with $\sigma = \omega\epsilon''$. Using these, we take the ratio, β/α, to find

$$\frac{\beta}{\alpha} = \frac{20}{0.5} = \frac{\omega\sqrt{\mu\epsilon'}\left[1 + (1/8)(\epsilon''/\epsilon')^2\right]}{(\omega\epsilon''/2)\sqrt{\mu/\epsilon'}} = 2\left(\frac{\epsilon'}{\epsilon''}\right) + \frac{1}{4}\left(\frac{\epsilon''}{\epsilon'}\right)$$

This becomes the quadratic equation:

$$\left(\frac{\epsilon''}{\epsilon'}\right)^2 - 160\left(\frac{\epsilon''}{\epsilon'}\right) + 8 = 0$$

The solution to the quadratic is $(\epsilon''/\epsilon') = 0.05$, which means that we can neglect the second term in Eq. (60b), so that $\beta \doteq \omega\sqrt{\mu\epsilon'} = (\omega/c)\sqrt{\epsilon'_r}$. With the given frequency of 100 MHz, and with $\mu = \mu_0$, we find $\sqrt{\epsilon'_r} = 20(3/2\pi) = 9.55$, so that $\epsilon'_r = 91.3$, and finally $\epsilon' = \epsilon'_r\epsilon_0 = \underline{8.1 \times 10^{-10}\text{ F/m}}$.

b) ϵ'': Using Eq. (60a), the set up is

$$\alpha = 0.5 = \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} \quad\Rightarrow\quad \epsilon'' = \frac{2(0.5)}{2\pi \times 10^8}\sqrt{\frac{\epsilon'}{\mu}} = \frac{10^{-8}}{2\pi(377)}\sqrt{91.3} = \underline{4.0 \times 10^{-11}\text{ F/m}}$$

c) η: Using Eq. (62b), we find

$$\eta \doteq \sqrt{\frac{\mu}{\epsilon'}}\left[1 + j\frac{1}{2}\left(\frac{\epsilon''}{\epsilon'}\right)\right] = \frac{377}{\sqrt{91.3}}(1 + j.025) = \underline{(39.5 + j0.99)\text{ ohms}}$$

d) $\mathbf{H}_s$: This will be a y-directed field, and will be

$$\mathbf{H}_s = \frac{E_s}{\eta}\mathbf{a}_y = \frac{4}{(39.5 + j0.99)}e^{-0.5z}e^{-j20z}\,\mathbf{a}_y = \underline{0.101e^{-0.5z}e^{-j20z}e^{-j0.025}\,\mathbf{a}_y\text{ A/m}}$$

e) $<\mathbf{S}>$: Using the given field and the result of part d, obtain

$$<\mathbf{S}> = \frac{1}{2}\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{(0.101)(4)}{2}e^{-2(0.5)z}\cos(0.025)\,\mathbf{a}_z = \underline{0.202e^{-z}\,\mathbf{a}_z\text{ W/m}^2}$$

f) the power in watts that is incident on a rectangular surface measuring 20m x 30m at $z = 10$m: At 10m, the power density is $<\mathbf{S}> = 0.202e^{-10} = 9.2 \times 10^{-6}$ W/m^2. The incident power on the given area is then $P = 9.2 \times 10^{-6} \times (20)(30) = \underline{5.5\text{ mW}}$.

11.19. Perfectly-conducting cylinders with radii of 8 mm and 20 mm are coaxial. The region between the cylinders is filled with a perfect dielectric for which $\epsilon = 10^{-9}/4\pi$ F/m and $\mu_r = 1$. If $\mathbf{E}$ in this region is $(500/\rho)\cos(\omega t - 4z)\mathbf{a}_\rho$ V/m, find:

a) ω, with the help of Maxwell's equations in cylindrical coordinates: We use the two curl equations, beginning with $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, where in this case,

$$\nabla \times \mathbf{E} = \frac{\partial E_\rho}{\partial z}\mathbf{a}_\phi = \frac{2000}{\rho}\sin(\omega t - 4z)\mathbf{a}_\phi = -\frac{\partial B_\phi}{\partial t}\mathbf{a}_\phi$$

So

$$B_\phi = \int \frac{2000}{\rho}\sin(\omega t - 4z)dt = \frac{2000}{\omega\rho}\cos(\omega t - 4z) \text{ T}$$

Then

$$H_\phi = \frac{B_\phi}{\mu_0} = \frac{2000}{(4\pi \times 10^{-7})\omega\rho}\cos(\omega t - 4z) \text{ A/m}$$

We next use $\nabla \times \mathbf{H} = \partial \mathbf{D}/\partial t$, where in this case

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z$$

where the second term on the right hand side becomes zero when substituting our H_ϕ. So

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho = -\frac{8000}{(4\pi \times 10^{-7})\omega\rho}\sin(\omega t - 4z)\mathbf{a}_\rho = \frac{\partial D_\rho}{\partial t}\mathbf{a}_\rho$$

And

$$D_\rho = \int -\frac{8000}{(4\pi \times 10^{-7})\omega\rho}\sin(\omega t - 4z)dt = \frac{8000}{(4\pi \times 10^{-7})\omega^2\rho}\cos(\omega t - 4z) \text{ C/m}^2$$

Finally, using the given ϵ,

$$E_\rho = \frac{D_\rho}{\epsilon} = \frac{8000}{(10^{-16})\omega^2\rho}\cos(\omega t - 4z) \text{ V/m}$$

This must be the same as the given field, so we require

$$\frac{8000}{(10^{-16})\omega^2\rho} = \frac{500}{\rho} \quad \Rightarrow \quad \omega = \underline{4 \times 10^8 \text{ rad/s}}$$

b) $\mathbf{H}(\rho, z, t)$: From part a, we have

$$\mathbf{H}(\rho, z, t) = \frac{2000}{(4\pi \times 10^{-7})\omega\rho}\cos(\omega t - 4z)\mathbf{a}_\phi = \underline{\frac{4.0}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\phi \text{ A/m}}$$

c) $\mathbf{S}(\rho, \phi, z)$: This will be

$$\mathbf{S}(\rho, \phi, z) = \mathbf{E} \times \mathbf{H} = \frac{500}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\rho \times \frac{4.0}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\phi$$
$$= \underline{\frac{2.0 \times 10^{-3}}{\rho^2}\cos^2(4 \times 10^8 t - 4z)\mathbf{a}_z \text{ W/m}^2}$$

11.19d) the average power passing through every cross-section $8 < \rho < 20$ mm, $0 < \phi < 2\pi$. Using the result of part *c*, we find $< \mathbf{S} >= (1.0 \times 10^3)/\rho^2 \mathbf{a}_z$ W/m^2. The power through the given cross-section is now

$$P = \int_0^{2\pi} \int_{.008}^{.020} \frac{1.0 \times 10^3}{\rho^2} \rho \, d\rho \, d\phi = 2\pi \times 10^3 \ln\left(\frac{20}{8}\right) = \underline{5.7\,\text{kW}}$$

11.20. Voltage breakdown in air at standard temperature and pressure occurs at an electric field strength of approximately 3×10^6 V/m. This becomes an issue in some high-power optical experiments, in which tight focusing of light may be necessary. Estimate the lightwave power in watts that can be focused into a cylindrical beam of 10μm radius before breakdown occurs. Assume uniform plane wave behavior (although this assumption will produce an answer that is higher than the actual number by as much as a factor of 2, depending on the actual beam shape).

The power density in the beam in free space can be found as a special case of Eq. (76) (with $\eta = \eta_0$, $\theta_\eta = \alpha = 0$):

$$| < \mathbf{S} > | = \frac{E_0^2}{2\eta_0} = \frac{(3 \times 10^6)^2}{2(377)} = 1.2 \times 10^{10} \text{ W/m}^2$$

To avoid breakdown, the power in a 10-μm radius cylinder is then bounded by

$$P < (1.2 \times 10^{10})(\pi \times (10^{-5})^2) = \underline{3.75 \text{ W}}$$

11.21. The cylindrical shell, 1 cm $< \rho <$ 1.2 cm, is composed of a conducting material for which $\sigma = 10^6$ S/m. The external and internal regions are non-conducting. Let $H_\phi = 2000$ A/m at $\rho = 1.2$ cm.

a) Find **H** everywhere: Use Ampere's circuital law, which states:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho(2000) = 2\pi(1.2 \times 10^{-2})(2000) = 48\pi \text{ A} = I_{encl}$$

Then in this case

$$\mathbf{J} = \frac{I}{Area}\mathbf{a}_z = \frac{48}{(1.44 - 1.00) \times 10^{-4}}\mathbf{a}_z = 1.09 \times 10^6\,\mathbf{a}_z \text{ A/m}^2$$

With this result we again use Ampere's circuital law to find **H** everywhere within the shell as a function of ρ (in meters):

$$H_{\phi 1}(\rho) = \frac{1}{2\pi\rho}\int_0^{2\pi}\int_{.01}^{\rho} 1.09 \times 10^6\ \rho\, d\rho\, d\phi = \underline{\frac{54.5}{\rho}(10^4\rho^2 - 1) \text{ A/m} \quad (.01 < \rho < .012)}$$

Outside the shell, we would have

$$H_{\phi 2}(\rho) = \frac{48\pi}{2\pi\rho} = \underline{24/\rho \text{ A/m} \quad (\rho > .012)}$$

Inside the shell ($\rho < .01$ m), $H_\phi = 0$ since there is no enclosed current.

b) Find **E** everywhere: We use

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{1.09 \times 10^6}{10^6}\mathbf{a}_z = \underline{1.09\,\mathbf{a}_z \text{ V/m}}$$

which is valid, presumeably, outside as well as inside the shell.

c) Find **S** everywhere: Use

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = 1.09\,\mathbf{a}_z \times \frac{54.5}{\rho}(10^4\rho^2 - 1)\,\mathbf{a}_\phi$$
$$= \underline{-\frac{59.4}{\rho}(10^4\rho^2 - 1)\,\mathbf{a}_\rho \text{ W/m}^2 \quad (.01 < \rho < .012\,\text{m})}$$

Outside the shell,

$$\mathbf{S} = 1.09\,\mathbf{a}_z \times \frac{24}{\rho}\,\mathbf{a}_\phi = \underline{-\frac{26}{\rho}\,\mathbf{a}_\rho \text{ W/m}^2 \quad (\rho > .012\,\text{m})}$$

11.22. The inner and outer dimensions of a copper coaxial transmission line are 2 and 7 mm, respectively. Both conductors have thicknesses much greater than δ. The dielectric is lossless and the operating frequency is 400 MHz. Calculate the resistance per meter length of the:

a) inner conductor: First

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(4\times10^8)(4\pi\times10^{-7})(5.8\times10^7)}} = 3.3\times10^{-6}\text{m} = 3.3\mu\text{m}$$

Now, using (90) with a unit length, we find

$$R_{in} = \frac{1}{2\pi a\sigma\delta} = \frac{1}{2\pi(2\times10^{-3})(5.8\times10^7)(3.3\times10^{-6})} = \underline{0.42 \text{ ohms/m}}$$

b) outer conductor: Again, (90) applies but with a different conductor radius. Thus

$$R_{out} = \frac{a}{b}R_{in} = \frac{2}{7}(0.42) = \underline{0.12 \text{ ohms/m}}$$

c) transmission line: Since the two resistances found above are in series, the line resistance is their sum, or $R = R_{in} + R_{out} = \underline{0.54 \text{ ohms/m}}$.

11.23. A hollow tubular conductor is constructed from a type of brass having a conductivity of 1.2×10^7 S/m. The inner and outer radii are 9 mm and 10 mm respectively. Calculate the resistance per meter length at a frequency of

a) dc: In this case the current density is uniform over the entire tube cross-section. We write:

$$R(\text{dc}) = \frac{L}{\sigma A} = \frac{1}{(1.2\times10^7)\pi(.01^2 - .009^2)} = \underline{1.4\times10^{-3}\ \Omega/\text{m}}$$

b) 20 MHz: Now the skin effect will limit the effective cross-section. At 20 MHz, the skin depth is

$$\delta(20\text{MHz}) = [\pi f \mu_0 \sigma]^{-1/2} = [\pi(20\times10^6)(4\pi\times10^{-7})(1.2\times10^7)]^{-1/2} = 3.25\times10^{-5}\text{ m}$$

This is much less than the outer radius of the tube. Therefore we can approximate the resistance using the formula:

$$R(20\text{MHz}) = \frac{L}{\sigma A} = \frac{1}{2\pi b\delta} = \frac{1}{(1.2\times10^7)(2\pi(.01))(3.25\times10^{-5})} = \underline{4.1\times10^{-2}\ \Omega/\text{m}}$$

c) 2 GHz: Using the same formula as in part b, we find the skin depth at 2 GHz to be $\delta = 3.25\times10^{-6}$ m. The resistance (using the other formula) is $R(2\text{GHz}) = \underline{4.1\times10^{-1}\ \Omega/\text{m}}$.

11.24 a) Most microwave ovens operate at 2.45 GHz. Assume that $\sigma = 1.2 \times 10^6$ S/m and $\mu_r = 500$ for the stainless steel interior, and find the depth of penetration:

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(2.45 \times 10^9)(4\pi \times 10^{-7})(1.2 \times 10^6)}} = 9.28 \times 10^{-6}\text{m} = 9.28\mu\text{m}$$

b) Let $E_s = 50\angle\, 0^\circ$ V/m at the surface of the conductor, and plot a curve of the amplitude of E_s vs. the angle of E_s as the field propagates into the stainless steel: Since the conductivity is high, we use (82) to write $\alpha \doteq \beta \doteq \sqrt{\pi f \mu \sigma} = 1/\delta$. So, assuming that the direction into the conductor is z, the depth-dependent field is written as

$$E_s(z) = 50e^{-\alpha z}e^{-j\beta z} = 50e^{-z/\delta}e^{-jz/\delta} = \underbrace{50\exp(-z/9.28)}_{\text{amplitude}}\exp(-j\,\underbrace{z/9.28}_{\text{angle}})$$

where z is in microns. Therefore, the plot of amplitude versus angle is simply a plot of e^{-x} versus x, where $x = z/9.28$; the starting amplitude is 50 and the $1/e$ amplitude (at $z = 9.28\ \mu$m) is 18.4.

11.25. A good conductor is planar in form and carries a uniform plane wave that has a wavelength of 0.3 mm and a velocity of 3×10^5 m/s. Assuming the conductor is non-magnetic, determine the frequency and the conductivity: First, we use

$$f = \frac{v}{\lambda} = \frac{3 \times 10^5}{3 \times 10^{-4}} = 10^9\ \text{Hz} = \underline{1\ \text{GHz}}$$

Next, for a good conductor,

$$\delta = \frac{\lambda}{2\pi} = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \Rightarrow \quad \sigma = \frac{4\pi}{\lambda^2 f \mu} = \frac{4\pi}{(9 \times 10^{-8})(10^9)(4\pi \times 10^{-7})} = \underline{1.1 \times 10^5\ \text{S/m}}$$

11.26. The dimensions of a certain coaxial transmission line are $a = 0.8$mm and $b = 4$mm. The outer conductor thickness is 0.6mm, and all conductors have $\sigma = 1.6 \times 10^7$ S/m.

a) Find R, the resistance per unit length, at an operating frequency of 2.4 GHz: First

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi (2.4 \times 10^8)(4\pi \times 10^{-7})(1.6 \times 10^7)}} = 2.57 \times 10^{-6}\text{m} = 2.57\mu\text{m}$$

Then, using (90) with a unit length, we find

$$R_{in} = \frac{1}{2\pi a \sigma \delta} = \frac{1}{2\pi(0.8 \times 10^{-3})(1.6 \times 10^7)(2.57 \times 10^{-6})} = 4.84 \text{ ohms/m}$$

The outer conductor resistance is then found from the inner through

$$R_{out} = \frac{a}{b} R_{in} = \frac{0.8}{4}(4.84) = 0.97 \text{ ohms/m}$$

The net resistance per length is then the sum, $R = R_{in} + R_{out} = \underline{5.81 \text{ ohms/m}}$.

b) Use information from Secs. 6.3 and 8.10 to find C and L, the capacitance and inductance per unit length, respectively. The coax is air-filled. From those sections, we find (in free space)

$$C = \frac{2\pi\epsilon_0}{\ln(b/a)} = \frac{2\pi(8.854 \times 10^{-12})}{\ln(4/.8)} = \underline{3.46 \times 10^{-11} \text{ F/m}}$$

$$L = \frac{\mu_0}{2\pi} \ln(b/a) = \frac{4\pi \times 10^{-7}}{2\pi} \ln(4/.8) = \underline{3.22 \times 10^{-7} \text{ H/m}}$$

c) Find α and β if $\alpha + j\beta = \sqrt{j\omega C(R + j\omega L)}$: Taking real and imaginary parts of the given expression, we find

$$\alpha = \text{Re}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = \frac{\omega\sqrt{LC}}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{R}{\omega L}\right)^2} - 1\right]^{1/2}$$

and

$$\beta = \text{Im}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = \frac{\omega\sqrt{LC}}{\sqrt{2}}\left[\sqrt{1 + \left(\frac{R}{\omega L}\right)^2} + 1\right]^{1/2}$$

These can be found by writing out

$$\alpha = \text{Re}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = (1/2)\sqrt{j\omega C(R + j\omega L)} + c.c.$$

where $c.c$ denotes the complex conjugate. The result is squared, terms collected, and the square root taken. Now, using the values of R, C, and L found in parts a and b, we find $\alpha = \underline{3.0 \times 10^{-2} \text{ Np/m}}$ and $\beta = \underline{50.3 \text{ rad/m}}$.

11.27. The planar surface at $z = 0$ is a brass-Teflon interface. Use data available in Appendix C to evaluate the following ratios for a uniform plane wave having $\omega = 4 \times 10^{10}$ rad/s:

a) $\alpha_{\text{Tef}}/\alpha_{\text{brass}}$: From the appendix we find $\epsilon''/\epsilon' = .0003$ for Teflon, making the material a good dielectric. Also, for Teflon, $\epsilon'_r = 2.1$. For brass, we find $\sigma = 1.5 \times 10^7$ S/m, making brass a good conductor at the stated frequency. For a good dielectric (Teflon) we use the approximations:

$$\alpha \doteq \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon'}} = \left(\frac{\epsilon''}{\epsilon'}\right)\left(\frac{1}{2}\right)\omega\sqrt{\mu\epsilon'} = \frac{1}{2}\left(\frac{\epsilon''}{\epsilon'}\right)\frac{\omega}{c}\sqrt{\epsilon'_r}$$

$$\beta \doteq \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)\right] \doteq \omega\sqrt{\mu\epsilon'} = \frac{\omega}{c}\sqrt{\epsilon'_r}$$

For brass (good conductor) we have

$$\alpha \doteq \beta \doteq \sqrt{\pi f \mu \sigma_{\text{brass}}} = \sqrt{\pi\left(\frac{1}{2\pi}\right)(4 \times 10^{10})(4\pi \times 10^{-7})(1.5 \times 10^7)} = 6.14 \times 10^5 \text{ m}^{-1}$$

Now

$$\frac{\alpha_{\text{Tef}}}{\alpha_{\text{brass}}} = \frac{1/2\,(\epsilon''/\epsilon')\,(\omega/c)\sqrt{\epsilon'_r}}{\sqrt{\pi f \mu \sigma_{\text{brass}}}} = \frac{(1/2)(.0003)(4 \times 10^{10}/3 \times 10^8)\sqrt{2.1}}{6.14 \times 10^5} = \underline{4.7 \times 10^{-8}}$$

b)

$$\frac{\lambda_{\text{Tef}}}{\lambda_{\text{brass}}} = \frac{(2\pi/\beta_{\text{Tef}})}{(2\pi/\beta_{\text{brass}})} = \frac{\beta_{\text{brass}}}{\beta_{\text{Tef}}} = \frac{c\sqrt{\pi f \mu \sigma_{\text{brass}}}}{\omega\sqrt{\epsilon'_{r\,\text{Tef}}}} = \frac{(3 \times 10^8)(6.14 \times 10^5)}{(4 \times 10^{10})\sqrt{2.1}} = \underline{3.2 \times 10^3}$$

c)

$$\frac{v_{\text{Tef}}}{v_{\text{brass}}} = \frac{(\omega/\beta_{\text{Tef}})}{(\omega/\beta_{\text{brass}})} = \frac{\beta_{\text{brass}}}{\beta_{\text{Tef}}} = \underline{3.2 \times 10^3} \text{ as before}$$

11.28. A uniform plane wave in free space has electric field given by $\mathbf{E}_s = 10e^{-j\beta x}\mathbf{a}_z + 15e^{-j\beta x}\mathbf{a}_y$ V/m.

a) Describe the wave polarization: Since the two components have a fixed phase difference (in this case zero) with respect to time and position, the wave has linear polarization, with the field vector in the yz plane at angle $\phi = \tan^{-1}(10/15) = 33.7°$ to the y axis.

b) Find $\mathbf{H}_s$: With propagation in forward x, we would have

$$\mathbf{H}_s = \frac{-10}{377}e^{-j\beta x}\mathbf{a}_y + \frac{15}{377}e^{-j\beta x}\mathbf{a}_z \text{ A/m} = \underline{-26.5e^{-j\beta x}\mathbf{a}_y + 39.8e^{-j\beta x}\mathbf{a}_z \text{ mA/m}}$$

c) determine the average power density in the wave in W/m^2: Use

$$\mathbf{P}_{avg} = \frac{1}{2}\text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{1}{2}\left[\frac{(10)^2}{377}\mathbf{a}_x + \frac{(15)^2}{377}\mathbf{a}_x\right] = 0.43\mathbf{a}_x \text{ W/m}^2 \text{ or } P_{avg} = \underline{0.43 \text{ W/m}^2}$$

11.29. Consider a left-circularly polarized wave in free space that propagates in the forward z direction. The electric field is given by the appropriate form of Eq. (100).

a) Determine the magnetic field phasor, $\mathbf{H}_s$:
We begin, using (100), with $\mathbf{E}_s = E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$. We find the two components of $\mathbf{H}_s$ separately, using the two components of $\mathbf{E}_s$. Specifically, the x component of $\mathbf{E}_s$ is associated with a y component of $\mathbf{H}_s$, and the y component of $\mathbf{E}_s$ is associated with a negative x component of $\mathbf{H}_s$. The result is

$$\mathbf{H}_s = \underline{\frac{E_0}{\eta_0}\left(\mathbf{a}_y - j\mathbf{a}_x\right)e^{-j\beta z}}$$

b) Determine an expression for the average power density in the wave in W/m^2 by direct application of Eq. (77): We have

$$\mathbf{P}_{z,avg} = \frac{1}{2}Re(\mathbf{E}_s \times \mathbf{H}_s^*) = \frac{1}{2}Re\left(E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z} \times \frac{E_0}{\eta_0}(\mathbf{a}_y - j\mathbf{a}_x)e^{+j\beta z}\right)$$
$$= \underline{\frac{E_0^2}{\eta_0}\mathbf{a}_z \text{ W/m}^2} \quad \text{(assuming } E_0 \text{ is real)}$$

11.30. In an *anisotropic* medium, permittivity varies with electric field *direction*, and is a property seen in most crystals. Consider a uniform plane wave propagating in the z direction in such a medium, and which enters the material with equal field components along the x and y axes. The field phasor will take the form:

$$\mathbf{E}_s(z) = E_0(\mathbf{a}_x + \mathbf{a}_y\, e^{j\Delta\beta z})\, e^{-j\beta z}$$

where $\Delta\beta = \beta_x - \beta_y$ is the difference in phase constants for waves that are linearly-polarized in the x and y directions. Find distances into the material (in terms of $\Delta\beta$) at which the field is:

a) Linearly-polarized: We want the x and y components to be in phase, so therefore

$$\Delta\beta z_{lin} = m\pi \;\Rightarrow\; \underline{z_{lin} = \frac{m\pi}{\Delta\beta}\;,\;(m = 1, 2, 3...)}$$

b) Circularly-polarized: In this case, we want the two field components to be in quadrature phase, such that the total field is of the form, $\mathbf{E}_s = E_0(\mathbf{a}_x \pm j\mathbf{a}_y)e^{-j\beta z}$. Therefore,

$$\Delta\beta z_{circ} = \frac{(2n+1)\pi}{2} \;\Rightarrow\; \underline{z_{circ} = \frac{(2n+1)\pi}{2\Delta\beta}\;,\quad (n = 0, 1, 2, 3...)}$$

c) Assume intrinsic impedance η that is approximately constant with field orientation and find $\mathbf{H}_s$ and $< \mathbf{S} >$: Magnetic field is found by looking at the individual components:

$$\mathbf{H}_s(z) = \underline{\frac{E_0}{\eta}\left(\mathbf{a}_y - \mathbf{a}_x e^{j\Delta\beta z}\right)e^{-j\beta z}} \text{ and}$$

$$< \mathbf{S} >= \frac{1}{2}\mathcal{R}e\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \underline{\frac{E_0^2}{\eta}\,\mathbf{a}_z \text{ W/m}^2}$$

where it is assumed that E_0 is real. η is real because the medium is evidently lossless.

11.31. A linearly-polarized uniform plane wave, propagating in the forward z direction, is input to a lossless *anisotropic* material, in which the dielectric constant encountered by waves polarized along y (ϵ_{ry}) differs from that seen by waves polarized along x (ϵ_{rx}). Suppose $\epsilon_{rx} = 2.15$, $\epsilon_{ry} = 2.10$, and the wave electric field at input is polarized at 45° to the positive x and y axes. *Assume free space wavelength* λ.

a) Determine the shortest length of the material such that the wave as it emerges from the output end is circularly polarized: With the input field at 45°, the x and y components are of equal magnitude, and circular polarization will result if the phase difference between the components is $\pi/2$. Our requirement over length L is thus $\beta_x L - \beta_y L = \pi/2$, or

$$L = \frac{\pi}{2(\beta_x - \beta_y)} = \frac{\pi c}{2\omega(\sqrt{\epsilon_{rx}} - \sqrt{\epsilon_{ry}})}$$

With the given values, we find,

$$L = \frac{(58.3)\pi c}{2\omega} = 58.3\,\frac{\lambda}{4} = \underline{14.6\,\lambda}$$

b) Will the output wave be right- or left-circularly-polarized? With the dielectric constant greater for x-polarized waves, the x component will lag the y component in time at the output. The field can thus be written as $\mathbf{E} = E_0(\mathbf{a}_y - j\mathbf{a}_x)$, which is left circular polarization.

11.32. Suppose that the length of the medium of Problem 11.31 is made to be *twice* that as determined in the problem. Describe the polarization of the output wave in this case: With the length doubled, a phase shift of π radians develops between the two components. At the input, we can write the field as $\mathbf{E}_s(0) = E_0(\mathbf{a}_x + \mathbf{a}_y)$. After propagating through length L, we would have,

$$\mathbf{E}_s(L) = E_0[e^{-j\beta_x L}\mathbf{a}_x + e^{-j\beta_y L}\mathbf{a}_y] = E_0 e^{-j\beta_x L}[\mathbf{a}_x + e^{-j(\beta_y - \beta_x)L}\mathbf{a}_y]$$

where $(\beta_y - \beta_x)L = -\pi$ (since $\beta_x > \beta_y$), and so $\mathbf{E}_s(L) = E_0 e^{-j\beta_x L}[\mathbf{a}_x - \mathbf{a}_y]$. With the reversal of the y component, the wave polarization is rotated by 90°, but is still linear polarization.

11.33. Given a wave for which $\mathbf{E}_s = 15e^{-j\beta z}\mathbf{a}_x + 18e^{-j\beta z}e^{j\phi}\mathbf{a}_y$ V/m, propagating in a medium characterized by complex intrinsic impedance, η.

a) Find $\mathbf{H}_s$: With the wave propagating in the forward z direction, we find:

$$\underline{\mathbf{H}_s = \frac{1}{\eta}\left[-18e^{j\phi}\mathbf{a}_x + 15\mathbf{a}_y\right] e^{-j\beta z} \text{ A/m}}$$

b) Determine the average power density in W/m^2: We find

$$P_{z,avg} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\text{Re}\left\{\frac{(15)^2}{\eta^*} + \frac{(18)^2}{\eta^*}\right\} = \underline{275\,\text{Re}\left\{\frac{1}{\eta^*}\right\} \text{ W/m}^2}$$

11.34. Given the general elliptically-polarized wave as per Eq. (93):

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{j\phi}\mathbf{a}_y]e^{-j\beta z}$$

a) Show, using methods similar to those of Example 11.7, that a linearly polarized wave results when superimposing the given field and a phase-shifted field of the form:

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{-j\phi}\mathbf{a}_y]e^{-j\beta z}e^{j\delta}$$

where δ is a constant: Adding the two fields gives

$$\begin{aligned}\mathbf{E}_{s,tot} &= \left[E_{x0}\left(1+e^{j\delta}\right)\mathbf{a}_x + E_{y0}\left(e^{j\phi}+e^{-j\phi}e^{j\delta}\right)\mathbf{a}_y\right]e^{-j\beta z} \\ &= \left[E_{x0}e^{j\delta/2}\underbrace{\left(e^{-j\delta/2}+e^{j\delta/2}\right)}_{2\cos(\delta/2)}\mathbf{a}_x + E_{y0}e^{j\delta/2}\underbrace{\left(e^{-j\delta/2}e^{j\phi}+e^{-j\phi}e^{j\delta/2}\right)}_{2\cos(\phi-\delta/2)}\mathbf{a}_y\right]e^{-j\beta z}\end{aligned}$$

This simplifies to $\mathbf{E}_{s,tot} = 2\left[E_{x0}\cos(\delta/2)\mathbf{a}_x + E_{y0}\cos(\phi-\delta/2)\mathbf{a}_y\right]e^{j\delta/2}e^{-j\beta z}$, which is linearly polarized.

b) Find δ in terms of ϕ such that the resultant wave is polarized along x: By inspecting the part a result, we achieve a zero y component when $\underline{2\phi-\delta=\pi}$ (or odd multiples of π).

CHAPTER 12

12.1. A uniform plane wave in air, $E_{x1}^+ = E_{x10}^+ \cos(10^{10}t - \beta z)$ V/m, is normally-incident on a copper surface at $z = 0$. What percentage of the incident power density is transmitted into the copper? We need to find the reflection coefficient. The intrinsic impedance of copper (a good conductor) is

$$\eta_c = \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\sqrt{\frac{10^{10}(4\pi \times 10^7)}{2(5.8 \times 10^7)}} = (1+j)(.0104)$$

Note that the accuracy here is questionable, since we know the conductivity to only two significant figures. We nevertheless proceed: Using $\eta_0 = 376.7288$ ohms, we write

$$\Gamma = \frac{\eta_c - \eta_0}{\eta_c + \eta_0} = \frac{.0104 - 376.7288 + j.0104}{.0104 + 376.7288 + j.0104} = -.9999 + j.0001$$

Now $|\Gamma|^2 = .9999$, and so the transmitted power fraction is $1 - |\Gamma|^2 = .0001$, or about <u>0.01%</u> is transmitted.

12.2. The plane $z = 0$ defines the boundary between two dielectrics. For $z < 0$, $\epsilon_{r1} = 9$, $\epsilon''_{r1} = 0$, and $\mu_1 = \mu_0$. For $z > 0$, $\epsilon'_{r2} = 3$, $\epsilon''_{r2} = 0$, and $\mu_2 = \mu_0$. Let $E_{x1}^+ = 10\cos(\omega t - 15z)$ V/m and find

a) ω: We have $\beta = \omega\sqrt{\mu_0 \epsilon'_1} = \omega\sqrt{\epsilon'_{r1}}/c = 15$. So $\omega = 15c/\sqrt{\epsilon'_{r1}} = 15 \times (3 \times 10^8)/\sqrt{9} = \underline{1.5 \times 10^9 \text{ s}^{-1}}$.

b) $< \mathbf{S}_1^+ >$: First we need $\eta_1 = \sqrt{\mu_0/\epsilon'_1} = \eta_0/\sqrt{\epsilon'_{r1}} = 377/\sqrt{9} = 126$ ohms. Next we apply Eq. (76), Chapter 11, to evaluate the Poynting vector (with no loss and consequently with no phase difference between electric and magnetic fields). We find $< \mathbf{S}_1^+ > = (1/2)|E_1|^2/\eta_1\, \mathbf{a}_z = (1/2)(10)^2/126\, \mathbf{a}_z = \underline{0.40\, \mathbf{a}_z \text{ W/m}^2}$.

c) $< \mathbf{S}_1^- >$: First, we need to evaluate the reflection coefficient:

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0/\sqrt{\epsilon'_{r2}} - \eta_0/\sqrt{\epsilon'_{r1}}}{\eta_0/\sqrt{\epsilon'_{r2}} + \eta_0/\sqrt{\epsilon'_{r1}}} = \frac{\sqrt{\epsilon'_{r1}} - \sqrt{\epsilon'_{r2}}}{\sqrt{\epsilon'_{r1}} + \sqrt{\epsilon'_{r2}}} = \frac{\sqrt{9} - \sqrt{3}}{\sqrt{9} + \sqrt{3}} = 0.27$$

Then $< \mathbf{S}_1^- > = -|\Gamma|^2 < \mathbf{S}_1^+ > = -(0.27)^2(0.40)\, \mathbf{a}_z = \underline{-0.03\, \mathbf{a}_z \text{ W/m}^2}$.

d) $< \mathbf{S}_2^+ >$: This will be the remaining power, propagating in the forward z direction, or $< \mathbf{S}_2^+ > = \underline{0.37\, \mathbf{a}_z \text{ W/m}^2}$.

12.3. A uniform plane wave in region 1 is normally-incident on the planar boundary separating regions 1 and 2. If $\epsilon_1'' = \epsilon_2'' = 0$, while $\epsilon_{r1}' = \mu_{r1}^3$ and $\epsilon_{r2}' = \mu_{r2}^3$, find the ratio $\epsilon_{r2}'/\epsilon_{r1}'$ if 20% of the energy in the incident wave is reflected at the boundary. There are two possible answers. First, since $|\Gamma|^2 = .20$, and since both permittivities and permeabilities are real, $\Gamma = \pm 0.447$. we then set up

$$\Gamma = \pm 0.447 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0\sqrt{(\mu_{r2}/\epsilon_{r2}')} - \eta_0\sqrt{(\mu_{r1}/\epsilon_{r1}')}}{\eta_0\sqrt{(\mu_{r2}/\epsilon_{r2}')} + \eta_0\sqrt{(\mu_{r1}/\epsilon_{r1}')}}$$

$$= \frac{\sqrt{(\mu_{r2}/\mu_{r2}^3)} - \sqrt{(\mu_{r1}/\mu_{r1}^3)}}{\sqrt{(\mu_{r2}/\mu_{r2}^3)} + \sqrt{(\mu_{r1}/\mu_{r1}^3)}} = \frac{\mu_{r1} - \mu_{r2}}{\mu_{r1} + \mu_{r2}}$$

Therefore

$$\frac{\mu_{r2}}{\mu_{r1}} = \frac{1 \mp 0.447}{1 \pm 0.447} = (0.382, 2.62) \quad \Rightarrow \quad \frac{\epsilon_{r2}'}{\epsilon_{r1}'} = \left(\frac{\mu_{r2}}{\mu_{r1}}\right)^3 = \underline{(0.056, 17.9)}$$

12.4. A 10-MHz uniform plane wave having an initial average power density of 5W/m^2 is normally-incident from free space onto the surface of a lossy material in which $\epsilon_2''/\epsilon_2' = 0.05$, $\epsilon_{r2}' = 5$, and $\mu_2 = \mu_0$. Calculate the distance into the lossy medium at which the transmitted wave power density is down by 10dB from the initial 5W/m^2:

First, since $\epsilon_2''/\epsilon_2' = 0.05 << 1$, we recognize region 2 as a good dielectric. Its intrinsic impedance is therefore approximated well by Eq. (62b), Chapter 11:

$$\eta_2 = \sqrt{\frac{\mu_0}{\epsilon_2'}}\left[1 + j\frac{1}{2}\frac{\epsilon_2''}{\epsilon_2'}\right] = \frac{377}{\sqrt{5}}\,[1 + j0.025]$$

The reflection coefficient encountered by the incident wave from region 1 is therefore

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{(377/\sqrt{5})[1 + j.025] - 377}{(377/\sqrt{5})[1 + j.025] + 377} = \frac{(1 - \sqrt{5}) + j.025}{(1 + \sqrt{5}) + j.025} = -0.383 + j0.011$$

The fraction of the incident power that is reflected is then $|\Gamma|^2 = 0.147$, and thus the fraction of the power that is transmitted into region 2 is $1 - |\Gamma|^2 = 0.853$. Still using the good dielectric approximation, the attenuation coefficient in region 2 is found from Eq. (60a), Chapter 11:

$$\alpha \doteq \frac{\omega\epsilon_2''}{2}\sqrt{\frac{\mu_0}{\epsilon_2'}} = (2\pi \times 10^7)(0.05 \times 5 \times 8.854 \times 10^{-12})\frac{377}{2\sqrt{5}} = 2.34 \times 10^{-2}\ \text{Np/m}$$

Now, the power that propagates into region 2 is expressed in terms of the incident power through

$$< S_2 > (z) = 5(1 - |\Gamma|^2)e^{-2\alpha z} = 5(.853)e^{-2(2.34\times 10^{-2})z} = 0.5\ \text{W/m}^2$$

in which the last equality indicates a factor of ten reduction from the incident power, as occurs for a 10 dB loss. Solve for z to obtain

$$z = \frac{\ln(8.53)}{2(2.34 \times 10^{-2})} = \underline{45.8\ \text{m}}$$

12.5. The region $z < 0$ is characterized by $\epsilon'_r = \mu_r = 1$ and $\epsilon''_r = 0$. The total **E** field here is given as the sum of the two uniform plane waves, $\mathbf{E}_s = 150e^{-j10z}\,\mathbf{a}_x + (50\angle 20°)e^{j10z}\,\mathbf{a}_x$ V/m.

a) What is the operating frequency? In free space, $\beta = k_0 = 10 = \omega/c = \omega/3\times10^8$. Thus, $\omega = 3\times10^9\ \text{s}^{-1}$, or $f = \omega/2\pi = \underline{4.7\times10^8\ \text{Hz}}$.

b) Specify the intrinsic impedance of the region $z > 0$ that would provide the appropriate reflected wave: Use

$$\Gamma = \frac{E_r}{E_{inc}} = \frac{50e^{j20°}}{150} = \frac{1}{3}e^{j20°} = 0.31 + j0.11 = \frac{\eta - \eta_0}{\eta + \eta_0}$$

Now

$$\eta = \eta_0\left(\frac{1+\Gamma}{1-\Gamma}\right) = 377\left(\frac{1+0.31+j0.11}{1-0.31-j0.31}\right) = \underline{691 + j177\ \Omega}$$

c) At what value of z ($-10\,\text{cm} < z < 0$) is the total electric field intensity a maximum amplitude? We found the phase of the reflection coefficient to be $\phi = 20° = .349$rad, and we use

$$z_{max} = \frac{-\phi}{2\beta} = \frac{-.349}{20} = -0.017\,\text{m} = \underline{-1.7\,\text{cm}}$$

12.6. In the beam-steering prism of Example 12.8, suppose the anti-reflective coatings are removed, leaving bare glass-to-air interfaces. Calcluate the ratio of the prism output power to the input power, assuming a single transit.

In making the transit, the light encounters two interfaces at normal incidence, at which loss will occur. The reflection coefficient at the front surface (air to glass) is

$$\Gamma_f = \frac{\eta_g - \eta_0}{\eta_g + \eta_0} = \frac{\eta_0/n_g - \eta_0}{\eta_0/n_g + \eta_0} = \frac{1-n_g}{1+n_g}$$

Taking the glass index, n_g, as 1.45, we find $\Gamma_f = -0.18$. The interface on exit from the prism is glass to air, and so the reflection coefficient there will be equal and opposite to Γ_f; i.e., $\Gamma_b = -\Gamma_f$.

Now, the wave power that makes it through (assuming total reflection at the 45° interface) will be

$$P_{out} = P_{in}(1-|\Gamma_f|^2)(1-|\Gamma_b|^2) = P_{in}(1-|0.18|^2)^2 = \underline{0.93}P_{in}$$

So we have 93% net transmission.

12.7. The semi-infinite regions $z < 0$ and $z > 1$ m are free space. For $0 < z < 1$ m, $\epsilon'_r = 4$, $\mu_r = 1$, and $\epsilon''_r = 0$. A uniform plane wave with $\omega = 4 \times 10^8$ rad/s is travelling in the $\mathbf{a}_z$ direction toward the interface at $z = 0$.

a) Find the standing wave ratio in each of the three regions: First we find the phase constant in the middle region,

$$\beta_2 = \frac{\omega\sqrt{\epsilon'_r}}{c} = \frac{2(4 \times 10^8)}{3 \times 10^8} = 2.67\,\text{rad/m}$$

Then, with the middle layer thickness of 1 m, $\beta_2 d = 2.67\,\text{rad}$. Also, the intrinsic impedance of the middle layer is $\eta_2 = \eta_0/\sqrt{\epsilon'_r} = \eta_0/2$. We now find the input impedance:

$$\eta_{in} = \eta_2 \left[\frac{\eta_0 \cos(\beta_2 d) + j\eta_2 \sin(\beta_2 d)}{\eta_2 \cos(\beta_2 d) + j\eta_0 \sin(\beta_2 d)}\right] = \frac{377}{2}\left[\frac{2\cos(2.67) + j\sin(2.67)}{\cos(2.67) + j2\sin(2.67)}\right] = 231 + j141$$

Now, at the first interface,

$$\Gamma_{12} = \frac{\eta_{in} - \eta_0}{\eta_{in} + \eta_0} = \frac{231 + j141 - 377}{231 + j141 + 377} = -.176 + j.273 = .325\angle 123^\circ$$

The standing wave ratio measured in region 1 is thus

$$s_1 = \frac{1 + |\Gamma_{12}|}{1 - |\Gamma_{12}|} = \frac{1 + 0.325}{1 - 0.325} = \underline{1.96}$$

In region 2 the standing wave ratio is found by considering the reflection coefficient for waves incident from region 2 on the second interface:

$$\Gamma_{23} = \frac{\eta_0 - \eta_0/2}{\eta_0 + \eta_0/2} = \frac{1 - 1/2}{1 + 1/2} = \frac{1}{3}$$

Then

$$s_2 = \frac{1 + 1/3}{1 - 1/3} = \underline{2}$$

Finally, $s_3 = \underline{1}$, since no reflected waves exist in region 3.

b) Find the location of the maximum $|\mathbf{E}|$ for $z < 0$ that is nearest to $z = 0$. We note that the phase of Γ_{12} is $\phi = 123^\circ = 2.15$ rad. Thus

$$z_{max} = \frac{-\phi}{2\beta} = \frac{-2.15}{2(4/3)} = \underline{-.81\,\text{m}}$$

12.8. A wave starts at point a, propagates 1m through a lossy dielectric rated at $\alpha_{dB} = 0.1\text{dB/cm}$, reflects at normal incidence at a boundary at which $\Gamma = 0.3 + j0.4$, and then returns to point a. Calculate the ratio of the final power to the incident power after this round trip: Final power, P_f, and incident power, P_i, are related through

$$P_f = P_i 10^{-0.1\alpha_{dB}L}|\Gamma|^2 10^{-0.1\alpha_{dB}L} \quad \Rightarrow \quad \frac{P_f}{P_i} = |0.3 + j0.4|^2 10^{-0.2(0.1)100} = \underline{2.5 \times 10^{-3}}$$

12.9. Region 1, $z < 0$, and region 2, $z > 0$, are both perfect dielectrics ($\mu = \mu_0$, $\epsilon'' = 0$). A uniform plane wave traveling in the $\mathbf{a}_z$ direction has a radian frequency of 3×10^{10} rad/s. Its wavelengths in the two regions are $\lambda_1 = 5$ cm and $\lambda_2 = 3$ cm. What percentage of the energy incident on the boundary is

a) reflected; We first note that

$$\epsilon'_{r1} = \left(\frac{2\pi c}{\lambda_1 \omega}\right)^2 \quad \text{and} \quad \epsilon'_{r2} = \left(\frac{2\pi c}{\lambda_2 \omega}\right)^2$$

Therefore $\epsilon'_{r1}/\epsilon'_{r2} = (\lambda_2/\lambda_1)^2$. Then with $\mu = \mu_0$ in both regions, we find

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0\sqrt{1/\epsilon'_{r2}} - \eta_0\sqrt{1/\epsilon'_{r1}}}{\eta_0\sqrt{1/\epsilon'_{r2}} + \eta_0\sqrt{1/\epsilon'_{r1}}} = \frac{\sqrt{\epsilon'_{r1}/\epsilon'_{r2}} - 1}{\sqrt{\epsilon'_{r1}/\epsilon'_{r2}} + 1} = \frac{(\lambda_2/\lambda_1) - 1}{(\lambda_2/\lambda_1) + 1}$$
$$= \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} = \frac{3-5}{3+5} = -\frac{1}{4}$$

The fraction of the incident energy that is reflected is then $|\Gamma|^2 = 1/16 = \underline{6.25 \times 10^{-2}}$.

b) transmitted? We use part a and find the transmitted fraction to be $1 - |\Gamma|^2 = 15/16 = \underline{0.938}$.

c) What is the standing wave ratio in region 1? Use

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 1/4}{1 - 1/4} = \frac{5}{3} = \underline{1.67}$$

12.10. In Fig. 12.1, let region 2 be free space, while $\mu_{r1} = 1$, $\epsilon''_{r1} = 0$, and ϵ'_{r1} is unknown. Find ϵ'_{r1} if

a) the amplitude of $\mathbf{E}_1^-$ is one-half that of $\mathbf{E}_1^+$: Since region 2 is free space, the reflection coefficient is

$$\Gamma = \frac{|\mathbf{E}_1^-|}{|\mathbf{E}_1^+|} = \frac{\eta_0 - \eta_1}{\eta_0 + \eta_1} = \frac{\eta_0 - \eta_0/\sqrt{\epsilon'_{r1}}}{\eta_0 + \eta_0/\sqrt{\epsilon'_{r1}}} = \frac{\sqrt{\epsilon'_{r1}} - 1}{\sqrt{\epsilon'_{r1}} + 1} = \frac{1}{2} \quad \Rightarrow \quad \epsilon'_{r1} = \underline{9}$$

.

b) $< \mathbf{S}_1^- >$ is one-half of $< \mathbf{S}_1^+ >$: This time

$$|\Gamma|^2 = \left|\frac{\sqrt{\epsilon'_{r1}} - 1}{\sqrt{\epsilon'_{r1}} + 1}\right|^2 = \frac{1}{2} \quad \Rightarrow \quad \epsilon'_{r1} = \underline{34}$$

c) $|\mathbf{E}_1|_{min}$ is one-half $|\mathbf{E}_1|_{max}$: Use

$$\frac{|\mathbf{E}_1|_{max}}{|\mathbf{E}_1|_{min}} = s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 2 \quad \Rightarrow \quad |\Gamma| = \Gamma = \frac{1}{3} = \frac{\sqrt{\epsilon'_{r1}} - 1}{\sqrt{\epsilon'_{r1}} + 1} \quad \Rightarrow \quad \epsilon'_{r1} = \underline{4}$$

12.11. A 150 MHz uniform plane wave in normally-incident from air onto a material whose intrinsic impedance is unknown. Measurements yield a standing wave ratio of 3 and the appearance of an electric field minimum at 0.3 wavelengths in front of the interface. Determine the impedance of the unknown material: First, the field minimum is used to find the phase of the reflection coefficient, where

$$z_{min} = -\frac{1}{2\beta}(\phi + \pi) = -0.3\lambda \quad \Rightarrow \quad \phi = 0.2\pi$$

where $\beta = 2\pi/\lambda$ has been used. Next,

$$|\Gamma| = \frac{s-1}{s+1} = \frac{3-1}{3+1} = \frac{1}{2}$$

So we now have

$$\Gamma = 0.5e^{j0.2\pi} = \frac{\eta_u - \eta_0}{\eta_u + \eta_0}$$

We solve for η_u to find

$$\eta_u = \eta_0(1.70 + j1.33) = \underline{641 + j501\ \Omega}$$

12.12. A 50MHz uniform plane wave is normally incident from air onto the surface of a calm ocean. For seawater, $\sigma = 4$ S/m, and $\epsilon'_r = 78$.

a) Determine the fractions of the incident power that are reflected and transmitted: First we find the loss tangent:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{2\pi(50 \times 10^6)(78)(8.854 \times 10^{-12})} = 18.4$$

This value is sufficiently greater than 1 to enable seawater to be considered a good conductor at 50MHz. Then, using the approximation (Eq. 85, Chapter 11), the intrinsic impedance is $\eta_s = \sqrt{\pi f \mu/\sigma}(1+j)$, and the reflection coefficient becomes

$$\Gamma = \frac{\sqrt{\pi f \mu/\sigma}\ (1+j) - \eta_0}{\sqrt{\pi f \mu/\sigma}\ (1+j) + \eta_0}$$

where $\sqrt{\pi f \mu/\sigma} = \sqrt{\pi(50 \times 10^6)(4\pi \times 10^{-7})/4} = 7.0$. The fraction of the power reflected is

$$\frac{P_r}{P_i} = |\Gamma|^2 = \frac{[\sqrt{\pi f \mu/\sigma} - \eta_0]^2 + \pi f \mu/\sigma}{[\sqrt{\pi f \mu/\sigma} + \eta_0]^2 + \pi f \mu/\sigma} = \frac{[7.0 - 377]^2 + 49.0}{[7.0 + 377]^2 + 49.0} = \underline{0.93}$$

The transmitted fraction is then

$$\frac{P_t}{P_i} = 1 - |\Gamma|^2 = 1 - 0.93 = \underline{0.07}$$

b) Qualitatively, how will these answers change (if at all) as the frequency is increased? Within the limits of our good conductor approximation (loss tangent greater than about ten), the reflected power fraction, using the formula derived in part *a*, is found to <u>decrease</u> with increasing frequency. The transmitted power fraction thus increases.

12.13. A right-circularly-polarized plane wave is normally incident from air onto a semi-infinite slab of plexiglas ($\epsilon'_r = 3.45$, $\epsilon''_r = 0$). Calculate the fractions of the incident power that are reflected and transmitted. Also, describe the polarizations of the reflected and transmitted waves. First, the impedance of the plexiglas will be $\eta = \eta_0/\sqrt{3.45} = 203\,\Omega$. Then

$$\Gamma = \frac{203 - 377}{203 + 377} = -0.30$$

The reflected power fraction is thus $|\Gamma|^2 = \underline{0.09}$. The total electric field in the plane of the interface must rotate in the same direction as the incident field, in order to continually satisfy the boundary condition of tangential electric field continuity across the interface. Therefore, the reflected wave will have to be left circularly polarized in order to make this happen. The transmitted power fraction is now $1 - |\Gamma|^2 = \underline{0.91}$. The transmitted field will be right circularly polarized (as the incident field) for the same reasons.

12.14. A left-circularly-polarized plane wave is normally-incident onto the surface of a perfect conductor.

a) Construct the superposition of the incident and reflected waves in phasor form: Assume positive z travel for the incident electric field. Then, with reflection coefficient, $\Gamma = -1$, the incident and reflected fields will add to give the total field:

$$\mathbf{E}_{tot} = \mathbf{E}_i + \mathbf{E}_r = E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z} - E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{+j\beta z}$$

$$= E_0 \left[\underbrace{\left(e^{-j\beta z} - e^{j\beta z}\right)}_{-2j\sin(\beta z)} \mathbf{a}_x + j \underbrace{\left(e^{-j\beta z} - e^{j\beta z}\right)}_{-2j\sin(\beta z)} \mathbf{a}_y \right] = \underline{2E_0 \sin(\beta z)\left[\mathbf{a}_y - j\mathbf{a}_x\right]}$$

b) Determine the real instantaneous form of the result of part a:

$$\mathbf{E}(z,t) = \mathrm{Re}\left\{\mathbf{E}_{tot}e^{j\omega t}\right\} = \underline{2E_0 \sin(\beta z)\left[\cos(\omega t)\mathbf{a}_y + \sin(\omega t)\mathbf{a}_x\right]}$$

c) Describe the wave that is formed: This is a standing wave exhibiting circular polarization in time. At each location along the z axis, the field vector rotates clockwise in the xy plane, and has amplitude (constant with time) given by $2E_0 \sin(\beta z)$.

12.15. Sulfur hexafluoride (SF_6) is a high-density gas that has refractive index, $n_s = 1.8$ at a specified pressure, temperature, and wavelength. Consider the retro-reflecting prism shown in Fig. 12.16, that is immersed in SF_6. Light enters through a quarter-wave antireflective coating and then totally reflects from the back surfaces of the glass. In principle, the beam should experience zero loss at the design wavelength ($P_{out} = P_{in}$).

a) Determine the minimum required value of the glass refractive index, n_g, so that the interior beam will totally reflect: We set the critical angle of total reflection equal to 45°, which gives

$$\sin\theta_c = \frac{n_s}{n_g} = \sin(45°) = \frac{1}{\sqrt{2}} \;\Rightarrow\; n_g = n_s\sqrt{2} = \underline{2.55}$$

b) Knowing n_g, find the required refractive index of the quarter-wave film, n_f: For a quarter-wave section, we know that the film intrinsic impedance will be

$$\eta_f = \sqrt{\eta_s \eta_g} \;\Rightarrow\; n_f = \sqrt{n_s n_g} = \sqrt{(1.80)(2.55)} = \underline{2.14}$$

c) With the SF_6 gas evacuated from the chamber, and with the glass and film values as previously found, find the ratio, P_{out}/P_{in}. Assume very slight misalignment, so that the long beam path through the prism is not re-traced by reflected waves. The beam loses power at the two normal-incidence boundaries, whereas the back reflections at 45° will still be lossless, as that angle is now greater than θ_c with the reduced surrounding index. At the first normal incidence boundary (from air to film to glass), the input intrinsic impedance is

$$\eta_{in1} = \frac{\eta_f^2}{\eta_g} = \frac{\eta_0^2/n_f^2}{\eta_0/n_g} = \eta_0\left(\frac{n_g}{n_f^2}\right)$$

At the second normal incidence boundary at the prism exit (glass to film to air), the input intrinsic impedance is

$$\eta_{in2} = \frac{\eta_f^2}{\eta_0} = \frac{\eta_0^2/n_f^2}{\eta_0} = \eta_0\left(\frac{1}{n_f^2}\right)$$

The reflection coefficients at the two boundaries will be

$$\Gamma_1 = \frac{\eta_{in1} - \eta_0}{\eta_{in1} + \eta_0} = \frac{n_g - n_f^2}{n_g + n_f^2} = \frac{2.55 - (2.14)^2}{2.55 + (2.14)^2} = -0.285$$

$$\Gamma_2 = \frac{\eta_{in2} - \eta_g}{\eta_{in2} + \eta_g} = \frac{n_g - n_f^2}{n_g + n_f^2} = \Gamma_1$$

The power ratio will be:

$$\frac{P_{out}}{P_{in}} = \left(1 - |\Gamma_1|^2\right)\left(1 - |\Gamma_2|^2\right) = \left(1 - (0.285)^2\right)^2 = \underline{0.845}$$

12.16. In Fig. 12.5, let regions 2 and 3 both be of quarter-wave thickness. Region 4 is glass, having refractive index, $n_4 = 1.45$; region 1 is air.

a) Find $\eta_{in,b}$: Since region 3 is a quarter-wave layer, $\beta_3 l_b = \pi/2$, and (47) reduces to

$$\eta_{in,b} = \frac{\eta_3^2}{\eta_4}$$

b) Find $\eta_{in,a}$: Again, with region 2 of quarter-wave thickness, $\beta_2 l_a = \pi/2$ and (48) becomes

$$\eta_{in,a} = \frac{\eta_2^2}{\eta_{in,b}} = \frac{\eta_2^2 \eta_4}{\eta_3^2}$$

c) Specify a relation between the four intrinsic impedances that will enable total transmission of waves incident from the left into region 4: At the front surface, we need to have a zero reflection coefficient, so the input impedance there must match that of free space:

$$\eta_{in,a} = \eta_0 \;\Rightarrow\; \underline{\eta_2^2 \eta_4 = \eta_3^2 \eta_0}$$

d) Specify refractive index values for regions 2 and 3 that will accomplish the condition of part c: We can rewrite the part c result as

$$\left(\frac{\eta_0^2}{n_2^2}\right)\left(\frac{\eta_0}{\eta_4}\right) = \left(\frac{\eta_0^2}{n_3^2}\right)\eta_0 \;\Rightarrow\; n_4 = \frac{n_3^2}{n_2^2}$$

So any combination of indices that satisfy this result will work. One combination, for example, would be $n_2 = 1.10$ and $n_3 = 1.33$. It is better to have the indices ascending (or descending) monotonically in value from layer to layer because the high transmission feature is then less sensitive to changes in wavelength (as an exercise for fun, show this).

e) Find the fraction of incident power transmitted if the two layers were of half-wave thickness instead of quarter-wave: For any half-wave layer, we know that the input impedance is equal to that of the load. Therefore, $\eta_{in,b} = \eta_{in,a} = \eta_4$. The reflection coefficient at the front surface is therefore

$$\Gamma_{in} = \frac{\eta_{in,a} - \eta_0}{\eta_{in,a} + \eta_0} = \frac{\eta_4 - \eta_0}{\eta_4 + \eta_0} = \frac{1 - n_4}{1 + n_4} = \frac{1 - 1.45}{1 + 1.45} = -0.184$$

The transmitted power fraction is then

$$\frac{P_t}{P_{in}} = 1 - |\Gamma_{in}|^2 = 1 - (0.184)^2 = \underline{0.97}$$

12.17. A uniform plane wave in free space is normally-incident onto a dense dielectric plate of thickness $\lambda/4$, having refractive index n. Find the required value of n such that exactly half the incident power is reflected (and half transmitted). Remember that $n > 1$.

In this problem, $\eta_1 = \eta_3 = \eta_0$, and the quarter-wave section input impedance is therefore

$$\eta_{in} = \frac{\eta_2^2}{\eta_3} = \frac{\eta_0^2/n^2}{\eta_0} = \frac{\eta_0}{n^2}$$

The reflection coefficient at the front surface is then

$$\Gamma_{in} = \frac{\eta_{in} - \eta_0}{\eta_{in} + \eta_0} = \frac{1 - n^2}{1 + n^2}$$

For half-power reflection, we require that $|\Gamma_{in}|^2 = 0.5$, or $\Gamma_{in} = \pm 1/\sqrt{2}$. Since n must be greater than 1, we choose the minus sign option and write:

$$\frac{1 - n^2}{1 + n^2} = -\frac{1}{\sqrt{2}} \;\Rightarrow\; n = \left[\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right]^{1/2} = \underline{2.41}$$

12.18. A uniform plane wave is normally-incident onto a slab of glass ($n = 1.45$) whose back surface is in contact with a perfect conductor. Determine the reflective phase shift at the front surface of the glass if the glass thickness is: (a) $\lambda/2$; (b) $\lambda/4$; (c) $\lambda/8$.

With region 3 being a perfect conductor, $\eta_3 = 0$, and Eq. (36) gives the input impedance to the structure as $\eta_{in} = j\eta_2 \tan \beta\ell$. The reflection coefficient is then

$$\Gamma = \frac{\eta_{in} - \eta_0}{\eta_{in} + \eta_0} = \frac{j\eta_2 \tan \beta\ell - \eta_0}{j\eta_2 \tan \beta\ell + \eta_0} = \frac{\eta_2^2 \tan^2 \beta\ell - \eta_0^2 + j2\eta_0\eta_2 \tan \beta\ell}{\eta_2^2 \tan^2 \beta\ell + \eta_0^2} = \Gamma_r + j\Gamma_i$$

where the last equality occurs by multiplying the numerator and denominator of the middle term by the complex conjugate of its denominator. The reflective phase is now

$$\phi = \tan^{-1}\left(\frac{\Gamma_i}{\Gamma_r}\right) = \tan^{-1}\left[\frac{2\eta_2\eta_0 \tan \beta\ell}{\eta_2^2 \tan^2 \beta\ell - \eta_0^2}\right] = \tan^{-1}\left[\frac{(2.90) \tan \beta\ell}{\tan \beta\ell - 2.10}\right]$$

where $\eta_2 = \eta_0/1.45$ has been used. We can now evaluate the phase shift for the three given cases. First, when $\ell = \lambda/2$, $\beta\ell = \pi$, and thus $\underline{\phi(\lambda/2) = 0}$. Next, when $\ell = \lambda/4$, $\beta\ell = \pi/2$, and

$$\phi(\lambda/4) \rightarrow \tan^{-1}[2.90] = \underline{71^\circ}$$

as $\ell \rightarrow \lambda/4$. Finally, when $\ell = \lambda/8$, $\beta\ell = \pi/4$, and

$$\phi(\lambda/8) = \tan^{-1}\left[\frac{2.90}{1 - 2.10}\right] = \underline{-69.2^\circ} \text{ (or } 291^\circ)$$

12.19. You are given four slabs of lossless dielectric, all with the same intrinsic impedance, η, known to be different from that of free space. The thickness of each slab is $\lambda/4$, where λ is the wavelength as measured in the slab material. The slabs are to be positioned parallel to one another, and the combination lies in the path of a uniform plane wave, normally-incident. The slabs are to be arranged such that the air spaces between them are either zero, one-quarter wavelength, or one-half wavelength in thickness. Specify an arrangement of slabs and air spaces such that

a) the wave is totally transmitted through the stack: In this case, we look for a combination of half-wave sections. Let the inter-slab distances be d_1, d_2, and d_3 (from left to right). Two possibilities are i.) $\underline{d_1 = d_2 = d_3 = 0}$, thus creating a single section of thickness λ, or ii.) $\underline{d_1 = d_3 = 0,\ d_2 = \lambda/2}$, thus yielding two half-wave sections separated by a half-wavelength.

b) the stack presents the highest reflectivity to the incident wave: The best choice here is to make $\underline{d_1 = d_2 = d_3 = \lambda/4}$. Thus every thickness is one-quarter wavelength. The impedances transform as follows: First, the input impedance at the front surface of the last slab (slab 4) is $\eta_{in,1} = \eta^2/\eta_0$. We transform this back to the back surface of slab 3, moving through a distance of $\lambda/4$ in free space: $\eta_{in,2} = \eta_0^2/\eta_{in,1} = \eta_0^3/\eta^2$. We next transform this impedance to the front surface of slab 3, producing $\eta_{in,3} = \eta^2/\eta_{in,2} = \eta^4/\eta_0^3$. We continue in this manner until reaching the front surface of slab 1, where we find $\eta_{in,7} = \eta^8/\eta_0^7$. Assuming $\eta < \eta_0$, the ratio η^n/η_0^{n-1} becomes smaller as n increases (as the number of slabs increases). The reflection coefficient for waves incident on the front slab thus gets close to unity, and approaches 1 as the number of slabs approaches infinity.

12.20. The 50MHz plane wave of Problem 12.12 is incident onto the ocean surface at an angle to the normal of 60°. Determine the fractions of the incident power that are reflected and transmitted for

a) s polarization: To review Problem 12, we first we find the loss tangent:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{2\pi(50\times10^6)(78)(8.854\times10^{-12})} = 18.4$$

This value is sufficiently greater than 1 to enable seawater to be considered a good conductor at 50MHz. Then, using the approximation (Eq. 85, Chapter 11), and with $\mu = \mu_0$, the intrinsic impedance is $\eta_s = \sqrt{\pi f \mu/\sigma}(1+j) = 7.0(1+j)$. Next we need the angle of refraction, which means that we need to know the refractive index of seawater at 50MHz. For a uniform plane wave in a good conductor, the phase constant is

$$\beta = \frac{n_{sea}\,\omega}{c} \doteq \sqrt{\pi f \mu \sigma} \quad \Rightarrow \quad n_{sea} \doteq c\sqrt{\frac{\mu\sigma}{4\pi f}} = 26.8$$

Then, using Snell's law, the angle of refraction is found:

$$\sin\theta_2 = \frac{n_{sea}}{n_1}\sin\theta_1 = 26.8\sin(60^\circ) \quad \Rightarrow \quad \theta_2 = 1.9^\circ$$

This angle is small enough so that $\cos\theta_2 \doteq 1$. Therefore, for s polarization,

$$\Gamma_s \doteq \frac{\eta_{s2}-\eta_{s1}}{\eta_{s2}+\eta_{s1}} = \frac{7.0(1+j) - 377/\cos 60^\circ}{7.0(1+j) + 377/\cos 60^\circ} = -0.98 + j0.018 = 0.98\angle 179^\circ$$

The fraction of the power reflected is now $|\Gamma_s|^2 = \underline{0.96}$. The fraction transmitted is then $\underline{0.04}$.

b) p polarization: Again, with the refracted angle close to zero, the relection coefficient for p polarization is

$$\Gamma_p \doteq \frac{\eta_{p2}-\eta_{p1}}{\eta_{p2}+\eta_{p1}} = \frac{7.0(1+j) - 377\cos 60^\circ}{7.0(1+j) + 377\cos 60^\circ} = -0.93 + j0.069 = 0.93\angle 176^\circ$$

The fraction of the power reflected is now $|\Gamma_p|^2 = \underline{0.86}$. The fraction transmitted is then $\underline{0.14}$.

12.21. A right-circularly polarized plane wave in air is incident at Brewster's angle onto a semi-infinite slab of plexiglas ($\epsilon'_r = 3.45$, $\epsilon''_r = 0$, $\mu = \mu_0$).

a) Determine the fractions of the incident power that are reflected and transmitted: In plexiglas, Brewster's angle is $\theta_B = \theta_1 = \tan^{-1}(\epsilon'_{r2}/\epsilon'_{r1}) = \tan^{-1}(\sqrt{3.45}) = 61.7°$. Then the angle of refraction is $\theta_2 = 90° - \theta_B$ (see Example 12.9), or $\theta_2 = 28.3°$. With incidence at Brewster's angle, all p-polarized power will be transmitted — only s-polarized power will be reflected. This is found through

$$\Gamma_s = \frac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}} = \frac{.614\eta_0 - 2.11\eta_0}{.614\eta_0 + 2.11\eta_0} = -0.549$$

where $\eta_{1s} = \eta_1 \sec\theta_1 = \eta_0 \sec(61.7°) = 2.11\eta_0$,
and $\eta_{2s} = \eta_2 \sec\theta_2 = (\eta_0/\sqrt{3.45})\sec(28.3°) = 0.614\eta_0$. Now, the reflected power fraction is $|\Gamma|^2 = (-.549)^2 = .302$. Since the wave is circularly-polarized, the s-polarized component represents one-half the total incident wave power, and so the fraction of the *total* power that is reflected is $.302/2 = 0.15$, or $\underline{15\%}$. The fraction of the incident power that is transmitted is then the remainder, or $\underline{85\%}$.

b) Describe the polarizations of the reflected and transmitted waves: Since all the p-polarized component is transmitted, the reflected wave will be entirely s-polarized (linear). The transmitted wave, while having all the incident p-polarized power, will have a reduced s-component, and so this wave will be right-elliptically polarized.

12.22. A dielectric waveguide is shown in Fig. 12.16 with refractive indices as labeled. Incident light enters the guide at angle ϕ from the front surface normal as shown. Once inside, the light totally reflects at the upper $n_1 - n_2$ interface, where $n_1 > n_2$. All subsequent reflections from the upper an lower boundaries will be total as well, and so the light is confined to the guide. Express, in terms of n_1 and n_2, the maximum value of ϕ such that total confinement will occur, with $n_0 = 1$. The quantity $\sin\phi$ is known as the *numerical aperture* of the guide.

From the illustration we see that ϕ_1 maximizes when θ_1 is at its minimum value. This minimum will be the critical angle for the $n_1 - n_2$ interface, where $\sin\theta_c = \sin\theta_1 = n_2/n_1$. Let the refracted angle to the right of the vertical interface (not shown) be ϕ_2, where $n_0 \sin\phi_1 = n_1 \sin\phi_2$. Then we see that $\phi_2 + \theta_1 = 90°$, and so $\sin\theta_1 = \cos\phi_2$. Now, the numerical aperture becomes

$$\sin\phi_{1max} = \frac{n_1}{n_0}\sin\phi_2 = n_1\cos\theta_1 = n_1\sqrt{1-\sin^2\theta_1} = n_1\sqrt{1-(n_2/n_1)^2} = \sqrt{n_1^2 - n_2^2}$$

Finally, $\phi_{1max} = \underline{\sin^{-1}\left(\sqrt{n_1^2 - n_2^2}\right)}$ is the numerical aperture angle.

12.23. Suppose that ϕ_1 in Fig. 12.16 is Brewster's angle, and that θ_1 is the critical angle. Find n_0 in terms of n_1 and n_2: With the incoming ray at Brewster's angle, the refracted angle of this ray (measured from the inside normal to the front surface) will be $90° - \phi_1$. Therefore, $\phi_1 = \theta_1$, and thus $\sin\phi_1 = \sin\theta_1$. Thus

$$\sin\phi_1 = \frac{n_1}{\sqrt{n_0^2 + n_1^2}} = \sin\theta_1 = \frac{n_2}{n_1} \quad \Rightarrow \quad n_0 = \underline{(n_1/n_2)\sqrt{n_1^2 - n_2^2}}$$

Alternatively, we could have used the result of Problem 12.22, in which it was found that $\sin\phi_1 = (1/n_0)\sqrt{n_1^2 - n_2^2}$, which we then set equal to $\sin\theta_1 = n_2/n_1$ to get the same result.

12.24. A *Brewster prism* is designed to pass p-polarized light without any reflective loss. The prism of Fig. 12.17 is made of glass ($n = 1.45$), and is in air. Considering the light path shown, determine the vertex angle, α: With entrance and exit rays at Brewster's angle (to eliminate reflective loss), the interior ray must be horizontal, or parallel to the bottom surface of the prism. From the geometry, the angle between the interior ray and the normal to the prism surfaces that it intersects is $\alpha/2$. Since this angle is also Brewster's angle, we may write:

$$\alpha = 2\sin^{-1}\left(\frac{1}{\sqrt{1+n^2}}\right) = 2\sin^{-1}\left(\frac{1}{\sqrt{1+(1.45)^2}}\right) = 1.21\,\text{rad} = \underline{69.2^\circ}$$

12.25. In the Brewster prism of Fig. 12.17, determine for s-polarized light the fraction of the incident power that is transmitted through the prism, and from this specify the dB *insertion loss*, defined as $10\log_{10}$ of that number:

We use $\Gamma_s = (\eta_{s2} - \eta_{s1})/(\eta_{s2} + \eta_{s1})$, where

$$\eta_{s2} = \frac{\eta_2}{\cos(\theta_{B2})} = \frac{\eta_2}{n/\sqrt{1+n^2}} = \frac{\eta_0}{n^2}\sqrt{1+n^2}$$

and

$$\eta_{s1} = \frac{\eta_1}{\cos(\theta_{B1})} = \frac{\eta_1}{1/\sqrt{1+n^2}} = \eta_0\sqrt{1+n^2}$$

Thus, at the first interface, $\Gamma = (1-n^2)/(1+n^2)$. At the second interface, Γ will be equal but of opposite sign to the above value. The power transmission coefficient through each interface is $1 - |\Gamma|^2$, so that for both interfaces, we have, with $n = 1.45$:

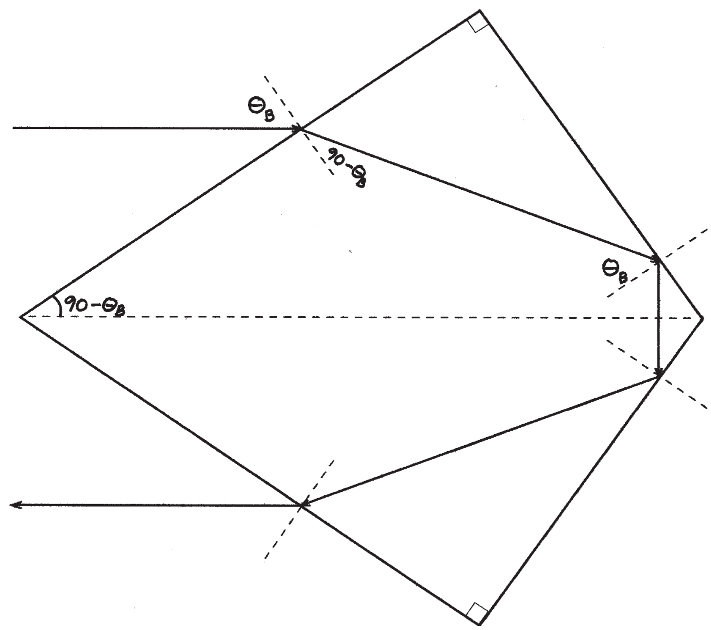
$$\frac{P_{tr}}{P_{inc}} = \left(1 - |\Gamma|^2\right)^2 = \left[1 - \left(\frac{n^2-1}{n^2+1}\right)^2\right]^2 = \underline{0.76}$$

The insertion loss is now

$$\ell_i(\text{dB}) = 10\log_{10}(0.76) = \underline{-1.19}\,\text{dB}$$

12.26. Show how a single block of glass can be used to turn a p-polarized beam of iight through 180°, with the light suffering, in principle, zero reflective loss. The light is incident from air, and the returning beam (also in air) may be displaced sideways from the incident beam. Specify all pertinent angles and use $n = 1.45$ for glass. More than one design is possible here.

The prism below is designed such that light enters at Brewster's angle, and once inside, is turned around using total reflection. Using the result of Example 12.9, we find that with glass, $\theta_B = 55.4°$, which, by the geometry, is also the incident angle for total reflection at the back of the prism. For this to work, the Brewster angle must be greater than or equal to the critical angle. This is in fact the case, since $\theta_c = \sin^{-1}(n_2/n_1) = \sin^{-1}(1/1.45) = 43.6°$.



12.27. Using Eq. (79) in Chapter 11 as a starting point, determine the ratio of the group and phase velocities of an electromagnetic wave in a good conductor. Assume conductivity does not vary with frequency: In a good conductor:

$$\beta = \sqrt{\pi f \mu \sigma} = \sqrt{\frac{\omega \mu \sigma}{2}} \qquad \rightarrow \qquad \frac{d\beta}{d\omega} = \frac{1}{2}\left[\frac{\omega\mu\sigma}{2}\right]^{-1/2}\frac{\mu\sigma}{2}$$

Thus

$$\frac{d\omega}{d\beta} = \left(\frac{d\beta}{d\omega}\right)^{-1} = 2\sqrt{\frac{2\omega}{\mu\sigma}} = v_g \qquad \text{and} \qquad v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega\mu\sigma/2}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

Therefore $v_g/v_p = \underline{2}$.

12.28. Over a small wavelength range, the refractive index of a certain material varies approximately linearly with wavelength as $n(\lambda) \doteq n_a + n_b(\lambda - \lambda_a)$, where n_a, n_b, and λ_a are constants, and where λ is the free space wavelength.

a) Show that $d/d\omega = -(2\pi c/\omega^2)d/d\lambda$: With λ as the free space wavelength, we use $\lambda = 2\pi c/\omega$, from which $d\lambda/d\omega = -2\pi c/\omega^2$. Then $d/d\omega = (d\lambda/d\omega)\, d/d\lambda = -(2\pi c/\omega^2)\, d/d\lambda$.

b) Using $\beta(\lambda) = 2\pi n/\lambda$, determine the wavelength-dependent (or independent) group delay over a unit distance: This will be

$$t_g = \frac{1}{v_g} = \frac{d\beta}{d\omega} = \frac{d}{d\omega}\left[\frac{2\pi n(\lambda)}{\lambda}\right] = -\frac{2\pi c}{\omega^2}\frac{d}{d\lambda}\left[\frac{2\pi}{\lambda}\left[n_a + n_b(\lambda - \lambda_a)\right]\right]$$
$$= -\frac{2\pi c}{\omega^2}\left[-\frac{2\pi}{\lambda^2}\left[n_a + n_b(\lambda - \lambda_a)\right] + \frac{2\pi}{\lambda}n_b\right]$$
$$= -\frac{\lambda^2}{2\pi c}\left[-\frac{2\pi n_a}{\lambda^2} + \frac{2\pi n_b \lambda_a}{\lambda^2}\right] = \underline{\frac{1}{c}(n_a - n_b\lambda_a)\ \text{s/m}}$$

c) Determine β_2 from your result of part *b*: $\beta_2 = d^2\beta/d\omega^2|_{\omega_0}$. Since the part *b* result is independent of wavelength (and of frequency), it follows that $\underline{\beta_2 = 0}$.

d) Discuss the implications of these results, if any, on pulse broadening: A wavelength-independent group delay (leading to zero β_2) means that there will simply be no pulse broadening at all. All frequency components arrive simultaneously. This sort of thing happens in most transparent materials – that is, there will be a certain wavelength, known as the *zero dispersion wavelength*, around which the variation of n with λ is locally linear. Transmitting pulses at this wavelength will result in no pulse broadening (to first order).

12.29. A $T = 5$ ps transform-limited pulse propagates in a dispersive channel for which $\beta_2 = 10\ \text{ps}^2/\text{km}$. Over what distance will the pulse spread to twice its initial width? After propagation, the width is $T' = \sqrt{T^2 + (\Delta\tau)^2} = 2T$. Thus $\Delta\tau = \sqrt{3}T$, where $\Delta\tau = \beta_2 z/T$. Therefore

$$\frac{\beta_2 z}{T} = \sqrt{3}T \text{ or } z = \frac{\sqrt{3}T^2}{\beta_2} = \frac{\sqrt{3}(5\,\text{ps})^2}{10\,\text{ps}^2/\text{km}} = \underline{4.3\ \text{km}}$$

12.30. A $T = 20$ ps transform-limited pulse propagates through 10 km of a dispersive channel for which $\beta_2 = 12\,\text{ps}^2/\text{km}$. The pulse then propagates through a second 10 km channel for which $\beta_2 = -12\,\text{ps}^2/\text{km}$. Describe the pulse at the output of the second channel and give a physical explanation for what happened.

Our theory of pulse spreading will allow for changes in β_2 down the length of the channel. In fact, we may write in general:

$$\Delta\tau = \frac{1}{T}\int_0^L \beta_2(z)\, dz$$

Having β_2 change sign at the midpoint, yields a zero $\Delta\tau$, and so the pulse emerges from the output unchanged! Physically, the pulse acquires a positive linear chirp (frequency increases with time over the pulse envelope) during the first half of the channel. When β_2 switches sign, the pulse begins to acquire a negative chirp in the second half, which, over an equal distance, will completely eliminate the chirp acquired during the first half. The pulse, if originally transform-limited at input, will emerge, again transform-limited, at its original width. More generally, complete *dispersion compensation* is achieved using a two-segment channel when $\beta_2 L = -\beta_2' L'$, assuming dispersion terms of higher order than β_2 do not exist.

CHAPTER 13

13.1. The conductors of a coaxial transmission line are copper ($\sigma_c = 5.8 \times 10^{-7}$ S/m) and the dielectric is polyethylene ($\epsilon'_r = 2.26$, $\sigma/\omega\epsilon' = 0.0002$). If the inner radius of the outer conductor is 4 mm, find the radius of the inner conductor so that (assuming a lossless line):

a) $Z_0 = 50\,\Omega$: Use

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon'}}\ln\left(\frac{b}{a}\right) = 50 \quad\Rightarrow\quad \ln\left(\frac{b}{a}\right) = \frac{2\pi\sqrt{\epsilon'_r}(50)}{377} = 1.25$$

Thus $b/a = e^{1.25} = 3.50$, or $a = 4/3.50 = \underline{1.142\,\text{mm}}$

b) $C = 100$ pF/m: Begin with

$$C = \frac{2\pi\epsilon'}{\ln(b/a)} = 10^{-10} \quad\Rightarrow\quad \ln\left(\frac{b}{a}\right) = 2\pi(2.26)(8.854 \times 10^{-2}) = 1.257$$

So $b/a = e^{1.257} = 3.51$, or $a = 4/3.51 = \underline{1.138\,\text{mm}}$.

c) $L = 0.2\,\mu$H/m: Use

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) = 0.2 \times 10^{-6} \quad\Rightarrow\quad \ln\left(\frac{b}{a}\right) = \frac{2\pi(0.2 \times 10^{-6})}{4\pi \times 10^{-7}} = 1$$

Thus $b/a = e^1 = 2.718$, or $a = b/2.718 = \underline{1.472\,\text{mm}}$.

13.2. Find R, L, C, and G for a coaxial cable with $a = 0.25$ mm, $b = 2.50$ mm, $c = 3.30$ mm, $\epsilon_r = 2.0$, $\mu_r = 1$, $\sigma_c = 1.0 \times 10^7$ S/m, $\sigma = 1.0 \times 10^{-5}$ S/m, and $f = 300$ MHz.

First, we note that the metal is a good conductor, as confirmed by the loss tangent:

$$\frac{\sigma_c}{\omega\epsilon'} = \frac{1.0 \times 10^7}{2\pi(3.00 \times 10^8)(2)(8.854 \times 10^{-12})} = 3.0 \times 10^8 >> 1$$

So the skin depth into the metal is

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_c}} = \sqrt{\frac{2}{2\pi(3.00 \times 10^8)(4\pi \times 10^{-7})(1.0 \times 10^7)}} = 9.2\,\mu\text{m}$$

The current can be said to exist in layers of thickness δ just beneath the inner conductor radius, a, and just inside the outer conductor at its inner radius, b. The outer conductor far radius, c, is of no consequence. Both conductors behave essentially as thin films of thickness δ. We are thus in the high frequency regime, and the equations in Sec. 13.1.1 apply. The calculations for the primary constants are as follows:

$$R = \frac{1}{2\pi\delta\sigma_c}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{1}{2\pi(9.2 \times 10^{-6})(1.0 \times 10^7)}\left(\frac{1}{0.25 \times 10^{-3}} + \frac{1}{2.5 \times 10^{-3}}\right) = \underline{7.6\,\Omega/\text{m}}$$

using Eq. (12).

$$L = L_{ext} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) = \frac{4\pi \times 10^{-7}}{2\pi}\ln\left(\frac{2.5}{0.25}\right) = \underline{0.46\,\mu\text{H/m}}$$

using Eq. (11).

$$C = \frac{2\pi\epsilon'}{\ln(b/a)} = \frac{2\pi(2)(8.854 \times 10^{-12})}{\ln(2.5/0.25)} = \underline{48\,\text{pF/m}}$$

using Eq. (9).

$$G = \frac{2\pi\sigma}{\ln(b/a)} = \frac{2\pi(1.0 \times 10^{-5})}{\ln(2.5/0.25)} = \underline{27\,\mu\text{S/m}}$$

using Eq. (10).

13.3. Two aluminum-clad steel conductors are used to construct a two-wire transmission line. Let $\sigma_{Al} = 3.8 \times 10^7$ S/m, $\sigma_{St} = 5 \times 10^6$ S/m, and $\mu_{St} = 100\,\mu$H/m. The radius of the steel wire is 0.5 in., and the aluminum coating is 0.05 in. thick. The dielectric is air, and the center-to-center wire separation is 4 in. Find C, L, G, and R for the line at 10 MHz: The first question is whether we are in the high frequency or low frequency regime. Calculation of the skin depth, δ, will tell us. We have, for aluminum,

$$\delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma_{Al}}} = \frac{1}{\sqrt{\pi(10^7)(4\pi \times 10^{-7})(3.8 \times 10^7)}} = 2.58 \times 10^{-5} \text{ m}$$

so we are clearly in the high frequency regime, where uniform current distributions cannot be assumed. Furthermore, the skin depth is considerably less than the aluminum layer thickness, so the bulk of the current resides in the aluminum, and we may neglect the steel. Assuming solid aluminum wires of radius $a = 0.5 + 0.05 = 0.55$ in $= 0.014$ m, the resistance of the two-wire line is now

$$R = \frac{1}{\pi a \delta \sigma_{Al}} = \frac{1}{\pi(.014)(2.58 \times 10^{-5})(3.8 \times 10^7)} = \underline{0.023\ \Omega/\text{m}}$$

Next, since the dielectric is air, no leakage will occur from wire to wire, and so $G = \underline{0 \text{ S/m}}$. Now the capacitance will be

$$C = \frac{\pi\epsilon_0}{\cosh^{-1}(d/2a)} = \frac{\pi \times 8.85 \times 10^{-12}}{\cosh^{-1}\left(4/(2 \times 0.55)\right)} = 1.42 \times 10^{-11} \text{ F/m} = \underline{14.2 \text{ pF/m}}$$

Finally, the inductance per unit length will be

$$L = \frac{\mu_0}{\pi}\cosh(d/2a) = \frac{4\pi \times 10^{-7}}{\pi}\cosh\left(4/(2 \times 0.55)\right) = 7.86 \times 10^{-7} \text{ H/m} = \underline{0.786\ \mu\text{H/m}}$$

13.4. Find R, L, C, and G for a two-wire transmission line in polyethylene at $f = 800$ MHz. Assume copper conductors of radius 0.50 mm and separation 0.80 cm. Use $\epsilon_r = 2.26$ and $\sigma/(\omega\epsilon') = 4.0 \times 10^{-4}$.

From the loss tangent, we find the conductivity of polyethylene:

$$\sigma = (4.0 \times 10^{-4})(2\pi \times 8.00 \times 10^8)(2.26)(8.854 \times 10^{-12}) = 4.0 \times 10^{-5}\,\text{S/m}$$

As polyethylene is a good dielectric, its penetration depth is (using Eq. (60a), Chapter 11):

$$\delta_p = \frac{1}{\alpha_p} = \frac{2}{\sigma}\sqrt{\frac{\epsilon'}{\mu}} = \frac{2\epsilon_r'}{\sigma\eta_0} = \frac{2\sqrt{2.26}}{(4.0 \times 10^{-5})(377)} = 199\,\text{m}$$

Therefore, we can assume field distributions over a any cross-sectional plane to be the same as those of the lossless line. On the other hand, within the copper conductors (for which $\sigma_c = 5.8 \times 10^7$) the skin depth will be (from Eq. (82), Chapter 11):

$$\delta_c = \frac{1}{\pi f \mu \sigma_c} = \frac{1}{\pi(8.00 \times 10^8)(4\pi \times 10^{-7})(5.8 \times 10^7)} = 2.3\,\mu\text{m}$$

As this value is much less than the overall conductor dimensions, the line operates in the high frequency regime. The primary constants are found using Eqs. (20) - (23). We have

$$C = \frac{\pi\epsilon'}{\cosh^{-1}(d/2a)} = \frac{\pi(2.26)(8.854 \times 10^{-12})}{\cosh^{-1}(8)} = \underline{22.7\,\text{pF/m}}$$

$$L = L_{ext} = \frac{\mu}{\pi}\cosh^{-1}(d/2a) = \frac{4\pi \times 10^{-7}}{\pi}\cosh^{-1}(8) = \underline{1.11\,\mu\text{H/m}}$$

$$G = \frac{\pi\sigma}{\cosh^{-1}(d/2a)} = \frac{\pi(4.0 \times 10^{-5})}{\cosh^{-1}(8)} = \underline{46\,\mu\text{S/m}}$$

$$R = \frac{1}{\pi a \delta_c \sigma_c} = \frac{1}{\pi(5.0 \times 10^{-4})(2.3 \times 10^{-6})(5.8 \times 10^7)} = \underline{4.8\,\text{ohms/m}}$$

13.5. Each conductor of a two-wire transmission line has a radius of 0.5mm; their center-to-center distance is 0.8cm. Let $f = 150MHz$ and assume σ and σ_c are zero. Find the dielectric constant of the insulating medium if

a) $Z_0 = 300\,\Omega$: Use

$$300 = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\epsilon'_r\epsilon_0}}\cosh^{-1}\left(\frac{d}{2a}\right) \quad\Rightarrow\quad \sqrt{\epsilon'_r} = \frac{120\pi}{300\pi}\cosh^{-1}\left(\frac{8}{2(.5)}\right) = 1.107 \quad\Rightarrow\quad \epsilon'_r = \underline{1.23}$$

b) $C = 20$ pF/m: Use

$$20 \times 10^{-12} = \frac{\pi\epsilon'}{\cosh^{-1}(d/2a)} \quad\Rightarrow\quad \epsilon'_r = \frac{20 \times 10^{-12}}{\pi\epsilon_0}\cosh^{-1}(8) = \underline{1.99}$$

c) $v_p = 2.6 \times 10^8$ m/s:

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu_0\epsilon_0\epsilon'_r}} = \frac{c}{\sqrt{\epsilon'_r}} \quad\Rightarrow\quad \epsilon'_r = \left(\frac{3.0 \times 10^8}{2.6 \times 10^8}\right)^2 = \underline{1.33}$$

13.6. The transmission line in Fig. 6.8 is filled with polyethylene. If it were filled with air, the capacitance would be 57.6 pF/m. Assuming that the line is lossless, find C, L, and Z_0.

The line cross-section is of little consequence here because we have its capacitance and we know that it is lossless. The capacitance with polyethylene added will be just the air-filled line capacitance multiplied by the dielectric constant of polyethylene, which is $\epsilon'_r = 2.26$, or

$$C_p = \epsilon'_r C_{air} = 2.26(57.6) = \underline{130\,\text{pF/m}}$$

The inductance is found from the capacitance and the wave velocity, which in the air-filled line is just the speed of light in vacuum:

$$v_p = c = 3\times10^8\,\text{m/s} = \frac{1}{\sqrt{LC_{air}}} \Rightarrow L = \frac{1}{c^2 C_{air}} = \frac{1}{(3.00 \times 10^8)^2(57.6 \times 10^{-12})} = \underline{0.193\,\mu\text{H/m}}$$

Finally, the characteristic impedance in the dielectric-filled line will be

$$Z_0 = \sqrt{\frac{L}{C_p}} = \sqrt{\frac{1.93 \times 10^{-7}}{1.30 \times 10^{-10}}} = \underline{38.5\text{ ohms}}$$

where the inductance is unaffected by the incorporation of the dielectric.

13.7. Pertinent dimensions for the transmission line shown in Fig. 13.2 are $b = 3$ mm, and $d = 0.2$ mm. The conductors and the dielectric are non-magnetic.

a) If the characteristic impedance of the line is 15 Ω, find ϵ'_r: We use

$$Z_0 = \sqrt{\frac{\mu}{\epsilon'}}\left(\frac{d}{b}\right) = 15 \quad \Rightarrow \quad \epsilon'_r = \left(\frac{377}{15}\right)^2 \frac{.04}{9} = \underline{2.8}$$

b) Assume copper conductors and operation at 2×10^8 rad/s. If $RC = GL$, determine the loss tangent of the dielectric: For copper, $\sigma_c = 5.8 \times 10^7$ S/m, and the skin depth is

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma_c}} = \sqrt{\frac{2}{(2 \times 10^8)(4\pi \times 10^{-7})(5.8 \times 10^7)}} = 1.2 \times 10^{-5} \text{ m}$$

Then

$$R = \frac{2}{\sigma_c \delta b} = \frac{2}{(5.8 \times 10^7)(1.2 \times 10^{-5})(.003)} = 0.98 \ \Omega/\text{m}$$

Now

$$C = \frac{\epsilon' b}{d} = \frac{(2.8)(8.85 \times 10^{-12})(3)}{0.2} = 3.7 \times 10^{-10} \text{ F/m}$$

and

$$L = \frac{\mu_0 d}{b} = \frac{(4\pi \times 10^{-7})(0.2)}{3} = 8.4 \times 10^{-8} \text{ H/m}$$

Then, with $RC = GL$,

$$G = \frac{RC}{L} = \frac{(.98)(3.7 \times 10^{-10})}{(8.4 \times 10^{-8})} = 4.4 \times 10^{-3} \text{ S/m} = \frac{\sigma_d b}{d}$$

Thus $\sigma_d = (4.4 \times 10^{-3})(0.2/3) = 2.9 \times 10^{-4}$ S/m. The loss tangent is

$$l.t. = \frac{\sigma_d}{\omega\epsilon'} = \frac{2.9 \times 10^{-4}}{(2 \times 10^8)(2.8)(8.85 \times 10^{-12})} = \underline{5.85 \times 10^{-2}}$$

13.8. A transmission line constructed from perfect conductors and an air dielectric is to have a maximum dimension of 8mm for its cross-section. The line is to be used at high frequencies. Specify its dimensions if it is:

a) a two-wire line with $Z_0 = 300\,\Omega$: With the maximum dimension of 8mm, we have, using (24):

$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon'}}\cosh^{-1}\left(\frac{8-2a}{2a}\right) = 300 \quad\Rightarrow\quad \frac{8-2a}{2a} = \cosh\left(\frac{300\pi}{120\pi}\right) = 6.13$$

Solve for a to find $a = \underline{0.56\,\text{mm}}$. Then $d = 8 - 2a = \underline{6.88\,\text{mm}}$.

b) a planar line with $Z_0 = 15\,\Omega$: In this case our maximum dimension dictates that $\sqrt{d^2+b^2} = 8$. So, using (8), we write

$$Z_0 = \sqrt{\frac{\mu}{\epsilon'}}\frac{\sqrt{64-b^2}}{b} = 15 \quad\Rightarrow\quad \sqrt{64-b^2} = \frac{15}{377}b$$

Solving, we find $b = \underline{7.99\,\text{mm}}$ and $d = \underline{0.32\,\text{mm}}$.

c) a $72\,\Omega$ coax having a zero-thickness outer conductor: With a zero-thickness outer conductor, we note that the outer radius is $b = 8/2 = 4$mm. Using (13), we write

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon'}}\ln\left(\frac{b}{a}\right) = 72 \quad\Rightarrow\quad \ln\left(\frac{b}{a}\right) = \frac{2\pi(72)}{120\pi} = 1.20 \quad\Rightarrow\quad a = be^{-1.20} = 4e^{-1.20} = 1.2$$

Summarizing, $a = \underline{1.2\,\text{mm}}$ and $b = \underline{4\,\text{mm}}$.

13.9. A microstrip line is to be constructed using a lossless dielectric for which $\epsilon'_r = 7.0$. If the line is to have a 50-Ω characteristic impedance, determine:

a) $\epsilon_{r,eff}$: We use Eq. (34) (under the assumption that $w/d > 1.3$) to find:

$$\epsilon_{r,eff} = 7.0\left[0.96 + 7.0(0.109 - 0.004\times 7.0)\log_{10}(10+50) - 1\right]^{-1} = \underline{5.0}$$

b) w/d: Still under the assumption that $w/d > 1.3$, we solve Eq. (33) for d/w to find

$$\frac{d}{w} = (0.1)\left[\frac{\epsilon'_r - 1}{2}\left(\epsilon_{r,eff} - \frac{\epsilon'_r + 1}{2}\right)^{-1}\right]^{1/0.555} - 0.10$$

$$= (0.1)\left[3.0\,(5.0-4.0)^{-1}\right]^{1/0.555} - 0.10 = 0.624 \quad\Rightarrow\quad \frac{w}{d} = \underline{1.60}$$

13.10. Two microstrip lines are fabricated end-to-end on a 2-mm thick wafer of lithium niobate ($\epsilon_r' = 4.8$). Line 1 is of 4mm width; line 2 (unfortunately) has been fabricated with a 5mm width. Determine the power loss in dB for waves transmitted through the junction.

We first note that $w_1/d_1 = 2.0$ and $w_2/d_2 = 2.5$, so that Eqs. (32) and (33) apply. As the first step, solve for the effective dielectric constants for the two lines, using (33). For line 1:

$$\epsilon_{r1,eff} = \frac{5.8}{2} + \frac{3.8}{2}\left[1 + 10\left(\frac{1}{2}\right)\right]^{-0.555} = 3.60$$

For line 2:

$$\epsilon_{r2,eff} = \frac{5.8}{2} + \frac{3.8}{2}\left[1 + 10\left(\frac{1}{2.5}\right)\right]^{-0.555} = 3.68$$

We next use Eq. (32) to find the characteristic impedances for the air-filled cases. For line 1:

$$Z_{01}^{air} = 60\ln\left[4\left(\frac{1}{2}\right) + \sqrt{16\left(\frac{1}{2}\right)^2 + 2}\right] = 89.6 \text{ ohms}$$

and for line 2:

$$Z_{02}^{air} = 60\ln\left[4\left(\frac{1}{2.5}\right) + \sqrt{16\left(\frac{1}{2.5}\right)^2 + 2}\right] = 79.1 \text{ ohms}$$

The actual line impedances are given by Eq. (31). Using our results, we find

$$Z_{01} = \frac{Z_{01}^{air}}{\sqrt{\epsilon_{r1,eff}}} = \frac{89.6}{\sqrt{3.60}} = \underline{47.2\ \Omega} \quad \text{and} \quad Z_{02} = \frac{Z_{02}^{air}}{\sqrt{\epsilon_{r2,eff}}} = \frac{79.1}{\sqrt{3.68}} = \underline{41.2\ \Omega}$$

The reflection coefficient at the junction is now

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} = \frac{47.2 - 41.2}{47.2 + 41.2} = 0.068$$

The transmission loss in dB is then

$$P_L(\text{dB}) = -10\log_{10}(1 - |\Gamma|^2) = -10\log_{10}(0.995) = \underline{0.02 \text{ dB}}$$

13.11. A parallel-plate waveguide is known to have a cutoff wavelength for the $m = 1$ TE and TM modes of $\lambda_{c1} = 0.4$ cm. The guide is operated at wavelength $\lambda = 1$ mm. How many modes propagate? The cutoff wavelength for mode m is $\lambda_{cm} = 2nd/m$, where n is the refractive index of the guide interior. For the first mode, we are given

$$\lambda_{c1} = \frac{2nd}{1} = 0.4 \text{ cm} \quad \Rightarrow \quad d = \frac{0.4}{2n} = \frac{0.2}{n} \text{ cm}$$

Now, for mode m to propagate, we require

$$\lambda \leq \frac{2nd}{m} = \frac{0.4}{m} \quad \Rightarrow \quad m \leq \frac{0.4}{\lambda} = \frac{0.4}{0.1} = 4$$

So, accounting for 2 modes (TE and TM) for each value of m, and the single TEM mode, we will have a total of $\underline{9}$ modes.

13.12. A parallel-plate guide is to be constructed for operation in the TEM mode only over the frequency range $0 < f < 3$ GHz. The dielectric between plates is to be teflon ($\epsilon'_r = 2.1$). Determine the maximum allowable plate separation, d: We require that $f < f_{c1}$, which, using (41), becomes

$$f < \frac{c}{2nd} \quad \Rightarrow \quad d_{max} = \frac{c}{2nf_{max}} = \frac{3 \times 10^8}{2\sqrt{2.1}\,(3 \times 10^9)} = \underline{3.45\,\text{cm}}$$

13.13. A lossless parallel-plate waveguide is known to propagate the $m = 2$ TE and TM modes at frequencies as low as 10GHz. If the plate separation is 1 cm, determine the dielectric constant of the medium between plates: Use

$$f_{c2} = \frac{c}{nd} = \frac{3 \times 10^{10}}{n(1)} = 10^{10} \;\Rightarrow\; n = 3 \;\text{ or }\; \epsilon_r = \underline{9}$$

13.14. A $d = 1$ cm parallel-plate guide is made with glass ($n = 1.45$) between plates. If the operating frequency is 32 GHz, which modes will propagate? For a propagating mode, we require $f > f_{cm}$ Using (41) and the given values, we write

$$f > \frac{mc}{2nd} \quad \Rightarrow \quad m < \frac{2fnd}{c} = \frac{2(32 \times 10^9)(1.45)(.01)}{3 \times 10^8} = 3.09$$

The maximum allowed m in this case is thus 3, and the propagating modes will be $\underline{TM_1}$, $\underline{TE_1}$, $\underline{TM_2}$, $\underline{TE_2}$, $\underline{TM_3}$, and $\underline{TE_3}$.

13.15. For the guide of Problem 13.14, and at the 32 GHz frequency, determine the difference between the group delays of the highest order mode (TE or TM) and the TEM mode. Assume a propagation distance of 10 cm: From Problem 13.14, we found $m_{max} = 3$. The group velocity of a TE or TM mode for $m = 3$ is

$$v_{g3} = \frac{c}{n}\sqrt{1 - \left(\frac{f_{c3}}{f}\right)^2} \quad \text{where} \quad f_{c3} = \frac{3(3 \times 10^{10})}{2(1.45)(1)} = 3.1 \times 10^{10} = 31\,\text{GHz}$$

Thus

$$v_{g3} = \frac{3 \times 10^{10}}{1.45}\sqrt{1 - \left(\frac{31}{32}\right)^2} = 5.13 \times 10^9 \text{ cm/s}$$

For the TEM mode (assuming no material dispersion) $v_{g,TEM} = c/n = 3 \times 10^{10}/1.45 = 2.07 \times 10^{10}$ cm/s. The group delay difference is now

$$\Delta t_g = z\left(\frac{1}{v_{g3}} - \frac{1}{v_{g,TEM}}\right) = 10\left(\frac{1}{5.13 \times 10^9} - \frac{1}{2.07 \times 10^{10}}\right) = \underline{1.5 \text{ ns}}$$

13.16. The cutoff frequency of the $m = 1$ TE and TM modes in an air-filled parallel-plate guide is known to be $f_{c1} = 7.5$ GHz. The guide is used at wavelength $\lambda = 1.5$ cm. Find the group velocity of the $m = 2$ TE and TM modes. First we know that $f_{c2} = 2f_{c1} = 15$ GHz. Then $f = c/\lambda = 3 \times 10^8/.015 = 20$ GHz. Now, using (57),

$$v_{g2} = \frac{c}{n}\sqrt{1 - \left(\frac{f_{c2}}{f}\right)^2} = \frac{c}{(1)}\sqrt{1 - \left(\frac{15}{20}\right)^2} = \underline{2 \times 10^8 \text{ m/s}}$$

13.17. A parallel-plate guide is partially filled with two lossless dielectrics (Fig. 13.25) where $\epsilon'_{r1} = 4.0$, $\epsilon'_{r2} = 2.1$, and $d = 1$ cm. At a certain frequency, it is found that the TM_1 mode propagates through the guide without suffering any reflective loss at the dielectric interface.

a) Find this frequency: The ray angle is such that the wave is incident on the interface at Brewster's angle. In this case

$$\theta_B = \tan^{-1}\sqrt{\frac{2.1}{4.0}} = 35.9°$$

The ray angle is thus $\theta = 90 - 35.9 = 54.1°$. The cutoff frequency for the $m = 1$ mode is

$$f_{c1} = \frac{c}{2d\sqrt{\epsilon'_{r1}}} = \frac{3 \times 10^{10}}{2(1)(2)} = 7.5\,\text{GHz}$$

The frequency is thus $f = f_{c1}/\cos\theta = 7.5/\cos(54.1°) = \underline{12.8\,\text{GHz}}$.

b) Is the guide operating at a single TM mode at the frequency found in part *a*? The cutoff frequency for the next higher mode, TM_2 is $f_{c2} = 2f_{c1} = 15$ GHz. The 12.8 GHz operating frequency is below this, so TM_2 will not propagate. So the answer is <u>yes</u>.

13.18. In the guide of Figure 13.25, it is found that $m = 1$ modes propagating from left to right totally reflect at the interface, so that no power is transmitted into the region of dielectric constant ϵ'_{r2}.

a) Determine the range of frequencies over which this will occur: For total reflection, the ray angle measured from the normal to the interface must be greater than or equal to the critical angle, θ_c, where $\sin\theta_c = (\epsilon'_{r2}/\epsilon'_{r1})^{1/2}$. The *maximum* mode ray angle is then $\theta_{1\,max} = 90° - \theta_c$. Now, using (39), we write

$$90° - \theta_c = \cos^{-1}\left(\frac{\pi}{k_{max}d}\right) = \cos^{-1}\left(\frac{\pi c}{2\pi f_{max} d\sqrt{4}}\right) = \cos^{-1}\left(\frac{c}{4df_{max}}\right)$$

Now

$$\cos(90 - \theta_c) = \sin\theta_c = \sqrt{\frac{\epsilon'_{r2}}{\epsilon'_{r1}}} = \frac{c}{4df_{max}}$$

Therefore $f_{max} = c/(2\sqrt{2.1}d) = (3\times 10^8)/(2\sqrt{2.1}(.01)) = 10.35\,\text{GHz}$. The frequency range is thus $\underline{f < 10.35\,\text{GHz}}$.

b) Does your part *a* answer in any way relate to the cutoff frequency for $m = 1$ modes in any region? We note that $f_{max} = c/(2\sqrt{2.1}d) = f_{c1}$ in guide 2. To summarize, as frequency is lowered, the ray angle in guide 1 decreases, which leads to the incident angle at the interface increasing to eventually reach and surpass the critical angle. At the critical angle, the refracted angle in guide 2 is 90°, which corresponds to a zero degree ray angle in that guide. This defines the cutoff condition in guide 2. So it would make sense that $f_{max} = f_{c1}$(guide 2).

13.19. A rectangular waveguide has dimensions $a = 6$ cm and $b = 4$ cm.

a) Over what range of frequencies will the guide operate single mode? The cutoff frequency for mode mp is, using Eq. (101):

$$f_{c,mn} = \frac{c}{2n}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{p}{b}\right)^2}$$

where n is the refractive index of the guide interior. We require that the frequency lie between the cutoff frequencies of the TE_{10} and TE_{01} modes. These will be:

$$f_{c10} = \frac{c}{2na} = \frac{3\times 10^8}{2n(.06)} = \frac{2.5\times 10^9}{n} \qquad f_{c01} = \frac{c}{2nb} = \frac{3\times 10^8}{2n(.04)} = \frac{3.75\times 10^9}{n}$$

Thus, the range of frequencies for single mode operation is $\underline{2.5/n < f < 3.75/n \text{ GHz}}$

b) Over what frequency range will the guide support *both* TE_{10} and TE_{01} modes and no others? We note first that f must be greater than f_{c01} to support both modes, but must be less than the cutoff frequency for the next higher order mode. This will be f_{c11}, given by

$$f_{c11} = \frac{c}{2n}\sqrt{\left(\frac{1}{.06}\right)^2 + \left(\frac{1}{.04}\right)^2} = \frac{30c}{2n} = \frac{4.5\times 10^9}{n}$$

The allowed frequency range is then

$$\underline{\frac{3.75}{n}\text{ GHz} < f < \frac{4.5}{n}\text{ GHz}}$$

13.20. Two rectangular waveguides are joined end-to-end. The guides have identical dimensions, where $a = 2b$. One guide is air-filled; the other is filled with a lossless dielectric characterized by ϵ'_r.

a) Determine the maximum allowable value of ϵ'_r such that single mode operation can be simultaneously ensured in *both* guides at some frequency: Since $a = 2b$, the cutoff frequency for any mode in either guide is written using (101):

$$f_{cmp} = \sqrt{\left(\frac{mc}{4nb}\right)^2 + \left(\frac{pc}{2nb}\right)^2}$$

where $n = 1$ in guide 1 and $n = \sqrt{\epsilon'_r}$ in guide 2. We see that, with $a = 2b$, the next modes (having the next higher cutoff frequency) above TE_{10} with be TE_{20} and TE_{01}. We also see that in general, $f_{cmp}(\text{guide 2}) < f_{cmp}(\text{guide 1})$. To assure single mode operation in both guides, the operating frequency must be above cutoff for TE_{10} in both guides, and below cutoff for the next mode in both guides. The allowed frequency range is therefore $f_{c10}(\text{guide 1}) < f < f_{c20}(\text{guide 2})$. This leads to $c/(2a) < f < c/(a\sqrt{\epsilon'_r})$. For this range to be viable, it is required that $\underline{\epsilon'_r < 4}$.

b) Write an expression for the frequency range over which single mode operation will occur in both guides; your answer should be in terms of ϵ'_r, guide dimensions as needed, and other known constants: This was already found in part *a*:

$$\underline{\frac{c}{2a} < f < \frac{c}{\sqrt{\epsilon'_r}\,a}}$$

where $\epsilon'_r < 4$.

13.21. An air-filled rectangular waveguide is to be constructed for single-mode operation at 15 GHz. Specify the guide dimensions, a and b, such that the design frequency is 10% higher than the cutoff frequency for the TE_{10} mode, while being 10% lower than the cutoff frequency for the next higher-order mode: For an air-filled guide, we have

$$f_{c,mp} = \sqrt{\left(\frac{mc}{2a}\right)^2 + \left(\frac{pc}{2b}\right)^2}$$

For TE_{10} we have $f_{c10} = c/2a$, while for the next mode (TE_{01}), $f_{c01} = c/2b$. Our requirements state that $f = 1.1 f_{c10} = 0.9 f_{c01}$. So $f_{c10} = 15/1.1 = 13.6$ GHz and $f_{c01} = 15/0.9 = 16.7$ GHz. The guide dimensions will be

$$a = \frac{c}{2f_{c10}} = \frac{3\times 10^{10}}{2(13.6\times 10^9)} = \underline{1.1\,\text{cm}} \quad \text{and} \quad b = \frac{c}{2f_{c01}} = \frac{3\times 10^{10}}{2(16.7\times 10^9)} = \underline{0.90\,\text{cm}}$$

13.22. Using the relation $P_{av} = (1/2)\text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\}$, and Eqs. (106) through (108), show that the average power density in the TE_{10} mode in a rectangular waveguide is given by

$$P_{av} = \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\, \mathbf{a}_z \qquad \text{W/m}^2$$

Inspecting (106) through (108), we see that (108) includes a factor of j, and so would lead to an imaginary part of the total power when the cross product with E_y is taken. Therefore, the real power in this case is found through the cross product of (106) with the complex conjugate of (107), or

$$P_{av} = \frac{1}{2}\text{Re}\{\mathbf{E}_{ys} \times \mathbf{H}_{xs}^*\} = \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\, \mathbf{a}_z \qquad \text{W/m}^2$$

13.23. Integrate the result of Problem 13.22 over the guide cross-section $0 < x < a$, $0 < y < b$, to show that the power in Watts transmitted down the guide is given as

$$P = \frac{\beta_{10} ab}{4\omega\mu} E_0^2 = \frac{ab}{4\eta} E_0^2 \sin\theta_{10} \;\; \text{W}$$

where $\eta = \sqrt{\mu/\epsilon}$, and θ_{10} is the wave angle associated with the TE_{10} mode. Interpret. First, the integration:

$$P = \int_0^b \int_0^a \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\, \mathbf{a}_z \cdot \mathbf{a}_z \, dx\, dy = \frac{\beta_{10} ab}{4\omega\mu} E_0^2$$

Next, from (54), we have $\beta_{10} = \omega\sqrt{\mu\epsilon}\sin\theta_{10}$, which, on substitution, leads to

$$P = \frac{ab}{4\eta} E_0^2 \sin\theta_{10} \;\; \text{W} \;\; \text{with } \eta = \sqrt{\frac{\mu}{\epsilon}}$$

The $\sin\theta_{10}$ dependence demonstrates the principle of group velocity as energy velocity (or power). This was considered in the discussion leading to Eq. (57).

13.24. Show that the group dispersion parameter, $d^2\beta/d\omega^2$, for given mode in a parallel-plate or rectangular waveguide is given by

$$\frac{d^2\beta}{d\omega^2} = -\frac{n}{\omega c}\left(\frac{\omega_c}{\omega}\right)^2 \left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}$$

where ω_c is the radian cutoff frequency for the mode in question (note that the first derivative form was already found, resulting in Eq. (57)). First, taking the reciprocal of (57), we find

$$\frac{d\beta}{d\omega} = \frac{n}{c}\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-1/2}$$

Taking the derivative of this equation with respect to ω leads to

$$\frac{d^2\beta}{d\omega^2} = \frac{n}{c}\left(-\frac{1}{2}\right)\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}\left(\frac{2\omega_c^2}{\omega^3}\right) = -\frac{n}{\omega c}\left(\frac{\omega_c}{\omega}\right)^2\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}$$

13.25. Consider a transform-limited pulse of center frequency $f = 10$ GHz and of full-width $2T = 1.0$ ns. The pulse propagates in a lossless single mode rectangular guide which is air-filled and in which the 10 GHz operating frequency is 1.1 times the cutoff frequency of the TE_{10} mode. Using the result of Problem 13.24, determine the length of the guide over which the pulse broadens to twice its initial width: The broadened pulse will have width given by $T' = \sqrt{T^2 + (\Delta\tau)^2}$, where $\Delta\tau = \beta_2 L/T$ for a transform limited pulse (assumed gaussian). β_2 is the Problem 13.24 result evaluated at the operating frequency, or

$$\beta_2 = \frac{d^2\beta}{d\omega^2}|_{\omega=10\,\mathrm{GHz}} = -\frac{1}{(2\pi \times 10^{10})(3 \times 10^8)}\left(\frac{1}{1.1}\right)^2\left[1 - \left(\frac{1}{1.1}\right)^2\right]^{-3/2}$$
$$= 6.1 \times 10^{-19}\ \mathrm{s^2/m} = 0.61\ \mathrm{ns^2/m}$$

Now $\Delta\tau = 0.61L/0.5 = 1.2L$ ns. For the pulse width to double, we have $T' = 1$ ns, and

$$\sqrt{(.05)^2 + (1.2L)^2} = 1 \quad \Rightarrow \quad L = 0.72\,\mathrm{m} = \underline{72\,\mathrm{cm}}$$

What simple step can be taken to reduce the amount of pulse broadening in this guide, while maintaining the same initial pulse width? It can be seen that β_2 can be reduced by increasing the operating frequency relative to the cutoff frequency; i.e., operate as far above cutoff as possible, without allowing the next higher-order modes to propagate.

13.26. A symmetric dielectric slab waveguide has a slab thickness $d = 10$ μm, with $n_1 = 1.48$ and $n_2 = 1.45$. If the operating wavelength is $\lambda = 1.3$ μm, what modes will propagate? We use the condition expressed through (141): $k_0 d\sqrt{n_1^2 - n_2^2} \geq (m-1)\pi$. Since $k_0 = 2\pi/\lambda$, the condition becomes

$$\frac{2d}{\lambda}\sqrt{n_1^2 - n_2^2} \geq (m-1) \quad \Rightarrow \quad \frac{2(10)}{1.3}\sqrt{(1.48)^2 - (1.45)^2} = 4.56 \geq m - 1$$

Therefore, $m_{max} = 5$, and we have TE and TM modes for which $m = 1, 2, 3, 4, 5$ propagating (ten total).

13.27. A symmetric slab waveguide is known to support only a single pair of TE and TM modes at wavelength $\lambda = 1.55$ μm. If the slab thickness is 5 μm, what is the maximum value of n_1 if $n_2 = 3.30$? Using (142) we have

$$\frac{2\pi d}{\lambda}\sqrt{n_1^2 - n_2^2} < \pi \quad \Rightarrow \quad n_1 < \sqrt{\frac{\lambda}{2d} + n_2^2} = \sqrt{\frac{1.55}{2(5)} + (3.30)^2} = \underline{3.32}$$

13.28. In a symmetric slab waveguide, $n_1 = 1.50$, $n_2 = 1.45$, and $d = 10$ μm.

a) What is the phase velocity of the $m = 1$ TE or TM mode at cutoff? At cutoff, the mode propagates in the slab at the critical angle, which means that the phase velocity will be equal to that of a plane wave in the upper or lower media of index n_2. Phase velocity will therefore be $v_p(\text{cutoff}) = c/n_2 = 3 \times 10^8/1.45 = \underline{2.07 \times 10^8\ \mathrm{m/s}}$.

b) What is the phase velocity of the $m = 2$ TE or TM modes at cutoff? The reasoning of part a applies to all modes, so the answer is the same, or $\underline{2.07 \times 10^8\ \mathrm{m/s}}$.

13.29. An *asymmetric* slab waveguide is shown in Fig. 13.26. In this case, the regions above and below the slab have unequal refractive indices, where $n_1 > n_3 > n_2$.

a) Write, in terms of the appropriate indices, an expression for the minimum possible wave angle, θ_1, that a guided mode may have: The wave angle must be equal to or greater than the critical angle of total reflection at *both* interfaces. The minimum wave angle is thus determined by the *greater* of the two critical angles. Since $n_3 > n_2$, we find $\theta_{min} = \theta_{c,13} = \underline{\sin^{-1}(n_3/n_1)}$.

b) Write an expression for the maximum phase velocity a guided mode may have in this structure, using given or known parameters: We have $v_{p,max} = \omega/\beta_{min}$, where $\beta_{min} = n_1 k_0 \sin\theta_{1,min} = n_1 k_0 n_3/n_1 = n_3 k_0$. Thus $v_{p,max} = \omega/(n_3 k_0) = \underline{c/n_3}$.

13.30. A step index optical fiber is known to be single mode at wavelengths $\lambda > 1.2$ μm. Another fiber is to be fabricated from the same materials, but is to be single mode at wavelengths $\lambda > 0.63$ μm. By what percentage must the core radius of the new fiber differ from the old one, and should it be larger or smaller? We use the cutoff condition, given by (159):

$$\lambda > \lambda_c = \frac{2\pi a}{2.405}\sqrt{n_1^2 - n_2^2}$$

With λ reduced, the core radius, a, must also be reduced by the same fraction. Therefore, the percentage *reduction* required in the core radius will be

$$\% = \frac{1.2 - .63}{1.2} \times 100 = \underline{47.5\%}$$

13.31. Is the mode field radius greater than or less than the fiber core radius in single-mode step-index fiber?

The answer to this can be found by inspecting Eq. (164). Clearly the mode field radius decreases with increasing V, so we can look at the extreme case of $V = 2.405$, which is the upper limit to single-mode operation. The equation evaluates as

$$\frac{\rho_0}{a} = 0.65 + \frac{1.619}{(2.405)^{3/2}} + \frac{2.879}{(2.405)^6} = 1.10$$

Therefore, ρ_0 is always greater than a within the single-mode regime, $V < 2.405$.

13.32. The mode field radius of a step-index fiber is measured as 4.5 μm at free space wavelength $\lambda = 1.30$ μm. If the cutoff wavelength is specified as $\lambda_c = 1.20$ μm, find the expected mode field radius at $\lambda = 1.55$ μm.

In this problem it is helpful to use the relation $V = 2.405(\lambda_c/\lambda)$, and rewrite Eq. (164) to read:

$$\frac{\rho_0}{a} = 0.65 + 0.434\left(\frac{\lambda}{\lambda_c}\right)^{3/2} + 0.015\left(\frac{\lambda}{\lambda_c}\right)^6$$

At $\lambda = 1.30\,\mu$m, $\lambda/\lambda_c = 1.08$, and at 1.55 μm, $\lambda/\lambda_c = 1.29$. Using these values, along with our new equation, we write

$$\rho_0(1.55) = 4.5\left[\frac{0.65 + 0.434(1.29)^{3/2} + 0.015(1.29)^6}{0.65 + 0.434(1.08)^{3/2} + 0.015(1.08)^6}\right] = \underline{5.3\ \mu\text{m}}$$

CHAPTER 14

14.1. A short dipole carrying current $I_0 \cos\omega t$ in the $\mathbf{a}_z$ direction is located at the origin in free space.

a) If $k = 1$ rad/m, $r = 2$ m, $\theta = 45°$, $\phi = 0$, and $t = 0$, give a unit vector in rectangular components that shows the instantaneous direction of $\mathbf{E}$: In spherical coordinates, the components of $\mathbf{E}$ are given by (13a) and (13b):

$$E_{rs} = \frac{I_0 d}{2\pi}\eta \cos\theta \, e^{-jkr}\left(\frac{1}{r^2} + \frac{1}{jkr^3}\right) \qquad (13a)$$

$$E_{\theta s} = \frac{I_0 d}{4\pi}\eta \sin\theta \, e^{-jkr}\left(\frac{jk}{r} + \frac{1}{r^2} + \frac{1}{jkr^3}\right) \qquad (13b)$$

Since we want a unit vector at $t = 0$, we need only the relative amplitudes of the two components, but we need the absolute phases. Since $\theta = 45°$, $\sin\theta = \cos\theta = 1/\sqrt{2}$. Also, with $k = 1 = 2\pi/\lambda$, it follows that $\lambda = 2\pi$ m. The above two equations can be simplified by these substitutions, while dropping all amplitude terms that are common to both. Obtain

$$A_r = \left(\frac{1}{r^2} + \frac{1}{jr^3}\right) e^{-jr}$$

$$A_\theta = \frac{1}{2}\left(j\frac{1}{r} + \frac{1}{r^2} + \frac{1}{jr^3}\right) e^{-jr}$$

Now with $r = 2$ m, we obtain

$$A_r = \left(\frac{1}{4} - j\frac{1}{8}\right) e^{-j2} = \frac{1}{4}(1.12)e^{-j26.6°}e^{-j2}$$

$$A_\theta = \left(j\frac{1}{4} + \frac{1}{8} - j\frac{1}{16}\right) e^{-j2} = \frac{1}{4}(0.90)e^{j56.3°}e^{-j2}$$

The total vector is now $\mathbf{A} = A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta$. We can normalize the vector by first finding the magnitude:

$$|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}^*} = \frac{1}{4}\sqrt{(1.12)^2 + (0.90)^2} = 0.359$$

Dividing the field vector by this magnitude and converting 2 rad to 114.6°, we write the normalized vector as

$$\mathbf{A}_{Ns} = 0.780e^{-j141.2°}\mathbf{a}_r + 0.627e^{-58.3°}\mathbf{a}_\theta$$

In real instantaneous form, this becomes

$$\mathbf{A}_N(t) = \text{Re}\left(\mathbf{A}_{Ns}e^{j\omega t}\right) = 0.780\cos(\omega t - 141.2°)\mathbf{a}_r + 0.627\cos(\omega t - 58.3°)\mathbf{a}_\theta$$

We evaluate this at $t = 0$ to find

$$\mathbf{A}_N(0) = 0.780\cos(141.2°)\mathbf{a}_r + 0.627\cos(58.3°)\mathbf{a}_\theta = -0.608\mathbf{a}_r + 0.330\mathbf{a}_\theta$$

14.1 a) (continued)

Dividing by the magnitude, $\sqrt{(0.608)^2 + (0.330)^2} = 0.692$, we obtain the unit vector at $t = 0$: $\mathbf{a}_N(0) = -0.879\mathbf{a}_r + 0.477\mathbf{a}_\theta$. We next convert this to rectangular components:

$$a_{Nx} = \mathbf{a}_N(0) \cdot \mathbf{a}_x = -0.879 \sin\theta \cos\phi + 0.477 \cos\theta \cos\phi = \frac{1}{\sqrt{2}}(-0.879 + 0.477) = -0.284$$

$$a_{Ny} = \mathbf{a}_N(0) \cdot \mathbf{a}_y = -0.879 \sin\theta \sin\phi + 0.477 \cos\theta \sin\phi = 0 \quad \text{since } \phi = 0$$

$$a_{Nz} = \mathbf{a}_N(0) \cdot \mathbf{a}_z = -0.879 \cos\theta - 0.477 \sin\theta = \frac{1}{\sqrt{2}}(-0.879 - 0.477) = -0.959$$

The final result is then

$$\mathbf{a}_N(0) = \underline{-0.284\mathbf{a}_x - 0.959\mathbf{a}_z}$$

b) What fraction of the total average power is radiated in the belt, $80° < \theta < 100°$? We use the far-zone phasor fields, (22) and (23), with $k = 2\pi/\lambda$, and first find the average power density:

$$P_{avg} = \frac{1}{2}\text{Re}[E_{\theta s} H^*_{\phi s}] = \frac{I_0^2 d^2 \eta}{8\lambda^2 r^2} \sin^2\theta \quad \text{W/m}^2$$

We integrate this over the given belt, an at radius r:

$$P_{belt} = \int_0^{2\pi} \int_{80°}^{100°} \frac{I_0^2 d^2 \eta}{8\lambda^2 r^2} \sin^2\theta \, r^2 \sin\theta \, d\theta \, d\phi = \frac{\pi I_0^2 d^2 \eta}{4\lambda^2} \int_{80°}^{100°} \sin^3\theta \, d\theta$$

Evaluating the integral, we find

$$P_{belt} = \frac{\pi I_0^2 d^2 \eta}{4\lambda^2} \left[-\frac{1}{3} \cos\theta \left(\sin^2\theta + 2 \right) \right]_{80}^{100} = (0.344) \frac{\pi I_0^2 d^2 \eta}{4\lambda^2}$$

The total power is found by performing the same integral over θ, where $0 < \theta < 180°$. Doing this, it is found that

$$P_{tot} = (1.333) \frac{\pi I_0^2 d^2 \eta}{4\lambda^2}$$

The fraction of the total power in the belt is then $f = 0.344/1.333 = \underline{0.258}$.

14.2. *For the Hertzian dipole*, prepare a curve, r vs. θ in polar coordinates, showing the locus in the $\phi = 0$ plane where:

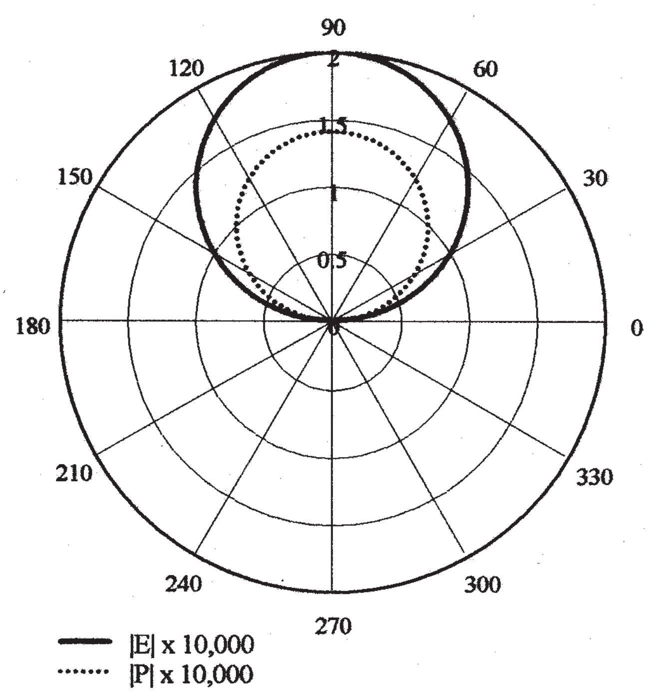
a) The radiation field $|E_{\theta s}|$ is one-half of its value at $r = 10^4$ m, $\theta = \pi/2$: Assuming the far field approximation, we use (22) with $k = 2\pi/\lambda$ to set up the equation:

$$|E_{\theta s}| = \frac{I_0 d\eta}{2\lambda r}\sin\theta = \frac{1}{2}\times\frac{I_0 d\eta}{2\times 10^4\lambda} \quad\Rightarrow\quad r = 2\times 10^4\sin\theta$$

b) The average radiated power density, $P_{r,av}$, is one-half of its value at $r = 10^4$ m, $\theta = \pi/2$. To find the average power, we use (22) and (23) in

$$P_{r,av} = \frac{1}{2}\text{Re}\{E_{\theta s}H^*_{\phi s}\} = \frac{1}{2}\frac{I_0^2 d^2\eta}{4\lambda^2 r^2}\sin^2\theta = \frac{1}{2}\times\frac{1}{2}\frac{I_0^2 d^2\eta}{4\lambda^2(10^8)} \quad\Rightarrow\quad r = \sqrt{2}\times 10^4\sin\theta$$

The polar plots for field ($r = 2\times 10^4\sin\theta$) and power ($r = \sqrt{2}\times 10^4\sin\theta$) are shown below. Both are circles.



14.3. Two short antennas at the origin in free space carry identical currents of $5\cos\omega t$ A, one in the $\mathbf{a}_z$ direction, one in the $\mathbf{a}_y$ direction. Let $\lambda = 2\pi$ m and $d = 0.1$ m. Find $\mathbf{E}_s$ at the distant point:

a) $(x = 0, y = 1000, z = 0)$: This point lies along the axial direction of the $\mathbf{a}_y$ antenna, so its contribution to the field will be zero. This leaves the $\mathbf{a}_z$ antenna, and since $\theta = 90°$, only the $E_{\theta s}$ component will be present (as (136) and (137) show). Since we are in the far zone, (138) applies. We use $\theta = 90°$, $d = 0.1$, $\lambda = 2\pi$, $\eta = \eta_0 = 120\pi$, and $r = 1000$ to write:

$$\mathbf{E}_s = E_{\theta s}\mathbf{a}_\theta = j\frac{I_0 d\eta}{2\lambda r}\sin\theta e^{-j2\pi r/\lambda}\,\mathbf{a}_\theta = j\frac{5(0.1)(120\pi)}{4\pi(1000)}e^{-j1000}\,\mathbf{a}_\theta$$
$$= j(1.5\times 10^{-2})e^{-j1000}\,\mathbf{a}_\theta = \underline{-j(1.5\times 10^{-2})e^{-j1000}\,\mathbf{a}_z \text{ V/m}}$$

b) $(0, 0, 1000)$: Along the z axis, only the $\mathbf{a}_y$ antenna will contribute to the field. Since the distance is the same, we can apply the part a result, modified such the the field direction is in $-\mathbf{a}_y$: $\mathbf{E}_s = \underline{-j(1.5\times 10^{-2})e^{-j1000}\,\mathbf{a}_y \text{ V/m}}$

c) $(1000, 0, 0)$: Here, both antennas will contribute. Applying the results of parts a and b, we find $\mathbf{E}_s = \underline{-j(1.5\times 10^{-2})(\mathbf{a}_y + \mathbf{a}_z)}$.

d) Find $\mathbf{E}$ at $(1000, 0, 0)$ at $t = 0$: This is found through

$$\mathbf{E}(t) = \text{Re}\left(\mathbf{E}_s e^{j\omega t}\right) = (1.5\times 10^{-2})\sin(\omega t - 1000)(\mathbf{a}_y + \mathbf{a}_z)$$

Evaluating at $t = 0$, we find
$\mathbf{E}(0) = (1.5\times 10^{-2})[-\sin(1000)](\mathbf{a}_y + \mathbf{a}_z) = \underline{-(1.24\times 10^{-2})(\mathbf{a}_y + \mathbf{a}_z) \text{ V/m}}$.

e) Find $|\mathbf{E}|$ at $(1000, 0, 0)$ at $t = 0$: Taking the magnitude of the part d result, we find $|\mathbf{E}| = \underline{1.75\times 10^{-2} \text{ V/m}}$.

14.4. Write the Hertzian dipole electric field, whose components are given in Eqs. (15) and (16), in the near-zone in free space, where $kr << 1$. In this case, only a single term in each of the two equations survives, and the phases, δ_r and δ_θ, simplify to a single value. Construct the resulting electric field vector and compare your result to the static dipole result (Eq. (35) (not (36)) in Chapter 4). What relation must exist between the static dipole charge, Q, and the current amplitude, I_0, so that the two results are identical?

First, we evaluate the phase terms under the approximation, $kr << 1$: Using (17b) and (18) we have

$$\delta_r \doteq \tan^{-1}(0) - \frac{\pi}{2} \;=\; -\frac{\pi}{2} \quad \text{and} \quad \delta_\theta \doteq \tan^{-1}(-\infty) \;=\; -\frac{\pi}{2}$$

These are used in (15) and (16). In those equations, the $(kr)^{-2}$ term is retained in (15) and the $(kr)^{-4}$ term is kept in (16). The results are

$$E_{rs} \doteq \frac{I_0 d}{2\pi r^2 (kr)} \eta \cos\theta \, \exp(-j\pi/2) = -j \frac{I_0 \eta d}{2\pi k r^3} \cos\theta$$

$$E_{\theta s} \doteq \frac{I_0 k d}{4\pi r (kr)^2} \eta \sin\theta \exp(-j\pi/2) = -j \frac{I_0 \eta d}{4\pi k r^3} \sin\theta$$

where, in free space

$$\frac{\eta}{k} = \frac{\sqrt{\mu_0/\epsilon_0}}{\omega\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\omega \epsilon_0}$$

The field vector can now be constructed as

$$\mathbf{E}_s = \frac{(-jI_0/\omega)d}{4\pi\epsilon_0 r^3} \; [2\cos\theta \, \mathbf{a}_r + \sin\theta \, \mathbf{a}_\theta]$$

This expression is equivalent to the static dipole field (Eq. (35) in Chapter 4), provided that $\underline{I_0 = j\omega Q}$, which is just a statement that the current is the time derivative of the time-harmonic charge.

14.5. Consider the term in Eq. (14) (or in Eq. (10)) that gives the $1/r^2$ dependence in the Hertzian dipole magnetic field. Assuming this term dominates and that $kr << 1$, show that the resulting magnetic field is the same as that found by applying the Biot-Savart law (Eq. (2), Chapter 7) to a current element of differential length d, oriented along the z axis, and centered at the origin.

We begin by writing Eq. (14) under the condition $kr << 1$, for which the phase term in (14) becomes, using Eq. (17a):

$$\exp[-j(kr - \delta_\phi)] = \exp[-j(kr - \tan^{-1}(kr))] \doteq \exp[-j(kr - kr)] = 0$$

With $kr << 1$, the $1/(kr)^2$ term in (14) dominates, and the field becomes:

$$H_\phi \doteq \frac{I_0 kd}{4\pi r}\frac{1}{(kr)}\sin\theta \exp(j0) = \frac{I_0 d}{4\pi r^2}\sin\theta$$

Now, using the Biot-Savart law for a short element of assumed differential length, d, we have

$$\mathbf{H} = \frac{I_0 d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

where, with the element at the origin and oriented along z, we have $R = r$, $\mathbf{a}_R = \mathbf{a}_r$, and $d\mathbf{L} = d\mathbf{a}_z$. With these subsitutions:

$$\mathbf{H} = \frac{I_0 d\,(\mathbf{a}_z \times \mathbf{a}_r)}{4\pi r^2} = \frac{I_0 d}{4\pi r^2}\sin\theta\,\mathbf{a}_\phi \quad \text{(done)}$$

14.6. Evaluate the time-average Poynting vector, $<\mathbf{S}> = (1/2)\mathcal{R}e\,\{\mathbf{E}_s \times \mathbf{H}_s^*\}$ for the Hertzian dipole, assuming the general case that involves the field components as given by Eqs. (10), (13a), and (13b). Compare your result to the far-zone case, Eq. (26).

With radial and theta components of $\mathbf{E}$, and with only a phi component of $\mathbf{H}$, the Poynting vector becomes

$$<\mathbf{S}> = \frac{1}{2}\mathcal{R}e\left\{E_{\theta s}H_{\phi s}^*\,\mathbf{a}_r \;-\; E_{rs}H_{\phi s}^*\,\mathbf{a}_\theta\right\}$$

Substituting (10), (13a) and (13b), this becomes:

$$<\mathbf{S}> = \frac{1}{2}\mathcal{R}e\left\{\left(\frac{I_0 d}{4\pi}\right)^2 \eta\sin^2\theta\left[\frac{k^2}{r^2} - j\frac{k}{r^3} - \frac{1}{r^4} + j\frac{k}{r^3} + \frac{1}{r^4} - j\frac{k}{r^5}\right]\mathbf{a}_r \right.$$
$$\left. - \frac{(I_0 d)^2}{8\pi^2}\sin\theta\cos\theta\left[-j\frac{k}{r^3} - \frac{1}{r^4} + \frac{1}{r^4} - j\frac{k}{r^5}\right]\mathbf{a}_\theta\right\}$$

In taking the real part, the imaginary terms that do not cancel are removed, other real terms cancel, and only the first term in the radial component remains. The final result is

$$<\mathbf{S}> = \frac{1}{2}\left(\frac{I_0 d\,k}{4\pi r}\right)^2 \eta\sin^\theta\,\mathbf{a}_r \ \mathrm{W/m^2}$$

which we see to be identical to Eq. (26).

14.7. A short current element has $d = 0.03\lambda$. Calculate the radiation resistance for each of the following current distributions:

a) Uniform: In this case, (30) applies directly and we find

$$R_{rad} = 80\pi^2 \left(\frac{d}{\lambda}\right)^2 = 80\pi^2(.03)^2 = \underline{0.711\,\Omega}$$

b) Linear: $I(z) = I_0(0.5d - |z|)/0.5d$: Here, the average current is $0.5I_0$, and so the average power drops by a factor of 0.25. The radiation resistance therefore is down to one-fourth the value found in part a, or $R_{rad} = (0.25)(0.711) = \underline{0.178\,\Omega}$.

c) Step: I_0 for $0 < |z| < 0.25d$ and $0.5I_0$ for $0.25d < |z| < 0.5d$: In this case the average current on the wire is $0.75I_0$. The radiated power (and radiation resistance) are down to a factor of $(0.75)^2$ times their values for a uniform current, and so $R_{rad} = (0.75)^2(0.711) = \underline{0.400\,\Omega}$.

14.8. Evaluate the time-average Poynting vector, $<\mathbf{S}> = (1/2)\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\}$ for the magnetic dipole antenna in the far-zone, in which all terms of order $1/r^2$ and $1/r^4$ are neglected in Eqs. (48), (49), and (50). Compare your result to the far-zone power density of the Hertzian dipole, Eq. (26). In this comparison, and assuming equal current amplitudes, what relation between loop radius, a, and dipole length, d, would result in equal radiated powers from the two devices?

First, neglecting the indicated terms in (48)-(50), only the terms in $1/r$ survive. We find:

$$E_{\phi s} \doteq -j\frac{\omega\mu_0\pi a^2 I_0 k}{4\pi r}\sin\theta\exp[-j(kr - \delta_\phi)]$$

$$H_{\theta s} \doteq j\frac{\omega\mu_0\pi a^2 I_0 k}{4\pi r}\frac{1}{\eta}\sin\theta\exp[-j(kr - \delta_\theta)]$$

where, in the approximation, $\delta_\phi \doteq \delta_\theta = \tan^{-1}(kr)$. So now

$$<\mathbf{S}> = (1/2)\mathcal{R}e\{\mathbf{E}_{\phi s} \times \mathbf{H}_{\theta s}^*\} = -\frac{1}{2}\mathcal{R}e\{E_{\phi s}H_{\theta s}^*\}\,\mathbf{a}_r$$

Using the above field expressions, we find

$$|<\mathbf{S}>| = S_{r(md)} = \frac{1}{2}\left(\frac{I_0 k}{4\pi r}\right)^2[\omega\mu_0\pi a^2]^2\,\frac{1}{\eta}\sin^2\theta$$

which we now compare to the power density from the Hertzian dipole, Eq. (26):

$$S_{r(Hd)} = \frac{1}{2}\left(\frac{I_0 k}{4\pi r}\right)^2 d^2\eta\sin^2\theta$$

For equal power densities from the two structures, we thus require (assuming free space):

$$[\omega\mu_0\pi a^2]^2\,\frac{1}{\eta} = d^2\eta \;\Rightarrow\; d = \pi a^2\,\frac{\omega\mu_0}{\eta} = \pi a^2\,(\omega\sqrt{\mu_0\epsilon_0}) = \pi a^2\left(\frac{2\pi}{\lambda}\right)$$

or

$$d = \frac{(2\pi a)^2}{2\lambda} \qquad \text{(the square of the loop circumference over twice the wavelength)}$$

14.9. A dipole antenna in free space has a linear current distribution. If the length is 0.02λ, what value of I_0 is required to:

a) provide a radiation-field amplitude of 100 mV/m at a distance of one mile, at $\theta = 90°$: With a linear current distribution, the peak current, I_0, occurs at the center of the dipole; current decreases linearly to zero at the two ends. The average current is thus $I_0/2$, and we use Eq. (138) to write:

$$|E_\theta| = \frac{I_0 d\eta_0}{4\lambda r}\sin(90°) = \frac{I_0(0.02)(120\pi)}{(4)(5280)(12)(0.0254)} = 0.1 \quad \Rightarrow \quad I_0 = \underline{85.4 \text{ A}}$$

b) radiate a total power of 1 watt? We use

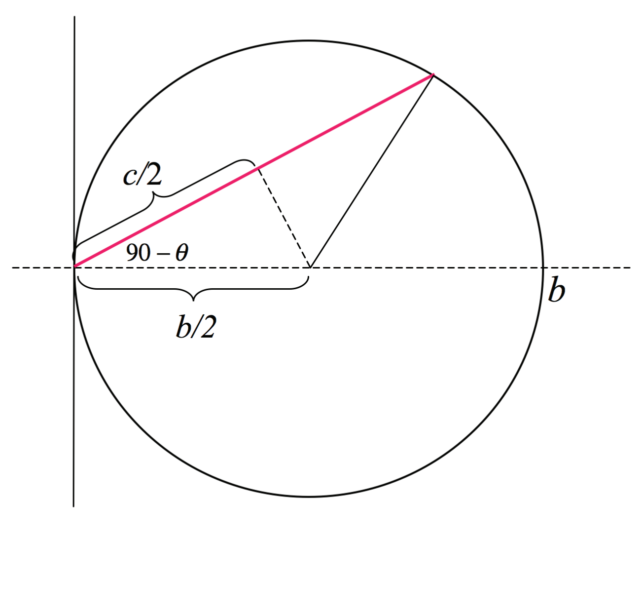
$$P_{avg} = \left(\frac{1}{4}\right)\left(\frac{1}{2}I_0^2 R_{rad}\right)$$

where the radiation resistance is given by Eq. (140), and where the factor of 1/4 arises from the average current of $I_0/2$: We obtain $P_{avg} = 10\pi^2 I_0^2(0.02)^2 = 1 \quad \Rightarrow \quad I_0 = \underline{5.03 \text{ A}}$.

14.10. Show that the chord length in the E-plane plot of Fig. 14.4 is equal to $b\sin\theta$, where b is the circle diameter.

Referring to the construction below, half the chord length, $c/2$, is given by

$$\frac{c}{2} = \frac{b}{2}\cos(90-\theta) = \frac{b}{2}\sin\theta \;\Rightarrow\; \underline{c = b\sin\theta} \;\; \text{(done)}$$



14.11. A monopole antenna in free space, extending vertically over a perfectly conducting plane, has a linear current distribution. If the length of the antenna is 0.01λ, what value of I_0 is required to

a) provide a radiation field amplitude of 100 mV/m at a distance of 1 mi, at $\theta = 90°$: The image antenna below the plane provides a radiation pattern that is identical to a dipole antenna of length 0.02λ. The radiation field is thus given by (22) in free space, where $k = 2\pi/\lambda$, $\theta = 90°$, and with an additional factor of 1/2 included to account for the linear current distribution:

$$|E_\theta| = \frac{1}{2}\frac{I_0 d\eta_0}{2\lambda r} \quad \Rightarrow \quad I_0 = \frac{4r|E_\theta|}{(d/\lambda)\eta_0} = \frac{4(5289)(12 \times .0254)(100 \times 10^{-3})}{(.02)(377)} = \underline{85.4\,\text{A}}$$

b) radiate a total power of 1W: For the monopole over the conducting plane, power is radiated only over the upper half-space. This reduces the radiation resistance of the equivalent dipole antenna by a factor of one-half. Additionally, the linear current distribution reduces the radiation resistance of a dipole having uniform current by a factor of one-fourth. Therefore, R_{rad} is one-eighth the value obtained from (30), or $R_{rad} = 10\pi^2(d/\lambda)^2$. The current magnitude is now

$$I_0 = \left[\frac{2P_{av}}{R_{rad}}\right]^{1/2} = \left[\frac{2(1)}{10\pi^2(d/\lambda)^2}\right]^{1/2} = \frac{\sqrt{2}}{\sqrt{10}\,\pi(.02)} = \underline{7.1\,\text{A}}$$

14.12. Find the zeros in θ for the E-plane pattern of a dipole antenna for which (using Fig. 14.8 as a guide):

a) $\ell = \lambda$: We look for zeros in the pattern function, Eq. (59), for which $k\ell = (2\pi/\lambda)\lambda = 2\pi$.

$$F(\theta) = \left[\frac{\cos(k\ell\cos\theta) - \cos(k\ell)}{\sin\theta}\right] = \left[\frac{\cos(2\pi\cos\theta) - \cos(2\pi)}{\sin\theta}\right]$$

Zeros will occur whenever

$$\cos(2\pi\cos\theta) = 1 \quad \Rightarrow \quad \underline{\theta = (0,\ 90°,\ 180°,\ 270°)}$$

You are encouraged to show (using small argument approximations for cosine and sine) that $F(\theta)$ does approach zero when θ approaches zero or 180° (despite the $\sin\theta$ term in the denominator).

b) $2\ell = 1.3\lambda$: In this case $k\ell = (2\pi/\lambda)(1.3\lambda/2) = 1.3\pi$. The pattern function becomes

$$F(\theta) = \left[\frac{\cos(1.3\pi\cos\theta) - \cos(1.3\pi)}{\sin\theta}\right]$$

Zeros of this function will occur at $\underline{\theta = (0,\ 57.42°,\ 122.58°,\ 180°)}$.

14.13. The radiation field of a certain short vertical current element is $E_{\theta s} = (20/r)\sin\theta\, e^{-j10\pi r}$ V/m if it is located at the origin in free space.

a) Find $E_{\theta s}$ at $P(r = 100, \theta = 90°, \phi = 30°)$: Substituting these values into the given formula, find

$$E_{\theta s} = \frac{20}{100}\sin(90°)e^{-j10\pi(100)} = \underline{0.2e^{-j1000\pi}\ \text{V/m}}$$

b) Find $E_{\theta s}$ at P if the vertical element is located at $A(0.1, 90°, 90°)$: This places the element on the y axis at $y = 0.1$. As a result of moving the antenna from the origin to $y = 0.1$, the change in distance to point P is negligible when considering the change in field *amplitude*, but is not when considering the change in *phase*. Consider lines drawn from the origin to P and from $y = 0.1$ to P. These lines can be considered essentially parallel, and so the difference in their lengths is $l \doteq 0.1\sin(30°)$, with the line from $y = 0.1$ being shorter by this amount. The construction and arguments are similar to those used in the discussion of the electric dipole in Sec. 4.7. The electric field is now the result of part a, modified by including a shorter distance, r, in the phase term only. We show this as an additional phase factor:

$$E_{\theta s} = 0.2e^{-j1000\pi}e^{j10\pi(0.1\sin 30} = \underline{0.2e^{-j1000\pi}e^{j0.5\pi}\ \text{V/m}}$$

c) Find $E_{\theta s}$ at P if identical elements are located at $A(0.1, 90°, 90°)$ and $B(0.1, 90°, 270°)$: The original element of part b is still in place, but a new one has been added at $y = -0.1$. Again, constructing a line between B and P, we find, using the same arguments as in part b, that the length of this line is approximately $0.1\sin(30°)$ *longer* than the distance from the origin to P. The part b result is thus modified to include the contribution from the second element, whose field will add to that of the first:

$$E_{\theta s} = 0.2e^{-j1000\pi}\left(e^{j0.5\pi} + e^{-j0.5\pi}\right) = 0.2e^{-j1000\pi}2\cos(0.5\pi) = \underline{0}$$

The two fields are out of phase at P under the approximations we have used.

14.14. For a dipole antenna of overall length $2\ell = \lambda$, evaluate the maximum directivity in dB, and the half-power beamwidth.

D_{max} is found using Eq. (64) with $k\ell = \pi$, and involves a numerical integration. This I did using a Mathematica code to find $D_{max} = 2.41$, and in decibels this is $10\log_{10}(2.41) = \underline{3.82\ \text{dB}}$, with the maximum occurring in the direction $\theta = 90°$.

With $k\ell = \pi$, the pattern function, Eq. (59), reaches a maximum value of 2 at $\theta = 90°$. The half-power beamwidth is found numerically by setting the pattern function equal to $\sqrt{2}$ and thus solving

$$F(\theta) = \left[\frac{\cos(\pi\cos\theta) - \cos(\pi)}{\sin\theta}\right] = \sqrt{2}$$

The result is $\theta = 66.1°$, which means that the deviation angle from the direction along the maximum is $\theta_{1/2}/2 = 90 - 66.1 = 23.9°$. The half-power beamwidth is therefore $\theta_{1/2} = 2 \times 23.9 = \underline{47.8°}$.

14.15. For a dipole antenna of overall length $2\ell = 1.3\lambda$, determine the locations in θ and the peak intensity of the sidelobes, expressed as a fraction of the main lobe intensity. Express your result as the sidelobe level in decibels, given by $S_s[\text{dB}] = 10\log_{10}(S_{r,main}/S_{r,sidelobe})$. Again, use Fig. 14.8 as a guide.

Here, $k\ell = (2\pi/\lambda)(1.3\lambda/2) = 1.3\pi$. The angular intensity distribution is given by the square of the pattern function, Eq. (59), with $k\ell = 1.3\pi$ substituted:

$$|<\mathbf{S}>| \propto [F(\theta)]^2 = \left[\frac{\cos(1.3\pi\cos\theta) - \cos(1.3\pi)}{\sin\theta}\right]^2$$

To find the maxima, the easiest way is to create a polar plot of the function, as shown in Fig. 14.8, and then numerically determine the relative intensities of the peaks by searching for local maxima at locations indicated in the plot. The main lobe occurs at $\theta = 90°$, and has value $[F(\theta)]^2 = 2.52$. The secondary maxima occur at $\theta = 33.8°$ and $146.2°$, and both peaks have values $[F(\theta)]^2 = 0.47$. The sidelobe level in decibels is thus

$$S_s[dB] = 10\log_{10}\left(\frac{2.52}{0.47}\right) = \underline{7.3 \text{ dB}}$$

14.16. For a dipole antenna of overall length, $2\ell = 1.5\lambda$:

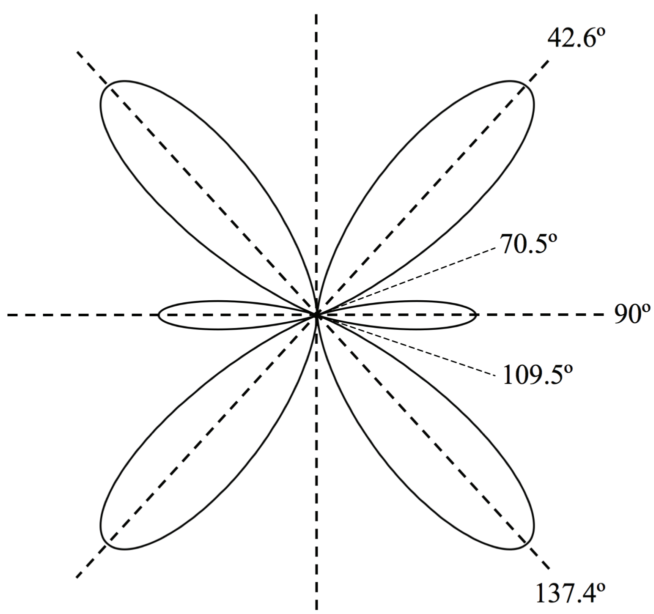
a) Evaluate the locations in θ at which the zeros and maxima in the E-plane pattern occur: In this case, $k\ell = (2\pi/\lambda)(1.5\lambda/2) = 1.5\pi$. The angular intensity distribution is given by the square of the pattern function, Eq. (59), with $k\ell = 1.3\pi$ substituted:

$$|<\mathbf{S}>| \propto [F(\theta)]^2 = \left[\frac{\cos(1.5\pi\cos\theta) - \cos(1.5\pi)}{\sin\theta}\right]^2 = \left[\frac{\cos(1.5\pi\cos\theta)}{\sin\theta}\right]^2$$

Zeros occur whenever $\cos(1.5\pi\cos\theta) = 0$, which in turn occurs whenever $1.5\pi\cos\theta$ is an odd multiple of $\pi/2$ (positive or negative). All zeros over the range $0 \le \theta \le \pi$ are found with just the first two odd multiples, or

$$\cos\theta_z = \pm\frac{1}{3},\ \pm 1 \quad \Rightarrow \quad \underline{\theta_z = (0, 70.5^\circ, 109.5^\circ, 180^\circ)}$$

The function does indeed zero at $\theta = 0$ and 180°, despite the presence of the $\sin\theta$ term in the denominator, as can be demonstrated using small argument approximations in the trig functions (show this!). The maxima are found numerically, as was done in Problem 14.15. The results are $\theta_{max,m} = (42.6^\circ, 137.4^\circ)$ (main lobes) and $\theta_{max,s} = 90^\circ$ (sidelobe). The pattern in plotted below.



b) Determine the sidelobe level, as per the definition in Problem 14.15: At the main lobe angle ($\theta = 42.6^\circ$), $[F(\theta)]^2$ evaluates as 1.96. The sidelobe at $\theta = 90^\circ$ gives a value of 1.00. The sidelobe level in decibles is thus

$$S_s[dB] = 10\log_{10}\left(\frac{1.96}{1.00}\right) = \underline{2.9 \text{ dB}}$$

c) Determine the maximum directivity: This is found evaluating Eq. (64) numerically (which I did using a Mathematica code), and the result is $D_{max} = 2.23$, or in decibels this becomes $10\log_{10}(2.23) = \underline{3.5 \text{ dB}}$.

14.17. Consider a lossless half-wave dipole in free space, with radiation resistance, $R_{rad} = 73$ ohms, and maximum directivity $D_{max} = 1.64$. If the antenna carries a 1-A current amplitude,

a) How much total power (in watts) is radiated? We use the definition of radiation resistance, Eq. (29), and write the radiated power:

$$P_r = \frac{1}{2} I_0^2 R_{rad} = \frac{1}{2}(1)^2(73) = \underline{36.5 \text{ W}}$$

b) How much power is intercepted by a 1 m^2 aperture situated at distance $r = 1$ km away. The aperture is on the equatorial plane and squarely faces the antenna. Assume uniform power density over the aperture: The maximum directivity is given by $D_{max} = 4\pi K_{max}/P_r$, where the maximum radiation intensity, K_{max}, is related to the power density in W/m^2 at distance r from the antenna as $S_r = K_{max}/r^2$. The power intercepted by the aperture is then the power density times the aperture area, or

$$P_{rec} = S_r \times Area = \frac{D_{max} P_r}{4\pi r^2} \times Area = \frac{1.64(36.5)}{4\pi(10^3)^2}(1)^2 = \underline{4.77\ \mu\text{W}}$$

14.18. Repeat Problem 14.17, but with a full-wave antenna ($2\ell = \lambda$). Numerical integrals may be necessary. In this case, $k\ell = \pi$, and the radiation resistance is found by applying Eq. (65):

$$R_{rad} = 60 \int_0^{\pi} \left[\frac{\cos(\pi \cos\theta) - \cos(\pi)}{\sin\theta} \right]^2 \sin\theta \, d\theta = 199 \text{ ohms}$$

where the answer was found by numerical integration. With the same 1-A current amplitude as before, the total radiated power is now

$$P_r = \frac{1}{2} I_0^2 R_{rad} = \frac{1}{2}(1)^2(199) = \underline{99.5 \text{ W}}$$

The maximum directivity is the result of Problem 14.14 (involving a numerical integration in Eq. (64) with $k\ell = \pi$), and the result was found to be $D_{max} = 2.41$. The received power is now found as in Problem 14.17:

$$P_{rec} = \frac{D_{max} P_r}{4\pi r^2} \times Area = \frac{2.41(99.5)}{4\pi(10^3)^2}(1)^2 = \underline{19.1\ \mu\text{W}}$$

14.19. Design a two-element dipole array that will radiate equal intensities in the $\phi = 0, \pi/2, \pi$, and $3\pi/2$ directions in the H-plane. Specify the smallest relative current phasing, ξ, and the smallest element spacing, d.

The array function is given by Eq. (81), with $n = 2$:

$$|A_2(\psi)| = \frac{1}{2}\left|\frac{\sin(\psi)}{\sin(\psi/2)}\right|$$

This has periodic maxima occurring at $\psi = 0, \pm 2m\pi$, and in the H-plane, $\psi = \xi + kd\cos\phi$. Now, for broadside operation, we have maxima at $\phi = \pm\pi/2$, at which $\psi = \xi$, so we set $\underline{\xi = 0}$ to get the array function principal maximum at $\psi = 0$. Having done this, we now have $\psi = kd\cos\phi$, which we need to have equal to $\pm 2\pi$ when $\phi = 0, \pi$, in order to get endfire operation. This will happen when $kd = 2\pi d/\lambda = 2\pi$, so we set $\underline{d = \lambda}$.

14.20. A two-element dipole array is configured to provide zero radiation in the broadside ($\phi = \pm 90°$) and endfire ($\phi = 0, 180°$) directions, but with maxima occurring at angles in between. Consider such a set-up with $\psi = \pi$ at $\phi = 0$ and $\psi = -3\pi$ at $\phi = \pi$, with both values determined in the H-plane.

a) Verify that these values give zero broadside and endfire radiation: For two elements, the array function is given by Eq. (81), with $n = 2$:

$$|A_2(\psi)| = \frac{1}{2}\left|\frac{\sin(\psi)}{\sin(\psi/2)}\right|$$

This is clearly zero at $\psi = \pi$ and at $\psi = -3\pi$.

b) Determine the required relative current phase, ξ: We know that $\psi = \xi + kd\cos\phi$ in the H-plane. Setting up the two given conditions, we have:

$$\pi = \xi + kd\cos(0) = \xi + kd$$

and

$$-3\pi = \xi + kd\cos(\pi) = \xi - kd$$

Adding these two equations gives $\underline{\xi = -\pi}$, which means that the contributions from the two elements will in fact completely cancel in the broadside direction, regardless of the element spacing.

c) Determine the required element spacing, d: We can now use the first condition, for example, substituting values that we know:

$$\psi(\phi = 0) = \pi = -\pi + kd\cos(0) = -\pi + kd$$

Therefore, $kd = 2\pi/\lambda = 2\pi$, or $\underline{d = \lambda}$.

d) Determine the values of ϕ at which maxima in the radiation pattern occur: With the values as found, the array function now becomes

$$|A_2(\phi)| = \frac{1}{2}\left|\frac{\sin[\pi(2\cos\phi - 1)]}{\sin[\frac{\pi}{2}(2\cos\phi - 1)]}\right|$$

This function maximizes at $\phi = 60°, 120°, 240°,$ and $300°$, as found numerically.

14.21. In the two-element endfire array of Example 14.4, consider the effect of varying the operating frequency, f, away from the original design frequency, f_0, while maintaining the original current phasing, $\xi = -\pi/2$. Determine the values of ϕ at which the maxima occur when the frequency is changed to

a) $f = 1.5f_0$: The original phase and length from the example are $\xi = -\pi/2$ and $d = \lambda_0/4$, where $\lambda_0 = c/f_0$. Thus $kd = (2\pi/\lambda_0)(\lambda_0/4) = \pi/2$. The H-plane array function in that case was found to be

$$A(\pi/2, \phi)\Big|_{\lambda_0} = \cos\left[\frac{\psi}{2}\right] = \cos\left[\frac{\xi}{2} + \frac{kd}{2}\cos\phi\right] = \cos\left[-\frac{\pi}{4} + \frac{\pi}{4}\cos\phi\right]$$

Now, if we change the frequency to $1.5f_0$, while leaving ξ fixed (and the antenna physical length is unchanged), we obtain $\lambda = \lambda_0/1.5$, and $kd = (2\pi/\lambda_0)(1.5)(\lambda_0/4) = 3\pi/4$. The new array function is then:

$$A(\pi/2, \phi)\Big|_{\lambda_0/1.5} = \cos\left[\frac{\xi}{2} + \frac{kd}{2}\cos\phi\right] = \cos\left[-\frac{\pi}{4} + \frac{3\pi}{8}\cos\phi\right] = \cos\left[\frac{\pi}{8}(3\cos\phi - 2)\right]$$

This function amplitude maximizes when $3\cos\phi - 2 = 0$, or $\phi = \cos^{-1}(2/3) = \underline{\pm 48.2^\circ}$.

b) $f = 2f_0$: Now, $\lambda = \lambda_0/2$ and $kd = (2\pi/\lambda_0)(2)(\lambda_0/4) = \pi$. The array function becomes

$$A(\pi/2, \phi)\Big|_{\lambda_0/2} = \cos\left[\frac{\xi}{2} + \frac{kd}{2}\cos\phi\right] = \cos\left[-\frac{\pi}{4} + \frac{\pi}{2}\cos\phi\right] = \cos\left[\frac{\pi}{4}(2\cos\phi - 1)\right]$$

This function amplitude maximizes when $2\cos\phi - 1 = 0$, or $\phi = \cos^{-1}(1/2) = \underline{\pm 60.0^\circ}$.

14.22. Revisit Problem 14.21, but with the current phase allowed to vary with frequency (this will automatically occur if the phase difference is established by a simple time delay between the feed currents). Now, the current phase difference will be $\xi' = \xi\, f/f_0$, where f_0 is the original (design) frequency. Under this condition, radiation will maximize in the $\phi = 0$ direction regardless of frequency (show this). Backward radiation (along $\phi = \pi$) will develop however as the frequency is tuned away from f_0. Derive an expression for the *front-to-back ratio*, defined as the ratio of the radiation intensities at $\phi = 0$ and $\phi = \pi$, expressed in decibels. Express this result as a function of the frequency ratio f/f_0.

With the current phase and wavenumber both changing with frequency, we would have

$$\psi' = \left(\frac{f}{f_0}\right)\xi + \left(\frac{f}{f_0}\right) kd\cos\phi$$

With $\xi = -\pi/2$ and $d = \lambda_0/4$, The H-plane array function at the original frequency, f_0, is

$$A(\pi/2, \phi)\Big|_{f_0} = \cos\left[\frac{\psi}{2}\right] = \cos\left[\frac{\xi}{2} + \frac{kd}{2}\cos\phi\right] = \cos\left[\frac{\pi}{4}(\cos\phi - 1)\right]$$

The effect on this of changing the frequency is then

$$A(\pi/2, \phi)\Big|_{f} = \cos\left[\frac{\psi'}{2}\right] = \cos\left[\frac{f}{f_0}\left(\frac{\xi}{2} + \frac{kd}{2}\cos\phi\right)\right] = \cos\left[\left(\frac{f}{f_0}\right)\frac{\pi}{4}(\cos\phi - 1)\right]$$

The condition for the zero argument (which maximizes A) is $(\cos\phi - 1) = 0$ This term is unaffected by changing the frequency, and so the array function will always maximize in the $\phi = 0$ direction.

In the forward direction ($\phi = 0$), the value of $|A|^2$ (proportional to the radiation intensity) is unity. In the backward direction ($\phi = 180°$), we find

$$|A(\phi = \pi)|^2 = \cos^2\left(-\frac{f}{f_0}\frac{\pi}{2}\right) = \cos^2\left(\frac{\pi f}{2 f_0}\right)$$

The front-to-back ratio is then

$$R_{fb} = 10\log_{10}\left[\frac{|A(\phi = 0)|^2}{|A(\phi = \pi)|^2}\right] = 10\log_{10}\left[\frac{1}{\cos^2\left(\pi f/2f_0\right)}\right]$$

Evaluate the front-to-back ratio for a) $f = 1.5 f_0$, b) $f = 2f_0$, c) $f = 0.75 f_0$. Substituting these values, we find:

$$f = 1.5 f_0: \quad R_{fb} = 10\log_{10}\left[\frac{1}{\cos^2(1.5\pi/2)}\right] = \underline{3\,\text{dB}}$$

$$f = 2 f_0: \quad R_{fb} = 10\log_{10}\left[\frac{1}{\cos^2(\pi)}\right] = \underline{0\,\text{dB}}$$

$$f = 0.75 f_0: \quad R_{fb} = 10\log_{10}\left[\frac{1}{\cos^2(\pi/2.67)}\right] = \underline{8.3\,\text{dB}}$$

14.23. A *turnstile* antenna consists of two crossed dipole antennas, positioned in this case in the xy plane. The dipoles are identical, lie along the x and y axes, and are both fed at the origin. Assume that equal currents are supplied to each antenna, and that a zero phase reference is applied to the x-directed antenna. determine the relative phase, ξ, of the y-directed antenna so that the net radiated electric field as measured on the positive z axis is a) left circularly-polarized; b) linearly polarized along the 45° axis between x and y.

When looking at the field along the z axis, the expression for $\mathbf{E}$ can be constructed using Eq. (57), evaluated at $\theta = \pi/2$ (so that we consider the direction normal to the antenna), and in which r is replaced by z. An additional "array" term includes the two polarization directions (unit vectors) which are out of phase by ξ:

$$\mathbf{E}(z, \theta = \pi/2) = J\frac{I_0\eta}{2\pi z}[1 - \cos(k\ell)]\, e^{-jkz}\left[\mathbf{a}_x + \mathbf{a}_y e^{j\xi}\right]$$

a) To achieve left circular polarization for propagation in the forward z direction, the vector array function must be

$$\left[\mathbf{a}_x + \mathbf{a}_y e^{j\xi}\right] = [\mathbf{a}_x + j\mathbf{a}_y] \quad \Rightarrow \quad \underline{\xi = \pi/2}$$

b) To achieve 45° polarization, the vector array function must be

$$\left[\mathbf{a}_x + \mathbf{a}_y e^{j\xi}\right] = [\mathbf{a}_x \pm \mathbf{a}_y] \quad \Rightarrow \quad \underline{\xi = 0, \pi}$$

14.24. Consider a linear endfire array, designed for maximum radiation intensity at $\phi = 0$, using ξ and d values as suggested in Example 14.5. Determine an expression for the front-to-back ratio (defined in Problem 14.22) as a function of the number of elements, n, if n is an odd number.

From Example 14.5, we use $\xi = -\pi/2$ and $d = \lambda/4$, so that $kd = \pi/2$. Then in the H-plane

$$\psi = \xi + kd\cos\phi = \frac{\pi}{2}(\cos\phi - 1)$$

The array function is then:

$$A(\psi) = \frac{1}{n}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right] = \frac{1}{n}\left[\frac{\sin[(n\pi/4)(\cos\phi - 1)]}{\sin[(\pi/4)(\cos\phi - 1)]}\right]$$

In the forward direction ($\phi = 0$), $A = 1$. In the backward direction ($\phi = \pi$), $A = \pm 1/n$, where n is an odd integer. The front-to-back ratio is therefore:

$$R_{fb} = 10\log_{10}\left(\frac{|A(\phi = 0)|^2}{|A(\phi = \pi)|^2}\right) = \underline{10\log_{10}(n^2)\ \text{dB}}$$

in which performance is clearly better as n increases.

14.25. A six-element linear dipole array has element spacing $d = \lambda/2$.

a) Select the appropriate current phasing, ξ, to achieve maximum radiation along $\phi = \pm 60°$: With $d = \lambda/2$, we have $kd = \pi$ and in the H-plane,

$$\psi = \xi + kd\cos\phi = \xi + \pi\cos\phi$$

We set the principal maximum to occur at $\psi = 0$, and with $\phi = 60°$, the condition for this becomes

$$\psi = 0 = \xi + \pi\cos(60°) = \xi + \frac{\pi}{2} \quad \Rightarrow \quad \underline{\xi = -\frac{\pi}{2}}$$

b) With the phase set as in part *a*, evaluate the intensities (relative to the maximum) in the broadside and endfire directions. With the above result, the array function (Eq. (81)) in the H-plane becomes

$$A(-\pi/2, \phi) = \frac{1}{6}\left[\frac{\sin[3(\xi + kd\cos\phi)]}{\sin[(1/2)(\xi + kd\cos\phi)]}\right] = \frac{1}{6}\left[\frac{\sin[(3\pi/2)(2\cos\phi - 1)]}{\sin[(\pi/4)(2\cos\phi - 1)]}\right]$$

In the broadside direction ($\phi = 90°$) the array function becomes

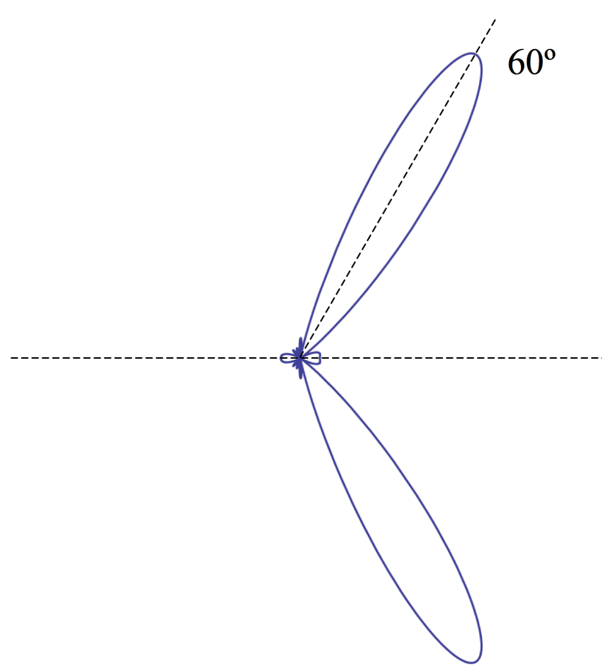
$$A(-\pi/2, \pi/2) = \frac{1}{6}\left[\frac{\sin[-3\pi/2]}{\sin[-\pi/4]}\right] = \frac{\sqrt{2}}{6}$$

The intensity in this direction (relative to that along $\phi = 60°$) is the square of this or $\underline{1/18}$

In the endfire direction ($\phi = 0°$) the array function becomes

$$A(-\pi/2, 0) = \frac{1}{6}\left[\frac{\sin[3\pi/2]}{\sin[\pi/4]}\right] = \frac{\sqrt{2}}{6}$$

The intensity in this direction (relative to that along $\phi = 60°$) is the square of this or $\underline{1/18}$, as before. The same result is found for $\phi = 180°$ and for $\phi = -90°$. The intensity pattern is shown below:



14.26. In a linear endfire array of n elements, a choice of current phasing that improves the directivity is given by the Hansen-Woodyard condition:

$$\xi = \pm \left(\frac{2\pi d}{\lambda} + \frac{\pi}{n} \right)$$

where the plus or minus sign choices give maximum radiation along $\phi = 180°$ and $0°$ respectively. Applying this phasing may not necessarily lead to unidirectional endfire operation (zero backward radiation), but will do so with the proper choice of element spacing, d.

a) Determine this required spacing as a function of n and λ: We first construct the expression for ψ in the H-plane, using the given current phase expression, and in which the minus sign is chosen:

$$\psi = \xi + kd\cos\phi = -\frac{2\pi d}{\lambda} - \frac{\pi}{n} + \frac{2\pi d}{\lambda}\cos\phi$$

Note that $\psi(\phi = 0) = -\pi/n$. The array function, Eq. (81) is:

$$A(\xi, \phi) = \frac{1}{n}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right] = \frac{1}{n}\left[\frac{\sin[(n\pi d/\lambda)(\cos\phi - 1) - \pi/2]}{\sin[(\pi d/\lambda)(\cos\phi - 1) - (\pi/2n)]}\right]$$

This function is equal to 1 when $\phi = 0$, regardless of the choice of d. When $\phi = \pi$, the array function becomes

$$A(\xi, \pi) = \frac{1}{n}\left[\frac{\sin[(-2n\pi d/\lambda) - (\pi/2)]}{\sin[(-2\pi d/\lambda) - (\pi/2n)]}\right]$$

which we require to be zero. This will happen when $\psi(\phi = \pi) = -\pi$. Using the above formula for ψ, we set up the condition:

$$\psi(\phi = \pi) = -\frac{4\pi d}{\lambda} - \frac{\pi}{n} = -\pi \quad \Rightarrow \quad \underline{d = \left(\frac{n-1}{n}\right)\frac{\lambda}{4}}$$

b) Show that the spacing as found in part a approaches $\lambda/4$ for a large number of elements: It is readily seen from the part a result that when $n \to \infty$, $d \to \lambda/4$.

c) Show that an even number of elements is required: This can be shown by substituting d as found in part a into the array function at $\phi = \pi$. The result is

$$A(\xi, \pi) = \frac{1}{n}\left[\frac{\sin(-n\pi/2)}{\sin(-\pi/2)}\right]$$

We require this result to be zero, which will only happen if n is an even integer. This can be more quickly seen from the general expression:

$$A(\xi, \phi) = \frac{1}{n}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right]$$

wherein if $\psi = -\pi$, we will get a zero result only if n is even.

14.27. Consider an n-element broadside linear array. Increasing the number of elements has the effect of narrowing the main beam. Demonstrate this by evaluating the separation in ϕ between the zeros on either side of the principal maximum at $\phi = 90°$. Show that for large n this separation is approximated by $\Delta\phi \doteq 2\lambda/L$, where $L \doteq nd$ is the overall length of the array.

For broadside operation, $\xi = 0$, and, with $kd = 2\pi d/\lambda$, we have

$$\psi = \xi + kd\cos\phi = \frac{2\pi d}{\lambda}\cos\phi$$

With this condition, the array function, Eq. (81) is:

$$A(\xi,\phi) = \frac{1}{n}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right] = \frac{1}{n}\left[\frac{\sin[(n\pi d/\lambda)\cos\phi]}{\sin[(\pi d/\lambda)\cos\phi]}\right]$$

Zeros in this function will occur whenever

$$\frac{n\pi d}{\lambda}\cos\phi = \pm m\pi$$

where m is an integer. Now, on either side of the principal maximum at $\psi = 0$, zeros will occur at $+\pi$ and $-\pi$ (giving a 2π separation), which we can associate with the two angles ϕ^+ and ϕ^- respectively, which lie on either side of $90°$. We can therefore write:

$$\frac{n\pi d}{\lambda}\left(\cos\phi^+ - \cos\phi^-\right) = 2\pi$$

Using a trig identity, this becomes:

$$\frac{2n\pi d}{\lambda}\underbrace{\sin\left[\frac{1}{2}(\phi^+ + \phi^-)\right]}_{1}\sin\left[\frac{1}{2}(\phi^- - \phi^+)\right] = 2\pi$$

Because we are considering the broadside direction, we have $(\phi^+ + \phi^-)/2 = \pi/2$ and so the first sine term is just unity. We find

$$\sin\left[\frac{1}{2}(\phi^- - \phi^+)\right] = \frac{\lambda}{nd}$$

Now as n gets large, both sides of the equation become $<< 1$, so that we may approximate the sine term by just its argument:

$$\frac{1}{2}(\phi^- - \phi^+) \doteq \frac{\lambda}{nd}$$

Defining $\Delta\phi = \phi^- - \phi^+$, the above expression becomes

$$\Delta\phi \doteq \frac{2\lambda}{nd} \doteq \frac{2\lambda}{L} \quad \text{(done)}$$

14.28. A large ground-based transmitter radiates 10kW, and communicates with a mobile receiving station that dissipates 1mW on the matched load of its antenna. The receiver (not having moved) now transmits back to the ground station. If the mobile unit radiates 100W, what power is received (at a matched load) by the ground station?

We can use Eq. (93) and the fact that $Z_{21} = Z_{12}$ to write:

$$\frac{P_{L2}}{P_{r1}} = \frac{|Z_{21}|^2}{R_{11}R_{22}} = \frac{|Z_{12}|^2}{R_{11}R_{22}} = \frac{P_{L1}}{P_{r2}}$$

We have

$$\frac{P_{L2}}{P_{r1}} = \frac{10^{-3}}{10^4} = 10^{-7} = \frac{P_{L1}}{P_{r2}}$$

So

$$P_{L1} = P_{r2} \times 10^{-7} = 100 \times 10^{-7} = 10^{-5} = \underline{10\,\mu\text{W}}$$

14.29. Signals are transmitted at a 1m carrier wavelength between two identical half-wave dipole antennas spaced by 1km. The antennas are oriented such that they are exactly parallel to each other.

a) If the transmitting antenna radiates 100 watts, how much power is dissipated by a matched load at the receiving antenna? To find the dissipated power, use Eq. (106):

$$P_{L2} = \frac{P_{r1}\lambda^2}{(4\pi r)^2}\, D_1(\theta_1\phi_1)\, D_2(\theta_2\phi_2)$$

Since the antennas face each other squarely, the directivities are both the maximum values for the two, where for the half-wave dipole, we know that $D_{max} = 1.64$ So

$$P_{L2} = \frac{100(1)^2}{(4\pi \times 10^3)^2}\,(1.64)^2 = \underline{1.7\ \mu\text{W}}$$

b) Suppose the receiving antenna is tilted by 45° while the two antennas remain in the same plane. What is the received power in this case? With the receiving antenna tilted, its directivity is reduced accordingly. Using Eq. (63), with the help of the results of Example 14.2, we have

$$D_2(\theta_2 = 45^\circ) = D_{max} \times |F(45^\circ)|^2 = 1.64 \times \left|\frac{\cos[(\pi/2)\cos(45^\circ)]}{\sin(45^\circ)}\right|^2 = 0.643$$

So now,

$$P_{L2}(45^\circ) = \frac{100(1)^2}{(4\pi \times 10^3)^2}\,(1.64)(0.643) = \underline{672\ \text{nW}}$$

14.30. A half-wave dipole antenna is known to have a maximum effective area, given as A_{max}.

a) Write the maximum directivity of this antenna in terms of A_{max} and wavelength λ: From Eq. (104), it follows that

$$D_{max} = \frac{4\pi}{\lambda^2} A_{max}$$

b) Express the current amplitude, I_0, needed to radiate total power, P_r, in terms of P_r, A_{max}, and λ: Combining Eqs. (64) and (65), we can write

$$\frac{2P_r}{I_0^2} = \frac{120[F(\theta)]^2_{max}}{D_{max}}$$

where, for a half-wave dipole, $[F(\theta)]^2_{max} = 1$. Then, using the part *a* result, we find

$$I_0 = \sqrt{\frac{\pi P_r A_{max}}{15\lambda^2}}$$

c) At what values of θ and ϕ will the antenna effective area be equal to A_{max}? These will be the same angles over which the half-wave dipole directivity maximizes, or at which $F(\theta)$ for the antenna reaches its maximum. We know from Example 14.2 that this happens at $\theta = \pi/2$, and at all values of ϕ.