تقدم لجنة ElCoM الاكاديمية

دفتر لمادة:

رياضيات تطبيقية

جزيل الشكر للطالب:

نمر عودة

Ch 13. Complex Number

1st

$x^2+1=0 \rightarrow x^2=-1 \rightarrow x=\pm\sqrt{-1} \rightarrow x=\pm i$

$i^2=-1$, $i^3=-i$, $i^4=1$

حقيقي

$Z=(x,y)$, $Z=x+iy$

تخيلي

$*$ $Z=(1,2)=1+2i$

∴ The complex number is an ordered pair of real numbers.

* الترتيب مهم

x is the real part of the complex number Re(Z)
y is the imaginary part of the complex number. Im(Z)

Ex) $Z=2-i$ Find $Re(Z)$, $Im(Z)$

Sol → $Re(Z)=2$

$Im(Z)=-1$

$(-i)$ x wrong

* Addition :

$Z_1=x_1+iy_1$, $Z_2=x_2+iy_2$

$Z_1+Z_2=(x_1+x_2\,,\,y_1+y_2)$

$=(x_1+x_2)+i(y_1+y_2)$

Ex) $Z_1=2-i$, $Z_2=-3+2i$

$Z_1+Z_2=-1+i$

$Z_1-Z_2=5-3i$

CH 13

* Multiplication :

$z_1 = x_1 + iy_1 \quad , \quad z_2 = x_2 + iy_2$

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1 x_2 + i^2 y_1 y_2 + i x_1 y_2 + i x_2 y_1$

$= \underbrace{(x_1 x_2 - y_1 y_2)}_{Re} + i \underbrace{(x_1 y_2 + x_2 y_1)}_{IM}$

$Re(z_1 z_2) = (x_1 x_2 - y_1 y_2)$

$IM(z_1 z_2) = (x_1 y_2 + x_2 y_1)$

Ex) $z_1 = 2 - i \quad , \quad z_2 = -3 + 2i$

$z_1 z_2 = (2 - i)(-3 + 2i)$

$= (-6 + 2) + 4i + 3i$

$z_1 z_2 = -4 + 7i$

* Division :

$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2}$ → conjugate

$= \frac{x_1 x_2 + y_1 y_2 + i x_2 y_1 - i x_1 y_2}{x_2^2 + y_2^2 + \cancel{i y_2 x_2} - \cancel{i y_2 x_2}}$

$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$

Ch.13

$z.\bar{z} = x^2 + y^2$

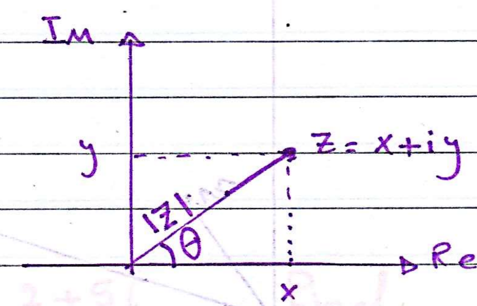
$\bar{z}$: conjugate

Ex) $z_1 = 8+3i$, $z_2 = 9-2i$

$$\frac{z_1}{z_2} = \frac{8+3i}{9-2i} \cdot \frac{9+2i}{9+2i} = \frac{(72-6)+(27+16)i}{81+4} = \frac{66}{85} + i\,\frac{43}{85}$$

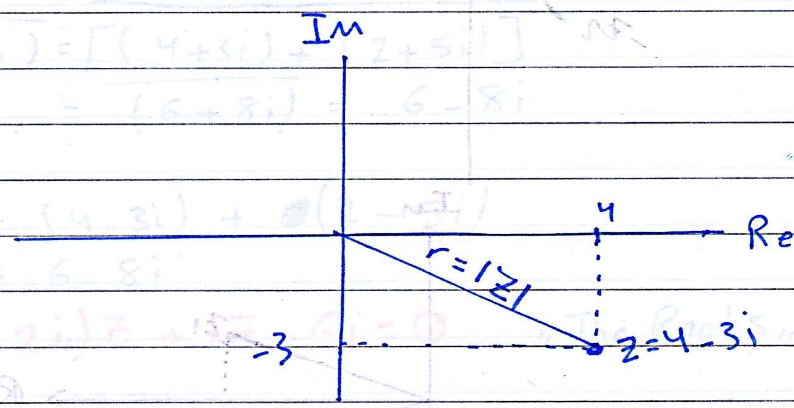
Ex) $\frac{1}{i} \cdot \frac{-i}{-i} = \frac{-i}{1} = -i$ # $\quad -i^2 = -(-1) = 1$

- Complex plane:-



$z = x + iy$

Ex) $z = 4-3i$



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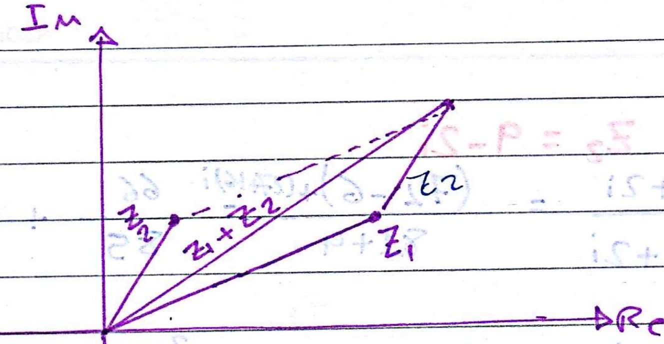
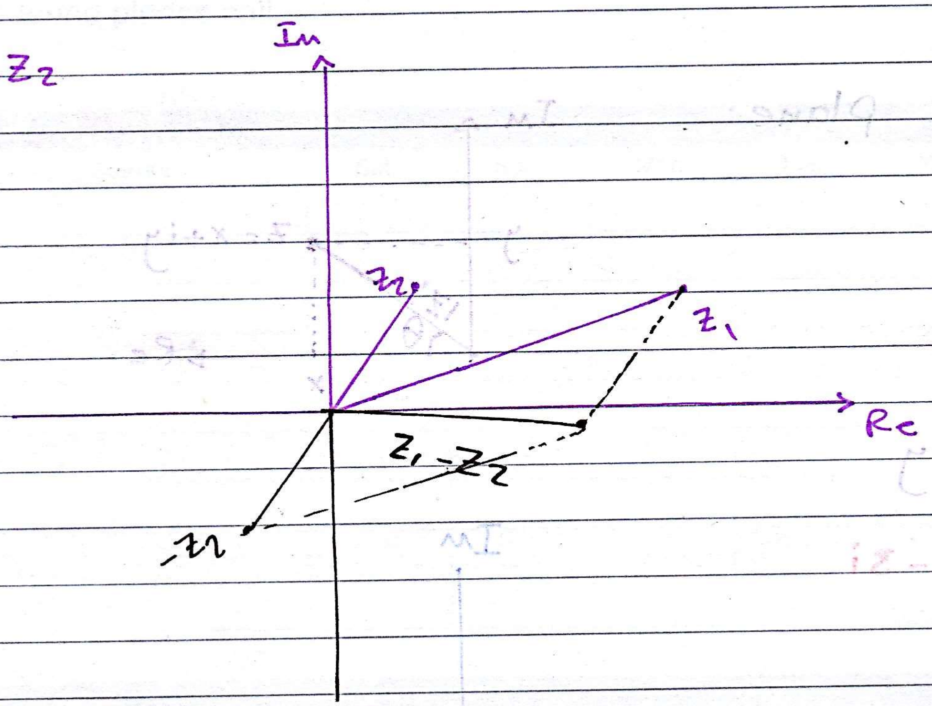
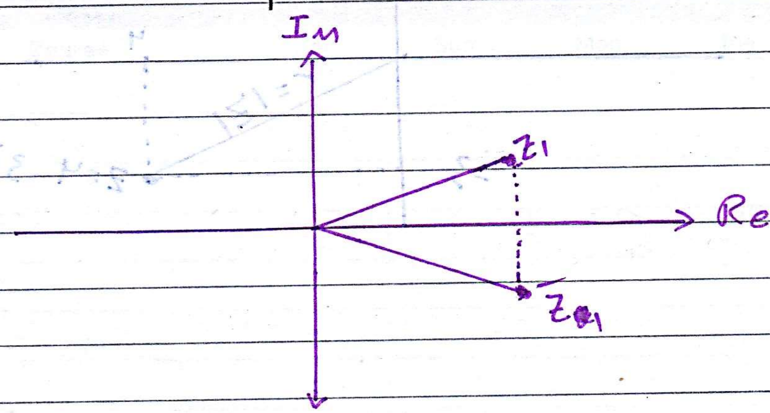
CH13

$Z_1 + Z_2$



$Z_1 - Z_2$





CH13

Complex conjugate:

By multipliation $z.\bar{z} = x^2 + y^2$

By Addition $z + \bar{z} = 2x \rightarrow x = \frac{z+\bar{z}}{2}$

By Substraction $z - \bar{z} = 2iy$ (pure Imaginary)

$(x+iy) - (x-iy)$

$= 2iy \rightarrow y = \frac{z-\bar{z}}{2i}$

* We have the following rules:

$\overline{(z_1+z_2)} = \bar{z}_1 + \bar{z}_2$ في كل العمليات

$\overline{(z_1-z_2)} = \bar{z}_1 - \bar{z}_2$

$\overline{(z_1 z_2)} = \bar{z}_1 \cdot \bar{z}_2$

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

Ex) $z_1 = 4+3i$, $z_2 = 2+5i$ find $\overline{(z_1+z_2)}$

Sol → $\overline{(z_1+z_2)} = \overline{[(4+3i)+(2+5i)]}$

$= \overline{(6+8i)} = 6-8i$

OR →

$= \bar{z}_1 + \bar{z}_2 = (4-3i) + (2-5i)$

$= 6-8i$

H.w) $z^2 - (6-2i)z + 17 - 6i = 0$ "The Roots"

$\frac{-B \pm \sqrt{B^2-4AC}}{2A}$

Ans: $z_1 = 3+2i$

$z_2 = 3-4i$

Ans: $z^2 - (6-2i)z + 17 - 6i = 0$

$$z = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{6-2i \pm \sqrt{(6-2i)^2 - 4(1)(17-6i)}}{2(1)}$$

$$= \frac{6-2i \pm \sqrt{36 \cancel{-24i} - 4 - 68 \cancel{+24i}}}{2}$$

$$z = \frac{6-2i \pm \sqrt{-36}}{2}$$

$$z = \frac{6-2i \pm 6i}{2} \Rightarrow z_1 = \frac{6+4i}{2} = 3+2i$$

$$z_2 = \frac{6-8i}{2} = 3-4i$$

#

CH 13

Ex)

توضيحي

$z = a + ib = \sqrt{4 - i}$

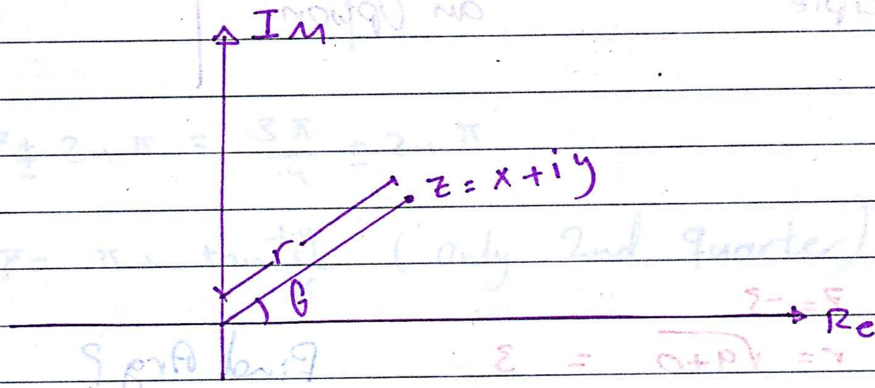
$(a+ib)^2 = 4 - i$ Then find a & b .....

$(a^2 - b^2) + 2abi = 4 - i$

$a^2 - b^2 = 4$ --- ①

$2ab = -1$ --- ②

13.2 Polar forms of complex numbers



$z = r(\cos\theta + i\sin\theta)$

$r = |z| =$ The absolute value of z or modulus

$r = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$

z: is the distance of point z from the Origin

θ: is the argument of z and denoted by

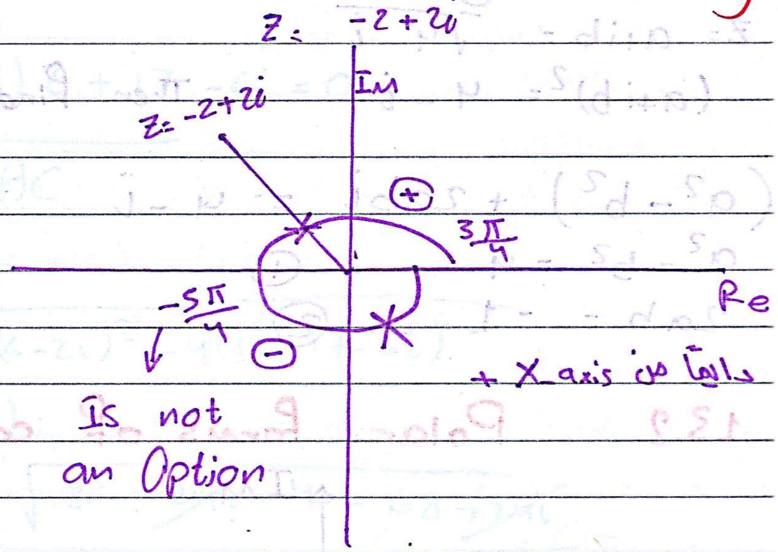
$\arg z \Rightarrow \tan^{-1}\frac{y}{x}$ (measured in radians)

(positive counter clockwise)

13.2

$\theta = \arg Z = \operatorname{Arg} Z \pm 2n\pi$

$n = 0, 1, 2, \ldots$



$-\pi < \operatorname{Arg} Z \leq \pi$

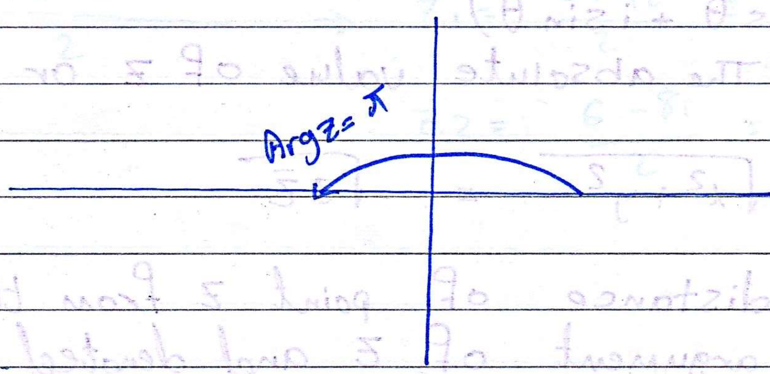
Principle Arg

is not an option

Ex) $Z = -3$

$r = \sqrt{9+0} = 3$

Find Arg?



$\arg Z = \operatorname{Arg} Z \pm 2n\pi$

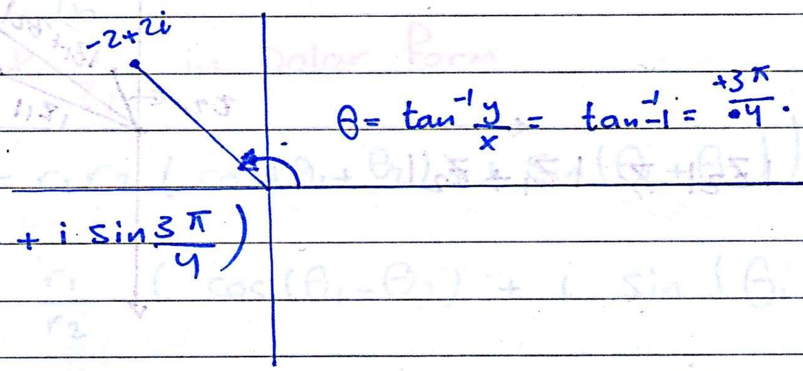
$= \pi \pm 2n\pi, \quad n = 0, 1, 2, \ldots$

$Z = 3(\cos\pi + i\sin\pi)$

13.2

Ex $z = -2+2i$

$r = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$



$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}-1 = \frac{+3\pi}{4}$.

$z = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$\theta = \operatorname{Arg} z \pm 2n\pi = \frac{3\pi}{4} \pm 2n\pi$

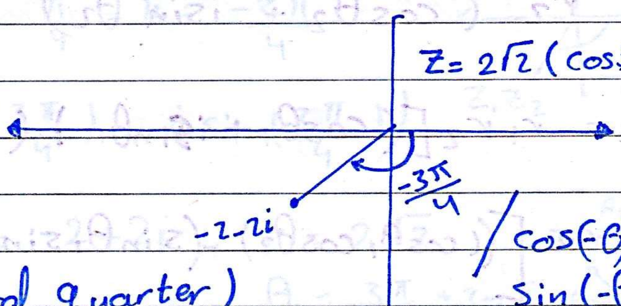
$\therefore \operatorname{Arg} z = \pi + \tan^{-1}\frac{y}{x}$ (Only 2nd quarter)

Ex $z = -2-2i$ $\quad r = 2\sqrt{2}$

$\theta = \arg z = \operatorname{Arg} z \pm 2n\pi$

$= \frac{-3\pi}{4} \pm 2n\pi$

$\operatorname{Arg} z = \tan^{-1}\frac{y}{x} - \pi$ (3rd quarter)



$z = 2\sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$ #

$\cos(-\theta) = \cos\theta$

$\sin(-\theta) = -\sin\theta$

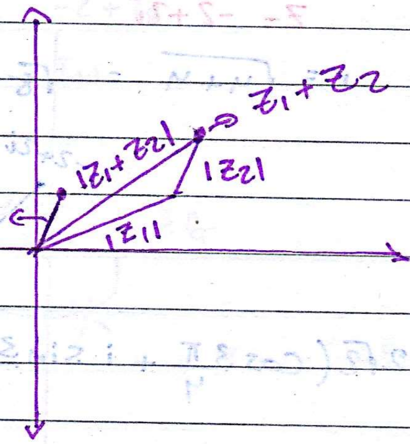
$\operatorname{Arg} z = \pi + \tan^{-1}\frac{y}{x}$ | $\operatorname{Arg} z = \tan^{-1}\frac{y}{x}$

$\operatorname{Arg} z = \tan^{-1}\frac{y}{x} - \pi$ | $\operatorname{Arg} z = \tan^{-1}\frac{y}{x}$

$\theta = \arg = \operatorname{Arg} \pm 2n\pi$

careful

* Triangle Inequality



$|z_1| + |z_2| \geq |z_1 + z_2|$

IM $\left|\frac{3-4i}{2+i}\right| = 0$

Always the result is Real

* Multiplication and division in polar Form.

$z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$
$z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$

$z_1 z_2 = r_1 r_2 [(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)]$

$= r_1 r_2 \left[(\cos\theta_1 \cos\theta_2) - (\sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1)\right]$

$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Try it home

13.2

Ex $z_1 = -2+2i$, $z_2 = 3i$,

write (a) $z_1 z_2$ & (b) $\frac{z_1}{z_2}$ in polar Form

Sol $\Rightarrow$ $z_1 z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right)$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left(\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right)$$

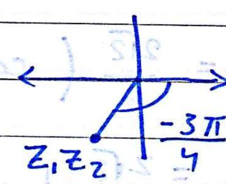
$\Rightarrow$ $\arg z_1 z_2 = \arg z_1 + \arg z_2$

$\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2$

$z_1 z_2 = (-2+2i)(3i) = -6i - 6 = -6 - 6i$

$|z_1 z_2| = \sqrt{72} = 6\sqrt{2}$

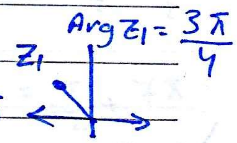
$\operatorname{Arg} z = \tan^{-1}\frac{-6}{-6} - \pi = \frac{\pi}{4} - \pi = \frac{-3\pi}{4}$



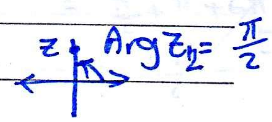
$\therefore$ $z_1 z_2 = 6\sqrt{2} \left(\cos\left(\frac{3\pi}{4}\right) - i \sin\left(\frac{3\pi}{4}\right) \right)$

Another way $\Rightarrow$ $z_1 = -2+2i$, $r_1 = 2\sqrt{2}$

$\theta_1 = \frac{3\pi}{4} \pm 2n\pi$



$z_2 = 3i$, $r_2 = 3$, $\theta_2 = \frac{\pi}{2} \pm 2n\pi$



$\therefore$ $z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]$

$= 6\sqrt{2} \left[\cos\left(\frac{3\pi}{4} + \frac{\pi}{2}\right) + i \sin\left(\frac{3\pi}{4} + \frac{\pi}{2}\right) \right]$

$z_1 z_2 = 6\sqrt{2} \left[\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) \right]$

$\hookrightarrow$ The same but Not the same Principle Argument

13.2

$\Theta(z_1 z_2) = \theta_1 + \theta_2$

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$\cos\left(\frac{-3\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right)$

$\pm 2n\pi$

(كدا)

$= \cos\left(-\frac{3\pi}{4} - 2\pi\right)$

b) $\frac{z_1}{z_2} = \frac{-2+2i}{3i} \times \frac{-3i}{-3i} = \frac{6i+6}{9} = \frac{2}{3} + \frac{2}{3}i$

$r = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

$\Theta = \frac{\pi}{4} \pm 2n\pi$

$\frac{z_1}{z_2} = \frac{2\sqrt{2}}{3}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Another way:

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$

$= \frac{2\sqrt{2}}{3}\left(\cos\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{2}\right)\right)$

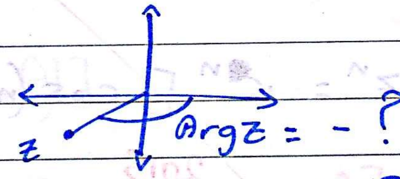
$= \frac{2\sqrt{2}}{3}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

13.2

Ex)

$z = 2(\cos 4 + i \sin 4)$

Find Arg z



$\text{Arg} z = 4 - 6.28 = -2.28$ $\qquad 2\pi = 6.28$

Ex) Find $(-2 + 2i)^{10}$ Sol →

$zz = z^2 = r^2(\cos 2\theta + i \sin 2\theta)$ #

$z^3 = r^3 (\cos 3\theta + i \sin 3\theta)$ #

$(-2+2i)^{10} = z^{10} = r^{10} [\cos(10\theta) + i \sin(10\theta)]$

$\theta = \frac{3\pi}{4} \pm 2n\pi$

$= (2\sqrt{2})^{10} \left[\cos \frac{30\pi}{4} + i \sin \frac{30\pi}{4}\right]$

$= (2\sqrt{2})^{10} \left[\cos \frac{15\pi}{2} + i \sin \frac{15\pi}{2}\right]$

$\frac{15\pi}{2} = \frac{\pi}{2} + \frac{14\pi}{2}$

$= (2\sqrt{2})^{10} \left[\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right]$

$= \frac{\pi}{2} + 7\pi$

$= \frac{\pi}{2} + \pi + \cancel{6\pi}$ $\pm 2n\pi$

$= \frac{3\pi}{2}$

13.3 Integer power of Complex Number

* $z^n = r^n [\cos(n\theta) + i \sin(n\theta)]$

Ex) $(1+i)^{2015} = (\sqrt{2})^{2015} [\cos 2015\frac{\pi}{4} + i \sin 2015\frac{\pi}{4}]$

For Arg z:

$2015\frac{\pi}{4} = (2012\frac{\pi}{4} + \frac{3\pi}{4})$

$= (503\pi + \frac{3\pi}{4} + \pi - \pi)$

$= (504\pi - \frac{\pi}{4})$

$= 2n\pi - \frac{\pi}{4}$

$\text{Arg}\, z = \frac{-\pi}{4}$

* Roots $\Rightarrow$ $z = w^n \dots ①$ $\Rightarrow$ $w = \sqrt[n]{z} \dots ②$

z & w are complex numbers

$w^4 + 324 = 0 \Rightarrow w^4 = -324 \Rightarrow w = \sqrt[4]{-324}$

To solve this eq. we write w & z in the polar form

$z = r(\cos\theta + i\sin\theta)$

$w = R(\cos\phi + i\sin\phi)$

Sub. z & w in eq 1.

$r(\cos\theta + i\sin\theta) = R^n(\cos n\phi + i\sin n\phi)$

$r = R^n \Rightarrow R = \sqrt[n]{r}$

CH. 13

$$\theta \pm 2K\pi = n\phi$$

$K = 0, 1, 2 \ldots$

$$\phi = \frac{\theta \pm 2\pi K}{n}$$

$$* \quad \phi = \frac{\theta + 2\pi K}{n}, \quad K = 0, 1, 2 \ldots, n-1$$

* Because I want distinct values.

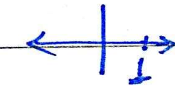
$\therefore$

$$\underset{0,1,2,3,\ldots,n-1}{w} = \sqrt[n]{r}\left[\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i \sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

$K = 0, 1, \ldots, n-1$

Ex) Find the 3rd root of $Z = 1 \Rightarrow \sqrt[3]{1}$

Sol $\Rightarrow$ For $Z = 1 \Rightarrow r = 1$, $\operatorname{Arg} Z = 0$



$\theta = 0 \pm 2n\pi$

$n = 0, 1, 2, \ldots$

$$K = 0 \Rightarrow w_0 = \sqrt[3]{1}\left[\cos\left(\frac{0 + 2(0)\pi}{3}\right) + i\sin\left(\frac{0 + 2(0)\pi}{3}\right)\right]$$

$$= 1[1 + 0] = 1$$

$$K = 1 \Rightarrow w_1 = \sqrt[3]{1}\left[\cos\left(\frac{0 + 2\pi}{3}\right) + i\sin\left(\frac{0 + 2\pi}{3}\right)\right]$$

$$w_1 = 1\left[\frac{-1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$$

$$K = 2 \Rightarrow w_2 = \sqrt[3]{1}\left[\cos\left(\frac{0 + 4\pi}{3}\right) + i\sin\left(\frac{0 + 4\pi}{3}\right)\right]$$

$$= 1\left[\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right] = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$$

* $\sqrt[n]{r}$ = Modulus

CH13

Thursday

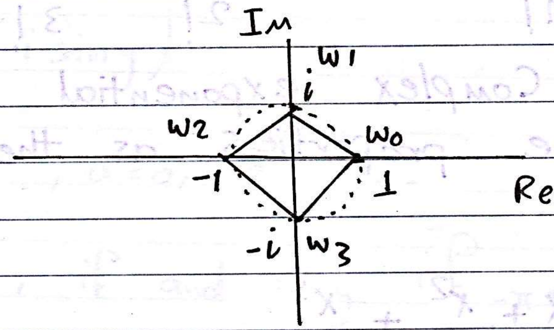
Ex) $z = \sqrt[4]{1}$

Sol $\Rightarrow w_0 = \sqrt[4]{1}\left[\cos\left(\frac{0+0}{4}\right) + i\left(\sin\frac{(0+0)}{4}\right)\right] = 1$

$w_1 = \sqrt[4]{1}\left[\cos\left(0+\frac{\pi}{2}\right) + i\sin\left(0+\frac{\pi}{2}\right)\right]$
$= i$

$w_2 = \sqrt[4]{1}\left[\cos(\pi) + i\sin\pi\right] = -1$

$w_3 = \sqrt[4]{1}\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right] = -i$



Ex) $\sqrt[4]{1+i}$ $r = \sqrt{2}$, $\theta = \frac{\pi}{4}$

$w_0 = \sqrt[4]{\sqrt{2}}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi(0)}{4}\right) + i\sin\left(\frac{\pi}{16}\right)\right]$

$\sqrt[8]{2}$

$w_1 = \sqrt[8]{2}\left[\cos\left(\frac{\frac{\pi}{4}+2\pi}{4}\right) + i\sin\left(\frac{9\pi}{16}\right)\right]$

$w_2 = \sqrt[8]{2}\left[\cos\left(\frac{\frac{\pi}{4}+4\pi}{4}\right) + i\sin\left(\frac{17\pi}{6}\right)\right]$

$w_3 = \sqrt[8]{2}\left[\cos\left(\frac{25\pi}{16}\right) + i\sin\left(\frac{25\pi}{16}\right)\right]$

CH 13

13.5 Exponential Function:

e^z, $\exp z$

$z = x + iy$

$e^z = e^{x+iy} = e^x . e^{iy} = e^x(\cos y + i \sin y)$

Using Taylor expausion series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

This Complex exponential function has the same properties as the real exponential function.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} \ldots$$

$$= \underbrace{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!}\right)}_{Re} + i\underbrace{\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} \ldots\right)}_{Im}$$

$i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$

$e^{iy} = \cos y + i \sin y$ Euler Formula

$$\boxed{e^{iy} = \cos y + i \sin y}$$

13.5

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$$

$$e^{(x_1+x_2)+i(y_1+y_2)} = e^{z_1} \cdot e^{z_2} = e^{(x_1+x_2)}\left[\cos(y_1+y_2) + i\sin(y_1+y_2)\right]$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$(e^{i\theta})^n = \cos(n\theta) + i\sin(n\theta) = e^{in\theta}$$

$$* \quad e^z = e^x(\cos y + i\sin y)$$

$$r = |e^z| = e^x$$

$$\arg(e^z) = y \pm 2n\pi \quad , \; n = 0, 1, 2, \ldots$$

$$\mathrm{Arg}(e^z) = y \quad , \text{ if and only if } -\pi \leq y \leq \pi$$

eg) $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$

eg) $e^{-2\pi i} = 1$

$* \quad e^{\pm 2n\pi i} = 1$

eg) $e^{\frac{\pi}{2}i} = i$

eg) $e^{-\frac{\pi}{2}i} = -i$

eg) $e^{-\pi i} = -1$

eg) $e^{\pi i} = -1$

The modulus for all the above = 1

CH13.5

Ex) $e^{1.4-0.6i} = e^{1.4}\left(\cos(0.6) - i\sin(0.6)\right)$

$= 3.347 - 2.289i$

$*\ \mathrm{Arg}\left(e^{1.4-0.6i}\right) = -0.6$ radians

$\left|e^{1.4-0.6i}\right| = e^{1.4} = 4.055$

Ex) $e^{2+i} \cdot e^{4-i} = e^{6}$

$e^{2}\left[\cos(1) + i\sin(1)\right] * e^{4}\left[\cos(1) - i\sin(1)\right]$

$= e^{2} \cdot e^{4}\left[\cos(1-1) + i\sin(1-1)\right]$

$= e^{6}$

Ex) Solve : $e^{z} = 3+4i$

$|e^{z}| = |e^{x}| = |3+4i| \Rightarrow e^{x} = 5 \Rightarrow x = \ln 5$

$e^{x} \cdot e^{iy} = e^{x}(\cos y + i\sin y) = 5(\cos y + i\sin y) = 3+4i$

$5\cos y + 5i\sin y = 3+4i$

$5\cos y = 3 \Rightarrow \cos y = 0.6$

$5\sin y = 4 \Rightarrow \sin y = 0.8$

$x = 1.609$

$y = 0.927 \pm 2n\pi$

$\therefore z = 1.609 + i(0.927 \pm 2n\pi)$

13.6 Trigonometric & Hyperbolic Functions.

$e^{ix} = \cos x + i\sin x$ ... (1)

$e^{-ix} = \cos x - i\sin x$ ... (2)

(1) + (2) $\rightarrow$ $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$, similarly $\boxed{\cos z = \dfrac{e^{iz}+e^{-iz}}{2}}$

(1) − (2) $\rightarrow$ $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$, Similarly $\boxed{\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}}$

- As Calculus we defined:

$\tan z = \dfrac{\sin z}{\cos z}$ $\qquad$ $\cot z = \dfrac{\cos z}{\sin z}$

$\sec z = \dfrac{1}{\cos z}$ $\qquad$ $\csc z = \dfrac{1}{\sin z}$

$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

- Hyperbolic Functions:-

$\boxed{\sinh z = \dfrac{e^{z}-e^{-z}}{2}}$, $\boxed{\cosh z = \dfrac{e^{z}+e^{-z}}{2}}$

- As Calculus we define:

$\tanh z = \dfrac{\sinh z}{\cosh z}$, $\coth z = \dfrac{\cosh z}{\sinh z}$, $\operatorname{sech} z = \dfrac{1}{\cosh z}$

$\operatorname{csch} z = \dfrac{1}{\sinh z}$

13.6

$$\cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} \times \frac{i}{i} = i \sin z$$

$$\tanh(iz) = \frac{\sinh(iz)}{\cosh(iz)} = \frac{i \sin z}{\cos z} = i \tan z$$

$$\cos(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{z}}{2}$$

$$= \frac{e^{z} + e^{-z}}{2} = \cosh(z)$$

$$\sin(iz) = \frac{e^{-z} - e^{z}}{2i} \cdot \frac{-i}{-i} = \frac{-(e^{z} - e^{-z})}{2} \cdot (-i)$$

$$= i\,\frac{(e^{z} - e^{-z})}{2}$$

$$\sin(iz) = i \sinh(z)$$

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i \sinh(z)}{\cosh(z)} = i \tanh(z)$$

Ex) $\cos(1+i) = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(1+i)} + e^{-i(1+i)}}{2}$

$$= \frac{1}{2}\left(e^{(-1+i)} + e^{(1-i)}\right)$$

$$= \frac{1}{2}\left[e^{-1}[\cos 1 + i \sin 1] + e^{1}[\cos 1 - i \sin 1]\right]$$

$$= \frac{1}{2}\left[\underbrace{e^{-1}\cos 1 + e\cos 1}_{Re} + i\,\underbrace{[e^{-1}\sin 1 - e \sin 1]}_{IM}\right]$$

13.6

$$\mathrm{IM}\left(\cos(1+i)\right) = \frac{1}{2}\left[e^{-1}\sin 1 - e^{1}\sin 1\right]$$

Ex) $\cos Z = -1$ $\qquad Z = x+iy$

$$\frac{e^{iZ}+e^{-iZ}}{2} = -1 \Rightarrow \left[e^{iZ}+e^{-iZ} = -2\right] e^{iZ}$$

$$= e^{2iZ} + 2e^{iZ} + 1 = 0 \Rightarrow e^{iZ} = u$$

$$= u^2 + 2u + 1 = 0$$

$$= (u+1)^2 = 0 \Rightarrow u = -1$$

$$e^{iZ} = -1$$

$$e^{ix-y} = -1 \Rightarrow e^{-y+ix} = -1$$

$$e^{-y} = 1 \Rightarrow y = 0$$

$$e^{-y+ix} = e^{-y}\left[\cos x + i\sin x\right] = -1$$

$$= 1\left[\cos x + i\sin x\right] = -1$$

$$\cos x = -1 \quad , \quad \sin x = 0$$

$$\boxed{x = \pi} \pm 2n\pi \quad , n = 0, 1 \ldots$$

$$Z = \pi \pm 2n\pi + i(0) = \pi \pm 2n\pi$$

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13.6

Ex) Show that:

a) $\cos z = \cos x \cosh y - i \sin x \sinh y$,

Sol → a) $\cos z = \dfrac{e^{iz} + e^{-iz}}{2} = \dfrac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right)$

$= \dfrac{1}{2}\left(e^{ix-y} + e^{-ix+y}\right)$

$= \dfrac{1}{2}\left(e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right)$

$= \dfrac{1}{2}\left[e^{-y}\cos x + e^{y}\cos x + i(e^{-y}\sin x - e^{y}\sin x)\right]$

$= \dfrac{1}{2}\left[\cos x\,(e^{y} + e^{-y}) - i\sin x\,(e^{y} - e^{-y})\right]$

$= \cos x\left(\dfrac{e^{y} + e^{-y}}{2}\right) - i\sin x\left(\dfrac{e^{y} - e^{-y}}{2}\right)$

$= \cos x\,\cancel{\sinh}\,\cosh y - i\sin x \sinh y$

Another solution:

$\cos(z) = \cos(x+iy) = \cos x \cos iy - i \sin x \sin iy$

$= \cos x \cosh y - i \sin x \sinh y$

b) $\sin z = \sin x \cosh y + i \cos x \sinh y$

$\sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + i \cos x \sinh y$

13.6

Ex) show that :-

a) $|\cos z|^2 = \cos^2 x + \sinh^2 y$

b) $|\sin z|^2 = \sin^2 x + \sinh^2 y$ → H.W

Sol → a) $\cos z = \cos x \cosh y - i \sin x \sinh y$

$$|\cos z|^2 = (\cos x \cosh y)^2 + (\sin x \sinh y)^2$$

$$= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$$

* $\boxed{\cosh^2 y - \sinh^2 y = 1}$

$$= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y$$

$$= \cos^2 x + \cos^2 x \sinh^2 y + \sin^2 x \sinh^2 y$$

$$= \cos^2 x + \sinh^2 y \, (\underbrace{\cos^2 x + \sin^2 x}_{1})$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

13.7 Logarithm General power, Principal Value.

$w = \ln Z$, w & Z are Complex numbers.

$e^{w} = Z \rightarrow w = u + vi \rightarrow Z = re^{i\theta}$

$e^{u+iv} = re^{i\theta} \rightarrow e^{u} \cdot e^{iv} = re^{i\theta}$

$\rightarrow$ ① $e^{u} = r \rightarrow u = \ln r$

② $v = \theta$

$\therefore \; w = \ln Z = \ln r + i\theta$

$= \ln r + i(\text{Arg}Z \pm 2n\pi)$

$w = \ln r + i\,\text{Arg}Z \pm 2in\pi$

(infinity many solution)

- principle value of $\ln Z$:

$\text{Ln}Z = \ln r + i\,\text{Arg}Z$ #

- General:

$\ln Z = \text{Ln}Z \pm 2n\pi i$

* $e^{\ln Z} = Z$
* $\ln e^{Z} = Z \pm 2n\pi i$

Ex) $\ln(1) \rightarrow Z = 1$, $r = 1$, $\theta = \pm 2n\pi$

$= \ln(1) \pm 2n\pi i$

$= \pm 2n\pi i$, $n = (0, 1, \ldots \infty)$

$\text{Ln}\,1 = 0$

13.7

Ex) $\ln(-1)$ $\quad z=-1 \quad r=1, \quad \theta = \pi \pm 2n\pi$

$\ln z = \underbrace{\ln 1}_{0} + \pi i \pm 2n\pi i \quad , \quad n = 0, 1, 2, \ldots$

$\operatorname{Ln} z = \pi i$

Ex) $\ln(i) =$ $\quad z = i, \quad r = 1, \quad \theta = \frac{\pi}{2} \pm 2n\pi$

$\ln(i) = \ln(1) + \frac{\pi}{2} i \pm 2n\pi i$

$\operatorname{Ln}(i) = \frac{\pi}{2} i$

H.w) $\ln(4)$, $\ln(-4)$, $\ln(4i)$, $\ln(-4i)$

13.7

Ex) $\ln(3-4i) = \ln r + \theta i$

$Z = 3-4i \rightarrow r = 5 \rightarrow \theta = \tan^{-1}\left(\frac{-4}{3}\right) \pm 2n\pi$

$= -0.927 \pm 2n\pi$

$\ln(3-4i) = \ln 5 + (-0.927)i \pm 2n\pi i$, $n = 0, 1, \ldots$

$\mathrm{Ln}(3-4i) = \ln 5 - 0.927i$

$= 1.609 - 0.927i$

Principle Value

Ex) $\ln Z = 3 + 4i \pm 2n\pi i$, Find Z in u+iv form.

$e^{\ln Z} = e^{3+4i \pm 2n\pi i}$

$Z = e^{3+4i} \cdot \underbrace{\cancel{e^{\pm 2n\pi i}}}_{1} = e^{3+4i}$

$Z = e^{3}(\cos 4 + i \sin 4) = e^{3}\cos 4 + i e^{3} \cos 4$

- General Power $\rightarrow Z^{c} = e^{c \ln Z}$, C is complex

Many solution $Z \neq 0$

Ex) $(i)^{i} = e^{i \ln i} = e^{i\left(\frac{\pi}{2} i \pm 2n\pi i\right)}$

$= \mathrm{Exp}\left[\frac{-\pi}{2} \pm 2n\pi\right]$

$= e^{\frac{-\pi}{2}} \cdot e^{\pm 2n\pi}$, $n = 0, 1, 2 \ldots$

(Infinity Many solution)

Principle Value $= e^{-\pi/2}$

principle value of $Z^{c} = e^{c \mathrm{Ln} Z}$

13.7

Ex) $(1+i)^3 = r^n(\cos n\theta + i\sin\theta)$ Integer power

$= e^{c\ln z}$ General power

Sol 1 → $(1+i)^3 = (\sqrt{2})^3\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$ //

Sol 2 → $(1+i)^3 = e^{3\ln(1+i)}$

$= e^{3[\ln\sqrt{2} + i\frac{\pi}{4} \pm 2n\pi i]}$

$= \exp\left[3\ln\sqrt{2} + i\frac{3\pi}{4} \pm 6n\pi i\right]$

$= e^{\ln\sqrt{2}^3} \cdot e^{i\frac{3\pi}{4}} \cdot \cancel{e^{\pm 6n\pi i}}_{1}$

$= (\sqrt{2})^3\, e^{\frac{3\pi}{4}i}$

$= (\sqrt{2})^3\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

1 solution

Ex) write $(1+i)^{2-i}$ in the form of $a+ib$.

$(1+i)^{2-i} = e^{(2-i)\ln(1+i)} = e^{(2-i)(\ln\sqrt{2} + \frac{\pi}{4}i \pm 2n\pi i)}$

$= e^{2\ln\sqrt{2} + (\frac{\pi}{4} \pm 2n\pi) + i(\frac{\pi}{2} \pm 4n\pi - \ln\sqrt{2})}$

$= e^{\ln 2} \cdot e^{\frac{\pi}{4} \pm 2n\pi} \cdot e^{\frac{\pi}{2}i} \cdot \cancel{e^{\pm 4n\pi i}}_{1} \cdot e^{-i\ln\sqrt{2}}$

$= 2 \cdot e^{\frac{\pi}{4} \pm 2n\pi} \cdot (i) \cdot (1) \cdot e^{-i\ln\sqrt{2}}$

$= 2 \cdot e^{\frac{\pi}{4} \pm 2n\pi}\, i\,[\cos\ln\sqrt{2} - i\sin\ln\sqrt{2}]$

$= 2e^{\frac{\pi}{4} \pm 2n\pi}[\sin\ln\sqrt{2} + i\cos\ln\sqrt{2}]$

13.7

$$(1+i)^{2-i} = \underbrace{2e^{\frac{\pi}{4}\pm 2n\pi}\sin \ln\sqrt{2}}_{Re} + \underbrace{i\,2e^{\frac{\pi}{4}\pm 2n\pi}\cos\ln\sqrt{2}}_{Im}$$

$$Re\,(1+i)^{2-i} = 2e^{\frac{\pi}{4}\pm 2n\pi}\sin\ln\sqrt{2}$$

$$Im\,(1+i)^{2-i} = 2e^{\frac{\pi}{4}\pm 2n\pi}\cos\ln\sqrt{2}$$

CH7

Linear Algebra, Matrix, Vectors, Determinants

entry

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \text{rows} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$A = [a_{jk}]$ a_{23} Column

upper case bold

$$\begin{bmatrix} e^{-x}, & 2x^2 \\ e^{6x}, & 4x \end{bmatrix}, \mathbf{a} = [a_1, a_2, a_3], \begin{bmatrix} 4 \\ 0.5 \end{bmatrix}$$

Lower case bold

row vector   Column vector

Square (Special Case)   , $j \to$ rows , $k \to$ Column

$B = [b_{jk}]$

$b_{21} = e^{6x}$

* Size of the matrix = $m \times n$ (m by n)

# of rows $\leftarrow m$, $n \to$ Number of Columns

Ex)

$$2x_1 - 3x_2 + x_3 = 20$$
$$3x_2 - 4x_3 = 10$$
$$-x_2 + 5x_1 + 3x_3 = 5$$

Sol $\Rightarrow$

$$\left[\begin{array}{ccc:c} 2 & -3 & 1 & 20 \\ 0 & 3 & -4 & 10 \\ 5 & -1 & 3 & 5 \end{array}\right] \Rightarrow \text{Size} = 3 \times 4$$

CH 7.

- Equality of matrices.

A = B if and Only if :-

1- Size of A = size of B

2- The Corresponding Entries are equal.

Ex) $A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 5 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 0 & 4 \end{bmatrix}$

[2 x 3] [3 x 2] [2 x 3]

$A \neq B$ (size of $A \neq$ size of B)

$A = C$.

- Addition of Matrices.

A + B is defined if and Only if size of A = size of B

Ex) $A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 5 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \end{bmatrix}$

A + B is not defined

$A + C = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 0 & 8 \end{bmatrix}$

- If A + B is defined then:

$C = A + B$, $c_{11} = a_{11} + b_{11}$, $c_{12} = a_{12} + b_{12}$, ...

CH7

Ex) $A = \overset{2\times 2}{\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}}$, $B = \overset{2\times 2}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}$

$A + B = \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}$

- Scalar Matrix Multiplication

Thursday
29-10-2015

$cA = [ca_{ik}]$ c: is scalar

Ex:

$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 0 & 4 \\ -5 & 0 & 3 \end{bmatrix} \rightarrow -A = \begin{bmatrix} -3 & -2 & 1 \\ -1 & 0 & -4 \\ 5 & 0 & -3 \end{bmatrix}$

$2A = \begin{bmatrix} 6 & 4 & -2 \\ 2 & 0 & 8 \\ -10 & 0 & 6 \end{bmatrix}$

- Matrix Multiplication, (Multiply Matrix by Matrix)

$C = \underset{[m\times n]}{A}\ \underset{[r\times p]}{B}$ is defined if and Only if $n = r$

The size of C is $m \times p$

Ex: $A = \underset{2\times 3}{\begin{bmatrix} \boxed{3 \quad 1 \quad 0} \\ 2 \quad 0 \quad 4 \end{bmatrix}}$, $B = \underset{3\times 3}{\begin{bmatrix} \boxed{\begin{matrix}2\\1\\3\end{matrix}} & \begin{matrix}3\\4\\-1\end{matrix} & \begin{matrix}6\\5\\-2\end{matrix} \end{bmatrix}}$ → C_{11}

size $[2 \times 3]$

$C = AB = \begin{bmatrix} 7 & 13 & C_{13} \to 5 \\ C_{21} \to 16 & C_{22} \to 2 & -8 \end{bmatrix}$

$C_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{3}$

$C_{11} = 6 + 1 + 0 = 7$

↔
Connected

CH 7

$C_{12} = a_{11} b_{12} + a_{12} b_{22} + a_{13} b_{32}$

$C_{12} = 9 + 4 + 0 = 13$

$C_{23} = a_{21} b_{13} + a_{22} b_{23} + a_{23} b_{33}$

$= 0 + 0 - 8 = -8$

$C_{jK} = \sum_{L=1}^{n} a_{jL}\, b_{LK}$

* BA is not defined

$\underset{[3\times 3]}{B}\ \underset{[2\times 3]}{A}$

* AB, generaly Not equal to BA

$AB \neq BA$ [In general]

* $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$

$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\substack{A \\ [3\times 3]}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\substack{X \\ [3\times 1]}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\substack{b \\ [3\times 1]}}$$

* This is Called Motivation Matrix behind multiplication.

CH7

Ex: $\underset{[1\times3]}{a = [\,1 \quad 2 \quad 3\,]}$, $\underset{[3\times1]}{b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}$

$ab = [1\times1] = [5]$

* Single entry Matrices will be Considered as Scalar.

$\therefore [5] = 5$

* Multiplication rules →

- $AB \neq BA$ In general
- $0A = 0$
- $1A = A$
- $A(B+C) = AB + AC$
- $(B+C)A = BA + CA$

* Rules of Addition →

$A + B = B + A$

$(A+B)+C = A + (B+C)$

$A + 0 = A$

$A - A = 0$

$c(A+B) = cA + cB$

$(c+k)A = cA + kA$, c, k → Scalers.

$1A = A$

$ckA = (ck)A$

CH7

- Rules of Matrix Multiplication

$(KA)B = K(AB) = A(KB)$

$A(BC) = (AB)C$

$C(A+B) = CA + CB$, A, B, C matrices
K scaler

* Pay attention to order.

- Product in terms of row and column products.

$$AB = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -26 \end{bmatrix}$$

* Prallel processing →

$$\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{matrix} 11 \\ -17 \end{matrix} \quad \Big| \quad \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \quad \Big| \quad \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 34 \\ -23 \end{bmatrix}$$

- Transposition

- The transpose of $m \times n$ matrix $A = [a_{jk}]$ is the $n \times m$ matrix $A^T = [a_{kj}]$

Ex) $\underset{2\times3}{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}$, $\underset{3\times2}{A^T} = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 6 \end{bmatrix}$

Ex) $\underbrace{[6 \;\; 2 \;\; 3]^T}_{\text{raw vector}} = \underbrace{\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}}_{\text{column vector}}$

CH 7

- Rules of transposition :

$[A^T]^T = A$

$(A+B)^T = A^T + B^T$

$(cA)^T = cA^T$, c is scaler

$\boxed{(AB)^T = B^T A^T}$

- Special Matrices :- (square) matrices

① Symmetric , skew symmetric matrices :-

Symmetric $A^T = A$

skew symmetric $A^T = -A$

Ex) $A = \begin{bmatrix} 200 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}$

Symetric matrix $\rightarrow A = A^T$

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Ex) $B = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$ skew symmetric Matrix

$A^T = -A$

② Triangular Matrices $\rightarrow$ upper Triangular

$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ upper Triangular

CH 7

Ex) $\begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ 7 & 6 & 8 \end{bmatrix}$ Lower Triangular matrix

Ex) $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ Diagonal matrix

Ex) $S = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ Scaler matrix
c, is constant

$(SA = AS = cA)$ #

Ex) $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, Identity matrix
(unit matrix)

$IA = AI = A$

7.3 Linear system of equations: Gauss Eliminations

$SA = AS = CA$

$$\underset{3\times 3}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}} = 2 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix}$$

$$\underset{3\times 1}{\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \text{notdefined}$$

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m$$

$\Rightarrow \quad Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & a_{m2} \\ a_{m1} & a_{m2} & \cdots & \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

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7.3

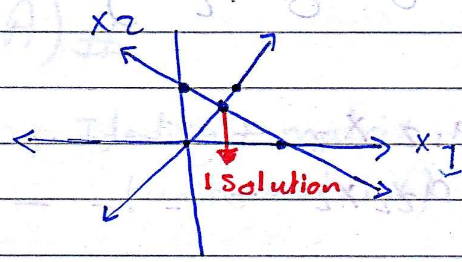
Angmented Matrix

$$\bar{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Ex) Geometric Representation Existance and uniqueness of solutions.

$\boxed{1}$ $x_1 + x_2 = 1$

$2x_1 - x_2 = 0$



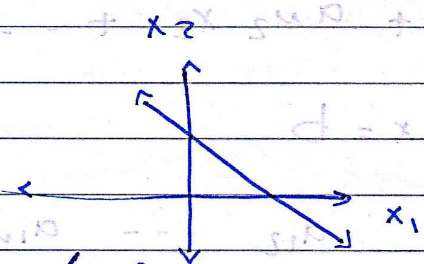
$x_1 + 2x_1 = 1$ 1 Solution

$x_1 = \frac{1}{3}$

$x_2 = \frac{2}{3}$

$\boxed{2}$ $x_1 + x_2 = 1$

$2x_1 + 2x_2 = 2$



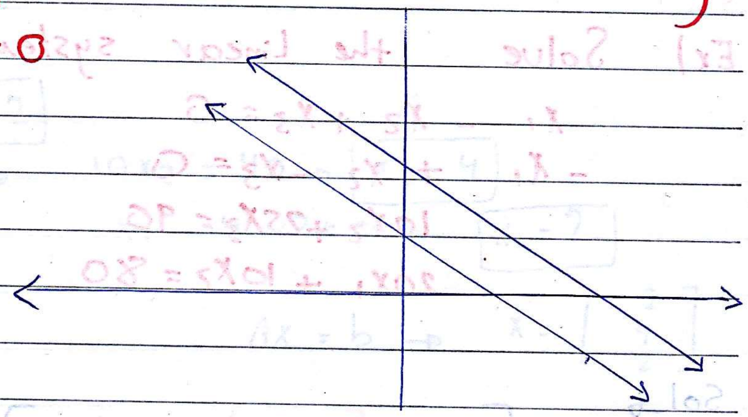
(Infinity Many solution)

$x_1 = 1 - x_2$

7.3

$\boxed{3}$ $x_1 + x_2 = 1$

$x_1 + x_2 = 0$



No solution
Not existed

Ex) Gauss Elimination and back substitutions:

$2x_1 + 5x_2 = 2$ $\qquad$ $2x_1 + 5x_2 = 2$

$13x_2 = -26$ $\xrightarrow[\text{eq ??}]{}$ $-4x_1 + 3x_2 = -30$

$$\begin{array}{c}\left[\begin{array}{cc|c} 2 & 5 & 2 \\ -4 & 3 & -30 \end{array}\right]\end{array} R_2$$

$R_2 - \frac{-4}{2}R_1 \Rightarrow R_2 + 2R_1$

$$\left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 13 & -26 \end{array}\right]$$

$x_2 = -2, \quad x_1 = 6$

7.3

Ex) Solve the linear system :-

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$10x_2 + 25x_3 = 90$$
$$20x_1 + 10x_2 = 80$$

Sol →

$$\begin{bmatrix} 1 & -1 & 1 & \vdots & 0 \\ -1 & 1 & -1 & \vdots & 0 \\ 0 & 10 & 25 & \vdots & 90 \\ 20 & 10 & 0 & \vdots & 80 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 10 & 25 & \vdots & 90 \\ 20 & 10 & 0 & \vdots & 80 \end{bmatrix} \quad \begin{array}{l} R - \frac{-1}{1} R_1 = R_2 + R_1 \\ \\ R_4 - \frac{20}{1} R_1 = R_4 - 20R_1 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 - \frac{30}{10} R_2 = R_3 - 3R_2$$

determined no. of eq = no. of unkown
underdetermined no. of eq < no of unkown

7.3

Back Substitution

$-95x_3 = -190 \rightarrow \boxed{x_3 = 2}$

$10x_2 + 25x_3 = 90 \rightarrow 10x_2 = 40 \rightarrow \boxed{x_2 = 4}$

$x_1 - x_2 + x_3 = 0 \rightarrow x_1 - 4 + 2 = 0 \rightarrow \boxed{x_1 = 2}$

(Unique Solution)

$Ax = b \rightarrow x = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

Ex)

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

Sol $\rightarrow$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \begin{array}{l} R_2 - \frac{0.6}{3}R_1 = R_2 - 0.2R_1 \\ R_3 - \frac{1.2}{3}R_1 = R_3 - 0.4R_1 \end{array}$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_3 - \frac{-1.1}{1.1}R_2 = R_3 + R_2$$

eq2 $\rightarrow$

$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1$

$x_2 + x_3 - 4x_4 = 1$

$\boxed{x_2 = 1 - x_3 + 4x_4}$

eq1 $\rightarrow$

$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$

$3x_1 + 2(1 - x_3 + 4x_4) + 2x_3 - 5x_4 = 8$

$3x_1 + 2 - \cancel{2x_3} + 8x_4 + \cancel{2x_3} - 5x_4 = 8$

$3x_1 + 3x_4 = 6 \rightarrow x_1 + x_4 = 2$

$\boxed{x_1 = 2 - x_4}$

$x = \begin{bmatrix} 2 - x_4 \\ 1 - x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix}$

7.3

Ex) Solve

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$\Downarrow$

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & \frac{-1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \begin{array}{l} R_2 - \frac{2}{3}R_1 \\ R_3 - \frac{6}{3}R_1 \end{array}$$

$\Downarrow$

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & \frac{-1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] R_3 - \frac{-2}{-\frac{1}{3}}R_2 = R_3 - 6R_2$$

$0 = 12$ X

There is no solution.

* Echelor Form & information from it:

At the end of G.E, the form of coefficient matrix, the angumented matrix, and the system itself are called row Echelor form.

① Rows of Zero's, if present are lost rows.

② In each non-Zero row, the left most non-Zero entry is farther to the right than in the previous row.

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Zeros

$$\left[\begin{array}{cccccc:c} r_{11} & r_{12} & \cdots & & & r_{1n} & f_1 \\ 0 & r_{22} & \cdots & & & r_{2n} & f_2 \\ 0 & 0 & r_{33} & \cdots & & r_{3n} & f_3 \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & r_r & r_{rn} & f_{r+1} \\ & & \text{Zero} & \text{row} & & & f_m \end{array}\right]$$

Ex)
$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$
$r=2 \rightarrow$ Non Zero rows
$m=3 \rightarrow$ Number of eq.
$n=3 \rightarrow$ Number of values.

- If $r=n$ & all $f_{r+1} \dots f_m$, if present, are Zero's. the Solution is unique.

- If $r<n$ & all $f_{r+1}, \dots f_m$, if present, are Zero's, then the solution is infinite.

- If $r<m$ & at least one of $f_{r+1}, f_m \neq$ Zero, then the solution $\rightarrow$ No solution (There is no solution)

Ex)
$$\left[\begin{array}{ccc:c} 2 & 1 & 0 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$r=3$
$n=3 \rightarrow$ a unique solution.
$m=4$

Ex)
$$\left[\begin{array}{ccc:c} 2 & 1 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{array}\right]$$
$r=3 \rightarrow$ No solution
$n=3$
$m=4$

7.3

Ex) $\left[\begin{array}{ccc|c} 3 & 4 & 5 & 13 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $r=2$, $n=3$, $m=3$ Infinite solution.

7.3

- Elementary Row operations.
Row equivalent systems.
- Interchange of two rows
- Addition of a constant multiple of one row to another row.
- Multiplication of a row by non-zero Constant

* These operations for <u>rows</u> not columns.

Thm: - Row equivalent systems have same set of solutions

- Linear systems ~~have same set~~

S_1 is row equivalent to S_2 if S_1 is obtained from S_2 by row operations

Linear system of eqs

Consistent (has solutions)

Inconsistent (no sols)

Unique solution

Infinity many solutions.

7.4 Linear Independence. Rank of a Matrix

. A set of m ~~xxxxx~~ vectors $a_1, a_2, a_3, \dots, a_m$ with same no. of components.

A linear combination of them:-

$c_1 a_1 + c_2 a_2 + c_3 a_3 + \dots + c_m a_m$ where $c_1, c_2, c_3, \dots, c_m$ are scalars.

If we consider this equations:-

$c_1 a_2 + c_2 a_2 + \dots + c_m a_m = 0$ * homogeneous

This equn holds when we choose c_j's all zero.

⇒ If this is the only case then the vectors are linearly independent.

⇒ If the equation holds with scalar not all zeroe's then the vectors are linearly dependent.

Ex) $a_1 = [3 \quad 0 \quad 2 \quad 2]$

$a_2 = [-6 \quad 42 \quad 24 \quad 54]$

$a_3 = [21 \quad -21 \quad 0 \quad -15]$

Find if the set of vectors is linearly independent.

⇒ $c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$

$$c_1 a_1 = [3c_1 \quad 0c_1 \quad 2c_1 \quad 2c_1]$$
$$c_2 a_2 = [-6c_2 \quad 42c_2 \quad 24c_2 \quad 54c_2]$$
$$c_3 a_3 = [21c_3 \quad -21c_3 \quad 0c_3 \quad -15c_3]$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

⇒ $3c_1 - 6c_2 + 21c_3 = 0$

$42c_2 - 21c_3 = 0$

$2c_1 + 24c_2 = 0$

$2c_1 + 54c_2 - 15c_3 = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} 3 & -6 & 21 & 0 \\ 0 & 42 & -21 & 0 \\ 2 & 24 & 0 & 0 \\ 2 & 54 & -15 & 0 \end{array}\right]$$

we don't need it.

7.4

①

$$\begin{bmatrix} 3 & -6 & 21 \\ 0 & 42 & -21 \\ 2 & 24 & 0 \\ 2 & 54 & -15 \end{bmatrix} \begin{matrix} \\ \\ R_3 - \frac{2}{3}R_1 \\ R_4 - \frac{2}{3}R_1 \end{matrix}$$

$$= \begin{bmatrix} 3 & -6 & 21 \\ 0 & 42 & -21 \\ 0 & 28 & -14 \\ 0 & 58 & -29 \end{bmatrix} \begin{matrix} \\ \\ R_3 - \frac{28}{42}R_2 \\ R_4 - \frac{58}{42}R_2 \end{matrix}$$

$$= \begin{bmatrix} 3 & -6 & 21 \\ 0 & 42 & -21 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$3C_1 - 6C_2 + 21C_3 = 0$

$42C_2 - 21C_3 = 0 \rightarrow C_2 = \frac{1}{2}C_3$

$C_1 = 2C_2 - 7C_3 = -6C_3$ # of Non Zero rows = 2

$C_2 = \frac{1}{2}C_3$

$$\begin{bmatrix} -6C_3 \\ \frac{1}{2}C_3 \\ C_3 \end{bmatrix}$$ ∴ The system Linearly dep.

②

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$\Downarrow$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ # of Non-Zero rows = 2

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7.4

Ex) $a_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 2 \end{bmatrix}$, $a_2 = \begin{bmatrix} -6 \\ 42 \\ 24 \\ 54 \end{bmatrix}$, $a_3 = \begin{bmatrix} 21 \\ -21 \\ 0 \\ -15 \end{bmatrix}$

Find the Linear dependencey of the following set of vectors & find the dimension.

$$\begin{bmatrix} 3 & -6 & 4 \\ 0 & 42 & -21 \\ 2 & 24 & 0 \\ 2 & 54 & -15 \end{bmatrix}$$

Dimension $\Rightarrow$ # of non zero rows.

$\Downarrow$

$$\begin{bmatrix} 3 & -6 & 4 \\ 0 & 42 & -21 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

* The system is Linearly dependent
* The dimension $= \underline{2}$

Ex) $a_1 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $a_3 = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix}$, $a_4 = \begin{bmatrix} 1/2 \\ 0 \\ 1/4 \end{bmatrix}$

Find the Linear dependencey of the Following set of vectors & Find the dimension.

$$\begin{bmatrix} 4 & 2 & 1 & 1/2 \\ 2 & 1 & 1/2 & 1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 - \frac{2}{4} R_1$$

$$= \begin{bmatrix} 4 & 2 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The system is Lineary dep.
Dimension $= 1$

7.4

Thm →
Dimension of A = Dimension of A^T
Rank of A = Rank of A^T

Dimension of set of vectors
= The rank of formed matrix

- Dimension of set of vectors is the # of linearly Independent vectors within the Origin set.

- To find the dimension of set of vectors:
1- We form a Matrix from the set of vectors & use G.E method till the end (Echelar form).
2- The Dimension = the # of non-zero rows.

- Rank of a matrix:-
The rank of matrix A is the maximum number of linearly Independent row vector (rank A).

Thm 1 → Row-equivalent matrices have the same rank

Thm 2 → Linear independence & dependence of vectors:-
Consider (P) vectors each vector consist of (n) components
- These vectors are Linearly independent if the matrix formed with these vectors has rank (P)

- These vectors are linearly dependent if rank < P
Thm3 → rank A = rank A^T

7.5 Solutions of linear systems:-
Existance & uniquness:-

1- Linear systems has a solution if & only if A & $\bar{A}$ have same rank ($\text{Rank } A = \text{Rank } \bar{A}$)

2- Uniquness: If $\text{rank } A = \text{rank } \bar{A} = n$ (no. of unknowns)

3- Infinity many solutions ($\text{rank } A = \text{rank } \bar{A} < n$)

Ex)
$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\text{rank } A = \text{rank } \bar{A} = 2$
$n = 3$

$2 < 3 \rightarrow$ Infinity many solutions.

7.6 Determinants. (square Matrices)

2nd Order

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Ex) $A = [-10] \rightarrow \det A = -10$

1st Order $|a_{11}| = a_{11}$

7.6

For 2nd Order

$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$$x_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Cramer's Rule.

Ex) $4x_1 + 3x_2 = 12$
$2x_1 + 5x_2 = -8$ $\longrightarrow \left[\begin{array}{cc|c} 4 & 3 & 12 \\ 2 & 5 & -8 \end{array}\right]$

$$x_1 = \frac{\begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{84}{14} = 6$$

$$x_2 = \frac{\begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{-56}{14} = -4$$

$(-1)^{j+k}$

3rd Order $\longrightarrow$ $\begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} \\ \overset{-}{a_{21}} & \overset{+}{a_{22}} & \overset{-}{a_{23}} \\ \overset{+}{a_{31}} & \overset{-}{a_{32}} & \overset{+}{a_{33}} \end{vmatrix}$

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

DATE 24-11-2015
Tuesday

"7.6"

Cramers Rule :-

$$X_1 = \frac{D_1}{D} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} / D$$

$$X_2 = \frac{D_2}{D} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} / D$$

$$X_3 = \frac{D_3}{D} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} / D$$

"7.7" Determinants (Camer's Rule)

* For $n=1 \Rightarrow D = a_{11}$
* For $n \geq 2 \Rightarrow D = a_{j1} C_{j1} + a_{j2} C_{j2} + \dots + a_{jn} C_{jn}$ $j = 0, 1, 2, 3 \dots$ or n

$$= \sum_{K=1}^{n} a_{jK} C_{jK} \quad j = 1, 2, 3, \dots \text{ or } n$$

OR $\Rightarrow D = a_{1K} C_{1K} + a_{2K} + C_{2K} + \dots + a_{nK} C_{nK}$ $K = 1, 2, 3, \dots$ or n

$$= \sum_{j=1}^{n} a_{jK} C_{jK} \quad K = 1, 2, 3 \dots \text{ or } n$$

$$C_{jK} = (-1)^{j+K} M_{jK}$$ where C_{jK} is Cofactor
M_{jK} is Minor

"7.7"

Ex) Evaluate

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1 \cdot \underbrace{\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix}}_{M_{11}} - 3 \underbrace{\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}}_{M_{12}} + 0 \underbrace{\begin{vmatrix} & \end{vmatrix}}_{M_{13}}$$

$$= 1(6 \times 2 - 4(0)) - 3(4+4) = 12 - 24 = -12$$

OR → $= 0 \underbrace{\begin{vmatrix} & \end{vmatrix}}_{M_{13}} - 4 \underbrace{\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}}_{M_{23}} + 2 \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}$

$$= -4(+3) + 2(0) = -12$$

※ $D = \sum_{K=1}^{n} (-1)^{j+K} a_{jK} M_{jK}$, $j = 0, 1, 2 \ldots$ or n

$D = \sum_{j=1}^{n} (-1)^{j+K} a_{jK} M_{jK}$ $K = 0, 1, 2 \ldots$ or n

※ $C_{jK} = (-1)^{j+K} M_{jK}$ → Cofactor

M_{jK} ⇒ Minor

Ex) $A = \begin{bmatrix} -1 & 1 & 6 \\ -3 & 7 & 0 \\ 0 & -2 & 5 \end{bmatrix}$, find cof A.

$C_{11} = (-1)^2 M_{11} = +1 \begin{vmatrix} 7 & 0 \\ -2 & 5 \end{vmatrix} = 35$

$C_{21} = (-1)^3 M_{21} = -1 \begin{vmatrix} 1 & 6 \\ -2 & 5 \end{vmatrix} = -17$

$C_{31} = (-1)^4 M_{31} = 1 \begin{vmatrix} 1 & 6 \\ 7 & 0 \end{vmatrix} = -42$

"7.7"

$C_{12} = -\begin{vmatrix} -3 & 6 \\ 0 & 5 \end{vmatrix} = 15$

$C_{22} = +\begin{vmatrix} -1 & 6 \\ 0 & 5 \end{vmatrix} = -5$

$C_{32} = -\begin{vmatrix} -1 & 6 \\ -3 & 0 \end{vmatrix} = -18$

$C_{13} = +| \quad |$

$C_{23} = -| \quad |$

$C_{33} = +| \quad |$

$$\text{cof } A = \begin{bmatrix} 35 & 15 & 6 \\ -17 & -5 & -2 \\ -42 & 18 & -4 \end{bmatrix}$$

Ex) $\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix} = -3\begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} + 0| \quad | - 0| \quad |$

$= -60$

OR $\rightarrow = 5\begin{vmatrix} -3 & 0 \\ 6 & 4 \end{vmatrix} = 5(-12) = -60$

※ $3|A| = 27(-60)$

Triangular Matrix $\rightarrow$ D = Multiplication of Main Diagonal

- General properties of determinant:

1) Interchange of two rows Multiplie the determinant by (-1)

2) Addition of a multiple of a row to another row does not alter the value of determinant.

3) Multiplication of a row by non-Zero Constant (c) $\rightarrow$ Multiplication the value of determinant by (c)

7.7

$3|A| = 3 \det A$

$|3A| = 3^n \det A$

4) The transposition Leaves the value of determinant unaltered

5) A Zero row or column renders the value of determinant to Zero

6) Proportional row or columns render the value of deteminant to zero.

Ex)

$$A = \begin{bmatrix} 3 & 1 & 0 & 5 \\ 3 & 1 & 4 & 2 \\ 2 & 1 & 4 & 0 \\ 1 & \frac{1}{2} & 2 & 0 \end{bmatrix}$$

Find det A=

det A=0 (row 3, row 4 are propotional).

$$B = \begin{bmatrix} 3 & 9 & 2 & 3 \\ 1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 6 & 18 & 5 & 5 \end{bmatrix}$$

$[\text{cof } A]^T = \text{Adj } A$ (Adjoint Matrix)

$A(\text{Adj } A) = (\text{Adj } A)A = (\det A)I$

$\Rightarrow A\left[\frac{\text{Adj } A}{\det A}\right] = \left[\frac{\text{Adj } A}{\det A}\right]A = I$

$A A^{-1} = A^{-1}A = I$

* Using G.E to find the determinant.

Ex)
$$\begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ \\ R_2 + \frac{3R_1}{2} \end{matrix}$$

$$\Rightarrow \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & \boxed{5} & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} \quad \begin{matrix} R_3 - \frac{2R_2}{5} \\ R_4 - \frac{8R_2}{5} \end{matrix}$$

$$\Rightarrow \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & \frac{12}{5} & \frac{19}{5} \\ 0 & 0 & \frac{-57}{5} & \frac{146}{5} \end{vmatrix} \quad R_4 + \frac{57}{12}R_3$$

$$\Rightarrow \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & \frac{12}{5} & \frac{19}{5} \\ 0 & 0 & 0 & 47.25 \end{vmatrix}$$

$\det = 2(5)(\frac{12}{5})(47.25)$

$= 1134$ ///

"7.8" Inverse of a Matrix.

The inverse ~~is~~ of $n \times n$ Matrix is denoted by A^{-1} which is also $n \times n$ matrix.

$AA^{-1} = A^{-1}A = I$, I is $n \times n$ unit or Identity Matrix.

One of the ways to find A^{-1} is:-

$$A^{-1} = \frac{\text{Adj} A}{\det A} = \frac{[\text{cof } A]^T}{\det A}$$

If A has an inverse, then A is called non-singular matrix
- If A has no inverse, then A is called Singular Matrix
- If A has inverse then the inverse is unique.

If B & C are inverse for A then:-
$AB = I$, $CA = I$

$\underline{\underline{B}} = IB = (CA)B = C(AB) = CI = \underline{\underline{C}}$

A has inverse if and only if it has maximum possible rank (n)

A

(non-Singular) has inverse → $\det A \neq 0$ → rank $= n$

has no inverse (singular) → $\det A = 0$ → rank $< n$

7.8

Ex) $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, find A^{-1} :-

$A^{-1} = \frac{Adj}{\det A} = \frac{[\text{cof} A]^T}{\det A}$ $\quad \det A = 10$

$\text{cof} A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} \Rightarrow [\text{cof} A]^T = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$ $\quad$ D.C. $AA^{-1} = 1$

Ex) $\begin{bmatrix} \overset{+}{-1} & \overset{-}{1} & \overset{+}{2} \\ \overset{-}{3} & \overset{+}{-1} & \overset{-}{1} \\ \overset{+}{-1} & \overset{-}{3} & \overset{+}{4} \end{bmatrix}$, find A^{-1}

$\det A = 10$ $\quad \text{cof} A = \begin{bmatrix} -7 & -13 & 8 \\ 2 & -2 & 2 \\ 3 & 7 & -2 \end{bmatrix}$

$[\text{cof} A]^T = \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix} = \text{adj} A$

$A^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$

2nd method to find A^{-1} :
Gauss - Jordan Elimination.

$[\underset{n\times n}{A} \;\vdots\; \underset{n\times n}{I}] \longrightarrow [I \;\vdots\; A^{-1}$

7.8

$$\underset{n\times n}{A} X = \underset{n\times n}{I}$$

$$A^{-1} A X = A^{-1} I \rightarrow I X = A^{-1} I$$

$$X = A^{-1}$$

Ex) $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$, Find A^{-1} using G.J Elimination

Sol →

$$\left[\begin{array}{ccc:ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2+3R_1 \\ R_3-R_1 \end{array}$$

$$\left[\begin{array}{ccc:ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] R_3-R_2$$

⇓

$$\left[\begin{array}{ccc:ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

⇓

$$\left[\begin{array}{ccc:ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{4}{5} & \frac{1}{5} & \frac{-1}{5} \end{array}\right] \begin{array}{l} R_1+2R_3 \\ R_2-3.5R_3 \end{array}$$

Pivot

7.8

⇓

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] R_1 + R_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

Reminder → $(AB)^T = B^T A^T$

$\det(AB) = \det A . \det B$

$(AB)^{-1} = B^{-1} A^{-1}$

$(ABC \ldots Q)^{-1} = Q^{-1} \ldots C^{-1} B^{-1} A^{-1}$

$AA^{-1} = I$

$\det AA^{-1} = \det I \rightarrow \det A . \det A^{-1} = 1$

$\det A^{-1} = \dfrac{1}{\det A}$

$(A^{-1})^{-1} = A$

→ Using Inverse Method to find the unique sol.

* $Ax = b$

$A^{-1}Ax = A^{-1}b \rightarrow x = A^{-1}b$

H.W → using Inverse Method to find the unknowns $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Ex)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

* For Diagonal matrix →
It has an inverse if $a_{jj} \neq 0$

A^{-1} is also diagonal with entries
$\frac{1}{a_{11}}, \frac{1}{a_{22}}, \ldots, \frac{1}{a_{nn}}$

has inverse (Invertible)

Ex) $A = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

, $A^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

* $\det(AB) = \det A . \det B$
$\det(A) = \det(A^T)$

unusual properties of matrix Multiplication.

1- Matrix Multiplication is not commutative

$$AB \neq BA$$

2- $AB = 0$, does not generally implies that $A=0$ or $BA=0$

Ex) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

7.8..

3) $AC = AB$, does not imply that $C = D$ even when $A \neq 0$

Cancelation Laws

Let A, B, C be $n \times n$ Matrices then:

* if $\operatorname{rank} A = n$ and $AB = AC$ then $B = C$

Proof $\rightarrow$ $A^{-1}AB = A^{-1}AC \rightarrow IB = IC \rightarrow B = C$

* if $\operatorname{rank} A = n$, then $AB = 0$ implies $B = 0$

Hence, If $AB = 0$, but $A \neq 0$ as well as $B \neq 0$ then $\operatorname{rank} A < n$, $\operatorname{rank} B < n$

Proof:

$AB = 0 \rightarrow A^{-1}AB = 0 \rightarrow IB = 0 \rightarrow B = 0$

CH8₁₁

Matrix Eigen Value Problems

→ They concern the solutions of vector equations

coef Matrix ← $Ax = \lambda x$

Scalar

vector of unknowns

$Ax - \lambda x = 0 \rightarrow [A - \lambda I]\, x = 0$ → Homogeneous system

infinite → $\det|A - \lambda I| = 0$ → eigen value

unique → $\det|A - \lambda I| \neq 0$ → trivial sol → $X = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

*1_ To find λ's :_ $\det(A - \lambda I) = 0$

2_ For each λ, we find the corresponding eigen vector.

Ex) $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$-5x_2 + 2x_2 = \lambda x_1 \rightarrow (-5 - \lambda)\, x_1 + 2x_2 = 0$

$2x_1 + (-2x_2) = \lambda x_2 \rightarrow 2x_1 + (-2 - \lambda)\, x_2 = 0$

$$\underbrace{\begin{bmatrix} -5 - \lambda & 2 \\ 2 & -2 - \lambda \end{bmatrix}}_{(A - \lambda I)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$$

"CH8"

* To Find λ's :-

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$(-5-\lambda)(-2-\lambda) - 4 = 0$

$10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$

$(\lambda+1)(\lambda+6) = 0 \Rightarrow \lambda_1 = -1$

$\lambda_2 = -6$

For $\lambda = -1$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Downarrow$ G.E

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow -4x_1 + 2x_2 = 0$$

$x_2 = 2x_1$

$\therefore x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t$

$\therefore$ For $\lambda = -1$ $\quad x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} \xRightarrow{\text{possible}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $x_1 = 1$

CH8

For $\lambda = -6$

$A + 6I$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Downarrow$ G.E

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$\therefore$ For $\lambda = -6$ Possible

$$X = \begin{bmatrix} x_1 \\ -\frac{1}{2} x_1 \end{bmatrix} \xRightarrow{\text{For } x=1} \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$$

Ex) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

* To find λ's $\rightarrow$ $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}$$

$(-2-\lambda)\left((1-\lambda)(-\lambda) - 12\right) - 2(-2\lambda - 6) - 3(-4 + (1-\lambda)) = 0$

$(-2-\lambda)(\lambda^2 - \lambda - 12) + 4\lambda + 12 + 12 + 3\lambda - 3 = 0$

$-2\lambda^2 + \cancel{2\lambda} + 24 - \lambda^3 + \cancel{\lambda^2} + \cancel{12\lambda} + \cancel{4\lambda} + 12 + 12 + \cancel{3\lambda} - 3 = 0$

$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$

$\lambda = 5 \rightarrow (\lambda - 5)(\text{2nd Order poly}) = 0$

	λ^3	λ^2	λ	λ^0
5	-1	-1	21	45
	0	-5	-30	-45
	-1	-6	-9	0

$(\lambda - 5)(-\lambda^2 - 6\lambda - 9) = 0$

$\lambda_1 = 5 \qquad \lambda_2 = \lambda_3 = -5$

"CH8"

For $\lambda = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \Rightarrow \text{G.E}$$

H.W

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\frac{24}{7} x_2 - \frac{48}{7} x_3 = 0 \rightarrow 24x_2 + 48x_3 = 0$$

$$x_3 = \frac{-1}{2} x_2$$

$$-7x_1 + 2x_2 + \frac{3}{2} x_2 = 0$$
$$-7x_1 + 7/2\, x_2 = 0 \rightarrow x_1 = \frac{1}{2} x_2$$

For $\lambda = 5 \rightarrow x = \begin{bmatrix} 0.5x_2 \\ x_2 \\ \frac{-x_2}{2} \end{bmatrix} \xRightarrow[\text{the sd}]{\text{one of}} \begin{bmatrix} 0.5 \\ 1 \\ -0.5 \end{bmatrix}$

For $\lambda = -3$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 + 2x_2 - 3x_3 = 0$$
$$x_1 = -2x_2 + 3x_3$$

For $\lambda = -3$

$$x = \begin{bmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Eigen vector basis

$$\underline{X} = \begin{bmatrix} \frac{1}{2} & 3 & -2 \\ 1 & 0 & 1 \\ \frac{-1}{2} & 1 & 0 \end{bmatrix}$$

"CH8"

Ex) Find the eigen value & eigen vectors for:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & -2 \\ 3 & 6 & 6 \end{bmatrix} \qquad (A - \lambda I)x = 0$$

→ To Find λ's :-

$$|A - \lambda I| = \det \begin{bmatrix} 3-\lambda & 0 & 0 \\ 2 & -1-\lambda & -2 \\ 3 & 6 & 6-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)\left((-1-\lambda)(6-\lambda) + 12\right) = 0$$

$$(3-\lambda)\ (-6 + \lambda - 6\lambda + \lambda^2 + 12) = 0$$

$$(3-\lambda)\ (\lambda^2 - 5\lambda - 6) = 0$$

$$\lambda_1 = \lambda_2 = 3, \quad \lambda_3 = 2$$

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & -2 \\ 3 & 6 & 3 \end{bmatrix}$$

$\Downarrow$ G.E

$$\begin{bmatrix} 2 & -4 & -2 \\ 3 & 6 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 - \frac{3}{2}R_1$$

$$= \begin{bmatrix} 2 & -4 & -2 \\ 0 & 12 & 6 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$$

for $\lambda = 3$

$$x = \begin{bmatrix} 2x_3 \\ -1/2\,x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cancel{0} \\ -\frac{1}{2}x_3 \\ x_3 \end{bmatrix}$$

$12x_2 + 6x_3 = 0$; $2x_1 - 4x_2 - 2x_3 = 0$
$x_3 = -2x_2$; $x_1 = -4x_3 - 2x_3 = -6x_3$

"CH8"

For $\lambda = 2$:-

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & -2 \\ 3 & 6 & 4 \end{bmatrix}$$

$\Downarrow$ G.E

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -2 \\ 0 & 6 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$

$-3x_2 - 2x_3 = 0 \rightarrow \quad x_2 = \frac{-2}{3} x_3$

$\therefore$ For $\lambda_3 = 2 \rightarrow \quad x = \begin{bmatrix} 0 \\ \frac{-2}{3} x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-2}{3} \\ 1 \end{bmatrix} t$

Eigen Vector basis

$$\underline{X} = \begin{bmatrix} 0 & 0 \\ \frac{-1}{2} & -2/3 \\ 1 & 1 \end{bmatrix}$$

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"CH8"

Ex) Show that $[1\ 1\ 1]^T$ is can eigen vector corresponding to the eigen value $\underline{\underline{2}}$ for the matrix:-

$$A = \begin{bmatrix} 4 & -1 & -1 \\ 0 & -1 & 3 \\ 1 & 3 & -2 \end{bmatrix}$$

$Ax = \lambda x$

$[A - \lambda I]x = 0$

$$A - \lambda I = \begin{bmatrix} 2 & -1 & -1 \\ 0 & -3 & 3 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

Ex) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$

$\lambda^2 + 1 = 0 \rightarrow \lambda = \pm i$

for $\lambda = i \rightarrow (A - \lambda I)$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} R_2 + \frac{-1}{i} R_1$$

$-1 + i(-i)$

$$= \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \rightarrow -ix_1 + x_2 = 0$$

$x_1 = -ix_2$

$\therefore$ for $\lambda = i \rightarrow x = \begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix}$

for $\lambda = -i \rightarrow \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \xrightarrow[G.E]{} \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \rightarrow ix_1 + x_2 = 0$

$x_1 = ix_2$

$\therefore$ for $\lambda = -i$

$x = \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} t$

CH8

The eigen vector basis

$$X = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{32} & a_{33} \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

$\therefore A^T = A^{-1} \rightarrow$ Orthogonal Matrix

$\frac{1}{2}(A+A^T) \rightarrow$ symmetric $\qquad \frac{1}{2}(A-A^T) \rightarrow$ Skew

Special Matrices:

- Symmetric $\quad A^T = A$
- Skew $\quad A^T = -A$
- Orthogonal $\quad A^T = A^{-1}$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \rightarrow \text{Orthogonal}$$

* Any Real square matrix A may be written as the sum of symmetric R and Skew-Symmetric (S)

$$R = \frac{1}{2}(A + A^T) \quad , \quad S = \frac{1}{2}(A - A^T)$$

$$A = R + S$$

"CH8"

- Theorem → a- the eigen values of Symmetric Matrix are real
  b- The eigen values of skew symmetric are zero or pure imaginary

Ex) $A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$ H.W ⇒ $\lambda_1 = 0$
$\lambda_2 = \lambda_3 = \pm 25i$

- Orthogonal matrices
- It's determinant +1 or -1
- Eigen values of an Orthogonal matrix A are real or complex conjugates is pairs and have absolute value 1

$$A^T = A^{-1}$$
$$\det A^T = \frac{1}{\det A} \rightarrow \det{}^2 A = 1$$
$$\det A = \pm 1$$

Ex) $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \end{bmatrix}$ $\det A = -1$

$-\lambda^3 + \frac{2}{3}\lambda^2 + \frac{2}{3}\lambda - 1 = 0$, $\lambda_1 = -1$, $\lambda_2 = \frac{5 + i\sqrt{11}}{6}$

$\lambda_3 = \frac{5 - i\sqrt{11}}{6}$

$|\lambda_1| = |\lambda_2| = |\lambda_3| = 1$

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CH8

- Diagolization of Matrices :-

Similar Matrices

A $n \times n$ matrix $\hat{A}$ is called Similar to an $n \times n$ Matrix A If $\boxed{\hat{A} = P^{-1} A P}$ where P is a non-singular $n \times n$ matrix

Thm → If $\hat{A}$ is similar to A, then $\hat{A}$ has the same eigen values as A

- If x is an eigen vector for A, then $y = P^{-1} x$ is the eigen vector for $\hat{A}$ corresponding for the same eigen value.

Ex) Eigen values & Eigen vectors for similar Matrices.

$A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ → $P^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$

$\hat{A} = P^{-1} A P = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} P$

$= \begin{bmatrix} 12 & -9 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

CH.8

Any P $\rightarrow$ $P = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $|P| = 2 \neq 0$ (non_Singular)

$P^{-1} = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

$\hat{A} = P^{-1}AP = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -2 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -7 & -10 \\ 9 & 12 \end{bmatrix}$

Many Similars

Back to the $\hat{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

for $A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$

1) $\begin{vmatrix} 6-\lambda & -3 \\ 4 & -1-\lambda \end{vmatrix} = 0$

$(6-\lambda)(-1-\lambda) + 12 = 0 \rightarrow +6 - 6\lambda + \lambda + \lambda^2 = 0$

$\lambda^2 - 5\lambda + 6 = 0 \rightarrow (\lambda - 3)(\lambda - 2) = 0$

$\lambda_1 = 3 \quad , \quad \lambda_2 = 2$

$\hat{A} - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} \Rightarrow |\hat{A}| = (3-\lambda)(2-\lambda) = 0$

$\lambda_1 = 3, \quad \lambda_2 = 2$

The Same.

2) Corresponding eigen vector:

$\lambda = 3 \rightarrow \begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix} \xrightarrow{G.E} \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \rightarrow 3x_1 - 3x_2 = 0$

$x_1 = x_2$

for $\lambda = 3 \rightarrow x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$

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For $\lambda_2 = 2 \rightarrow \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix} \xrightarrow{G.E} \begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}$

$4x_1 - 3x_2 = 0 \rightarrow x_1 = \frac{3x_2}{4}$

$\therefore$ For $\lambda_2 = 2$

$x = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} t = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

2) eigen vectors for $\hat{A}$

$\lambda_1 = 3$ $\quad \hat{A} - 3I$

$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow 0x_1 - x_2 = 0$

$x_2 = 0$

For $\lambda_1 = 3$ $\quad y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$

$\lambda_2 = 2$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 + 0x_2 = 0$

$x_1 = 0$

$\therefore$ For $\lambda_2 = 2$

$y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$

OR $\rightarrow$ By Thm

For $\lambda = 3$ $\quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow y = P^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda = 2 \rightarrow x = \begin{bmatrix} \cancel{4} \; 3 \\ 4 \cancel{3} \end{bmatrix} \rightarrow y = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ +1 \end{bmatrix}$

"CH8"

Similar Matrices

$\hat{A} = P^{-1} A P$

- The same λ's

$y = P^{-1} x$

- Special case for selecting P

$\underset{n\times n}{P} = \underset{n\times n}{X}$ (Eigen vector basis of A)

$\hat{A} = D = X^{-1} A X = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$

IF X is not $n\times n$ then Matrix A cannot be diagonalized.

Ex) Diagonalize $A = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$

$\det A = 0 \rightarrow -\lambda^3 - \lambda^2 + 12\lambda = 0$

$-\lambda(\lambda^2 + \lambda - 12) = 0$

$\lambda_1 = 3, \quad \lambda_2 = -4, \quad \lambda_3 = 0$

$\lambda_1 = 3$ $\quad \lambda_2 = -4$ $\quad \lambda_3 = 0$

$X = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

$X = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ Diagonalizable $X^{-1} = \begin{bmatrix} -0.2 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

. CH 8 .

Ex) Diagonalize

The examples in this page have been Solved before.

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \rightarrow \det A = 0$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\lambda_1 = 5 \quad , \quad \lambda_2 = \lambda_3 = -3$$

$$\lambda_1 = 5 \rightarrow x = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \lambda_3 = -3 \rightarrow x = \begin{bmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow X^{-1} = \begin{bmatrix} \text{H.W} \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Ex) Diagonalize $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & -2 \\ 3 & 6 & 6 \end{bmatrix}$

$$|A| = 0 \rightarrow \lambda_1 = \lambda_2 = 3 \qquad \lambda_3 = 2 \rightarrow \lambda_1, \lambda_2 \Rightarrow x = \begin{bmatrix} 0 \\ -\frac{x_3}{2} \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & -\frac{2}{3} \\ 0 & 1 \end{bmatrix} \quad 3 \times 2$$

Cannot be diagolizer

$$\lambda_3 \rightarrow x = \begin{bmatrix} 0 \\ -\frac{2x_3}{3} \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$