ElCoM
ELECTRICAL COMPUTER MECHATRONICS

تقدم لجنة ElCoM الاكاديمية

تلخيص فيديوهات:

رياضيات تطبيقية

جزيل الشكر للطالب:

قتيبة الكعابنة

Complex number

$x^2+1=0$

$x^2=-1$

$x=\sqrt{-1}$

$\sqrt{-1}=i, j$

$i^2=-1$

$i^3=i^2 * i=-i$

$i^4=i^2 * i^2=1$

$\frac{1}{i}=-i \quad =\frac{i^4}{i}=-i$

Ex:-

$\sqrt{-25}=\sqrt{-1} * \sqrt{25}=5i$

$\sqrt{-9}=\sqrt{-1} * \sqrt{9}=3i$

$z=x+iy$

x :- Real part

y :- imaginary part

=> العمليات الحسابية

1) جمع, طرح

$z_1=x_1+iy_1$

$z_2=x_2+iy_2$

$(z_1 \pm z_2)=(x_1 \pm x_2)+i(y_1 \pm y_2)$

Ex:-

$z_1=8+3i$

$z_2=9-2i$

$z_1+z_2=17+i$

$z_1-z_2=-1+5i$

* ضرب

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$(x_1 + iy_1) * (x_2 + iy_2)$

$x_1 x_2 + \underline{i y_2 x_1} + \underline{x_2 y_1 i} + \underbrace{i^2}_{-1} y_1 y_2$

$(x_1 x_2 - y_1 y_2) + i(y_2 x_1 + x_2 y_1)$

Ex!.

$z_1 = 8+3i \qquad (z_1 * z_2)$

$z_2 = 9-2i$

$(8+3i) * (9-2i)$

$72 - 16i + 27i - 6\underbrace{i^2}_{-1}$

$(72+6) + 11i$

$(78 + 11i)$

Ex!.

$(4+5j)(3-2j)$

$\cancel{21}\ 12 - 8j + 15j - 10\underbrace{j^2}_{-1}$

$12 + 10 + 7j$

$22 + 7j$

$*$ القسمة

$$\frac{(x_1+iy_1)}{x_2+iy_2} * \frac{(x_2-iy_2)}{x_2-iy_2}$$

$$\frac{x_1x_2 - iy_2x_1 + iy_1x_2 - y_1y_2i^2}{x_2^2 - \cancel{iy_2x_2} + \cancel{iy_2x_2} - y_2^2i^2}$$

$$\frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - y_2x_1)}{x_2^2 + y_2^2}$$

Ex :-

$z_1 = 8+3i$ $\qquad \frac{z_1}{z_2}$

$z_2 = 9-2i$

$$\frac{(8+3i)}{(9-2i)} * \frac{(9+2i)}{(9+2i)}$$

$$\frac{72 + 18i + 27i - 6}{81 + \cancel{18i} - \cancel{18i} + 4}$$

$$\frac{66+43i}{85} = \frac{66}{85} + \frac{43i}{85}$$

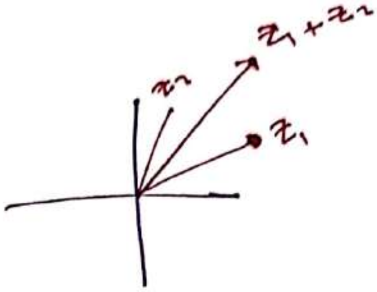
Ex:- $-3 + 4i$

$x + iy$

$4 = y$

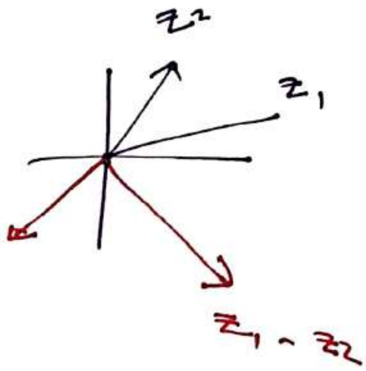
$-3 = x$

$z_1 + z_2$

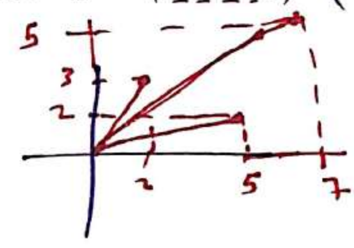


$z_1 - z_2$

$z_1 + (-z_2)$



Ex:- $(\overset{x}{5} + \overset{y}{2}j) + (\overset{x}{2} + \overset{y}{3}j)$

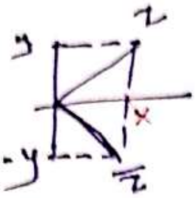


$\underline{7 + 5j}$

Complex conjugate num

$z = x + iy$

$\bar{z} = x - iy$



Ex!. find $\bar{z}$

$z = 3 + 5i \qquad \bar{z} = 3 - 5i$

Ex!. prove the following

1) $z * \bar{z} = x^2 + y^2$

2) $z + \bar{z} = 2x$

3) $z - \bar{z} = 2iy$

sol:.

1)

$z = x + iy \qquad (z * \bar{z})$

$\bar{z} = x - iy \qquad (x+iy) * (x-iy)$

$x^2 \cancel{- iyx} + \cancel{iyx} - y^2 \underset{-1}{i^2}$

$x^2 + y^2$ \#

2) $x + \cancel{iy} + x - \cancel{iy} = 2x$

3) $\cancel{x} + iy - \cancel{x} + iy = 2iy$

1) $\operatorname{Re} z = x = \frac{z+\bar{z}}{2}$ $\quad z = \underline{x} + iy$

2) $\operatorname{Img} z = y = \frac{z-\bar{z}}{2i}$

* $z = x$

$\bar{z} = x = z$

* $z = iy$

$\bar{z} = -iy = -z$

Ex!-

$z_1 + \bar{z}_1 = 8$ $\quad z_2 + \bar{z}_2 = -2$

$\frac{z_1 - \bar{z}_1}{i} = 6$ $\quad \frac{z_2 - \bar{z}_2}{i} = 22$

find $(z_2 - z_1)^2$

$z = x + iy$

$x_1 = \frac{z+\bar{z}}{2} = \frac{8}{2} = \underline{\underline{4}}$

$y_1 = \frac{z-\bar{z}}{2i} =$ $\quad 2y_1 = \frac{z-\bar{z}}{i}$

$2y_1 = 6$

$y_1 = 3$

$\boxed{z_1 = 4 + 3i}$

$x_2 = \frac{z_2 + \bar{z}_2}{2} = \frac{-2}{2} = -1$

$2y_2 = \frac{z_2 - \bar{z}_2}{i}$ $\quad 2y_2 = 22$

$y_2 = 11$

$\boxed{z_2 = -1 + 11i}$

$-1 + 11i - 4 - 3i$

$-5 + 8i$

$(z_2 - z_1)^2 = (-5 + 8i)^2$

$25 - 80i + 64i^2$ $\quad i^2 = -1$

$25 - 80i - 64$

$\boxed{-39 - 80i}$

1) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

2) $\overline{(z_1 * z_2)} = \bar{z}_1 * \bar{z}_2$

3) $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$

Ex:-

$z_1 = 4+3i$

$z_2 = 2+5i$

find 1) $\overline{(z_1 + z_2)}$

2) $\overline{(z_1 * z_2)}$

sol:-

$\bar{z}_1 = 4-3i$

$\bar{z}_2 = 2-5i$

1) $\bar{z}_1 + \bar{z}_2 = 6-8i$

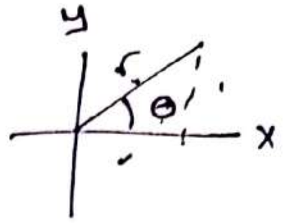
2) $(4-3i)(2-5i) = 8-20i-6i+15i^2$

$= 8-15-26i$

$= -7-26i$

polar

$$z = x + iy$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$r = \sqrt{x^2+y^2}$$

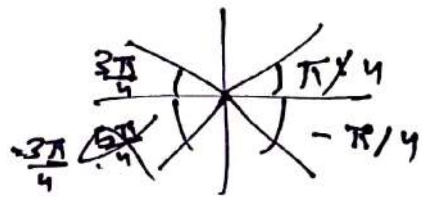
$$z = r\cos\theta + i r\sin\theta$$
$$= r[\cos\theta + i\sin\theta]$$

$$z = re^{i\theta}$$

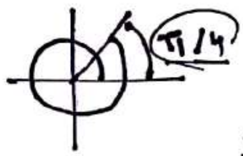
Argument $\xrightarrow{}$ θ

Arg , arg

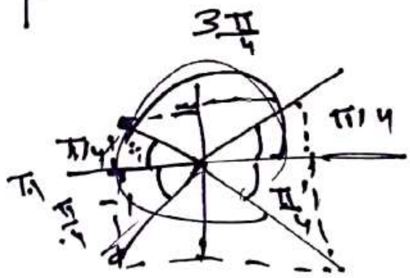
Arg $\Rightarrow$ θ $\quad -\pi \leq \text{Arg} \leq \pi$



$\underline{\underline{\text{arg}}} = \text{Arg} \underline{\pm 2n\pi}$



Ex:-



1) $z = 1+i$ $\quad$ 1) $\text{Arg} = \theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$

2) $z = -1+i$ $\quad$ 2) $\text{Arg} = \theta = \tan^{-1} 1 = \frac{\pi}{4}$

$$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

3) $z = -1-i$

4) $z = 1-i$ $\quad$ 3) $\text{Arg} = \theta = \tan^{-1} 1 = \frac{\pi}{4}$

$$-\frac{3\pi}{4}$$

4) $\text{Arg} = \theta = \tan^{-1} 1 = \frac{\pi}{4}$

$$-\frac{\pi}{4}$$

Conj polar

$$z = re^{i\theta}$$

$$\bar{z} = re^{-i\theta}$$

$$\bar{z} = r[\cos\theta - \sin\theta]$$

Ex:- 1) $z = 1+i$

2) $z = 3+3\sqrt{3}i$

find the polar

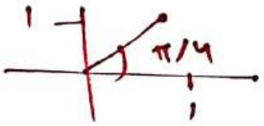
Sol:-

1) $z = r[\cos\theta + i\sin\theta] = re^{i\theta}$

$r = \sqrt{x^2+y^2}$

$= \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$

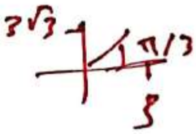


$$z = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \sqrt{2}\,e^{i\frac{\pi}{4}}$$

2)

$r = \sqrt{9+(9\times 3)} = \sqrt{36} = 6$

$\theta = \tan^{-1}\frac{3\sqrt{3}}{3} = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$



$$z = 6\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

$$= 6e^{i\frac{\pi}{3}}$$

Ex:. $|z_1 + z_2| \leq |z_1| + |z_2|$ اثبت

$r = \sqrt{x^2 + y^2}$

$z_1 = 1 + i$

$z_2 = -2 + 3i$

sol:.

$z_1 + z_2 = -1 + 4i$

$|z_1 + z_2| = \sqrt{(1)^2 + (4)^2} = \sqrt{17}$

$|z_1| = \sqrt{2}$

$|z_2| = \sqrt{4 + 9} = \sqrt{13}$

$|z_1| + |z_2| = \sqrt{2} + \sqrt{13} = 5,02$

$\sqrt{17} = 4,12$

$|z_1| + |z_2| \geq |z_1 + z_2|$

$5,02 > 4,12$

$\Rightarrow$ الضرب

$z_1 = r_1 [\cos\theta_1 + i\sin\theta_1]$

$z_2 = r_2 [\cos\theta_2 + i\sin\theta_2]$

$z_1 * z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$|z_1 * z_2| = r_1 r_2$

$\mathrm{Arg}(z_1 * z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$

* القسمة

$z_1 = r_1 [\cos\theta_1 + i\sin\theta_2]$

$z_2 = r_2 [\cos\theta_2 + i\sin\theta_2]$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$ $\quad$ $\operatorname{Arg}\frac{z_1}{z_2} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$

Ex!.

$z_1 = -2 + 2i$ $\quad$ find $z_1 * z_2$

$z_2 = 3i$ $\quad$ z_1 / z_2

in polar

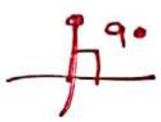
Sol!-

$r_1 = \sqrt{(-2)^2 + (2)^2} = \sqrt{8}$

$r_2 = \sqrt{(3)^2} = 3$

$\theta_1 = \tan^{-1}\frac{2}{-2} = \cancel{\frac{\pi}{4}}$

$\frac{3\pi}{4}$



$\theta_2 = \frac{\pi}{2}$

1) $z_1 * z_2 = r_1 r_2 [\cos[\theta_1 + \theta_2] + i\sin(\theta_1 + \theta_2)]$

$3\sqrt{8}\ [\cos(\frac{3\pi}{4} + \frac{\pi}{2}) + i\sin(\frac{3\pi}{4} + \frac{\pi}{2})]$

2) $z_1 / z_2 = \frac{\sqrt{8}}{3}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{2}\right)\right]$

$\frac{\pi}{4}$

power

$z^n = r^n[\cos n\theta + i\sin n\theta] = r^n e^{in\theta}$

Ex!. find z^{10}

$z = (\frac{1}{2} + \frac{1}{2}i)$

sol!-

$r = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2}}$

$\theta = \tan^{-1}\frac{\frac{1}{2}}{\frac{1}{2}} = \pi/4$

$z = re^{i\theta}$

$z = \sqrt{\frac{1}{2}}\, e^{i\frac{\pi}{4}} \Rightarrow z^{10} = (\sqrt{\frac{1}{2}})^{10} * e^{\cancel{10}^{5}\frac{\pi}{\cancel{4}_{2}} i}$

$= \frac{1}{32} * e^{\frac{\pi}{2}i}$

$-\pi < Arg \leq \pi$ $\quad \frac{5\pi}{2}$

-2π

$+2\pi$

$\frac{1}{32} * e^{i\frac{\pi}{2}} = \frac{1}{32}[\underset{}{\overset{\text{zero}}{\cos\frac{\pi}{2}}} + i\overset{1}{\sin\frac{\pi}{2}}]$

$\frac{5\pi}{2} - 2\pi = \frac{\pi}{2}$

$z^{10} = \frac{i}{32}$

Root

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\frac{\theta+2\pi K}{n} + i\sin\frac{\theta+2\pi K}{n}\right] \qquad K = 0, 1 \cdots n-1$$

$n = 3$

$K = 0, 1, 2$

$n = 4$

$K = 0, 1, 2, 3$

Ex:- find $\sqrt[3]{z}$

$\cancel{z = \sqrt[3]{1+i}}$

$z = 1 + i$

$\sqrt[3]{z} = \sqrt[3]{1+i}$

$r = \sqrt{2}$

$\theta = \tan^{-1}\frac{1}{1} = \pi/4$

$n = 3$

$K = 0, 1, 2$

$K = 0 \quad \sqrt[3]{(\sqrt{2})}\left[\cos\frac{\frac{\pi}{4} + \cancel{2\pi\times 0}}{3} + i\sin\frac{\frac{\pi}{4}}{3}\right]$

$\sqrt[6]{2}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]$

$K = 1 \quad \sqrt[6]{2}\left[\cos\frac{\frac{\pi}{4} + 2\pi}{3} + i\sin\frac{\frac{\pi}{4} + 2\pi}{3}\right]$

$K = 2 \quad \sqrt[6]{2}\left[\cos\frac{\frac{\pi}{4} + 4\pi}{3} + i\sin\frac{\frac{\pi}{4} + 4\pi}{3}\right]$

Ex:- $z = \sqrt{3} - i$

find second root of $\sqrt[3]{z}$,

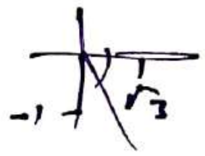
$n = 3$

$k = 0, 1, 2$

sol.

$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

$\theta = \tan^{-1}\frac{1}{\sqrt{3}} = \pi/6$



$-\frac{\pi}{6}$

$n = 3$

$k = 1$

$$\sqrt[3]{z} = \sqrt[3]{2}\left[\cos\frac{-\frac{\pi}{6}+2\pi}{3} + i\sin\frac{-\frac{\pi}{6}+2\pi}{3}\right]$$

Exponential function

$z = x + iy$ → Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{z} = e^{x+iy} = e^{x} * e^{iy} = e^{x}[\cos y + i\sin y]$

* properties of e^{z}

1) $(e^{z})' = e^{z}$

2) $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$

$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

3) $|e^{iy}| = 1$

$\cos y + i\sin y$

$\sqrt{(\cos y)^2 + (\sin y)^2}$

$\sqrt{1} = 1$

4) $|e^{z}| = e^{x}$

$|e^{x} \cdot e^{iy}| = e^{x}$

5) $\arg z = y \pm 2\pi n$

Ex!-

1) $e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1$

2) $e^{\frac{\pi}{2}i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

3) $e^{\pi i} = \cos\pi + i\sin\pi = -1$

4) $e^{-\frac{\pi}{2}i} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$

cos − sin +	cos + sin +
cos − sin −	cos + sin −

Ex:- $Z = 1,4 - 0,6i$

find e^{z}

$$e^{z} = e^{1,4-0,6i} = e^{1,4} \cdot e^{-0,6i} = e^{1,4}[\cos \underset{\text{rad}}{0,6} - i\sin 0,6]$$

$$= 3,347 - 2,289i$$

Ex:- $Z = 5+2i$

find e^{z}

$$e^{z} = e^{5+2i} = \underline{e^{5}[\cos 2 + i\sin 2]}$$

Ex:- solve $e^{z} = 3+4i$

$Z = x+iy$

$$|e^{z}| = e^{x} = \sqrt{x^2+y^2} = \sqrt{9+16} = 5 \qquad e^{x} = 5$$

$$x = \ln 5$$

$$\arg e^{z} = y \pm 2\pi n$$

$$\tan^{-1}\frac{4}{3} = 53,1^{\circ} = y$$

$$53,1 * \frac{\pi}{180} = y \qquad Z = \ln 5 + [0,927 \pm 2\pi n]i$$

$$\underline{y = 0,927 \pm 2\pi n}$$

Trigonometric Function

$$e^{iz} = \cos z + i\sin z$$

$$e^{-iz} = \cos z - i\sin z$$

$$\rightarrow e^{iz} + e^{-iz} = \cos z + \cancel{i\sin z} + \cos z - \cancel{i\sin z} = 2\cos z$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{---} \; 1$$

$$\rightarrow e^{iz} - e^{-iz} = \cancel{\cos z} + i\sin z - \cancel{\cos z} + i\sin z = 2i\sin z$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{---} \; 2$$

1) $\tan z = \dfrac{\sin z}{\cos z}$

3) $\sec z = \dfrac{1}{\cos z}$

2) $\cot z = \dfrac{\cos z}{\sin z}$

4) $\csc z = \dfrac{1}{\sin z}$

Hyperbolic Function

1) $\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)$

2) $\cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$

3) $\cosh iz = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos z$

4) $\sinh iz = \frac{1}{2}\left(e^{iz} - e^{iz}\right) = i\sin z$

5) $\cos iz = \cosh z$

6) $\sin iz = i\sinh z$

7) $\tanh z = \dfrac{\sinh z}{\cosh z}$

8) $\cot z = \dfrac{\cosh z}{\sinh z}$

9) $\operatorname{sech} = \dfrac{1}{\cosh z}$

10) $\csc z = \dfrac{1}{\sinh z}$

11) $\cos z = \cos x \cosh y - i\sinh y \sin x$

12) $\sin z = \sin x \cosh y + i\cos x \sinh y$

13) $|\cos z|^2 = \cos^2 x + \sinh^2 y$

14) $|\sin z^2| = \sin^2 x + \sinh^2 y$

Ex:-

solve $\cos z = 5$

sol:- $\underline{z = x + iy}$

$\cancel{z = x}$

$$5 = \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$(2\cos z = e^{iz} + e^{-iz}) \times e^{iz}$$

$$(10 = e^{iz} + e^{-iz}) \times e^{iz}$$

$$10 e^{iz} = \cancel{10} e^{2iz} + 1$$

$$e^{2iz} - 10 e^{iz} + 1 = 0 \qquad w = e^{iz}$$

$$w^2 - 10w + 1 = 0$$

$$w = \frac{10 \pm \sqrt{100 - 4 * 1 * 1}}{2 * 1} = 5 \pm \sqrt{24} = e^{iz}$$

$$e^{iz} = 5 \pm \sqrt{24}$$

$$e^{ix - y} = 5 \pm \sqrt{24}$$

$$e^{ix} \cdot e^{-y} = (5 \pm \sqrt{24}) * \frac{e^{i0}}{1}$$

$$e^{ix} = 1 \Rightarrow \underline{x = 0 \pm 2\pi n}$$

$$e^{-y} = 5 \pm \sqrt{24}$$

$$y = -\ln(5 + \sqrt{24}) = -2,29$$

$$y = -\ln(5 - \sqrt{24}) = 2,29$$

$$\underline{\underline{y = \pm 2,29}}$$

$$z = \pm 2\pi n + [\pm 2,29] i$$

log function

$z = x + iy$

$\ln z = \ln r + i\theta$ $\quad$ arg $\pm 2\pi n$

$r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\frac{y}{x}$

Principal

$Ln\, z = \ln r + \text{Arg}\,\theta$

*

1) $\ln(z_1 * z_2) = \ln z_1 + \ln z_2$

2) $\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$

Ex:- $\quad x+iy \quad r = \sqrt{(1)^2} = 1 \quad$ zero

1) $\ln(1) = \ln 1 \mp 2\pi n i$
$= \pm 2\pi n i$

$Ln(1) = \ln 1 + 0i$
$= 0$

2) $\ln 4 = \ln 4 \mp 2\pi n i$

$Ln(4) = \ln 4$
$= \ln 4$

3) $\ln(-1) = \overset{0}{\ln 1} + [\pi \pm 2\pi n]i$
$(2n+1)\pi i$

$Ln(-1) = \ln 1 + \pi i$
$= \pi i$

4) $\ln(i) = \ln 1 + [\frac{\pi}{2} \pm 2\pi n]i$

$Ln(i) = \ln 1 + \frac{\pi}{2} i$

5) $\ln(3-4i) = \ln 5 + [-0,927 \pm 2\pi n]i$; $Ln(3-4i) = \ln 5 - 0,927i$

$r = \sqrt{(3)^2 + (4)^2}$
$= 5$

$\theta = \tan^{-1}\frac{-4}{3} = -0,927$
$-53,1^\circ$

CamScanner المسوحة ضوئيا بـ

General power

$$z^c = e^{\ln z^c} = e^{c\ln z}$$

Ex!. i^i

$$e^{\ln i^i} = e^{i\ln i}$$

$$\ln i = \ln 1 + \left[\frac{\pi}{2} \pm 2\pi n\right] i$$

$$\ln i = \left[\frac{\pi}{2} \pm 2\pi n\right] i$$

$$e^{i\left[\frac{\pi}{2} \pm 2\pi n\right] i}$$

$$e^{-\frac{\pi}{2} \mp 2\pi n}$$

1) $Z = \underset{x - yi}{\pi - 3i}$ find $\sec z$

$$\sec z = \frac{1}{\cos z} = \frac{1}{\cos x \cosh y - i \sinh y \sin x}$$

$$= \frac{1}{\underset{-1}{\cos\pi}\cosh(-3) - i\,\cancel{\sinh(-3)\sin\pi}}$$

$$= \frac{1}{-\cosh(-3)}$$

2) $\csc z = \frac{1}{2}$ find e^{-iz}

$$\frac{1}{\sin z} = \frac{1}{2} \qquad \sin z = 2$$

$$\frac{e^{iz} - e^{-iz}}{2i} = 2 \Rightarrow e^{iz} - e^{-iz} = 4i \quad * e^{-iz}$$

$$1 - e^{-2iz} = 4i\, e^{-iz}$$

$$e^{-2iz} + 4e^{-iz} - 1 = 0 \qquad w = e^{-iz}$$

$$w^2 + 4iw - 1 = 0$$

$$w = \frac{-4i \pm \sqrt{-16 - 4 \times -1}}{2}$$

$$= -2i \pm i\sqrt{3}$$

$$= [-2 \pm \sqrt{3}]\, i$$

find

3) $(a+bi)^{4n+2} + (b-ai)^{4n+2} + 2i$ a, b real

and n is integer positive number

Sol:-

$z = a + bi$

$-z = -a - bi$

$-iz = b - ai$

$(z)^{4n+2} + (-iz)^{4n+2} + 2i$ $n=1$, $n=2$

$z^6 + (iz)^6 + 2i$ $i^6 = i^4 * i^2 = -1$

$z^6 + -1\, z^6 + 2i$

$\cancel{z^6} - \cancel{z^6} + 2i$

$\underline{\underline{2i}}$

4) ~~$(1+i)^{5i}$ fin~~

$z = (1+i)^{5i}$ find z

$e^{\ln(1+i)^{5i}} = e^{5i\ln(1+i)}$

$\theta = \tan^{-1}\frac{1}{1} = \pi/4$

$\ln(1+i) = \ln r + i\theta$

$= \ln\sqrt{2} + \left[\frac{\pi}{4} \pm 2\pi n\right] i$

$= \frac{1}{2}\ln 2 + \left[\frac{\pi}{4} \pm 2\pi n\right] i$

$5i \times \ln[1+i] = \frac{5i}{2}\ln 2 - \frac{5\pi}{4} \mp 10\pi n$

$\underline{e^{\frac{5i}{2}\ln 2 - \frac{5\pi}{4} \mp 10\pi n}}$

$\frac{5i}{2}\ln 2 - \frac{5\pi}{4} \mp 10\pi n$

5) $z = \dfrac{3-3i}{1+i}$

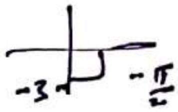
Find $\ln z$

sol:-

$$\frac{3-3i}{1+i} \times \frac{1-i}{1-i}$$

$$\frac{\cancel{3} - 3i - 3i - \cancel{3}}{2} = -3i$$

$$\ln(-3i) = \ln r + i\theta$$



$$= \ln 3 + \left[-\frac{\pi}{2} \pm 2\pi n\right] i$$

6) $a = 3-3i$, $b = 1-5i$, $c = -2-i$

Find $(2a-c)^2$

sol:-

$$6 - 6i + 2 + i = 8 - 5i$$

$$2a - c = \cancel{8 - 8i} \;\; 8 - 5i$$

$$(2a-c)^2 = (8-5i)^2 = 64 - 80i - 25$$

$$= 39 - 80i$$

7) $a = 5\sqrt{2}\, e^{-\frac{\pi}{4}i}$ find $|a-d|$

$d = 3\sqrt{2}\, e^{-\frac{\pi}{4}i}$

$c = 1 - i$

sol:-

$a = 5\sqrt{2}\, e^{-\frac{\pi}{4}i} - 3\sqrt{2}\, e^{-\frac{\pi}{4}i}$

$|a-d| = \overset{r}{2\sqrt{2}}\, e^{-\frac{\pi}{4}i}$ $= r e^{i\theta}$

$= \underline{\underline{2\sqrt{2}}}$

=

$a = 5\sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$

$= 5 - 5i$

$d = 3\sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$

$= 3 - 3i$

$= 2 - 2i$

$\sqrt{(2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8}$

$= \sqrt{2 \times 4}$

$= \sqrt{2} \times 2$

$= \underline{\underline{2\sqrt{2}}}$

8) $z_1 = 5+3i$ find k = ??

$z_2 = 1+ki$

$\arg \frac{z_2}{z_1} = \frac{\pi}{4}$

sol:-

$$\frac{z_2}{z_1} = \frac{1+ki}{5+3i} * \frac{5-3i}{5-3i} = \frac{5-3i+5ki+3k}{25+9} = \frac{5+3k}{36} + i\left[\frac{5k-3}{36}\right]$$

$$\tan^{-1} \frac{5k-3}{5+3k} = \frac{\pi}{4}$$

$$\tan \tan^{-1} \frac{5k-3}{5+3k} = \tan\frac{\pi}{4}$$

$$\frac{5k-3}{5+3k} = \frac{1}{1} \qquad 5+3k = 5k-3$$

$$8 = \cancel{8}2k$$

$$\underline{k=4}$$

9) $z_1 = 3-\sqrt{3}i$

$z_2 = \sqrt{3}-i$

find $\ln(z_1) - \ln(z_2)$

$r = \sqrt{x^2+y^2}$

$\theta = 0 \pm 2\pi n$

$\ln \frac{z_1}{z_2}$

$\ln\sqrt{3} = \ln r + i\theta$

$\ln\sqrt{3} = \underline{\ln\sqrt{3} \mp 2\pi n}$

$$\frac{3-\sqrt{3}i}{\sqrt{3}-i} * \frac{\sqrt{3}+i}{\sqrt{3}+i}$$

$$\frac{3\sqrt{3}+\cancel{3i}-\cancel{3i}+\sqrt{3}}{3+1} = \frac{4\sqrt{3}}{4} = \sqrt{3}$$

10) if $(3+i)$ is a root for the quadric equation

$z^2 - (a+2i)z + (2b+i)$ then b is

Sol:-

$z = 3+i$

$(3+i)^2 - (a+2i)(3+i) + 2b+i = 0$

$9 + 6i - 1 - [3a + ai + 6i - 2] + 2b + i = 0$

$8 + \cancel{6i} - 3a - ai - \cancel{6i} + 2 + 2b + i$

$[10 - 3a + 2b] + i[1-a] = 0 + 0i$

x $\qquad$ y

$1 - a = 0$

$a = 1$

$10 - 3a + 2b = 0$

$7 + 2b = 0 \qquad b = -3{,}5$

11) z find $2\left[\frac{1+i}{\sqrt{2}}\right]^{8n} + 2\left[\frac{1-i}{\sqrt{2}}\right]^{8n}$ $\qquad$ n:- integer number

$n = 1$

$$2\left[\frac{[1+i]^8}{16} + \frac{(1-i)^8}{16}\right]$$

$z = [1+i]^8 =$

$r = \sqrt{2}$ $\qquad [\sqrt{2} \cdot e^{\frac{i\pi}{4}}]^8 = 16e^{2\pi i} = 16[\cos 2\pi + i\sin 2\pi]$

$\theta = \frac{\pi}{4}$ $\qquad = 16$

$z = 1-i$ $\qquad [\sqrt{2}\, e^{-\frac{\pi}{4}i}]^8 = 16e^{-2\pi i} = 16[\cos 2\pi - i\sin 2\pi]$

$r = \sqrt{2}$ $\qquad = 16$

$\theta = -\frac{\pi}{4}$

$$2\left[\frac{16}{16} + \frac{16}{16}\right] = \underline{4}$$

Matrices

$$\begin{bmatrix} 0.3 & 4 & 6 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

* الأعداد الموجودة داخل الماتريكس
تسمى ب elements

* أهم تطبيقات الماتريكس
حل المعادلات الخطية

$$4x_1 + 6x_2 + 9x_3 = 6$$
$$6x_1 - 2x_3 = 20$$
$$5x_1 - 8x_2 + x_3 = 10$$

$$A = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix}$$

Augmented matrix $$\tilde{A} = \left[\begin{array}{ccc|c} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{array}\right]$$

* ملاحظات

1) يجب ان نشير إلى الماتريكس ب أحرف كبيرة A, B, C

أو $A = [a_{jk}]$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{array}$$

3×3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

2×3

2) $m \times n$

صفوف m :- rows

أعمدة n :- columns

3) $m \times n \Rightarrow$ size

4) $A = B \Rightarrow$ جميع elements متساوية

Ex:-

Find the elements of the matrix A if $A = B$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 \\ 5 & 3 \end{bmatrix}$$

sol:-

$$A = \begin{bmatrix} 4 & 0 \\ 5 & 3 \end{bmatrix}$$

Ex:-

Find the size of the matrix

1) $\begin{bmatrix} 0,3 & 1 & -3 \\ 2 & 4 & 3 \end{bmatrix} \Rightarrow 2 \times 3$ صفوف × أعمدة

3) $\begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow 2 \times 1$

2) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow 3 \times 3$

4) $[a_1 \ a_2 \ a_3] \Rightarrow 1 \times 3$

$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Rightarrow 3 \times 1$

* ملاحظات

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

2×2

1) squar matrix ⇐ $m = n$

2) $a_{11}\ a_{22}\ a_{33} \dots$ ⇐ diagonal

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

2×3

3) rectangular matrix ⇐ $m \neq n$

4) Vector matrix ,

1) one column $= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

2) one rowe $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$

* جمع

شرط جمع الماتريكس

نفس الحجم same size

Ex:-

$$A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}_{2\times3} \qquad B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}_{2\times3}$$

$$A + B = ??$$

sol:-

$$A + B = \begin{bmatrix} -4+5 & 6+(-1) & 3+0 \\ 0+3 & 1+1 & 2+0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} -1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}_{2\times3}$$

* الضرب ب ثابت

Ex:- $A = \begin{bmatrix} 2,7 & -1,8 \\ 0 & 0,9 \\ 9 & -4,5 \end{bmatrix}$

1) $-1A = \begin{bmatrix} -2,7 & 1,8 \\ 0 & -0,9 \\ -9 & 4,5 \end{bmatrix}$

3) $0A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

2) $\frac{10}{9}A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}$

* قواعد الجمع

1) $A + B = B + A$

2) $A + (B + C) = (A + B) + C$

3) $A + 0 = A$

4) $A + (-A) = 0$

* قواعد الضرب بـ ثابت

1) $\cancel{A}\,(A + B)c = Ac + Bc$

2) $(c + K)A = cA + KA$

3) $1A = A$

4) $(cK)A = cKA$

*

⑥

* ضرب الماتريكس

$$\overset{\text{الصف}\quad\text{عمود}}{A \Rightarrow [m * n]}$$

$$B \Rightarrow [\overset{\text{صف}}{n} * \overset{\text{عمود}}{p}] \qquad n = n$$

Ex:-

$$A = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}$$

$$3 \times 3 \qquad\qquad 3 \times 4$$

$$A \times B$$

$$\begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

$$3 \times 4$$

$a_{11} = 3\times2 + 5\times5 + -1\times9$
$= 6 + 25 + 9 = 22$

$a_{12} = -6 + 0 + 4 = -2$

$a_{13} = 9 + 35 - 1 = 43$

$a_{14} = 3 + 40 - 1 = 42$

$a_{21} = 8 + 0 + 18$
$= 26$

$a_{22} = -8 - 8 = -16$

$a_{23} = 12 + 0 + 2 = 14$

$a_{24} = 4 + 0 + 2 = 6$

$a_{31} = -12 - 15 + 18 = -9$

$a_{32} = 12 + 0 - 8 = 4$

$a_{33} = -18 - 21 + 2 = -37$

$a_{34} = -6 - 24 + 2 = -28$

⑦ Ex

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$$

$\underbrace{2 \times \underline{\underline{2}} \qquad \underline{\underline{2}}}\times 1 \qquad 2 \times 1$

Ex:-

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}$$

$2 \times \underline{1} \qquad \underline{\underline{2}} \times 2$

Ex:-

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \end{bmatrix}$$

$1 \times \underline{\underline{3}} \qquad \underline{\underline{3}} \times 1 \qquad 1 \times 1$

$3 + 12 + 4 = 19$

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}$$

$\underbrace{\underline{\underline{3}} \times \underline{1} \qquad \underline{1} \times \underline{\underline{3}}}$

3×3

Ex:-

$$\underset{\substack{2\times 2 \\ A}}{\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}} \underset{\substack{2\times 2 \\ B}}{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$B \times A \neq 0$$

$$\underset{2\times 2}{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}$$

* قواعد الضرب

1) $(kA)B = k(AB) = A(kB)$

2) $A(BC) = (AB)C$

3) $(A+B)C = AC + BC$

4) $C(A+B) = CA + CB$

* $AB = 0$

مش شرط ان يكون

$A = 0$

او $\quad \cancel{A}\, BA = 0$

$B = 0$

Transposition

$$A = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow A^T = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \end{array}$$

$[3 \times 3]$ $\qquad$ $[3 \times 3]$

Ex:-

$$\begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

$[2 \times 3]$ $\qquad$ $[3 \times 2]$

Ex:-

$$\begin{bmatrix} 3 & 0 & 7 \\ 8 & -1 & 5 \\ 1 & -9 & 4 \end{bmatrix}^T \Rightarrow \begin{bmatrix} 3 & 8 & 1 \\ 0 & -1 & -9 \\ 7 & 5 & 4 \end{bmatrix}$$

$[3 \times 3]$ $\qquad$ $[3 \times 3]$

Ex:-

$$\begin{bmatrix} 6 & 2 & 3 \end{bmatrix}^T$$

$[1 \times 3]$

$$= \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

$[3 \times 1]$

Ex:-

$$\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}^T$$

$[3 \times 1]$

$$= \begin{bmatrix} 6 & 2 & 3 \end{bmatrix}$$

$[1 \times 3]$

* قواعد

1) $(A^T)^T = A$

2) $(A+B)^T = A^T + B^T$

3) $(cA)^T = cA^T$

4) $(AB)^T = B^T * A^T$

$AB \neq BA$

* Symmetric and skew-symmetric

1) $A^T = A \Rightarrow$ symmetric

2) $A^T = -A \Rightarrow$ skew-symmetric

Ex:-

$$A = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}$$

symmetric

Ex:-

$$A = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

$A = -A^T$ skew-Symmetric

(11)

Triangular Matrices *

1) $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ ⇒ upper triangular

2) $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$ ⇒ upper triangular

3) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}$ ⇒ lower triangular

4) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}$ ⇒ lower triangular

*

1) Diagonal matrix

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2) scaler matrix

$S = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$

3) unit, identity matrix

$AI = IA = A$ $\quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12

Linear system of equation

$$a_{11}x_1 + \cdots\cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + \cdots\cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + \cdots\cdots + a_{mn}x_n = b_m$$

* $a_{11} \cdots a_{mn} \Rightarrow$ Cofficient of the system
* if all $b = 0 \Rightarrow$ homogeneous system

* Gauss

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

pivot

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix}$$

$\overset{3}{\cancel{8}}\, y\,Row_1 + Row_2$

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ 0 & a'_{22} & b'_2 \end{array}\right]$$

$$0 + a'_{22}x_2 = b'_2$$

$$x_2 = \frac{b'_2}{a'_{22}}$$

13

Ex:- solve the linear system

$$2x_1 + 5x_2 = 2$$

$$-4x_1 + 3x_2 = -30$$

sol:-

pivot

$$\tilde{A} = \begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix} \quad 2\,Row_1 + Row_2$$

$$\begin{array}{ccc} 4 & 10 & 4 \\ -4 & 3 & -30 \\ \hline 0 & 13 & -26 \end{array}$$

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$

$$13x_2 = -26$$

$$\boxed{x_2 = -2}$$

$$2x_1 + (-10) = 2$$

$$2x_1 = 12$$

$$\boxed{x_1 = 6}$$

Ex:- solve the linear system

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

$$10x_2 + 25x_3 = 90$$

$$20x_1 + 10x_2 = 80$$

sol:-

pivot

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix} \quad \begin{array}{l} Row_1 + Row_2 \\ \\ -20\,Row_1 + Row_4 \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -3\,Row_2 + Row_3$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & b \\ \end{array}$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row 3

$$-95x_3 = -190$$

$$x_3 = 2$$

Row 2

$$10x_2 + 25x_3 = 90$$

$$x_2 = 4$$

$$x_1 - x_2 + x_3 = 0$$

$$x_1 - 4 + 2 = 0$$

$$x_1 = 2$$

Ex:-

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$

$$0{,}6x_1 + 1{,}5x_2 + 1{,}5x_3 - 5{,}4x_4 = 2{,}7$$

$$1{,}2x_1 - 0{,}3x_2 - 0{,}3x_3 + 2{,}4x_4 = 2{,}1$$

Sol:-

$$-\frac{0{,}6}{3} = -0{,}2 \qquad -\frac{1{,}2}{3} = -0{,}4$$

pivot

$$\left[\begin{array}{ccccc} 3 & 2 & 2 & -5 & 8 \\ \boxed{0{,}6} & 1{,}5 & 1{,}5 & -5{,}4 & 2{,}7 \\ \boxed{1{,}2} & -0{,}3 & -0{,}3 & 2{,}4 & 2{,}1 \end{array}\right] \begin{array}{l} \\ -0{,}2\,Row_1 + Row_2 \\ -0{,}4\,Row_1 + Row_3 \end{array}$$

$$\left[\begin{array}{ccccc} 3 & 2 & 2 & -5 & 8 \\ 0 & 1{,}1 & 1{,}1 & -4{,}4 & 1{,}1 \\ 0 & -1{,}1 & -1{,}1 & 4{,}4 & -1{,}1 \end{array}\right] \begin{array}{l} \\ \\ Row_2 + Row_3 \end{array}$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left[\begin{array}{ccccc} 3 & 2 & 2 & -5 & 8 \\ 0 & 1{,}1 & 1{,}1 & -4{,}4 & 1{,}1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$1{,}1x_2 + 1{,}1x_3 - 4{,}4x_4 = 1{,}1$$

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$

infinitly many solution

(15)

Ex:-

$$3x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + x_2 + x_3 = 0$$
$$6x_1 + 2x_2 + 4x_3 = 6$$

sol:-

$$-\frac{2}{3}$$

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ \boxed{2} & 1 & 1 & 0 \\ \boxed{6} & 2 & 4 & 6 \end{bmatrix} \begin{array}{l} -\frac{2}{3}\,Row_1 + Row_2 \\ -2\,Row_1 + Row_3 \end{array}$$

$$-1 \times \frac{-2}{-\frac{1}{3}} = -6$$

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & \boxed{-2} & 2 & 0 \end{bmatrix} \quad 6\,Row_2 + Row_3$$

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix} \quad \text{no solution}$$

$$\cancel{0x_1 + 0x_2 + 0x_3} = 12$$

$$0 = 12$$

$$\times$$

* 3 حالات للأجوبة

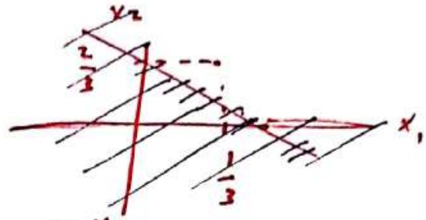
1) one solution / unique solution

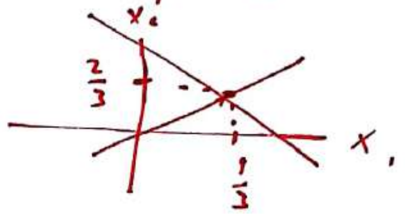
$x_1 + x_2 = 1$

$2x_1 - x_2 = 0$

$x_1 = \frac{1}{3}$

$x_2 = \frac{2}{3}$

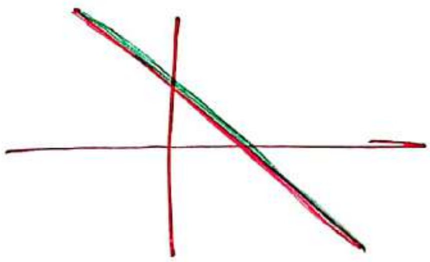




2) infinitly many solution

$x_1 + x_2 = 1$

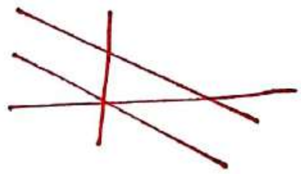
$2x_1 + 2x_2 = 2$



3) No solution

$x_1 + x_2 = 1$

$x_1 + x_2 = 0$



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ملاحظات

*
1) A linear system is called [overdetermined]
if equation more than unknown

$m > n$

m :- عدد المعادلات
n :- عدد المجاهيل

2) determined
$m = n$

3) underdetermined
$m < n$

* A system is called :-

1) consistent
A) one solution
B) infinitely solution

2) in consistant
no solution

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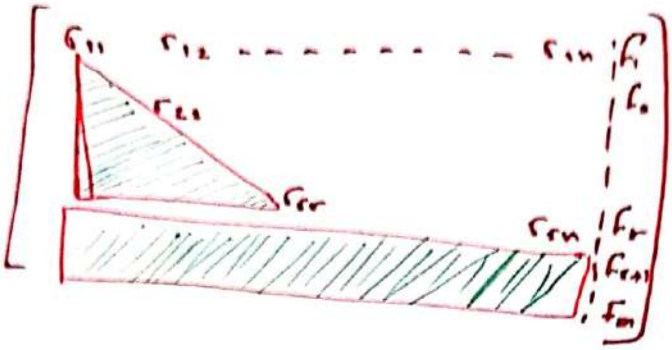
Row Echelon Form



1) Unique solution :. $r = n$

f_{r+1} to f_m ~~are~~ zero

r :. عدد الصفوف اللي ما تصفرت

n :. عدد المجاهيل

m :. عدد المعادلات

2) infinitely many solution :. $r < n$

f_{r+1} to f_m = zero

3) No solution $r < m$

f_{r+1} to f_m is non zero

Rank of a matrix

1) غاوس

2) عدد الصفوف التي لم تصفر = rank

Ex:-

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Sol:-

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \begin{matrix} Row_2 + 2\,Row_1 \\ Row_3 - 7\,Row_1 \end{matrix}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} Row_3 + \frac{1}{2} Row_2$$

rank A = 2

Ex :-

1) $\begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix} \rightarrow \begin{array}{ccc} x_1 & x_2 & \\ \end{array} \begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$

n = 2
rank(A) = 2
ran($\tilde{A}$) = 2
unique solution

rank = 2
vector = 2

ind

2) $\begin{array}{cccc} x_1 & x_2 & x_3 & \end{array}$
$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

rank = 3
vector = 4

dep

3) $\begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0,6 & 1,5 & 1,5 & -5,4 & 2,7 \\ 1,2 & -0,3 & -0,3 & 2,4 & 2,1 \end{bmatrix} \rightarrow \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \begin{bmatrix} 3 & 2 & 2 & -5 & 8 \\ 0 & 1,1 & 1,1 & -4,4 & 1,1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

dep

rank = 2
vector = 3

rank(A) = 3
rank($\tilde{A}$) = 3
n = 4
infinitly

4) $\begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & \frac{-1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$

ind

rank = 3
vector = 3

rank(A) = 2
rank($\tilde{A}$) = 3
No solution

* ملاحظات

1) - Rank = vector → independent

- Rank < Vector → dependent

او بصيغة أخرى
اذا لم يكن هناك اي صف صفري بعد عملية خاوس
((independent))

2) A and A^T have the same rank

3)

عدد المجاهيل :- n

- unique solution → $rank(A) = rank(\tilde{A}) = n$

- infinitely many solution → $rank(A) = rank(\tilde{A}) < n$

- no solution → $rank(A) \neq rank(\tilde{A})$

Determinants

$$D = \det A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{m} \end{vmatrix}$$

- Second - order Determinants

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11} a_{22}) - (a_{21} a_{12})$$

- Threed order Determinants

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{12}(a_{21} a_{33} - a_{23} a_{31}) + a_{13}(a_{21} a_{32} - a_{22} a_{31})$$

Ex:-

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1 \begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0 \cancel{\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}}$$

$$1 \times (12 - 0) - 3(4 + 4)$$

$$12 - 24 = -12$$

(23)

Minors and Cofactors

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \qquad C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$$

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$C_{11} = M_{11}$ $\qquad$ $C_{21} = -M_{21}$ $\qquad$ $C_{31} = M_{31}$

$C_{12} = -M_{12}$ $\qquad$ $C_{22} = M_{22}$ $\qquad$ $C_{32} = -M_{32}$

$C_{13} = M_{13}$ $\qquad$ $C_{23} = -M_{23}$ $\qquad$ $C_{33} = M_{33}$

(24)

Ex:-

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$m_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -4-3 = -7 \qquad m_{21} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

$$m_{12} = \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = 12+1 = 13 \qquad m_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2$$

$$m_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 9-1 = 8 \qquad m_{23} = \begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = -2$$

$$m_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1+2 = 3 \qquad m_{32} = \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = -7$$

$$m_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2$$

$$\text{cof}\, A = \begin{bmatrix} -7 & -13 & 8 \\ 2 & -2 & 2 \\ 3 & 7 & -2 \end{bmatrix}$$

(25)

Adjoint (Adjugate) matrix

$\text{adj } A = (\text{cof } A)^T$

$A(\text{adj } A) = (\text{adj } A)A = (\det A) I$

* 1) Determinat
- Tringular matrix
- diagonal matrix / scaler / I

فقط نضرب قطر المصفوفة

Ex:

$$\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix} = -3 * 4 * 5 = -60$$

*

- اذا كان صفا كامل او عمود يساوي صفر فإنه det تساوي صفر
- اذا كان صفا من الصفوف يساوي يساوي صفا وضربنا هذا الصف بثابت فإن قيمة det = صفر

- اذا تم تبديل صفا ~~بصف~~ بدل صف تضرب قيمة det ب -1
- قيمة det لا تتغير بعملية غاوس
- اذا تم ضرب صفا ب ثابت فإن قيمة det تساوي قيمتها قبل ضرب الثابت مضروبة بالثابت

$\det cA = c^n \det(A)$

n: عدد الصفوف

- عملية Transposition لا تؤثر على قيمة det

(26)

Ex:.

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{bmatrix} = \cancel{-60} \quad \det A = -60$$

$$\begin{bmatrix} -1 & 2 & 5 \\ 6 & 4 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \det = ??$$

$$\det = 60$$

Ex:.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{-1}{3} & \frac{-1}{3} \\ 0 & 0 & 0 \end{bmatrix} \quad \det = ??$$

$$\det = 0$$

Ex:.

$$\begin{bmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \\ 7 & 13 & 2 \end{bmatrix} \quad \det = ??$$

det = صفر

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Ex:.

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} \begin{matrix} \\ Row_2 - 2Row_1 \\ \\ Row_4 + 1,5\,Row_1 \end{matrix}$$

$$\begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2,4 & 3,8 \\ 0 & 0 & -11,4 & 29,2 \end{vmatrix} \begin{matrix} \\ \\ Row_3 - 0,4\,Row_2 \\ Row_4 - 1,6\,Row_2 \end{matrix}$$

$$\begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2,4 & 3,8 \\ 0 & 0 & 0 & 47,25 \end{vmatrix} \begin{matrix} \\ \\ \\ Row_4 + 4,75\,Row_3 \end{matrix}$$

$$2 * 5 * 2,4 * 47,25 = 1134$$

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Cramer's Rule

1) $a_{11} x_1 + a_{12} x_2 = b_1$

2) $a_{21} x_1 + a_{22} x_2 = b_2$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$x_1 = \frac{D_1}{D} \qquad D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}$$

$$x_2 = \frac{D_2}{D} \qquad D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

Ex:-

$4x_1 + 3x_2 = 12$

$2x_1 + 5x_2 = -8$

$$D = \begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} = 20 - 6 = 14$$

$$x_1 = \frac{D_1}{D} \qquad D_1 = \begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix} = 60 - (-24) = 84$$

$$x_1 = \frac{84}{14} = 6$$

$$x_2 = \frac{D_2}{D} \qquad D_2 = \begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix} = -32 - 24 = -56$$

$$= -\frac{56}{14} = -4$$

* Cramer's Rule for linear system of Three equation

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = b_1$$

$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 = b_2$$

$$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = b_3$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$X_1 = \frac{D_1}{D} \qquad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$X_2 = \frac{D_2}{D}$$

$$X_3 = \frac{D_3}{D} \qquad D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} \qquad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

* if the system is homogeneous and $D \neq 0$
all b_j = zero

Then the solution

$X_1 = 0$, $X_2 = 0$, $X_n = 0$

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 = 0$$

$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 = 0 \qquad D \neq 0$$

$$a_{31}X_1 + a_{32}X_2 + a_{33}X_3 = 0$$

$X_1 = 0$

$X_2 = 0$

$X_3 = 0$

inverse of a Matrix

- for a matrix to have ~~aniver~~ an inverse it has to be a square matrix

- $AA^{-1} = A^{-1}A = I$ $\quad AB \neq BA$

- if A has an inverse, then A is called a non-singular (invertible) matrix

- if A has an inverse then the ~~an~~ inverse is unique

- A has an ~~inf~~ inverse if rank = n

- A has an inverse if $\det(A) \neq 0$

Ex:-

Determine the inverse A^{-1} of

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ Row_2 + 3\,Row_1 \\ Row_3 - Row_1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ Row_3 - Row_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \begin{array}{l} -Row_1 \\ 0{,}5\,Row_2 \\ -0{,}2\,Row_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3{,}5 & 1{,}5 & 0{,}5 & 0 \\ 0 & 0 & 1 & 0{,}8 & 0{,}2 & -0{,}2 \end{array}\right] \begin{array}{l} Row_1 + 2\,Row_3 \\ Row_2 - 3{,}5\,Row_3 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & \boxed{-1} & 0 & 0{,}6 & 0{,}4 & -0{,}4 \\ 0 & 1 & 0 & -1{,}3 & -0{,}2 & 0{,}7 \\ 0 & 0 & 1 & 0{,}8 & 0{,}2 & -0{,}2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & \boxed{-1} & 0 & 0{,}6 & 0{,}4 & -0{,}4 \\ 0 & 1 & 0 & -1{,}3 & -0{,}2 & 0{,}7 \\ 0 & 0 & 1 & 0{,}8 & 0{,}2 & -0{,}2 \end{array}\right] \quad Row_1 + Row_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0{,}7 & 0{,}2 & 0{,}3 \\ 0 & 1 & 0 & -1{,}3 & -0{,}2 & 0{,}7 \\ 0 & 0 & 1 & 0{,}8 & 0{,}2 & -0{,}2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -0{,}7 & 0{,}2 & 0{,}3 \\ -1{,}3 & -0{,}2 & 0{,}7 \\ 0{,}8 & 0{,}2 & -0{,}2 \end{bmatrix}$$

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$$- \quad A^{-1} = \frac{1}{\det A}[C]^{T}$$

$$* \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{is} \quad A^{-1} = \frac{1}{\det}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$[\text{cof}(A)]^{T} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Ex:-

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \quad \text{find } A^{-1}$$

$$A^{-1} = \frac{1}{\det(A)} * [\text{cof}]^{T} \qquad \det(A) = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= 12 - 2 = 10$$

$$[\text{cof}]^{T} = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0,4 & -0,1 \\ -0,2 & 0,3 \end{bmatrix}$$

33

Ex:- Find the inverse of

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$A^{-1} = \frac{1}{\det(A)} * (\text{cof})^T$$

$$\det(A) = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{vmatrix}$$

$$= -1(-4-3) - 1(12+1) + 2(9-1)$$

$$= 10$$

$$C_{11} = -7$$

$$C_{12} = -(12+1) = -13$$

$$C_{13} = (9-1) = 8$$

$$C_{21} = -(4-6) = 2$$

$$C_{22} = (-4+2) = -2$$

$$C_{23} = -(-3+1) = 2$$

$$C_{31} = (1+2) = \cancel{4}\ 3$$

$$C_{32} = -(-1-6) = 7$$

$$C_{33} = (1-3) = -2$$

$$A^{-1} = \frac{1}{10} \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -0,7 & 0,2 & 0,3 \\ -1,3 & -0,2 & 0,7 \\ 0,8 & 0,2 & -0,2 \end{bmatrix}$$

* inverse of diagonal matrix

A^{-1} is also diagonal with entries $\frac{1}{a_{11}}, \frac{1}{a_{22}} \dots \frac{1}{a_{nn}}$

Ex:.

$$A = \begin{bmatrix} -0,5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^{-1} = ??$$

$$A^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0,25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* properties of Matrix operations

1) $(AC)^{-1} = C^{-1}A^{-1}$

2) $(AC \dots PQ)^{-1} = Q^{-1}P^{-1} \dots C^{-1}A^{-1}$

3) $AC = AD \Rightarrow$ ليس بالضرورة ان $C = D$

if $rank(A) = n$

$AC = AB \Rightarrow B = C$

4) $AB = 0 \Rightarrow$ ليس بالضرورة ان $B = 0 , A = 0$

if $rank(A) = n \quad AB = 0 \Rightarrow B = 0$

$A, B \Rightarrow (n \times n)$

- if A is singular

BA and $AB \Rightarrow$ singular

- $\det(AB) = \det(BA) = \det A \cdot \det B$

* solving systems of linear Equations using matrix inverse

Ex:-

$$x_1 + x_2 - x_3 = 3$$
$$-x_1 + x_2 + x_3 = -1$$
$$x_1 - x_2 + x_3 = 5$$

1) $AX = b$

2) $A^{-1}AX = A^{-1}b$

$X = A^{-1}b$

$$A^{-1} = \frac{1}{\det(A)} * [cof]^T$$

$$\det = \begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$\det = 4$

$C_{11} = 2$

$C_{12} = \cancel{0}\ 2$

$C_{13} = 0$

$C_{21} = 0$

$C_{22} = 2$

$C_{23} = 2$

$C_{31} = 2$

$C_{32} = 0$

$C_{33} = 2$

$$cof = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

$$[cof]^T = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0,5 & 0 & 0,5 \\ 0,5 & 0,5 & 0 \\ 0 & 0,5 & 0,5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underset{3\times3}{\begin{bmatrix} 0,5 & 0 & 0,5 \\ 0,5 & 0,5 & 0 \\ 0 & 0,5 & 0,5 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$x_1 = 4$

$x_2 = 1$

$x_3 = 2$

1)

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \qquad 29A^{-1}$$

A) a $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ B) $\begin{bmatrix} -3 & 7 & -1 \\ 15 & -6 & 5 \\ -5 & 2 & 8 \end{bmatrix}$ c) $\begin{bmatrix} -3 & 4 & -1 \\ 15 & -6 & 5 \\ -5 & 2 & 3 \end{bmatrix}$

D) $\begin{bmatrix} 0 & 4 & -4 \\ 3 & -10 & 11 \\ -1 & -2 & 3 \end{bmatrix}$ E) $\begin{bmatrix} 0 & 4 & -4 \\ 15 & -6 & 11 \\ -5 & 2 & 3 \end{bmatrix}$ F) $\begin{bmatrix} 0 & 2 & -2 \\ 6 & -5 & 7 \\ -2 & 1 & 3 \end{bmatrix}$

G) $\begin{bmatrix} 3 & -2 & 3 \\ -6 & 5 & -7 \\ -2 & 1 & -1 \end{bmatrix}$ H) $\begin{bmatrix} -6 & 13 & -15 \\ 12 & -10 & 14 \\ 4 & -14 & 26 \end{bmatrix}$

Sol:-

$$\frac{1}{\det(A)}[\operatorname{cof} A]^T \qquad D = \begin{vmatrix} 2 & 2 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

$2 \times 3 - 2 * 15 - 1 * 5$

$\det = -29$

$$A^{-1} = \frac{\cancel{29}}{\cancel{-29}}[\operatorname{cof}]^T$$

$C_{11} = 3 \quad C_{21} = -7$

$C_{12} = -15$

$C_{13} = 5$

$$\operatorname{cof} = \begin{bmatrix} 3 & -15 & 5 \\ -7 & -- & -- \\ -- & -- & -- \end{bmatrix}$$

$$A^{-1} = -[\operatorname{cof}]^T = \begin{bmatrix} -3 & 7 & -- \\ 15 & -- & -- \\ -5 & -- & -- \end{bmatrix}$$

B

2)

$$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \qquad A^{16} = ??$$

$$A^{16}$$
$$\downarrow$$
$$A^{8} A^{8}$$
$$\downarrow$$
$$A^{4} A^{4}$$
$$\downarrow$$
$$A^{2} A^{2}$$
$$\downarrow$$
$$A A$$

$$A^2 = A A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

$$A^4 = A^2 A^2 = \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 5 & -8 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -16 \\ 4 & -7 \end{bmatrix}$$

$$A^8 = A^4 A^4 = \begin{bmatrix} 9 & -16 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} 9 & -16 \\ 4 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & -32 \\ 8 & -15 \end{bmatrix}$$

G

$$A^{16} = \begin{bmatrix} 17 & -32 \\ 8 & -15 \end{bmatrix} \begin{bmatrix} 17 & -32 \\ 8 & -15 \end{bmatrix} = \begin{bmatrix} 33 & -64 \\ 16 & -31 \end{bmatrix}$$

3)

$$A = \begin{bmatrix} i & 5 & 2 & 4 \\ -2 & 1+i & 1 & 0 \\ -3 & 2 & 2+i & 1 \\ 9 & 8 & 7 & 1-3i \end{bmatrix}$$ and D is the adjoint of A then D_{13} is :-

sol:-

$$adj_{13} = [cof_{31}]^T$$

$$D_{13} = [cof_{31}]^T$$

$$cof_{31} = + \begin{vmatrix} 5 & 2 & 4 \\ 1+i & 1 & 0 \\ 8 & 7 & 1-3i \end{vmatrix}$$

$$= 5(1-3i) - 2\left((1+i)(1-3i)\right) + 4\left((1+i)7 - 8\right)$$

$$= 5 - 15i - 8 + 4i - 4 + 28i$$

$$= -7 + 17i$$

D

4)

$$A = \begin{bmatrix} 1 & -1 & -1 & 3 \\ 1 & 1 & -2 & 1 \\ 4 & -2 & 4 & 1 \end{bmatrix}$$

rank of A = ??

sol:-

$$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 3 \\ \boxed{1} & 1 & -2 & 1 \\ \boxed{4} & -2 & 4 & 1 \end{array}\right]$$

Row₂ - Row₁

Row₃ - 4Row₁

$$\begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & \boxed{2} & 8 & -11 \end{bmatrix}$$

Row₃ - Row₂

$$\begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

E

rank = 3

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5) $Ax = b$

$$A = \begin{bmatrix} 1 & -8 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} \qquad \text{find } x_3$$

$$\tilde{A} = [A \,\vdots\, b]$$

$$\left[\begin{array}{ccc|c} 1 & -8 & 1 & 4 \\ -1 & 2 & 1 & 2 \\ 1 & -1 & 2 & -1 \end{array}\right] \begin{array}{l} \\ Row_2 + Row_1 \\ Row_3 - Row_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -8 & 1 & 4 \\ 0 & -6 & 2 & 6 \\ 0 & 7 & 1 & -5 \end{array}\right] \begin{array}{l} \\ \\ Row_3 + \frac{7}{6} Row_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -8 & 1 & 4 \\ 0 & -6 & 2 & 6 \\ 0 & 0 & \frac{10}{3} & 2 \end{array}\right]$$

$$\frac{10}{3} x_3 = 2$$

$$x_3 = \frac{6}{10} \qquad \text{(H)}$$

$$= 0{,}6$$

b)

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{4} & x \\ y & 3/8 \end{bmatrix}$$

$$(A+B)^2 = A^2 + B^2 + I \qquad y = ??$$

Sol:-

$$\cancel{A^2} + 2AB + \cancel{B^2} = \cancel{A^2} + \cancel{B^2} + I$$

$$2AB = I$$

$$2 \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & x \\ y & 3/8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2 \begin{bmatrix} \frac{3}{4} - 2y & 3x - \frac{3}{4} \\ -\frac{1}{4} + 2y & -x + \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} - 4y & 6x - \frac{3}{2} \\ -\frac{1}{2} + 4y & -2 + \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-\frac{1}{2} + 4y = 0$$

$$4y = \frac{1}{2}$$

$$y = \frac{1}{8} \qquad F$$

7)

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad A + A^{-1} = 0,2\,I$$

$\theta = ??$

sol:-

$$A^{-1} = \frac{1}{det}[cof]^T \qquad \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$det A = 1$$

$$[cof^T] = \begin{bmatrix} \cancel{\cos\theta} & +\sin\theta \\ \sin\theta & \cancel{\cos\theta} \end{bmatrix}$$

$$[cof A]^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} + \underset{A^{-1}}{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}} = 0,2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0,2 & 0 \\ 0 & 0,2 \end{bmatrix}$$

$$2\cos\theta = 0,2$$

$$\cos\theta = 0,1 \qquad \cos^{-1}(0,1) = 84,26^\circ$$

D

8)

$$A = \begin{bmatrix} i & 1+i & 2 \\ 3-i & 1 & 2 \\ 4 & 5 & 6-2i \end{bmatrix}$$ $\det(A) = ??$

sol:-

$$\det A = \begin{vmatrix} i & 1+i & 2 \\ 3-i & 1 & 2 \\ 4 & 5 & 6-2i \end{vmatrix}$$

$$= i \begin{vmatrix} 1 & 2 \\ 5 & 6-2i \end{vmatrix} - (1+i) \begin{vmatrix} 3-i & 2 \\ 4 & 6-2i \end{vmatrix} + 2 \begin{vmatrix} 3-i & 1 \\ 4 & 5 \end{vmatrix}$$

$$i\left((6-2i) - 10\right) - (1+i)\left[[3-i][6-2i] - 8\right]$$

$$+ 2\left[[3-i]5 - 4\right]$$

$$= 4 - 10i \quad A$$

10) Which of the following vectors are linerly independent

A) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

B) $\propto$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 4 & 5 \end{bmatrix}$

C) $\begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -1 \end{bmatrix}$

D) $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 7 & 1 \\ 3 & -1 & 12 \end{bmatrix}$

E) $\begin{bmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{bmatrix}$

F) $\begin{bmatrix} 3 & 5 & 2 \\ 1 & 7 & 6 \\ 2 & 5 & 3 \end{bmatrix}$

G) $\propto$ $\begin{bmatrix} 2 & 5 & 2 \\ 1 & 6 & 6 \\ 4 & 10 & 4 \end{bmatrix}$

H) $\begin{bmatrix} -9 & 5 & 2 \\ 1 & -5 & 6 \\ 2 & 5 & -9 \end{bmatrix}$

C

* اذا وجدنا اي علاقة بين 2Row
ضرب او انقاص او بعد عملية جاوس

اذا صف تصفر dep

غير ذلك ind

$\det = 0 \Rightarrow$ dep

$\det \neq 0 \Rightarrow$ ind

A) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Row2 - 4Row1
Row3 - 7Row1

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$ dep

C) $\begin{bmatrix} 2 & 2 & -1 \\ 4 & 0 & 2 \\ 0 & 6 & -1 \end{bmatrix}$ Row2 - 45Row1

$\begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & \frac{5}{2} \\ 0 & 6 & -1 \end{bmatrix}$ Row3 + 6Row2

$\begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & \frac{5}{2} \\ 0 & 0 & 14 \end{bmatrix}$ ind

D) $\begin{bmatrix} 1 & 3 & -2 \\ 4 & 7 & 1 \\ 3 & -1 & 12 \end{bmatrix}$ Row2 - 4Row1
Row3 - 3Row1

$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -5 & 9 \\ 0 & -10 & 18 \end{bmatrix}$ dep

E) $\begin{bmatrix} 1 & 2 & 1 \\ 9 & 5 & 2 \\ 7 & 1 & 0 \end{bmatrix}$ Row2 - 9Row1
Row3 - 7Row1

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -13 & -7 \\ 0 & -13 & -7 \end{bmatrix}$ dep

F) $\begin{bmatrix} 3 & 5 & 2 \\ 1 & 7 & 6 \\ 2 & 5 & 3 \end{bmatrix}$ Row2 - $\frac{1}{3}$ Row1
Row3 - $\frac{2}{3}$ Row1

$\begin{bmatrix} 3 & 5 & 2 \\ 0 & \frac{16}{3} & \frac{16}{3} \\ 0 & \frac{5}{3} & \frac{5}{3} \end{bmatrix}$ dep

H) $\begin{bmatrix} -9 & 5 & 2 \\ 1 & -5 & 6 \\ 2 & 5 & -9 \end{bmatrix}$ Row2 + $\frac{1}{9}$ Row1
Row3 + $\frac{2}{9}$ Row1

$\begin{bmatrix} -9 & 5 & 2 \\ 0 & -\frac{40}{9} & \frac{56}{9} \\ 0 & \frac{55}{9} & -\frac{77}{9} \end{bmatrix}$ Row3 + $\frac{55}{40}$ Row2

$\begin{bmatrix} -9 & 5 & 2 \\ 0 & -\frac{40}{9} & \frac{56}{9} \\ 0 & 0 & 0 \end{bmatrix}$ dep

©

Matrix eigenvalue problems

$$Ax = b$$

Ex:-

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$Ax = b$$

$$\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = b$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = b \Rightarrow b = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{x} = \underbrace{2}_{\lambda} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{x}$$

$$Ax = \lambda x$$

A:- squar matrix

λ:- eigenvalue

x:- eigenvector

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \underline{\underline{A}}\ \underline{x} = \lambda x$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{1}{2} x_1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & 0 \\ a_{21} & a_{22} - \lambda & 0 \end{bmatrix} \qquad \lambda_1 = --$$

$$x_1 = \begin{cases} x \\ x \end{cases}$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12} a_{21} = 0$$

$\lambda_1 =$

$\lambda_2 =$

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②

* steps for solving Eigenvalue ~~problems~~ problems

- $D(\lambda) = \det(A - I\lambda) = 0$

- $\lambda = ??$ $\begin{cases} x_1 \\ x_2 \end{cases}$

Ex

$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

sol:

$\det(A - I\lambda) = 0$

$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$(-5-\lambda)(-2-\lambda) - 4 = 0$

$10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$

$\lambda^2 + 7\lambda + 6 = 0$

$(\lambda+1)(\lambda+6)$

$\lambda_1 = -1$

$\lambda_2 = -6$

$Ax = \lambda x$

$\lambda_1 = -1$

$A - I\lambda = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$

$\begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$ $Row_2 + \frac{1}{2} Row_1$

$\begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$-4x_1 + 2x_2 = 0$

$\underline{\underline{x_2 = 2x_1}}$

$\underline{\underline{x = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}}}$

$\lambda = -6$

$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$ $\underline{\underline{x = \begin{bmatrix} x_1 \\ -\frac{1}{2}x_1 \end{bmatrix}}}$

* General Case :.

$$D(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & \cdots \\ a_{21} & a_{22} - \lambda & \cdots & \cdots \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} - \lambda \end{vmatrix}$$

* Definitions

1) $A - \lambda I \Rightarrow$ characteristic matrix

2) $D(\lambda) = \det(A - \lambda I) \Rightarrow$ characteristic ~~matrix~~ determinant of A

3) $D(\lambda) = 0 \Rightarrow$ characteristic equation of A

* Eigenvalues of the transpose

The transpose A^T of a square matrix A has the same eigenvalues as A

④

Ex:- find the eigenvalues and eigenvector of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)\left[(1-\lambda)(-\lambda) - 12\right] - 2\left[-2\lambda - 6\right] + -3\left[(2 * -2) + (1-\lambda)\right] = 0$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$\lambda_1 = +5$

$\lambda_2 = \lambda_3 = -3$

$\lambda_1 = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right]$$

$$X = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$$

$\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & -3 \end{bmatrix}$$

$$X = \begin{bmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

Ex:-

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \mp i$$

$\lambda_1 = i$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$$

$$X = \begin{bmatrix} x_1 \\ i x_1 \end{bmatrix}$$

$\lambda = -i$

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right]$$

$$X = \begin{bmatrix} x_1 \\ -i x_1 \end{bmatrix}$$

6

Symmetric, skew-symmetric and orthognal matrix

- symmetric matrix: $A^T = A$
- skew-symmetric matrix: $A^T = -A$
- orthogonal matrix:- $A^T = A^{-1}$

Ex:-

symmetric

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

skew-symmetric

$$\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

orthogonal

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{5} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

*

$$A = R + S$$

$R = \frac{1}{2}(A + A^T)$ — R: symmetric

$S = \frac{1}{2}(A - A^T)$ — S: skew-symmetric

Ex:. Find $\underline{R}$ and $\underline{S}$ of a square matrix

$$A = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix}$$

$$R = \frac{1}{2}(A + A^T)$$

$$= \begin{bmatrix} 9 & 3,5 & 3,5 \\ 3,5 & 3 & -2 \\ 3,5 & -2 & 3 \end{bmatrix}$$

$$S = \frac{1}{2}(A - A^T)$$

$$S = \begin{bmatrix} 0 & 1,5 & -1,5 \\ -1,5 & 0 & -6 \\ 1,5 & 6 & 0 \end{bmatrix}$$

$$A = R + S$$

$$= \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}$$

$\underline{\underline{x}} + i\underline{y}$

*1) The eigenvalues of asymmetric matrix are real

2) The eigenvalues of skew-symmetric matrix are pure imaginary or zero

3) The determinant of orthogonal matrix has the Value $\underline{\underline{+1}}$ or $\underline{\underline{-1}}$

4) The eigenvalues of an orthogonal matrix A are real or complex conjugates in pairs and have absolute value 1

$$\sqrt{x^2 + y^2} = 1$$

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⑧ similar matrices

$\hat{A} = P^{-1}AP$

P: is non-singular nxn matrix

*

1) eigenvalues for A = eigenvalues for $\hat{A}$

$\lambda_A = \lambda_{\hat{A}}$

2) eigenvector for $\hat{A}$ = y

eigenvector for A = x

$y = P^{-1}x$

Ex:- Eigen values and vector of similar matrices

$A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

$\hat{A} = P^{-1}AP \qquad P^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 12 & 9 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

$\hat{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

(a)

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix}$$

$$(3-\lambda)(2-\lambda) = 0$$

$\lambda = 3$

$\lambda = 2$

$$\begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$$

$$\begin{vmatrix} 6-\lambda & -3 \\ 4 & -1-\lambda \end{vmatrix}$$

$$(6-\lambda)(-1-\lambda) + 12 =$$

$$-6 - 6\lambda + \lambda + \lambda^2 + 12 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-3)(\lambda-2)$$

$\lambda = 3$

$\lambda = 2$

$\lambda = 3$

$$\begin{bmatrix} 3 & -3 & 0 \\ 4 & -4 & 0 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$y_1 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 4 & -3 & 0 \\ 4 & -3 & 0 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ \frac{4}{3} x_1 \end{bmatrix}$$

$$y = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \frac{4}{3} x_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{1}{3} x_1 \end{bmatrix}$$

10

Diagonalization of matrix

* converting asquare matrix into a diagonal matrix

$$D = X^{-1} A X \qquad A = X D X^{-1}$$

$$D X^{-1} = X^{-1} A \cancel{X} \cancel{X^{-1}}$$

$$X D X^{-1} = \cancel{X} \cancel{X^{-1}} A$$

$$X D X^{-1} = A$$

D:. Diagonal matrix

X:. eigen vector for A

Ex:.

$$A = \begin{bmatrix} 7,3 & 0,2 & -3,7 \\ -11,5 & 1 & 5,5 \\ 17,7 & 1,8 & -9,3 \end{bmatrix}$$

$$\begin{vmatrix} 7,3-\lambda & 0,2 & -3,7 \\ -11,5 & 1-\lambda & 5,5 \\ 17,7 & 1,8 & -9,3-\lambda \end{vmatrix}$$

$$-\lambda^3-\lambda^2+12\lambda=0$$

$\lambda_1 = 3$

$\lambda_2 = -4$

$\lambda_3 = 0$

$\lambda_1 = 3$

$$\left[\begin{array}{ccc|c} 4,3 & 0,2 & -3,7 & 0 \\ -11,5 & -2 & 5,5 & 0 \\ 17,7 & 1,8 & -12,3 & 0 \end{array}\right] \quad X = \begin{bmatrix} -\frac{1}{3}x_2 \\ x_2 \\ -\frac{1}{3}x_2 \end{bmatrix}$$

$\lambda_2 = -4$

$$\left[\begin{array}{ccc|c} 11,3 & 0,2 & -3,7 & 0 \\ -11,5 & 5 & 5,5 & 0 \\ 17,7 & 1,8 & -5,3 & 0 \end{array}\right] \quad X = \begin{bmatrix} -x_2 \\ x_2 \\ -3x_2 \end{bmatrix}$$

$\lambda = 0$

$$\left[\begin{array}{ccc|c} 7,3 & 0,2 & -3,7 & 0 \\ -11,5 & 1 & 5,5 & 0 \\ 17,7 & 1,8 & -9,3 & 0 \end{array}\right] \quad X = \begin{bmatrix} 2x_2 \\ x_2 \\ 4x_2 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & -1 & 2 \\ 3 & 1 & 1 \\ -1 & -3 & 4 \end{bmatrix} \qquad X^{-1} = \begin{bmatrix} -0,7 & 0,2 & 0,3 \\ -1,3 & -0,2 & 0,7 \\ 0,8 & 0,2 & -0,2 \end{bmatrix}$$

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$$D = \begin{bmatrix} -0,7 & 0,2 & 0,3 \\ -1,3 & -0,2 & 0,7 \\ 0,8 & 0,2 & -0,2 \end{bmatrix} \begin{bmatrix} 7,3 & 0,2 & -3,2 \\ -11,5 & 1 & 5,5 \\ 17,7 & 1,8 & -9,3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 2 \\ 3 & 1 & 1 \\ -1 & -3 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Vector

Two kinds of quantites

1) scaler :- determined by its magnitude

2) vector :- has both ~~any~~ a magnitude and directon

* نعبر عن الفيكتور ب سهم ↗

وطول السهم = القيمة = $|a|$

إتجاه السهم هو إتجاه الفيكتور

- tail : ~~النقطة~~ نقطة البداية

tip : نقطة النهاية

* كتابة الفيكتور

- خط غامق **a**
- $\vec{a}, \vec{b}, \vec{c}$

* فيكتور طوله (1) نسميه ~~unit~~ unit Vector

*

1) 

a = b

2) 

- same length
- different direction

3) 

- same direction
- different length

4) 

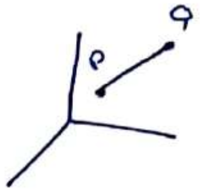
- different length and direction

(14)

- Components of vector

initial point $P(x_1, y_1, z_1)$

terminal point $Q(x_2, y_2, z_2)$



$a_1 = x_2 - x_1$

$a_2 = y_2 - y_1$

$a_3 = z_2 - z_1$

$$a = [a_1, a_2, a_3]$$

$$|a| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$$

Ex:-

initial point $P(4, 0, 2)$

terminal point $Q(6, -1, 2)$

find components of vector

$a_1 = 6 - 4 = 2$

$a_2 = -1 - 0 = -1$

$a_3 = 2 - 2 = 0$

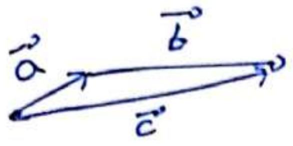
$$a = [2, -1, 0]$$

- Position Vector

The position vector r of a point $A(x, y, z)$ is the vector with the origin $(0, 0, 0)$ as the initial point and A as the terminal point

- vector addition

$\vec{a} + \vec{b} = \vec{c}$



$\vec{a} = [a_1, a_2, a_3]$

$\vec{b} = [b_1, b_2, b_3]$

$\vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$

$\vec{a} - \vec{b} = [a_1 - b_1, a_2 - b_2, a_3 - b_3]$

* خصائص ل إضافة الفيكتور

1) $a + b = b + a$

2) $(u + v) + w = u + (v + w)$

3) $a + 0 = 0 + a = a$

4) $a + (-a) = 0$

- scalar multiplication

$a = [a_1, a_2, a_3]$

$c\,a = [c a_1, c a_2, c a_3]$

* خصائص الضرب بثابت

1) $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$

2) $(c + k)\vec{a} = c\vec{a} + k\vec{a}$

3) $c(k\vec{a}) = (ck)\vec{a}$

4) $1a = a$

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Ex:- $a = [4, 0, 1]$

$b = [2, -5, \frac{1}{3}]$

find

1) $-a$ 3) $a+b$

2) $7a$ 4) $2(a-b)$

sol:-

$\frac{3}{3} + \frac{1}{3}$

1) $-a = [-4, 0, -1]$

2) $7a = [28, 0, 7]$

3) $(a+b) = [6, -5, \frac{4}{3}]$

- Unit vector

i, j, k

$5j = [0, 5, 0]$

$i = [1, 0, 0]$

$j = [0, 1, 0]$

$k = [0, 0, 1]$

$a = a_1 i + a_2 j + a_3 k$

Ex:- $a = [4, 0, 1]$

$b = [2, -5, \frac{1}{3}]$

$a = 4i + k$

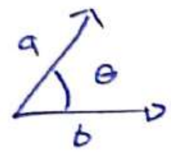
$b = 2i - 5j + \frac{1}{3}k$

- Dot Product (inner Product)

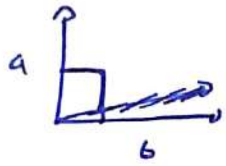
$a \cdot b = |a|\,|b| \cos\theta$

θ :. الزاوية بين a و b

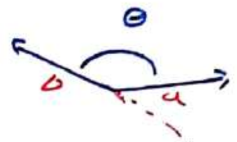
$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

* 

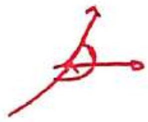
$a \cdot b > 0$



$a \cdot b = 0$



$a \cdot b < 0$



$|a| = \sqrt{a \cdot a}$

$$\cos\theta = \frac{a \cdot b}{|a|\,|b|}$$

Ex:.

Find inner product and the length of ~~a and b~~ $a = [1, 2, 0]$

$b = [3, -2, 1]$ as well as the angle between these vectors

$a \cdot b = 3 \times 1 + 2 \times -2 + 0 * 1$

$= 3 - 4 = -1$

$a = \sqrt{(1)^2 + (2)^2 + (0)^2} = \sqrt{5}$ $\Big|$ $\sqrt{1 + 4 + 0} = \sqrt{5}$

$b = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{14}$

$0 < \theta < \pi$

$\cancel{\theta = \frac{-1}{}}$ $\cos^{-1} \frac{-1}{\sqrt{5} * \sqrt{14}} = 96,86$

(18)

خصائص الضرب النقطي *

1) $(q_1 \vec{a} + q_2 \vec{b}) \cdot \vec{c} = q_1\, a \cdot c + q_2\, b \cdot c$

2) $a \cdot b = b \cdot a$

3) $a \cdot a \geq 0$

4) $a \cdot a = 0 \qquad a = 0$

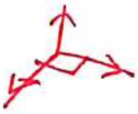
5) $|a \cdot b| \leq |a||b|$

6) $|a+b| \leq |a| + |b|$

7) $|a+b|^2 + |a-b|^2 = 2(|a|^2 + |b|^2)$

8) $i \cdot i = j \cdot j = k \cdot k = 1$

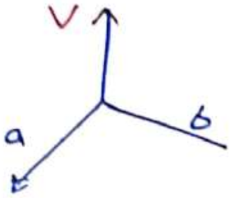
9) $i \cdot j = j \cdot k = k \cdot i = 0$



- Cross Product

$$a \times b = |a|\,|b| \sin\theta$$

* The direction of v is perpendicular both a and b



$$a = [a_1, a_2, a_3] \qquad b = [b_1, b_2, b_3]$$

$$a \times b = v$$

$$v = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & b_1 \\ b_1 & b_2 \end{vmatrix}$$

Ex:. (20)

$V = a \times b \qquad a = [1, 1, 0]$

$\qquad\qquad\qquad b = [3, 0, 0]$

$$V = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = i \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= i0 - j0 + 3k$$

$$= -3K$$

* خصائص ضرب المتجهي

1) $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$

2) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

3) $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$

4) $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$

5) $a \times (b \times c) \neq (a \times b) \times c$

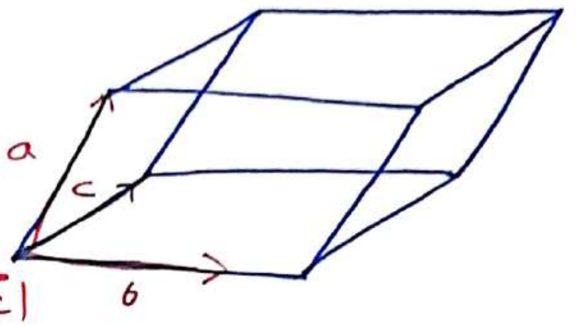
(21)

Scalar triple product

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

- Volume of the parallelepiped

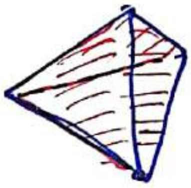
$$V = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$



- Area of the parallelpiped $= |\vec{b}\times\vec{c}|$

- Volume of tetrahedron

$$V = \frac{1}{6}\,|\vec{a}\cdot(\vec{b}\times\vec{c})|$$



Ex1. A tetrahedron is determined by three edge vectors $\vec{a}, \vec{b}, \vec{c}$

Find the Volume when $\vec{a} = [2, 0, 3]$, $\vec{b} = [0, 4, 1]$, $\vec{c} = [5, 6, 0]$

$$\begin{vmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \\ 5 & 6 & 0 \end{vmatrix} = 2\begin{vmatrix} 4 & 1 \\ 6 & 0 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 5 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= -12 - 60 = -72$$

$$V = \left|\frac{1}{6} * -72\right| = 12$$

Vector and scalar functions

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* Vector function $\quad \vec{V} = 3xy\,i + 3z\,j + yzx\,K$

$$\vec{V} = \vec{V}(P) = [v_1(P),\ v_2(P),\ v_3(P)]$$

in Cartesian coordinate system

$$\vec{V}(x, y, z) = [v_1(x, y, z),\ v_2(x, y, z),\ v_3(x, y, z)]$$

* scalar function

$f = f(P) \qquad f = 2xyz^2 + 3y$

in cartesian coordinate system

$f(P) = f(x, y, z)$

- Derivative of a vector function

$$\vec{V}(t) = [v_1(t),\ v_2(t),\ v_3(t)] = v_1(t)\,i + v_2(t)\,j + v_3(t)\,K$$

$$\vec{V}'(t) = [v_1'(t),\ v_2'(t),\ v_3'(t)]$$

Ex!. $V(t) = [t, t^2, 0]$

$$\vec{V}'(t) = [1, 2t, 0]$$

Rules:

1) $(c\vec{V})' = c\vec{V}'$

2) $(\vec{u}+\vec{v})' = \vec{u}' + \vec{v}'$

3) $(\vec{u}\cdot\vec{v})' = \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$

4) $(\vec{u}\times\vec{v})' = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$

5) $(\vec{u}\ \vec{v}\ \vec{w})' = (\vec{u}'\ \vec{v}\ \vec{w}) + (\vec{u}\ \vec{v}'\ \vec{w}) + (\vec{u}\ \vec{v}\ \vec{w}')$

- Partial Derivatives of a vector function

$$\frac{\partial v}{\partial t_m} = \frac{\partial u}{\partial t_m} i + \frac{\partial v_2}{\partial t_m} j + \frac{\partial v_3}{\partial t_m} k$$

second partial derivatives

$$\frac{\partial^2 \vec{V}}{\partial t_l\, \partial t_m} = \frac{\partial^2 v_1}{\partial t_l\, \partial t_m} i + \frac{\partial^2 v_2}{\partial t_l\, \partial t_m} j + \frac{\partial^2 v_3}{\partial t_l\, \partial t_m} k \qquad f_{xy} =$$

Ex:-

$$r(t_1, t_2) = a\cos t_1 i + a\sin t_1 j + t_2 k$$

find $\frac{dr}{dt_1}$, $\frac{dr}{dt_2}$, $\frac{dr}{dt_1\, dt_2}$

sol:-

$$\frac{dr}{dt_1} = -a\sin t_1\, i + a\cos t_1\, j$$

$$\frac{dr}{dt_1\, dt_2} = \cancel{1k}\ 0$$

$$\frac{dr}{dt_2} = 1k$$

– Gradient

$f_{(P)} = f(x, y, z)$ $\quad\quad \nabla$:- Nabla

$$\text{grad} f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$$

$$= \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Ex:-

find gradient of a scalar function

$$f(x, y, z) = \cancel{2y^3 + 4x} \quad 2y^3 + 4xz + 3x$$

$$\nabla f = \left[\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \right]$$

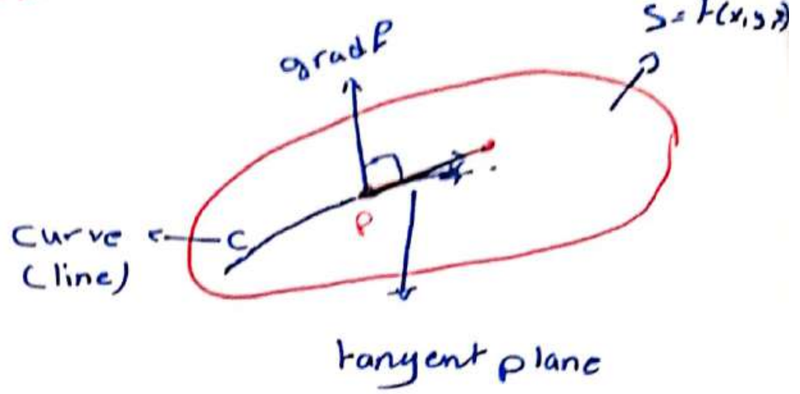
$$\frac{\partial f}{\partial x} = 4z + 3$$

$$\frac{\partial f}{\partial y} = 6y^2 \quad\quad \nabla f = (4z + 3) i + 6y^2 j + 4x k$$

$$\frac{\partial f}{\partial z} = 4x$$

(25)

Gradient as a Surface Normal Vector



- curve C $r(t) = [x(t), y(t), z(t)]$

- tangent vector of C

$r'(t) = [x'(t), y'(t), z'(t)]$

- tangent vector of c = tangent plane of s

- surface normal vector of $s(f(x,y,z))$ at point P equals $\nabla f(P)$

Ex: find unit normal vector n of the cone of revolution

$z^2 = 4(x^2+y^2)$ at point $P(1,0,2)$

$4(x^2+y^2) - z^2 = 0$

$\frac{\nabla f}{|\nabla f|}$

$\frac{\partial f}{\partial x} = 8x$ $\quad \nabla f = [8x, 8y, -2z]$

$\frac{\partial f}{\partial y} = 8y$ $\quad \nabla f = [8, 0, -4]$

$|\nabla f| = \sqrt{64+16} = \sqrt{80}$

$\frac{\partial f}{\partial z} = -2z$ $\quad \left[\frac{8}{\sqrt{80}}, 0, \frac{-4}{\sqrt{80}}\right]$

(26)

- Laplacian of a Scalar field

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

∇^2 : Nabla squared : laplacian

f : scalar function

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Ex: Find the Laplacian of the scalar function

$f = 4xy^2z^3$

sol:

$$\nabla f = 4y^2z^3\,i + 8xyz^3\,j + 12xy^2z^2\,K$$

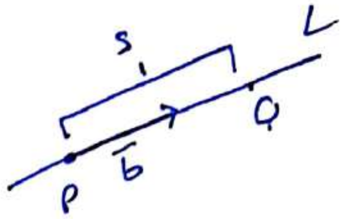
$$\nabla^2 f = 0 + 8xz^3 + 24xy^2z$$

$$\nabla^2 f = 8xz^3 + 24y^2z$$

(27)

- Directional Derivative of a Scalar field

$$D_b f(P) = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}$$



$|s|$ is the distance between P and Q

1) $s > 0$ if Q lies in the direction of $\vec{b}$

2) $s < 0$ if Q lies in the direction of $-\vec{b}$

3) $s = 0$, if $P = Q$

$$D_a f = \frac{1}{|a|} \vec{a} \cdot \text{grad} f$$

Ex:- find the direction derivative of

$f(x, y, z) = 2x^2 + 3y^2 + z^2$

at $P(2, 1, 3)$ in the direction of $a = [1, 0, -2]$

$$\frac{\vec{a}}{|a|} = \frac{[1, 0, -2]}{\sqrt{5}}$$

$$\nabla f = 4x\,i + 6y\,j + 2z\,k$$
$$= [8, 6, 6]$$

$$D_a f = \frac{[1, 0, -2] \cdot [8, 6, 6]}{\sqrt{5}}$$

$$= \frac{8 - 12}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

(28)

Divergence of a vector field

$\vec{V} = [v_1, v_2, v_3]$

$$\operatorname{div} \vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Ex:

$\vec{V} = [3xz, 2xy, -yz^2]$

find div

$$\operatorname{div} \vec{V} = 3z + 2x - 2yz$$

*

$\vec{V} = \operatorname{grad} f$

$$\operatorname{div} \vec{V} = \operatorname{div}(\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 f = \operatorname{div}(\operatorname{grad} f)$$

(24)

- Curl of a vector field

$\nabla \times \vec{V} = \text{Curl } \vec{V}$

$\vec{V} = [v_1, v_2, v_3]$ $\quad \nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

$$\nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left[\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right] i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right] j + \left[\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right] k$$

Ex:

$\vec{V} = [yz, 3zx, z]$

Curl $\vec{V}$

$$\nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3xz & z \end{vmatrix}$$

$$i\left[\overset{0}{\cancel{\frac{\partial}{\partial y}(z)}} - \frac{\partial}{\partial z}(3xz)\right] - j\left[\overset{0}{\cancel{\frac{\partial}{\partial x}(z)}} - \frac{\partial}{\partial z}(yz)\right]$$

$$+ k\left[\frac{\partial}{\partial x}(3xz) - \frac{\partial}{\partial y}(yz)\right]$$

$-3x i + y j + [3z - z] k =$

$-3x i + y j + 2z k$

CamScanner المسموحة ضوئيا بـ

* (3)

1) Scalar $\xrightarrow{\nabla f}$ Vector

2) Scalar $\xrightarrow{\nabla^2 f}$ Scalar

3) vector $\xrightarrow{\text{div } f}$ Scalar

4) Vector $\xrightarrow{\text{curl } f}$ Vector

(31)

Basic Formulas for Grad, Div, & Curl

1) $\nabla(fg) = f\nabla g + g\nabla f$

2) $\nabla(f/g) = \dfrac{g\nabla f - f\nabla g}{g^2}$

3) $div(f\vec{V}) = f\, div\,\vec{V} + \vec{V}\cdot\nabla f$

4) $div(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$

5) $\nabla^2 f = div(\nabla f)$

6) $\nabla^2(fg) = g\nabla^2 f + 2\nabla f\cdot\nabla g + f\nabla^2 g$

7) $curl(f\vec{V}) = \nabla f \times \vec{V} + f\, curl\,\vec{V}$

8) $div(\vec{u}\times\vec{V}) = \vec{V}\cdot curl\,\vec{u} - \vec{u}\cdot curl\,\vec{V}$

9) $curl(\nabla f) = 0$

10) $div(curl\,\vec{V}) = 0$