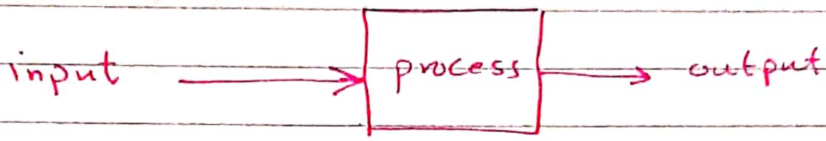
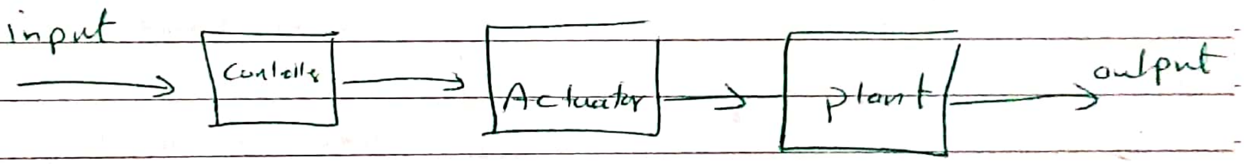
Chapter 1: Introduction to control system

→ Control system is an interconnection of components forming a system configuration that will provide a desired system response.



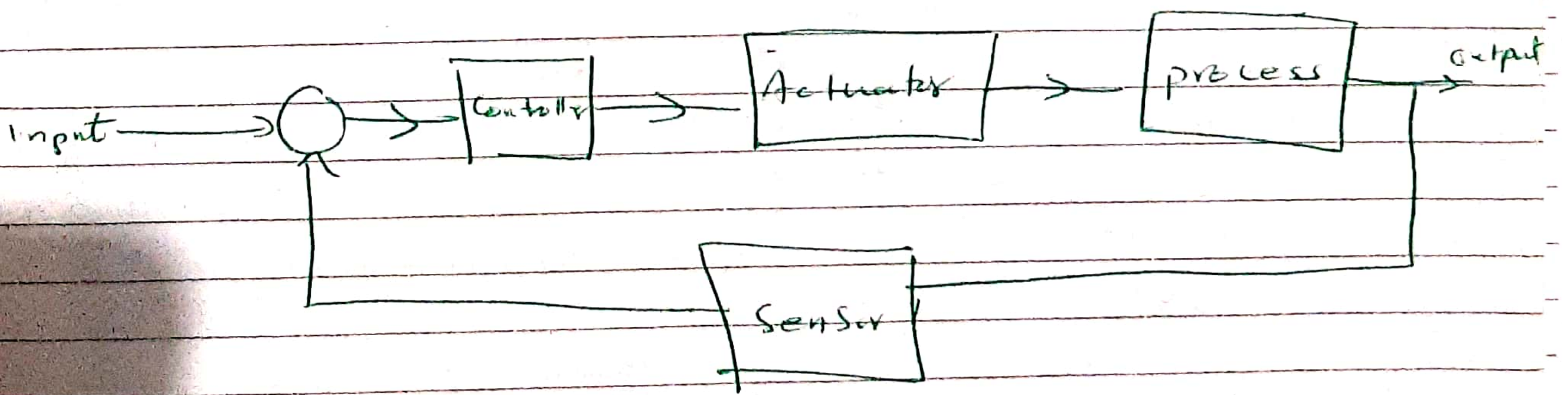
→ Control system types:-

1. open loop system:-



e.g.:- microwave
building lights

2. Closed-Loop System:-

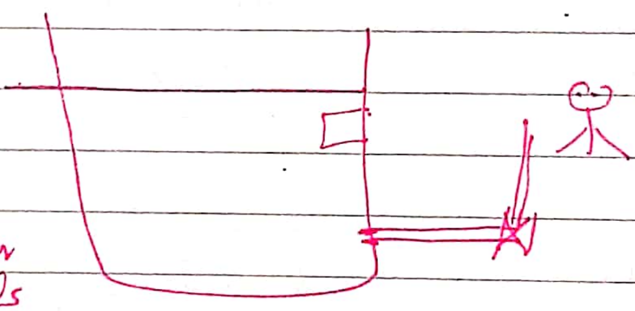


→ Control system Components:-

1. plant → to be Controlled
2. Actuators → Converts input to power Signal
3. Reference input / desired output
4. Controller
5. Error detector
6. Sensors

Examples:- ① Manual - level control system.

1. plant → tank ~~level~~
2. Controller Variable →
tank level



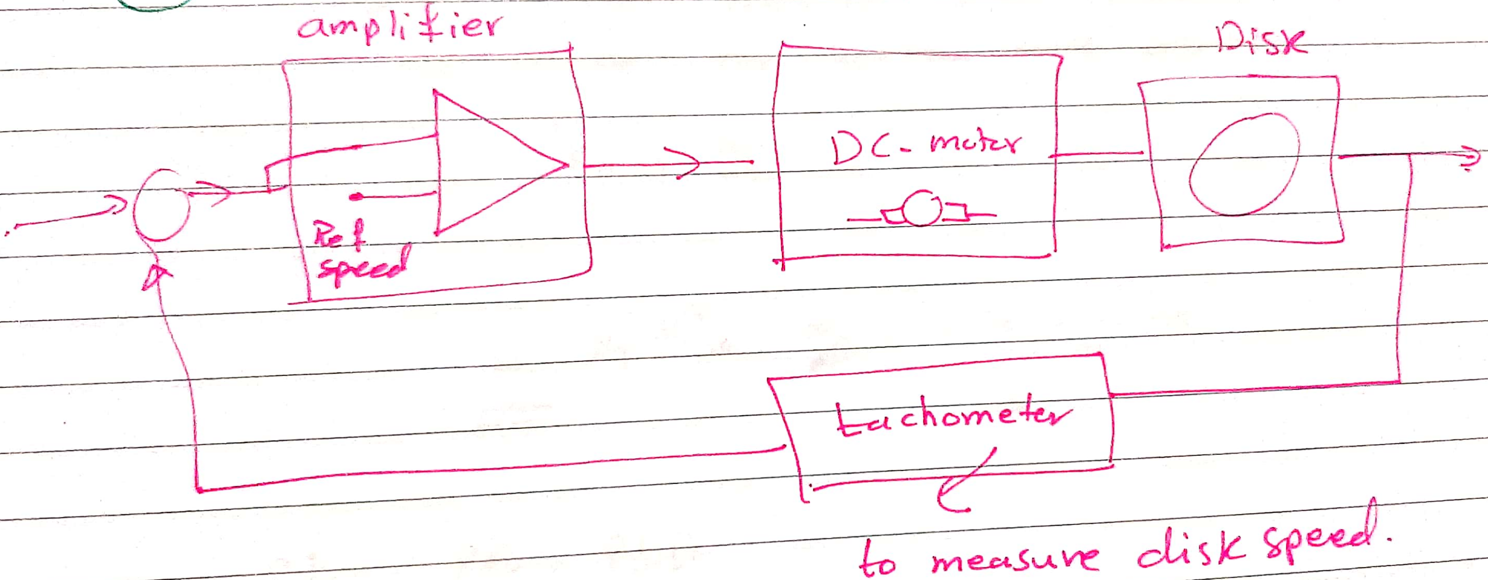
3. Actuator → Valve → operator hands
4. Controller → operator brain
5. Sensor → operator eyes.

speed / direction.

(3) Car

1. Car → plant
2. driver eyes → ~~Con~~ sensor
3. driver hands + stearing → actuater
4. driver brain → controller
5. output → direction or speed.

(4) Disk Drive Read system



EX 5: car → speed control

1. Controlled variable / desired output / Reference input
   (car speed)

2. Controller: (driver brain ~~...~~)

3. Actuator: (driver legs + brake + ...)

4. plant / process: (car)

5. Sensor if any: (driver eyes).

Ex 6: water tank system.

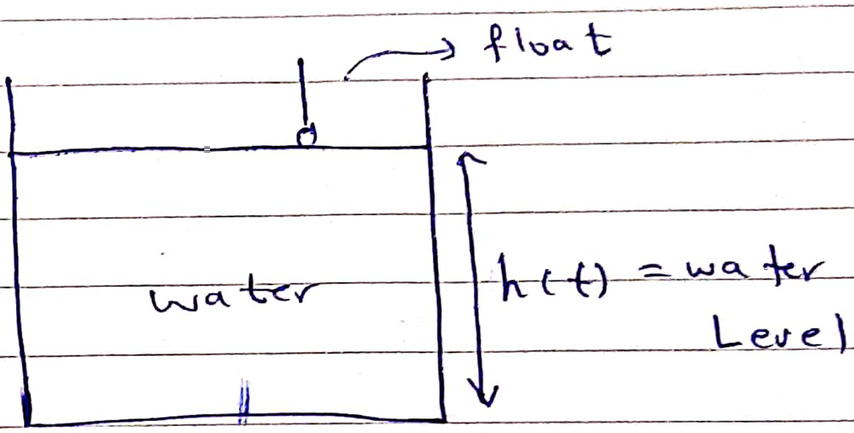
1. desired output : water Level

2. Controller : float

3. Acutator : float + Valve

4. process : water tank

5. Sensor : float



chapter 2: Mathematical Models of system

→ Mathematical Model is describing the system by set of differential equations that relate between the input and the output.

- systems
  - Mechanical
    - Translational motion
      - $\sum F = ma$
    - Rotational Motion
      - $\sum T = J\ddot{\theta}$
  - Electrical
    - KCL
      - $\sum I = 0$
    - KVL
      - $\sum V = 0$

F = Force

M = mass

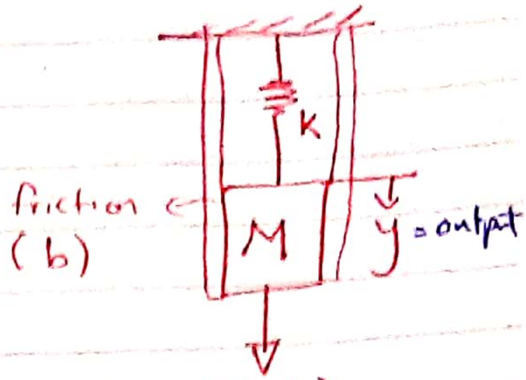
a = acceleration

T = Torque

J = inertia

$\ddot{\theta}$ = angular acceleration

Ex 1: Derive the equation of motion of the system below:-



input = $F(t)$.

Sol:

Free body diagram

$$\downarrow + \sum F = m a = m\ddot{y}$$

$$\underbrace{F(t)}_{\text{input}} - b\dot{y} - ky = m\ddot{y}$$

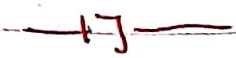
note:
y = displacement
$\dot{y}$ = velocity
$\ddot{y}$ = acceleration

① spring force:

spring

$F_s = Ky$ where K = spring constant
y = spring deflection

direction: against deformation direction

② Damper force :  b

(or friction)

$F_d = b\dot{y}$

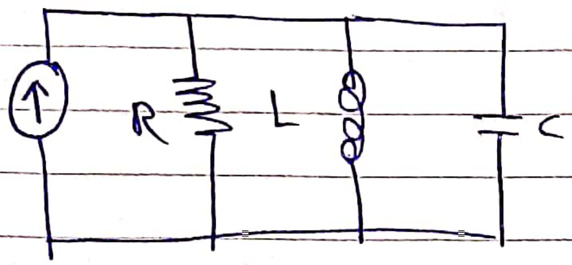
b = damping constant

$\dot{y}$ = velocity

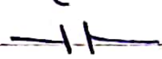
direction: against speed direction.

EX2: RLC circuit

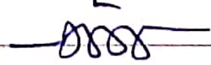
I(t) current source (input)



$v(t)$ = output.

Remember: ①  C capacitor

$$I_c(t) = C\frac{dv(t)}{dt}$$

②  L inductor

$$v(t) = L\frac{dI}{dt}$$

③  Resistor $V = RI$

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Ex2:

$I(t) = I_R + I_L + I_C$ → parallel

$= \frac{V(t)}{R} + \frac{1}{L}\int V(t) + C\frac{dV(t)}{dt}$

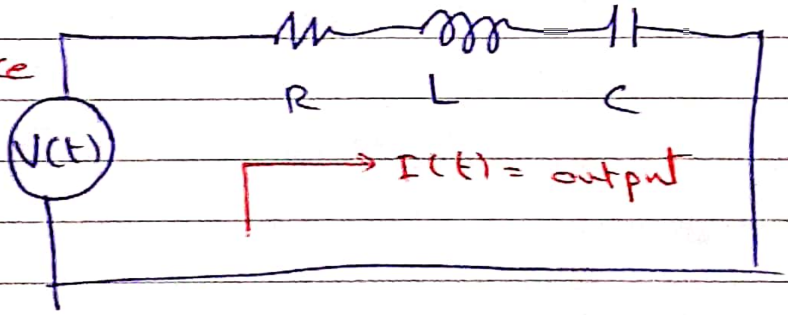
but $V_C = V_R = V_R$

~~so~~ → also, derive the equation above to get rid of integration

$$\therefore \; \dot{I}(t) = \frac{\dot{V}(t)}{R} + \frac{V(t)}{L} + C\ddot{V}(t)$$

Ex 3: RLC – Circuit

voltage source (input)



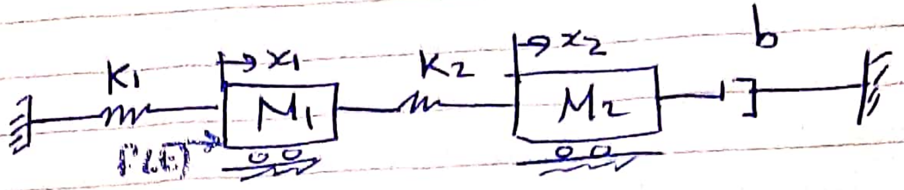
$I_C = I_R = I_L = I$ and

$V(t) = V_C + V_L + V_R$

$$V(t) = \frac{1}{C}\int I(t)\,dt + \frac{L\,dI}{dt} + RI(t)$$

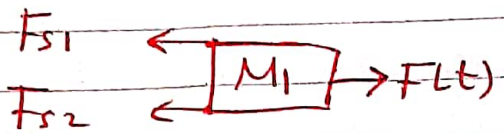
$$\boxed{\therefore \quad V(t) = \frac{I(t)}{C} + L\ddot{I}(t) + R\dot{I}}$$

Ex4:



obtain the governing equation for the system ~~below~~ above.

Free body diagram



$$\overset{+}{\rightarrow} \sum F = M_1 \ddot{x}_1$$

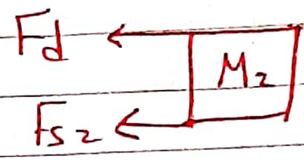
$$F(t) - F_{s_1} - F_{s2} = m_1 \ddot{x}_1$$

$$F(t) = m_1 \ddot{x}_1 + F_{s_1} + F_{s2}$$

$$\boxed{F(t) = M_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2)}$$

input = F(t)

x_1, x_2 = outputs



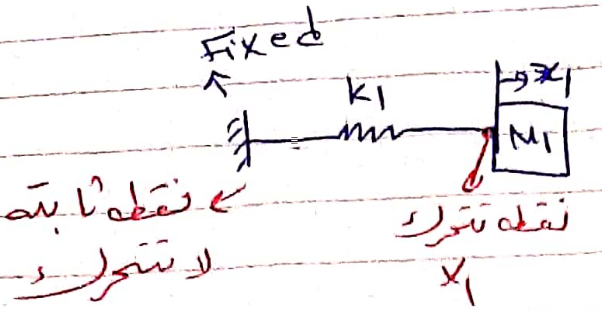
$$\overset{+}{\rightarrow} \sum F = m_2 \ddot{X}_2$$

$$-F_{s_2} - F_d = m_2 \ddot{x}_2$$

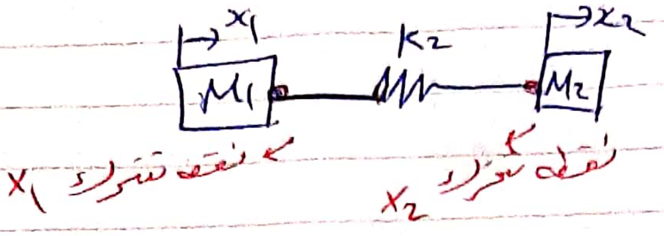
$$M_2\ddot{x}_2 + F_d + F_{s2} = 0$$

$$M_2\ddot{x}_2 + b\dot{x}_2 + K_2(x_2 - x$$

$$= 0$$

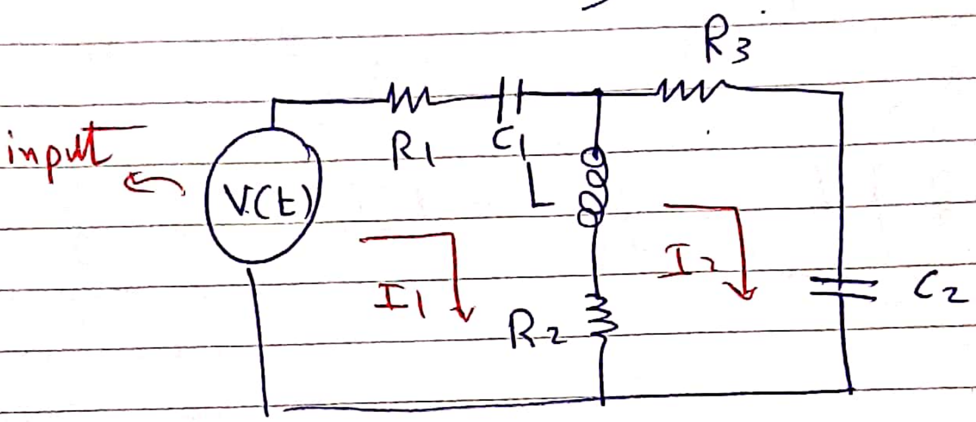


$$\Rightarrow \quad k_1(x_1 - 0) = k_1 x_1$$



$$k_2(x_1 - x_2)$$

Ex 5 :- obtain the differential equation of the following system:



ccw +ve

cw −ve

$$V(t) = I_1 R_1 + \frac{1}{C_1}\int I_1\,dt + L\frac{d(I_1 - I_2)}{dt}$$

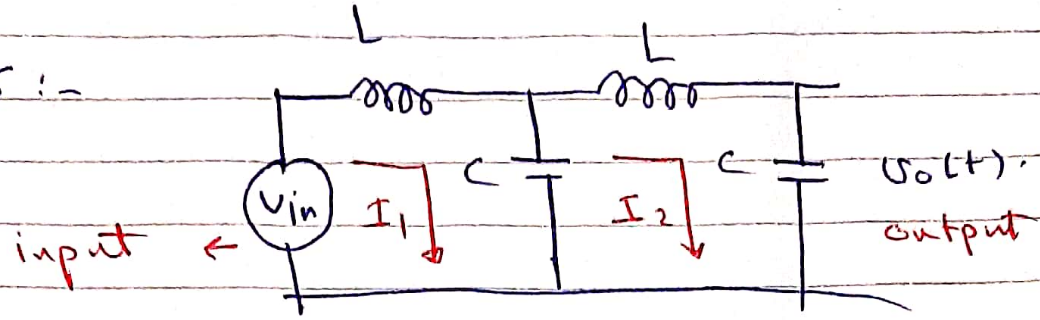
$$+ R_2(I_1 - I_2)$$

نشتق للتخلص من التكامل

$$0 = \frac{L(dI_2 - dI_1)}{dt} + R_2(I_2 - I_1) + R_3 I_2$$

$$+ \frac{1}{C_2} \int I_2 \, dt.$$

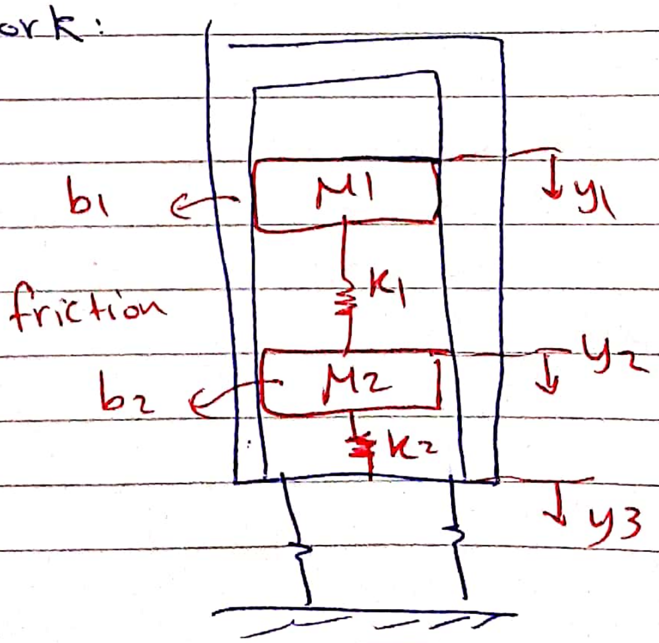
EX 6 :-



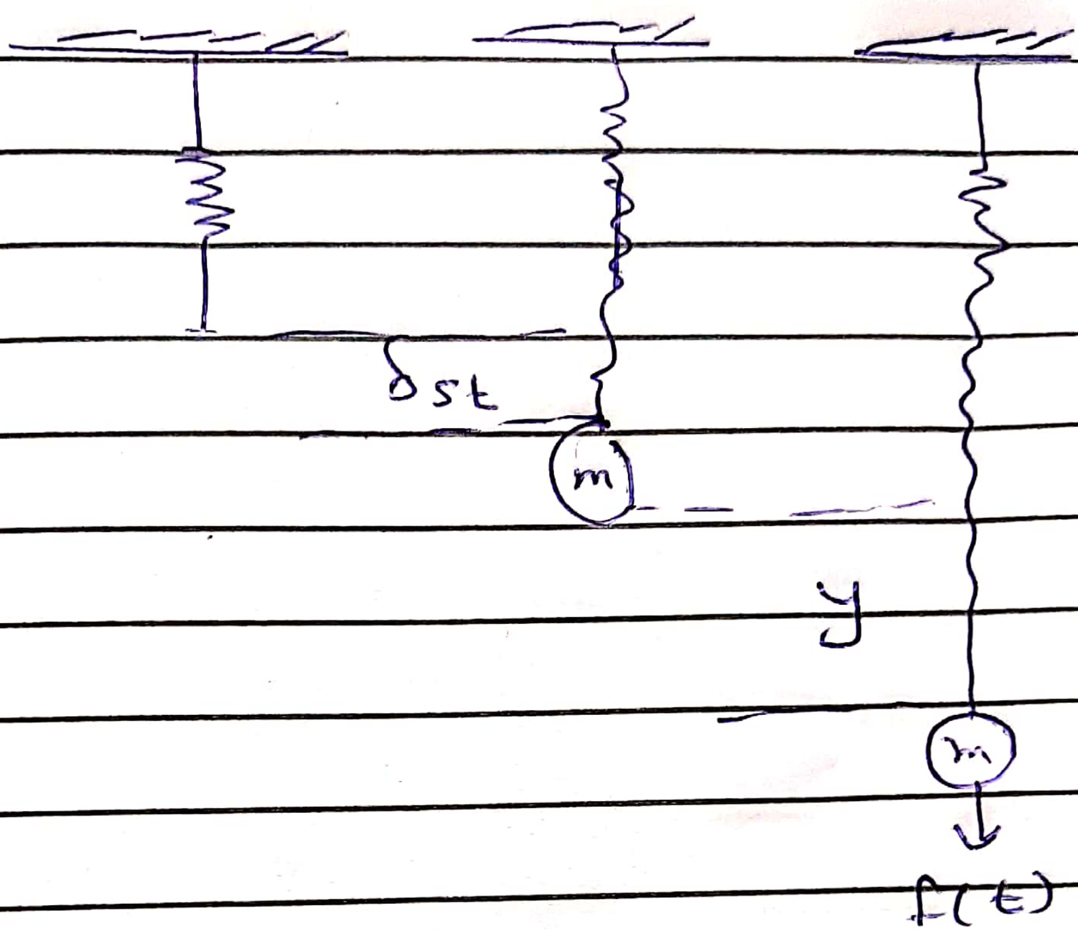
$$V_{in} = L \frac{dI_1(t)}{dt} + \frac{1}{C} \underbrace{\int}_{=} (I_1 - I_2) \, dt \quad \ldots\ldots (1)$$

$$0 = \frac{1}{C} \underbrace{\int}_{=} (I_2 - I_1) \, dt + L \frac{dI_2}{dt} + \frac{1}{C} \underbrace{\int}_{=} I_2(t) \, dt \ldots (2)$$

EX: 7: Home work:



free length



$$\sum F = m\ddot{y}$$

$$mg + F(t) - k(\delta_{st} + y) = m\ddot{y} \quad \text{---} \quad (1)$$

but at equilibrium

$$\sum F = 0$$

$$mg - k\delta_{st} = 0 \quad \text{---} \quad (2)$$

δ_{st} = static deflection

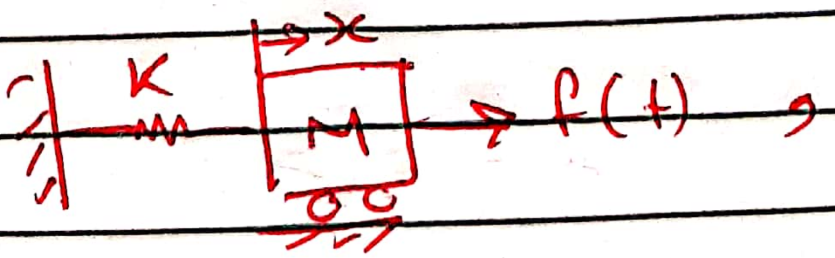
$$mg = k\delta_{st}$$

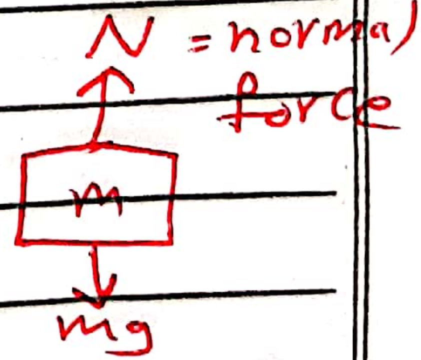
⇒ Subistute 2 in 1

$$\cancel{mg} + f(t) - \cancel{k\delta_{st}} - ky = m\ddot{y}$$

$$\boxed{f(t) = m\ddot{y} + ky}$$

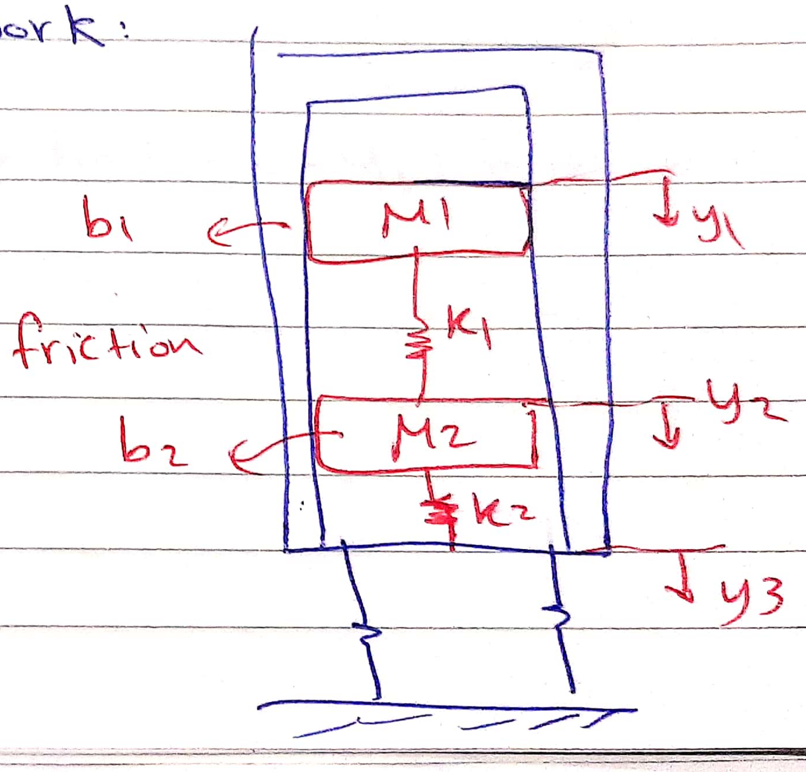
also: in x- direction



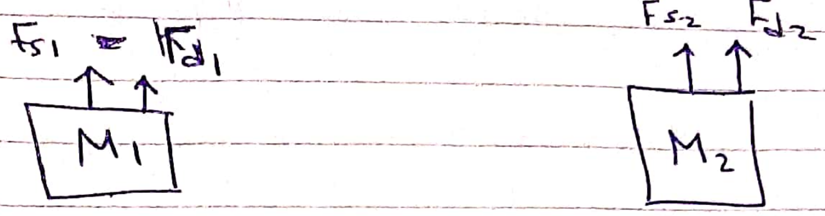
,


$$\sum F_y = 0 \Rightarrow N = mg$$

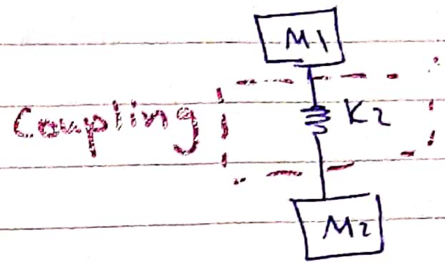
Ex: 7: Home work:



Ex 7: Free body diagram



$$\Sigma F = M_1 \ddot{y}_1$$



$$-F_{s1} - F_{d1} = M_1 \ddot{y}_1$$

$$M_1 \ddot{y}_1 + F_{d1} + F_{s1} = 0$$

$$M_1 \ddot{y}_1 + b_1(\dot{y}_1 - \dot{y}_3) + k_1(y_1 - y_2) = 0$$

also

$$\Sigma F = M_2 \ddot{y}_2$$

$$-F_{s2} - F_{d2} = M_2 \ddot{y}_2$$

$$M_2 \ddot{y}_2 + F_{d2} + F_{s2} = 0$$

$$M_2 \ddot{y}_2 + b_2(\dot{y}_2 - \dot{y}_3) + k_2(y_2 - y_1) = 0$$

* Linear System:

The Linear system should satisfy two conditions:

① Superposition : $y(X_1 + X_2) = y(X_1) + y(X_2)$

e.g: Let $x_1 = 1$ and $X_2 = 2$

$$y(x) = 3X$$

$$y(1+2) \overset{?}{=} y(1) + y(2)$$

$$y(3) \overset{?}{=} y(1) + y(2)$$

$$3(3) \overset{?}{=} 3(1) + 3(2)$$

$$9 \overset{?}{=} 3 + 6$$

$$9 = 9$$

② Scaling : $y(\alpha X_1 + \beta X_2) = y(\alpha x_1) + y(\beta x_2)$

e.g: $\alpha = 1$, $\beta = 2$

$$y(1\times 1 + 2(2)) \overset{?}{=} y(1\times 1) + y(2\times 2)$$

$$y(5) \overset{?}{=} y(1) + y(4)$$

$$15 = 3(5) = 3(1) + 3(4) = 15$$

* Taylor series expansion:

For $y = g(x)$ where $g(x)$ non-linear

$$y = g(x_0) + \left.\frac{dg}{dx_1}\right|_{x=x_0} (x - x_0) + \frac{1}{2!} \left.\frac{d^2 g}{dx^2}\right|_{x=x_0} (x - x_0)^2$$

$$+ \cdots + \frac{1}{n!} \left.\frac{d^n g}{dx^n}\right|_{x=x_0} (x - x_0)^n$$

EX: Let $T = mgL \sin\theta$ → non-linear

use two terms, and $\theta_0 = 0$

$$T = T(\theta_0) + \left.\frac{dT}{d\theta}\right|_{\theta=\theta_0} (\theta - \theta_0)$$

$$= mgL \sin(0) + (mgL \cos(0))(\theta - 0)$$

$$= 0 + mgL\,(1)\,(\theta)$$

$$T = mgL\theta$$

* Laplace Transforms: is a mathematical tool for solving linear time invariant differential equation.

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt \qquad f(t),\ t > 0$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

See Table 2-3 → important

$f(t) = A \rightarrow F(s) = \frac{A}{s}$

$f(t) = \sin\omega t \rightarrow F(s) = \frac{\omega}{s^2+\omega^2}$

$f(t) = \cos\omega t \rightarrow F(s) = \frac{s}{s^2+\omega^2}$

$f(t) = t^n \rightarrow F(s) = \frac{n!}{s^{n+1}}$

unit impulse ← $f(t) = \delta(t) \rightarrow F(s) = 1$

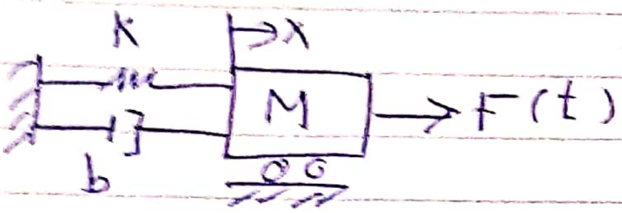
⋮ ⋮

etc etc

$$\mathcal{L}\left\{\frac{dF}{dt}\right\} = sF(s) - F(0)$$

$$\mathcal{L}\left\{\frac{d^2F}{dt^2}\right\} = s^2F(s) - sF(0) - F'(0)$$

Ex1: Find $x(t) = ?$ with $x(0) = \dot{x}(0) = 0$



$$M\ddot{x} + b\dot{x} + kx = f(t)$$

Let $\frac{k}{m} = 2$, $\frac{b}{m} = 3$, $\frac{f(t)}{m} = 1$

divide eq(1) by m:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{f(t)}{m}$$

$$\ddot{x} + 3\dot{x} + 2x = 1$$

$$\mathcal{L}\{\ \ddot{x} + 3\dot{x} + 2x = 1\}$$

$$\left[s^2 X(s) - sX(0) - \dot{x}(0)\right] + 3\left[sX(s) - X(0)\right] + 2X(s) = \frac{1}{s}$$

$$s^2 X(s) + 3sX(\cancel{s}) + 2X(s) = \frac{1}{s}$$

$$X(s)\left[s^2 + 3s + 2\right] = \frac{1}{s}$$

$$X(s) = \frac{1}{s}\left[\frac{1}{s^2+3s+2}\right]$$

take laplace inverse.

we have to make partial fraction

$$X(s) = \frac{1}{(s^2+3s+2)s} = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{s+1}$$

$$\mathcal{L}^{-1}\{X(s)\} = x(t) = A + Be^{-2t} + Ce^{-t}$$

$$\frac{1}{s(s^2+3s+2)} = \frac{A(s+1)(s+2)+B(s)(s+1)+Cs(s+2)}{s(s^2+3s+2)}$$

$$1 = A(s+1)(s+2) + Bs(s+1) + Cs(s+2)$$

$s=0 \Rightarrow 1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$

$s=-1 \Rightarrow -C(1) = 1 \quad \boxed{C=-1}$

$s=-2 \Rightarrow 1 = -2B(-1) \quad \boxed{B = \frac{1}{2}}$

$$\Rightarrow \boxed{x(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}}$$

EX 2 : $Y(s) = \dfrac{2}{(s+1)(s+2)^2}$, Find $y(t) = ?$

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$\boxed{Y(t) = A e^{-t} + B e^{-2t} + C t e^{-2t}}$$

$$2 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

at $s = -1 \Rightarrow \boxed{A = 2}$

at $s = -2 \Rightarrow \boxed{C = -2}$

at $s = 0 \Rightarrow 4A + 2B + C = 2 \Rightarrow 4(2) + 2B - 2 = 2$

$$\boxed{B = -2}$$

$$\boxed{y(t) = 2e^{-t} - 2e^{-2t} - 2te^{-2t}}$$

Ex 3: $y(s) = \dfrac{3}{s^2 + 2s + 5}$

$s^2 + 2s + 5 = 0 \Rightarrow s_{1,2} = -1 \mp j2$

Complex roots

$$\underbrace{s^2+2s+1}+4$$

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$$s^2+2s+5 = (s+1)^2+4$$

$\Rightarrow$ note : Table 2.3

$$f(t) = e^{-\alpha t}\sin\omega t \rightarrow F(s) = \frac{\omega}{(s+\alpha)^2+\omega^2}$$

$$f(t) = e^{-\alpha t}\cos\omega t \rightarrow F(s) = \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$$

$$\therefore \left.\begin{array}{l}(s+\alpha)^2+\omega^2\\ (s+1)^2+4\end{array}\right\} \Rightarrow \begin{array}{l}\omega=2\\ \alpha=1\end{array}$$

$$Y(s) = \frac{3}{(s+1)^2+4} \Rightarrow Y(s) = 3\left(\frac{\frac{2}{2}}{(s+1)^2+4}\right)$$

$$y(t) = \frac{3}{2}e^{-t}\sin 2t$$

* Final Value Theorm: to find the steady-state value

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} s\,Y(s)$$

Ex: Find the steady-state value of the system response.

$$Y(s) = \left(\frac{2}{s+1} - \frac{1}{s+2}\right) y_o$$

Sol:

$$Y_{ss} = \lim_{s \to 0} s\,Y(s)$$

$$Y_{ss} = \lim_{s \to 0} s\left(\frac{2}{s+1} - \frac{1}{s+2}\right) y_o$$

$$y_{ss} = 0 \quad \text{Zero}$$

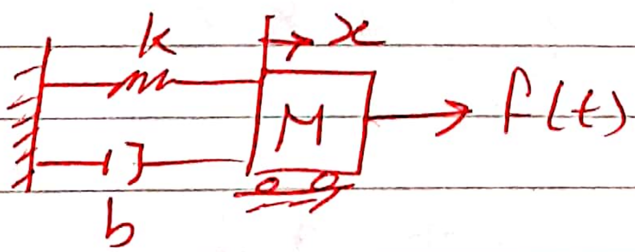
* The transfer function of linear system:
the ratio of the laplace transform of the output variable to the laplace transform of the input variable with the initial conditions assumed to be zero.

$Y(s) \rightarrow$ output

$R(s) \rightarrow$ input

$$\boxed{T(s) = \frac{Y(s)}{R(s)}}$$

Ex1: Find the transfer function of the system below.



$X(s)$ = output

$F(s)$ = input

$$T(s) = \frac{X(s)}{F(s)}$$

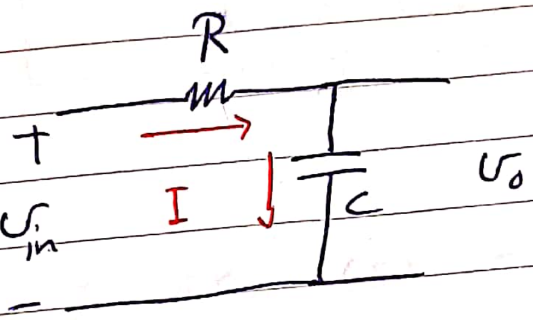
$$M\ddot{x} + b\dot{x} + kx = F(t)$$

$$M(s^2 X(s)) + bsX(s) + kX(s) = F(s)$$

$$X(s)\,[Ms^2 + bs + k] = F(s)$$

$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + bs + k}}$$

EX2 :



Find $T(s) = ?$

V_{in} = input

V_o = output

$$\frac{V_o(s)}{V_{in}(s)} = ?$$

$$V_{in} = IR + \frac{1}{C}\int I\,dt$$

$$\dot{V}_{in} = \dot{I}R + \frac{I}{C}$$

$$sV_{in}(s) = RsI(s) + \frac{I(s)}{C}$$

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$$\boxed{s\,V_{in}(s) = I(s)\left[Rs + \frac{1}{c}\right]} \quad \text{--- } 1$$

but $V_o = \frac{1}{c}\int I\,dt$

$$\dot{V}_o = \frac{I}{c}$$

$$s\,V_o(s) = \frac{I(s)}{c}$$

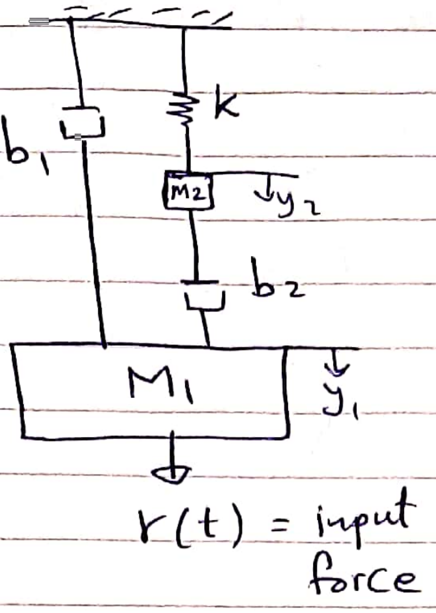
$$\boxed{I(s) = c\,s\,V_o(s)} \quad \text{--- } 2$$

$$\cancel{s}\,V_{in}(s) = c\,\cancel{s}\,V_o(s)\left[Rs + \frac{1}{c}\right]$$

$$\frac{V_o(s)}{V_{in}(s)} = \frac{1}{c\left[Rs + \frac{1}{c}\right]}$$

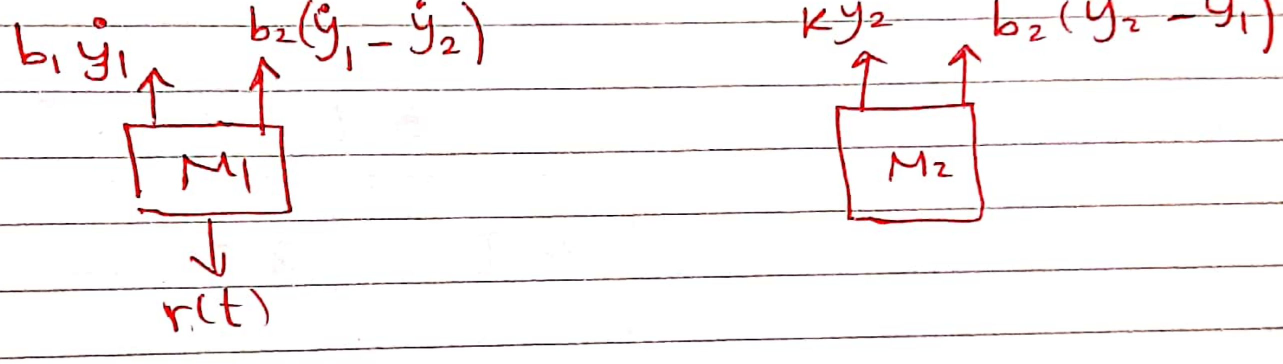
$$\boxed{\frac{V_o(s)}{V_{in}(s)} = \frac{1}{RCs + 1}}$$

Ex 3:



Find $\frac{Y_2(s)}{R(s)}$, $\frac{Y_1(s)}{R(s)}$

Free body diagram



$$M_1 \ddot{y}_1 + b_1 \dot{y}_1 + b_2(\dot{y}_1 - \dot{y}_2) = r(t)$$

$$M_2 \ddot{y}_2 + b_2(\dot{y}_2 - \dot{y}_1) + k y_2 = 0$$

} → take the laplace for two equations

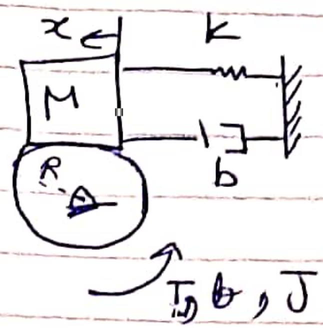
$$\Rightarrow \boxed{Y_2(s) = \frac{b_2 s}{M_2 s^2 + b_2 s + k_2} Y_1(s)}$$

⋮

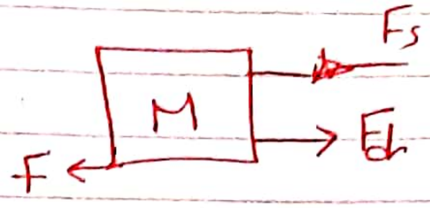
homework.

Ex 4:- Rack and pinion

Find $\dfrac{X(s)}{T_{in}(s)} = ??$

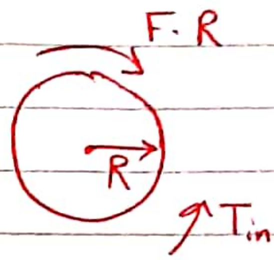


$+\leftarrow \sum F = m\ddot{x}$



$$\boxed{M\ddot{x} + b\dot{x} + kx = F} \quad \text{--- (1)}$$

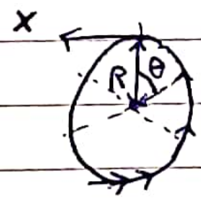
$\sum T = J\ddot{\theta}$



$$\boxed{T_{in} - F.R = J\ddot{\theta}} \quad \text{--- (2)}$$

also,

$x = R\theta$



$\dot{x} = R\dot{\theta}$

$\boxed{\ddot{x} = R\ddot{\theta}}$

$\ddot{\theta} = \dfrac{\ddot{x}}{R}$

$$\Rightarrow M\ddot{x} + b\dot{x} + kx = \frac{T_{in}}{R} - \frac{J\ddot{\theta}}{R}$$

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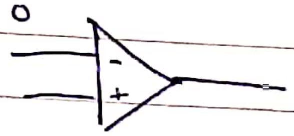
$$\boxed{\frac{X(s)}{T_{in}(s)} = \frac{1}{\left(\frac{J}{R} + RM\right)s^2 + Rb\,s + RK}}$$

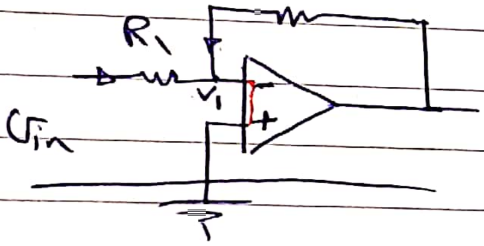
* Operational Amplifier

$\Sigma I_- = 0$

$\Sigma I_+ = 0$



Ex1:



$\Sigma I_- = 0$

$$\frac{V_1 - V_{in}}{R_1} + \frac{-V_o + V_1}{R_2} = 0$$

but $V_1 = 0$

$$\Rightarrow \cancel{0} - \frac{V_{in}(s)}{R_1} = + \frac{V_o(s)}{R_2}$$

$$\boxed{\frac{V_o(s)}{V_{in}(s)} = -\frac{R_2}{R_1}}$$

note: $Z_R = R$ and $Z_L = Ls$

$Z_C = \frac{1}{Cs}$

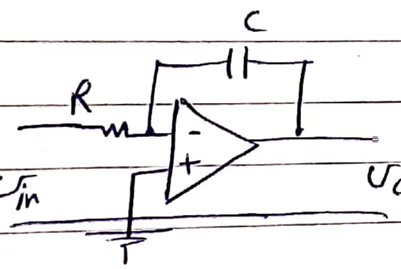
For inverting amplifiers

$$\frac{V_o}{V_{in}} = -\frac{Z_f}{Z_{in}}$$

→ from example 1:

$$\frac{V_o(S)}{V_{in}(S)} = -\frac{R_2}{R_1}$$ (propotional Controller)

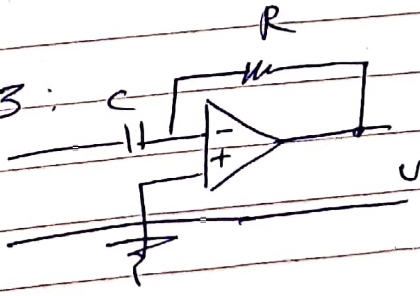
Ex2:



$$\frac{V_o(S)}{V_{in}(S)} = \frac{-Z_f}{Z_{in}} = -\frac{Z_c}{Z_R} = -\frac{\frac{1}{CS}}{R} = -\frac{1}{RCS}$$

(Integral Controller)

Ex3:



$$\frac{V_o(S)}{V_{in}(S)} = -\frac{Z_R}{Z_c}$$

$$= -RCS$$

(Derivative Controller)

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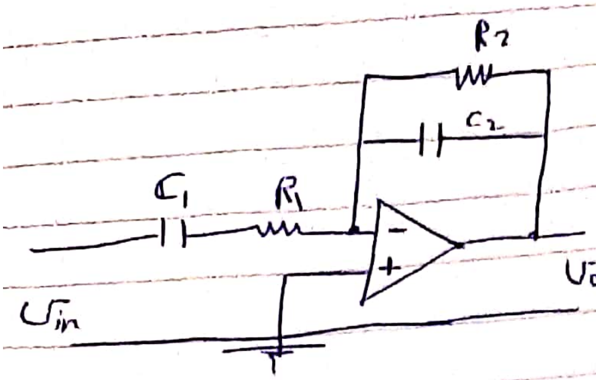
EX 4: Find $\frac{V_o(s)}{V_{in}(s)} = ??$



$$\frac{V_o(s)}{V_{in}(s)} = -\frac{Z_f}{Z_{in}}$$

$$Z_f = Z_{C_2} \parallel Z_{R_2}$$

$$= \frac{R_2\left(\frac{1}{C_2 s}\right)}{R_2 + \frac{1}{C_2 s}} = \frac{R_2(1)}{R_2 C_2 s + 1}$$

$Z_{in} = Z_{C_1}$ series with Z_{R_1}

$$= R_1 + \frac{1}{C_1 s} = \frac{C_1 R_1 s + 1}{C_1 s}$$

$$\frac{V_o(s)}{V_{in}(s)} = -\frac{\frac{R_2}{R_2 C_2 s + 1}}{\frac{C_1 R_1 s + 1}{C_1 s}} = -\frac{R_2 C_1 s}{(R_2 C_2 s + 1)(R_1 C_1 s + 1)}$$

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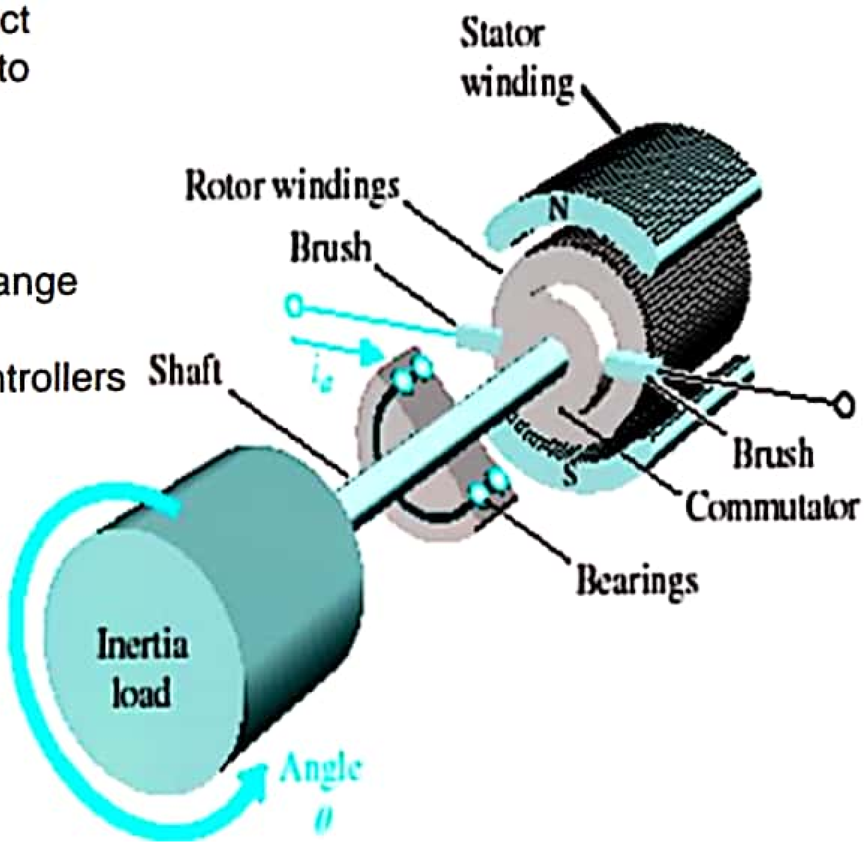
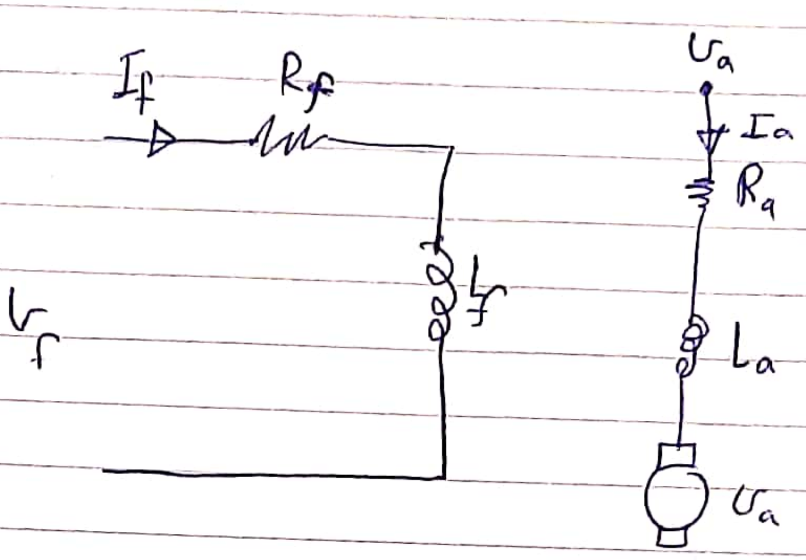
Examples: TF's of DC motors

- A DC motor is used to move loads and is called an **actuator**.
- The DC motor converts direct current (DC) electrical energy into rotational mechanical energy
- DC motors features:
  - High output torque
  - Speed controllability over a wide range
  - Portability
  - Adaptability to various types of controllers

DC motors are widely used in numerous control applications such as robotic manipulators, tape transport mechanisms, disk drives, and machine tools



* DC- motor :



Field (stator) Armature (Rotor)

→ $\boxed{T_m = K_1 \phi I_a I_f}$

T_m = motor torque , ϕ = air gap flux

I_a = armature current , I_f = field current

$\boxed{T_m = J\ddot{\theta} + b\dot{\theta}}$

(1) Field - Controlled DC-motor: $\left(\frac{\theta(s)}{V_f(s)}\right)$

$$T_m = J\ddot{\theta} + b\dot{\theta} = K_m I_f$$

$$J s^2 \theta(s) + b s \theta(s) = K_m I_f(s)$$

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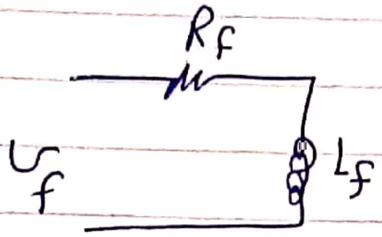
$$\boxed{\frac{\theta(s)}{I_f(s)} = \frac{k_m}{Js^2 + bs}}$$



$$v_f = R_f I_f + L_f \frac{dI_f}{dt}$$

$$v_f(s) = R_f I_f(s) + L_f s I_f(s)$$

$$v_f(s) = I_f(s)[R_f + L_f s]$$

$$\boxed{I_f(s) = \frac{v_f(s)}{R_f + L_f s}}$$

$$\boxed{\frac{\theta(s)}{v_f(s)} = \frac{k_m}{(Js^2 + bs)(R_f + L_f s)}}$$

$$\frac{\omega(s)}{I_f(s)} = ?$$

Let $\dot{\theta} = \omega \Rightarrow \dot{\omega} = \ddot{\theta}$

$$J\dot{\omega} + b\omega = k_m I_f$$

$$J s\, \omega(s) + b\, \omega(s) = K_m I_f(s)$$

$$\boxed{\frac{\omega(s)}{I_f(s)} = \frac{K_m}{Js + b}}$$

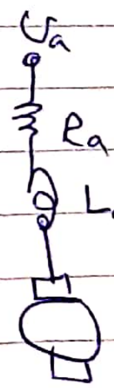
② Armature - controlled DC - motor $\left(\frac{\theta(s)}{V_a(s)} \right)$

$$T_m = J\ddot{\theta} + b\dot{\theta} = K_m I_a$$

$$(Js^2 + bs)\, \theta(s) = K_m I_a(s)$$

$$\frac{\theta(s)}{I_a(s)} = \frac{K_m}{Js^2 + bs}$$

$$V_a = R_a I_a + L_a \frac{dI_a}{dt} + V_b$$



$V_b = K_b \dot{\theta}$ induced voltage

$$V_a - V_b = R_a I_a(s) + L_a s I_a(s)$$

$$V_a(s) - V_b(s) = I_a(s)\left[R_a + L_a s\right]$$

$$\boxed{I_a(s) = \frac{V_a(s) - V_b(s)}{(R_a + L_a s)}}$$

~~$\theta(s)$~~

$$(Js^2 + bs)\,\theta(s) = K_m\left(\frac{V_a - V_b}{R_a + L_a s}\right)$$

$$(Js^2 + bs)\,\theta(s) + \frac{K_m K_b\, s\, \theta(s)}{R_a + L_a s} = \frac{K_m V_a(s)}{R_a + L_a s}$$

$$\theta(s)\left[\left[Js^2 + bs\right] + \frac{K_m K_b s}{R_a + L_a s}\right] = \frac{K_m V_a}{R_a + L_a s}$$

$$\boxed{\frac{\theta(s)}{V_a(s)} = \frac{K_m}{(Js^2 + bs)(R_a + L_a s) + K_m K_b s}}$$

Find $\dfrac{\omega(s)}{I_a(s)} = ?$

$$\frac{\omega(s)}{V_a(s)} = ?$$

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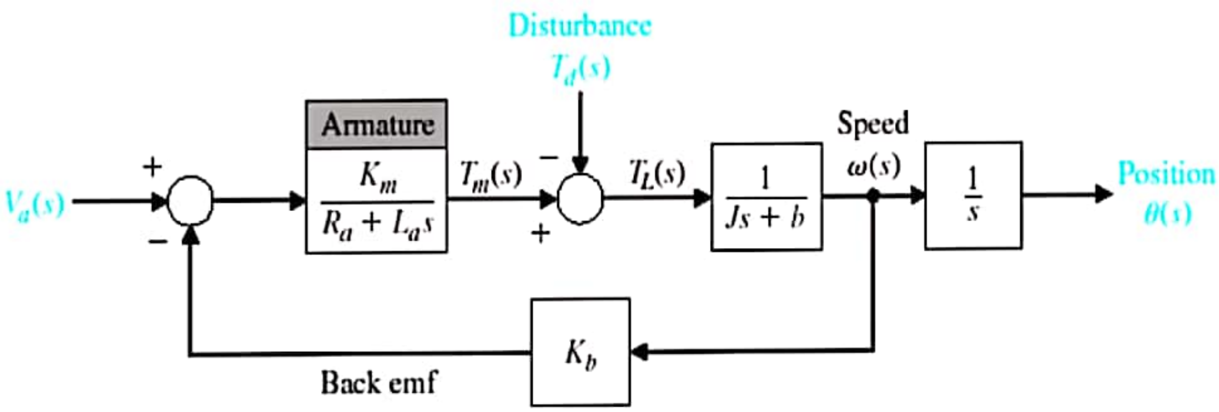
Block Diagram (BD) Models

Again:

- Control systems consists of elements that are represented mathematically by a set of simultaneous differential equations
- Laplace transformation reduces the problem of differential equations to the solution of a set of linear algebraic equations.
- Since control systems are concerned with the control of specific variables, the controlled variables must relate to the controlling variables
  - This relationship is typically represented by the TF of the subsystem relating the input and output variables
  - The importance of the TF is evidenced by the ability to represent the relationship of system variables by diagrammatic means called BD

Hence, the control system with all its elements can be represented by one BD showing all variables relations

Armature controlled DC motor BD

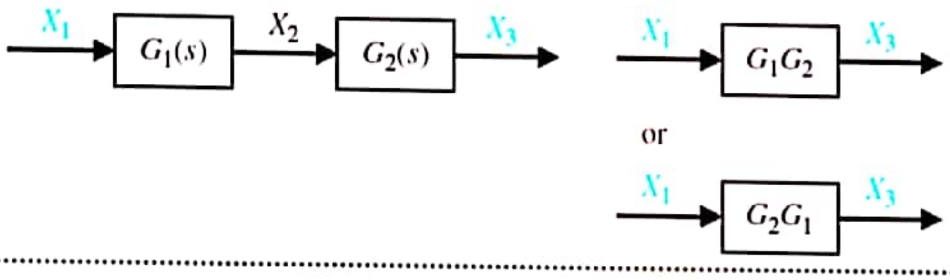


In order to find the cause-effect relationship of a system BD, we simplify the BD (reduction) by applying the rules of BD algebra.

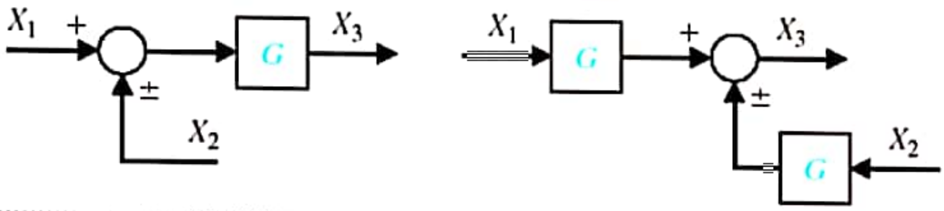
Block Diagram (BD) Algebra

Original Diagram **Equivalent Diagram**

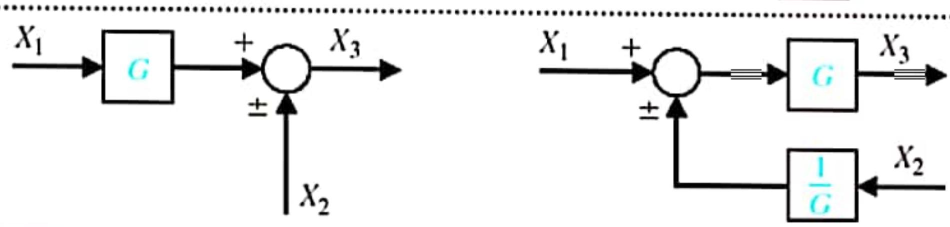
(1)



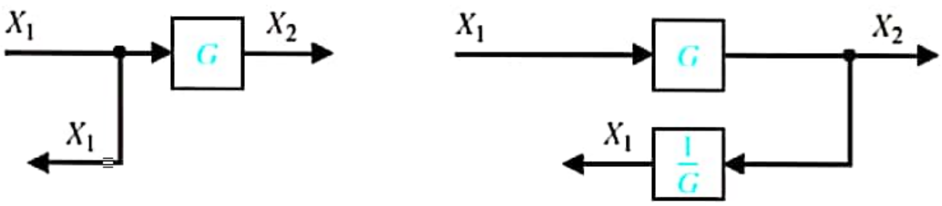
(2)



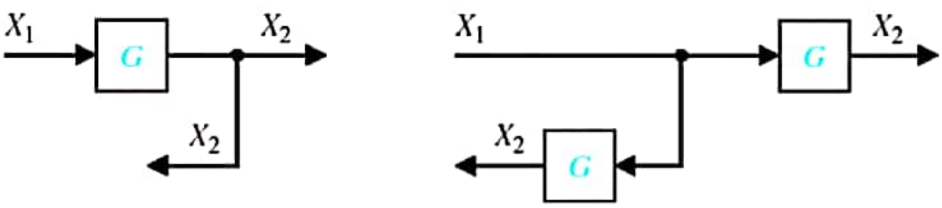
(3)



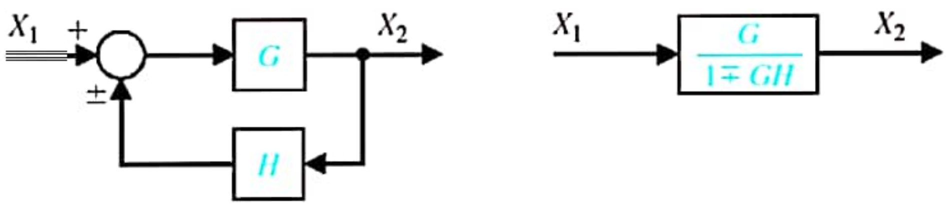
(4)



(5)

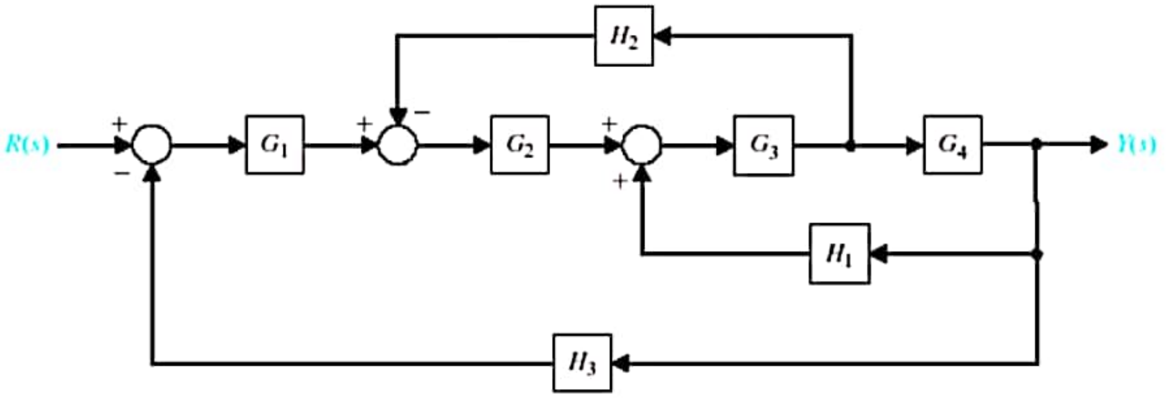


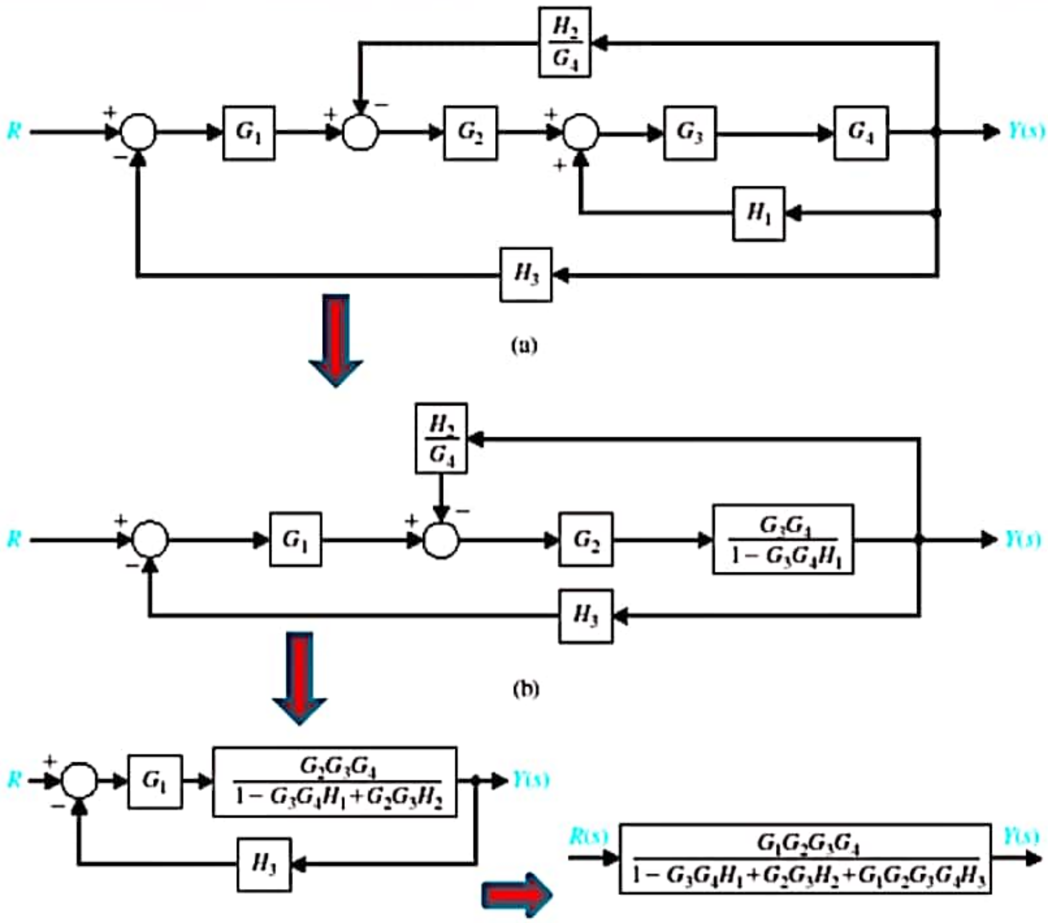
(6)



Example

For the following control system, find the input-output relationship (i.e. TF) relation the output variable Y(s) to the input variable R(s).





(a) (b) (c) (d)

Signal-Flow (SF) Graph Models

- Block diagrams are adequate for the representation of the system interrelationships. However, for a system with reasonably complex interrelationships, the block diagram reduction procedure is often quite difficult to complete.

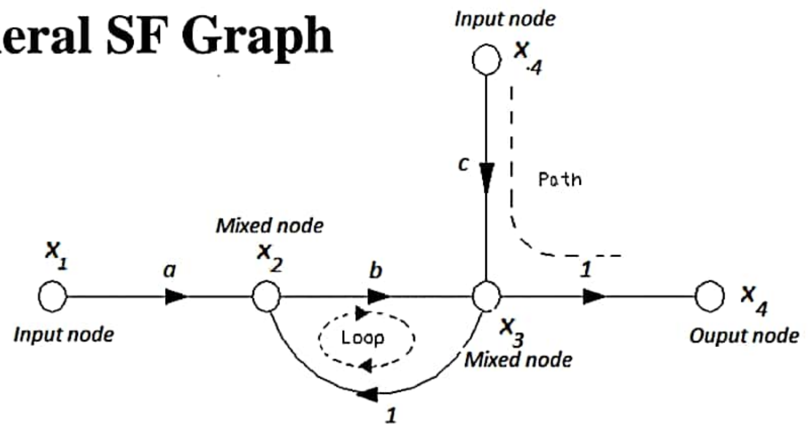
  ⟹ An alternative method for determining the relationship between system variables has been developed by Mason which is called the signal-flow graph method

- A signal-flow graph is a diagram consisting of nodes that are connected by several directed branches and is a graphical representation of a set of linear relations.

- The reduction procedure (used in the BD method) is not necessary to determine TF (input-output relationship) of a system represented by SF graph.

  ⟹ **We apply Mason`s Gain Formula to find the TF**

General SF Graph



Node: acts like a summing point and also represents a system variable

Transmittance: real or complex gain between two nodes.

Branch: directed line segment joining two nodes.

Input node (source): only outgoing branches.

Output node (sink): only incoming branches.

Mixed node: both incoming and outgoing branches

Path: traversal of connected branches in the direction of arrows.

Loop: closed path.

Loop gain: product of branch transmittance at a loop.

Loop gain: product of branch transmittance at a loop.

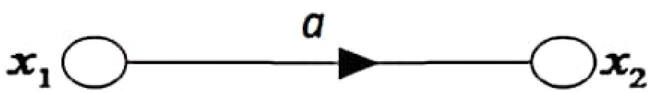
Non touching loops: they do not posses any common nodes.

Forward path: path from an input to an output node that does not cross any node more than once.

Forward path gain: product of transmittances of a forward path

SF Graph Algebra

1)

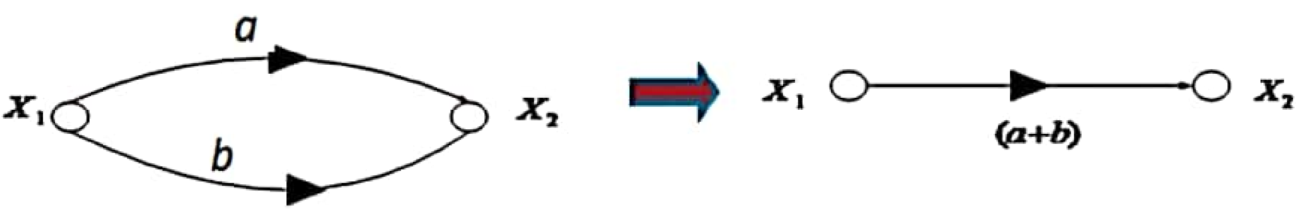


$$x_2 = a x_1$$

2)

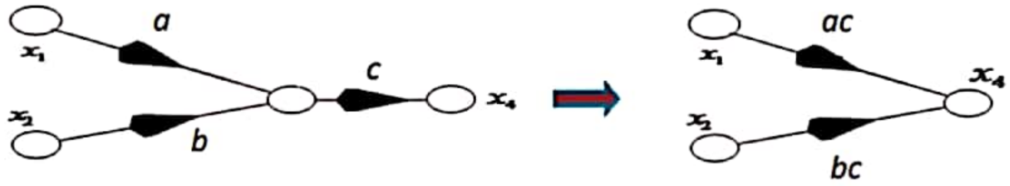
$$x_3 = b x_2 = a b x_1$$

3)

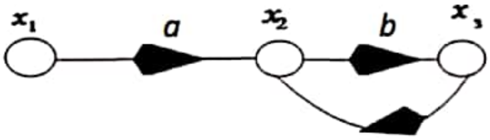


$$x_2 = a x_1 + b x_1 = (a+b) x_1$$

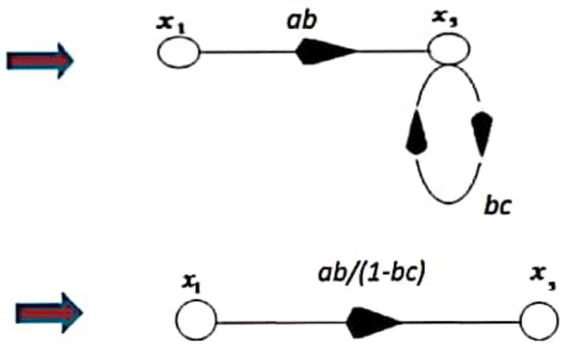
4)



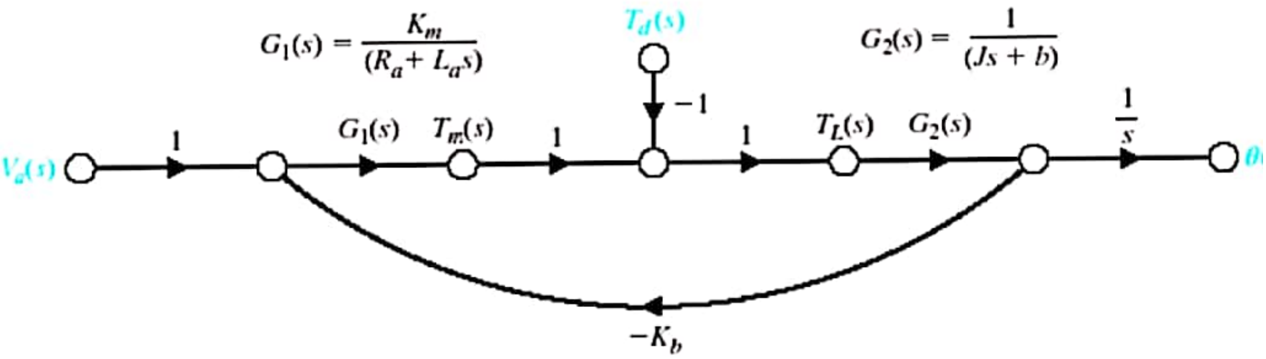
5)



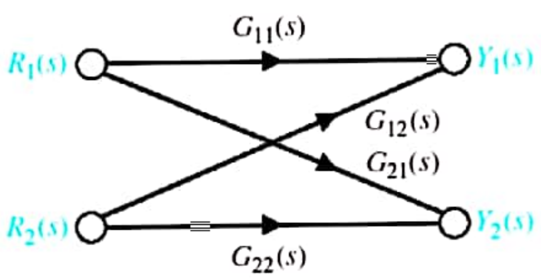
$$x_3 = bx_2$$
$$x_2 = ax_1 + cx_3$$
$$x_3 = b(ax_1 + cx_3)$$
$$x_3 = abx_1 + bcx_3$$



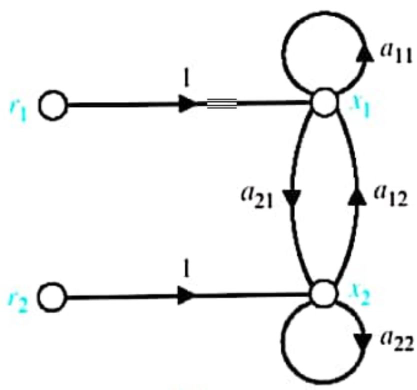
Examples



(1)



(2)



(3)

Mason`s Gain Formula

The formula is often used to relate the output variable Y(s) to the input variable R(s) (i.e. finding the TF) and is given by

$$TF = \frac{\sum_K P_K \Delta_K}{\Delta}$$

where,

P_K is the gain of path K from input node to output node in the direction of the arrows and without passing node than once.

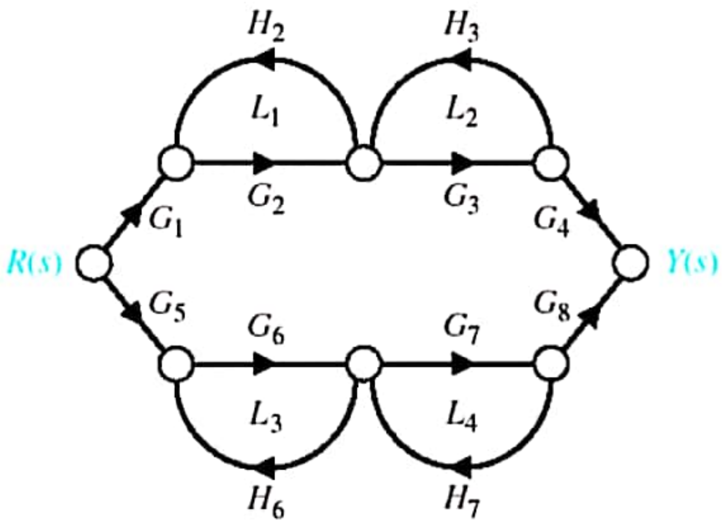
Δ_K: Cofactor o the path P_K

Δ : determinant of the graph

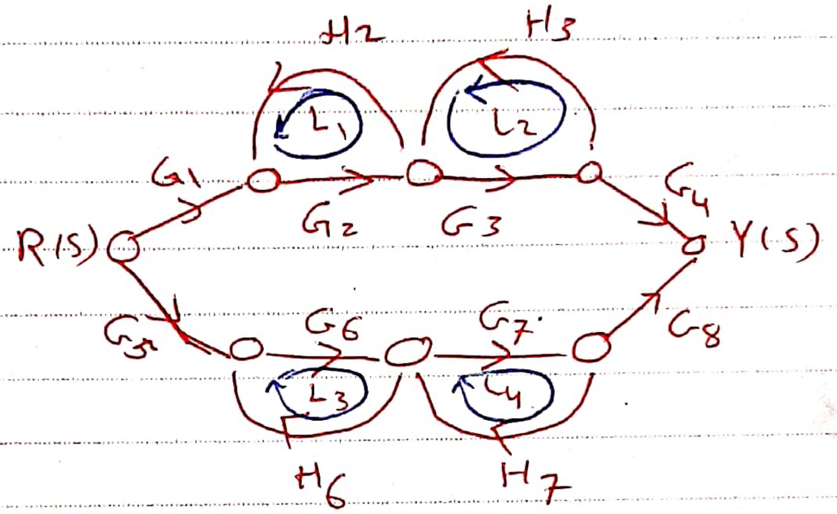
Δ = 1 – (sum of all different loop gains) + (sum of the gain products of all combinations of two non touching loops) – (sum of the gain products of all combinations of three non touching loops)

Example

For the following control system, find the input-output relationship (i.e. TF) relation the output variable Y(s) to the input variable R(s).



Example:— Find the T.F $\frac{Y(S)}{R(S)}$



⇒ (1) Forward path: P_1 and P_2

$P_1 = G_1 G_2 G_3 G_4$, $P_2 = G_5 G_6 G_7 G_8$

(2) Loops: L_1, L_2, L_3 and L_4

$L_1 = G_2 H_2$, $L_2 = G_3 H_3$, $L_3 = G_6 H_6$

$L_4 = G_7 H_7$

non-touching loops
L_1 with L_3, L_4
L_2 with L_3, L_4

(3) $\Delta_1 = 1 - L_3 - L_4$

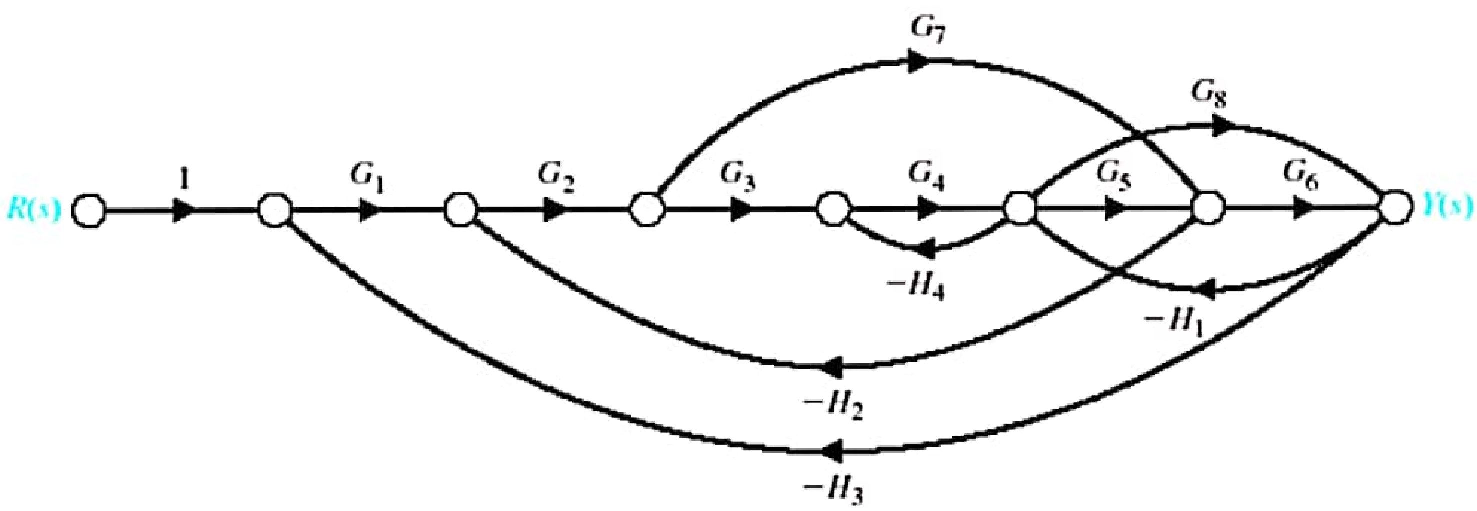
$\Delta_2 = 1 - L_1 - L_2$

(4) $\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1 L_3 + L_1 L_4 + L_2 L_3 + L_2 L_4)$

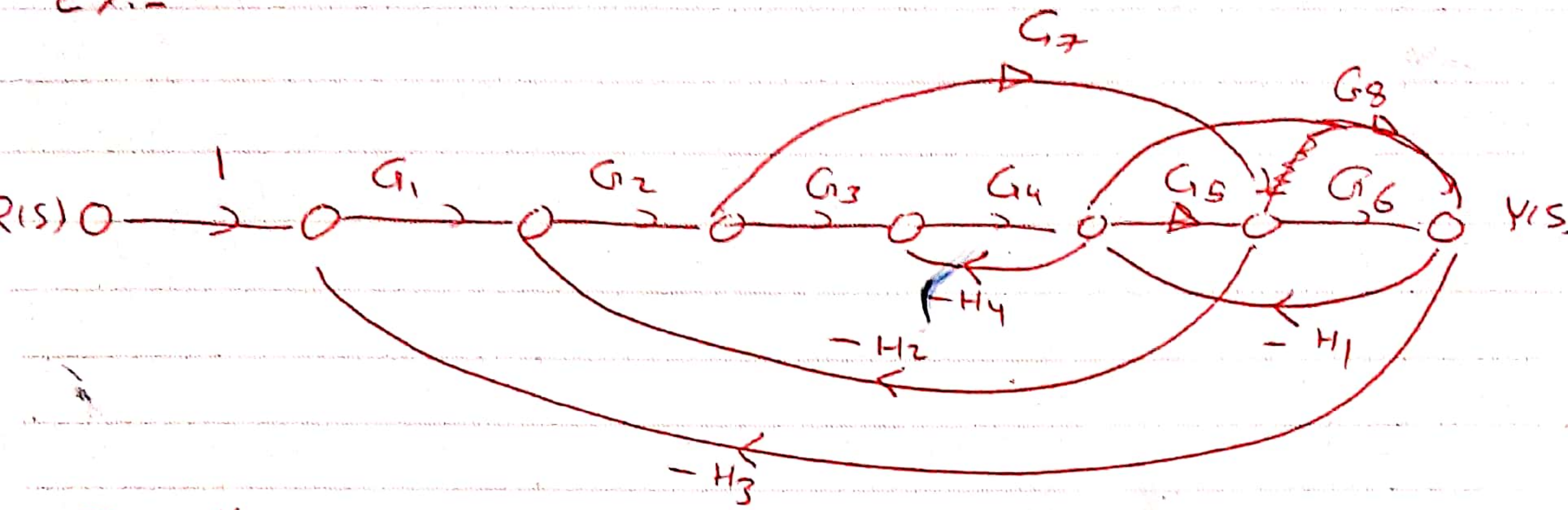
$$\boxed{T.F = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}}$$

Example

For the following control system, find the input-output relationship (i.e. TF) relation the output variable Y(s) to the input variable R(s).



Ex:-



① path

$p_1 = G_1 G_2 G_3 G_4 G_5 G_6$

$p_2 = G_1 G_2 G_7 G_6$

$p_3 = G_1 G_2 G_3 G_4 G_8$

non-touching loops

L_5 doesn't touch L_4, L_7.

L_3 doesn't touch L_4

Loops

$L_1 = -G_2 G_3 G_4 G_5 H_2$

$L_2 = -G_5 G_6 H_1$

$L_3 = -G_8 H_1$

$L_4 = -G_2 G_7 H_2$

$L_5 = -G_4 H_4$

$L_6 = -G_1 G_2 G_3 G_4 G_5 G_6 H_3$

$L_7 = -G_1 G_2 G_7 G_6 H_3$

$L_8 = -G_1 G_2 G_3 G_4 G_8 H_3$

② $\Delta_1 = 1 - 0$

$\Delta_2 = 1 - L_5 = 1 + G_4 H_4$

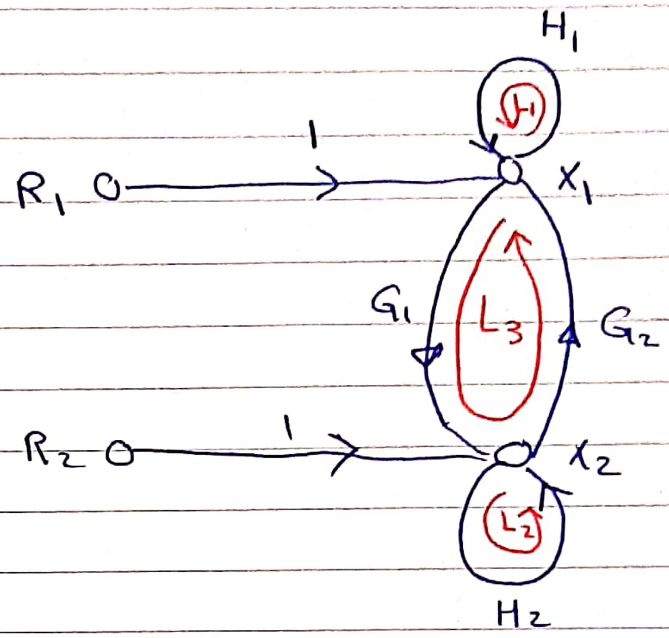
$\Delta_3 = 1 - 0$

$$T.F = \frac{Y(s)}{R(s)} = \cancel{P_1 + P_2 \Delta_2} \frac{P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3}{\Delta}$$

③

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8) + (L_5 L_4 + L_5 L_7 + L_3 L_4)$$

Ex: Find $\dfrac{X_1(s)}{R_1(s)} = ?$



① $\dfrac{X_1(s)}{R_1(s)} = \dfrac{(1)\ (1 - L_2)}{1-(L_1+L_2+L_3)+(L_1 L_2)} = \dfrac{P_1 \Delta_1}{\Delta}$

$$= \frac{1 - H_2}{1-(H_1+H_2+G_1G_2)+(H_1H_2)}$$

② $\dfrac{X_1(s)}{R_2(s)} = \dfrac{G_2\ (1)}{\Delta}$

④ $\dfrac{X_2(s)}{R_2(s)} = \dfrac{(1)(1-H_1)}{\Delta}$

③ $\dfrac{X_2(s)}{R_1(s)} = \dfrac{G_1(1)}{\Delta}$

chapter 4:- Feedback control system characteristics

1. open loop vs closed loop system.

2. error signal (closed loop) & steady state error

3. Sensitivity of control system to parameter variation

4. Disturbance signals in a feedback control system.

→ ① An open loop signal operates without feedback and directly generates the output in response to an input signal.

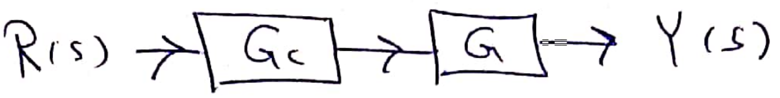
② a closed loop system uses a measurement of the output signal and a comparison with the desired output to generate an error signal that is used by the controller to adjust the actuator.

⇒ Sensitivity = is the ratio of the change in the system T.F to change of a process T.F for a small incremental change

$$S_{G}^{T} = \frac{1}{1 + G_c(S)G(S)}$$

* Sensitivity:-

① open loop control system



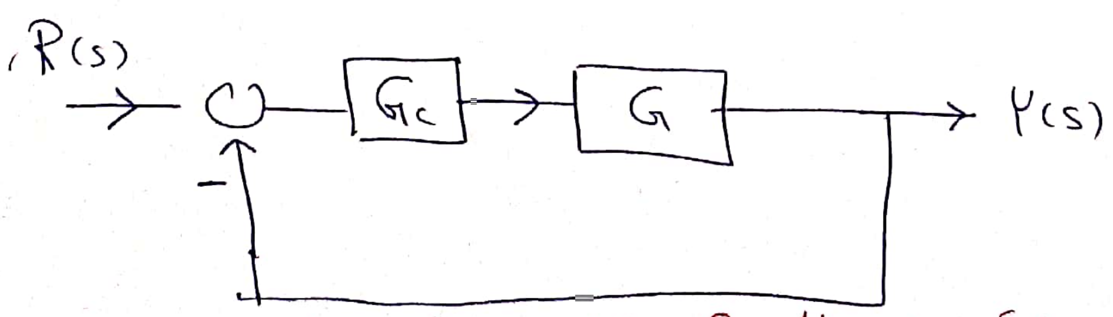
$$S_G^T = \frac{dT}{dG} \cdot \frac{G}{T}$$

$$T = G_c G$$

$$\frac{dT}{dG} = G_c$$

$$\Rightarrow \boxed{S_G^T = G_c \cdot \frac{G}{G_c G} = 1}$$

② Closed loop control system:-



negative unity feedbak Signal

$$S_G^T = \frac{dT}{dG} \cdot \frac{G}{T}$$

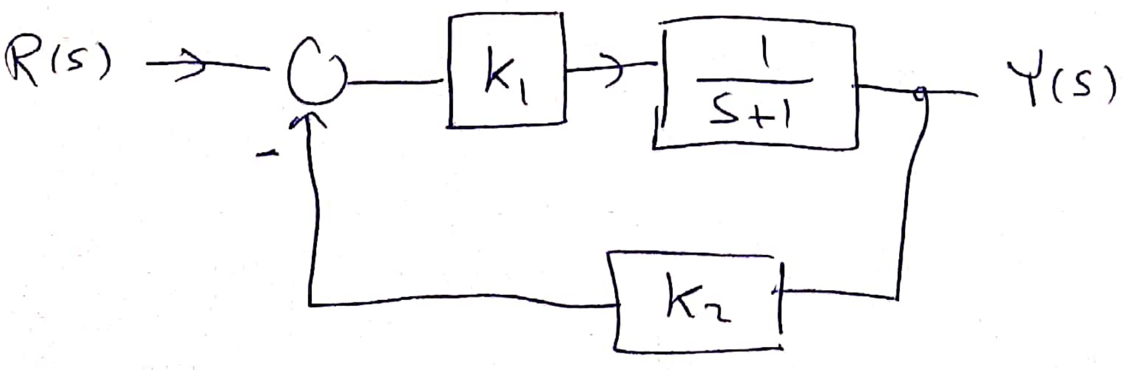
$$T = \frac{G_c\, G}{1 + G_c G}$$

$$\frac{dT}{dG} = \frac{(1 + G_c G)\, G_c - G_c G\,(G_c)}{(1 + G_c G)^2}$$

$$\frac{dT}{dG} = \frac{G_c}{(1 + G_c G)^2}$$

$$\boxed{S^T_G = \frac{G_c}{(1+G_c G)^2} \cdot \frac{\cancel{G}}{\dfrac{G_c \cancel{G}}{\cancel{1+G_c G}}} = \frac{\cancel{G}\ 1}{1 + G_c G}}$$

EX1: Find $S^T_{k_1}$ and $S^T_{k_2}$ = ?



$$S_{k_1}^{T} = \frac{dT}{dk_1} \cdot \frac{k_1}{T}$$

$$T = \frac{k_1}{s+1+k_1k_2} \quad , \quad \frac{dT}{dk_1} = \frac{s+1}{(s+1+k_1k_2)^2}$$

$$S_{k_1}^{T} = \frac{s+1}{((s+1)+k_1k_2)^{\cancel{2}}} \cdot \frac{\cancel{k_1}}{\frac{\cancel{k_1}}{\cancel{s+1+k_1k_2}}} = \frac{s+1}{s+1+k_1k_2}$$

(b) $$S_{k_2}^{T} = \frac{dT}{dk_2} \cdot \frac{k_2}{T}$$

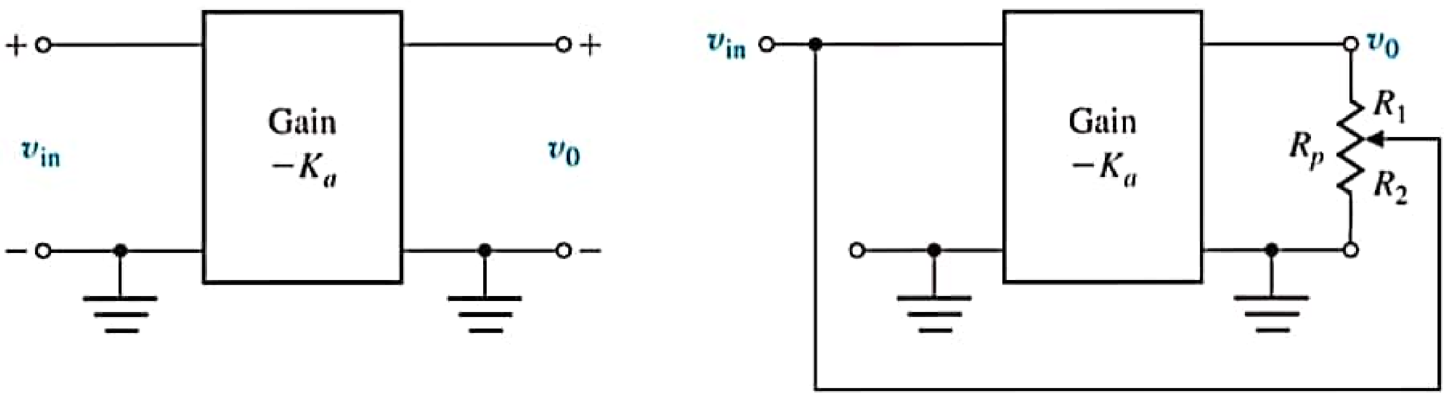
$$\frac{dT}{dk_2} = \frac{-k_1^2}{(s+1+k_1k_2)^2}$$

$$S^{T}_{K_2} = \frac{-K_1^{2}}{(s+1+K_1K_2)^{2}} \cdot \frac{K_2}{K_1 / \cancel{K}s+1+K_1K_2}$$

$$\boxed{S^{T}_{K_2} = \frac{-K_1K_2}{s+1+K_1K_2}}$$

Example: Feedback Amplifier

Study the sensitivity changes for the two cases: open-loop and closed-loop.



* Error Signal and steady-state error

$$\underset{\text{desired}}{} \qquad \underset{\text{actual}}{}$$

$$E(s) = \overset{\text{desired}}{R(s)} - \overset{\text{actual}}{Y(s)}$$

$$e_{ss} = \lim_{s \to 0} s\, E(s)$$

→ open loop control system:

$$E(s) = R(s) - Y(s)$$

$$= R(s) - G_c G\, R(s)$$

$$\boxed{E(s) = R(s)\,[1 - G_c G]}$$

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$$e_{ss} = \lim_{s \to 0} s\, E(s).$$

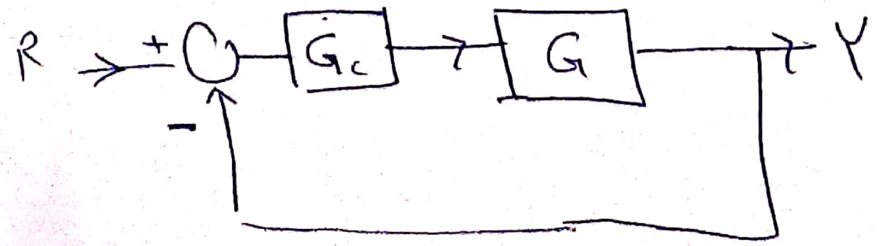
→ For a unit-step input $\left(R(s) = \frac{1}{s}\right)$

$$e_{ss} = \lim_{s \to 0} s\, R(s) \left[1 - G_c(s)\, G(s)\right]$$

$$= \lim_{s \to 0} s \left(\frac{1}{s}\right) \left[1 - G_c G\right]$$

$$\boxed{e_{ss} = 1 - \underbrace{G_c(0)\, G(0)}_{\text{loop gain}}}$$

→ Closed loop control system:



$$E(s) = R(s) - Y(s)$$

$$= R(s) - \frac{G_c G}{1 + G_c G} R$$

$$= R(s) \left[\frac{1}{1 + G_c G} \right]$$

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \left(\frac{1}{s}\right) \left[\frac{1}{1 + G_c G} \right]$$

$$\boxed{E(s) = \frac{1}{1 + G_c(0) G(0)}} < 1 - G_c(0) G(0)$$

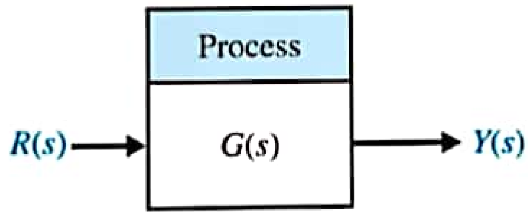
Steady-State Error

The steady-state error is the error after the transient response has decayed, leaving only the continuous steady response.

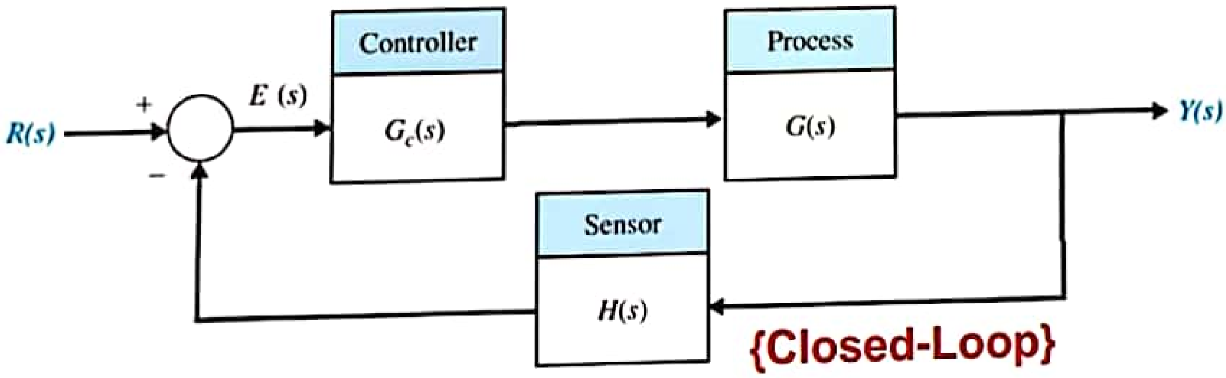
$$E(s) = R(s) - Y(s)$$

$$\frac{E(s)}{R(s)} = 1 - \frac{Y(s)}{R(s)}$$

$$= 1 - G(s)$$



{Open-Loop}



{Closed-Loop}

For unity-feedback control system, i.e., $H(s) = 1$

$$E(s) = R(s) - Y(s)$$

$$\frac{E(s)}{R(s)} = 1 - \frac{Y(s)}{R(s)}$$

$$= 1 - T(s)\ , \qquad T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}$$

$$= \frac{1}{1 + G_c(s)G(s)}$$

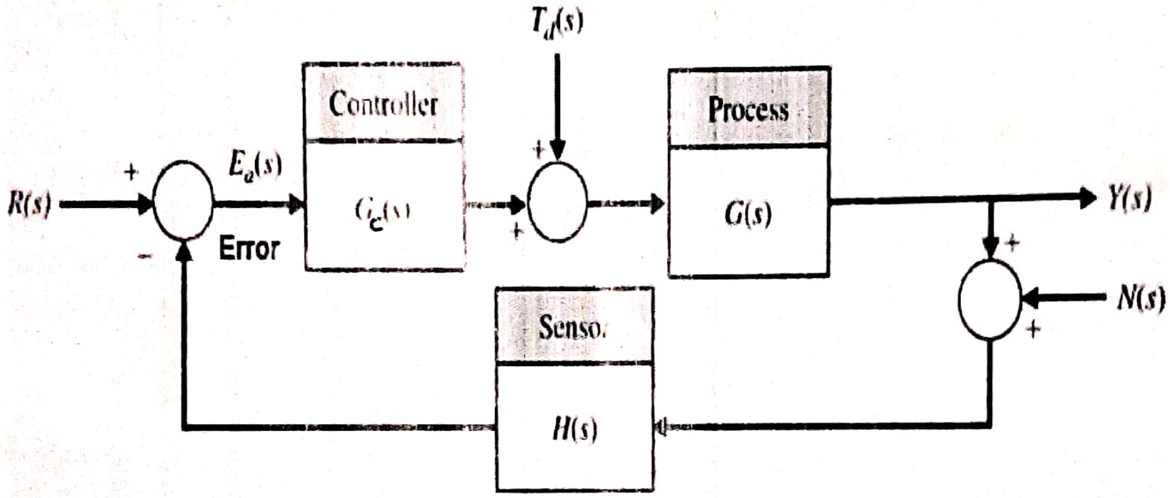
and

$$e_{ss} = e(\infty) = \lim_{s \to \infty} s\, E(s)$$

DISTURBANCE SIGNALS IN FEEDBACK CONTROL SYSTEMS

Disturbance signals represent unwanted inputs which affect the control-system's output, and result in an increase of the system error. It is the job of the control-system engineer to properly design the control system to partially eliminate the affects of disturbances on the output and system error.

A disturbance signal is commonly found in control systems. For example, wind gusts hitting the antenna dish of a tracking radar create large unwanted torques which affect the position of the antenna. Another example, are sea waves hitting a hydrofoil's foil which create very large unwanted torques which affect the foil's position.



① $T_d(s) = 0$ and $N(s) = 0$, $H(s) = 1$

$$Y(s) = \frac{G_c\, G}{1 + G_c\, G} R(s)$$

② $R(s) = 0$ and $N(s) = 0$, $H(s) = 1$

$$Y(s) = \frac{G(s)}{1 + G_c\, G} T_d(s)$$

③ $R(s) = 0$ and $T_d(s) = 0$, $H(s) = 0$

$$Y(s) = \frac{-G_c\, G}{1 + G_c\, G} N(s)$$

Error signal analysis:

$E(s) = R(s) - Y(s)$

But,

Desired Input   Disturbance input   Noise input

$$Y(s) = \frac{G_C(s)G(s)}{1+G_C(s)G(s)}R(s) + \frac{G(s)}{1+G_C(s)G(s)}T_d(s) - \frac{G_C(s)G(s)}{1+G_C(s)G(s)}N(s)$$

$$\therefore\ E(s) = \frac{1}{1+G_C(s)G(s)}R(s) - \frac{G(s)}{1+G_C(s)G(s)}T_d(s) + \frac{G_C(s)G(s)}{1+G_C(s)G(s)}N(s)$$

where,

① Error due to desired input $R(s)$:-

$$\underset{\text{due to } R(s) \text{ only}}{E(s)\downarrow} = \frac{1}{1+G_c G}R(s)$$

② Error due to disturbance only:

$$E(s) = -\frac{G(s)}{1+G_c G}T_d(s)$$

note: $E(s) = R(s) - Y(s)$ but $R(s) = 0$

$\therefore E(s) = 0 - Y(s) = -Y(s)$

③ Error due to noise only:

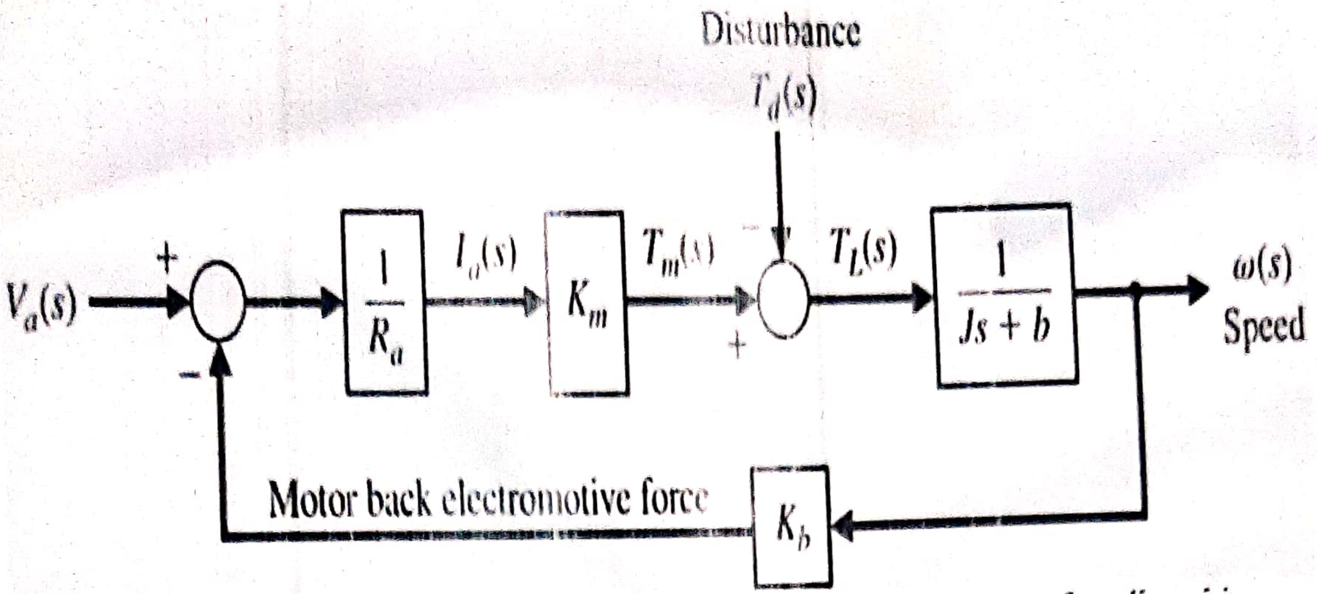
$$E(s) = R(s) - Y(s) \qquad \text{but } R(s) = 0$$

$$E(s) = 0 - Y(s) = + \frac{G_c G}{1 + G_c G} N(s)$$

→ See DC- motor block Diagram

then, calculate $E(s)$ due to

disturbance only.



→ Assuming very small inductance and only disturbance input } → $V_a(s) = 0$ and $T_d(s) = \frac{D}{s}$

$$\therefore \frac{\omega(s)}{T_d(s)} = \frac{-1}{Js + b + \frac{K_m K_b}{R_a}}$$

Steady-state speed due to a step disturbance $T_d(s) = \frac{D}{s}$ is

$$\omega_{ss} = \lim_{s \to \infty} s\omega(s)$$

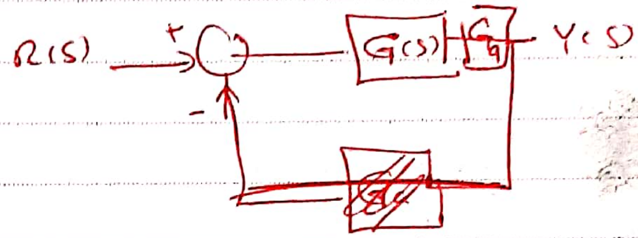
$$= \lim_{s \to \infty} s \frac{-1}{Js + b + \frac{K_m K_b}{R_a}} \frac{D}{s}$$

$$(\omega_{ss})_{open} = -\frac{D}{b + \frac{K_m K_b}{R_a}}$$

Ex:- $G(s) = \frac{10}{s(0.001s+1)}$ for step input what is e_{ss}

$G_a = K$

$$e_{ss} = \lim_{s \to 0} s\, E(s)$$



$$T(s) = \frac{G(s)}{1 + G_a G(s)}$$

$$E(s) = R(s) - \frac{G G_a}{1 + G_a G} R(s)$$

$$E(s) = R(s)\left[\frac{s(0.001s+1)}{s(0.001s+1) + 10K}\right]$$

$$= \frac{1}{s}\left[1 - \frac{G\,\cancel{G_a}K}{KG+1}\right]$$

$$\frac{1 + KG(0) - KG(0)}{1 + KG(0)}$$

$$\lim_{s \to 0} s \times \frac{1}{s}\left[1 - \frac{G(0)K}{KG(0)+1}\right] = \frac{1}{K\left(\frac{10}{0}\right)} =$$

$$= \frac{1}{1 + KG(0)} = 0$$

if the error $= 0.1$ mm and $R(s) =$ ramp $= 10$ m/s

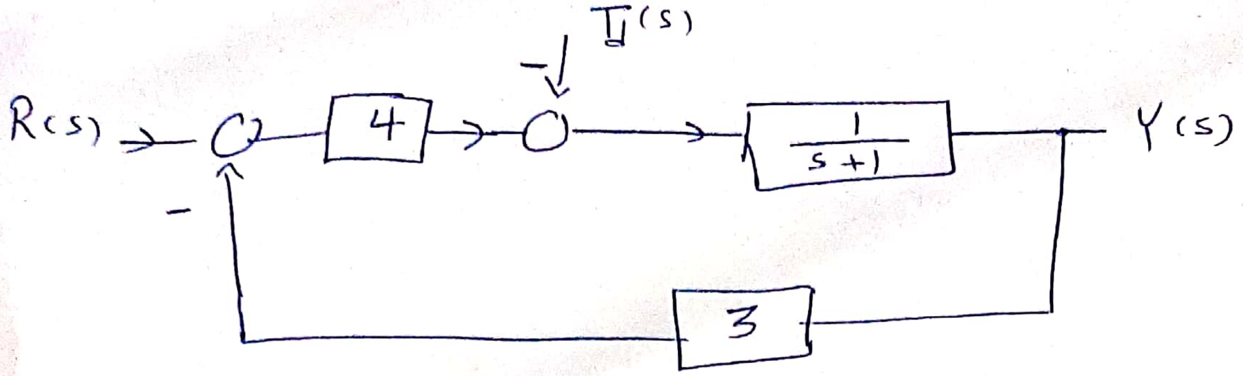
Find K_a?

$$R(t) = 10t \Rightarrow R(s) = \frac{10}{s^2}$$

$$E_{ss} = 0.1 * 10^{-3} \text{ m} = \lim_{s \to 0} s\, E(s) =$$

$$\Rightarrow K_a = 10{,}000$$

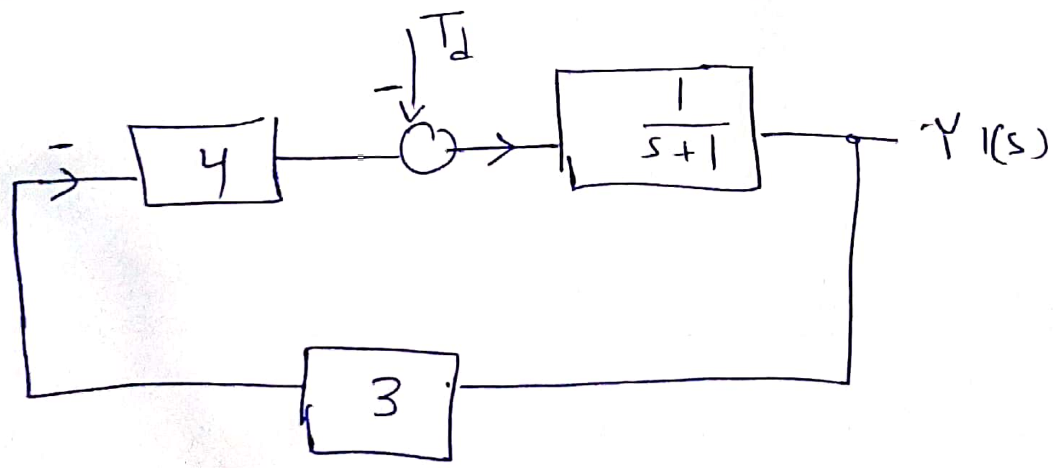
Ex: Find the steady-state error due to disturbance.



Solution: due to disturbance $R(s) = 0$

the block diagram when $R(s) = 0$

will be



→ forward path is $\frac{1}{s+1}$

→ feedback signal is $-3 * 4$

→ $T_d(s)$ is negative → $-T_d(s)$

$$Y(s) = \frac{\frac{1}{s+1}}{1 + \frac{1}{s+1}(12)} \left(-T_d(s)\right)$$

$$Y(s) = \frac{1}{(s+1)+12} \left(-T_d(s)\right)$$

So $\therefore$ $E(s)\Big|_{\text{due to } T_d} = R(s) - Y(s)$

but $R(s) = 0$

$$\Rightarrow E(s) = 0 - Y(s)$$

$$= -\left(\frac{-T_d(s)}{s+1+12}\right)$$

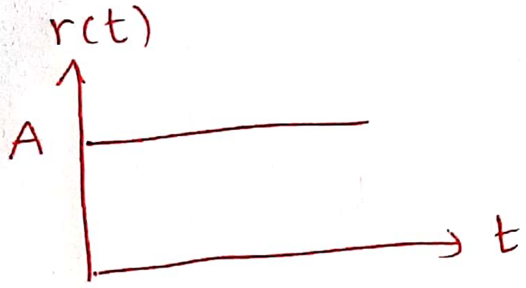
$$E(s) = + \frac{T_d(s)}{s+13}$$

chapter 5: The performance of feedback control system.

→ Test input signals for the time response of control system.

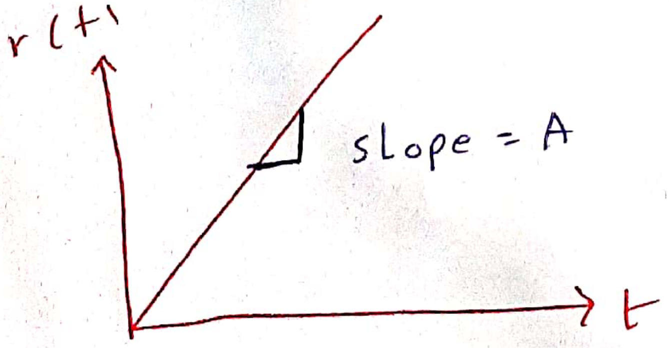
① step input

$$R(t) = A \Rightarrow R(s) = \frac{A}{s}$$



$$r(t) = \begin{cases} A, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

② Ramp input



$$r(t) = \begin{cases} At, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

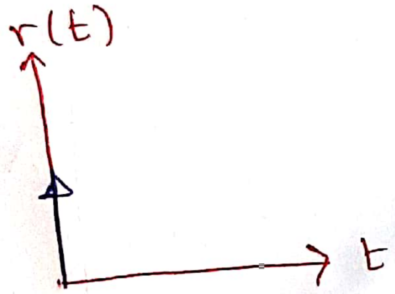
$$R(s) = \frac{A}{s^2}$$

③ parabolic input

$$r(t) = \begin{cases} A t^2 & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

$$R(s) = \frac{2A}{s^3}$$

④ unit-impulse input



unit → 1

→ for a unit-impulse input always $R(s) = 1$

→ remember table 2.3

→ order System Response:

① zero-order system.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K}{s^0} = K$$

(2) First-order System

$$R(s) \rightarrow \boxed{\frac{K}{\tau s+1}} \rightarrow Y(s)$$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K}{\tau s+1}$$ ... standard form of first-order transfer function

where : K is the gain of the system

τ is the time constant of the system.

* * * *

DC - gain $\quad K = \dfrac{y_{ss}}{R_{ss}}$ ... (1)

$$y(\tau) = 0.63\, y_{ss} \quad \text{--- (2)}$$

→ for a unit-step input $R(s) = \frac{1}{s}$

$$y(t) = K(1 - e^{-t/\tau}) \quad \text{--- (3)}$$

note: → For first - order system

$$\frac{Y(s)}{R(s)} = \frac{K}{\tau s+1}$$

wher $R(s) = \frac{1}{s} \Rightarrow y(s) = \frac{1}{s}\left(\frac{k}{\tau s+1}\right)$

take Laplace inverse of $y(s)$

$$y(t) = k(1 - e^{-t/\tau})$$

$$y(s) = \frac{A}{s} + \frac{B}{s+\frac{1}{\tau}} = \frac{k/\tau}{s(\tau s+1)}$$

$$y(t) = A + B e^{-t/\tau}$$

when $s = 0 \Rightarrow A(s+\frac{1}{\tau}) + Bs = \frac{k}{\tau}$

$$\boxed{A = k}$$

$$s = \frac{-1}{\tau} \Rightarrow B = -k$$

$$\Rightarrow y(t) = K - k e^{-t/\tau}$$

$$\boxed{y(t) = k(1 - e^{-t/\tau})}$$

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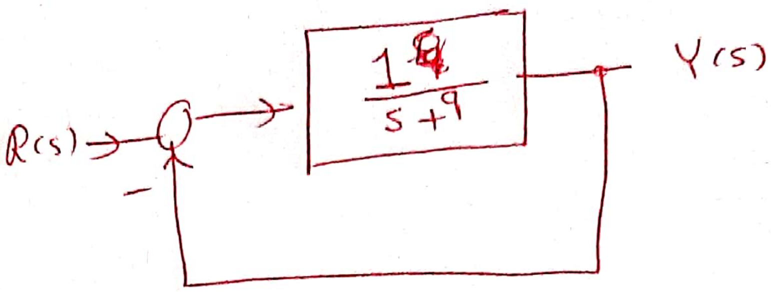
EX 1 : For a unit-step input, Find $y(t)$ for system below.



$$\frac{Y(s)}{R(s)} = \frac{\frac{1}{s+9}}{1+\left(\frac{1}{s+9}\right)(1)} = \frac{1}{s+10} = \frac{1}{s+10}$$

standard form $\frac{K}{\tau s+1}$

$$\Rightarrow \frac{\frac{1}{10}}{\frac{s}{10}+\frac{10}{10}} = \frac{0.1}{0.1s+1}$$

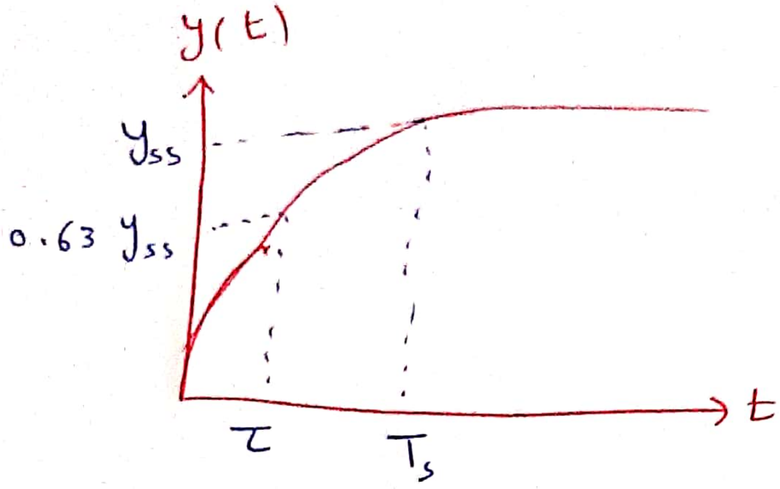
$\Rightarrow$ $\tau = 0.1$ second

$K = 0.1$

$\therefore$ $y(t) = K\left(1 - e^{-t/\tau}\right)$

$= 0.1\left(1 - e^{-t/0.1}\right)$

→ plot first-order response:

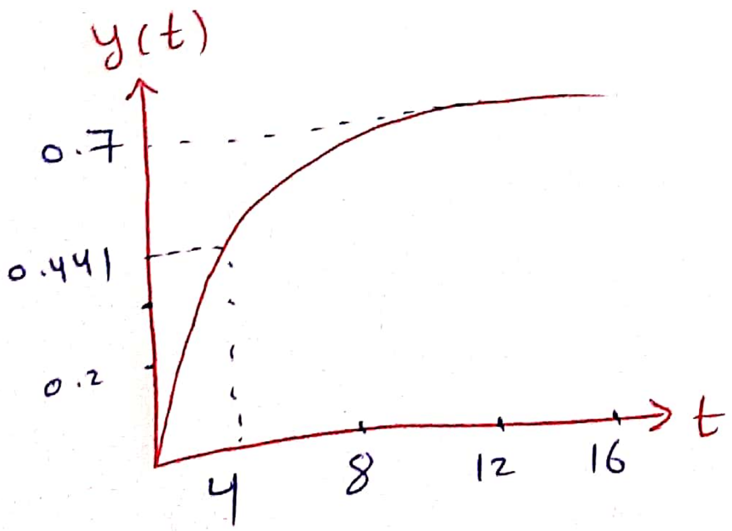


T_s = settling time

$= 4\tau$

T_s = is the time required to reach the steady-state the value of the response.

Ex2:- Find $y(t)$ and T_s for the following response and $R(s) = \frac{1}{s}$



Solution:—

$y_{ss} = 0.7$ from plote

$R_{ss} = \lim_{s \to 0} s\,R(s)$

$= \lim_{s \to 0} s\,\frac{1}{s} = 1$

then, $k = \frac{y_{ss}}{R_{ss}} = \frac{0.7}{1}$

Ex2: Continue ----

$y(\tau) = 0.63\, y_{ss} = 0.63(0.7) = 0.441$

from plot when $y(t) = 0.441$

$t = 4$ seconds $= \tau$

$\Rightarrow y(t) = k(1 - e^{-t/\tau})$

$$\boxed{y(t) = 0.7(1 - e^{-t/4})}$$

$T_s = 4\tau = 4(4) = 16$ seconds.

* * * *

3 Second-order system:-

$$R(s) \rightarrow \boxed{\frac{K a}{s^2 + 2bs + a^2}} \rightarrow Y(s)$$

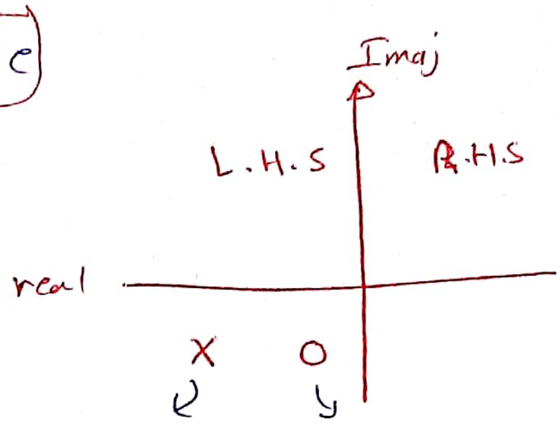
or $\dfrac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

ω_n = natural frequency

ζ = damping ratio

* stabitiy is an important concept.

① The system is stable if all $\boxed{\text{Poles}}$ are on the Left hand side of $\boxed{\text{s-plane}}$



Transfer function لل أصفار المقام ← Pole

zero

أصفار البسط لل Transfer function

* * *

L.H.S = Left hand side

R.H.S = Right hand side

* * *

* * * → damping ration can be determined depnding on s-plane

→ A second-order system can be set into one of the following four cases:

$\boxed{1}$ Undamped system $(\zeta = 0)$

ζ : damping ratio $= \dfrac{c}{2 m \omega_n}$

where c = damping coefficient

m = mass

ω_n = natural frequency

→ For undamped 2^{nd} - order system:

$$T(s) = \frac{K \omega_n^2}{s^2 + \omega_n^2} \quad , \zeta = 0$$

← بعد تعويض قيمة ζ
صفر في القانون العام
لل 2^{nd} - order

$$s^2 + \omega_n^2 = 0 \Rightarrow s_{1,2} = \mp j\omega_n$$

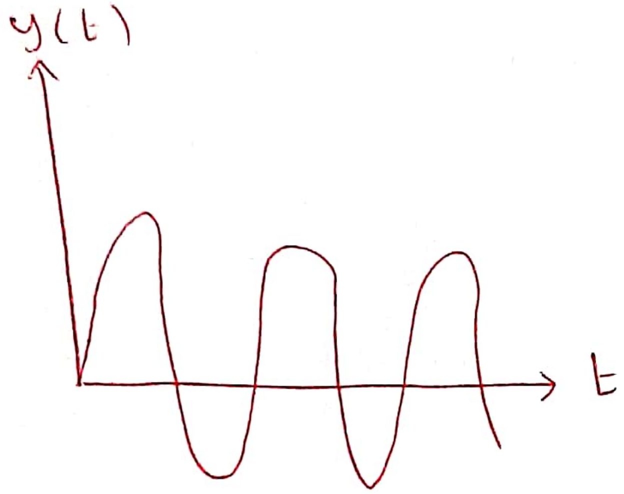
imaginary roots

$\Rightarrow y(t)$ will be sine or cosine

For a unit-step input ($R(s) = \frac{1}{s}$)

$$\frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + \omega_n^2}$$

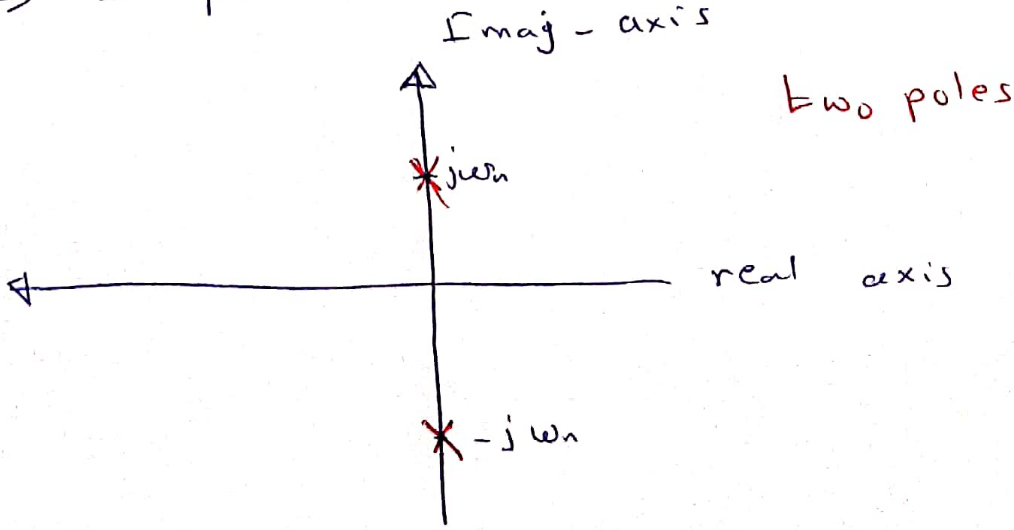
$$y(t) = 1 - \cos(\omega_n t)$$



← y(t) Bounded Input Bounded Output

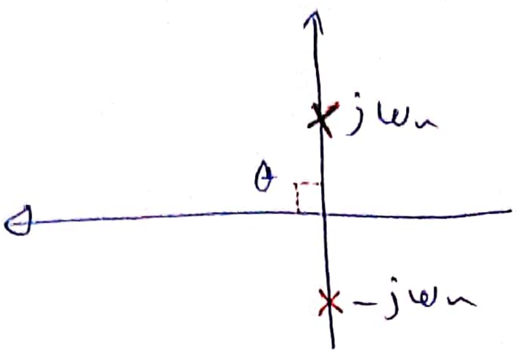
(stable system)

→ s-plane



How???

For undamped system



$\zeta = \cos\theta$

$\zeta = \cos(90^\circ)$

$\boxed{\zeta = 0}$

→ θ is measured from negative real axis

$\boxed{2}$ underdamped - 2^{nd} order system

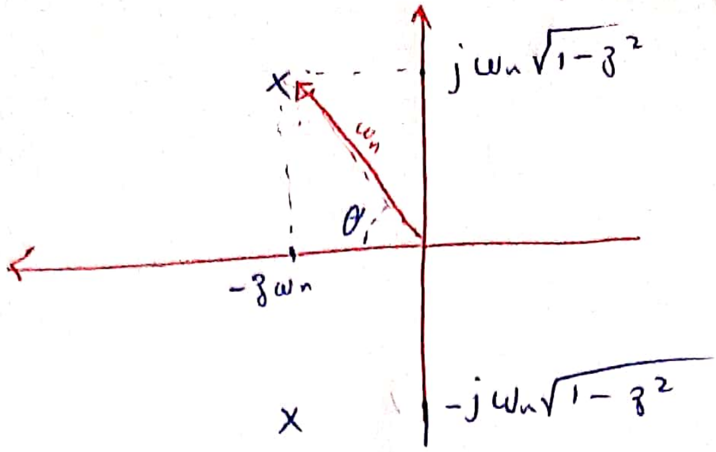
$$\boxed{0 < \zeta < 1}$$

$$T(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

تحليل العبارة التربيعية باستخدام القانون العام ⇐

$$s_{1,2} = -\zeta\omega_n \mp j\omega_n\sqrt{1-\zeta^2}$$

L.H.S R.H.S



$\Rightarrow \zeta = \cos\theta$

$0 < \theta < 90^\circ$

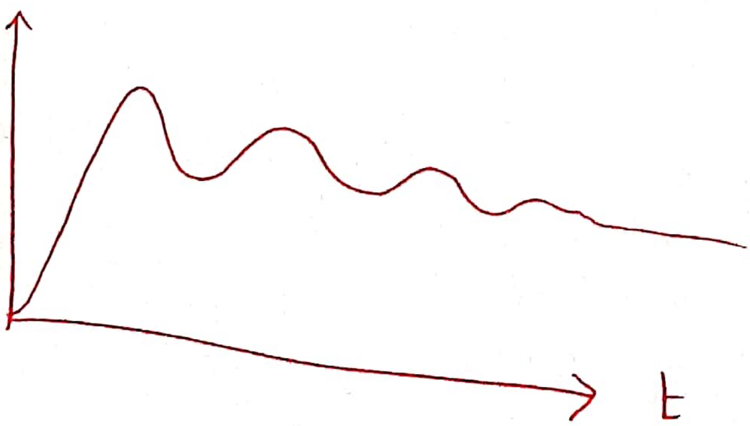
$\Rightarrow 0 < \zeta < 1$

for $R(s) = \frac{1}{s}$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \cos^{-1}(\zeta))$$

ω_d = damped frequency $= \omega_n\sqrt{1-\zeta^2}$

y(t)

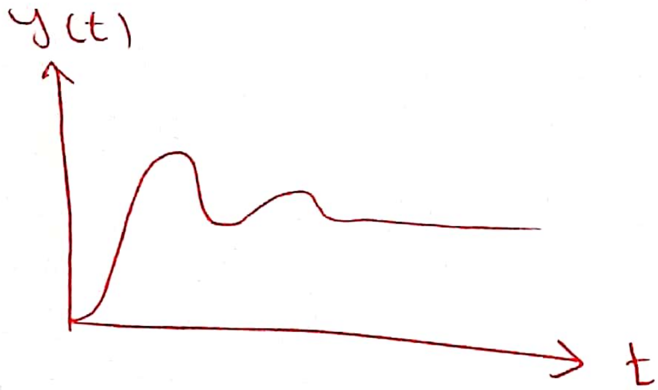


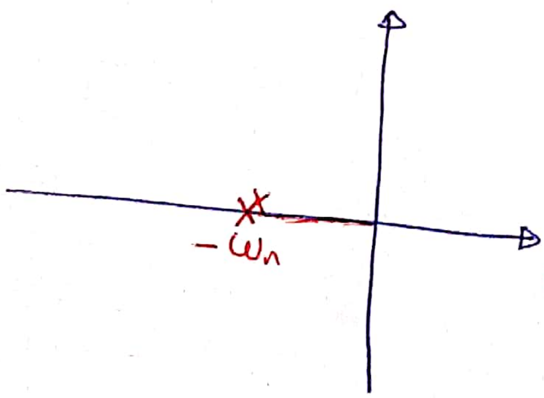
$\boxed{3}$ critically damped ($\zeta = 1$)

$$T(s) = \frac{k\,\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$S_{1,2} = -\omega_n$ (repeated and real roots)

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$





$\zeta = \cos\theta$

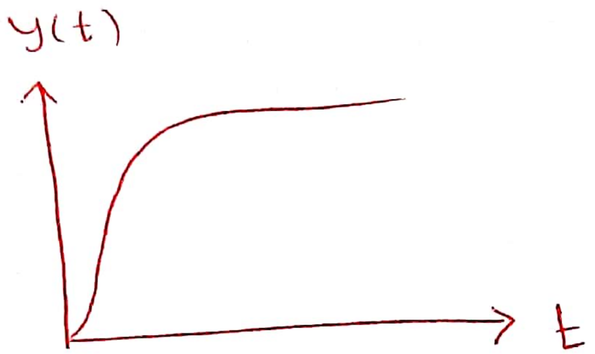
$\zeta = \cos(0)$

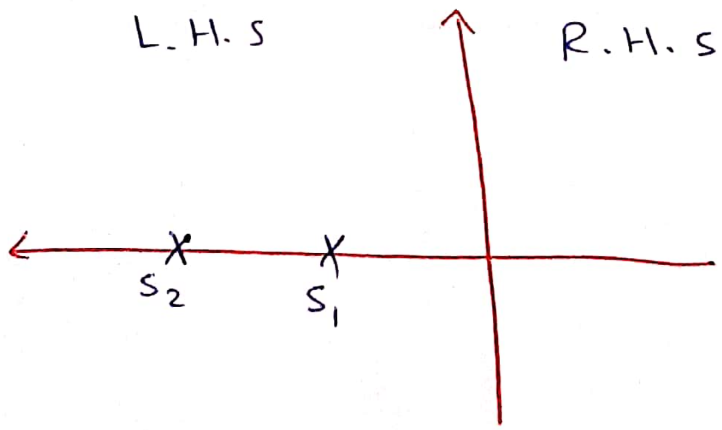
$\boxed{\zeta = 1}$

$\boxed{4}$ overdamped system ($\zeta > 1$)

$$S_{1,2} = -\zeta\omega_n \mp \omega_n\sqrt{\zeta^2-1}$$

$$y(t) = 1 + \frac{\omega_n}{\sqrt{\zeta^2-1}}\left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right]$$





Ex 1: For a unit-step input, Find

(a) $y(t) = ??$

$\boxed{a}$ $T(s) = \dfrac{9}{s^2 + \underbrace{9}_{=\omega_n^2}}$

$\omega_n^2 = 9 \Rightarrow \omega_n = +3$ rad/s

$\omega_n = -3$ X canceled

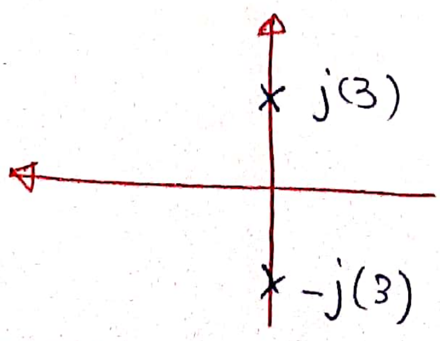
$K \omega_n^2 = 9$

$\boxed{K = 1}$

معامل s ← $2\zeta\omega_n = 0$

$\boxed{\zeta = 0} \rightarrow$ undamped ~~sqst~~ response

$\Rightarrow \boxed{y(t) = 1 - \cos 3t}$



Marginally stable

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$\boxed{b}$ $T(s) = \dfrac{9}{s^2 + 2s + 9}$

$\omega_n = \sqrt{9} = 3 \text{ rad/s}$

$2\zeta\omega_n = 2$

$2\zeta(3) = 2 \Rightarrow \boxed{\zeta = \frac{1}{3} < 1}$

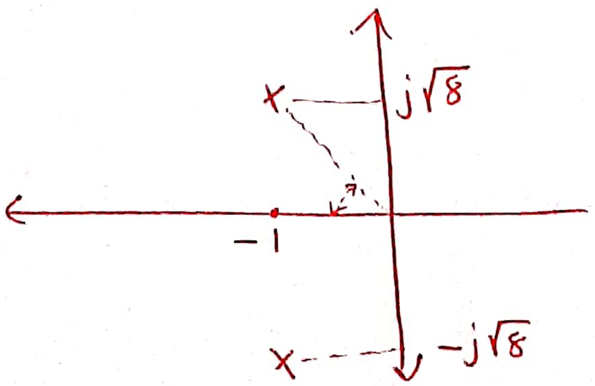
underdamped response

$k\omega_n^2 = 9 \Rightarrow \boxed{k = 1}$

$y(t) = 1 - 1.06\, e^{-t} \cos(\sqrt{8}\, t - 19.47)$

— OR —

$y(t) = 1 - 1.06\, e^{-t} \sin(\sqrt{8} t - 70.5)$



c

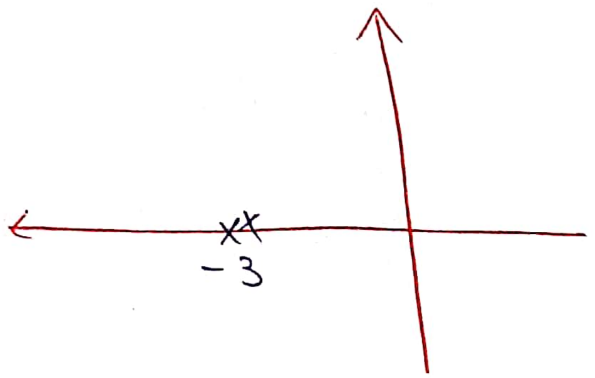
$$T(s) = \frac{9}{s^2 + 6s + 9}$$

$\omega_r = 3$ rad/s and $2\zeta\omega_n = 6$

$$2\zeta(3) = 6$$

critically damped $\leftarrow \boxed{\zeta = 1}$

$$\Rightarrow \boxed{y(t) = 1 - e^{-3t} - 3t\,e^{-3t}}$$



d

$$T(s) = \frac{9}{s^2 + 9s + 9}$$

$\omega_r = 3$ rad/s and $2\zeta\omega_n = 9$

$$2(3)\zeta = 9$$

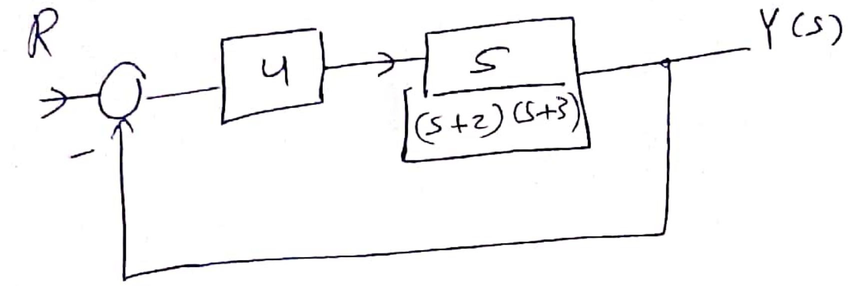
overdamped response $\leftarrow \zeta = 1.5 > 1$

$$y(t) = 1 + \frac{3}{\sqrt{1.5^2 - 1}} \left[\frac{e^{-s_1}}{s_1} - \frac{e^{-s_2}}{s_2} \right]$$

$$s_{1,2} = -(1.5)(3) \mp 3\sqrt{1.5^2 - 1}$$

* * *

Ex2:



Find $y(t) = ?$ for a unit-step input

Sol:-

$$T(s) = \frac{\frac{20}{s^2+5s+6}}{1 + \frac{20}{s^2+5s+6}} = \frac{20}{s^2+5s+26}$$

$\omega_n = \sqrt{26}$ and $2\zeta\omega_n = 5$

$2\zeta(\sqrt{26}) = 5$

$\zeta < 1 \Rightarrow$ underdamped

Ex3: Find system response ??

$$T(s) = \frac{4}{2s^2 + 8s + 8}$$

should be 1 ⇐

معامل s^2 ← 1

نقسم المعادلة على 2 ⇐

ليصبح معامل $s^2 = 1$

$$\Rightarrow \quad T(s) = \frac{2}{s^2 + 4s + 4}$$

$$\omega_n = \sqrt{4} = 2 \text{ rad/s}$$

$$2\zeta\omega_n = 4 \Rightarrow 2(2)\zeta = 4$$

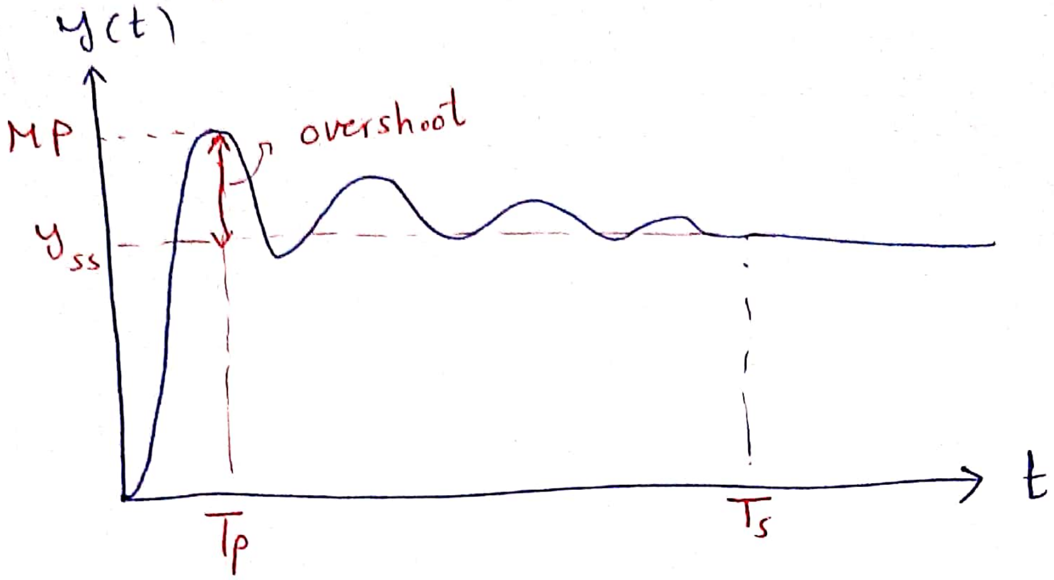
critically damped ← $\boxed{\zeta = 1}$

$$K\omega_n^2 = 2 \Rightarrow K(4) = 2$$

$\boxed{K = \frac{1}{2}}$ ←

important ← $K = \dfrac{y_{ss}}{R_{ss}}$

* The performance of Underdamped Response



Settling time = $T_s = \dfrac{4}{\zeta \omega_n}$ using 2% criterion

$T_s = \dfrac{3}{\zeta \omega_n}$ Using 5% criterion

$T_p = \dfrac{\pi}{\omega_n \sqrt{1-\zeta^2}}$ peak time

$$M_p\% = \frac{y(T_p) - y(\infty)}{y(\infty)} * 100\% = 100\, e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

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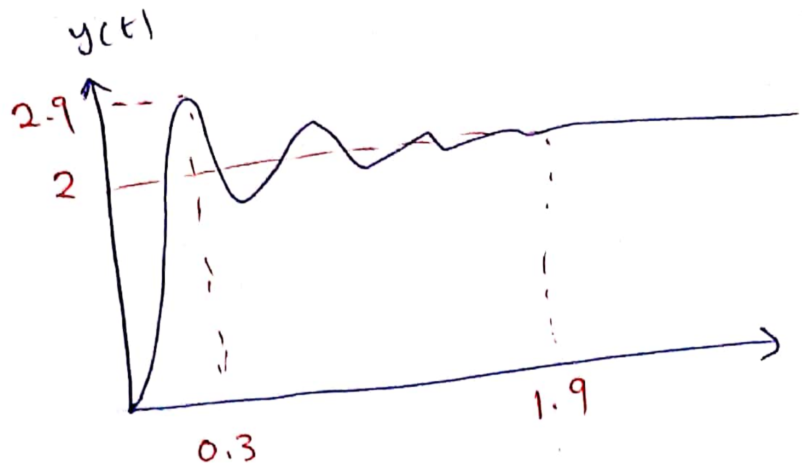
Ex1: For a unit_step input, Find $T(s) = ?$



Using 2% criterion

① $\zeta = ??$

$$M_P = \frac{y(T_p) - y(\infty)}{y(\infty)} \times 100\% = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$= \frac{2.9 - 2}{2} \times 100\% = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$\Rightarrow \boxed{\zeta = 0.246}$$

$0 < \zeta < 1 \rightarrow$ underdamped

② $T_p = 0.3$ seconds

(3) $T_s = \frac{4}{\zeta \omega_n} = 1.9 = \frac{4}{0.246(\omega_n)}$

$$\boxed{\omega_n = 7.87 \text{ rad/s}}$$

(4) $T(s) = \frac{k\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$k = \frac{y_{ss}}{R_{ss}} = \frac{2}{1} = 2$

$y_{ss} \rightarrow$ from plot $= 2$

$R_{ss} \rightarrow$ (unit-step input $= \frac{1}{s}$)

$R_{ss} = \lim_{s \to 0} s\,R(s) = \lim_{s \to 0} s\left(\frac{1}{s}\right)$

$= 1$

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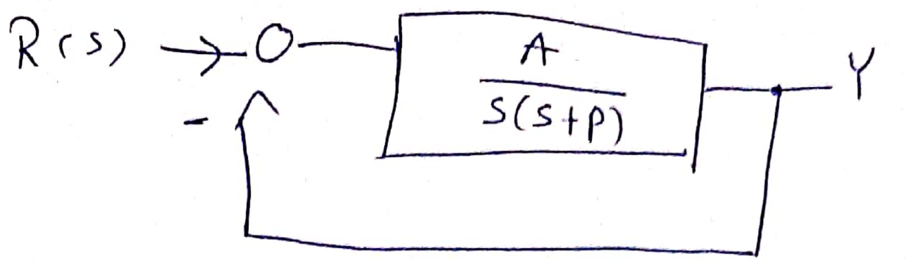
Ex2:



Select A and P where

$T_s < 4$ sec Using 2% criterion

$\zeta = 0.707$

Sol:

$$\frac{Y(s)}{R(s)} = \frac{\frac{A}{s(s+P)}}{1 + \frac{A}{s(s+P)}}$$

$$T(s) = \frac{A}{s^2 + Ps + A}$$

$$\boxed{\omega_n^2 = A} \quad \dots (1)$$

$$\boxed{2\zeta\omega_n = P} \quad \dots (2)$$

$$T_s < 4 \Rightarrow \frac{4}{0.707\,\omega_n} < 4$$

$$1 < 0.707\,\omega_n$$

$$\frac{1}{0.707} < \omega_n$$

$$\boxed{1.414 < \omega_n}$$

$\Rightarrow \boxed{A = 2}$ from eq(1)

also,

$$2(0.707)(1.414) = P$$

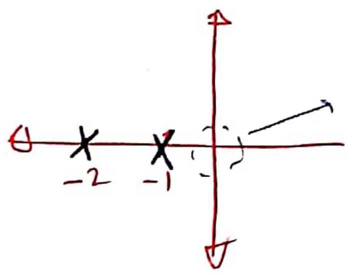
$\boxed{2 = P}$ from eq (2) ~~eq~~

→ steady - state error analysis :-

System type:

① Type zero system

e.g: $\dfrac{1}{(s+1)(s+2)}$



لايوجد pole عند نقطه الصفر

$s = 0$ ← لايوجد

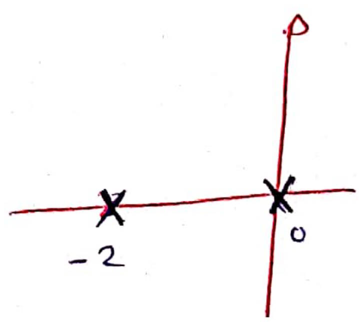
② Type one - system.

e.g: $\dfrac{1}{s(s+2)}$

← يوجد pole عند $s = 0$

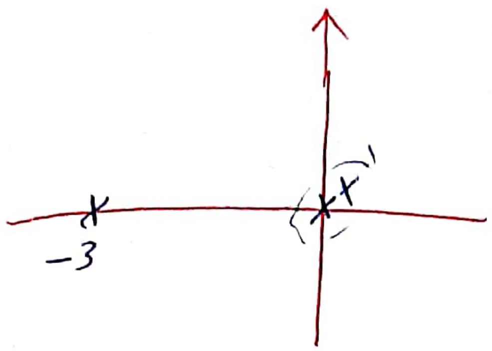
$\Leftarrow$

type 1



③ Type Two System

e.g.: $\dfrac{1}{s^2(s+3)}$



Two poles يوجد

at $s=0$

Type	input Step A/s	Ramp A/s^2	Acceleraton A/s^3
$N=0$	$\dfrac{A}{1+K_p}$	∞	∞
$N=1$	0	A/K_v	∞
$N=2$	0	0	$\dfrac{A}{K_a}$

note: step input $\Rightarrow r(t) = A\,t^{0}$

when $N=0 \Rightarrow e_{ss} = \dfrac{A}{1+K_p}$

but if $r(t) = A t^{0}$

and $N = 1$

$\Rightarrow 0 < 1 \Rightarrow e_{ss} = 0$

also

$N = 2 > t^{0} \Rightarrow e_{ss} = 0$

e.g : if $r(t) = A t^{1} \rightarrow$ Ramp

and $N = 0 \Rightarrow N < t^{1}$

$0 < 1$

$\Rightarrow e_{ss} = \infty$

but $N = 2$

$2 > t^{1}$

$e_{ss} = 0$

Remember

$$e_{ss} = \lim_{s \to 0} s\,E(s)$$

and

$$E(s) = R(s) - Y(s)$$

chapter 6: The stability of Linear feedback System

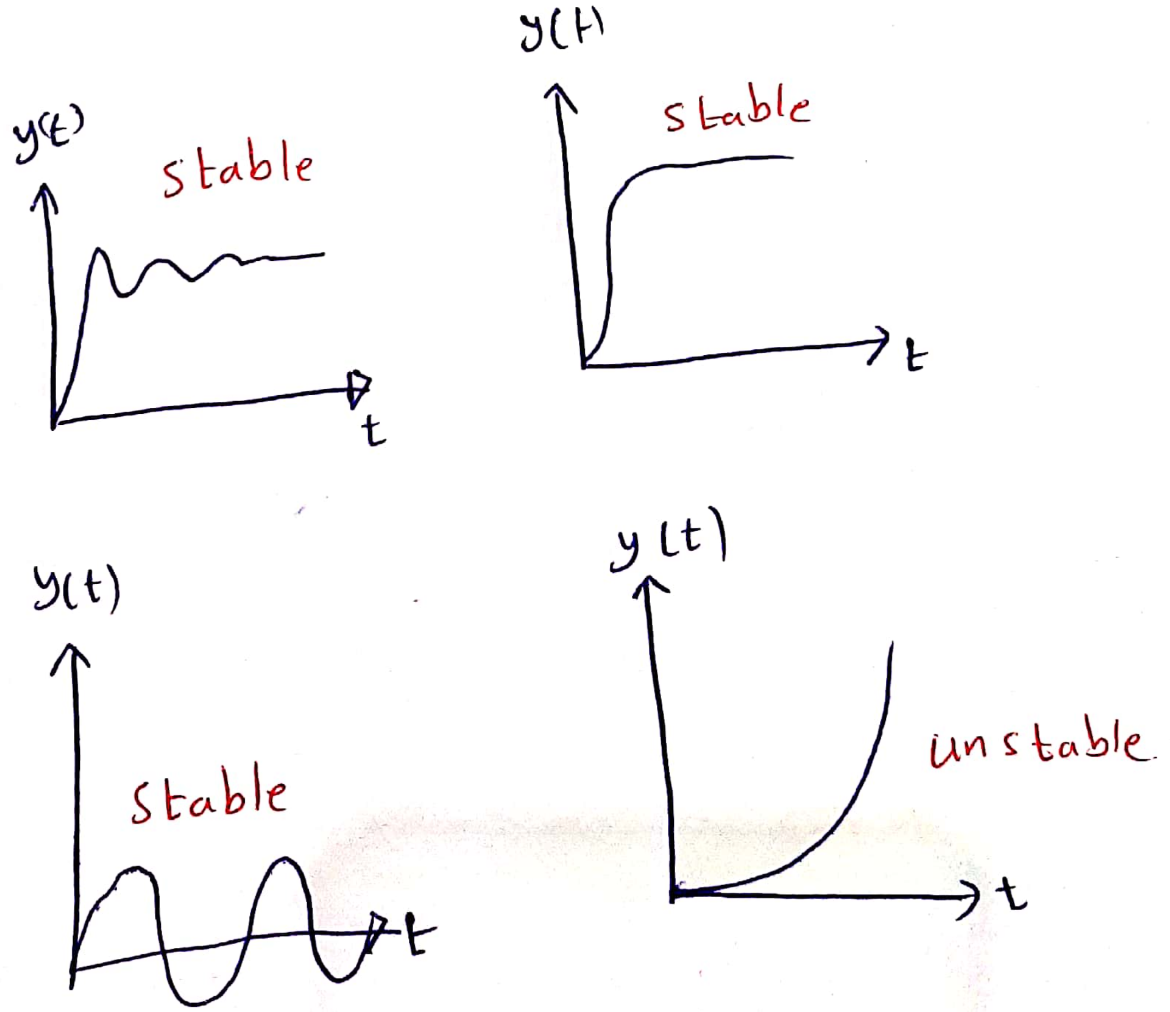
→ Stability : Bounded input Bounded output.

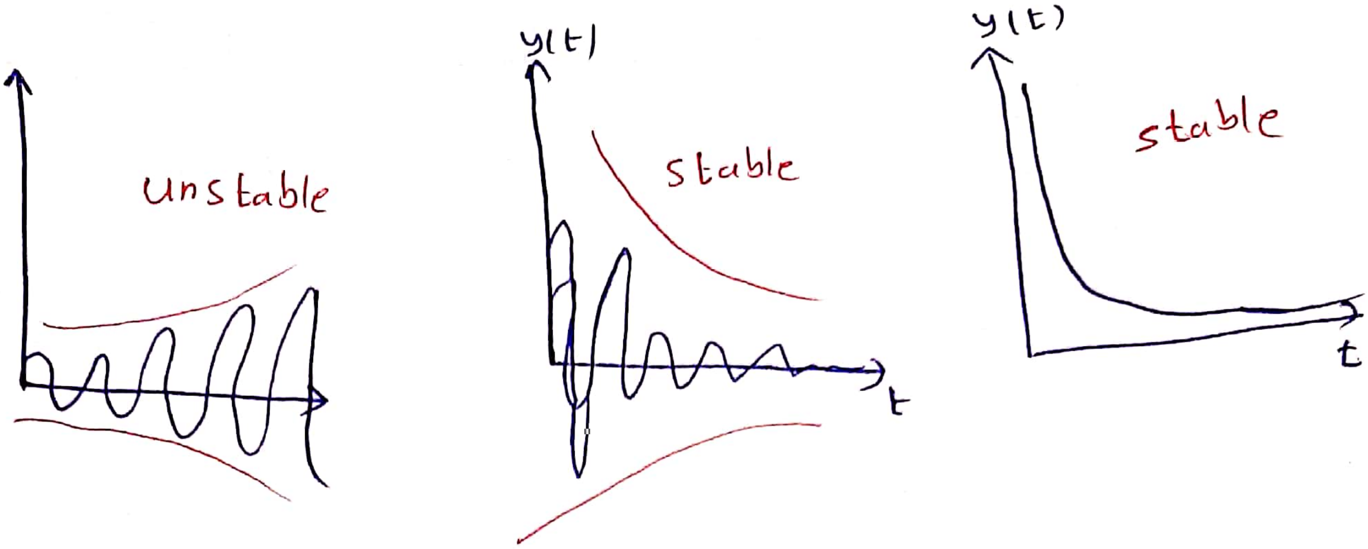
① stable: all poles on left hand side of s-plane

② unstable: one or more from the poles on the right hand side of the s-plane

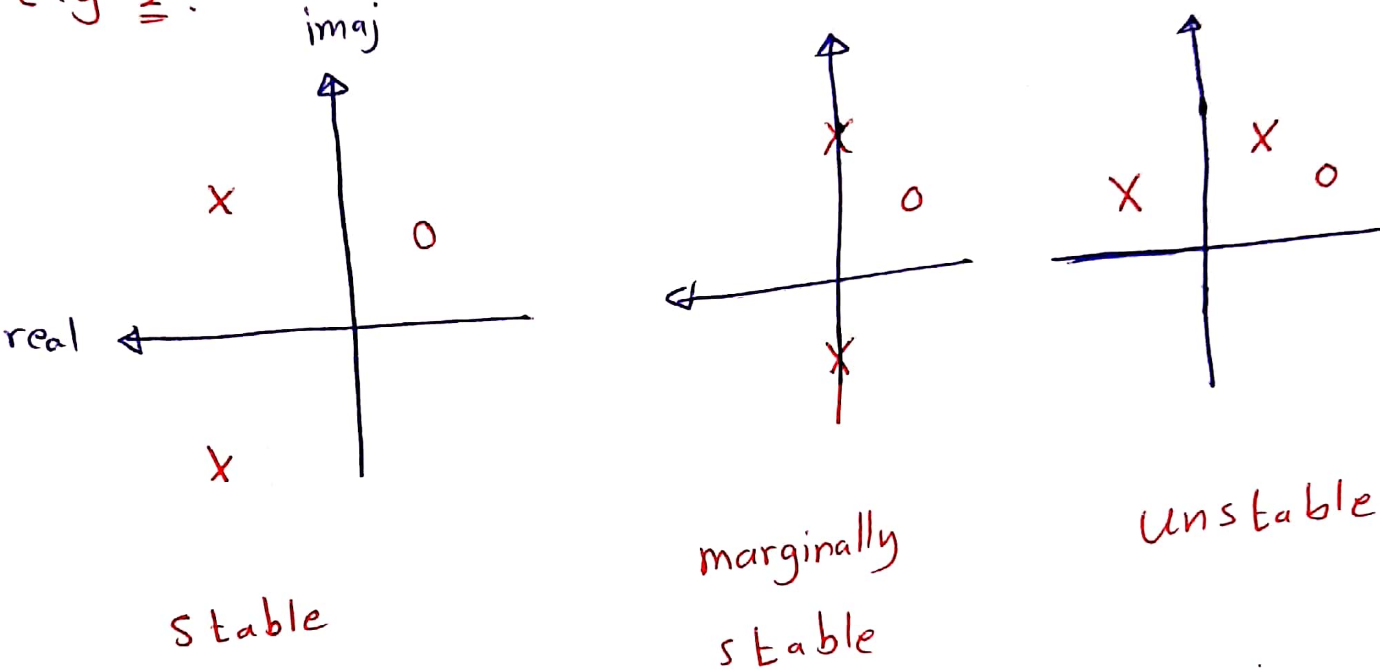
③ Maginally stable: There is poles on $\mp j\omega$ axis

e.g:





e.g 2:



* Routh-Hurwitz stability criterion:

→ for $T(s) = \frac{Z(s)}{P(s)} = \frac{\text{zeros}}{\text{poles}}$

$$P(s) = a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + \cdots + a_n$$

$$\begin{array}{c|cccc}
S^{N} & a_0 & a_2 & a_4 & \cdots \\
S^{N-1} & a_1 & a_3 & a_5 & \cdots \\
\hline
S^{N-2} & b_1 & b_2 & b_3 & \cdots \\
\vdots & c_1 & c_2 & c_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
S^{0} & a_N & 0 & 0 &
\end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

ex1: $P(s) = s^3 + 2s^2 + 4s + K$

s^3	1	4	0
s^2	2	K	0
s^1	$\frac{8-K}{2}$	0	0
s^0	K	0	0

$$b_1 = \frac{2*4 - 1*K}{2} = \frac{8-K}{2}$$

$$b_2 = \frac{2*0 - 1*0}{2} = 0$$

$$c_1 = \frac{\left(\frac{8-K}{2}\right)(K) - 2*0}{\frac{8-K}{2}}$$

$$= K$$

to be stable $\Rightarrow$ all of the first coloum ~~must be~~ should have the same sign and doesn't equal to zero.

$\Rightarrow$
$1 > 0$
$2 > 0$
$\frac{8-K}{2} > 0 \Rightarrow K < 8$
$K > 0$

$0 < K < 8$

the system is stable in this range

Ex2: $P(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

s^5	1	2	11	0
s^4	2	4	10	0
s^3	$\cancel{0}$ ϵ	6	0	0
s^2	$\frac{-12}{\epsilon}$	10	0	0
s^1	6	0	0	0
s^0	10	0	0	0

$$b_1 = \frac{2*2 - 4*1}{2} = 0$$

$$b_2 = \frac{2*11 - 10*1}{2} = 6$$

in the third row, the first element is zero but not all row is zero

$\Rightarrow$ replace the zero with ~~sm~~ positive small value ϵ

$$\therefore\ c_1 = \frac{\overset{\text{Zero}}{4*\epsilon} - 12}{\epsilon} = \frac{-12}{\epsilon}$$

$$c_2 = \frac{10*\epsilon - \overset{\text{Zero}}{2*0}}{\epsilon} = \frac{10\cancel{\epsilon}}{\cancel{\epsilon}}$$

$$= 10$$

$$d_1 = \frac{\frac{-\cancel{12}}{\cancel{\epsilon}}(6) - \overset{\text{Zero}}{10\epsilon}}{\frac{-\cancel{12}}{\cancel{\epsilon}}}$$

$$= 6$$

$$e_1 = \frac{6*10 - \frac{-12}{\epsilon}(0)}{6}$$

$$= 10$$

$\therefore$ The first coloum has different sign (+ and −) so, the system is unstable

The system has two sign changes

→ There are two poles on the R.H.S of the s-plane

* * * * *

Ex 3: $p(s) = s^4 + s^3 + s^2 + s + K$

s^4	1	1	K
s^3	1	1	0
s^2	$\cancel{0}$ ϵ	K	0
s^1	$-\frac{K}{\epsilon}$	0	0
s^0	K	0	0

$$b_1 = \frac{1\times 1 - 1\times 1}{1} = 0$$

$$b_2 = \frac{1\times K - 1\times 0}{1} = K$$

$$c_1 = \frac{1\times \underset{}{\overset{\text{zero}}{\epsilon}} - 1\times K}{\epsilon} = \frac{-K}{\epsilon}$$

$$d_1 = \frac{-\frac{\cancel{K}}{\epsilon}(K) - \overset{\text{zero}}{\epsilon\times 0}}{-\frac{\cancel{K}}{\epsilon}} = K$$

→ There are two sign changes

∴ " " " poles on the R.H.S

and the system unstable for all values of K

$\frac{-k}{\epsilon} > 0 \Rightarrow$ no

$k > 0$

$\Rightarrow$ what is the rang of k that make the system marginall stable?

System is unstable for all values of k

Ex 4: $P(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$

s^5	1	2	1	0
s^4	1	2	1	0
s^3				
s^2				
s^1				
s^0				

$b_1 = \frac{2\times1 - 1\times2}{1} = 0$

$b_2 = \frac{1\times1 - 1\times1}{1} = 0$

~~The first row~~

all elements in the third row are zero

$\Rightarrow$

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all values of k

Ex 4: $P(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$ / Fifth-order

s^5	1	2	1	0
s^4	1	2	1	0
s^3	$\overset{4}{\cancel{0}}$	$\overset{4}{\cancel{0}}$	0	0
s^2	1	1	0	0
s^1	$\overset{2}{\cancel{0}}$	$\overset{0}{\cancel{1}}$	0	0
s^0	1	0	0	0

$$b_1 = \frac{2\times1 - 1\times2}{1} = 0$$

$$b_2 = \frac{1\times1 - 1\times1}{1} = 0$$

~~The first row~~

all elements in the third row are zero

$\Rightarrow$

$s^4 \quad + \quad 2s^2 \quad + 1$

$4s^3 \quad + 4s$

$c_1 = \frac{4 * 2 - 1 * 4}{4} = 1 \quad , \quad c_2 = \frac{4 * 1 - 1 * 0}{4} = 1$

$d_1 = \frac{4 * 1 - 1 \times 4}{1} = 0 \quad , \quad d_2 = 0$

$\Rightarrow \quad s^2 + 1$

$2s$

$e_1 = \frac{2 * 1 - 1 \times 0}{2} = 1$

The system is stable.

Ex5: $P(s) = s^4 + 2s^3 - 2s - 1$ / Fourth-order

s^4	1	0	-1
s^3	2	-2	0
s^2	1	-1	0
s^1	$\cancel{0}$ 2	0	0
s^0	-1	0	0

$$b_1 = \frac{2\times 0 - 1\times -2}{2} = 1$$

$$b_2 = \frac{2(-1) - (1\times 0)}{2} = -1$$

$$c_1 = \frac{1(-2) - (2(-1))}{1}$$

$$c_1 = 0$$

$$c_2 = \frac{1\times 0 - 2(0)}{1} = 0$$

$$d_1 = \frac{2(-1) - (1\times 0)}{2}$$

$$= -1$$

$$\boxed{\begin{array}{l} s^2 - 1 \\ 2s \end{array}} \rightarrow$$

The system is unstable

one sign change (unstable roots)

1 pole on R.H.S of s-plane

3 poles " L.H.S " "

Ex 6: $P(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$

s^5	1	4	3	0
s^4	1	24	63	0
s^3	-20	-60	0	0
s^2	21	63	0	0
s^1	$\cancel{0}$ 42	0	0	0
s^0	63	0	0	0
	e_1			

The system is unstable

There are tow sign changes $\Rightarrow$

There are two unstable roots

$$b_1 = \frac{1\times 4 - 1\times 24}{1} = -20 \quad , \quad b_2 = \frac{1\times 3 - 1\times 63}{1} = -60$$

$$c_1 = \frac{-20(24) - (1 * -60)}{-20} = 21 \quad , \quad c_2 = \frac{-20(63) - 0}{-20} = 63$$

$$d_1 = \frac{21(-60) - (-20)(63)}{21} = 0 \quad , \quad d_2 = 0$$

$\therefore$ $21s^2 + 63 \Rightarrow 42s$

$$e_1 = \frac{42 * 63 - 21 * 0}{42} = 63$$

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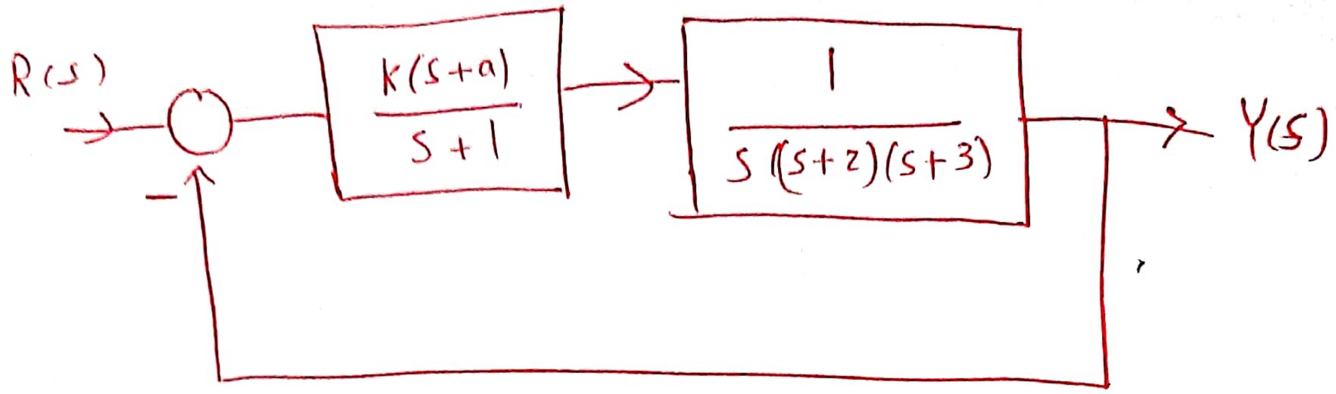
Ex 7: Determine the range of K and a for the system is stable?



$$\frac{Y(s)}{R(s)} = \frac{\dfrac{K(s+a)}{s(s+2)(s+3)(s+1)}}{1 + \dfrac{K(s+a)}{s(s+1)(s+2)(s+3)}}$$

$$\frac{Y(s)}{R(s)} = \frac{K(s+a)}{s(s+1)(s+2)(s+3) + K(s+a)}$$

The system has four poles and one zero

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$$Z(s) = K(s+a)$$

$$P(s) = s^4 + 6s^3 + 11s^2 + (K+6)s + Ka$$

s^4	1	11	$K \cdot a$
s^3	6	$K+6$	0
s^2	$\frac{60-K}{6}$	$K \cdot a$	0
s^1	C_1	0	0
s^0	$K \cdot a$	0	0

$$C_1 = \frac{\left(\frac{60-K}{6}\right)(K+6) - 6Ka}{\frac{60-K}{6}}$$

* * *

To be stable $\rightarrow$ $\frac{60-K}{6} > 0$

$\rightarrow$ $C_1 > 0$

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P1 :

$$\frac{Y(s)}{R(s)} = \frac{\dfrac{K}{s^2(s+P)}}{1+\dfrac{K}{s^2(s+P)}} = \frac{K}{s^2(s+P)+K}$$

$$\Rightarrow P(s) = s^3 + Ps^2 + K$$

s^3	1	0	0
s^2	P	K	0
s^1	$-K$	0	0
s^0	K	0	0

$\Rightarrow$ $P > 0$

$-K > 0$?

$K > 0$

The system is unstable for all values of K

*

type two system

third-order system.

Pk Pk

P_2 :

(a) $$\frac{Y(s)}{R(s)} = \frac{K}{(s+5)(s+2)^2 + K}$$

system type (zero), third-order

(b)

$$P(s) = s^3 + 9s^2 + 24s + 20 + K$$

s^3	1	24	0
s^2	9	20+K	0
s^1	$\frac{196-K}{9}$	0	0
s^0	20 + K		

$$\frac{196-K}{9} > 0$$

$$20 + K > 0$$

$\therefore$ $K > -20$ and $K < 196$

$$0 < K < 196$$

(c) $R(s) = \frac{R}{s}$ $\Rightarrow$ $e_{ss} = \lim_{s \to 0} s\,E(s)$

$$= \lim_{s \to 0} s \frac{R}{s} \left(\frac{(s+5)(s+2)^2}{(s+5)(s+2)^2 + K} \right)$$

$$= \frac{20R}{20+K}$$

(d) Ramp $\Rightarrow$ $R(s) = \frac{R}{s^2}$ $\Rightarrow$ $e_{ss} = \infty$

$\boxed{P_3}$: $P(s) = s^6 + s^5 + 5s^4 + s^3 + 2s^2 - 2s - 8$

s^6	1	5	2	-8
s^5	1	1	-2	0
s^4	4	4	-8	0
s^3	$\cancel{0}$ 16	$\cancel{0}$ 8	0	0
s^2	2	-8	0	0
s^1	72	0	0	0
s^0	-8	0	0	0

$\rightarrow 4s^4 + 4s^2 - 8$

$\rightarrow 16s^3 + 8s$

The system is unstable

$\boxed{P_4}$

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{(s+1)(s+2)}}{1 + \dfrac{K}{(s+1)(s+2)(s+3)}}$$

$$\frac{C(s)}{R(s)} = \frac{K(s+3)}{(s+1)(s+2)(s+3) + K}$$

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$$P(s) = s^3 + 6s^2 + 11s + 6 + K$$

s^3	1	11	0
s^2	6	$6+K$	0
s^1	$\frac{60-K}{6}$	0	0
s^0	K	0	0

$$\left.\begin{array}{l} K > 0 \\ \\ \frac{60-K}{6} > 0 \\ \\ K < 60 \end{array}\right\} \quad 0 < K < 60$$

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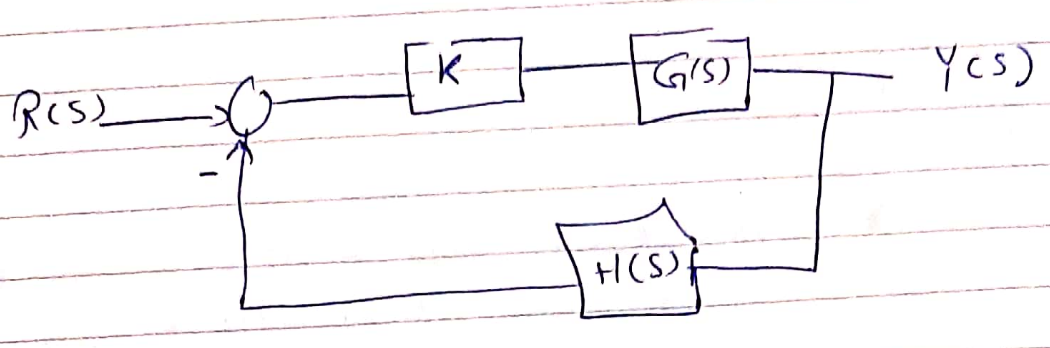
Chapter 7:- The root Locus method.



$$T.F = \frac{K\,G(s)}{1 + K\,G(s)\,H(s)}$$

$$1 + K\,G(s)\,H(s) = 0$$

$$\text{Loop gain} = G(s)\,H(s) = \frac{K\,a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n}{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n}$$

$$0 = 1 + K \text{ ——}$$

eg: $\frac{K}{(s+1)(s+2)} \Rightarrow G(s)\,H(s)$

$$1 + \frac{K}{(s+1)(s+2)} = 0 \Rightarrow (s+1)(s+2) + K = 0$$

Let $K = 0 \Rightarrow S_{1,2} = -1, -2$

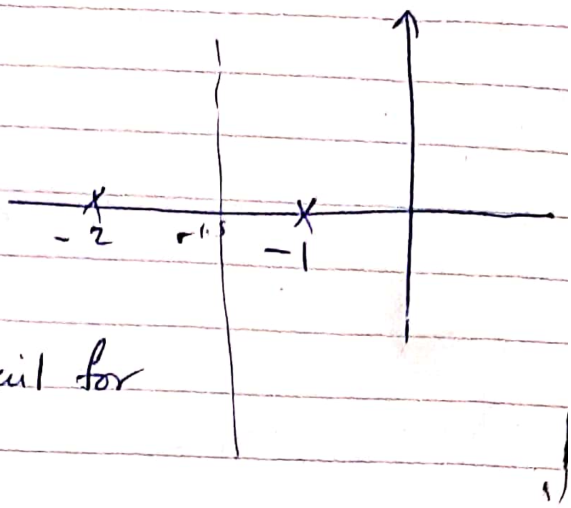
$K = 0.25 \Rightarrow S_{1,2} = -1.5$

$K = 0.5 \Rightarrow S_{1,2} = -1.5 \mp 0.5j$

$K = 1 \Rightarrow S_{1,2} = -1.5 \mp 0.866j$

$\vdots$

$K = 1000$



* The conditions that must prevail for any point on the root locus

① Magnitude condition

$$\left|\frac{K\,|s+z_1|\,|s+z_2| \cdots |s+z_m|}{|s+p_1|\,|s+p_2| \cdots |s+p_n|}\right|_{s=p} = 1$$

② Angle condition

$$\angle G(s)H(s) = \mp 180^\circ (2k+1) \quad , k = 0, 1, 2 \ldots$$

$$\angle (s+z_1) + \ldots \angle s+z_m - \angle s+p_1 \ldots\ldots - \angle s+p_n$$

$$= \mp 180 (2k+1)$$

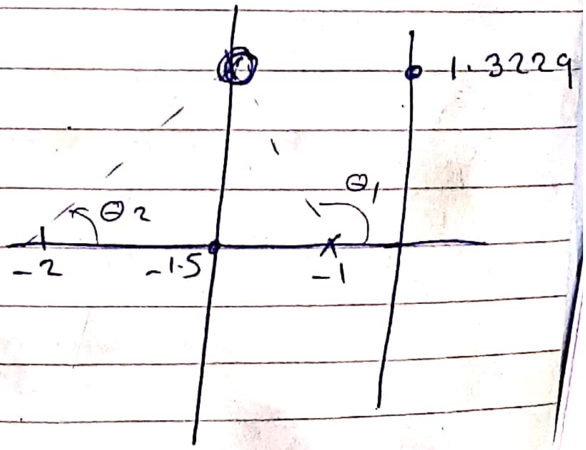
according to the previous example

$$s = -1.5 \mp j\,1.3229$$

⇒ angle condition

$$\angle s+z - \angle (s+p_1) - \angle s+p_2$$

$$= \mp 180 (2k+1)$$



o -

$$\theta_1 = \tan^{-1}\left(\frac{1.3229}{-0.5}\right) = 110.7^\circ$$

$$\theta_2 = \tan^{-1}\left(\frac{1.3229}{+0.5}\right) = 69.29^\circ$$

$$0 - \theta_1 - \theta_2 \Rightarrow 0 - 110.7 - 69.29 = -180 \checkmark$$

$\Rightarrow$ Magnitude condition

$$\left|\frac{K(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}\right| = 1 \Rightarrow K = \left|(s+p_1)(s+p_2)\right|$$

$$K = \left|(-1.5 + 1.3229j \cancel{+} + 1)(-1.5 + 1.3229j + 2)\right| =$$

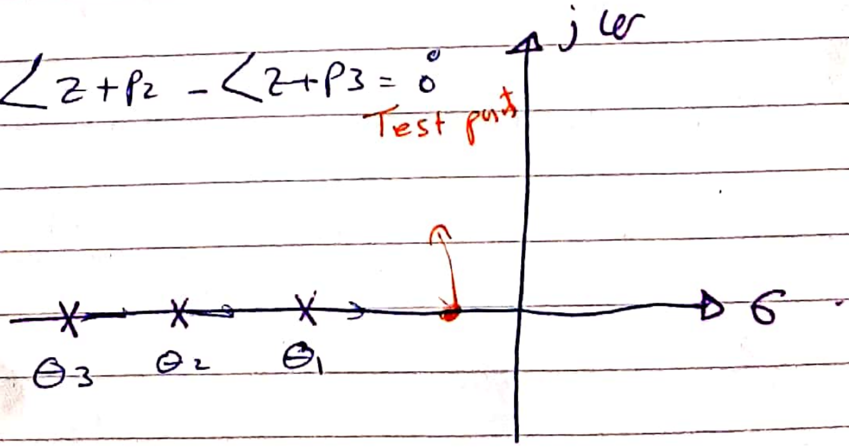
① Root locus starts at the system poles when $k = 0$ and ends at the system zeros when $k = \infty$

② number of segments (branches) is equal to the number of poles.

$$\angle s+z_1 - \angle z+p_1 - \angle z+p_2 - \angle z+p_3 = 0^\circ$$

$$-\theta_1 - \theta_2 - \theta_3 = 0$$

$$\circ \quad \circ \quad \circ$$



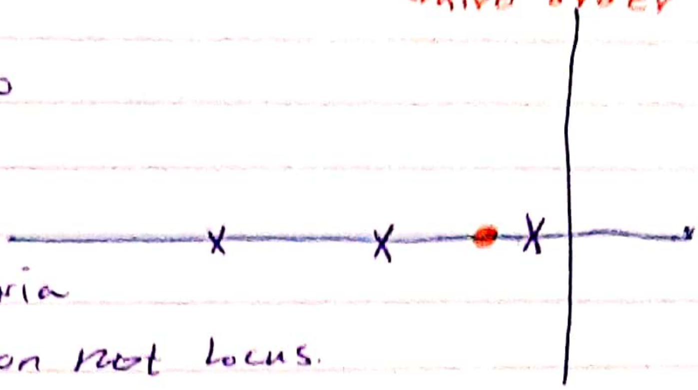
$\cancel{\theta_0 = -180}$.

this point violates the angle criteria and cannot be on segment of root locus.

type zero
third order

$\theta_1 = 180$

$\theta_2 = 0$, $\theta_3 = 0$



$\Rightarrow -180$ ✓

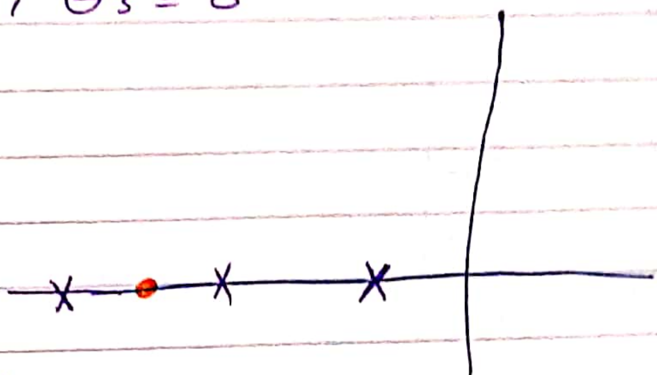
Confirms angle criteria

k. is a segment on root locus.

$\theta_1 = 180$, $\theta_2 = 180$, $\theta_3 = 0$

$\Rightarrow -360$ ✗

لا تحقق الشرط

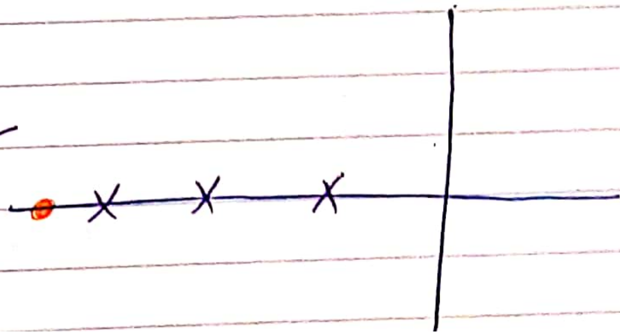


$\theta_1 = -180 = \theta_2 = \theta_3$

-540

$\Rightarrow \mp 180(2k+1)$

$k = 1$ ✓



Steps of Constructing Root Locus of a System

1- Write the characteristic equation of the system in the following standard form

$$\Delta = 1 + K \ \frac{(s+z_1)(s+z_2)\ldots\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots\ldots(s+p_n)}$$

Where K might be a controller gain (or system gain) and is the parameter of interest. $-z_1, -z_2, \ldots, -z_m$ are the zeros of the open loop and $-p_1, -p_2, \ldots\ldots\ldots, -p_n$ are the poles of the open-loop.

2- Locate all poles and zeros in s-plane.
3- Locate root locus segments on real axis.
4- Determine the asymptotes of the root locus: if the number of the open-loop poles (n) is greater than the number of open-loop zeros (m), then

number of asymptotes = *n-m*

Intersection of asymptotes with real axis

$$\sigma_a = \frac{\sum poles - \sum zeros}{n-m}$$

Angles of asymptotes

$$\alpha_a = \frac{\pm 180(2k+1)}{n-m}, \quad k = n-m-1$$

5- Find the breakaway / in points if any

$$\frac{dK}{dS} = 0$$

6- Find the points of intersection with *imaginary* axis by applying Routh-Hurwitz criteria.
7- Determine the departure angle (arrival angle) of there is complex poles (complex zeros) by applying the angle condition.
8- Calculate the desired gain K that corresponds to a particular desired closed loop poles by applying the magnitude condition.

$$K = \frac{product\ of\ lengths\ between\ s\ and\ the\ open-loop\ poles}{product\ of\ lengths\ between\ s\ and\ the\ open-loop\ zeros}$$

Ex 1 :- Draw the root Locus

$$1 + KGH = 0 = 1 + \frac{k}{(s+1)(s+2)}$$

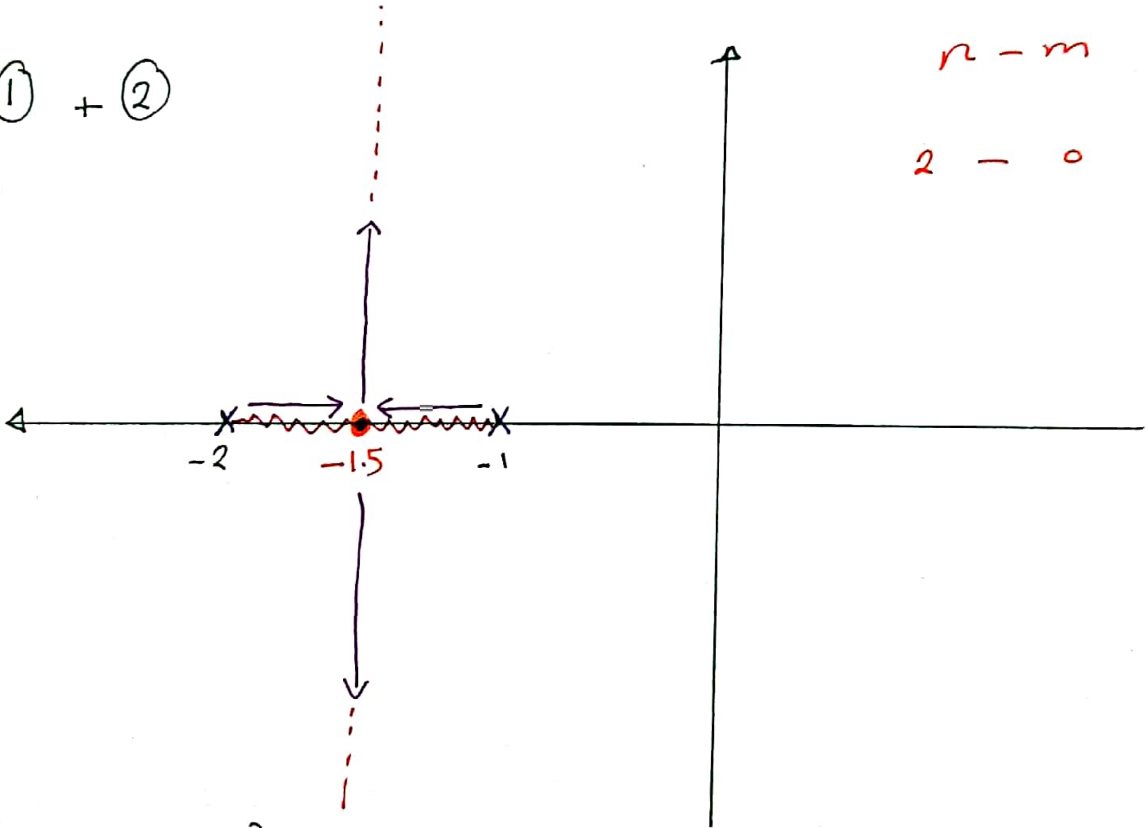
$$s^2 + 3s + 2 + k = 0$$

\# of asymptotes

$n - m$

$2 - 0 = \underline{\underline{2}}$

① + ②



③ $\sigma_a = \frac{-3 - 0}{2 - 0} = -1.5$

$k = n - m - 1 = 2 - 0 - 1 \Rightarrow$

$$\theta_0 = \frac{-180(2(0)+1)}{2-0} = 90^\circ$$

$$\theta_1 = \frac{-180(2(1)+1)}{2-0} = 270^\circ$$

(4) break-awy point $\frac{dk}{ds} = 0$

$-k$

$0 = s^2 + 3s + 2 + k \Rightarrow \frac{dk}{ds} = -(2s + 3) = 0$

$s = \frac{-3}{2}$

angle of departure لا يوجد

$0 < K < \infty \rightarrow$ System is stable.

K at $s = -1$ is zero

$\therefore$ system order is second-order

$\therefore$ type = zero type.

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Ex 2: Draw the root Locus for the system shown:

$$1 + \frac{K(s+1)}{s(s+2)(s+4)^2}$$

(1) $s(s+2)(s+4)^2 + k(s+1) = 0$

The system has one Zero at $s = -1$

and Four poles at $s = 0$

$s = -2$

$s = -4, -4$ repeated roots

③ # of asymptotes: $n - m = 4 - 1 = 3$

$$\sigma_a = \frac{(0 - 2 - 4 - 4) - (-1)}{4 - 1} = -3$$

$$k = n - m - 1 = 4 - 1 - 1 = 2$$

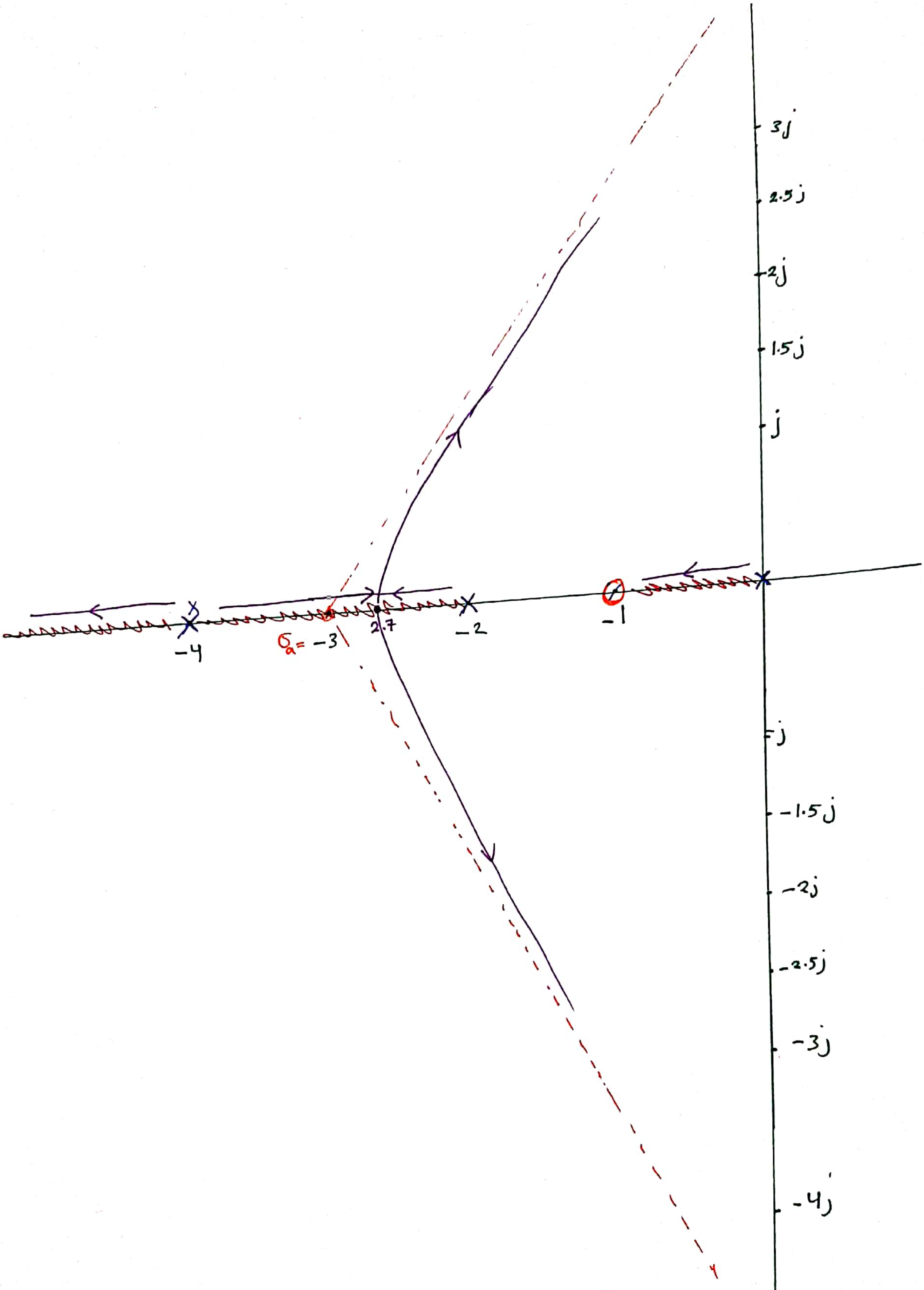
$\therefore$ at $k = 0 \Rightarrow \theta_0 = \dfrac{-180\,(2(0)+1)}{4-1} = \cancel{1}60^\circ$

at $k = 1 \Rightarrow \theta_1 = \dfrac{-180\,(2(1)+1)}{4-1} = -180^\circ$

at $k = 2 \Rightarrow \theta_2 = \dfrac{-180\,(2(2)+1)}{4-1} = 300^\circ$

④ * break-away point : $\dfrac{dK}{ds} = 0$

$s \cong 2.7$



$$\theta = 60^\circ \Rightarrow \tan\theta = \frac{y}{3} = \tan 60^\circ$$

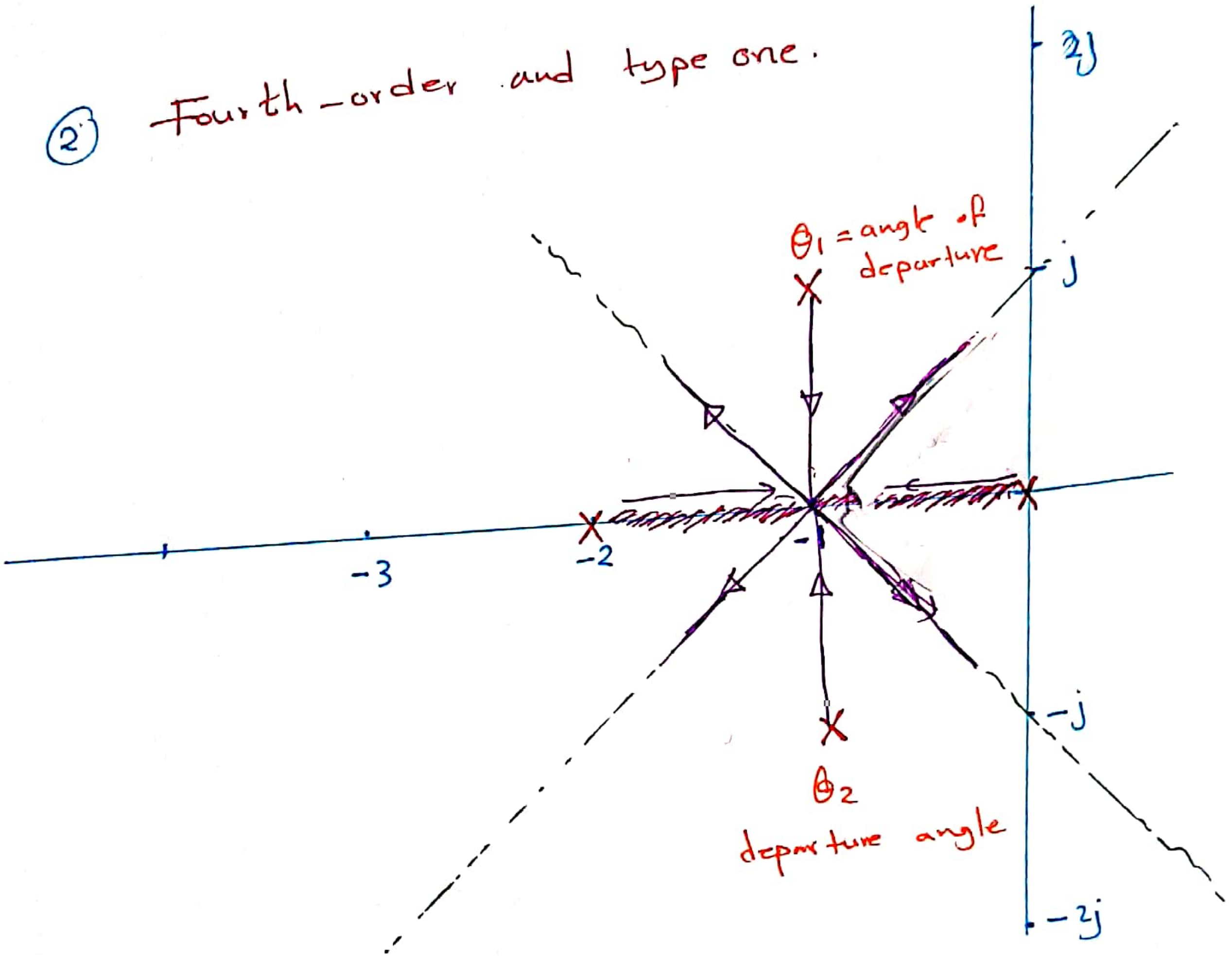
نقطة التقاطع مع Imajenary axis ل asymptote $\quad y = (1.732) * 3 = 5.19$

Ex 3: Draw the root locus.

note
$s=0$
$s=-2$
$s=-1-j$
$s=-1+j$

(1) $$1+KGH = 1+\frac{K}{s(s+2)((s+1)^2+1)}$$

(2) Fourth-order and type one.



(4) $n-m = \#$ of asymptotes
$4-0 = 4$

$$\sigma_a = \frac{(0-2-1+j-1-j)-(0)}{4-0} = -1$$

$$k = n-m-1 = 4-0-1 = 3$$

$$\alpha_0 = \frac{180(1)}{4} = 45^\circ$$

$$\alpha_1 = \frac{180(3)}{4} = 135^\circ$$

$$\alpha_2 = \frac{180(5)}{4} = 225^\circ$$

$$\alpha_3 = \frac{180(7)}{4} = 315^\circ$$

(5) break-away point $\frac{dK}{ds} = 0$

$$(s^2+2s)(s^2+2s+2) \pm K = 0$$

$$s^4 + 4s^3 + 6s^2 + 4s + k = 0$$

$$4s^3 + 12s^2 + 12s + 4 = -\frac{dk}{ds} = 0$$

$$s^3 + 3s^2 + 3s + 1 = \frac{dk}{ds} = 0$$

$$s \simeq -1$$

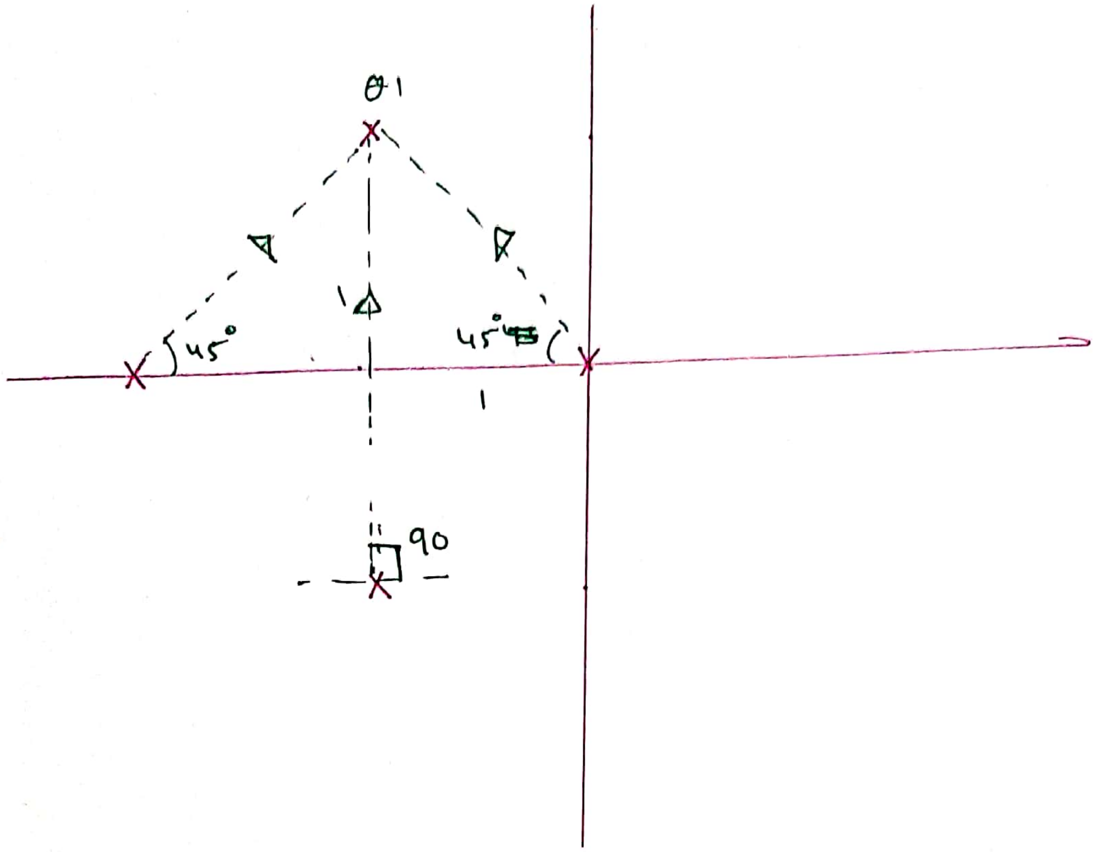
s^4	1	6	K
s^3	4	4	0
s^2	5	$\frac{4K}{4}$	0
s^1	$\frac{20-4K}{5}$	0	0
s^0	K		

when $\frac{20-4K}{4} = 0 \Rightarrow$ The system is marginally stable

$\Rightarrow K = 5$ at $5s^2 + K = 0$

$5s^2 + 5 = 0 \Rightarrow s^2 = -1$

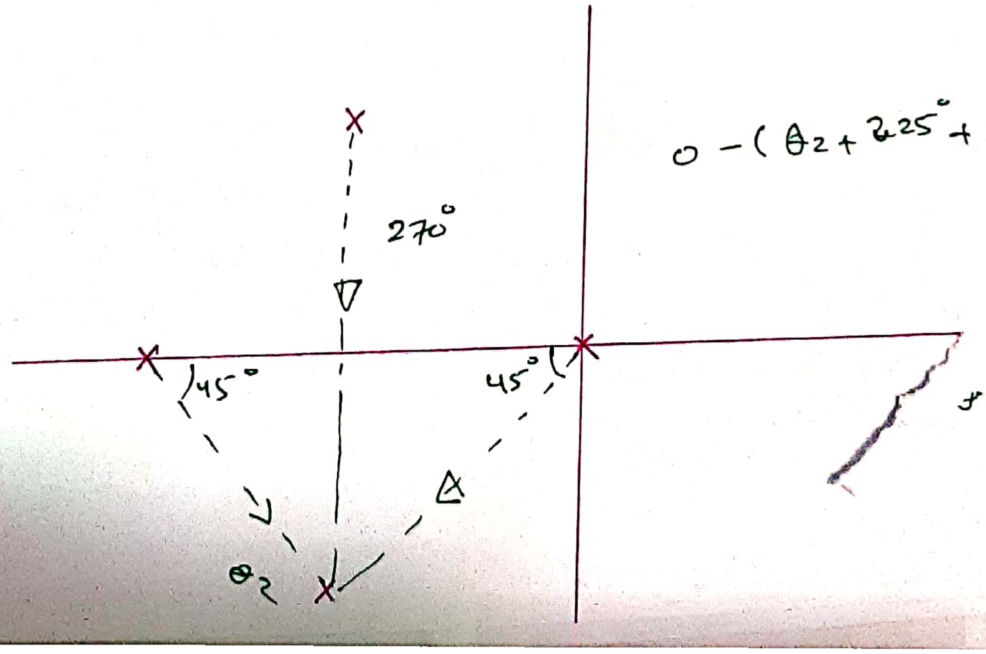
θ_1 and θ_2 → angle of departure.



$$\Sigma \angle \text{zeros} - \Sigma \angle \text{poles} = -180(2K+1)$$

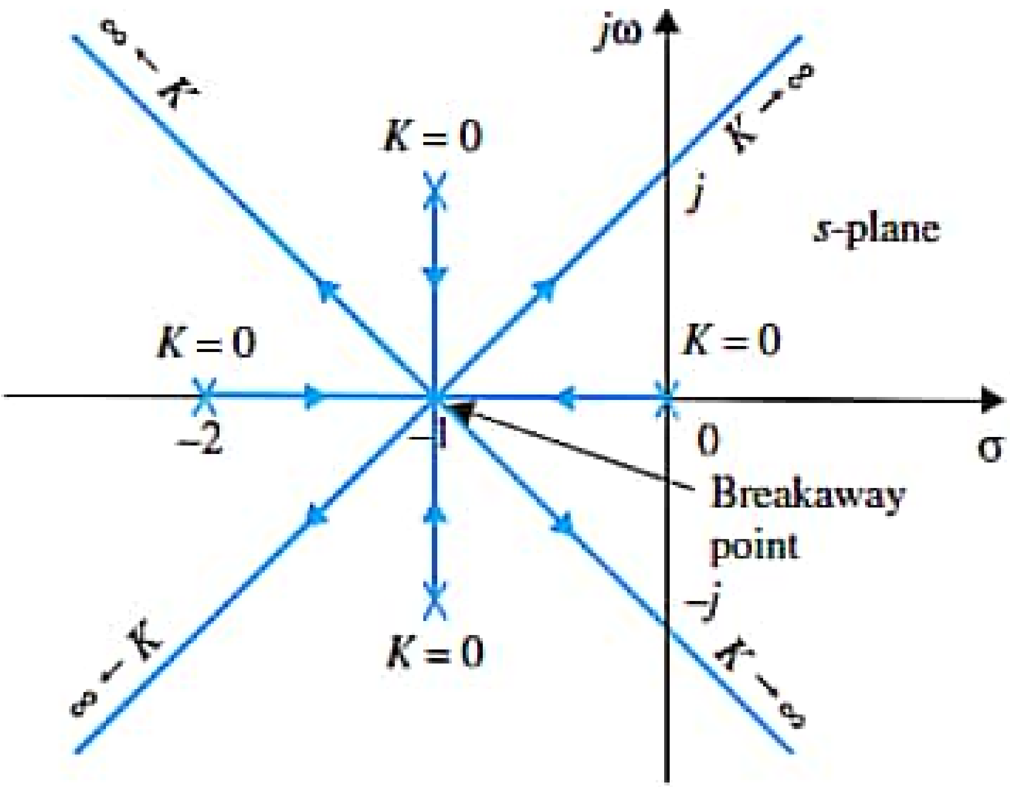
$$0 - (\theta_1 + 135^\circ + 45^\circ + 90) = -180$$

$$\theta_1 = -90^\circ = 270^\circ$$

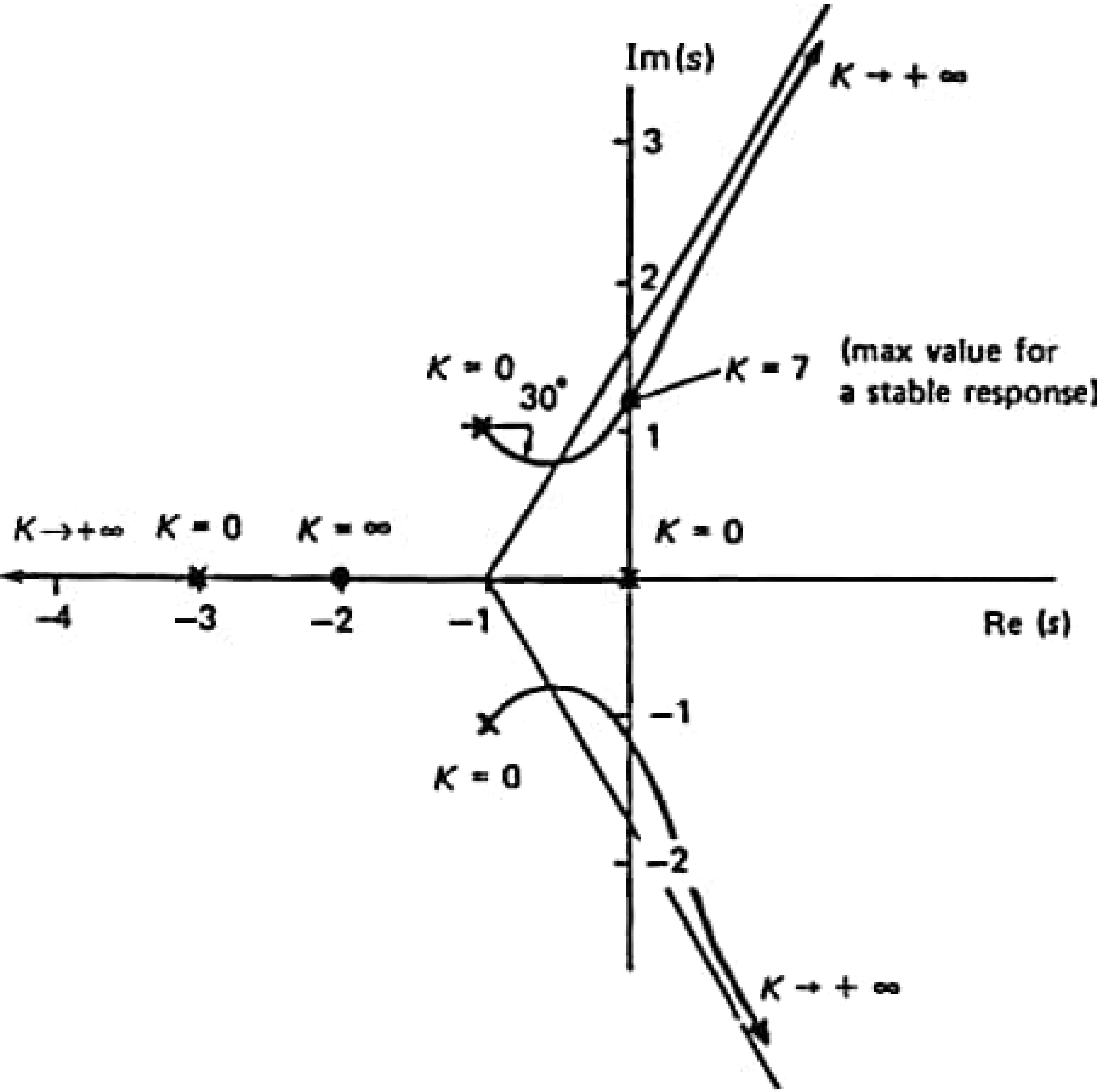


$$0 - (\theta_2 + 225^\circ + 315^\circ + 270) = -180$$

$$\theta_2 = -270^\circ = +90^\circ$$



(c)



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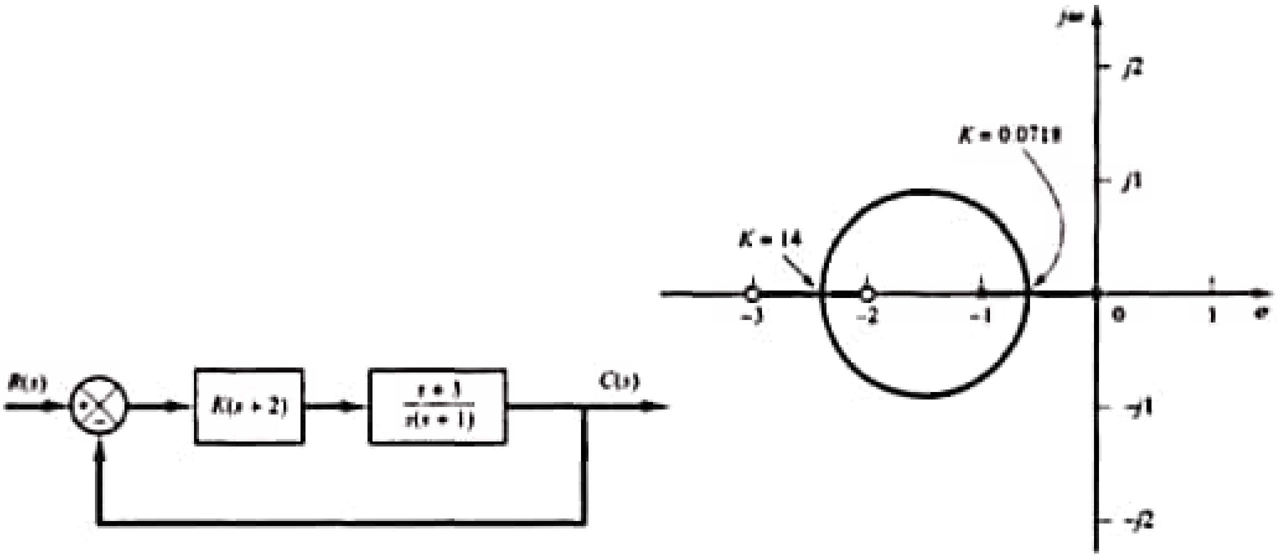
EXAMPLE PROBLEMS AND SOLUTIONS

A-6-1. Sketch the root loci for the system shown in Figure 6–39(a). (The gain K is assumed to be positive.) Observe that for small or large values of K the system is overdamped and for medium values of K it is underdamped.

Solution. The procedure for plotting the root loci is as follows:

1. Locate the open-loop poles and zeros on the complex plane. Root loci exist on the negative real axis between 0 and -1 and between -2 and -3.
2. The number of open-loop poles and that of finite zeros are the same. This means that there are no asymptotes in the complex region of the s plane.



(a) (b)

Figure 6–39
(a) Control system; (b) root-locus plot.

Ex4: Draw a root Locus for the characteristic equation shown.

① $1 + \dfrac{K}{s^4 + 12s^3 + 64s^2 + 128s}$

② no zeros, 4 poles at

$s = 0$

$s = -4$

$s = -4 + j4$

$s = -4 - j4$

System type one

Fourth order

③ # of segments on s-plane

④ # of asymptotes $n - m = 4 - 0 = 4$

$$\sigma_a = \frac{(0 - 4 - 4 + j4 - 4 - j4) - (0)}{4 - 0} = -3$$

$K = n - m - 1 = 4 - 0 - 1 = 3$

$k = 0 \Rightarrow \partial_0 = \frac{180\,(1)}{4} = 45^\circ$

$K = 1 \Rightarrow \partial_1 = \frac{180\,(3)}{4} = 135^\circ$

$K = 2 \Rightarrow \partial_2 = \frac{180\,(5)}{4} = 225^\circ$

$K = 3 \Rightarrow \partial_3 = \frac{180\,(7)}{4} = 315^\circ$

(5) break-away point $\frac{dK}{ds} = 0 \Rightarrow S = -1.5$

(6) Routh-Herwitz, $p(s) = s^4 + 12s^3 + 64s^2 + 128s + K = 0$

find zero row on Routh-array

zero-row → marginally stable

and there ~~are~~ poles on $j\omega$-axis

s^4	1	64	K
s^3	12	128	0
s^2	53.3	K	0
s^1	c_1	0	0
s^0	K	0	0

$\Rightarrow$ if $c_1 = 0 \Rightarrow$ we have zero row

$$c_1 = \frac{(53.3)(128) - (12)(K)}{53.3} = 0$$

$$\Rightarrow \quad c_1 = \frac{6826.67 - 12K}{53.33} = 0$$

(8) الخطوة رقم (8) $\Rightarrow$ $\boxed{K = 568.89}$

$$53.3\, s^2 + K = 0$$

$$53.3\, s^2 + 568.89 = 0 \Rightarrow \boxed{s_{1,2} = \mp j3.266}$$

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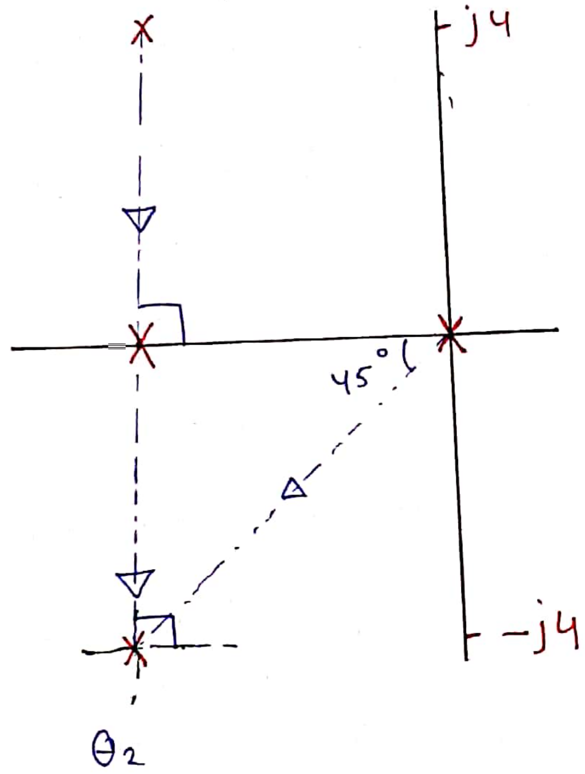
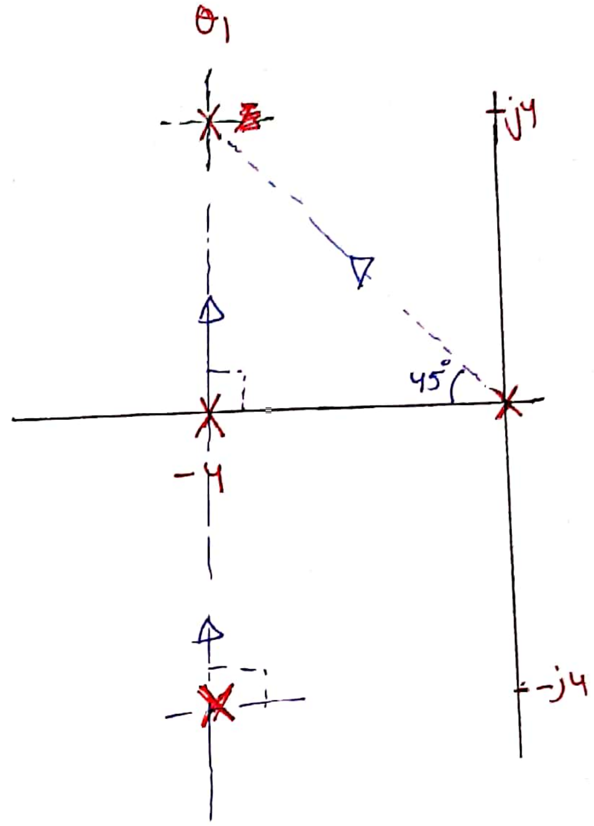
The system is stable $0 < K < 568.89$

(7) departure angle.





by applying angle criterion

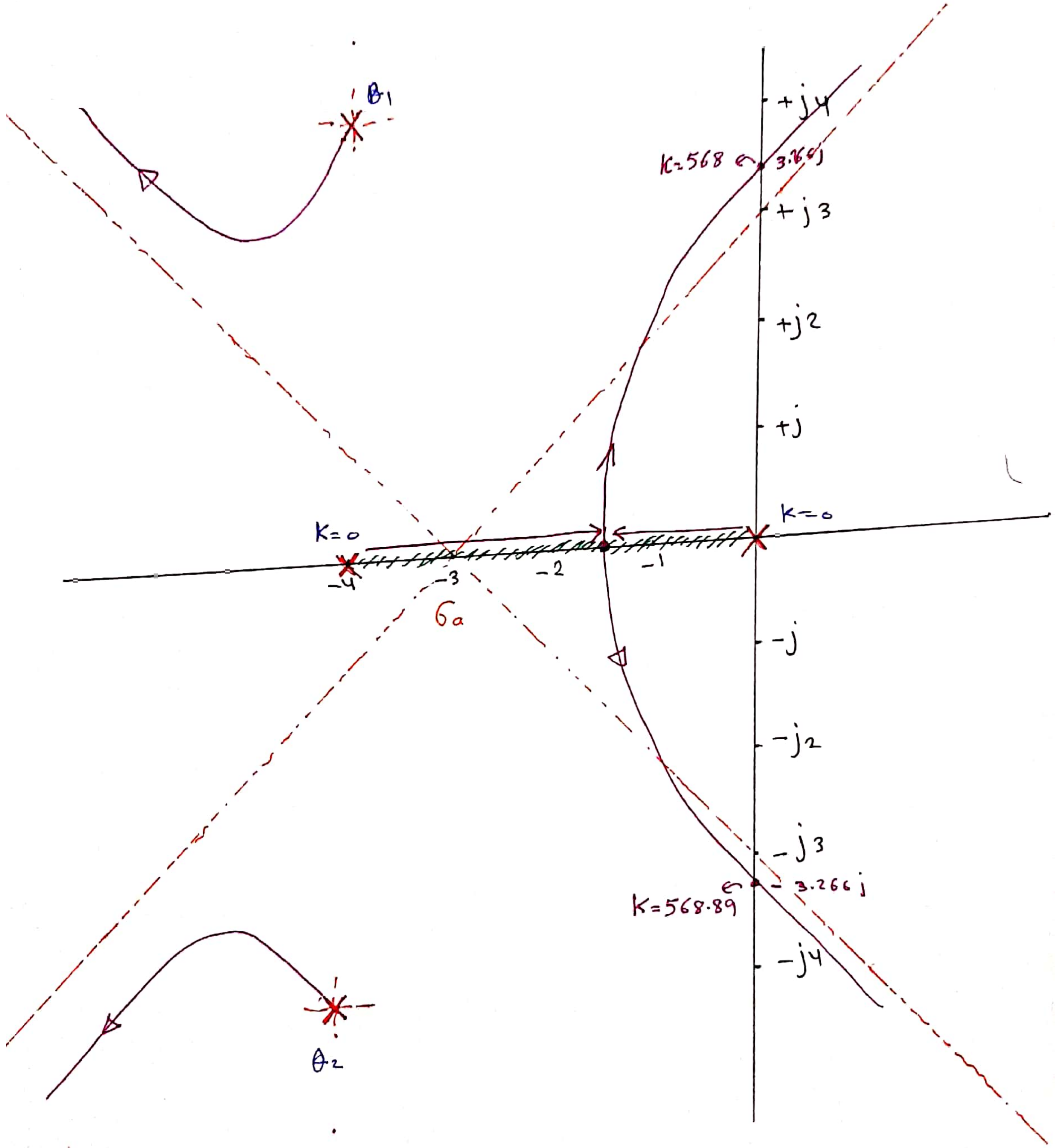
$$0 - (\theta_2 + 225^\circ + 270^\circ + 270^\circ) = -180(2K+1)$$

$$\boxed{\theta_2 = 135^\circ}$$

also,

$$0 - (\theta_1 + 135^\circ + 90^\circ + 90^\circ) = -180(2K+1)$$

$$\boxed{\theta_1 = 225^\circ}$$



θ1
θ2
+j4
+j3
+j2
+j
-j
-j2
-j3
-j4
K=568
3.266j
K=568.89
-3.266j
K=0
K=0
-4
-3
-2
-1
σa

A–6–6. Sketch the root loci for the system shown in Figure 6–68(a).

Solution. The open-loop poles are located at $s = 0$, $s = -1$, $s = -2 + j3$, and $s = -2 - j3$. A root locus exists on the real axis between points $s = 0$ and $s = -1$. The angles of the asymptotes are found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180°(2k+1)}{4} = 45°, -45°, 135°, -135°$$

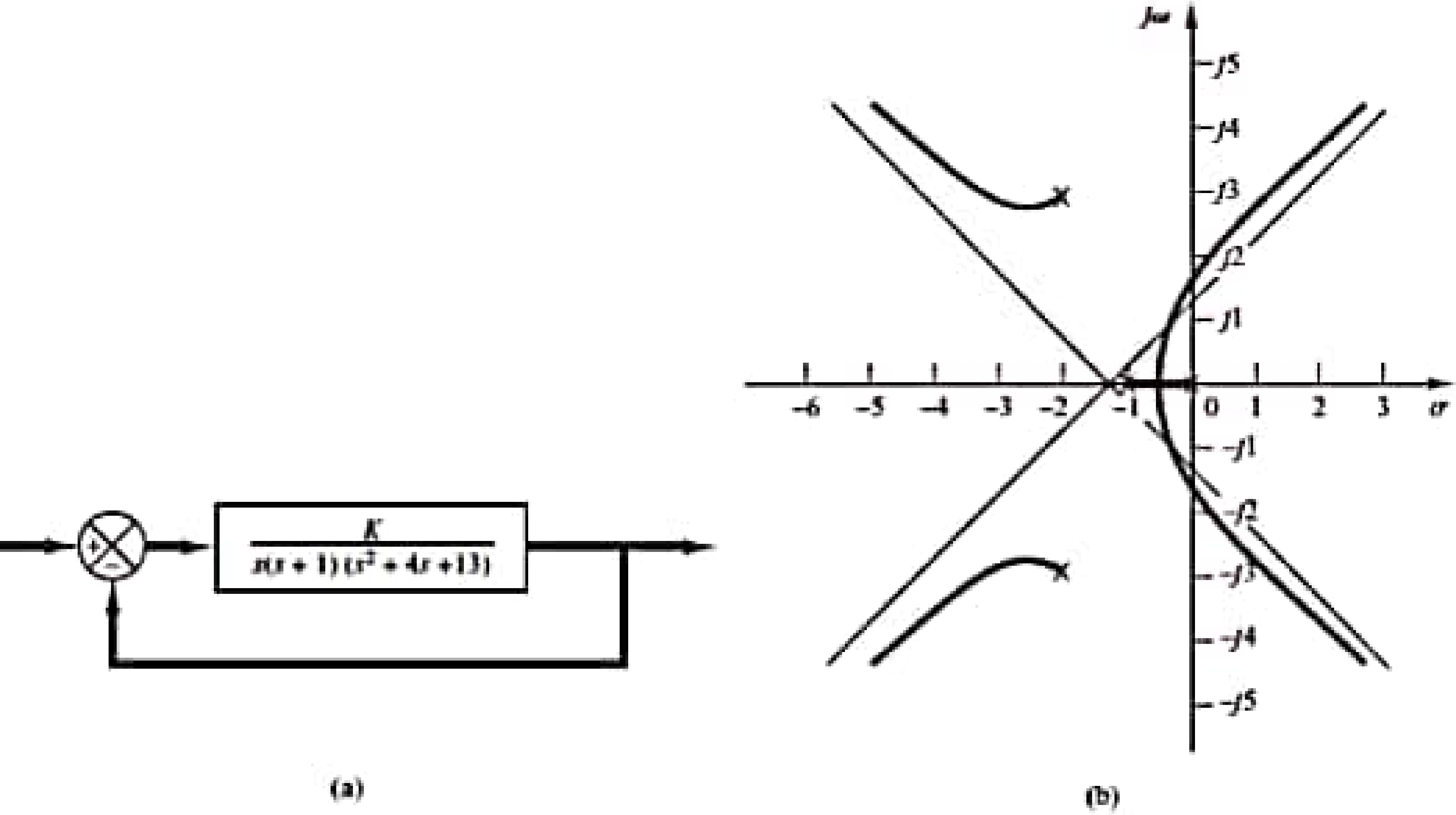
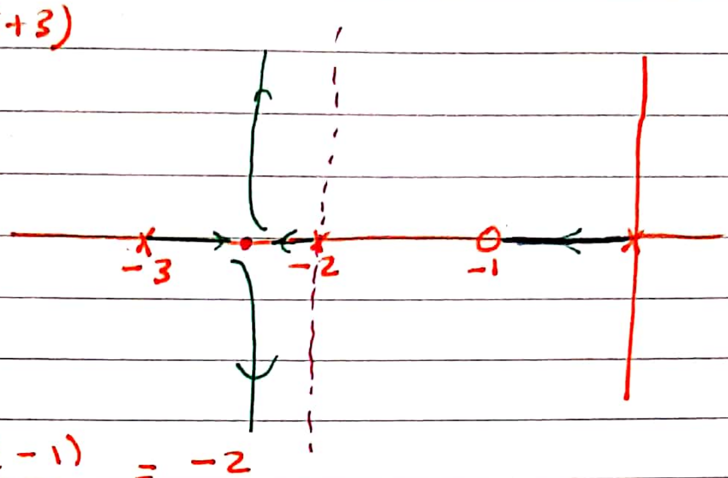


Figure 6–68
(a) Control system; (b) root-locus plot.

type one
Third order

Ex: $1 + \dfrac{K(s+1)}{s(s+2)(s+3)}$



$n - m = 3 - 1 = 2$

$\sigma_A = \dfrac{(0 - 2 - 3) - (-1)}{3 - 1} = -2$

$K = n - m - 1 = 3 - 1 - 1 = 1$

$\alpha_0 = \dfrac{180(1)}{2} = 90°$

$\alpha_1 = \dfrac{180(3)}{2} = 270°$

$\rightarrow s(s+2)(s+3) + K(s+1) = 0 \Rightarrow (s^2 + 2s)(s+3) = -K(s+1)$

$-K = \dfrac{s^3 + 3s^2 + 2s^2 + 6s}{s+1} = \dfrac{s^3 + 5s^2 + 6s}{s+1} = 0$

$-\dfrac{\partial K}{\partial s} = 0 \Rightarrow 3s^2 + 10s + 6 = 0$

$s = -2.46\ldots$ breakaway point

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→ PID Controller: propotinal - Integral - Derivative Controller

P ~~Kp~~ : Decrease the steady - state error

I ~~KI~~ : Eliminate the steady - state error (add a pole in the origin $S=0$)

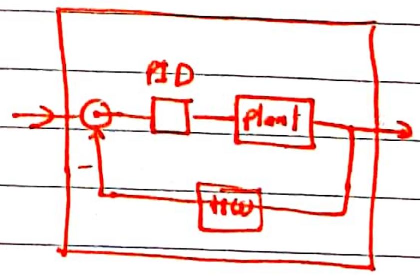
D ~~KD~~ : Decrease the overshoot, and settling time
- the system become more stable.
or makes the system more stable

$$T(s)\Big|_{PID} = K_P + \frac{K_I}{s} + K_D s$$

$$= \frac{\cancel{K_P s^2} + K_D s^2 + K_P s + K_I}{s}$$

PI , PD, PID

→ $T(s)\Big|_{PI} = K_P + \frac{K_I}{s}$



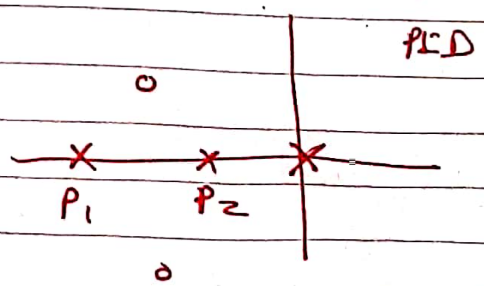
→ $T(s)\Big|_{PD} = K_P + K_D s$

$$\frac{1}{(S+P_1)(S+P_2)}$$

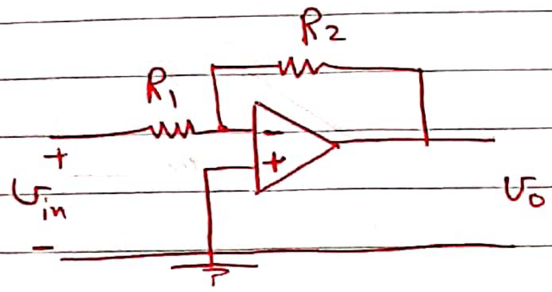
K_P : propotional gain

K_D = Dorivative gain

K_I = integral gain

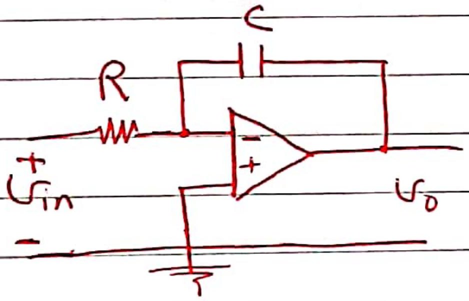


* PID - Controllers:



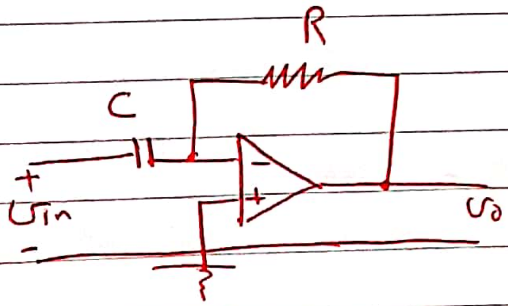
→ propotional controller (P)

$$\frac{v_o(s)}{v_{in}(s)} = -\frac{R_2}{R_1}$$



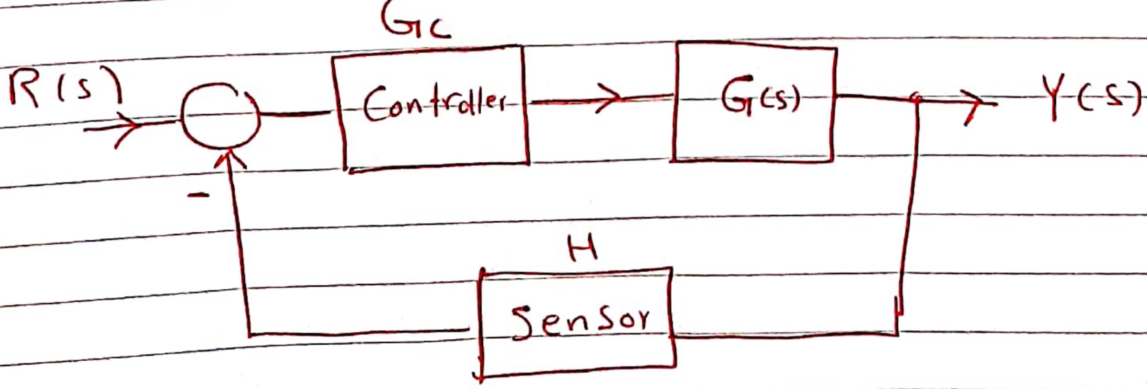
→ Integral Controller (I)

$$\frac{v_o(s)}{v_{in}(s)} = -\frac{1}{RCs}$$



→ Derivative Controller (D)

$$\frac{v_o(s)}{v_{in}(s)} = -RCs$$



$$\frac{Y(s)}{R(s)} = \frac{G_c\, G}{1 + G_c\, G\, H}$$

* * * G_c = Controller transfer function

G_c for PID $= K_P + \frac{K_I}{S} + K_D S$

$$= \frac{K_D S^2 + K_P S + K_I}{S}$$

PID added one pole at $S = 0$

and Two Zeros depending on

K_D, K_P, K_I values

K_P = propotional gain

K_I = Integral gain

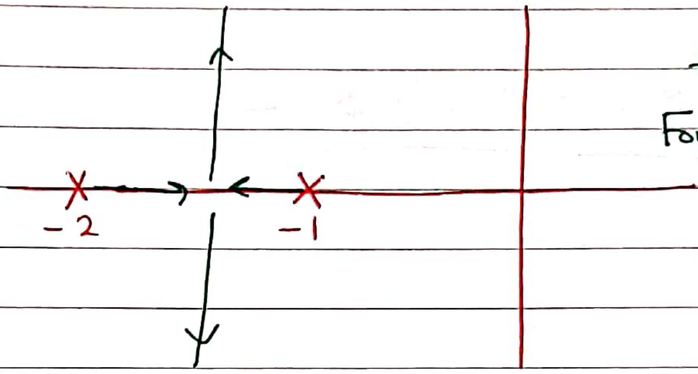
K_D = Derivative gain

Table 1: Effect of increasing parameter independently

Parameter	Rise Time	Overshoot	Settling Time	Steady-State Error	Stability
K_p	Decrease	Increase	Small Change	Decrease	Degrade
K_i	Decrease	Increase	Increase	Eliminate	Degrade
K_d	Minor Change	Decrease	Decrease	No Effect	Improve if K_d small

Ex: $T(s) = \dfrac{1}{(s+1)(s+2)}$

before



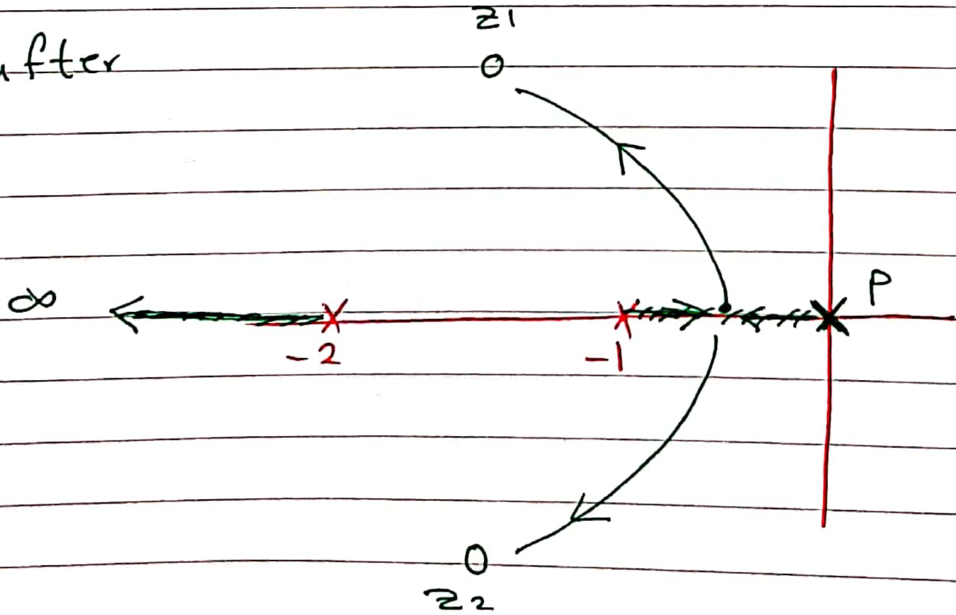
type zero

For a unit-step input

$e_{ss} = \dfrac{A}{1+K_p}$

add PID

after



type one

for a unit-step input →

$\boxed{e_{ss} = 0}$

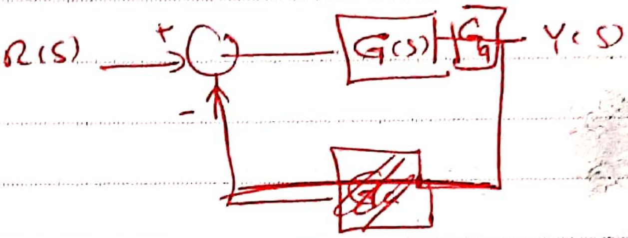
adding one pole at $s = 0$

→ eliminate the steady state error

Ex:- $G(s) = \dfrac{10}{s(0.001s+1)}$ for step input what is e_{ss}

$G_a = K$

$$e_{ss} = \lim_{s \to 0} s\,E(s)$$



$$T(s) = \frac{G(s)}{1 + G_a G(s)}$$

$$E(s) = R(s) - \frac{G\,G_a}{1+G_a G} R(s)$$

$$E(s) = R(s)\left[\frac{s(0.001s+1)}{s(0.001s+1)+10K}\right]$$

$$= \frac{1}{s}\left[1 - \frac{G\,K}{KG+1}\right]$$

$$\lim_{s\to 0} s \times \frac{1}{s}\left[1 - \frac{G(0)K}{KG(0)+1}\right] = \frac{1 + KG(0) - KG(0)}{1+KG(0)} = \frac{1}{K\frac{10}{0}} = \frac{1}{1+KG(0)} = 0$$

if the error = 0.1 mm and R(s) = ramp = 10 m/s

Find K_a?

$$R(t) = 10\,t \Rightarrow R(s) = \frac{10}{s^2}$$

$$E_{ss} = 0.1 * 10^{-3}\ \text{m} = \lim_{s\to 0} s\,E(s) =$$

$$\Rightarrow K_a = 10{,}000$$