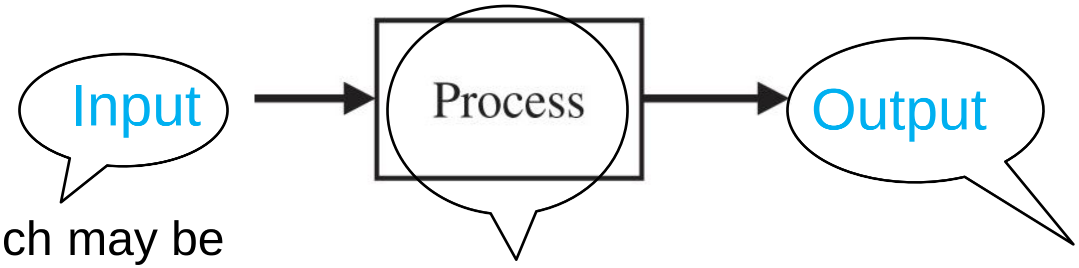
Ch.1 Introduction to Control Systems

- What is Control System?
- Control systems types
- Examples
- Control systems objectives
- Design Process

What is Control System?

- A control system is an interconnection of components forming a system configuration that will provide a desired system response.

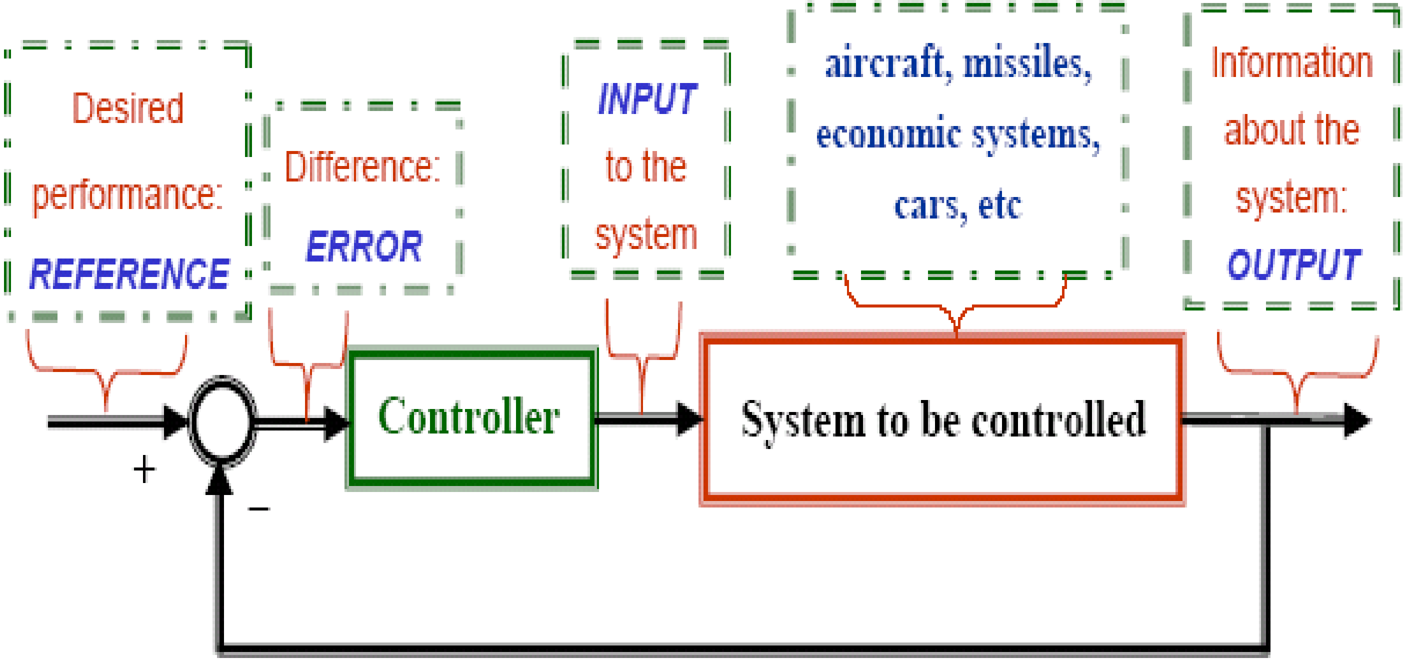
linear system theory assumes a cause–effect relationship for the components of a system and is the basis of system analysis



The variable which may be adjusted to bring about the required control action (also know on the actuating Signal).

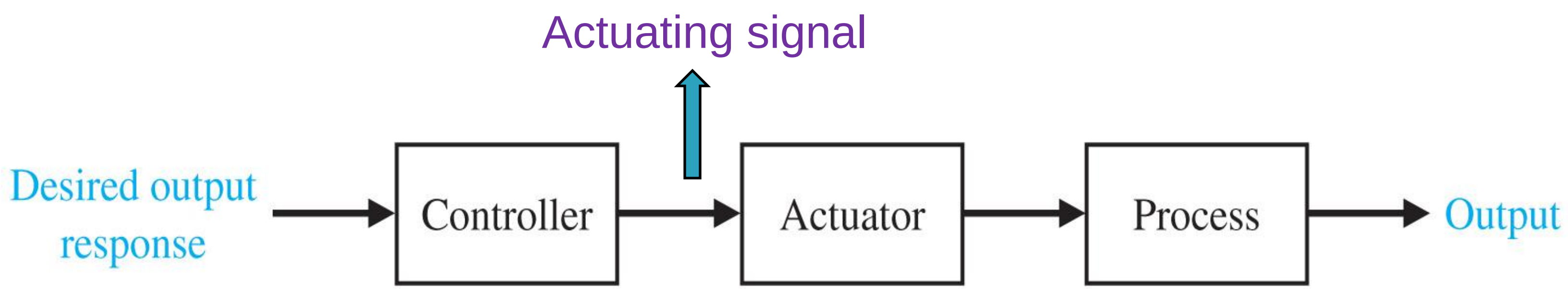
The physical system which is to be controlled.

The variable to be controlled.



Objective: To make the system *OUTPUT* and the desired *REFERENCE* as close as possible, i.e., to make the *ERROR* as small as possible.

Open loop and closed loop control systems



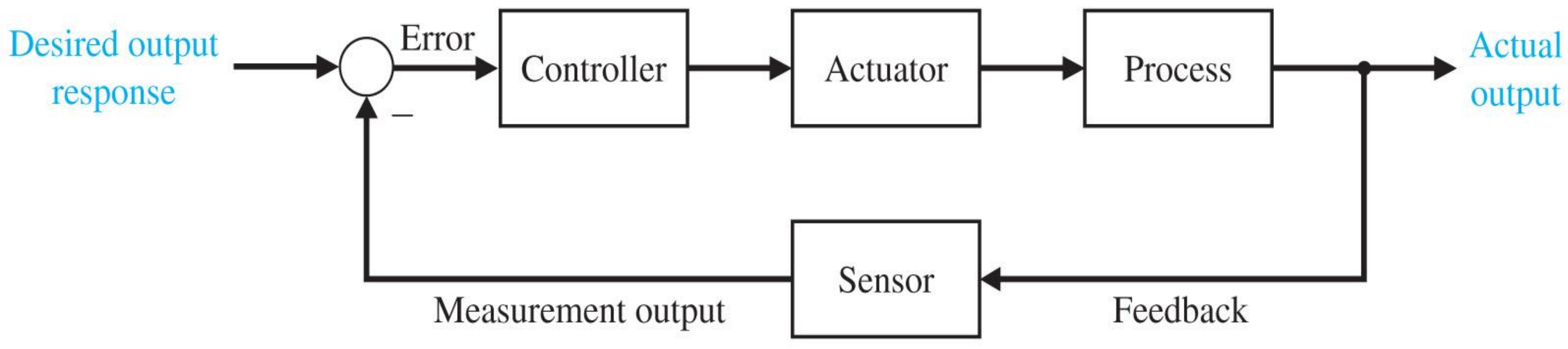
Open Loop System

An open-loop control system utilizes an actuating device to control the process directly without using feedback

Disadvantages:

Open-loop control system cannot compensate for **disturbance Inputs** to the process or for process **parameter variations.**

Closed loop control system



Closed Loop System

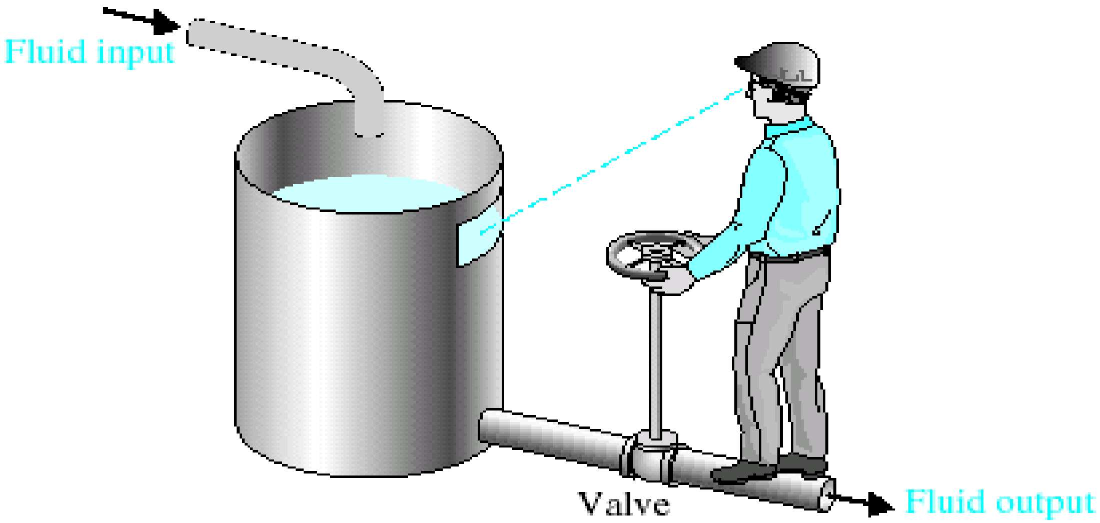
A closed-loop control system uses a measurement of the output and feedback of this signal to compare it with the desired output (reference or command)

The error can be compensated for by the controller which generates a correcting signal *u(t)*

Control system components

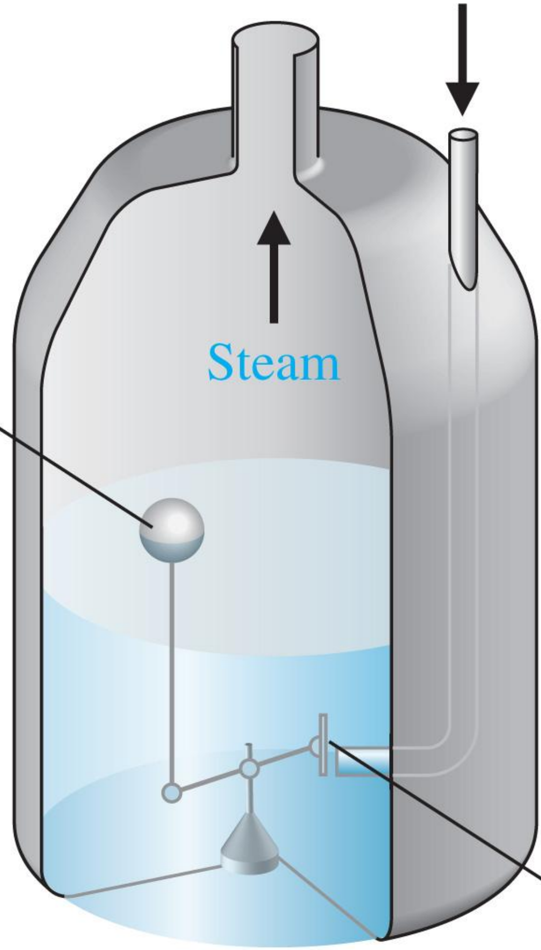
1. System/Plant ⇒ to be controlled
2. Actuators ⇒ Converts *u(t)* to power signal
3. Sensors ⇒ gives measurement of system output
4. Reference input ⇒ desired output
5. Error detector
6. Controller ⇒ operates on the error signal to form the required control signal *u(t)*

Example: Maual Level control system

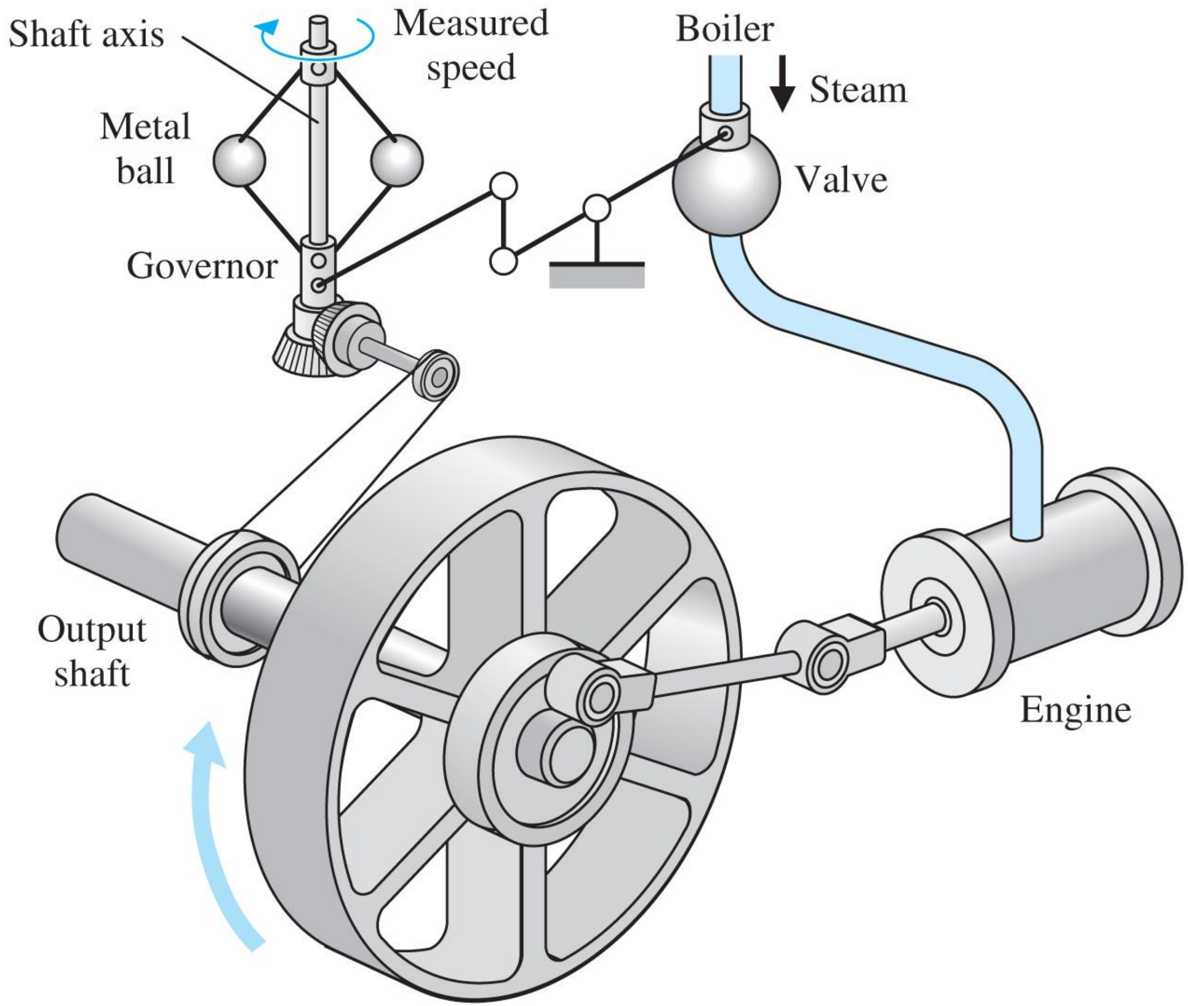


Control system components…!!

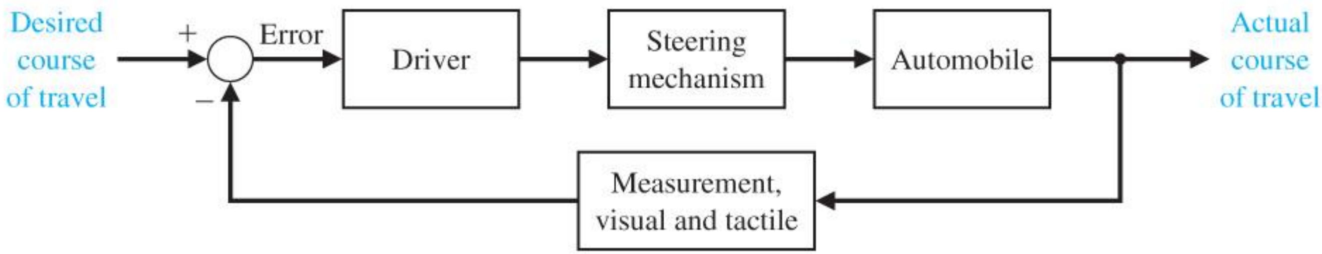
Example: Polzunov's Water level float regulator



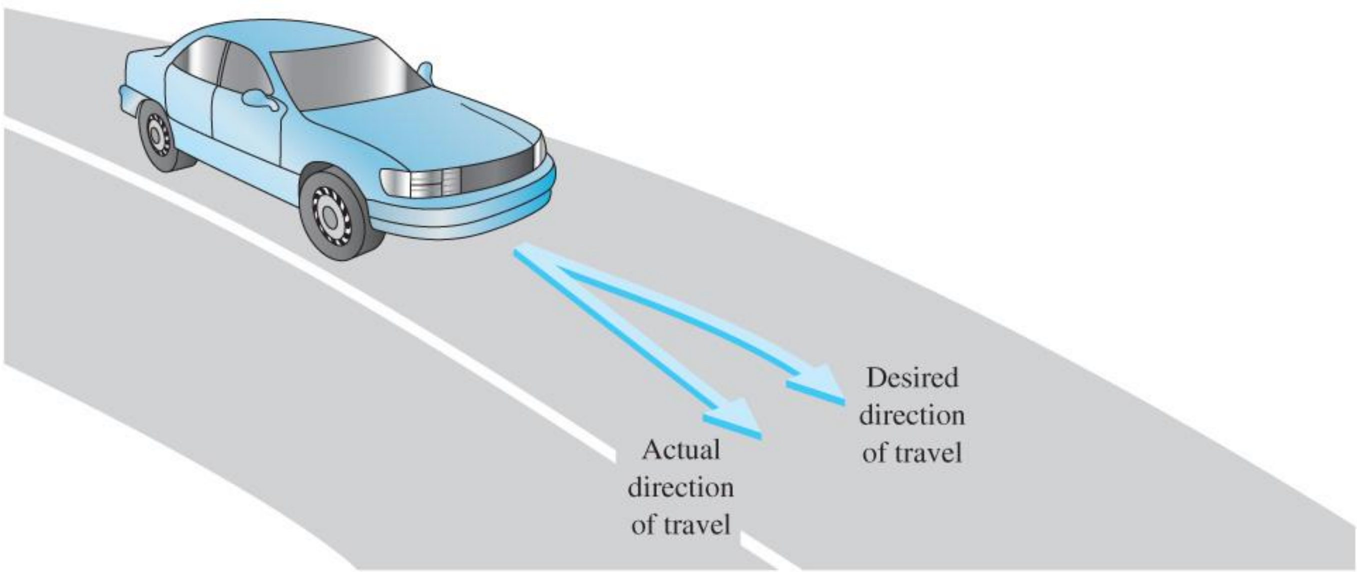
Example: James Watt's flyball governor



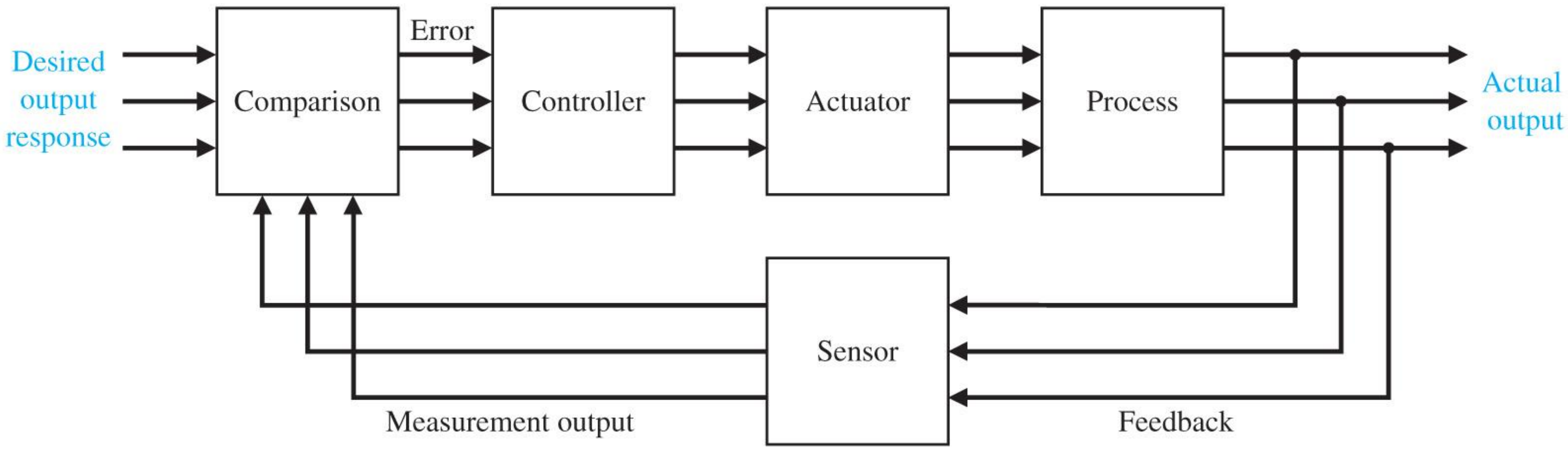
Example: Feedback in everyday life



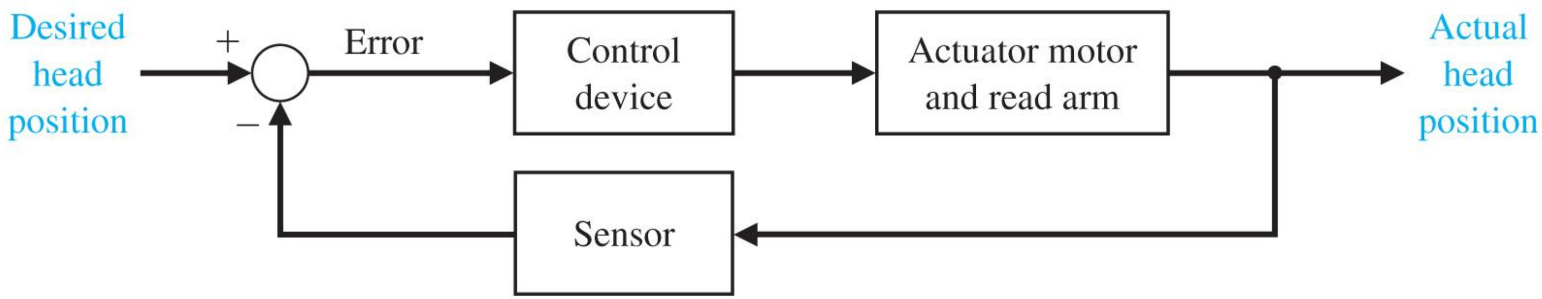
(a)

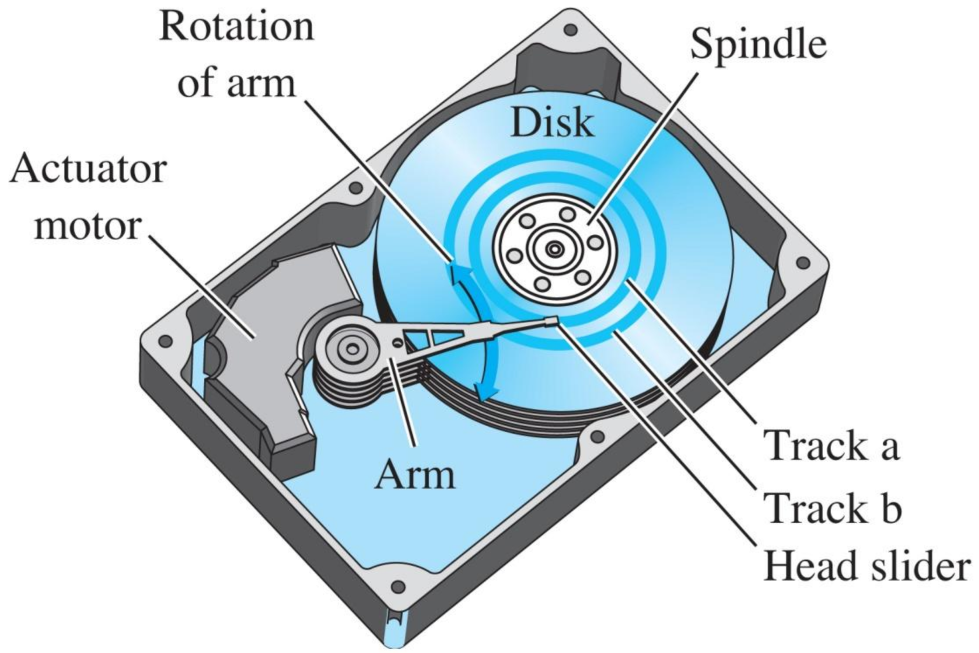


Multivariable Control System Model

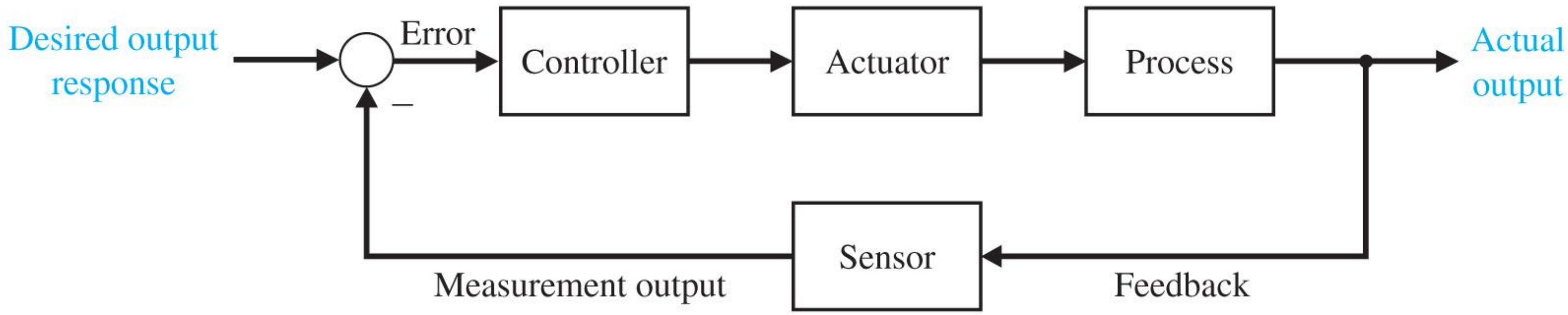


Example: Disk Drive





Feedback Control System



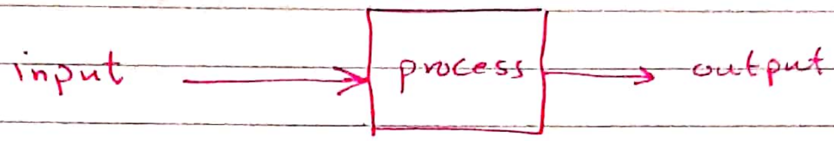
- Stabilizing closed loop system
- Accuracy
- Achieving proper transient and steady-state response
- Reduction of sensitivity to process parameters
- Disturbance rejection
- Performance and robustness

The control system design process

1. Establishment of goals, variables to be controlled, and specifications.
2. System definition and modeling.
3. Control system design, simulation, and analysis.
4. If the performance meets the specifications, then finalize the design.
5. Otherwise iterate.

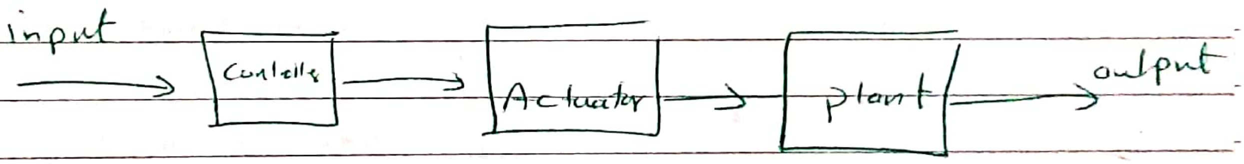
Chapter 1: Introduction to control system

→ Control system is an interconnection of components forming a system configuration that will provide a desired system response.



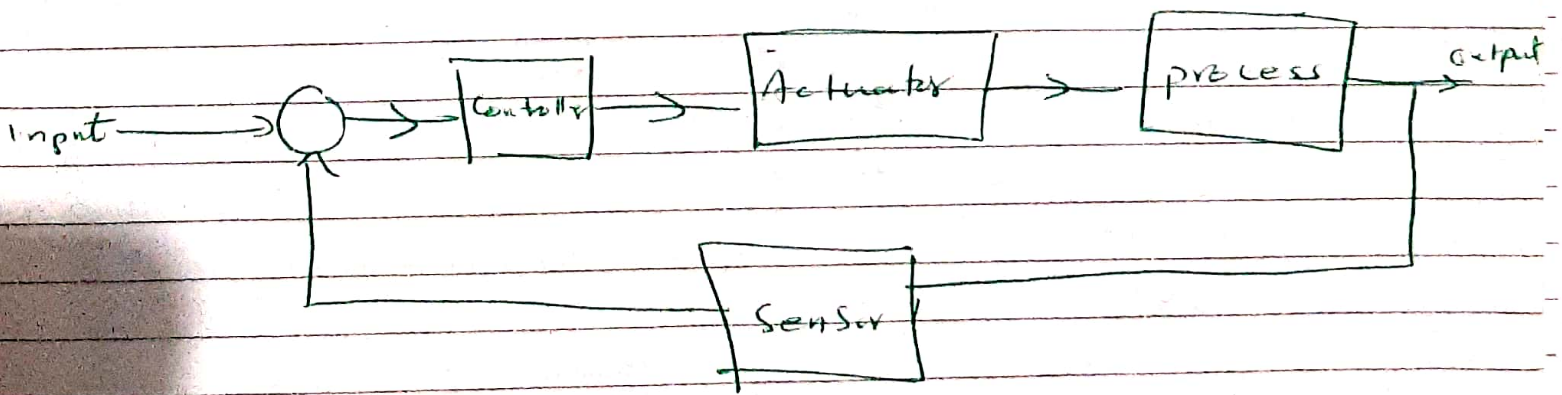
→ Control system types:-

1. Open loop system:-



e.g:- microwave
building lights

2. Closed-Loop System:-

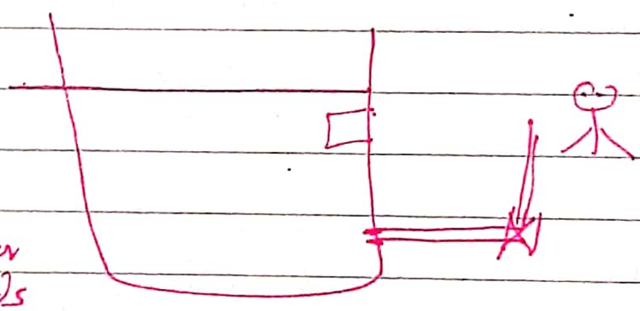


→ Control system components:-

1. plant → to be Controlled
2. Actuators → Converts input to power signal
3. Reference input / desired output
4. Controller
5. Error detector
6. Sensors

Examples:- (1) Manual - level control system.

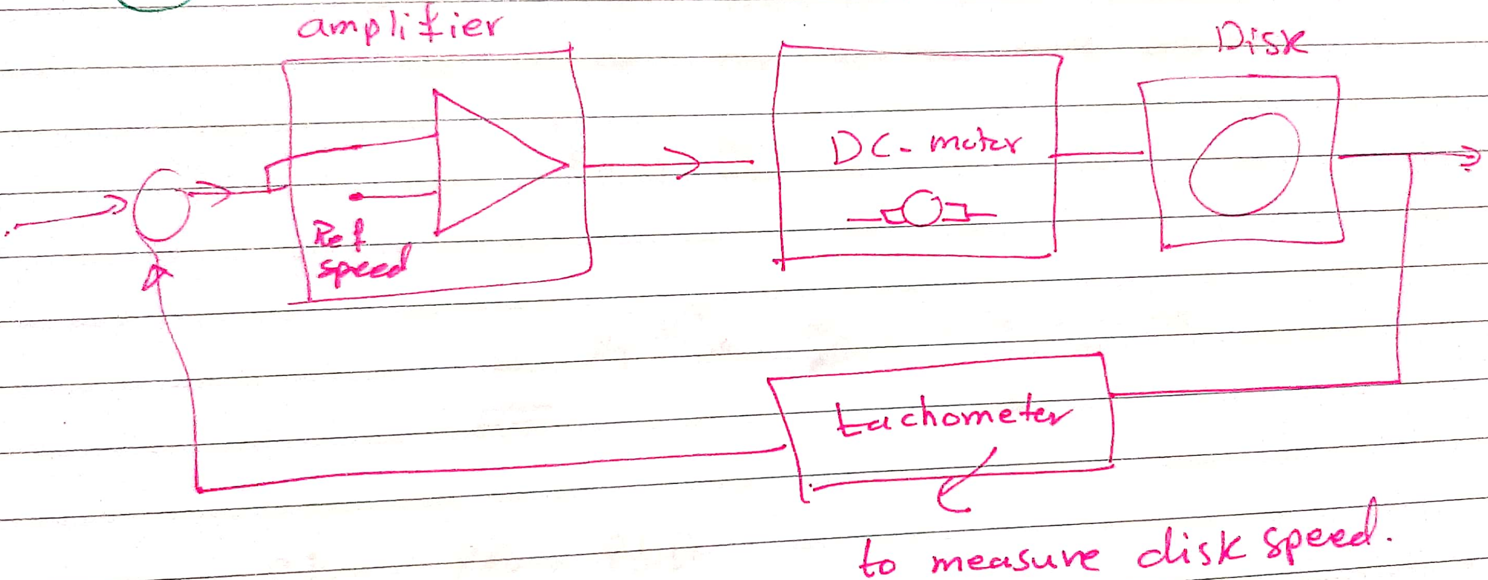
1. plant → tank ~~level~~
2. Controller Variable → tank level
3. Actuator → Valve → operator hands
4. Controller → operator brain
5. Sensor → operator eyes.



(3) Car speed / direction.

1. Car → plant
2. driver eyes → sensor
3. driver hands + stearing → actuater
4. driver brain → controller
5. output → direction or speed.

(4) Disk Drive Reel system



EX 5: car → speed control

1. Controlled variable / desired output / Reference input
   (car speed)

2. Controller: (driver brain ~~.......~~)

3. Actuator: (driver legs + brake + ...)

4. plant / process: (car)

5. Sensor if any: (driver eyes).

Ex 6: water tank system.

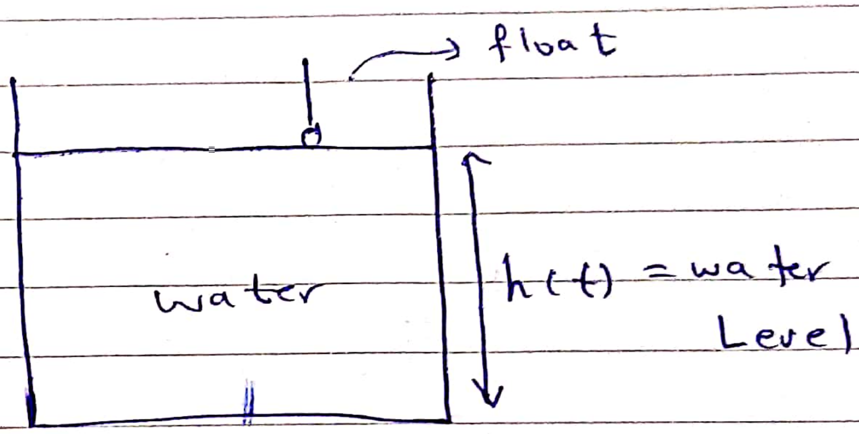
1. desired output : water Level

2. Controller : float

3. Acutator : float + Valve

4. process : water tank

5. Sensor : float



Ch.2 Mathematical Models of Systems

- Mathematical models of physical systems are key elements in the design and analysis of control systems.
- We will consider electrical and mechanical systems
- Obtain the input-output relationship for components and subsystems of the system in the form of transfer functions using Laplace transforms.
- Forming Different graphical representations of the system model (Block diagram and signal flow).

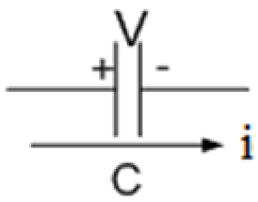
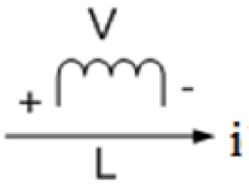
Expected outcomes

- Recognize that differential equations can describe the dynamic behavior of physical systems.
- Be able to utilize linearization approximations through the use of Taylor series expansions.
- Understand the application of Laplace transforms and their role in obtaining transfer functions.
- Be aware of block diagrams (and signal-flow graphs) and their role in analyzing control systems.
- Understand the important role of modeling in the control system design process

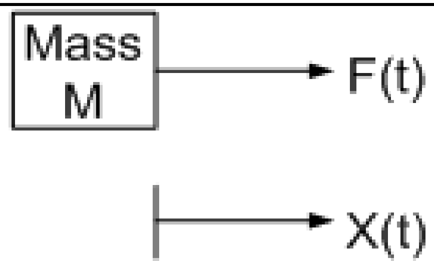
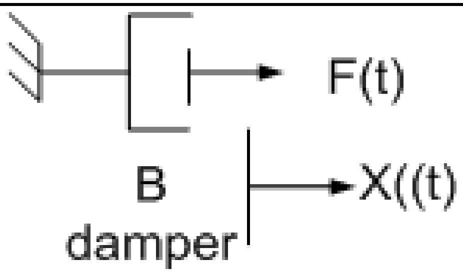
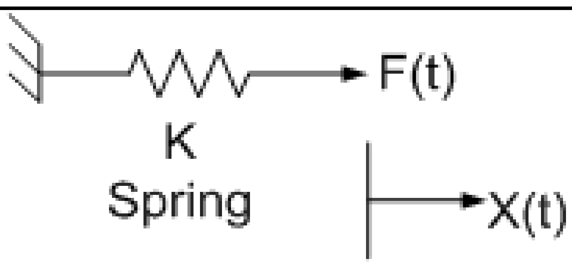
Electrical and Mechanical Systems

The differential equations describing the dynamic performance of a physical system are obtained by utilizing the physical laws of the process like Kirchhoff's laws (KCL, KVL) and Newton's second law

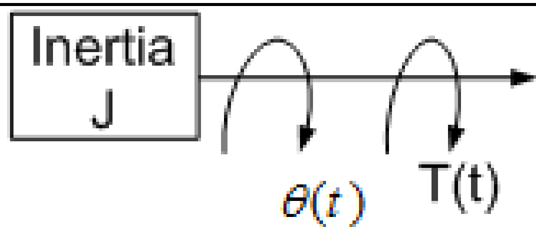
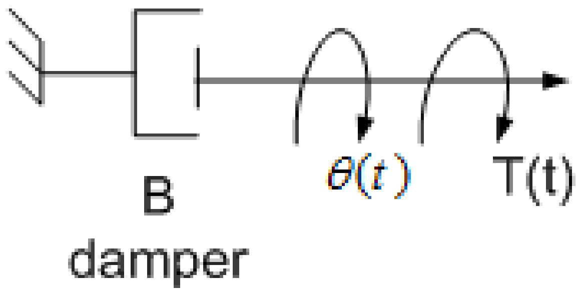
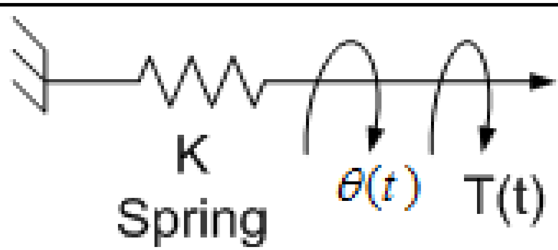
Electrical Components

Components	Time relation
	$i(t) = C\dfrac{dv(t)}{dt}$
	$v(t) = L\dfrac{di(t)}{dt}$
	$v(t)=R\,i(t)$

Mechanical Components: Translational motion

Component s	Time relation
	$F(t) = m\dfrac{d^2x(t)}{dt^2}$
	$F(t) = B\dfrac{dx(t)}{dt}$
	$F(t) = kx(t)$

Mechanical Components: Rotational motion

Components	Time relation
	$T(t) = J\frac{d^2\theta(t)}{dt}$
	$T(t) = B\frac{d\ \theta(t)}{dt}$
	$T(t) = K\theta(t)$

Element Type	Physical Element	Describing Equation
Inductive storage	Electrical Inductance	$v = L\ di/dt$
	Translational spring	$F = kx$
	Rotational spring	$T = k\theta$
Capacitive storage	Electrical capacitance	$i = C\ dv/dt$
	Translational mass	$F = M\ d^2x/dt^2$
	Rotational mass	$T = J\ d\omega/dt$
Energy dissipators	Electrical resistance	$v = iR$
	Translational damper	$F = b\ dx/dt$
	Rotational damper	$T = b\omega$

Examples

Laplace Transforms

- The Laplace transform is a mathematical tool for solving linear time invariant differential equation.
- It allows a time domain differential equation model of a system to be transformed in to algebraic model

Therefore simplifying the analysis and design of a control system

- Definition

$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt \quad \text{for} \quad f(t), t > 0$$

$$f(t) = L^{-1}\{F(s)\}$$

x(t) can be found by applying the inverse Laplace transform of X(s)

A linear system satisfies the principle superposition

- In general, a necessary condition for a linear system can be determined in terms of an excitation *x(t) and a response y(t)*

  ➡ *When the* system at rest is subjected to an excitation $x_1(t)$, it provides a response $y_1(t)$ and when the system is subjected to an excitation $x_2(t)$ *, it provides a corresponding* response $y_2(t)$;

⬇

For a linear system, it is necessary that the excitation $x_1(t) + x_2(t)$ *result in a response* $y_1(t) + y_2(t)$

This is usually called the principle of superposition

- If the system is nonlinear a linear one can be obtained using Taylor series expansion around a known operating conditions

Taylor Series Expansion

Consider a system whose input variable is x(t) and output variable is y(t) where the relationship between them is nonlinear given by y=f(x); If the operation conditions corresponds to $(\bar{x}, \bar{y})$, then a linear relationship around this point can be found using the Taylor series as follows:

$$y = f(x)$$

$$y = f(\bar{x}) + \frac{df}{dx}(x-\bar{x}) + \frac{1}{2!}\frac{d^2 f}{dx^2}(x-\bar{x})^2 + \ldots\ldots$$

Where the derivatives $\frac{df}{dx}, \frac{d^2 f}{dx^2}, \ldots\ldots$ are evaluated $x = \bar{x}$.

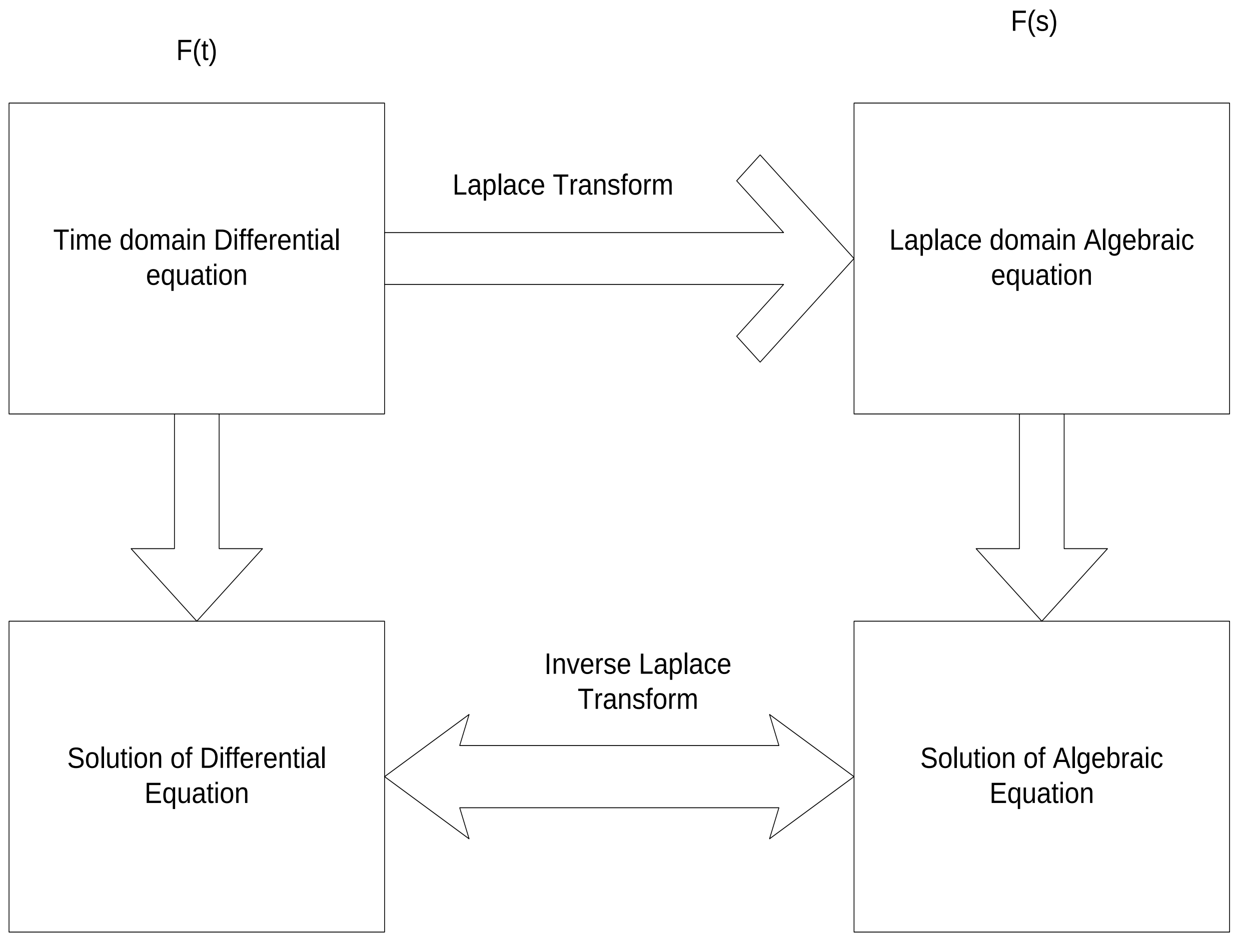
Eliminating higher order terms from Taylor series gives:

$$(y-\bar{y}) = k(x-\bar{x})$$

$$\Delta y = k\,\Delta x$$

Linear relationship

Where, $\bar{y} = f(\bar{x})$, $k = \left.\frac{df}{dx}\right|_{x=\bar{x}}$


F(t)
F(s)
Laplace Transform
Time domain Differential equation
Laplace domain Algebraic equation
Inverse Laplace Transform
Solution of Differential Equation
Solution of Algebraic Equation

Laplace transform of the unit step (u(t)=1)

$$L[u(t)] = \int_0^{\infty} 1 e^{-st} dt = \frac{-1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$$

Laplace transform of time Differentiation

$$L[\frac{df(t)}{dt}] = sF(s) - f(0)$$

We can extend the time differentiation to be:

$$L\left[\frac{df(t)^2}{dt^2}\right] = s^2 F(s) - sf(0) - f'(0)$$

$$L\left[\frac{df(t)^3}{dt^3}\right] = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

general case

$$L\left[\frac{df(t)^n}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0)$$

$$- \ldots - f^{(n-1)}(0)$$

	f(t)	$\mathcal{L}$(f)		f(t)	$\mathcal{L}$(f)
1	Unit-impulse $\delta(t)$	1	7	cos ωt	$\frac{s}{s^2+\omega^2}$
2	Unit-step 1	1/s	8	sin ωt	$\frac{\omega}{s^2+\omega^2}$
3	Unit-ramp t	$1/s^2$	9	cosh at	$\frac{s}{s^2-a^2}$
4	t^2	$2!/s^3$	10	t e^{at}	$\frac{1}{(s-a)^2}$
5	t^n (n is +ve integer)	$\frac{n!}{s^{n+1}}$	11	e^{at} cos ωt	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	e^{at} sin ωt	$\frac{\omega}{(s-a)^2+\omega^2}$

For most engineering purposes the inverse Laplace transformation can be accomplished simply by referring to Laplace transform tables

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The Laplace Transform….cont

- The Laplace variable *s* can be considered to be the differential operator so that → $s = \frac{d}{dt}$

  And we also have the integral operator → $\frac{1}{s} = \int_0^{\infty} dt$

- *s*-operator is a complex quantity has a real and imaginary parts → $s = a + jb$

- Initial Value Theorem

$$f(0) = \lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t)$$

- Final Value Theorem

$$f(\infty) = \lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)$$

For function f(t)

The final value theorem is very useful for analysis and design of control systems, since it gives the final value of a time function **f(t)**

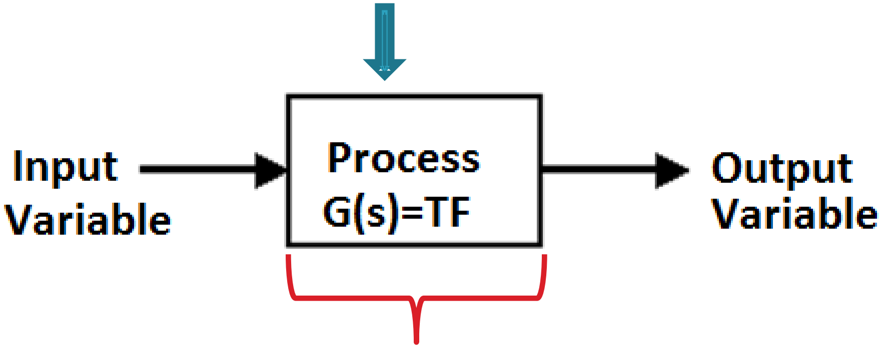
Examples

The Transfer Function (T.F) of Linear Systems

- The transfer function of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, **with all initial conditions assumed to be zero.**

- The transfer function of a system (or element) represents the relationship describing the dynamics of the system under consideration

$$T.F = \frac{output}{input} = G(s)$$



Represents system dynamics in s-domain

$$T.F = \frac{output}{input} = G(s)$$

$$= \frac{p(s)}{q(s)}$$

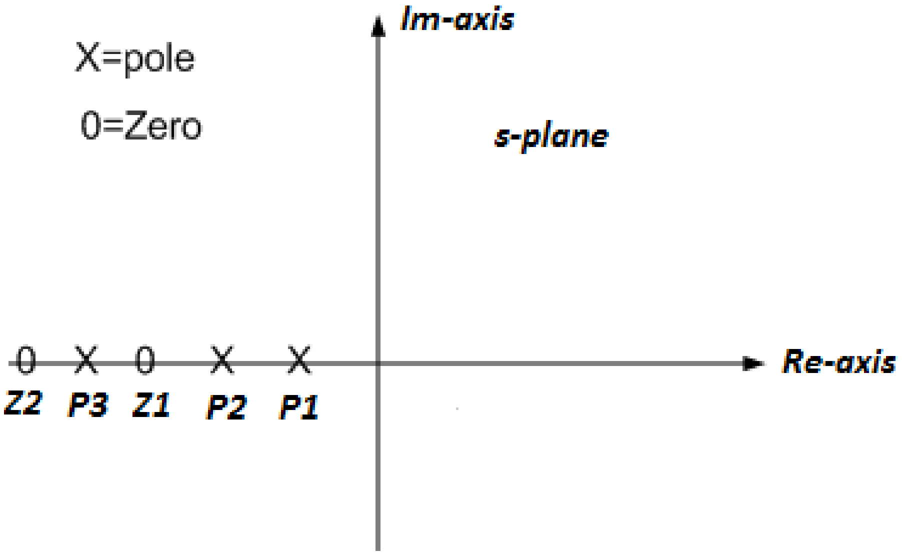
- **Where *p(s)* and *q(s)* are polynomials**
- **The roots of *p(s)* are called the zeros of the system where the roots of *q(s)* are called the poles of the system**

***q(s)* is also known as the characteristic equation of the systems**

- **The location of the roots of *q(s)* in s-plane gives a character to the system performance**

$$G(s) = \frac{p(s)}{q(s)}$$

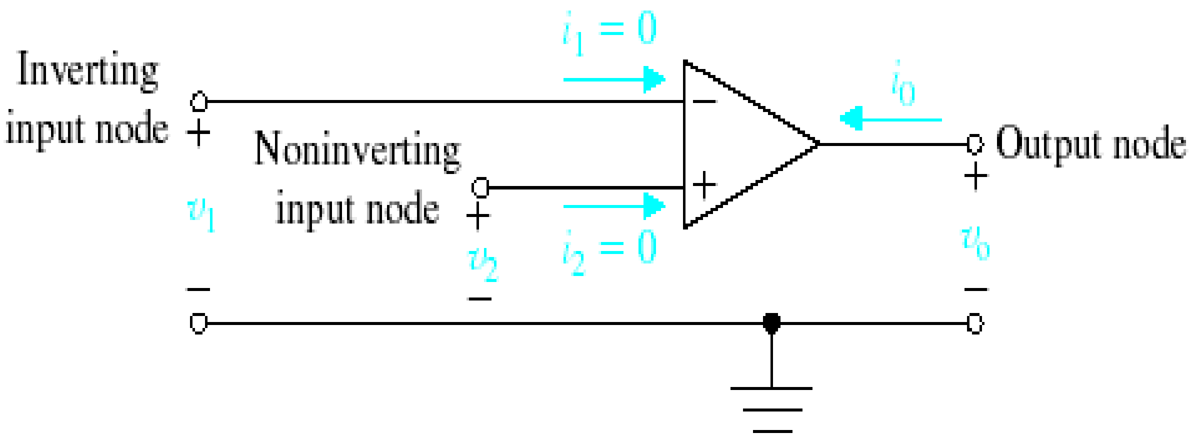
$$= \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)(s+p_3)}$$



Example: TF's of Operational Amplifier circuits

The operational amplifier (op-amp) belongs to an important class of analog integrated circuits commonly used as building blocks in the implementation of control systems and in many other important applications.

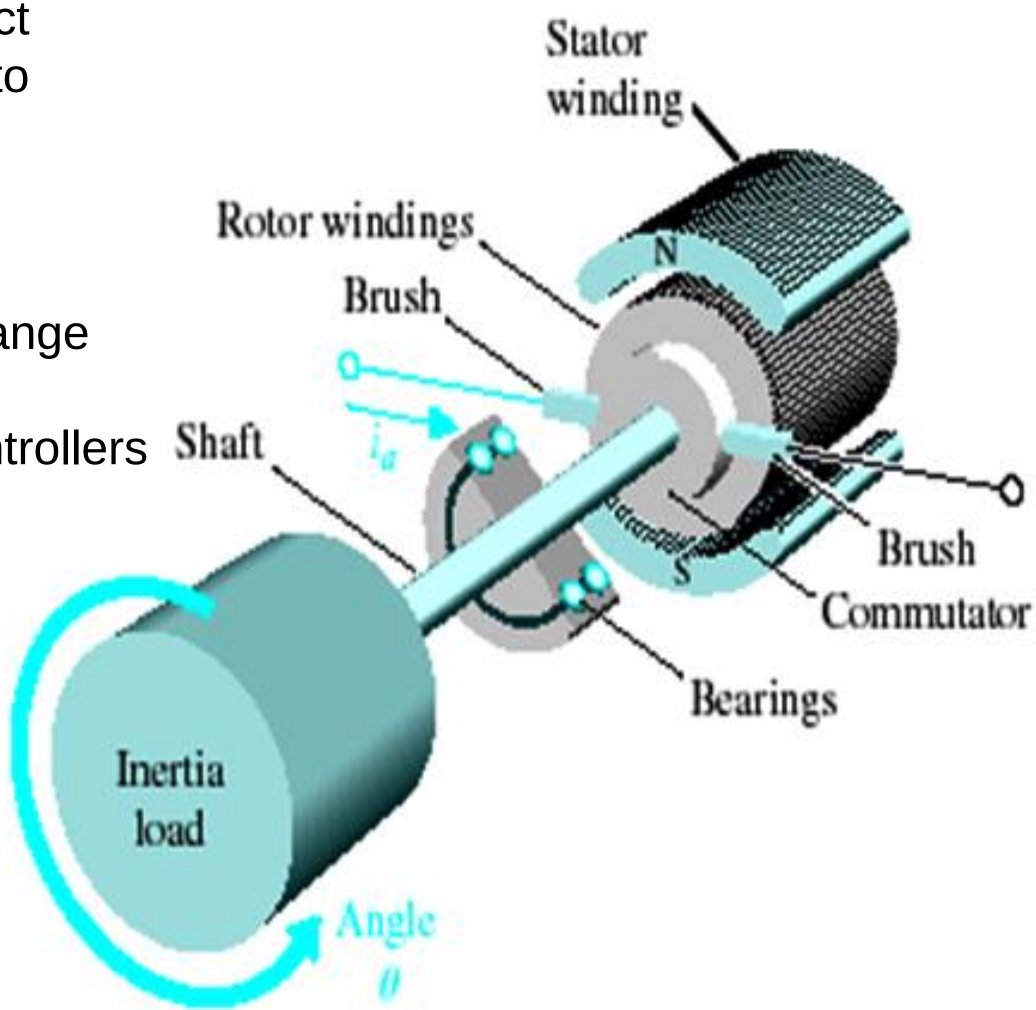
- Compensators
- Control laws
- Filters
- Comparator/summing elements

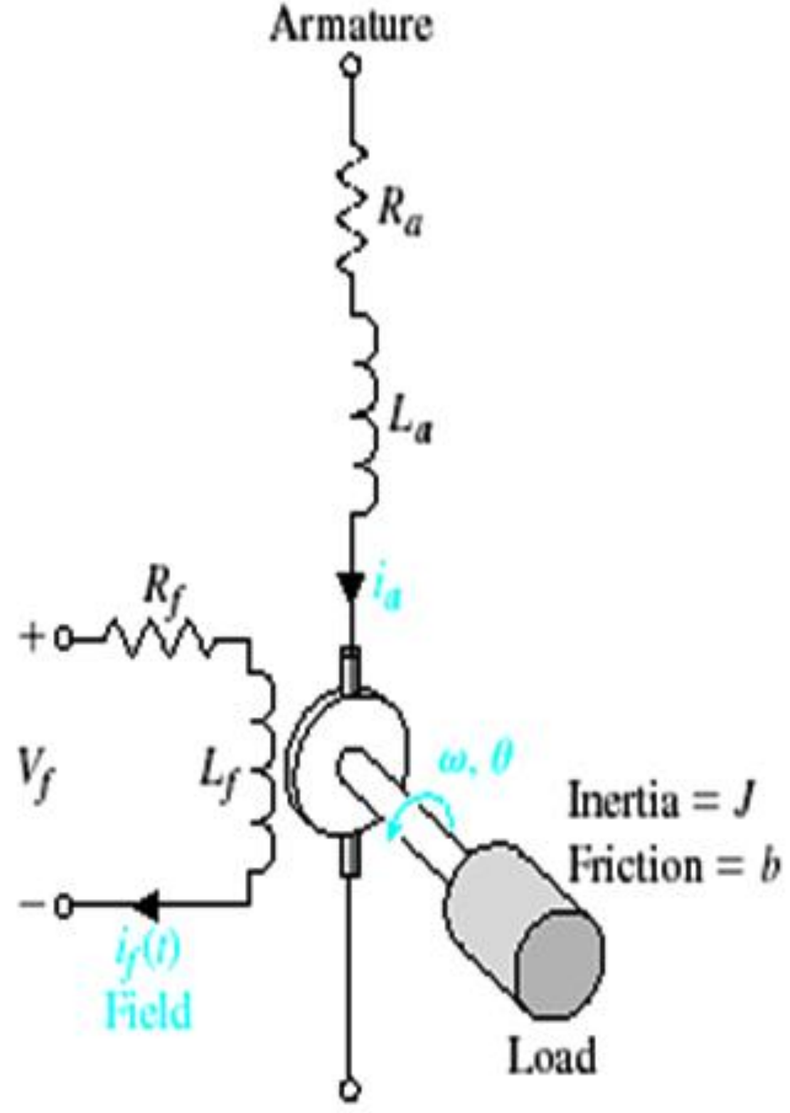


Examples: TF's of DC motors

- A DC motor is used to move loads and is called an **actuator**.
- The DC motor converts direct current (DC) electrical energy into rotational mechanical energy
- DC motors features:
  - High output torque
  - Speed controllability over a wide range
  - Portability
  - Adaptability to various types of controllers

DC motors are widely used in numerous control applications such as robotic manipulators, tape transport mechanisms, disk drives, and machine tools




Armature
R_a
L_a
i_a
R_f
+
V_f
L_f
−
$i_f(t)$
Field
ω, θ
Inertia = J
Friction = b
Load

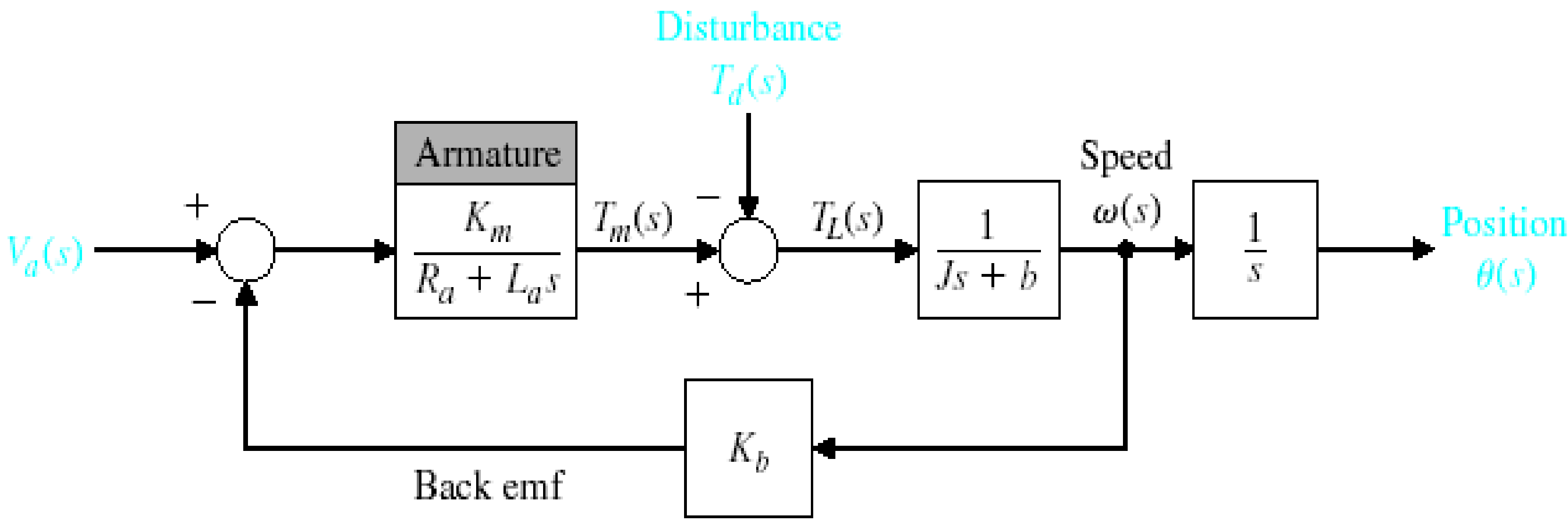
Block Diagram (BD) Models

Again:

- Control systems consists of elements that are represented mathematically by a set of simultaneous differential equations
- Laplace transformation reduces the problem of differential equations to the solution of a set of linear algebraic equations.
- Since control systems are concerned with the control of specific variables, the controlled variables must relate to the controlling variables
  - This relationship is typically represented by the TF of the subsystem relating the input and output variables
  - The importance of the TF is evidenced by the ability to represent the relationship of system variables by diagrammatic means called BD

Hence, the control system with all its elements can be represented by one BD showing all variables relations

Armature controlled DC motor BD



In order to find the cause-effect relationship of a system BD, we simplify the BD (reduction) by applying the rules of BD algebra.

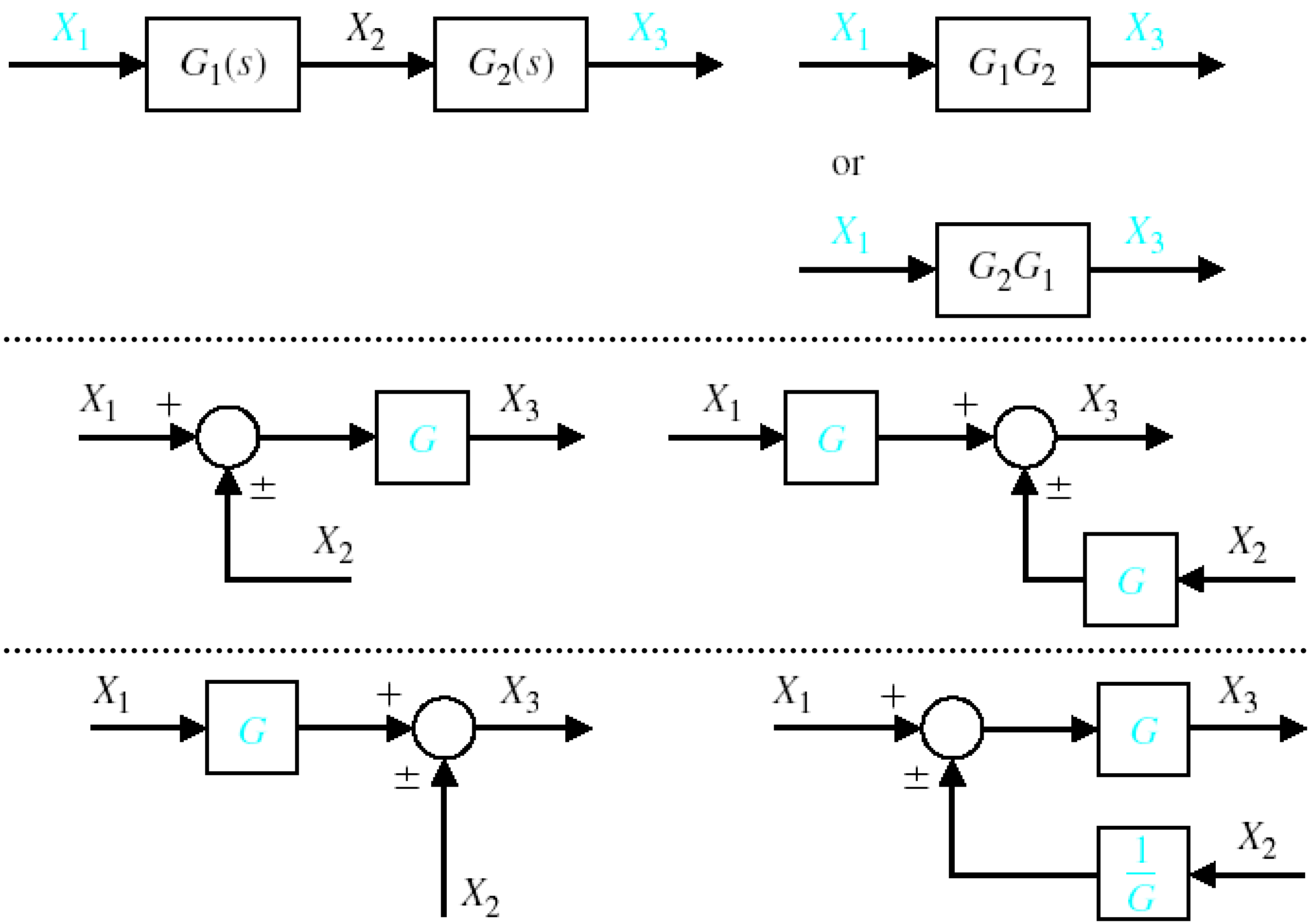
Block Diagram (BD) Algebra

Original Diagram | Equivalent Diagram

(1)

(2)

(3)



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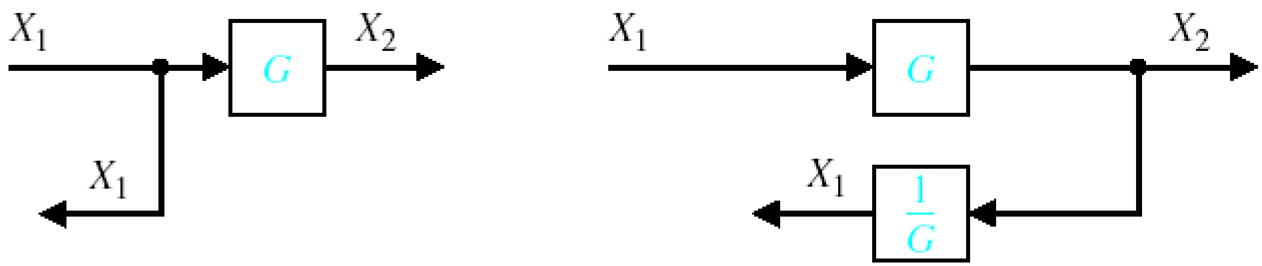
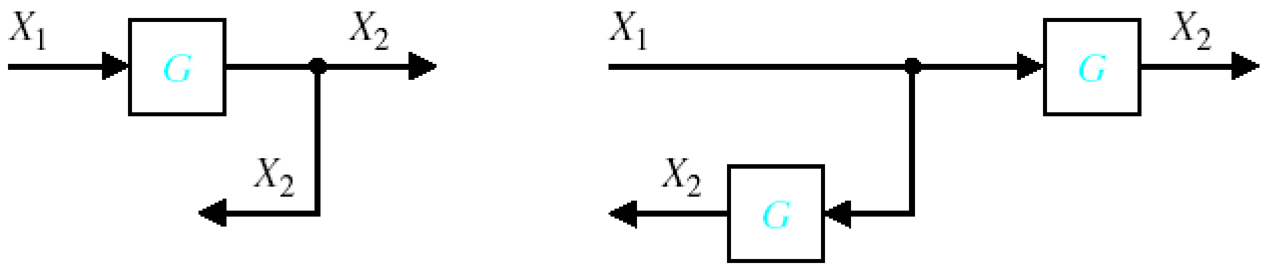
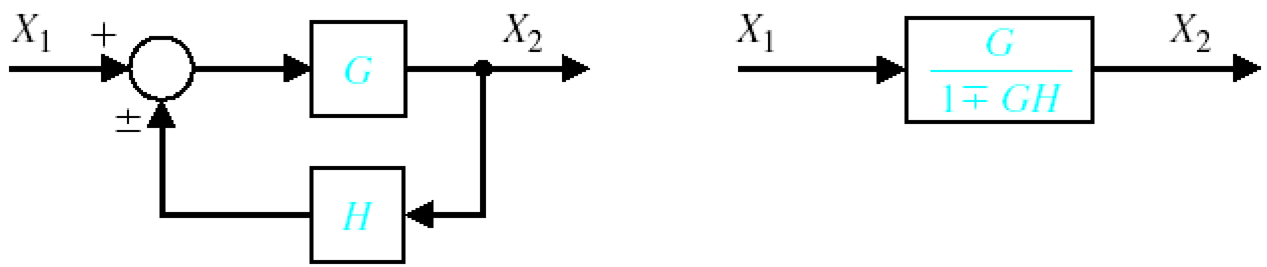


(5)

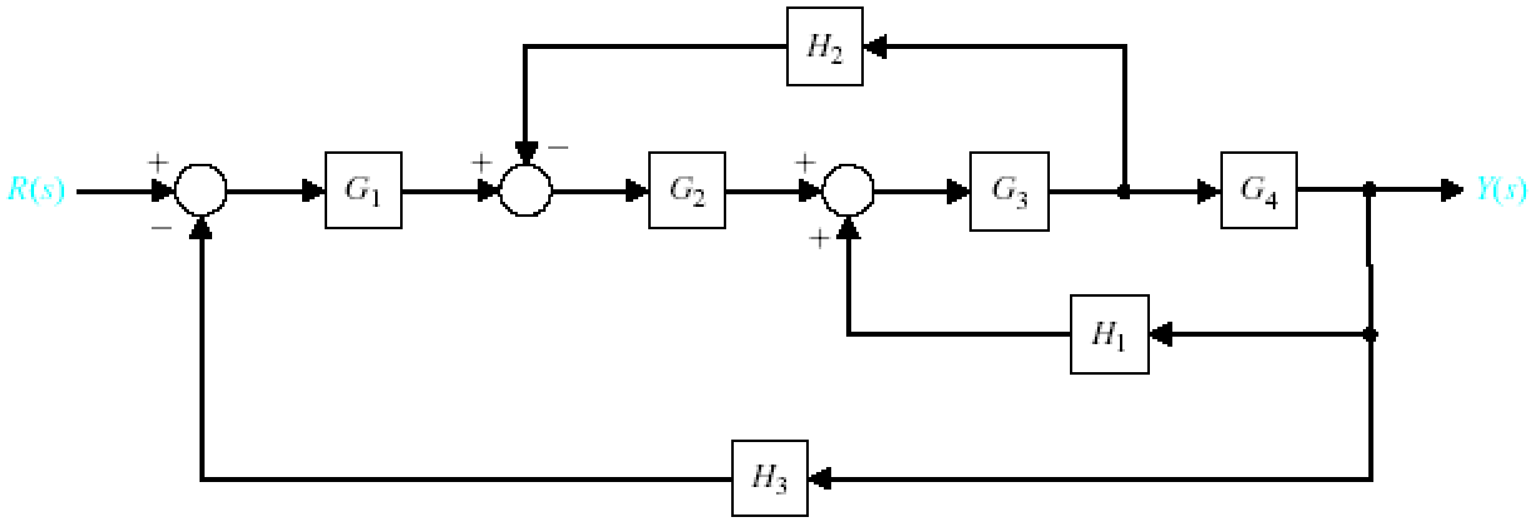


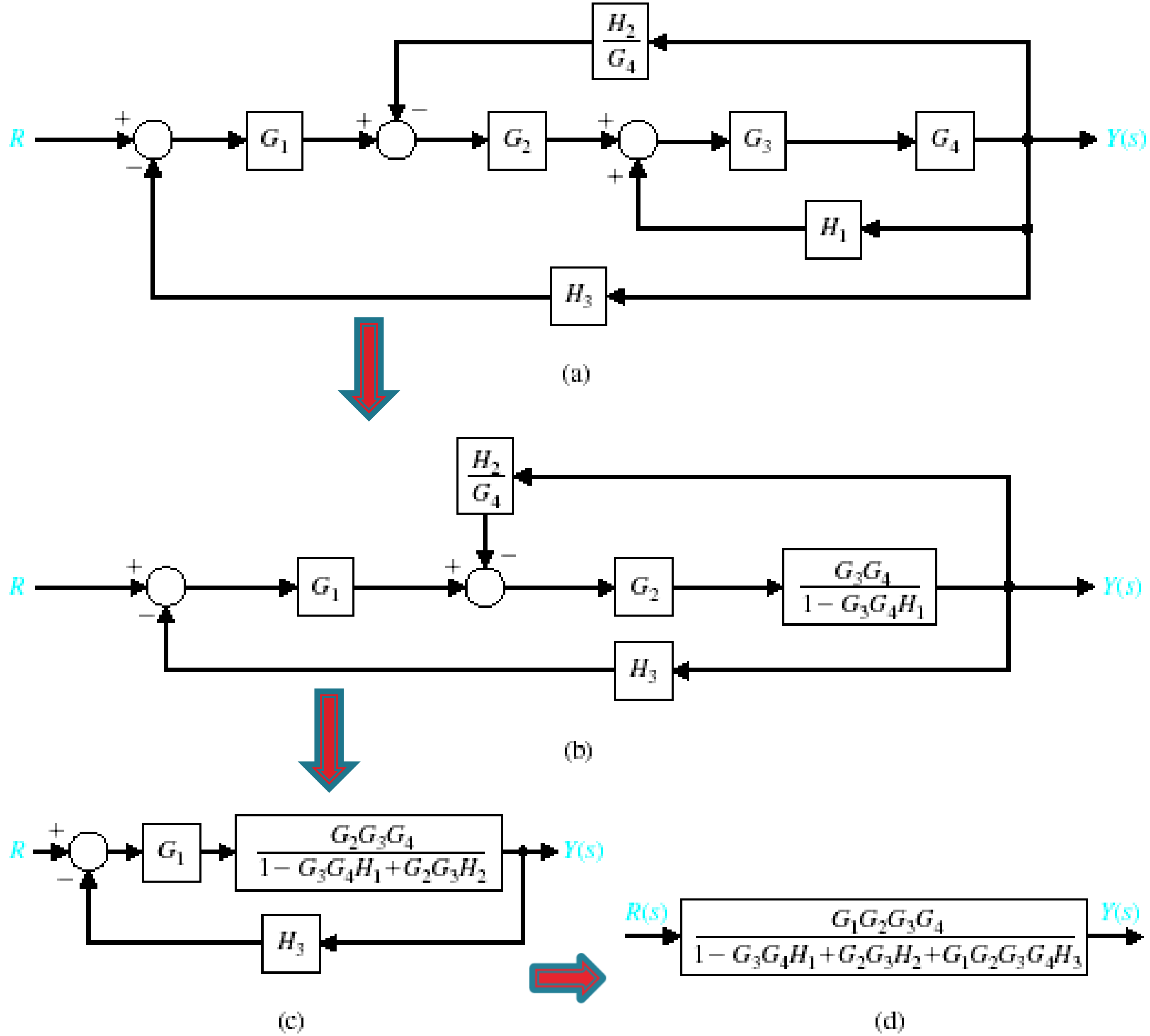
(6)



Example

For the following control system, find the input-output relationship (i.e. TF) relation the output variable Y(s) to the input variable R(s).




R
Y(s)
G1
G2
G3
G4
H1
H2/G4
H3
(a)
G3G4 / (1 − G3G4H1)
(b)
G2G3G4 / (1 − G3G4H1 + G2G3H2)
(c)
R(s)
G1G2G3G4 / (1 − G3G4H1 + G2G3H2 + G1G2G3G4H3)
(d)

Signal-Flow (SF) Graph Models

- Block diagrams are adequate for the representation of the system interrelationships. However, for a system with reasonably complex interrelationships, the block diagram reduction procedure is often quite difficult to complete.

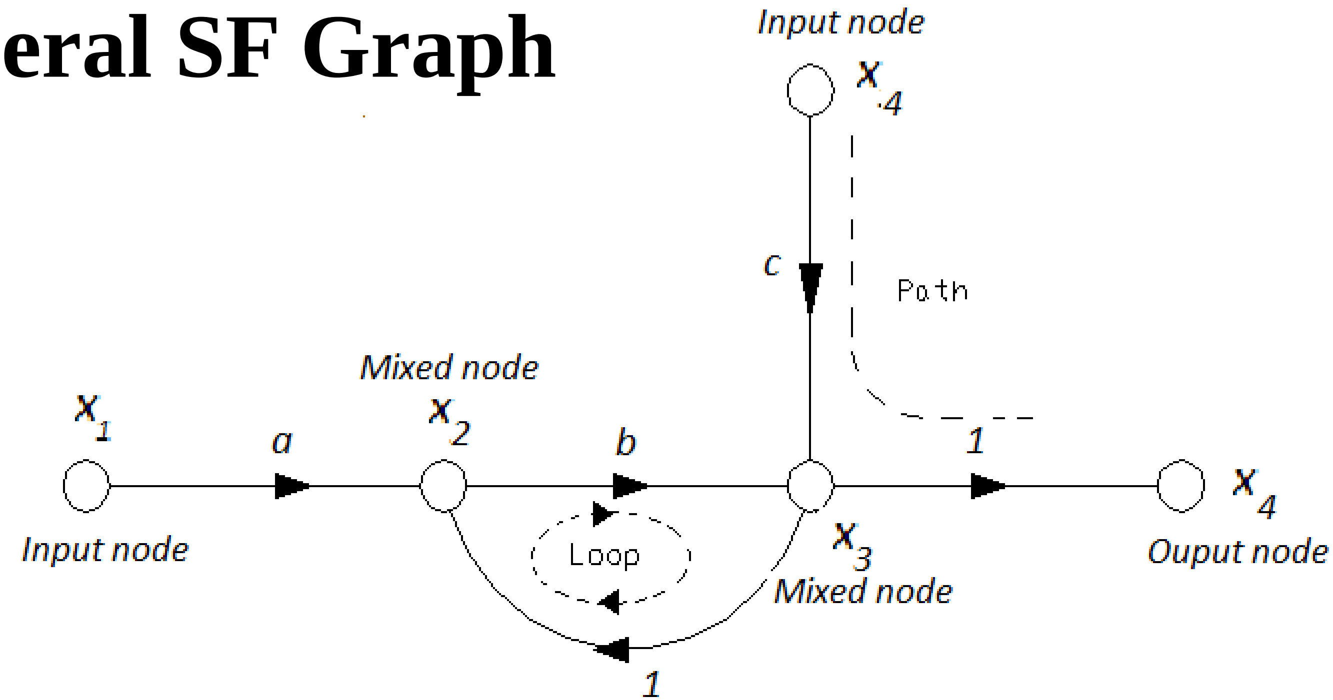
  - An alternative method for determining the relationship between system variables has been developed by Mason which is called the signal-flow graph method

- A signal-flow graph is a diagram consisting of nodes that are connected by several directed branches and is a graphical representation of a set of linear relations.

- The reduction procedure (used in the BD method) is not necessary to determine TF (input-output relationship) of a system represented by SF graph.

  - **We apply Mason`s Gain Formula to find the TF**

General SF Graph



Node: acts like a summing point and also represents a system variable.

Transmittance: real or complex gain between two nodes.

Branch: directed line segment joining two nodes.

Input node (source): only outgoing branches.

Output node (sink): only incoming branches.

Mixed node: both incoming and outgoing branches

Path: traversal of connected branches in the direction of arrows.

Loop: closed path.

Loop gain: product of branch transmittance at a loop.

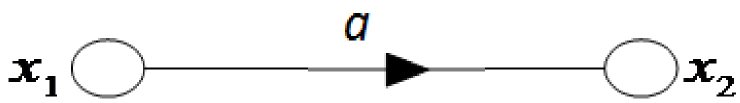
Non touching loops: they do not posses any common nodes.

Forward path: path from an input to an output node that does not cross any node more than once.

Forward path gain: product of transmittances of a forward path

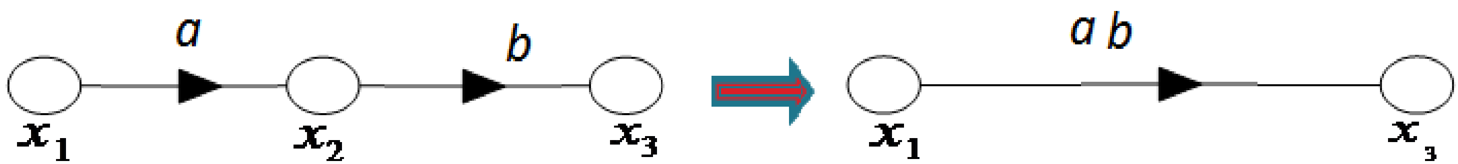
SF Graph Algebra

1)



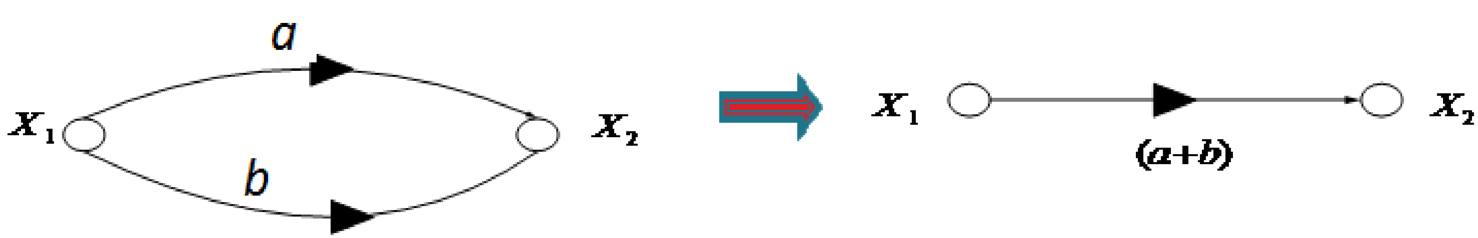
$$x_2 = ax_1$$

2)



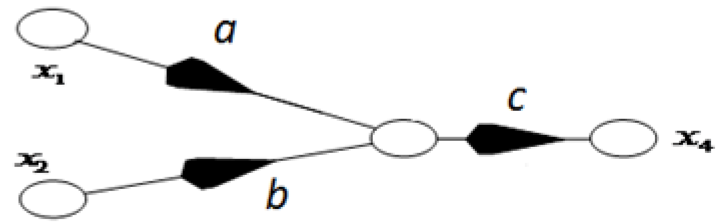
$$x_3 = bx_2 = abx_1$$

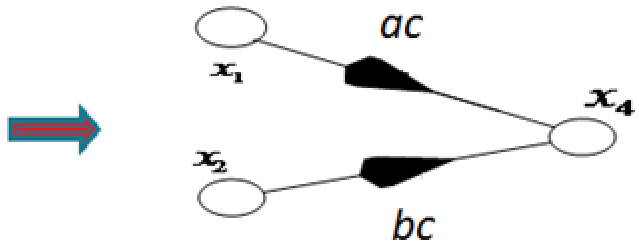
3)



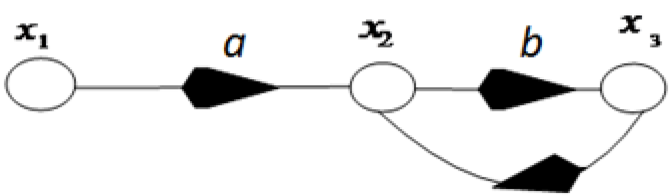
$$x_2 = ax_1 + bx_1 = (a+b)x_1$$

4)

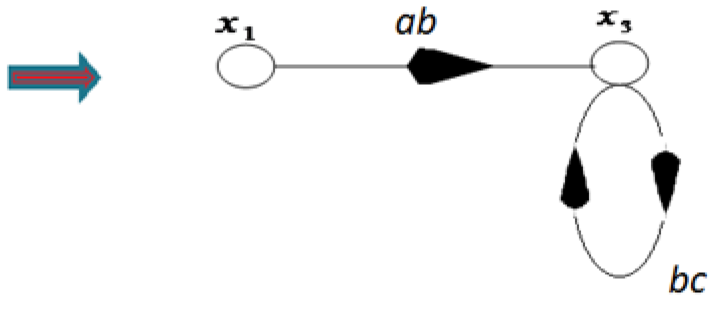


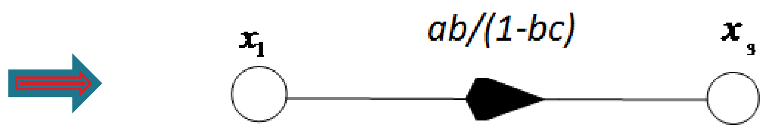


5)

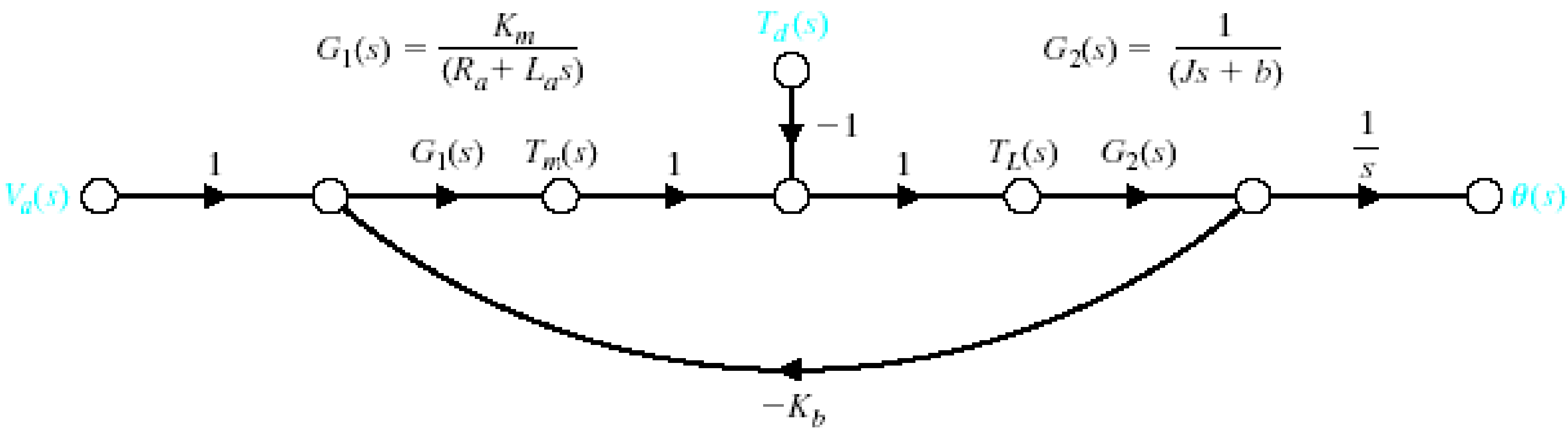


$$x_3 = bx_2$$
$$x_2 = ax_1 + cx_3$$
$$x_3 = b(ax_1 + cx_3)$$
$$x_3 = abx_1 + bcx_3$$

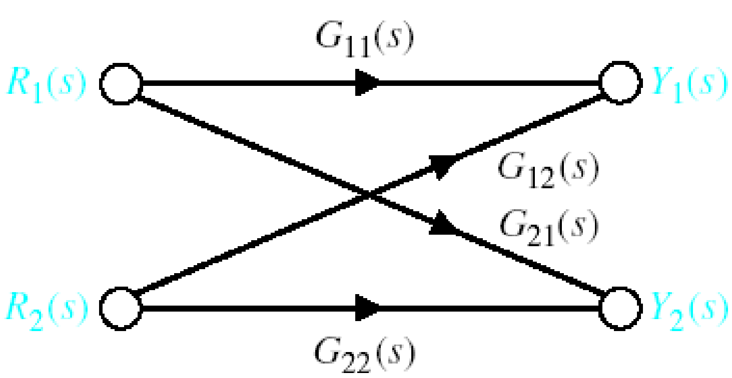




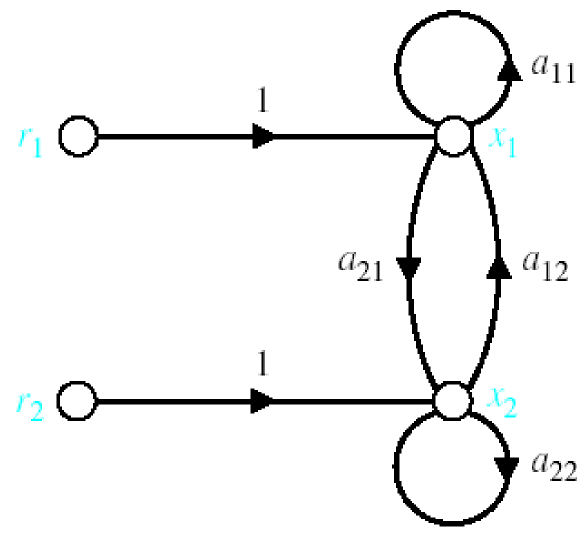
Examples



(1)



(2)



(3)

Mason`s Gain Formula

The formula is often used to relate the output variable Y(s) to the input variable R(s) (i.e. finding the TF) and is given by

$$TF = \frac{\sum_K P_K \Delta_K}{\Delta}$$

where,

P_K is the gain of path *K* from input node to output node in the direction of the arrows and without passing node than once.

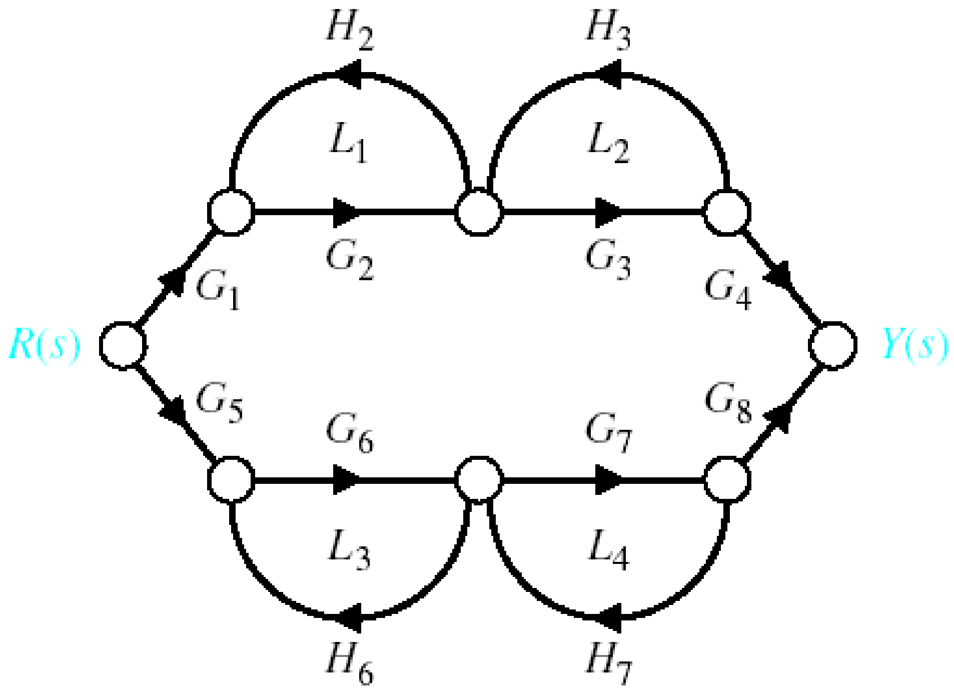
Δ_K: Cofactor o the path P_K

Δ : determinant of the graph

Δ = 1 – (sum of all different loop gains) + (sum of the gain products of all combinations of two non touching loops) – (sum of the gain products of all combinations of three non touching loops)

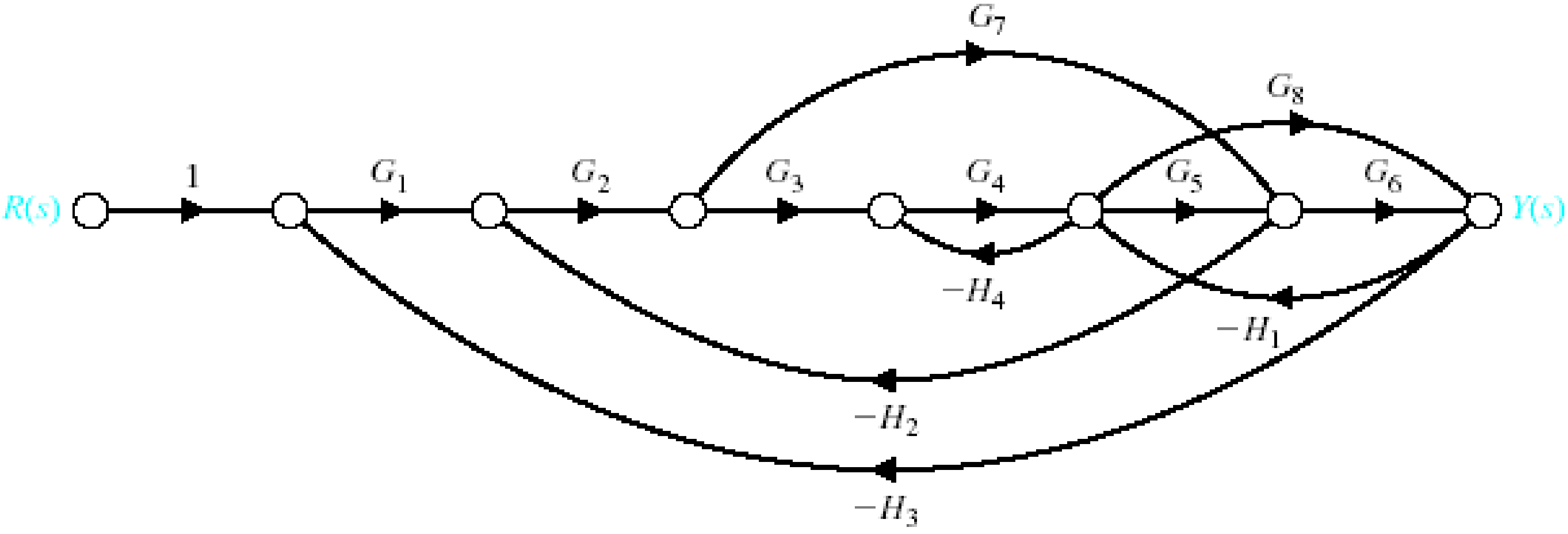
Example

For the following control system, find the input-output relationship (i.e. TF) relation the output variable Y(s) to the input variable R(s).

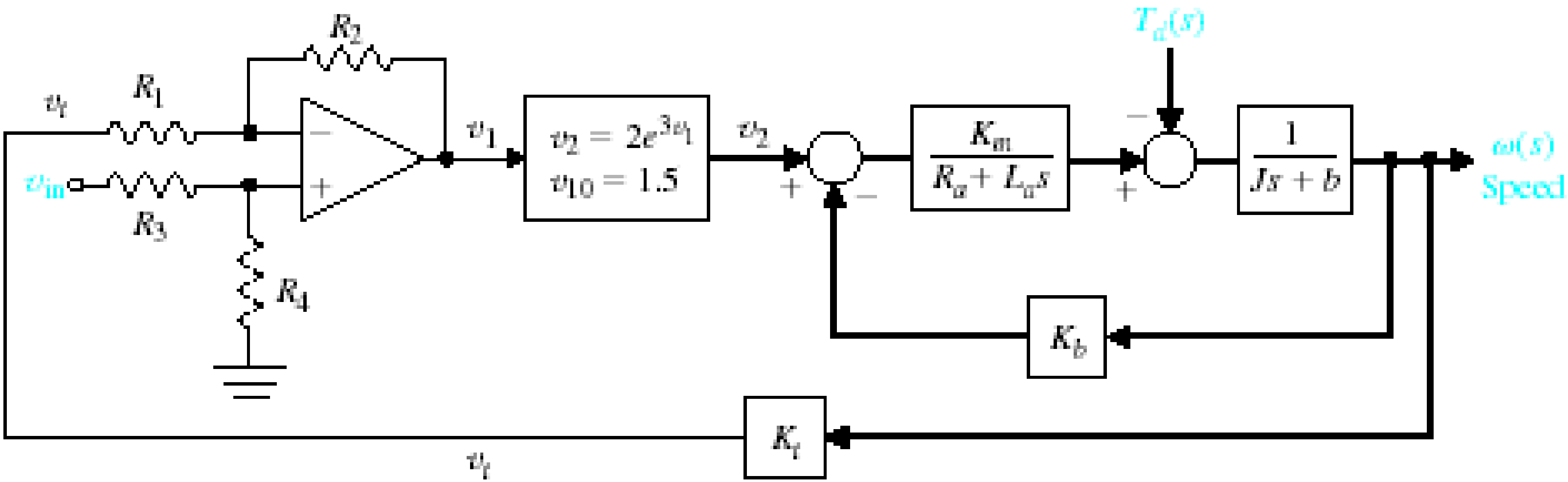


Example

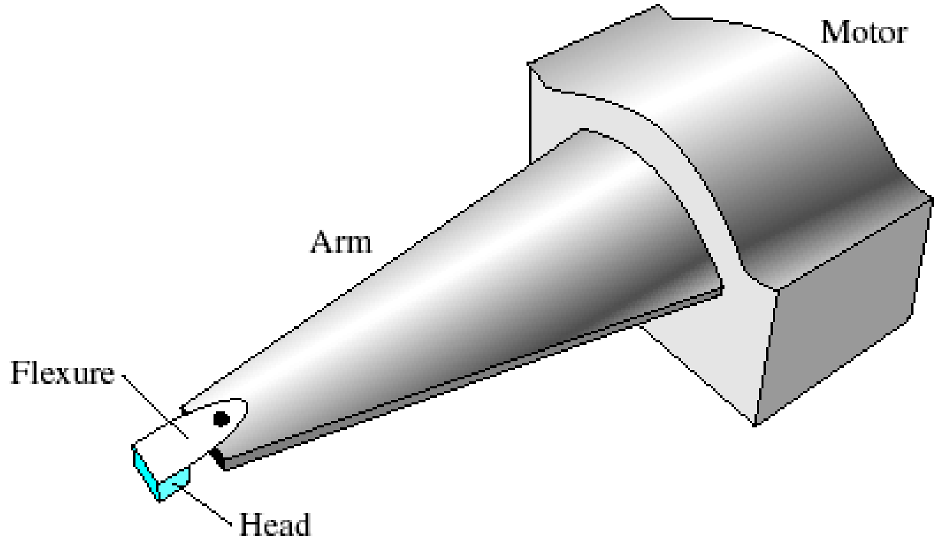
For the following control system, find the input-output relationship (i.e. TF) relation the output variable Y(s) to the input variable R(s).

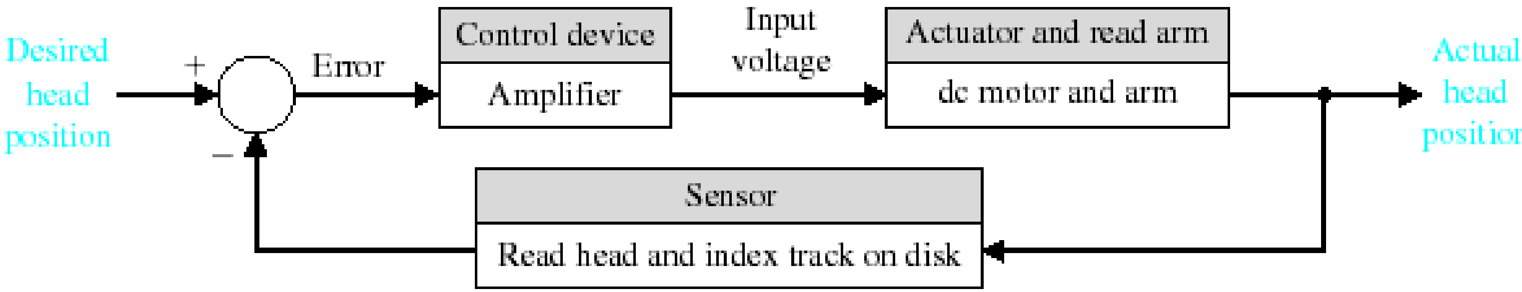


Example: Armature Controlled DC Motor (page 94)

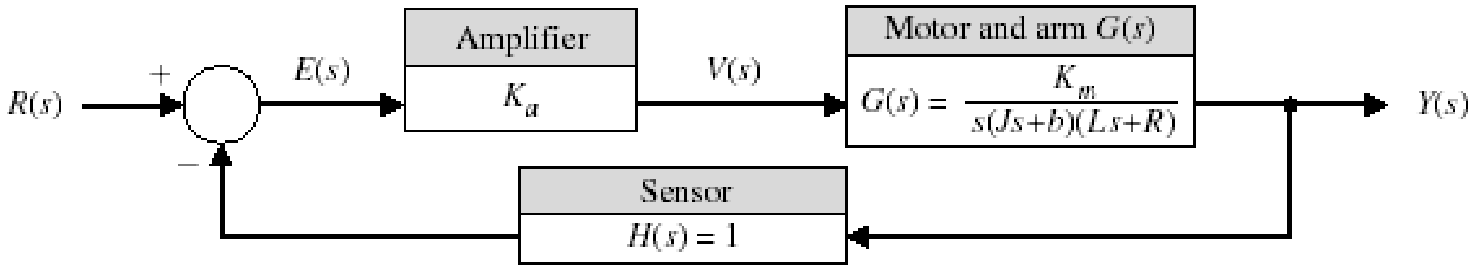


Example: Disk Drive Read System (Page 118)




Desired head position
+
−
Error
Control device
Amplifier
Input voltage
Actuator and read arm
dc motor and arm
Actual head position
Sensor
Read head and index track on disk

(a)


R(s)
+
−
E(s)
Amplifier
K_a
V(s)
Motor and arm G(s)
$G(s) = \frac{K_m}{s(Js+b)(Ls+R)}$
Y(s)
Sensor
H(s) = 1

(b)

chapter 2: Mathematical Models of system

→ Mathematical Model is describing the system by set of differential equations that relate between the input and the output.

Systems
- Mechanical
  - Translational motion → $\sum F = ma$
  - Rotational Motion → $\sum T = J\ddot{\theta}$
- Electrical
  - KCl → $\sum I = 0$
  - KVL → $\sum V = 0$

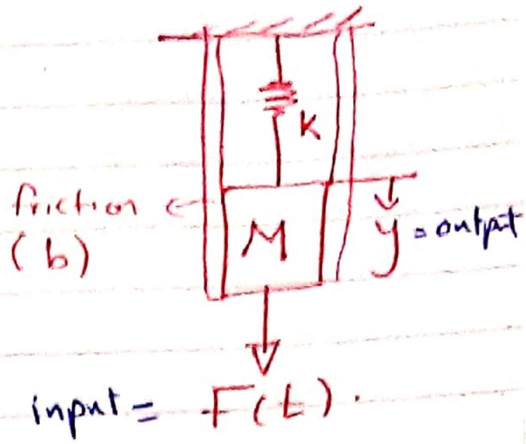
F = Force
M = mass
a = acceleration

T = Torque
J = inertia
$\ddot{\theta}$ = angular acceleration

Ex 1: Derive the equation of motion of the system below:-

Sol:

Free body diagram



note:

y = displacement

$\dot{y}$ = velocity

$\ddot{y}$ = acceleration

$$\downarrow + \sum F = m\,a = m\ddot{y}$$

$$F(t) - b\dot{y} - ky = m\ddot{y}$$

input

---

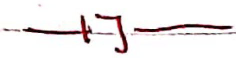
① spring force:

spring

$F_s = Ky$ where K = spring constant

y = spring deflection

direction: against deformation direction

(2) Damper force:  b

(or friction)

$F_d = b\dot{y}$

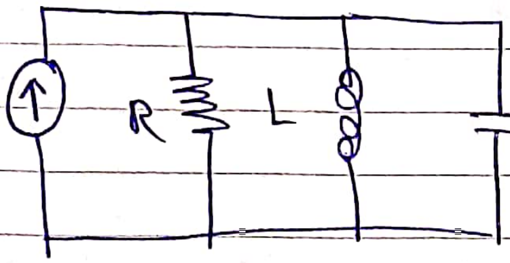
b = damping constant

$\dot{y}$ = velocity

direction: against speed direction.

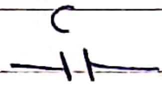
EX 2: RLC circuit

I(t) current source (input)

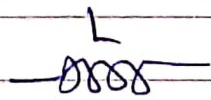


v(t) = output.

Remember:

(1)  Capacitor

$I_c(t) = C \frac{dv(t)}{dt}$

(2)  inductor

$v(t) = L \frac{dI}{dt}$

(3)  Resistor $V = RI$

Ex2:

$I(t) = I_R + I_L + I_C$ → parallel

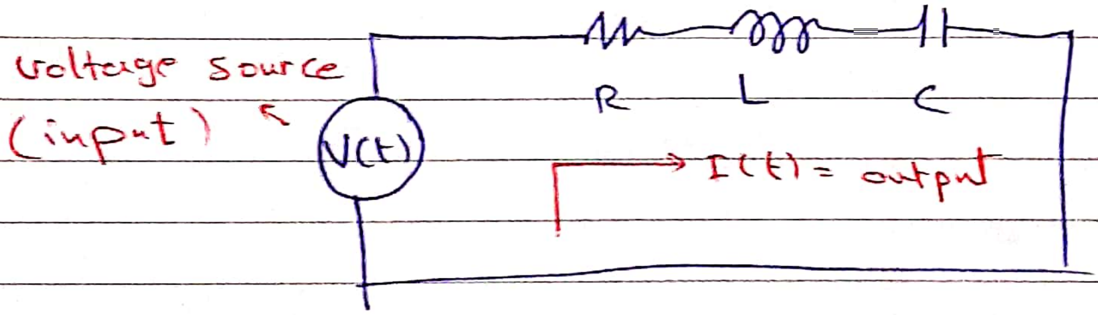
$$= \frac{V(t)}{R} + \frac{1}{L}\int V(t) + C\frac{dV(t)}{dt}$$

but $V_C = V_R = V_R$

also, derive the equation above to get rid of integration

$$\therefore \dot{I}(t) = \frac{\dot{V}(t)}{R} + \frac{V(t)}{L} + C\ddot{V}(t)$$

Ex 3: RLC - Circuit



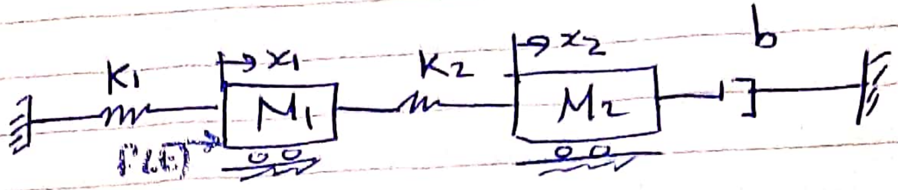
$I_C = I_R = I_L = I$ and

$V(t) = V_C + V_L + V_R$

$$V(t) = \frac{1}{C}\int I(t)\,dt + L\frac{dI}{dt} + RI(t)$$

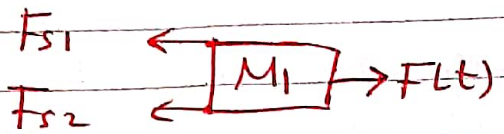
$$\therefore \quad V(t) = \frac{I(t)}{C} + L\ddot{I}(t) + R\dot{I}$$

Ex4:



obtain the governing equation for the system ~~below~~ above.

Free body diagram



$$\overset{+}{\rightarrow} \sum F = M_1 \ddot{x}_1$$

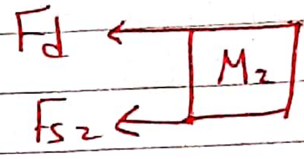
$$F(t) - F_{s_1} - F_{s2} = m_1 \ddot{x}_1$$

$$F(t) = m_1 \ddot{x}_1 + F_{s_1} + F_{s2}$$

$$F(t) = M_1 \ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2)$$

input = F(t)

x_1, x_2 = outputs



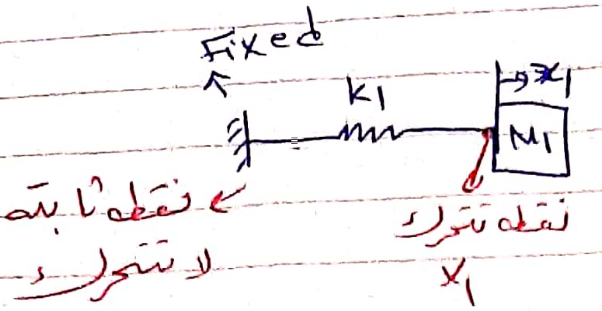
$$\overset{+}{\rightarrow} \sum F = m_2 \ddot{X}_2$$

$$-F_{s_2} - F_d = m_2 \ddot{x}_2$$

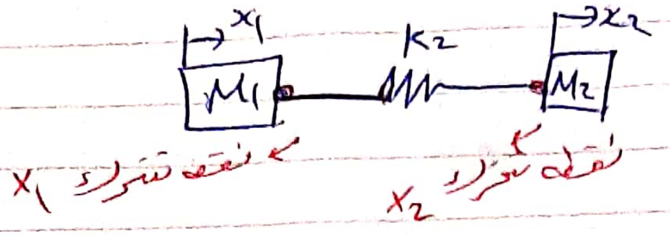
$$M_2 \ddot{x}_2 + F_d + F_{s2} = 0$$

$$M_2 \ddot{x}_2 + b\dot{x}_2 + K_2(x_2 - x$$

$$= 0$$

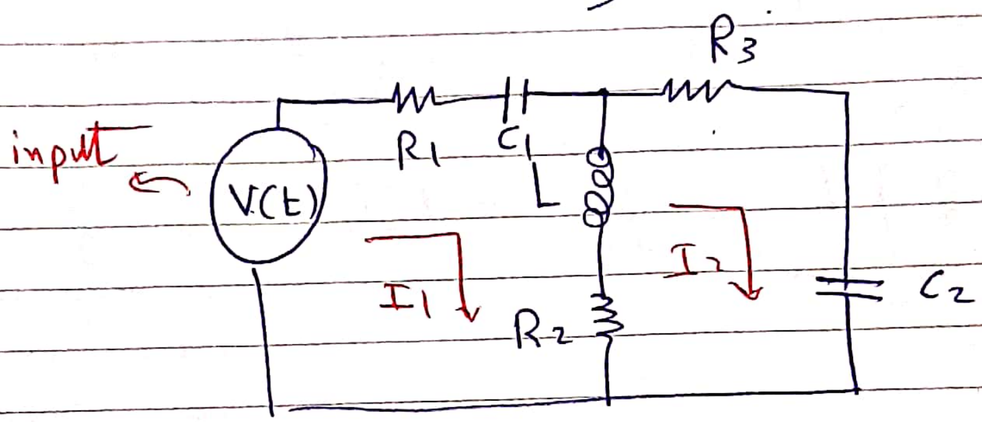


$\Rightarrow \quad k_1(x_1 - 0) = k_1 x_1$



$K_2(x_1 - x_2)$

Ex 5:- obtain the differential equation of the following system:



ccw +ve

cw −ve

$$V(t) = I_1 R_1 + \frac{1}{C_1}\int I_1\,dt + L\frac{d(I_1 - I_2)}{dt}$$

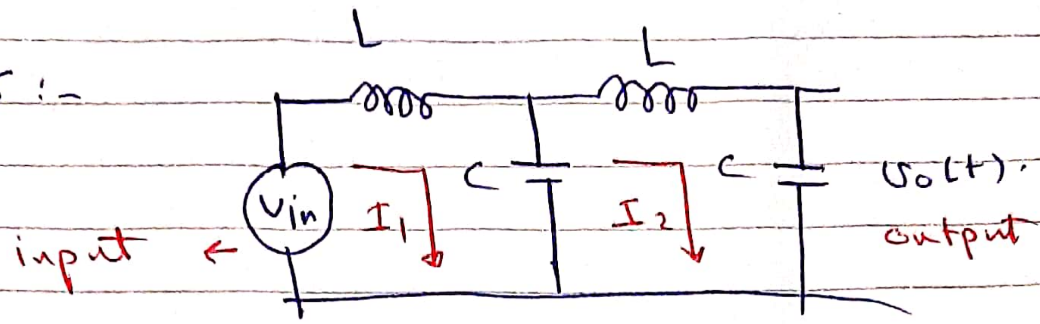
$$+ R_2(I_1 - I_2)$$

نشتق للتخلص من التكامل

$$0 = L\frac{(dI_2 - dI_1)}{dt} + R_2(I_2 - I_1) + R_3 I_2$$

$$+ \frac{1}{C_2}\int I_2\, dt.$$

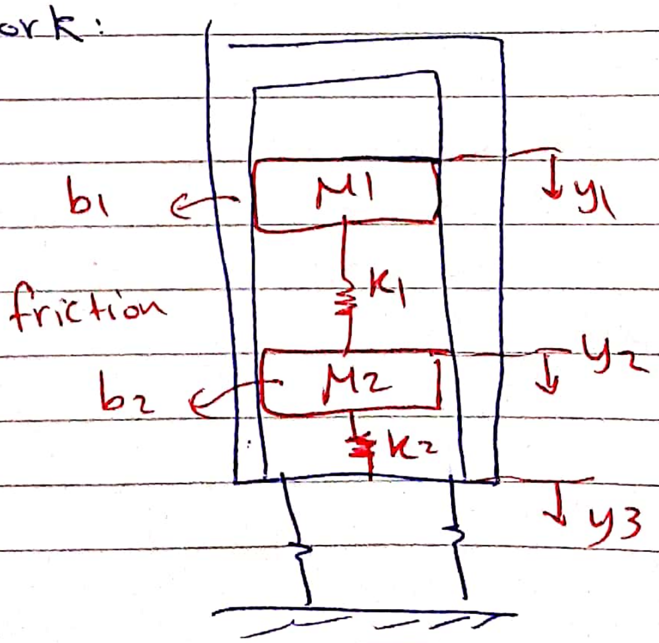
EX 6 :-



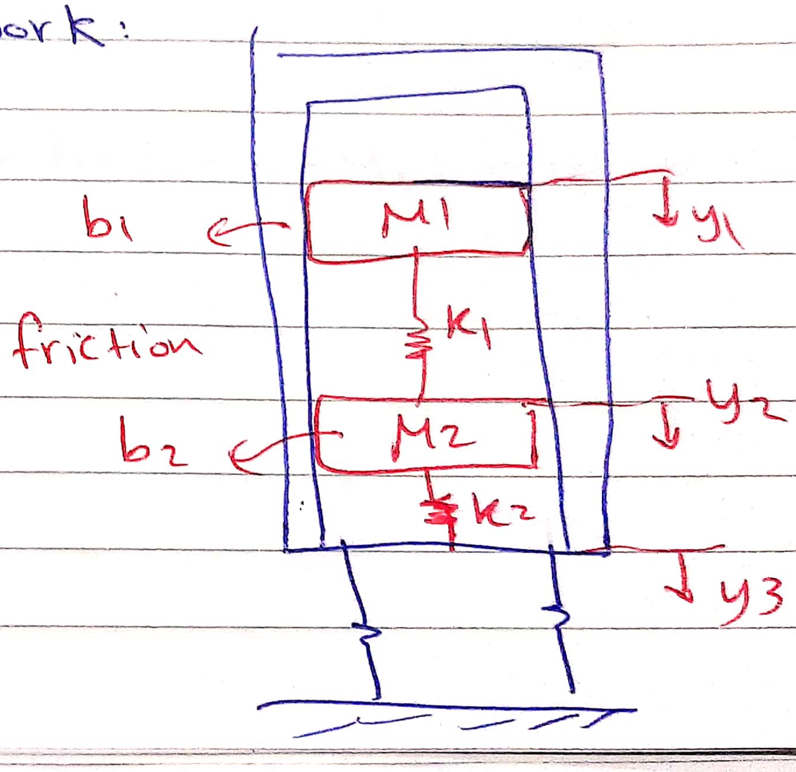
$$V_{in} = L\frac{dI_1(t)}{dt} + \frac{1}{C}\int (I_1 - I_2)\, dt \quad \ldots\ (1)$$

$$0 = \frac{1}{C}\int (I_2 - I_1)\, dt + L\frac{dI_2}{dt} + \frac{1}{C}\int I_2(t)\, dt \ldots\ (2)$$

EX: 7: Home work:



Ex: 7:   Homework:



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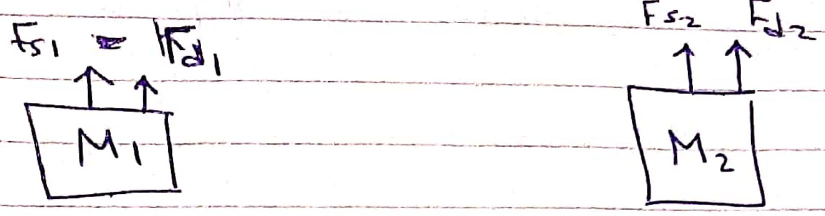
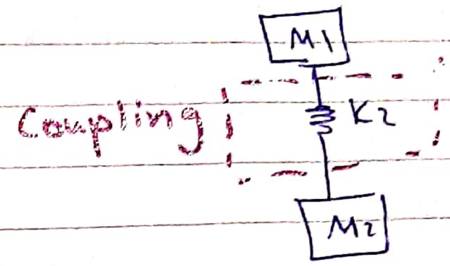
Ex 7:

Free body diagram



$$\Sigma F = M_1 \ddot{y}_1$$



$$-F_{s1} - F_{d1} = M_1 \ddot{y}_1$$

$$M_1 \ddot{y}_1 + F_{d1} + F_{s1} = 0$$

$$M_1 \ddot{y}_1 + b_1(\dot{y}_1 - \dot{y}_3) + k_1(y_1 - y_2) = 0$$

also

$$\Sigma F = M_2 \ddot{y}_2$$

$$-F_{s2} - F_{d2} = M_2 \ddot{y}_2$$

$$M_2 \ddot{y}_2 + F_{d2} + F_{s2} = 0$$

$$M_2 \ddot{y}_2 + b_2(\dot{y}_2 - \dot{y}_3) + k_2(y_2 - y_1) = 0$$

* Linear System:

The Linear system should satisfy two conditions:

① Superposition: $y(X_1 + X_2) = y(X_1) + y(X_2)$

e.g: let $x_1 = 1$ and $X_2 = 2$

$y(x) = 3x$

$y(1+2) \stackrel{?}{=} y(1) + y(2)$

$y(3) \stackrel{?}{=} y(1) + y(2)$

$3(3) \stackrel{?}{=} 3(1) + 3(2)$

$9 \stackrel{?}{=} 3 + 6$

$9 = 9$ ✓

② Scaling: $y(\alpha X_1 + \beta X_2) = y(\alpha x_1) + y(\beta x_2)$

e.g: $\alpha = 1$, $\beta = 2$

$y(1 \times 1 + 2(2)) \stackrel{?}{=} y(1 \times 1) + y(2 \times 2)$

$y(5) \stackrel{?}{=} y(1) + y(4)$

✓ $15 = 3(5) = 3(1) + 3(4) = 15$

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* Taylor series expansion:

For $y = g(x)$ where $g(x)$ non-linear

$$y = g(x_0) + \left.\frac{dg}{dx_1}\right|_{x=x_0}(x - x_0) + \frac{1}{2!}\left.\frac{d^2 g}{dx^2}\right|_{x=x_0}(x - x_0)^2$$

$$+ \cdots\cdots + \frac{1}{n!}\left.\frac{d^n g}{dx^n}\right|_{x=x_0}(x - x_0)^n$$

EX: Let $T = mgL\sin\theta$ → non-linear

use two terms, and $\theta_0 = 0$

$$T = T(\theta_0) + \left.\frac{dT}{d\theta}\right|_{\theta=\theta_0}(\theta - \theta_0)$$

$$= mgL\sin(0) + (mgL\cos(0))(\theta - 0)$$

$$= 0 + mgL(1)(\theta)$$

$$T = mgL\theta$$

* Laplace Transforms : is a mathematical tool for solving linear time invariant differential equation.

$$F(s) = \int_0^{\infty} e^{-st} f(t) \, dt \qquad f(t), t > 0$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

See Table 2-3 → important

$f(t) = A \rightarrow F(s) = \frac{A}{s}$

$f(t) = \sin\omega t \rightarrow F(s) = \frac{\omega}{s^2 + \omega^2}$

$f(t) = \cos\omega t \rightarrow F(s) = \frac{s}{s^2 + \omega^2}$

$f(t) = t^n \rightarrow F(s) = \frac{n!}{s^{n+1}}$

unit impulse ← $f(t) = \delta(t) \rightarrow F(s) = 1$

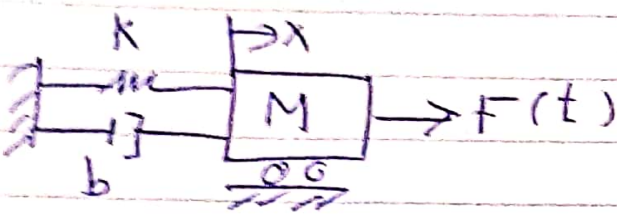
⋮ ⋮

etc etc

$$\mathcal{L}\left\{\frac{dF}{dt}\right\} = sF(s) - F(0)$$

$$\mathcal{L}\left\{\frac{d^2F}{dt^2}\right\} = s^2F(s) - sF(0) - F'(0)$$

Ex1: Find $x(t) = ?$ with $x(0) = \dot{x}(0) = 0$



$$M\ddot{x} + b\dot{x} + kx = f(t)$$

Let $\frac{k}{m} = 2$, $\frac{b}{m} = 3$, $\frac{f(t)}{m} = 1$

divide eq(1) by m:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{f(t)}{m}$$

$$\ddot{x} + 3\dot{x} + 2x = 1$$

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$$\mathcal{L}\{\ddot{x} + 3\dot{x} + 2x = 1\}$$

$$[s^2 X(s) - sX(0) - \dot{x}(0)] + 3[sX(s) - X(0)] + 2X(s) = \frac{1}{s}$$

$$s^2 X(s) + 3sX(s) + 2X(s) = \frac{1}{s}$$

$$X(s)[s^2 + 3s + 2] = \frac{1}{s}$$

$$X(s) = \frac{1}{s}\left[\frac{1}{s^2 + 3s + 2}\right]$$

take laplace inverse.

we have to make partial fraction

$$X(s) = \frac{1}{(s^2 + 3s + 2)s} = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{s+1}$$

$$\mathcal{L}^{-1}\{X(s)\} = x(t) = A + Be^{-2t} + Ce^{-t}$$

$$\frac{1}{s(s^2+3s+2)} = \frac{A(s+1)(s+2)+B(s)(s+1)+Cs(s+2)}{s(s^2+3s+2)}$$

$$1 = A(s+1)(s+2) + Bs(s+1) + Cs(s+2)$$

$s=0 \Rightarrow 1 = 2A \Rightarrow \boxed{A = \frac{1}{2}}$

$s=-1 \Rightarrow -C(1) = 1 \quad \boxed{C = -1}$

$s=-2 \Rightarrow 1 = -2B(-1) \quad \boxed{B = \frac{1}{2}}$

$$\Rightarrow \boxed{x(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}}$$

EX 2: $Y(s) = \dfrac{2}{(s+1)(s+2)^2}$, Find $y(t)$ = ?

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$Y(t) = A e^{-t} + B e^{-2t} + C t e^{-2t}$$

$$2 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$$

at $s = -1 \Rightarrow$ $\boxed{A = 2}$

at $s = -2 \Rightarrow$ $\boxed{C = -2}$

at $s = 0 \Rightarrow 4A + 2B + C = 2 \Rightarrow 4(2) + 2B - 2 = 2$

$$\boxed{B = -2}$$

$$\boxed{y(t) = 2e^{-t} - 2e^{-2t} - 2te^{-2t}}$$

Ex 3: $y(s) = \dfrac{3}{s^2 + 2s + 5}$

$s^2 + 2s + 5 = 0 \Rightarrow s_{1,2} = -1 \mp j2$

Complex roots

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$$\underbrace{s^2+2s+1}+4$$

إكمال مربع

$$s^2+2s+5 = (s+1)^2+4$$

⇒ note : <u>Table 2.3</u>

$$f(t) = e^{-\alpha t}\sin\omega t \rightarrow F(s) = \frac{\omega}{(s+\alpha)^2+\omega^2}$$

$$f(t) = e^{-\alpha t}\cos\omega t \rightarrow F(s) = \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$$

$$\therefore \left.\begin{array}{l}(s+\alpha)^2+\omega^2\\(s+1)^2+4\end{array}\right\} \Rightarrow \begin{array}{l}\omega=2\\\alpha=1\end{array}$$

$$Y(s) = \frac{3}{(s+1)^2+4} \Rightarrow Y(s) = 3\left(\frac{\frac{2}{2}}{(s+1)^2+4}\right)$$

$$y(t) = \frac{3}{2}e^{-t}\sin 2t$$

* Final Value Theorm: to find the steady-state value

$$y_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} s\,Y(s)$$

Ex: Find the steady-state value of the system response.

$$Y(s) = \left(\frac{2}{s+1} - \frac{1}{s+2} \right) y_0$$

Sol:

$$Y_{ss} = \lim_{s \to 0} s\,Y(s)$$

$$Y_{ss} = \lim_{s \to 0} s \left(\frac{2}{s+1} - \frac{1}{s+2} \right) y_0$$

$$y_{ss} = 0$$ Zero

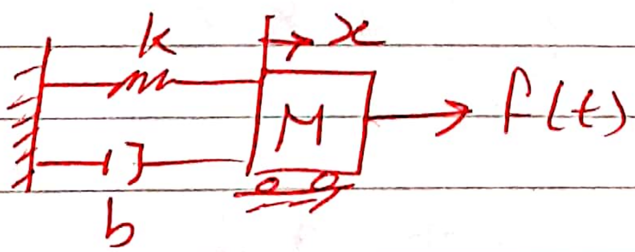
* The transfer function of linear system:
the ratio of the laplace transform of the output variable to the laplace transform of the input variable with the initial conditions assumed to be zero.

$Y(s) \rightarrow$ output

$R(s) \rightarrow$ input

$$T(s) = \frac{Y(s)}{R(s)}$$

Ex1: Find the transfer function of the system below.



$X(s)$ = output

$F(s)$ = input

$$T(s) = \frac{X(s)}{F(s)}$$

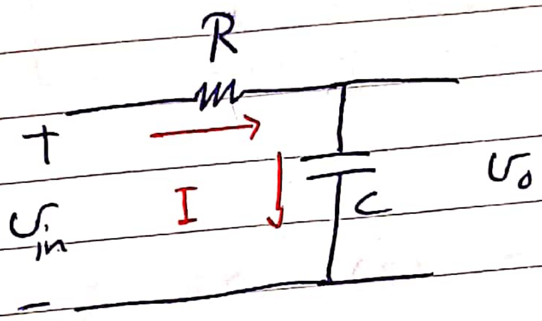
$$M\ddot{x} + b\dot{x} + kx = F(t)$$

$$M(s^2 X(s)) + bsX(s) + kX(s) = F(s)$$

$$X(s)\,[Ms^2 + bs + k] = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + bs + k}$$

EX2 :



Find $T(s) = ?$

V_{in} = input

V_o = output

$$\frac{V_o(s)}{V_{in}(s)} = ?$$

$$V_{in} = IR + \frac{1}{C}\int I\,dt$$

$$\dot{V}_{in} = \dot{I}R + \frac{I}{C}$$

$$sV_{in}(s) = RsI(s) + \frac{I(s)}{C}$$

$$s\,V_{in}(s) = I(s)\left[Rs + \frac{1}{c}\right] \quad \text{--- (1)}$$

but $$V_o = \frac{1}{c}\int I\,dt$$

$$\dot{V}_o = \frac{I}{c}$$

$$s\,V_o(s) = \frac{I(s)}{c}$$

$$I(s) = c\,s\,V_o(s) \quad \text{--- (2)}$$

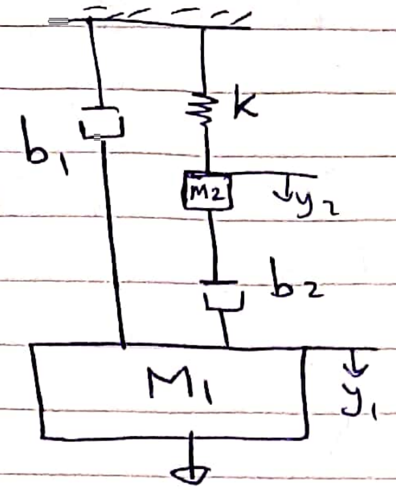
$$\not{s}\,V_{in}(s) = c\not{s}\,V_o(s)\left[Rs + \frac{1}{c}\right]$$

$$\frac{V_o(S)}{V_{in}(S)} = \frac{1}{c\left[Rs + \frac{1}{c}\right]}$$

$$\frac{V_o(S)}{V_{in}(S)} = \frac{1}{RCs + 1}$$

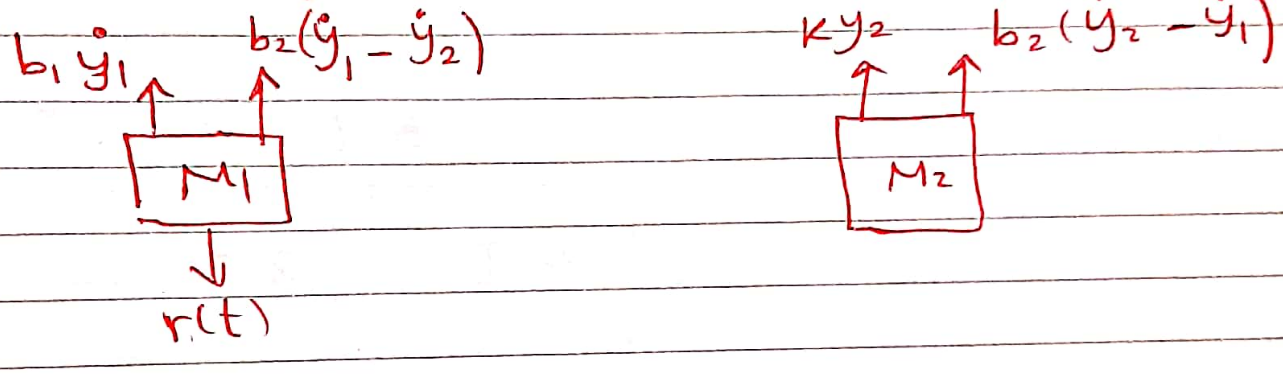
Ex 3:

Find $\dfrac{Y_2(s)}{R(s)}$, $\dfrac{Y_1(s)}{R(s)}$



$r(t)$ = input force

Free body diagram



$$M_1\ddot{y}_1 + b_1\dot{y}_1 + b_2(\dot{y}_1 - \dot{y}_2) = r(t)$$

$$M_2\ddot{y}_2 + b_2(\dot{y}_2 - \dot{y}_1) + k y_2 = 0$$

→ take the laplace for two equations

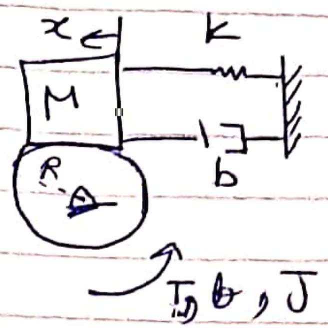
$$\Rightarrow \quad Y_2(s) = \frac{b_2 s}{M_2 s^2 + b_2 s + k_2} \, Y_1(s)$$

⋮

homework.

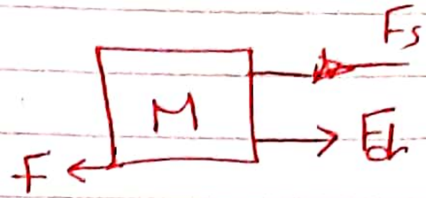
EX 4:- Rack and pinion

Find $\frac{X(s)}{T_{in}(s)} = ??$



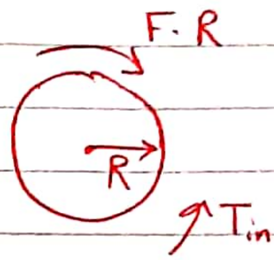
$+\leftarrow \sum F = m\ddot{x}$

$$M\ddot{x} + b\dot{x} + kx = F \quad \text{--- (1)}$$



$\sum T = J\ddot{\theta}$

$$T_{in} - F.R = J\ddot{\theta} \quad \text{--- (2)}$$

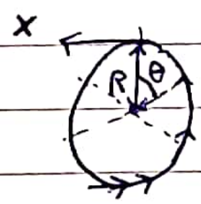


also,

$x = R\theta$

$\dot{x} = R\dot{\theta}$

$\ddot{x} = R\ddot{\theta}$



$\ddot{\theta} = \frac{\ddot{x}}{R}$

$$\Rightarrow M\ddot{x} + b\dot{x} + kx = \frac{T_{in}}{R} - \frac{J\ddot{\theta}}{R}$$

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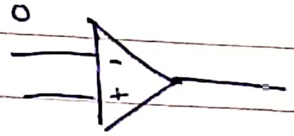
$$\frac{X(s)}{T_{in}(s)} = \frac{1}{\left(\frac{J}{R} + RM\right)s^2 + Rb\,s + RK}$$

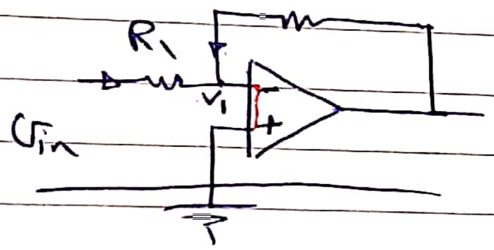
* Operational Amplifier

$\sum I_- = 0$

$\sum I_+ = 0$



Ex1:



$\sum I_- = 0$

$$\frac{v_1 - v_{in}}{R_1} + \frac{-v_o + v_1}{R_2} = 0$$

but $v_1 = 0$

$$\Rightarrow \frac{-v_{in}(s)}{R_1} = +\frac{v_o(s)}{R_2}$$

$$\frac{v_o(s)}{v_{in}(s)} = -\frac{R_2}{R_1}$$

note: $Z_R = R$ and $Z_L = Ls$

$Z_C = \frac{1}{Cs}$

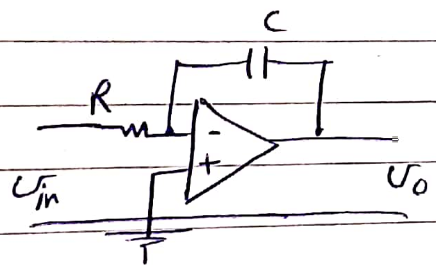
For inverting amplifiers

$$\frac{V_o}{V_{in}} = -\frac{Z_f}{Z_{in}}$$

→ from example 1:

$$\frac{V_o(S)}{V_{in}(S)} = -\frac{R_2}{R_1}$$ (propotional controller)

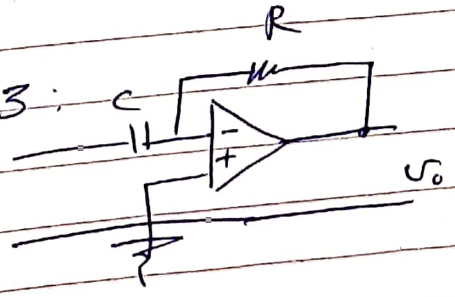
Ex2:



$$\frac{V_o(S)}{V_{in}(S)} = \frac{-Z_f}{Z_{in}} = -\frac{Z_c}{Z_R} = -\frac{\frac{1}{CS}}{R} = -\frac{1}{RCS}$$

(Integral controller)

Ex3:

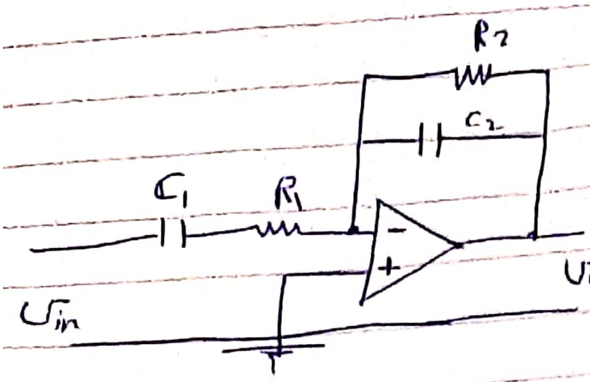


$$\frac{V_o(S)}{V_{in}(S)} = -\frac{Z_R}{Z_c}$$

$$= -RCS$$

(Derivative Controller)

EX 4: Find $\frac{V_o(s)}{V_{in}(s)} = ??$



$$\frac{V_o(s)}{V_{in}(s)} = -\frac{Z_f}{Z_{in}}$$

$$Z_f = Z_{C_2} \parallel Z_{R_2}$$

$$= \frac{R_2\left(\frac{1}{C_2 s}\right)}{R_2 + \frac{1}{C_2 s}} = \frac{R_2(1)}{R_2 C_2 s + 1}$$

$Z_{in} = Z_{C_1}$ series with Z_{R_1}

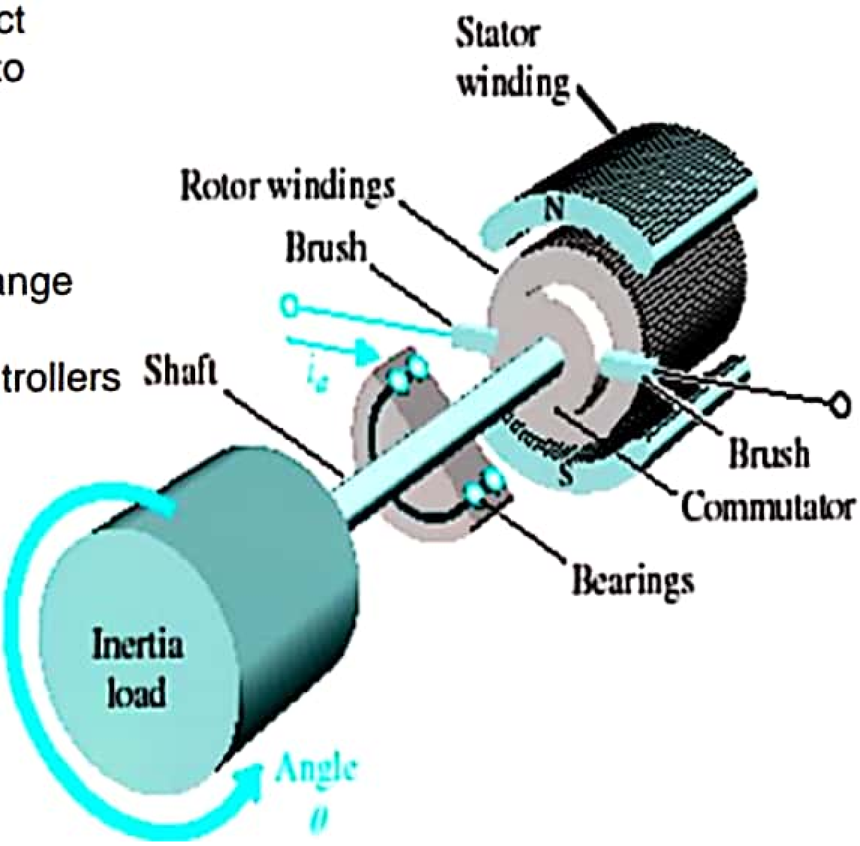
$$= R_1 + \frac{1}{C_1 s} = \frac{C_1 R_1 s + 1}{C_1 s}$$

$$\frac{V_o(s)}{V_{in}(s)} = -\frac{\frac{R_2}{R_2 C_2 s + 1}}{\frac{C_1 R_1 s + 1}{C_1 s}} = -\frac{R_2 C_1 s}{(R_2 C_2 s + 1)(R_1 C_1 s + 1)}$$

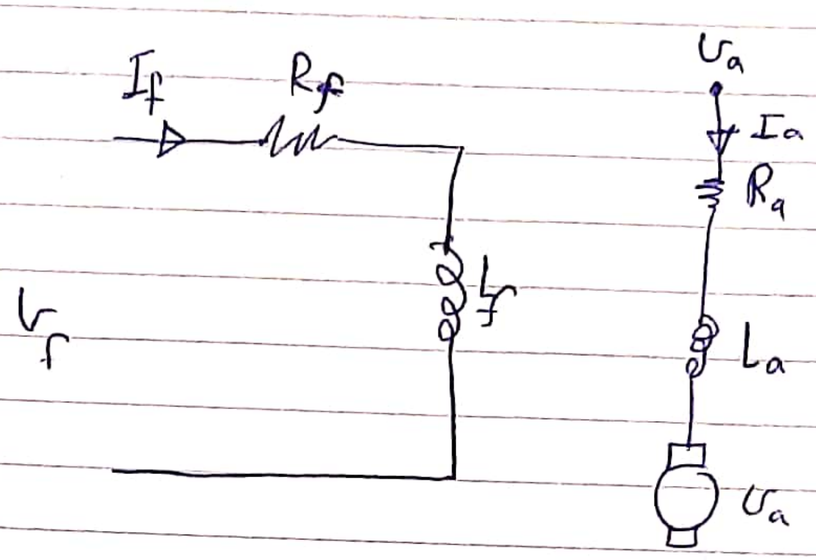
Examples: TF's of DC motors

- A DC motor is used to move loads and is called an **actuator**.
- The DC motor converts direct current (DC) electrical energy into rotational mechanical energy
- DC motors features:
  - High output torque
  - Speed controllability over a wide range
  - Portability
  - Adaptability to various types of controllers

DC motors are widely used in numerous control applications such as robotic manipulators, tape transport mechanisms, disk drives, and machine tools



* DC- motor :



Field (stator) Armature (Rotor)

→ $$T_m = K_1 \phi I_a I_f$$

T_m = motor torque , ϕ = air gap flux

I_a = armature current , I_f = field current

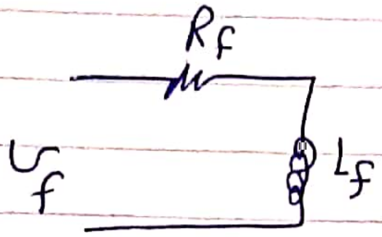
$$T_m = J\ddot{\theta} + b\dot{\theta}$$

① Field - Controlled DC - motor: $\left(\frac{\theta(s)}{V_f(s)} \right)$

$$T_m = J\ddot{\theta} + b\dot{\theta} = k_m I_f$$

$$J s^2 \theta(s) + b s \theta(s) = k_m I_f(s)$$

$$\frac{\theta(s)}{I_f(s)} = \frac{k_m}{Js^2 + bs}$$



$$v_f = R_f I_f + L_f \frac{dI_f}{dt}$$

$$v_f(s) = R_f I_f(s) + L_f s I_f(s)$$

$$v_f(s) = I_f(s)[R_f + L_f s]$$

$$I_f(s) = \frac{v_f(s)}{R_f + L_f s}$$

$$\frac{\theta(s)}{v_f(s)} = \frac{k_m}{(Js^2 + bs)(R_f + L_f s)}$$

$$\frac{\omega(s)}{I_f(s)} = ?$$

Let $\dot{\theta} = \omega \Rightarrow \dot{\omega} = \ddot{\theta}$

$$J\dot{\omega} + b\omega = k_m I_f$$

$$J s\,\omega(s) + b\,\omega(s) = K_m I_f(s)$$

$$\frac{\omega(s)}{I_f(s)} = \frac{K_m}{Js + b}$$

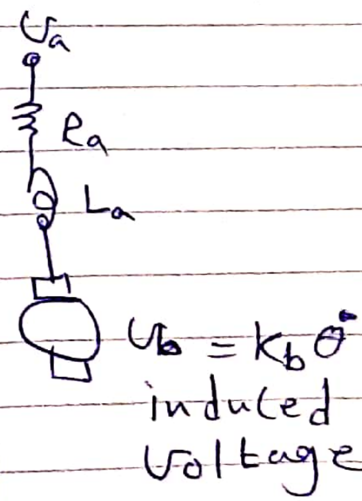
② Armature – controlled DC-motor $\left(\frac{\theta(s)}{V_a(s)} \right)$

$$T_m = J\ddot{\theta} + b\dot{\theta} = K_m I_a$$

$$(Js^2 + bs)\,\theta(s) = K_m I_a(s)$$

$$\frac{\theta(s)}{I_a(s)} = \frac{K_m}{Js^2 + bs}$$

$$V_a = R_a I_a + L_a \frac{dI_a}{dt} + V_b$$



$$V_a - V_b = R_a I_a(s) + L_a s\, I_a(s)$$

$$V_a(s) - V_b(s) = I_a(s)\left[R_a + L_a s\right]$$

$$I_a(s) = \frac{V_a(s) - V_b(s)}{(R_a + L_a s)}$$

~~$\theta(s)$~~

$$(J s^2 + b s)\,\theta(s) = K_m \left(\frac{V_a - V_b}{R_a + L_a s} \right)$$

$$(J s^2 + b s)\,\theta(s) + \frac{K_m K_b\, s\, \theta(s)}{R_a + L_a s} = \frac{K_m V_a(s)}{R_a + L_a s}$$

$$\theta(s)\left[[J s^2 + b s] + \frac{K_m K_b s}{R_a + L_a s} \right] = \frac{K_m V_a}{R_a + L_a s}$$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{(J s^2 + b s)(R_a + L_a s) + K_m K_b s}$$

Find $\frac{\omega(s)}{I_a(s)} = ?$

$$\frac{\omega(s)}{V_a(s)} = ?$$

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Block Diagram (BD) Models

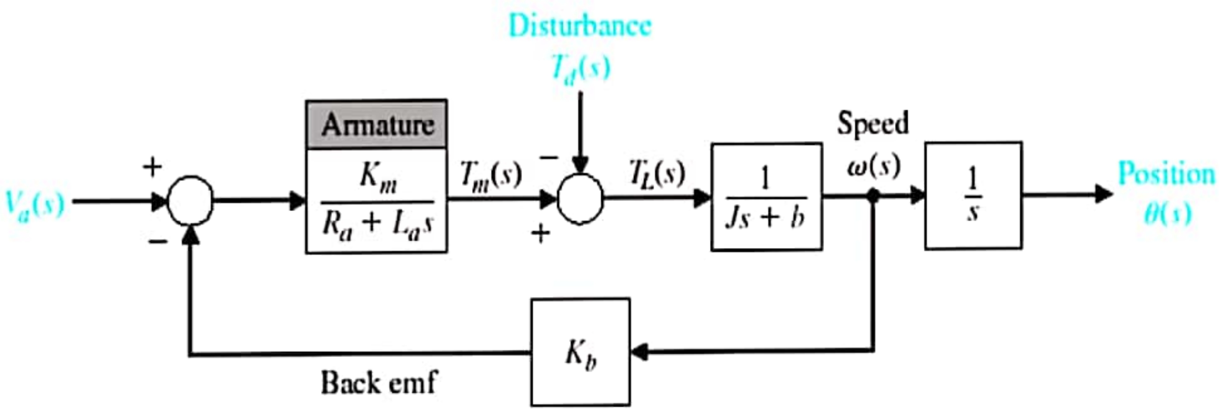
Again:

- Control systems consists of elements that are represented mathematically by a set of simultaneous differential equations
- Laplace transformation reduces the problem of differential equations to the solution of a set of linear algebraic equations.
- Since control systems are concerned with the control of specific variables, the controlled variables must relate to the controlling variables
  - This relationship is typically represented by the TF of the subsystem relating the input and output variables
  - The importance of the TF is evidenced by the ability to represent the relationship of system variables by diagrammatic means called BD

Hence, the control system with all its elements can be represented by one BD showing all variables relations

Armature controlled DC motor BD

21-22/38

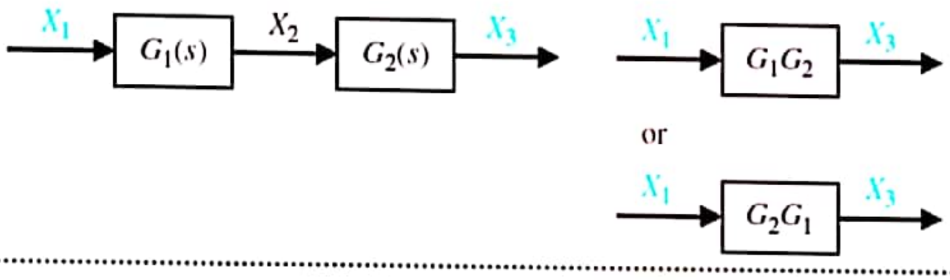


In order to find the cause-effect relationship of a system BD, we simplify the BD (reduction) by applying the rules of BD algebra.

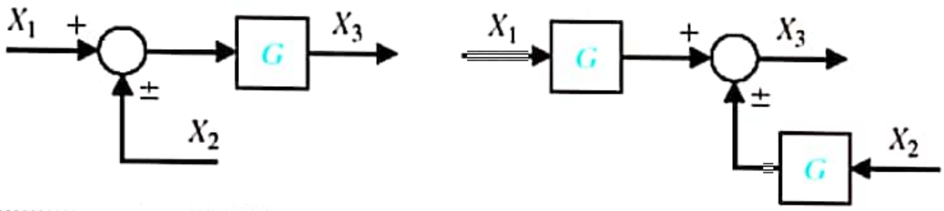
Block Diagram (BD) Algebra

Original Diagram | **Equivalent Diagram**

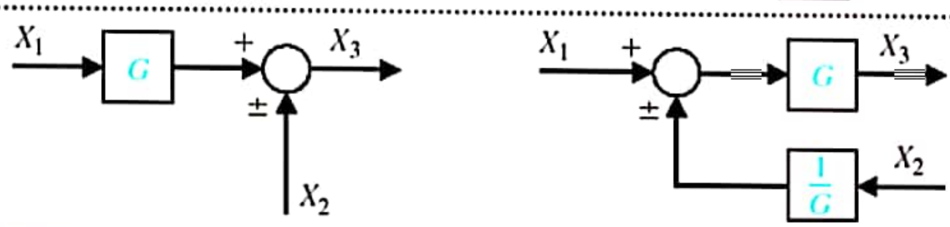
(1)



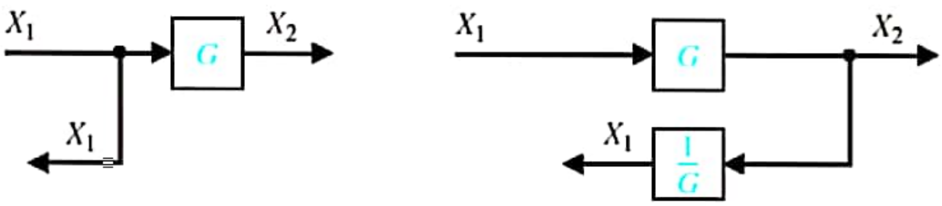
(2)



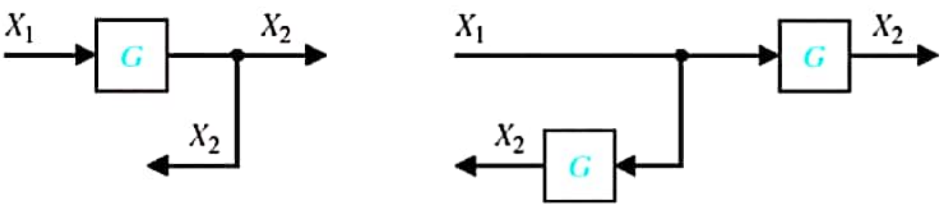
(3)



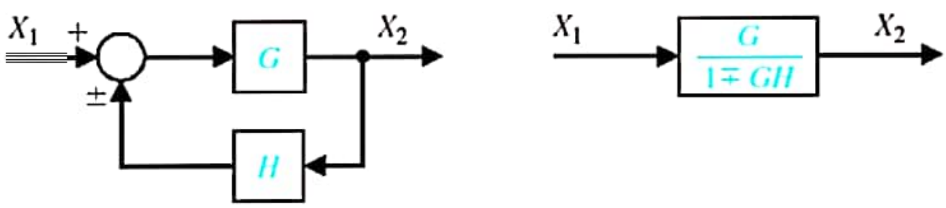
(4)



(5)

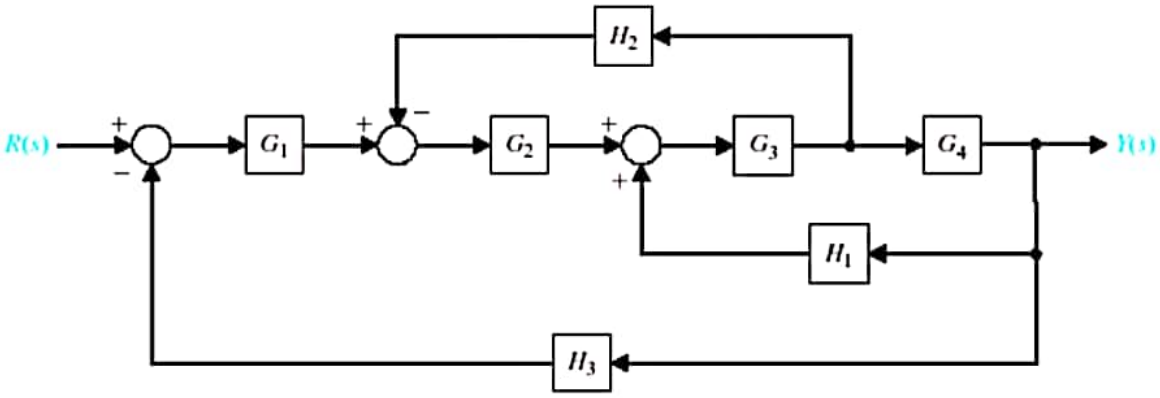


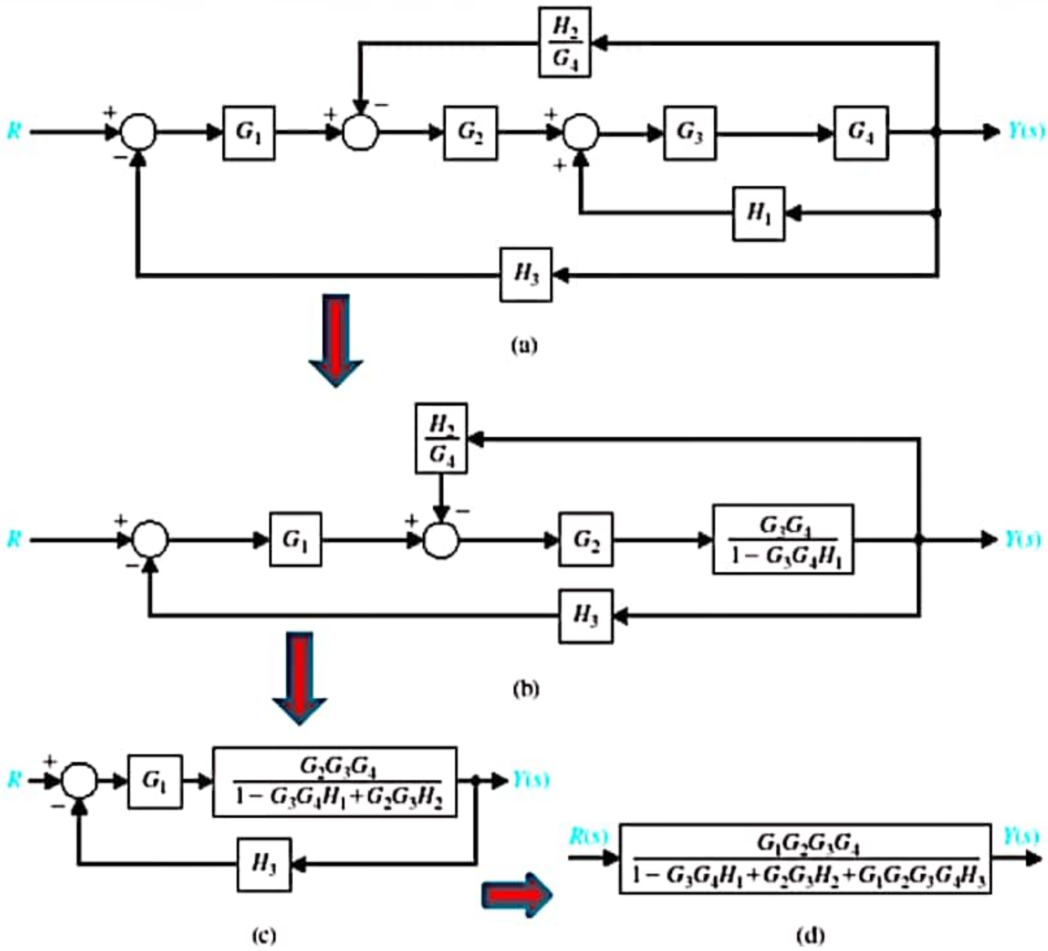
(6)



Example

For the following control system, find the input-output relationship (i.e. TF) relation the output variable Y(s) to the input variable R(s).





$$\frac{G_1G_2G_3G_4}{1-G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4H_3}$$

Signal-Flow (SF) Graph Models

- Block diagrams are adequate for the representation of the system interrelationships. However, for a system with reasonably complex interrelationships, the block diagram reduction procedure is often quite difficult to complete.

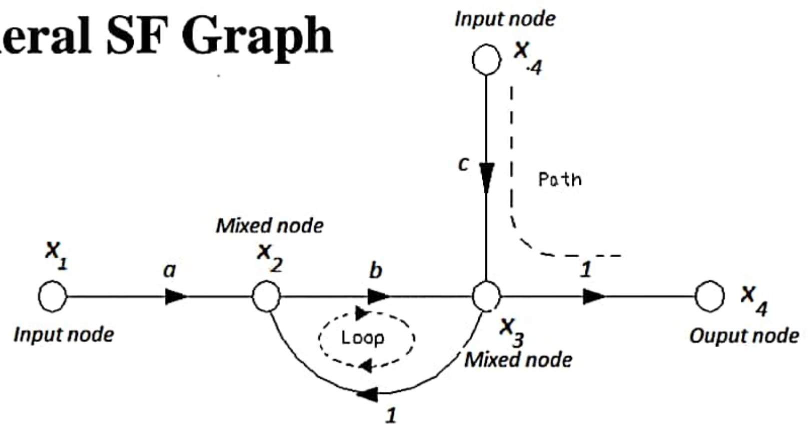
  → An alternative method for determining the relationship between system variables has been developed by Mason which is called the signal-flow graph method

- A signal-flow graph is a diagram consisting of nodes that are connected by several directed branches and is a graphical representation of a set of linear relations.

- The reduction procedure (used in the BD method) is not necessary to determine TF (input-output relationship) of a system represented by SF graph.

  → **We apply Mason`s Gain Formula to find the TF**

General SF Graph



Node: acts like a summing point and also represents a system variable

Transmittance: real or complex gain between two nodes.

Branch: directed line segment joining two nodes.

Input node (source): only outgoing branches.

Output node (sink): only incoming branches.

Mixed node: both incoming and outgoing branches

Path: traversal of connected branches in the direction of arrows.

Loop: closed path.

Loop gain: product of branch transmittance at a loop.

Loop gain: product of branch transmittance at a loop.

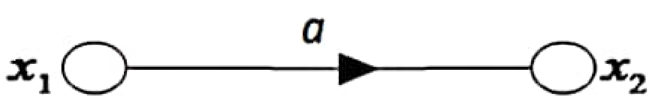
Non touching loops: they do not posses any common nodes.

Forward path: path from an input to an output node that does not cross any node more than once.

Forward path gain: product of transmittances of a forward path

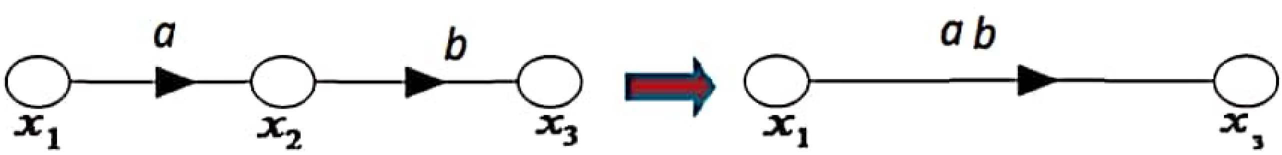
SF Graph Algebra

1)



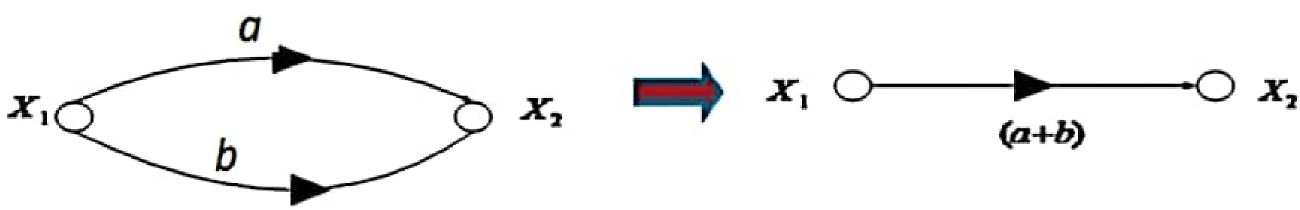
$$x_2 = a x_1$$

2)



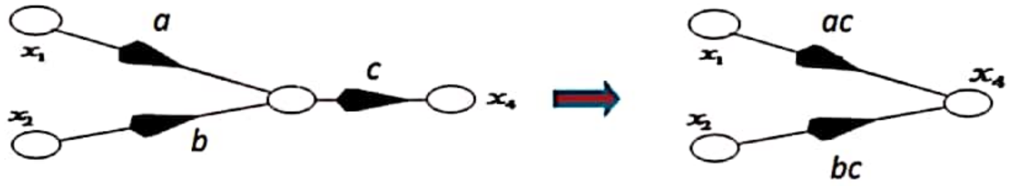
$$x_3 = b x_2 = a b x_1$$

3)

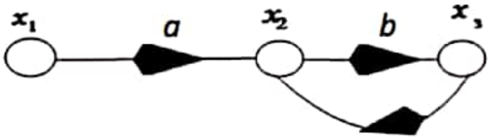


$$x_2 = a x_1 + b x_1 = (a+b) x_1$$

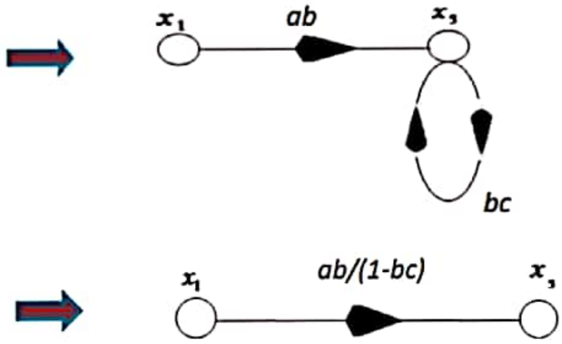
4)



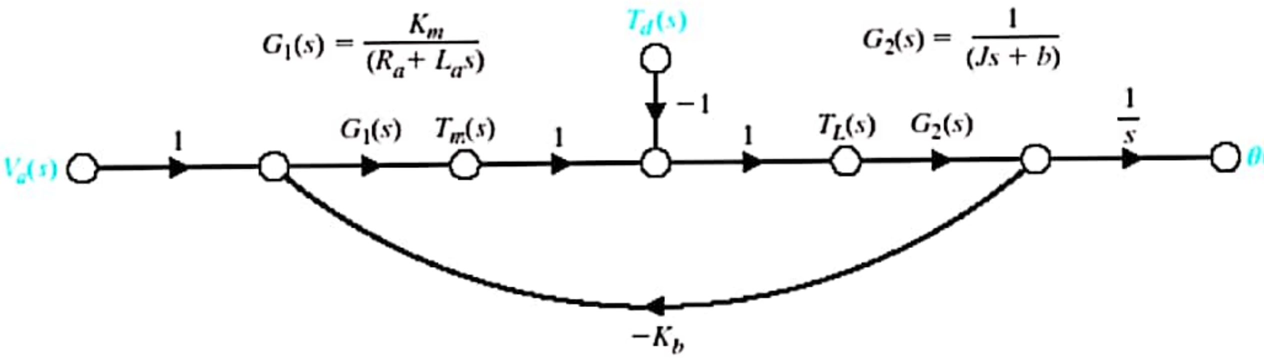
5)



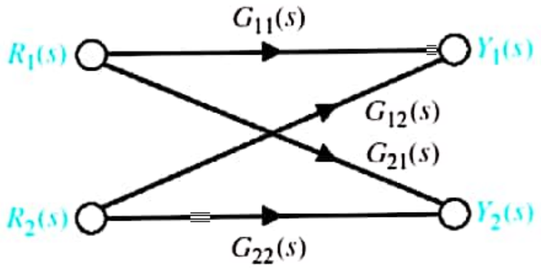
$x_3 = bx_2$
$x_2 = ax_1 + cx_3$
$x_3 = b(ax_1 + cx_3)$
$x_3 = abx_1 + bcx_3$



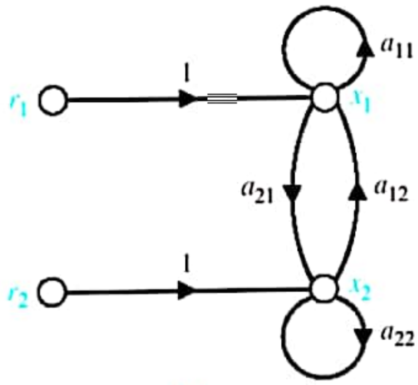
Examples



(1)



(2)



(3)

Mason`s Gain Formula

The formula is often used to relate the output variable Y(s) to the input variable R(s) (i.e. finding the TF) and is given by

$$TF = \frac{\sum_K P_K \Delta_K}{\Delta}$$

where,

P_K is the gain of path K from input node to output node in the direction of the arrows and without passing node than once.

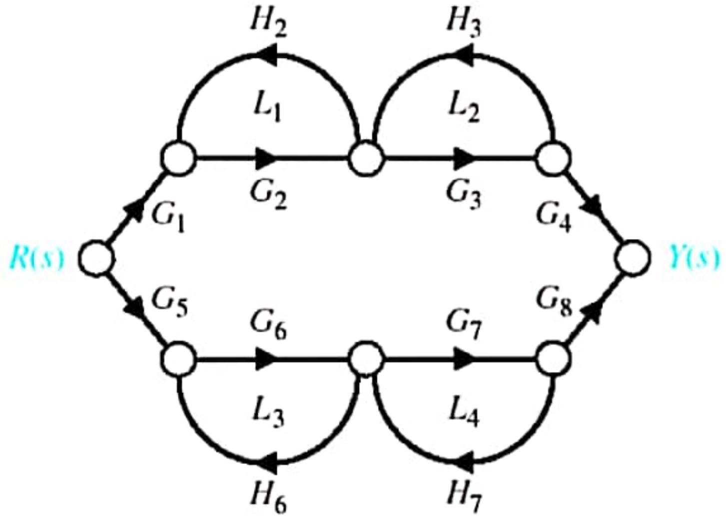
Δ_K: Cofactor o the path P_K

Δ : determinant of the graph

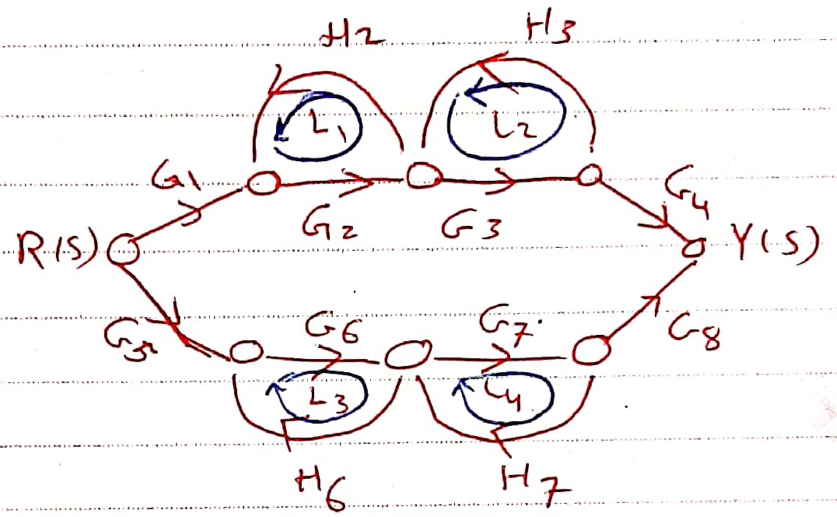
Δ = 1 – (sum of all different loop gains) + (sum of the gain products of all combinations of two non touching loops) – (sum of the gain products of all combinations of three non touching loops)

Example

For the following control system, find the input-output relationship (i.e. TF) relation the output variable Y(s) to the input variable R(s).



Example:— Find the T.F $\frac{Y(S)}{R(S)}$



⇒ (1) Forward path: P_1 and P_2

$P_1 = G_1 G_2 G_3 G_4$, $P_2 = G_5 G_6 G_7 G_8$

(2) Loops: L_1, L_2, L_3 and L_4

$L_1 = G_2 H_2$, $L_2 = G_3 H_3$, $L_3 = G_6 H_6$

$L_4 = G_7 H_7$

non-touching loops
L_1 with L_3, L_4
L_2 with L_3, L_4

(3) $\Delta_1 = 1 - L_3 - L_4$

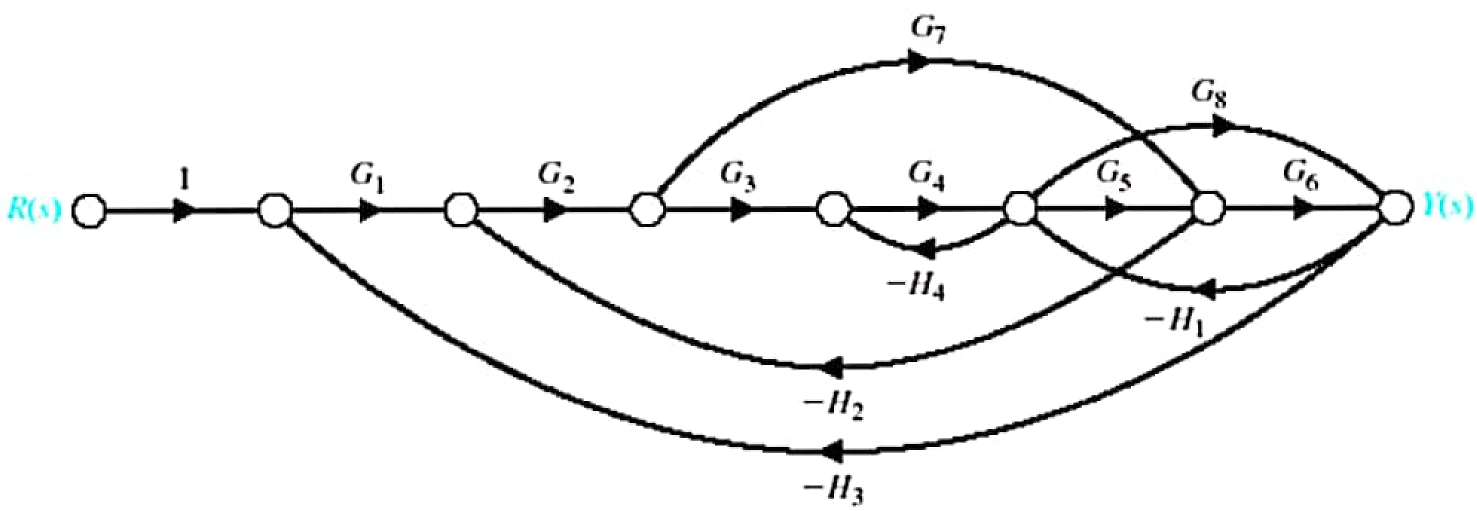
$\Delta_2 = 1 - L_1 - L_2$

(4) $\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1 L_3 + L_1 L_4 + L_2 L_3 + L_2 L_4)$

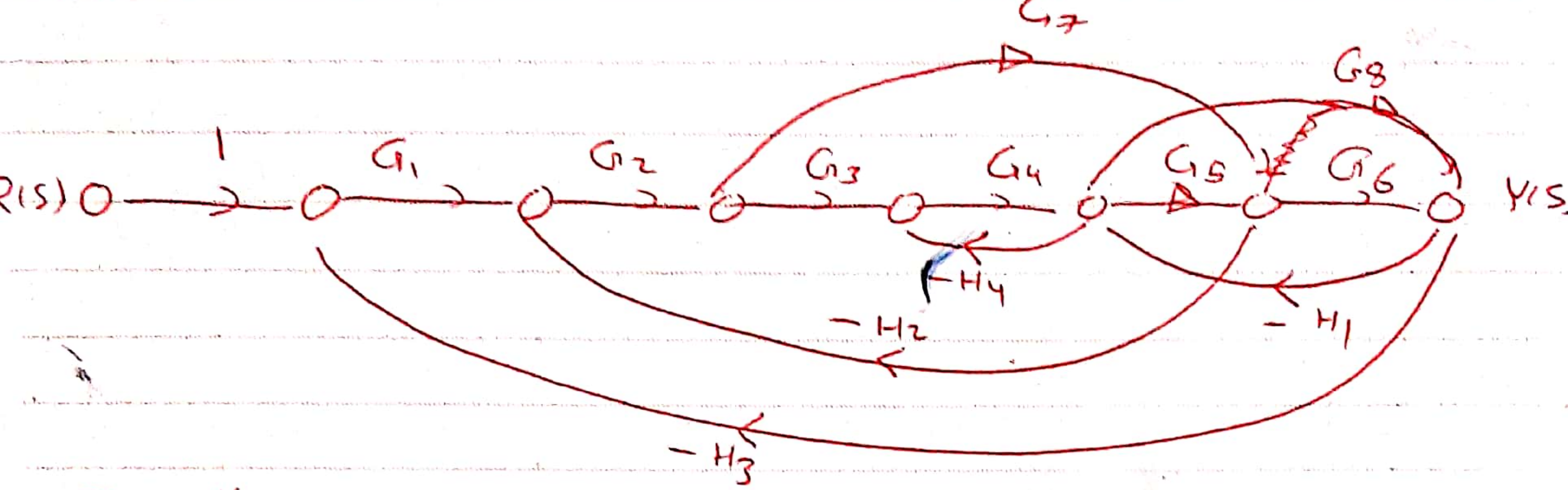
$$T.F = \frac{P_1 \Delta_1 + P_2 \Delta_1}{\Delta}$$

Example

For the following control system, find the input-output relationship (i.e. TF) relation the output variable Y(s) to the input variable R(s).



Ex:-



① path

$P_1 = G_1 G_2 G_3 G_4 G_5 G_6$

$P_2 = G_1 G_2 G_7 G_6$

$P_3 = G_1 G_2 G_3 G_4 G_8$

non-touching loops

L_5 doesn't touch L_4, L_7.

L_3 doesn't touch L_4

Loops

$L_1 = -G_2 G_3 G_4 G_5 H_2$

$L_2 = -G_5 G_6 H_1$

$L_3 = -G_8 H_1$

$L_4 = -G_2 G_7 H_2$

$L_5 = -G_4 H_4$

$L_6 = -G_1 G_2 G_3 G_4 G_5 G_6 H_3$

$L_7 = -G_1 G_2 G_7 G_6 H_3$

$L_8 = -G_1 G_2 G_3 G_4 G_8 H_3$

② $\Delta_1 = 1 - 0$

$\Delta_2 = 1 - L_5 = 1 + G_4 H_4$

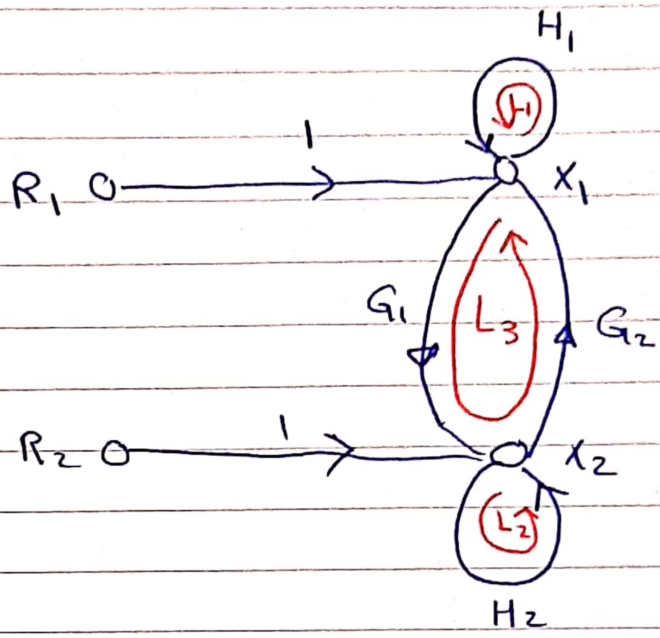
$\Delta_3 = 1 - 0$

$$T.F = \frac{Y(s)}{R(s)} = \frac{P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3}{\Delta}$$

③

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8) + (L_5 L_4 + L_5 L_7 + L_3 L_4)$$

Ex: Find $\frac{X_1(s)}{R_1(s)} = ?$



① $\frac{X_1(s)}{R_1(s)} = \frac{(1)\ (1 - L_2)}{1 - (L_1 + L_2 + L_3) + (L_1 L_2)} = \frac{P_1 \Delta_1}{\Delta}$

$= \frac{1 - H_2}{1 - (H_1 + H_2 + G_1 G_2) + (H_1 H_2)}$

② $\frac{X_1(s)}{R_2(s)} = \frac{G_2\ (1)}{\Delta}$

④ $\frac{X_2(s)}{R_2(s)} = \frac{(1)(1 - H_1)}{\Delta}$

③ $\frac{X_2(s)}{R_1(s)} = \frac{G_1\ (1)}{\Delta}$

chapter 4 :- Feedback control system characteristics

1. open loop vs closed loop system.

2. error signal (closed loop) & steady state error

3. Sensitivity of control system to parameter variation

4. Disturbance signals in a feedback control system.

→ ① An open loop signal operates without feedback and directly generates the output in response to an input signal.

② a closed loop system uses a measurement of the output signal and a comparison with the desired output to generate an error signal that is used by the controller to adjust the actuator.

⇒ Sensitivity = is the ratio of the change in the system T.F to change of a process T.F for a small incremental change

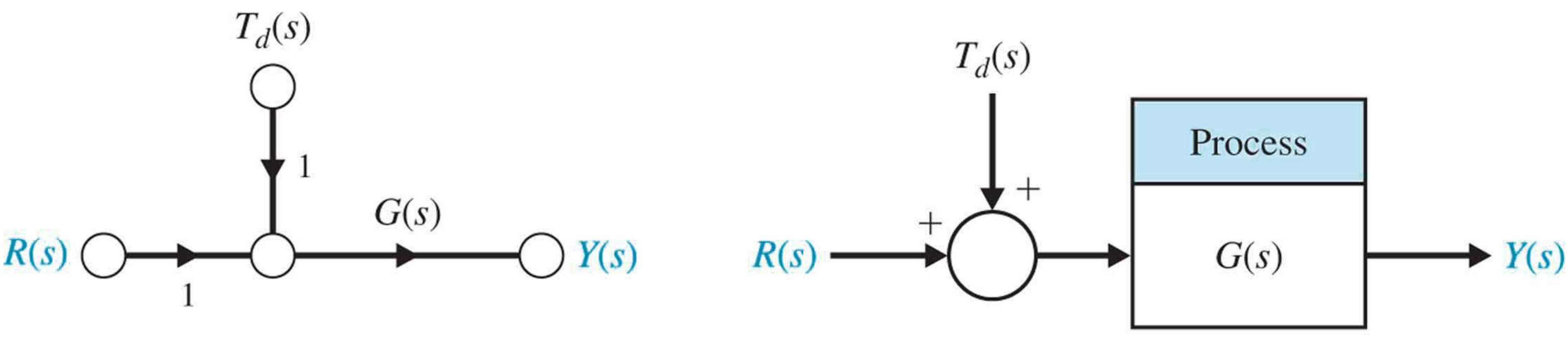
$$S_G^T = \frac{1}{1 + G_c(S)G(S)}$$

Ch.4 Feedback Control System Characteristics

- This chapter will emphasize on the advantages of the feedback closed-loop control system compared to the open-loop.
- Introducing a feedback in a control system is often necessary to improve the control system.
- Generally speaking, the areas of interest in a control system response are:
  - Minimizing the error signal.
  - Reducing the effect of system parameter uncertainties or changes (i.e. reducing the system sensitivity to parameter uncertainties/variations).
  - Reducing the effect of unwanted signals like disturbance or noise.
  - Improving the transient and steady-state performance of a system.

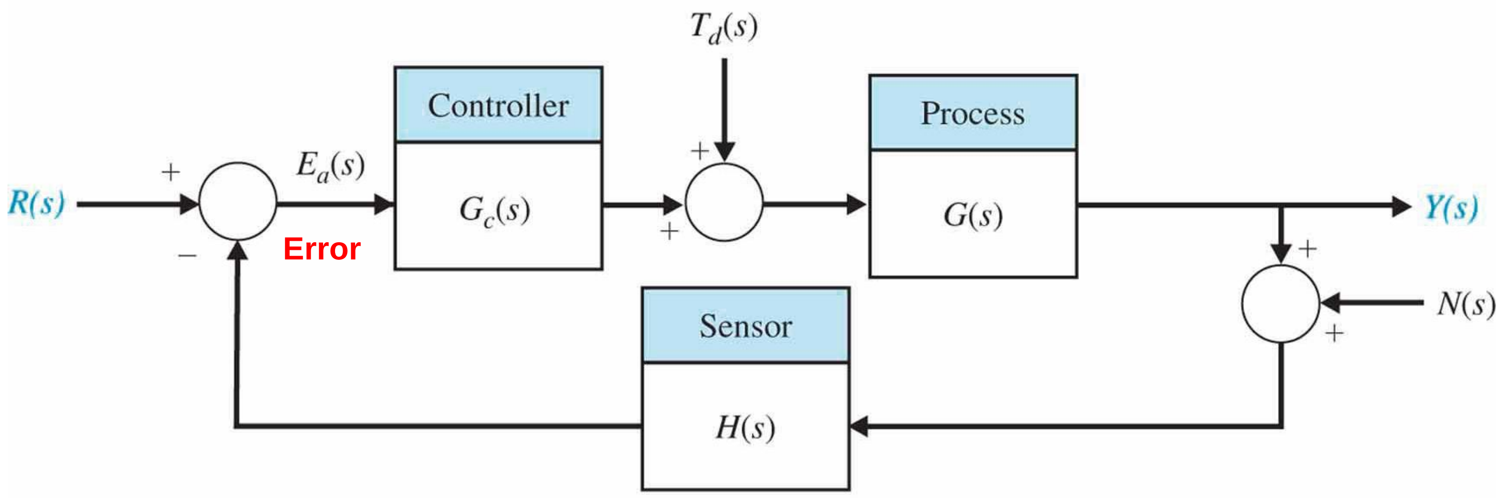
Open-loop control system

An open-loop system operates without feedback and directly generates the output in response to an input signal.



The disturbance $T_d(s)$ directly influence the output; without feedback, the control system is highly sensitive to disturbance.

Closed-loop control system



A closed-loop system uses a measurement of the output signal and a comparison with the desired output to generate an error signal [*E(s)*=desired response-actual response] that is used by the controller to adjust the actuator be generating a control signal *u(t).*

Advantages of closed-loop system:

- Reducing the error.
- Reducing the sensitivity to parameter variations.
- Improving rejection of disturbances
- Improving the transient response.

Error signal analysis:

$$E(s) = R(s) - Y(s)$$

$$But,$$

Desired Input — Disturbance input — Noise input

$$Y(s) = \frac{G_C(s)G(s)}{1+G_C(s)G(s)}R(s) + \frac{G(s)}{1+G_C(s)G(s)}T_d(s) - \frac{G_C(s)G(s)}{1+G_C(s)G(s)}N(s)$$

$$\therefore \quad E(s) = \frac{1}{1+G_C(s)G(s)}R(s) - \frac{G(s)}{1+G_C(s)G(s)}T_d(s) + \frac{G_C(s)G(s)}{1+G_C(s)G(s)}N(s)$$

Define

$$L(s) = G_C(s)G(s)$$

$$F(s) = 1 + L(s)$$

$$S(s) = \frac{1}{F(s)} = \frac{1}{1+L(s)} = \frac{1}{1+G_C(s)G(s)}$$

Where,

L(s) is loop gain

S(s) is the sensitivity function

$$\therefore \quad E(s) = S(s)R(s) - S(s)G(s)T_d(s) + C(s)N(s)$$

Where *C(s)* is the complementary sensitivity function

$$C(s) = \frac{L(s)}{1 + L(s)}$$

Note that sensitivity can be reduced by increasing the controller gain and the error can be reduced by reducing the sensitivity *S(s)* and *C(s)*, but

$$S(s) + C(s) = 1$$

i.e. S(s) and *C(s)* can not reduced simultaneously; so, design compromises must be made.

Sensitivity of Control Systems to Parameter Variations

- System sensitivity *S(s)* is the ratio of the percentage change in the system transfer function to the percentage change of a process transfer function (or parameter).

- Assume the system TF to be

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G_C(s)G(s)}{1 + G_C(s)G(s)}$$

Where *T(s)* is the system TF, $G_C(s)$ is the controller, and G(s) is the process TF.

$$\therefore S(s) = \frac{\Delta T(s) / T(s)}{\Delta G(s) / G(s)}$$

For small incremental changes, given the previous T(s), the sensitivity of the closed-loop system T(s) with respect to a small changes in the process G(s) becomes

$$S(s) = \frac{\partial T / T}{\partial G / G}$$

$$S(s) = \frac{\partial T}{\partial G} \cdot \frac{G}{T}$$

$$S_G^T = \frac{G_C}{(1 + G_C G)^2} \cdot \frac{G}{G_C G / (1 + G_C G)}$$

$$S_G^T = \frac{1}{1 + G_C(s) G(s)}$$

If we seek to determine S_α^T, where α is a parameter within G(s), using the chain rule gives

$$S_\alpha^T = S_G^T \ S_\alpha^G$$

Note:

Uncertainties in the system model might come from : Aging , changing environment, and ignorance of exact values of the system parameters which all affect the control process.

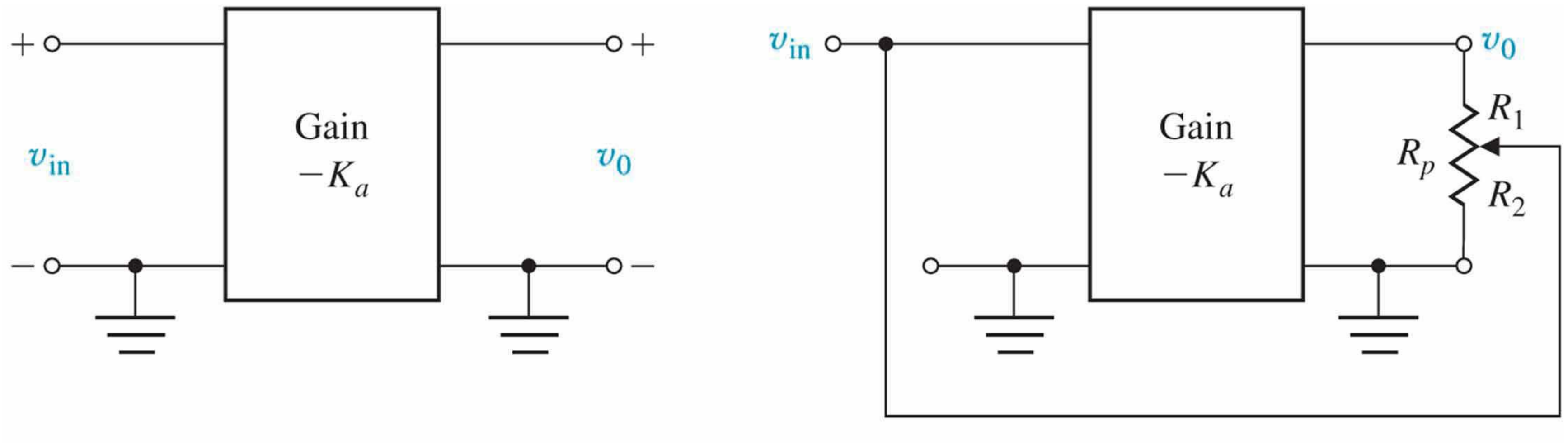
- In open-loop : inaccurate output result from these effects.

- In closed-loop: the system attempts to compensate and correct for these effects by use of the controller

Sensitivity is reduced in closed-loop by increasing $G_C(s)G(s)$ system compared to the open-loop case where *S=1* in the case of *T(s)=G(s)*

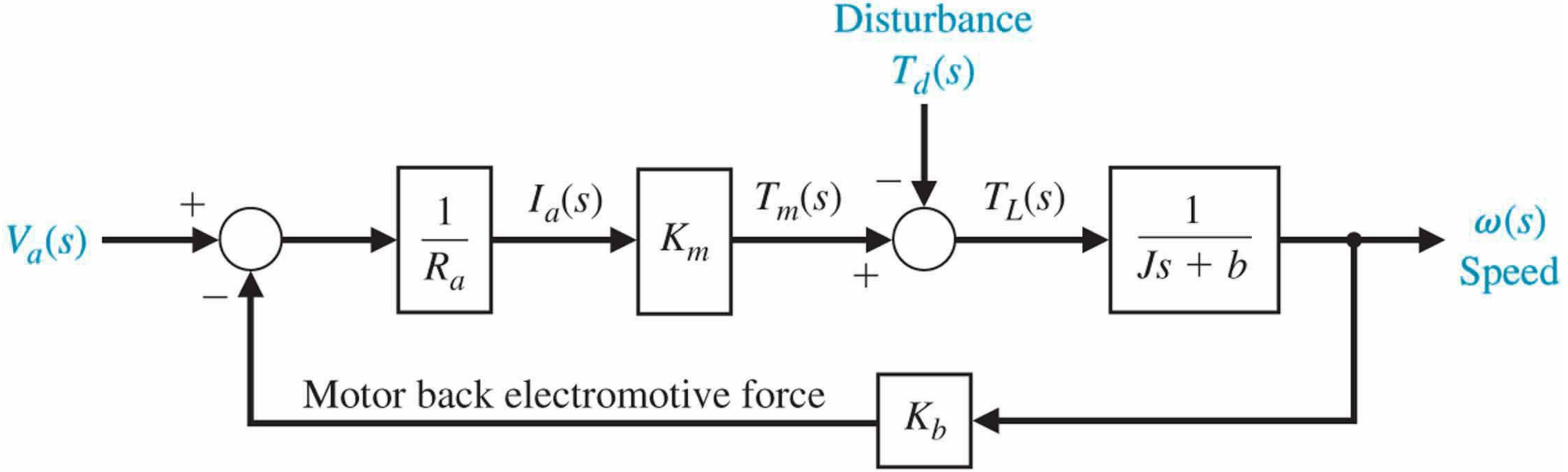
Example: Feedback Amplifier

Study the sensitivity changes for the two cases: open-loop and closed-loop.



Disturbance Signals in a Feedback Control System

Disturbance signal is unwanted undesired signal that affects the output signal



Open-loop speed control system (without tachometer feedback)

$$\frac{\omega(s)}{T_d(s)} = \frac{-1/(Js+b)}{1+\frac{K_m K_b}{R_a}\cdot\frac{1}{Js+b}}$$

Assuming very small inductance and only disturbance input

$L_a = 0 \qquad V_a = 0$

$$\therefore \quad \frac{\omega(s)}{T_d(s)} = \frac{-1}{Js + b + \dfrac{K_m K_b}{R_a}}$$

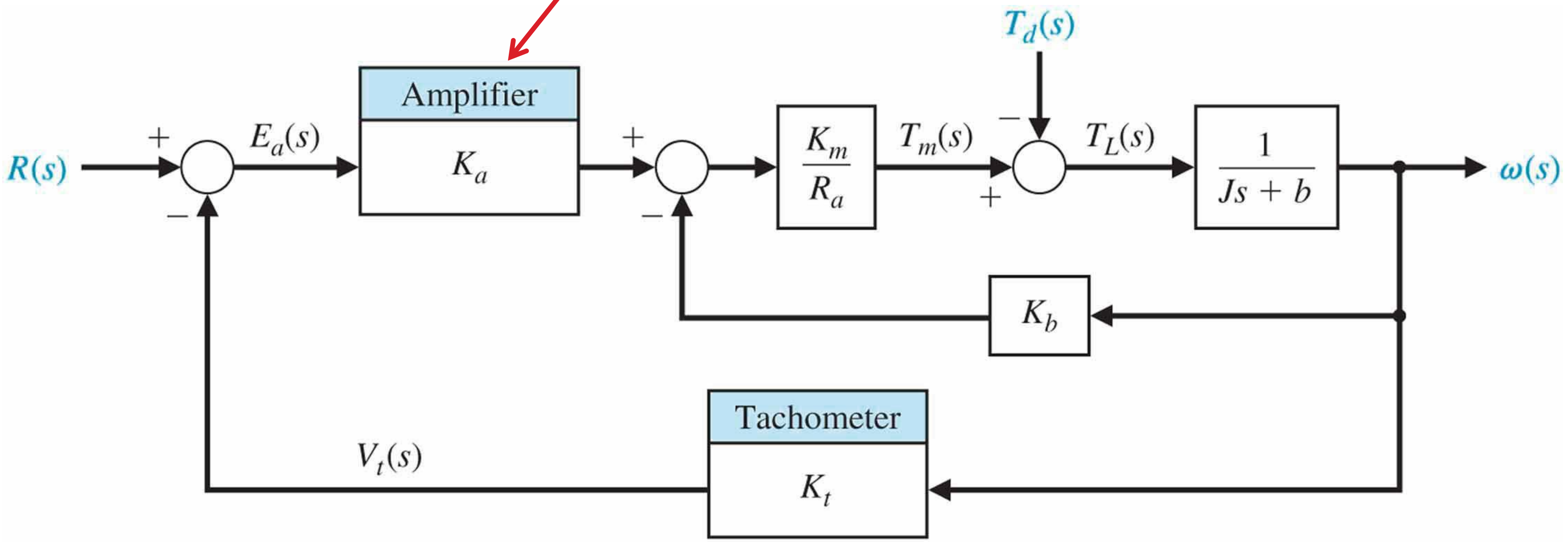
Steady-state speed due to a step disturbance $T_d(s) = \dfrac{D}{s}$ is

$$\omega_{ss} = \lim_{s \to \infty} s\,\omega(s)$$

$$= \lim_{s \to \infty} s \frac{-1}{Js + b + \dfrac{K_m K_b}{R_a}} \frac{D}{s}$$

$$(\omega_{ss})_{open} = -\frac{D}{b + \dfrac{K_m K_b}{R_a}}$$

controller



Closed-loop speed tachometer control system

$$\frac{\omega(s)}{T_d(s)} = \frac{-1/(Js+b)}{1+\frac{1}{Js+b}\frac{K_m K_a}{R_a}\left(K_t + \frac{K_b}{K_a}\right)}$$

$$= \frac{-1}{Js+b+\frac{K_m}{R_a}(K_t K_a + K_b)}$$

$$\omega_{ss} = \lim_{s \to \infty} s\omega(s)$$

$$= \lim_{s \to \infty} s \frac{-1}{Js + b + \frac{K_m}{R_a}(K_t K_a + K_b)} \frac{D}{s}$$

$$(\omega_{ss})_{closed} = -\frac{D}{b + \frac{K_m}{R_a}(K_t K_a + K_b)}$$

$$\frac{(\omega_{ss})_{closed}}{(\omega_{ss})_{open}} = \frac{R_a b + K_m K_b}{K_a K_m K_t} < 0.02$$

For high controller gain K_a

Control of the Transient Response

The system TF is

$$\frac{\omega(s)}{V_a(s)} = \frac{K_m}{R_a Js + R_a b + K_m K_b}$$

$$= G(s) = \frac{K_1}{\tau_1 s + 1}$$

where,

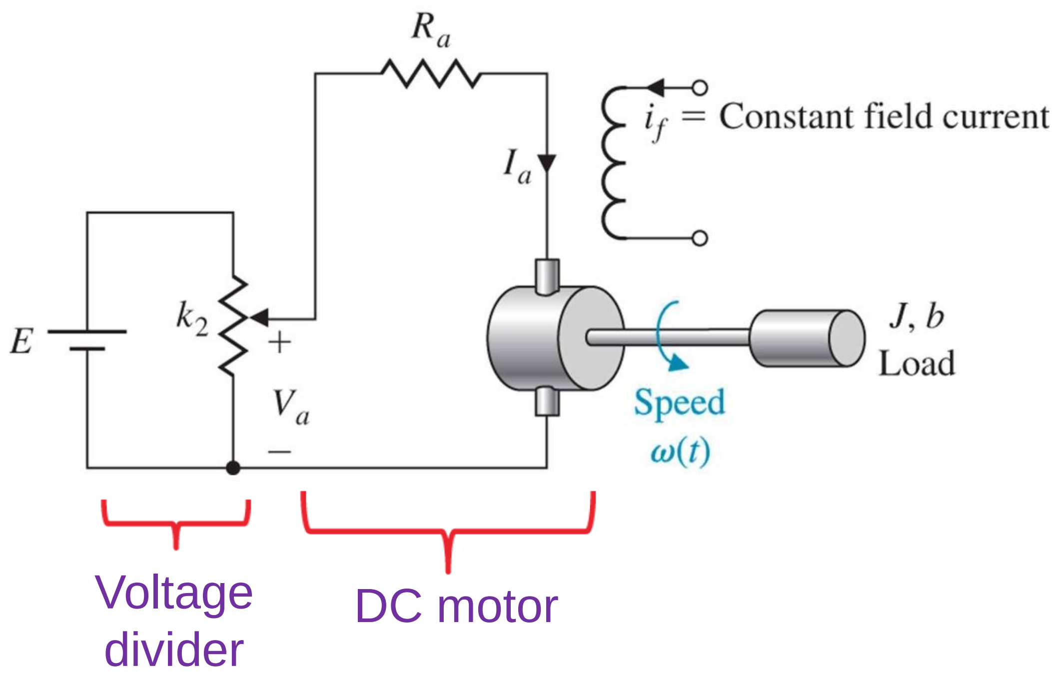
$$\tau_1 = \frac{R_a J}{R_a b + K_b K_m}$$

$$K_1 = \frac{K_m}{R_a b + K_b K_m}$$

τ_1 is known as the time constant

And the input is : $v_a(t) = k_2 E$

$$\therefore V_a(s) = \frac{k_2 E}{s}$$



{Open-Loop}

To find the response $\omega(t)$:

$$\omega(s) = G(s)V_a(s) = \frac{K_1}{\tau_1 s + 1}\frac{k_2 E}{s}$$

$$= K_1 k_2 E\left(\frac{1}{s} - \frac{1}{s + \frac{1}{\tau_1}}\right)$$

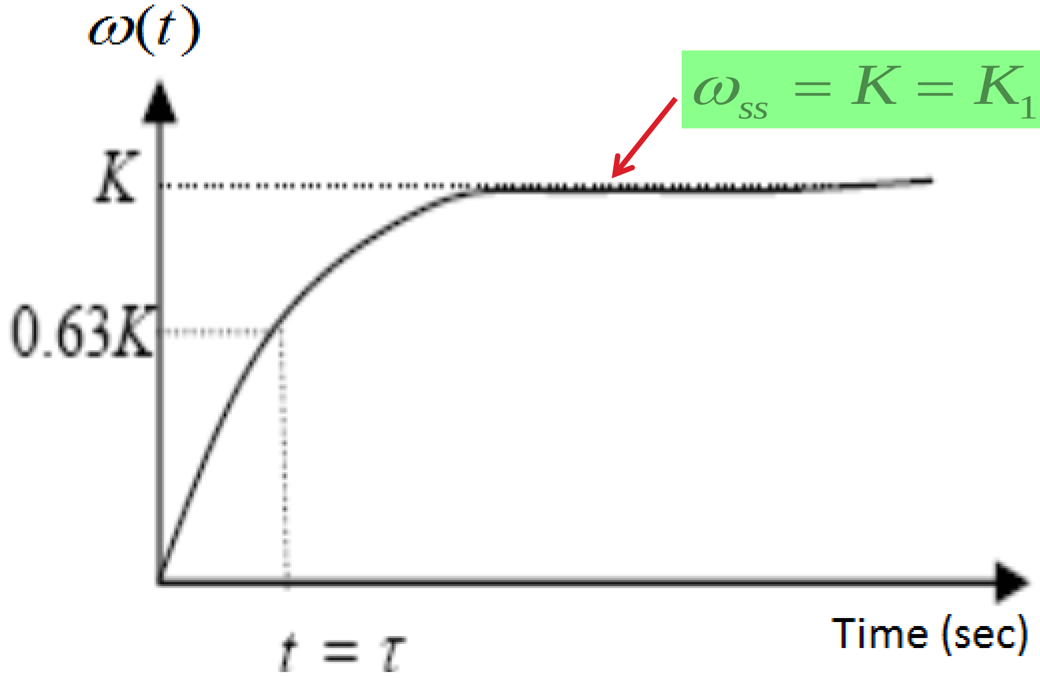
$$\omega(t) = L^{-1}\{\omega(s)\}$$

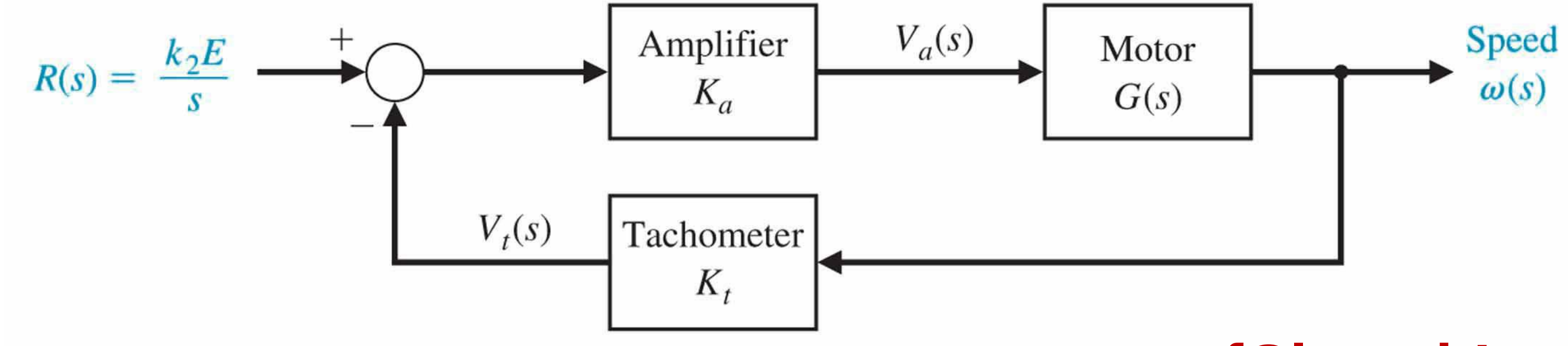
$$= K_1 k_2 (1 - e^{-t/\tau_1})$$

If the speed $\omega(t)$ is too slow, we reduce the time Constant of the motor

i.e. Choose another motor with different time constant

$\omega_{ss} = K = K_1 k_2$





{Closed-Loop}

$$\frac{\omega(s)}{R(s)} = \frac{K_a G(s)}{1 + K_a K_t G(s)}$$

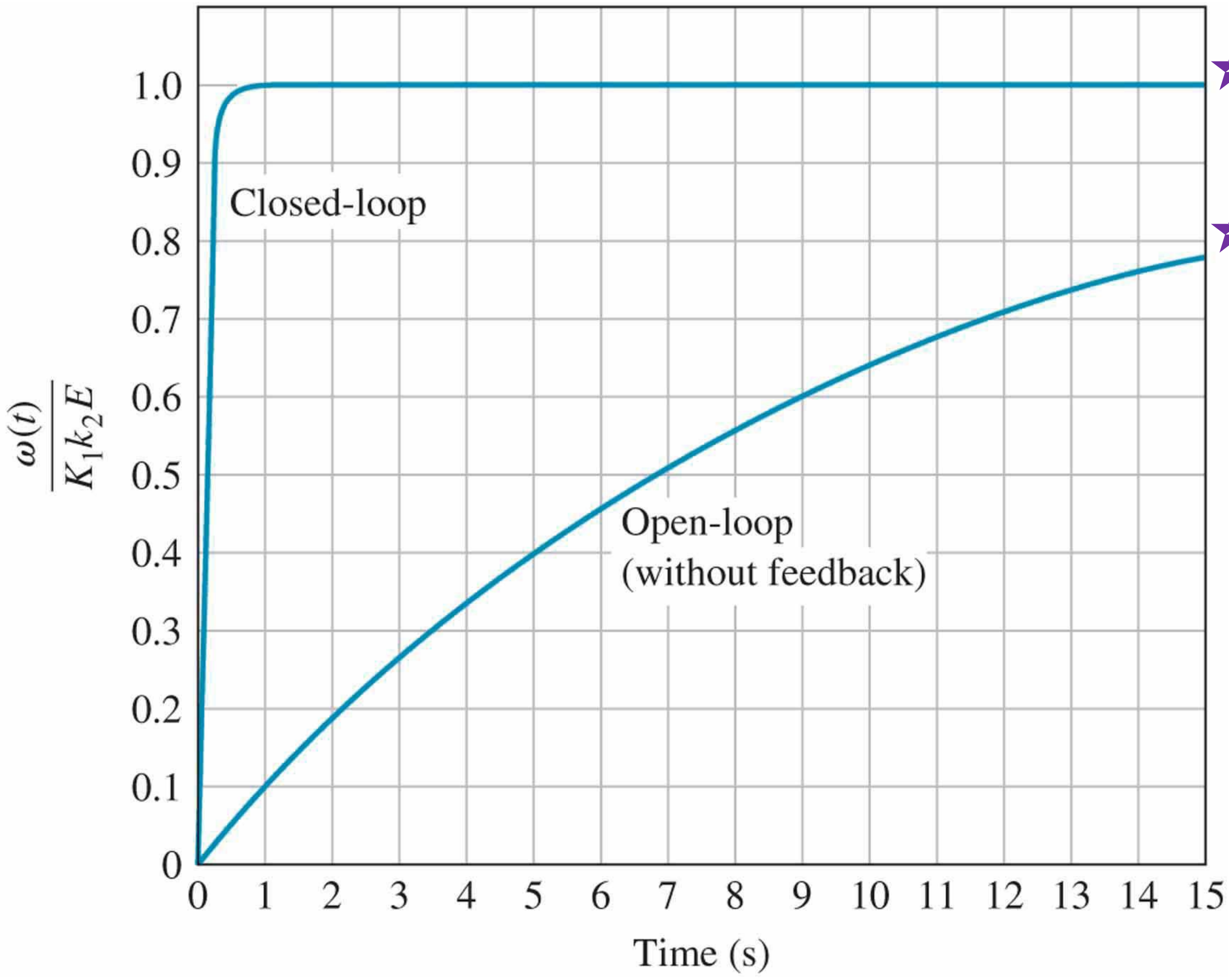
$$= \frac{K_1 K_a}{\tau_1 s + 1 + K_1 K_a K_t} = \frac{K_1 K_a / \tau_1}{s + \frac{(1 + K_1 K_a K_t)}{\tau_1}} = \frac{K_1 K_a / \tau_1}{s + \frac{1}{\tau_c}}$$

where,

$$\tau_c = \frac{\tau_1}{1 + K_1 K_a K_t} \Rightarrow \tau_1 > \tau_c$$

i.e. the Closed-loop system has a faster response compared to the Open-loop system

Also, τ_c can be reduced by increasing the controller gain K_a



- The response is For High amplifier gain K_a
- Noting different steady-state values for the open-loop and closed-loop

$$\omega(t) = L^{-1}\{\omega(s)\}$$

$$= K(1 - e^{-t/\tau_C})$$

where,

$$K = \frac{K_1 K_a k_2 E}{1 + K_1 K_a K_t}$$

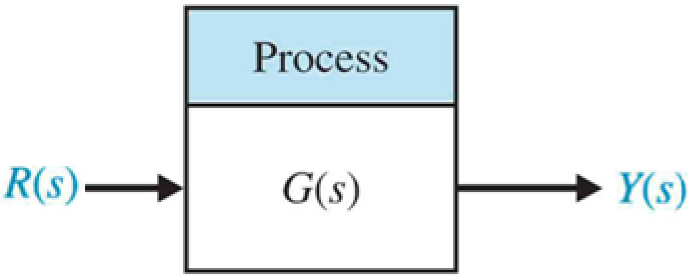
Steady-State Error

The steady-state error is the error after the transient response has decayed, leaving only the continuous steady response.

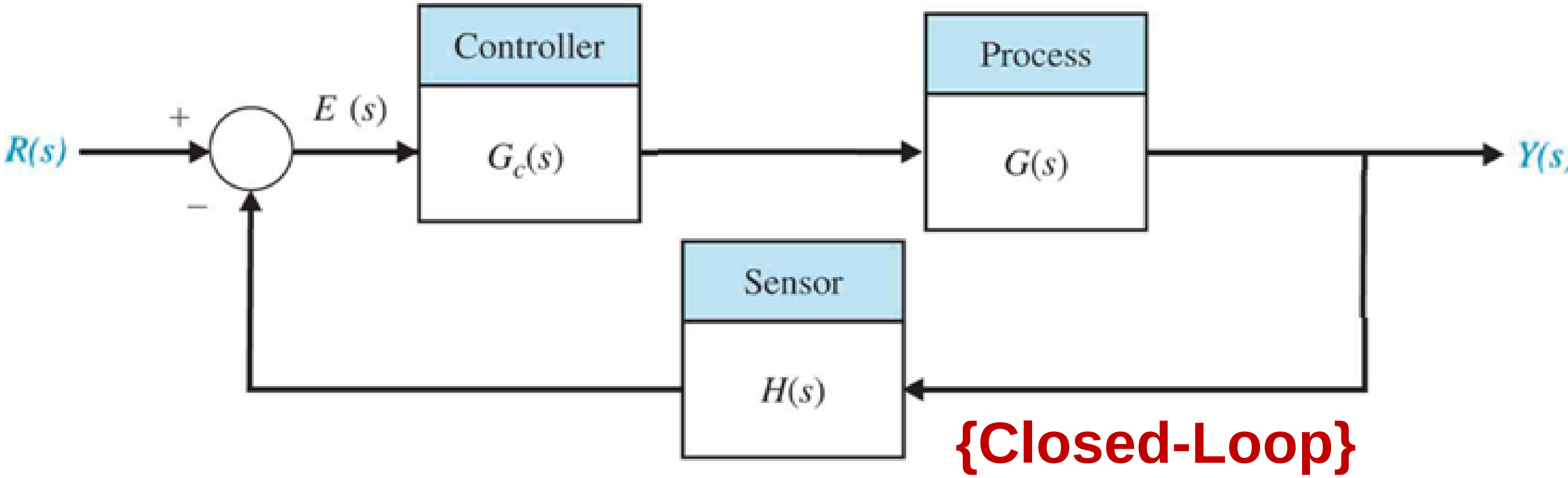
$$E(s) = R(s) - Y(s)$$

$$\frac{E(s)}{R(s)} = 1 - \frac{Y(s)}{R(s)}$$

$$= 1 - G(s)$$



{Open-Loop}



{Closed-Loop}

For unity-feedback control system, i.e., $H(s) = 1$

$$E(s) = R(s) - Y(s)$$

$$\frac{E(s)}{R(s)} = 1 - \frac{Y(s)}{R(s)}$$

$$= 1 - T(s) \;, \qquad T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}$$

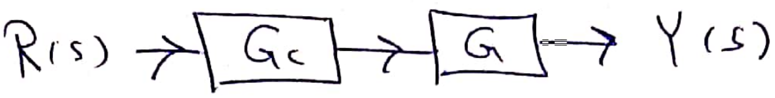
$$= \frac{1}{1 + G_c(s)G(s)}$$

and

$$e_{ss} = e(\infty) = \lim_{s \to \infty} s\, E(s)$$

* Sensitivity :-

① Open loop control system



$$S_G^T = \frac{dT}{dG} \cdot \frac{G}{T}$$

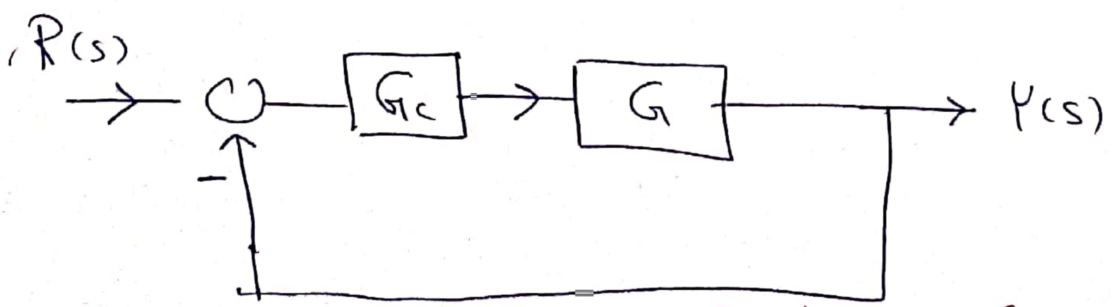
$$T = G_c G$$

$$\frac{dT}{dG} = G_c$$

$$\Rightarrow \boxed{S_G^T = G_c \cdot \frac{G}{G_c G} = 1}$$

* * * *

② Closed loop control system :-



negative unity feedback signal

$$S_G^T = \frac{dT}{dG} \cdot \frac{G}{T}$$

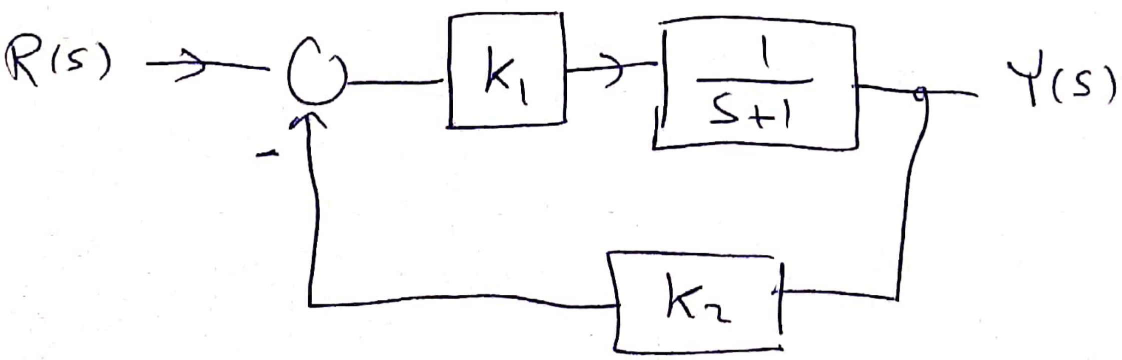
$$T = \frac{G_c \, G}{1 + G_c G}$$

$$\frac{dT}{dG} = \frac{(1 + G_c G)\, G_c - G_c G\, (G_c)}{(1 + G_c G)^2}$$

$$\frac{dT}{dG} = \frac{G_c}{(1 + G_c G)^2}$$

$$S_G^T = \frac{G_c}{(1 + G_c G)^2} \cdot \frac{G}{\frac{G_c G}{1 + G_c G}} = \frac{1}{1 + G_c G}$$

EX1: Find $S_{k_1}^T$ and $S_{k_2}^T = ?$



$$S_{k_1}^{T} = \frac{dT}{dk_1} \cdot \frac{k_1}{T}$$

$$T = \frac{k_1}{s+1+k_1 k_2} \quad , \quad \frac{dT}{dk_1} = \frac{s+1}{(s+1+k_1 k_2)^2}$$

$$S_{k_1}^{T} = \frac{s+1}{((s+1)+k_1 k_2)^2} \cdot \frac{k_1}{\frac{k_1}{s+1+k_1 k_2}} = \frac{s+1}{s+1+k_1 k_2}$$

(b) $$S_{k_2}^{T} = \frac{dT}{dk_2} \cdot \frac{k_2}{T}$$

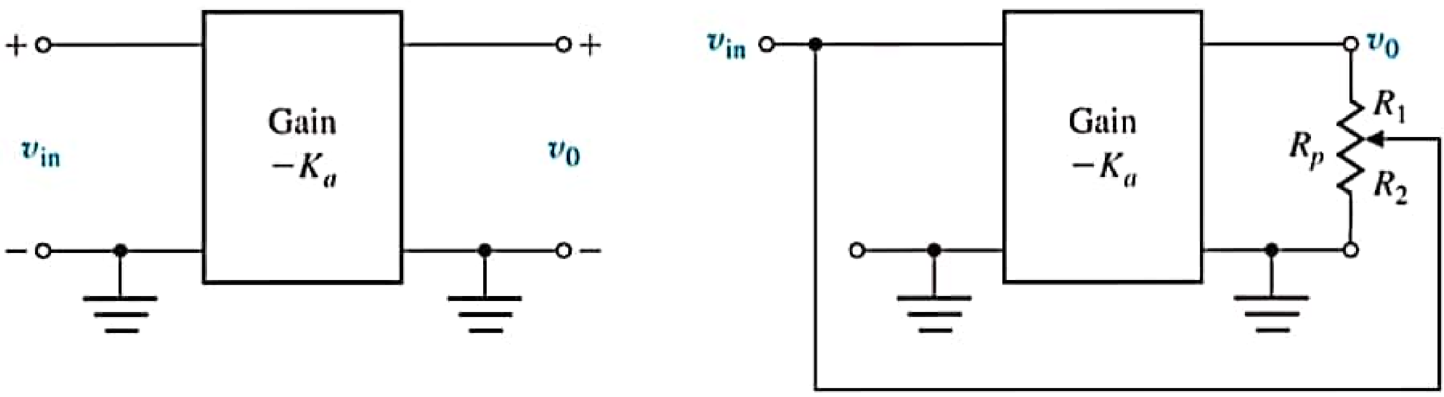
$$\frac{dT}{dk_2} = \frac{-k_1^2}{(s+1+k_1 k_2)^2}$$

$$S_{K_2}^{T} = \frac{-K_1^2}{(s+1+K_1K_2)^2} \cdot \frac{K_2}{K_1 / s+1+K_1K_2}$$

$$\boxed{S_{K_2}^{T} = \frac{-K_1K_2}{s+1+K_1K_2}}$$

Example: Feedback Amplifier

Study the sensitivity changes for the two cases: open-loop and closed-loop.



* Error Signal and steady-state error

$$E(s) = \underset{\text{desired}}{R(s)} - \underset{\text{actual}}{Y(s)}$$

$$e_{ss} = \lim_{s \to 0} s\, E(s)$$

→ open loop control system:

$$E(s) = R(s) - Y(s)$$

$$= R(s) - G_c G\, R(s)$$

$$\boxed{E(s) = R(s)\,[1 - G_c G]}$$

$$e_{ss} = \lim_{s \to 0} s\, E(s).$$

→ For a unit-step input $\left(R(s) = \frac{1}{s} \right)$

$$e_{ss} = \lim_{s \to 0} s\, R(s) \left[1 - G_c(s)\, G(s) \right]$$

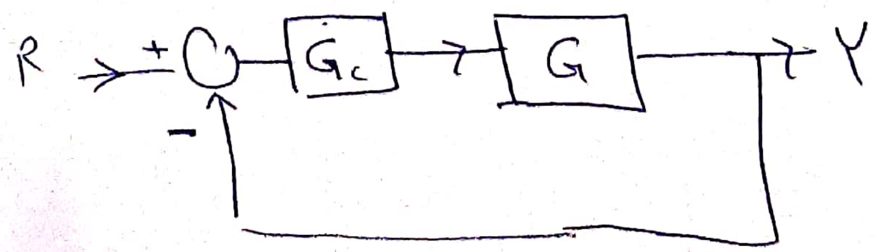
$$= \lim_{s \to 0} s \left(\frac{1}{s} \right) \left[1 - G_c G \right]$$

$$\boxed{e_{ss} = 1 - G_c(0)\, G(0)}$$

loop gain

* * * *

→ Closed loop control system:



$$E(s) = R(s) - Y(s)$$

$$= R(s) - \frac{G_c G}{1 + G_c G} R$$

$$= R(s) \left[\frac{1}{1 + G_c G} \right]$$

$$e_{ss} = \lim_{s \to 0} s E(s) = \lim_{s \to 0} s \left(\frac{1}{s} \right) \left[\frac{1}{1 + G_c G} \right]$$

$$E(s) = \frac{1}{1 + G_c(0) G(0)}$$

$$< 1 - G_c(0) G(0)$$

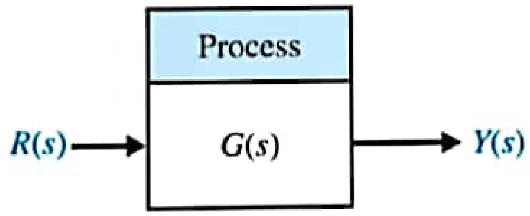
Steady-State Error

The steady-state error is the error after the transient response has decayed, leaving only the continuous steady response.

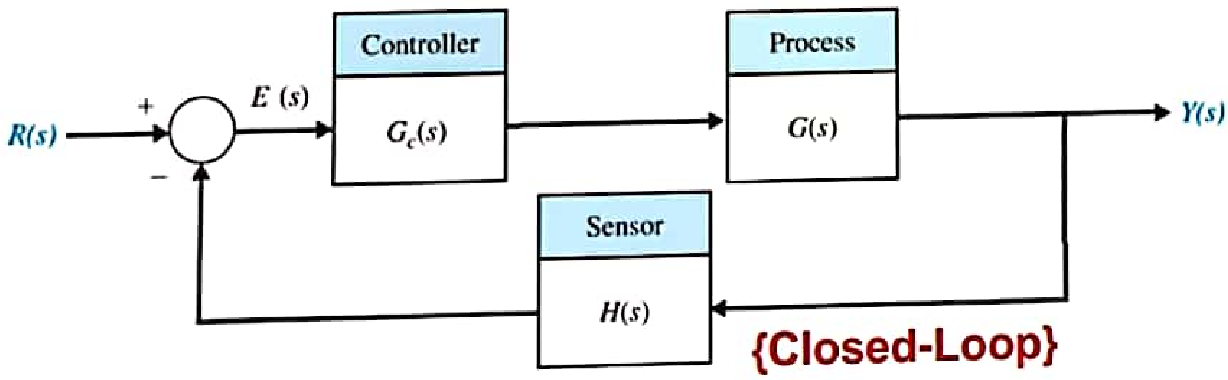
$$E(s) = R(s) - Y(s)$$

$$\frac{E(s)}{R(s)} = 1 - \frac{Y(s)}{R(s)}$$

$$= 1 - G(s)$$



{Open-Loop}



{Closed-Loop}

For unity-feedback control system, i.e., $H(s) = 1$

$$E(s) = R(s) - Y(s)$$

$$\frac{E(s)}{R(s)} = 1 - \frac{Y(s)}{R(s)}$$

$$= 1 - T(s)\ , \qquad T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}$$

$$= \frac{1}{1 + G_c(s)G(s)}$$

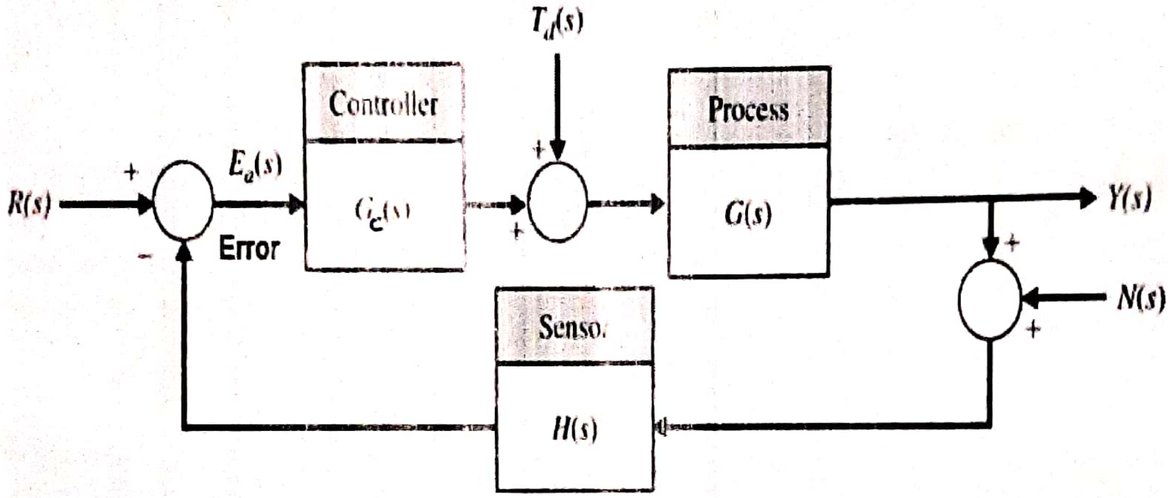
and

$$e_{ss} = e(\infty) = \lim_{s \to \infty} s\,E(s)$$

DISTURBANCE SIGNALS IN FEEDBACK CONTROL SYSTEMS

Disturbance signals represent unwanted inputs which affect the control-system's output, and result in an increase of the system error. It is the job of the control-system engineer to properly design the control system to partially eliminate the affects of disturbances on the output and system error.

A disturbance signal is commonly found in control systems. For example, wind gusts hitting the antenna dish of a tracking radar create large unwanted torques which affect the position of the antenna. Another example, are sea waves hitting a hydrofoil's foil which create very large unwanted torques which affect the foil's position.



① $T_d(s) = 0$ and $N(s) = 0$, $H(s) = 1$

$$Y(s) = \frac{G_c G}{1 + G_c G} R(s)$$

② $R(s) = 0$ and $N(s) = 0$, $H(s) = 1$

$$Y(s) = \frac{G(s)}{1 + G_c G} T_d(s)$$

③ $R(s) = 0$ and $T_d(s) = 0$, $H(s) = 0$

$$Y(s) = \frac{-G_c G}{1 + G_c G} N(s)$$

Error signal analysis:

$E(s) = R(s) - Y(s)$

But,

Desired Input — Disturbance input — Noise input

$$Y(s) = \frac{G_C(s)G(s)}{1+G_C(s)G(s)} R(s) + \frac{G(s)}{1+G_C(s)G(s)} T_d(s) - \frac{G_C(s)G(s)}{1+G_C(s)G(s)} N(s)$$

$$\therefore\ E(s) = \frac{1}{1+G_C(s)G(s)} R(s) - \frac{G(s)}{1+G_C(s)G(s)} T_d(s) + \frac{G_C(s)G(s)}{1+G_C(s)G(s)} N(s)$$

where,

① Error due to desired input $R(s)$:-

$$\underset{\text{due to } R(s) \text{ only}}{E(s)} = \frac{1}{1 + G_c G} R(s)$$

② Error due to disturbance only:

$$E(s) = - \frac{G(s)}{1 + G_c G} T_d(s)$$

note : $E(s) = R(s) - Y(s)$ but $R(s) = 0$

$\therefore E(s) = 0 - Y(s) = -Y(s)$

(3) Error due to noise only:

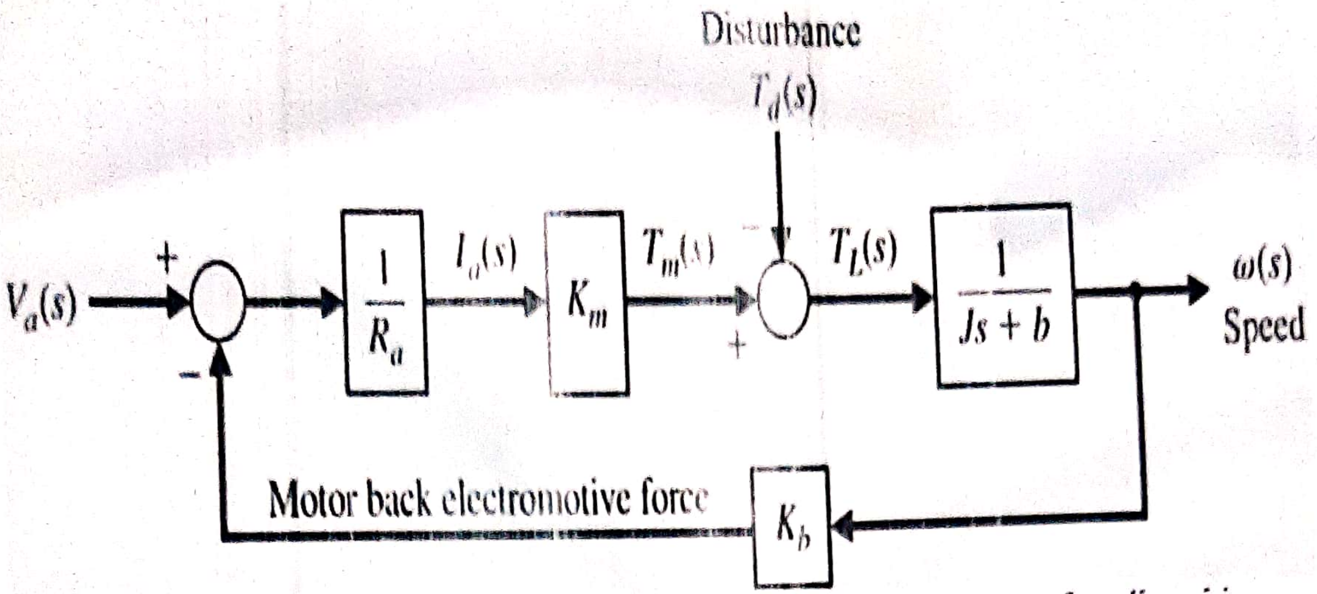
$E(s) = R(s) - Y(s)$ but $R(s) = 0$

$$E(s) = 0 - Y(s) = + \frac{G_c G}{1 + G_c G} N(s)$$

→ See DC-motor block Diagram

then, calculate $E(s)$ due to

disturbance only.



→ Assuming very small inductance and only disturbance input } → $V_a(s) = 0$ and $T_d(s) = \frac{D}{s}$

$$\therefore \frac{\omega(s)}{T_d(s)} = \frac{-1}{Js + b + \frac{K_m K_b}{R_a}}$$

Steady-state speed due to a step disturbance $T_d(s) = \frac{D}{s}$ is

$$\omega_{ss} = \lim_{s \to \infty} s\omega(s)$$

$$= \lim_{s \to \infty} s \frac{-1}{Js + b + \frac{K_m K_b}{R_a}} \frac{D}{s}$$

$$(\omega_{ss})_{open} = -\frac{D}{b + \frac{K_m K_b}{R_a}}$$

Ch.5 The performance of Feedback Control Systems

- One of the first steps in the design process is to specify the measures of performance (performance specifications). In this chapter we introduce the common time-domain specifications such as percent overshoot, settling time, peak time, rise time and steady-state error.

- Time domain performance specifications can be found from the response of the system to a given input.
  $\underbrace{\text{response}}_{??}$ ... $\underbrace{\text{input}}_{??}$

- The time response of a control system is usually divided into two parts. The transient response and the steady-state response.

Let $Y(t)$ denote the time response of a continuous data system

$$Y(t) = Y_{ss}(t) + Y_t(t)$$

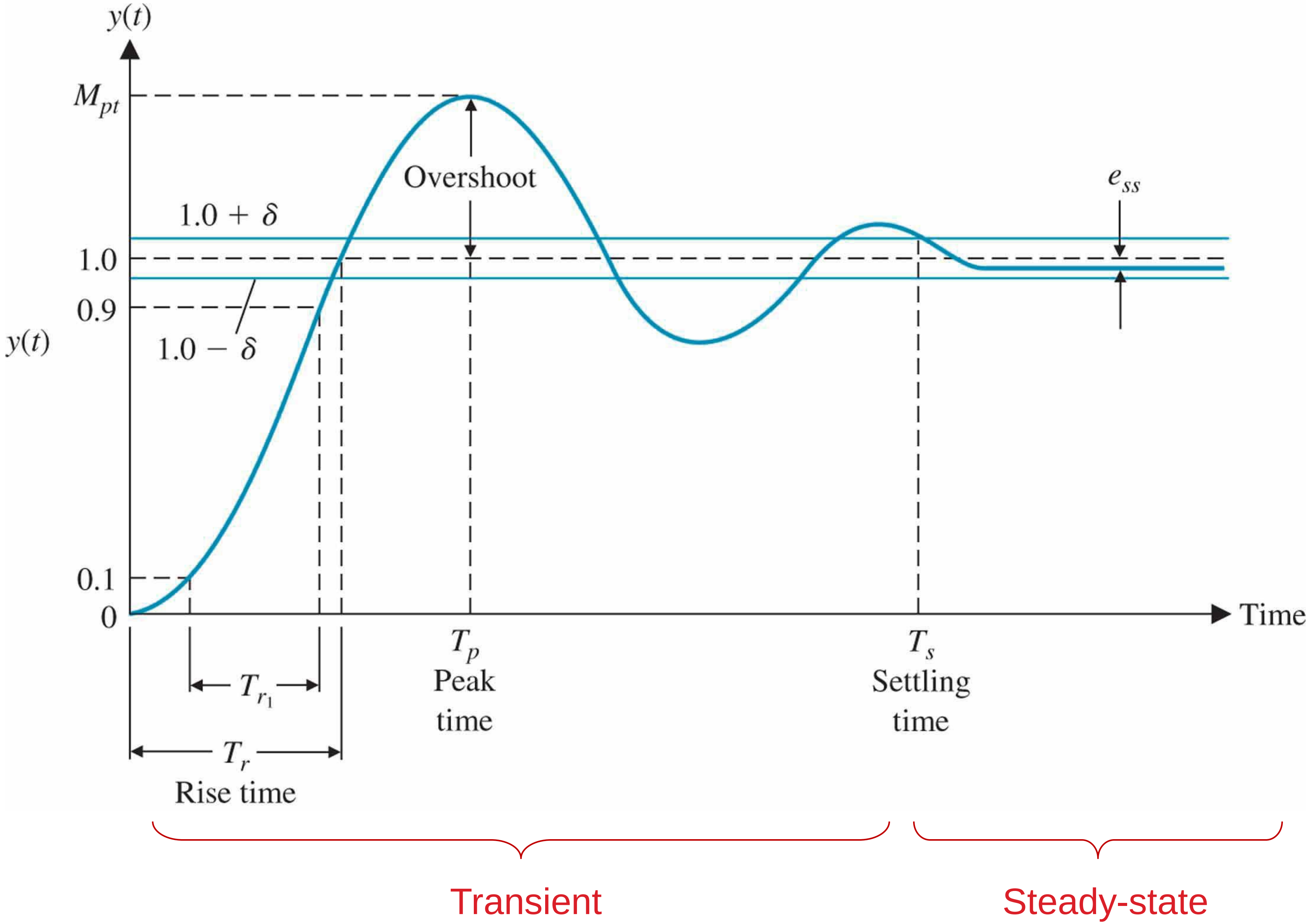
$$Y(t) = 5 - 5e^{-t}\cos 2t$$

Where

Y_t = transient response $\qquad$ Y_{ss} = steady state response

- In control systems transient response is defined as the part of the time response that goes to zero as time becomes very large $\lim_{t\to\infty} y_t(t) = 0$

- The steady state response is simply the part of the total response that remains after the transient has died out.



y(t)
M_pt
Overshoot
e_ss
1.0 + δ
1.0
0.9
y(t)
1.0 − δ
0.1
0
Time
T_r1
T_r
Rise time
T_p
Peak
time
T_s
Settling
time
Transient
Steady-state

Test Input Signals for the Time Response of Control Systems

- For the purposes of analysis and design it is necessary to assume some basic types of test input signals so that the performance of a system can be evaluated

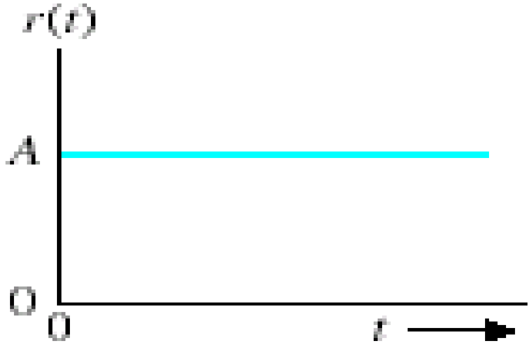
- These input signals are close to a real life input signals

Step input , ramp input, parabolic input, and unit-impulse input

I. Step input *r(t)* :

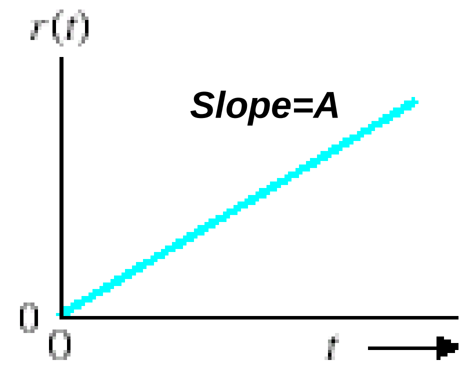
$$r(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases} \Rightarrow R(s) = \frac{A}{s}$$

Where *A* is the amplitude of the step input



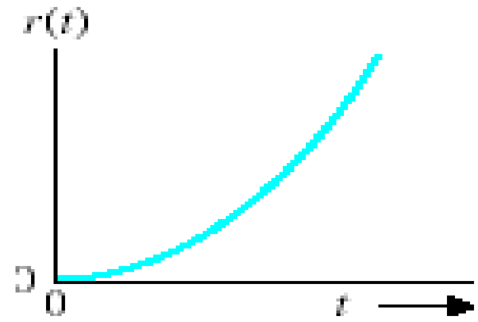
II. Ramp input *r(t)* :

$$r(t) = \begin{cases} At & t \ge 0 \\ 0 & t < 0 \end{cases} \Rightarrow R(s) = \frac{A}{s^2}$$



III. Parabolic input *r(t)* :

$$r(t) = \begin{cases} At^2 & t \ge 0 \\ 0 & t < 0 \end{cases} \Rightarrow R(s) = \frac{2A}{s^3}$$



IV. Unit impulse input $\delta(t)$:

Unit impulse is a special case from rectangular function *f(t)*

$$f(t) = \begin{cases} 1/\varepsilon & -\frac{\varepsilon}{2} \le t \le \frac{\varepsilon}{2} \\ 0 & otherwise \end{cases}$$

where $\varepsilon > 0$

As ε approaches to zero, the function *f(t)* approaches the unit-impulse function

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \Rightarrow R(s) = 1$$

Performance of Control Systems

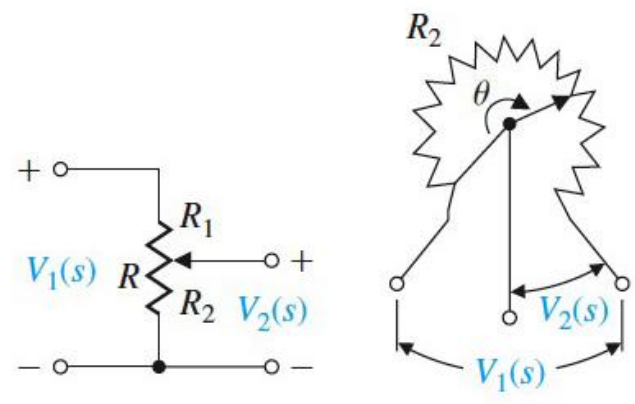
We will study the performance specification of the following control systems:

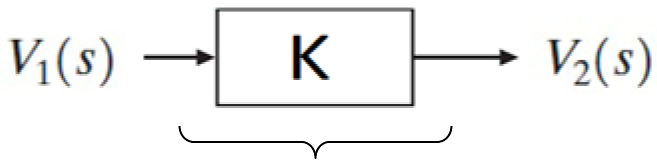
I. Zero order system (static TF)

II. First order system

III. Second order system

IV. Higher order system



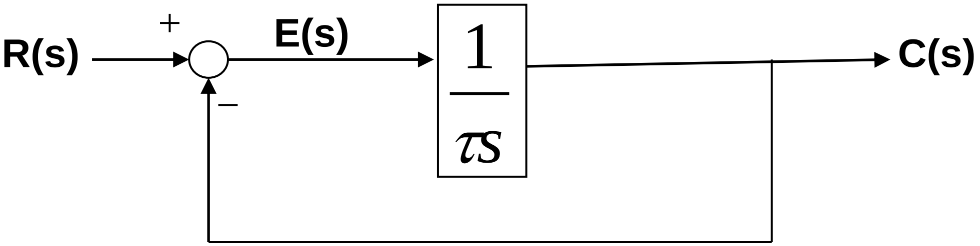


Static TF

$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R} = \frac{R_2}{R_1 + R_2}$$

II. First order system:

A first-order system without zeros can be represented by the following TF:



$$\frac{C(s)}{R(s)} = \frac{1}{\tau s + 1}$$

Test signal is a unit-step function, R(s)=1/s

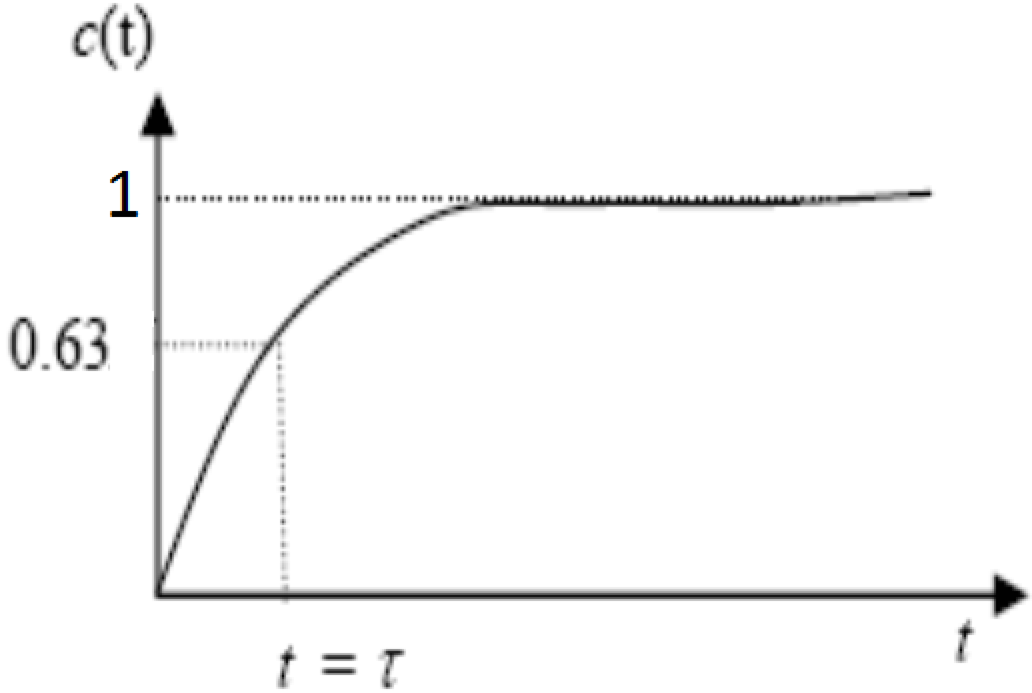
To find the response

$$C(s) = \frac{1}{\tau s + 1} R(s) = \frac{1}{s(\tau s + 1)} = \frac{1}{s} - \frac{1}{s + \frac{1}{\tau}}$$

$$c(t) = 1 - e^{-\frac{t}{\tau}}$$

If t= τ , so the step response is

$C(\tau) = (1 - 0.37) = 0.63$

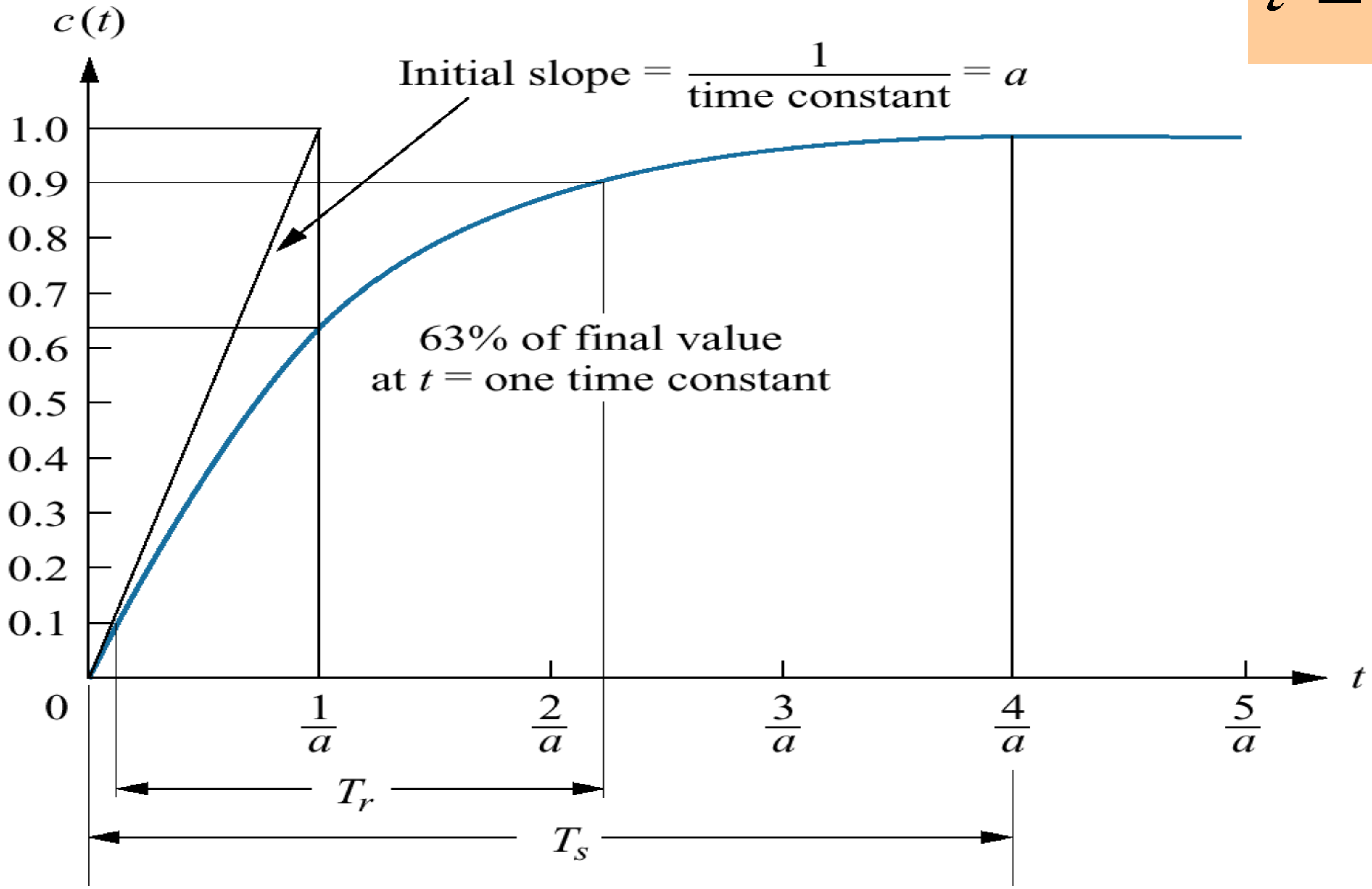


Settling time (t_s) is defined to be the time taken for the step response to come to within **2% of the final value** of the step response (i.e. entering the steady-state region).

$$t_s = 4\tau$$ ⇒ If t= 4τ , so the step response is

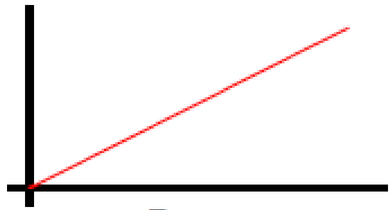
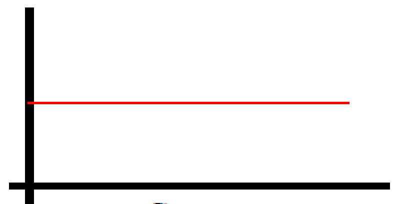
$C(4\tau) = (1 - 0.018) = 0.982$

$$\tau = \frac{1}{a}$$



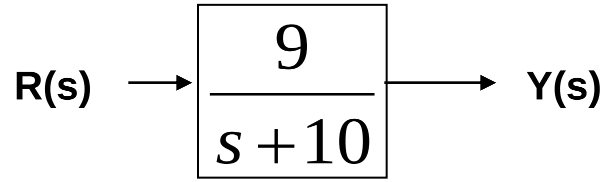
c(t)
Initial slope = 1 / time constant = a
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
63% of final value
at t = one time constant
t
1/a
2/a
3/a
4/a
5/a
T_r
T_s

First order system response for different input signals:

GRAPHICAL REPRESENTATION	INPUT	OUTPUT
 Ramp	$r(t) = t$	$c(t) = t - T + Te^{-t/T}$
 Step	$r(t) = 1$	$c(t) = 1 - e^{-t/T}$
 Impulse	$r(t) = \begin{cases} 1 & for\ t = 0 \\ 0 & for\ t \neq 0 \end{cases}$	$c(t) = \frac{1}{T} e^{-t/T}$

Example:

The following system, find the unit-step response of the system and the steady-state error.



$$Y(s) = G(s)R(s)$$

$$= \frac{9}{s(s+10)}$$

Again, what is the time constant??

$$\therefore\ y(t) = L^{-1}\{Y(s)\}$$

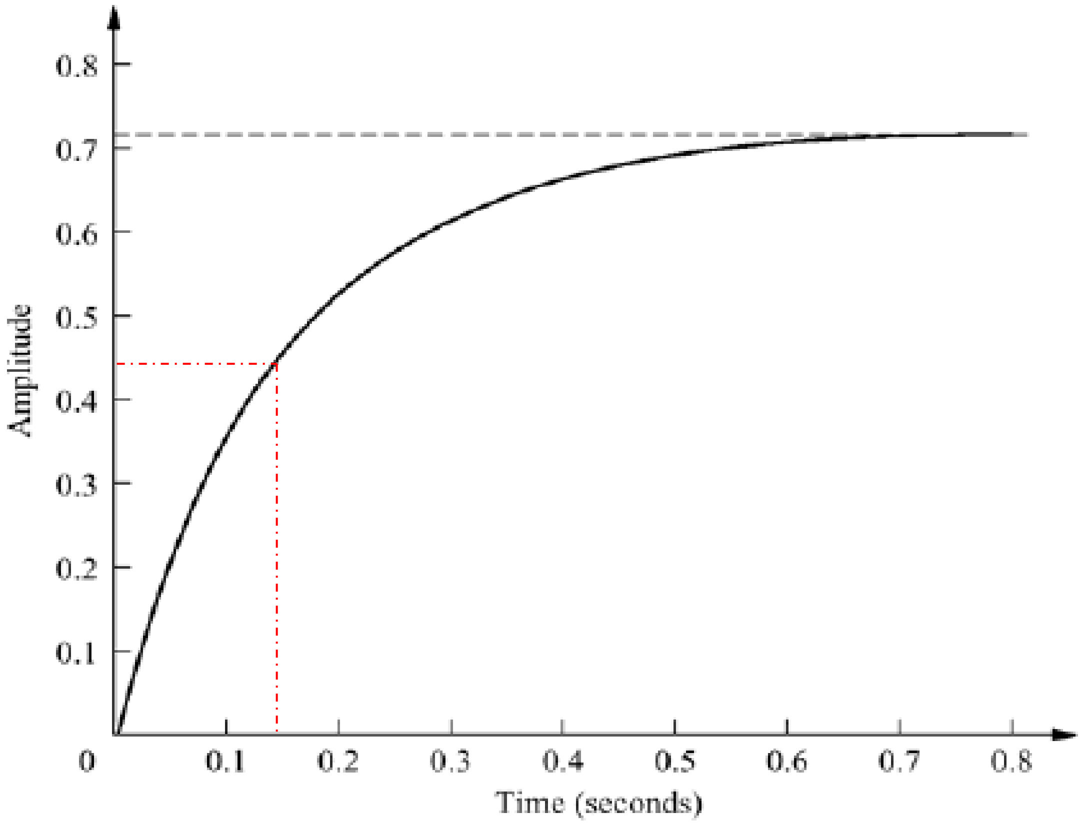
$$= L^{-1}\left\{\frac{0.9}{s} - \frac{0.9}{s+10}\right\} = 0.9(1 - e^{-10t})$$

And $$y_{ss} = y(\infty) = \lim_{t \to \infty} y(t) = 0.9$$

$$\therefore\ e_{ss} = 1 - 0.9 = 0.1$$

Example:

The following figure gives the measurements of the step response of a first-order system, find the transfer function of the system.



III. Second order system:

- *Second-order systems* exhibit a wide range of responses which must be analyzed and described.

- For a *first-order system*, varying a single parameter changes the speed of response,

- Changes in the parameters of a *second order system* can change the form of the response not only the speed of the response.

For example: a second-order system can display characteristics much like a first-order system or, depending on the system's parameters values, pure oscillations or damped behavior might result for its transient response.

Assume the following transfer function of a general closed-loop second-order system:

$$T(s) = \frac{b}{s^2 + as + b}$$

We can re-write the above transfer function in the following form of a standard second order system closed-loop transfer function:

$$T(s) = \frac{\omega_n^{\ 2}}{s^2 + 2\xi\omega_n\ s + \omega_n^{\ 2}}$$

ω_n $(\omega_n = \sqrt{b})$ ⇨ is referred to as *the* *un-damped natural frequency* of the system, which is the frequency of oscillation of the system without damping.

ζ ($\zeta = \frac{a}{2\sqrt{b}}$) → is referred to as *the damping ratio* of the second order system, which is a measure of the degree of resistance to change in the system output.

And the poles of the closed loop system (roots of the characteristic equation)are: →

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$
$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

According the value of ζ, a second-order system can be set into one of the following four cases:

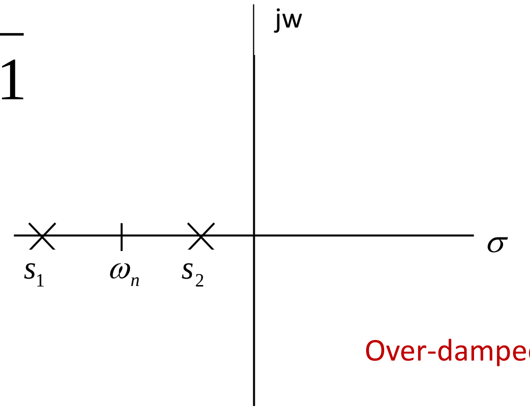
1. *Overdamped* - when the system has two real distinct poles ($\zeta > 1$).
2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)
3. *Undamped* - when the system has two imaginary poles ($\zeta = 0$).
4. *Critically damped* - when the system has two real but equal poles ($\zeta = 1$).

Assume a unit-step input for the coming slides,

Case I: Over damped case (Stable) $\xi \rangle 1$

Real different poles $s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

$$y(t) = 1 + \frac{\omega_n}{2\sqrt{\xi - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right)$$



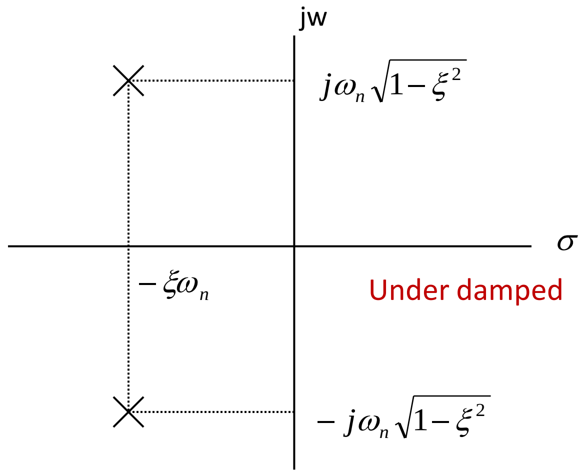
Case II: Underdamped case (Stable) $0 \langle \xi \langle 1$

Complex poles $s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

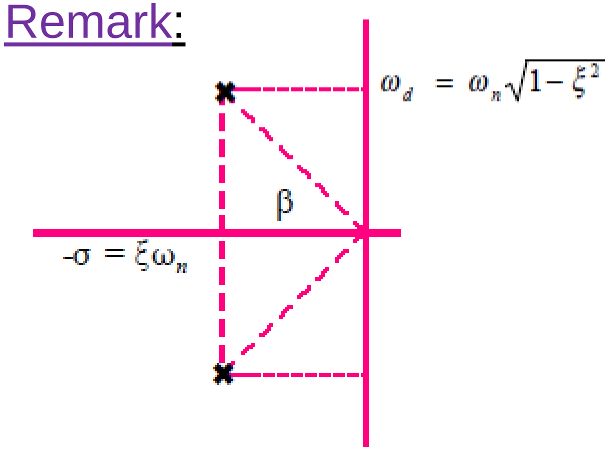
$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \beta)$$

$$where \quad \beta = \tan^{-1}\frac{\omega_d}{\xi\omega_n} = \tan^{-1}\frac{\omega_n\sqrt{1-\xi^2}}{\xi\omega_n}$$

$\omega_d = \omega_n\sqrt{1-\xi^2}$: is the damped natural frequency



Remark:

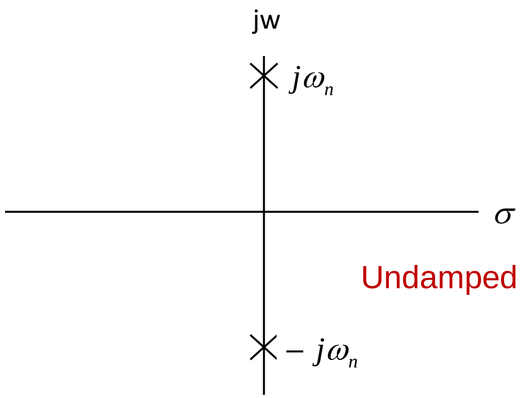


Case III:

Zero damped (*Undamped*) case (Stable)

$\xi = 0$, Imaginary poles $s_{1,2} = \pm j\omega_n$

$$y(t) = 1 - \cos(\omega_n t)$$

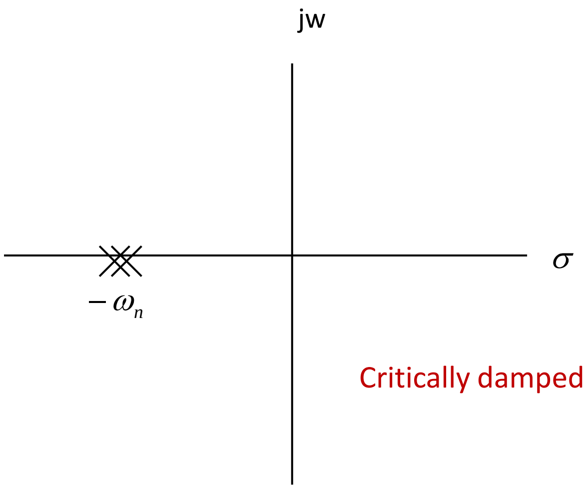


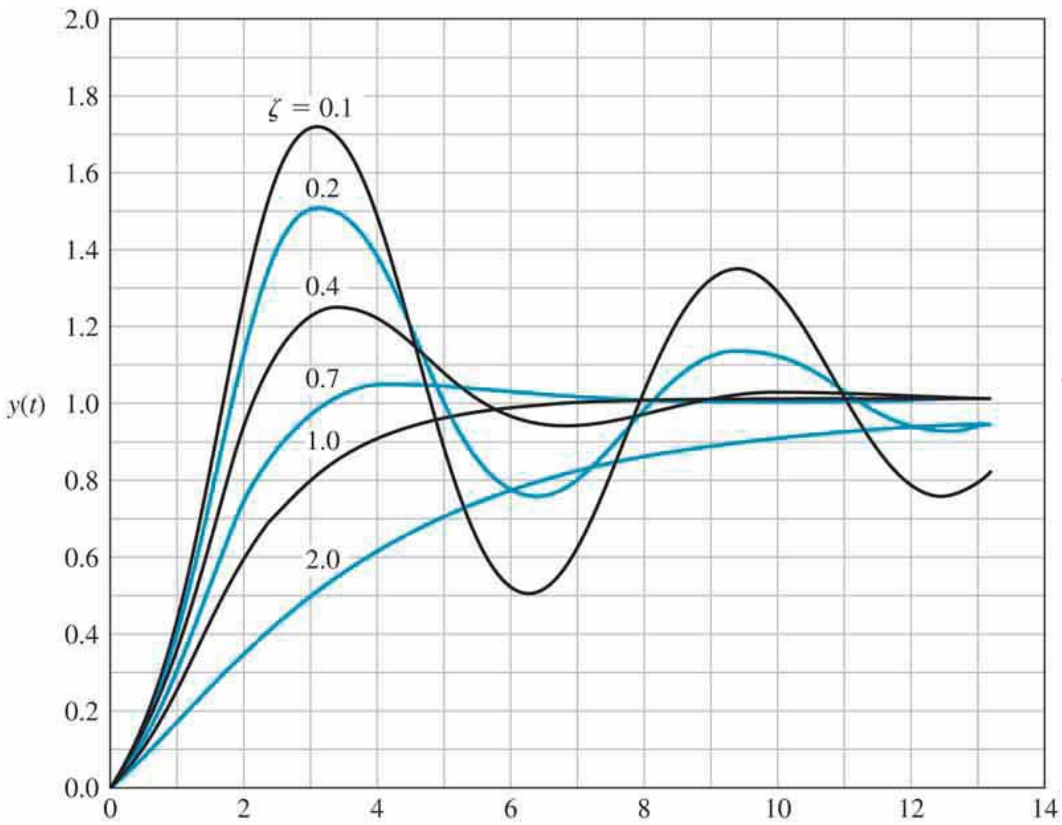
Case IV:

Critically damped (Stable) $\xi = 1$

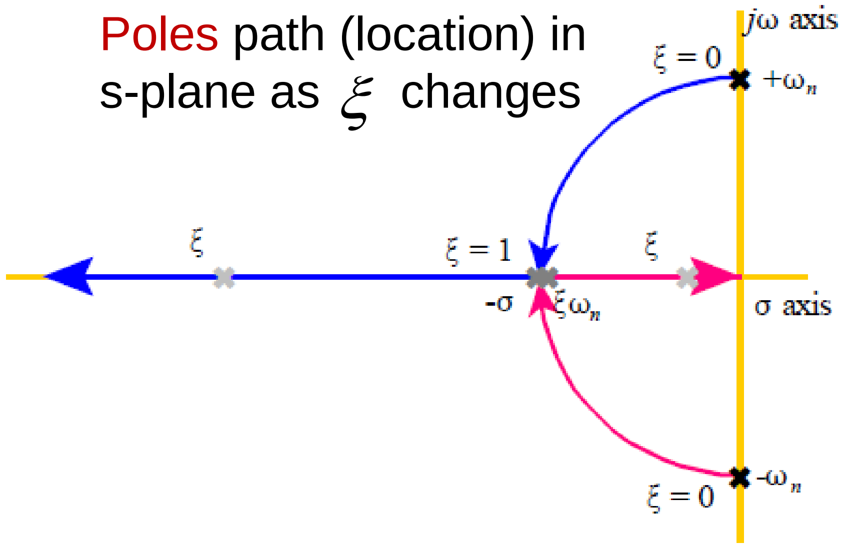
Real equall double pole $-\omega_n$

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$



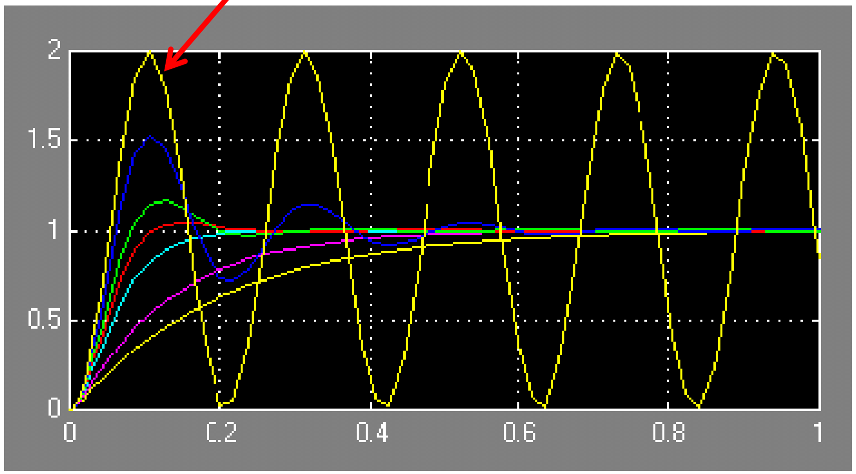


Poles path (location) in s-plane as ξ changes

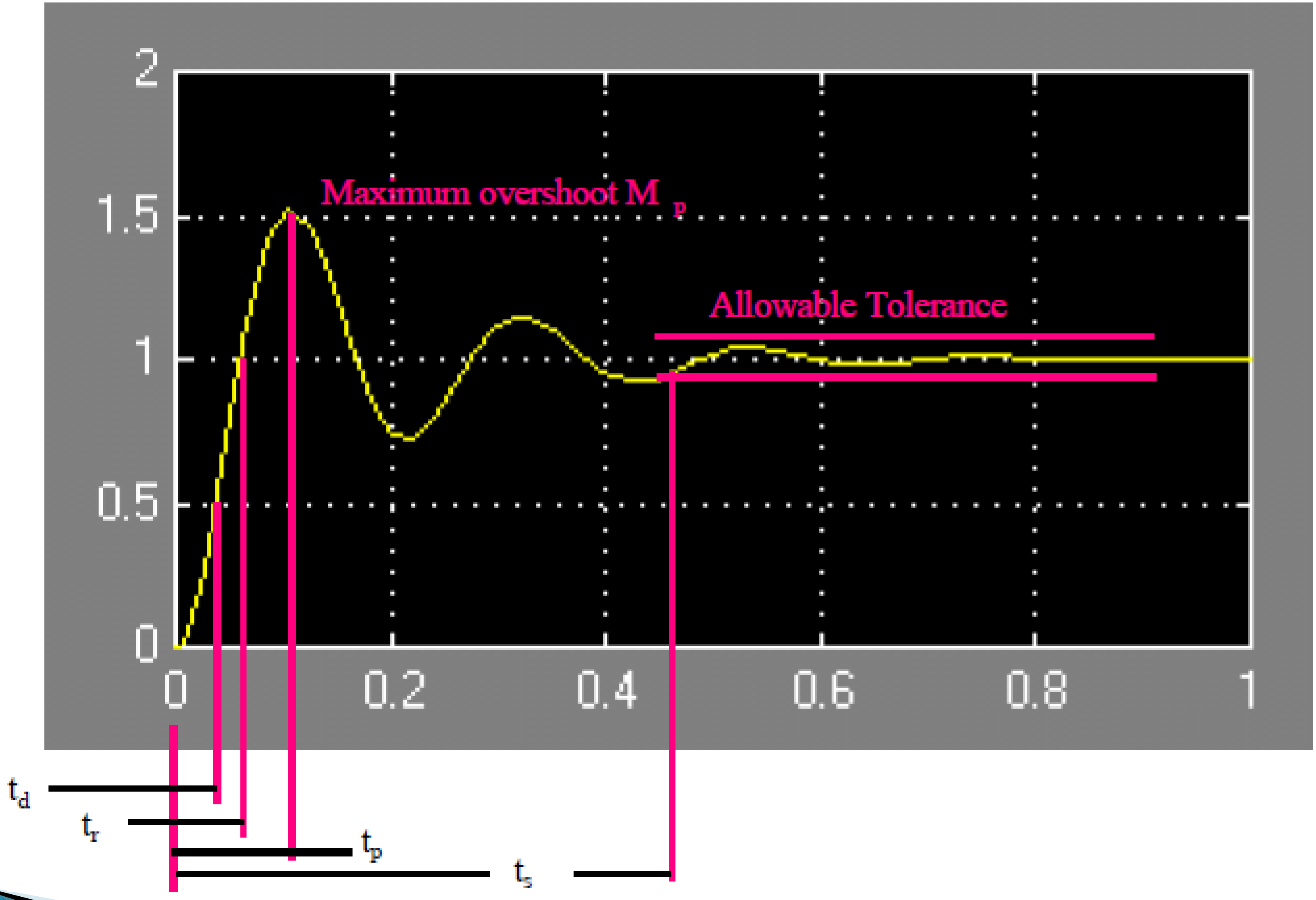


$\xi = 0$

Second order system responses for a unit-step input and different values of ξ



Transient response specifications of an underdamped second order system for a unit-step input:



All performance specifications are derived from:

$$y(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \beta)$$

1. Rise time(t_r) : rise time is the time required for the response to change from a lower prescribed value to a higher one.

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \beta}{\omega_n \sqrt{1-\zeta^2}}$$

2. Peak time(t_p): the peak time is the time required for the response to reach the first peak

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\varsigma^2}}$$

3. Settling time (t_s): the settling time is the time required for the amplitude of the sinusoid to decay to 2% or 5% of the steady-state value.

$$t_s = \frac{4}{\varsigma\omega_n}$$ 2% criterion

$$t_s = \frac{3}{\varsigma\omega_n}$$ 5% criterion

4. Maximum overshoot percentage: the percent overshoot is defined as the amount that the waveform at the peak time overshoots the steady-state value.

Maximum Peak Percentage MP%

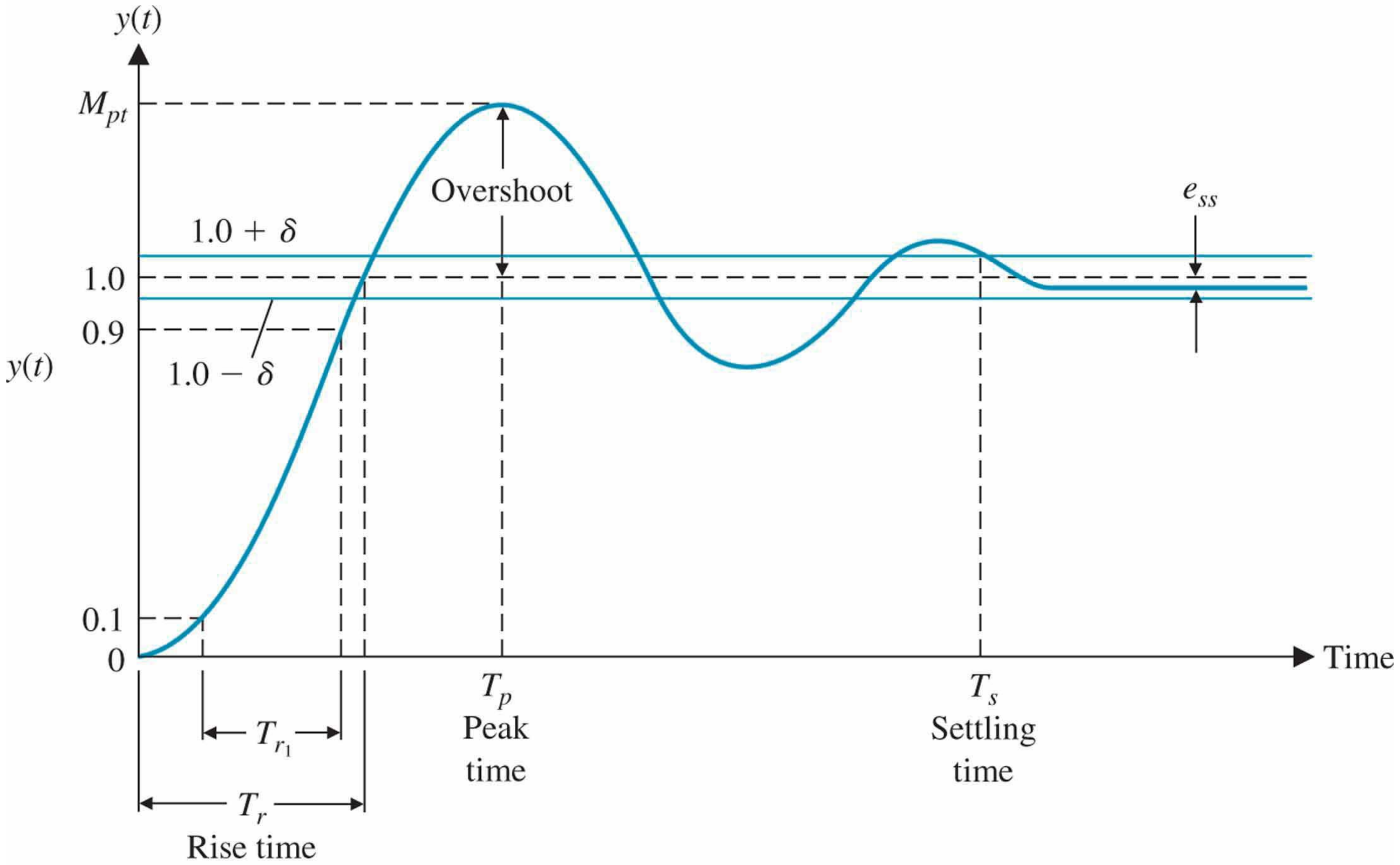
$$MP\% \equiv \frac{y(t_p) - y(\infty)}{y(\infty)} x100\%$$

OR

Overshoot Percentage OS%

$$OS\% = \frac{y\_\max - y\_final}{y\_final} \; x\,100\%$$

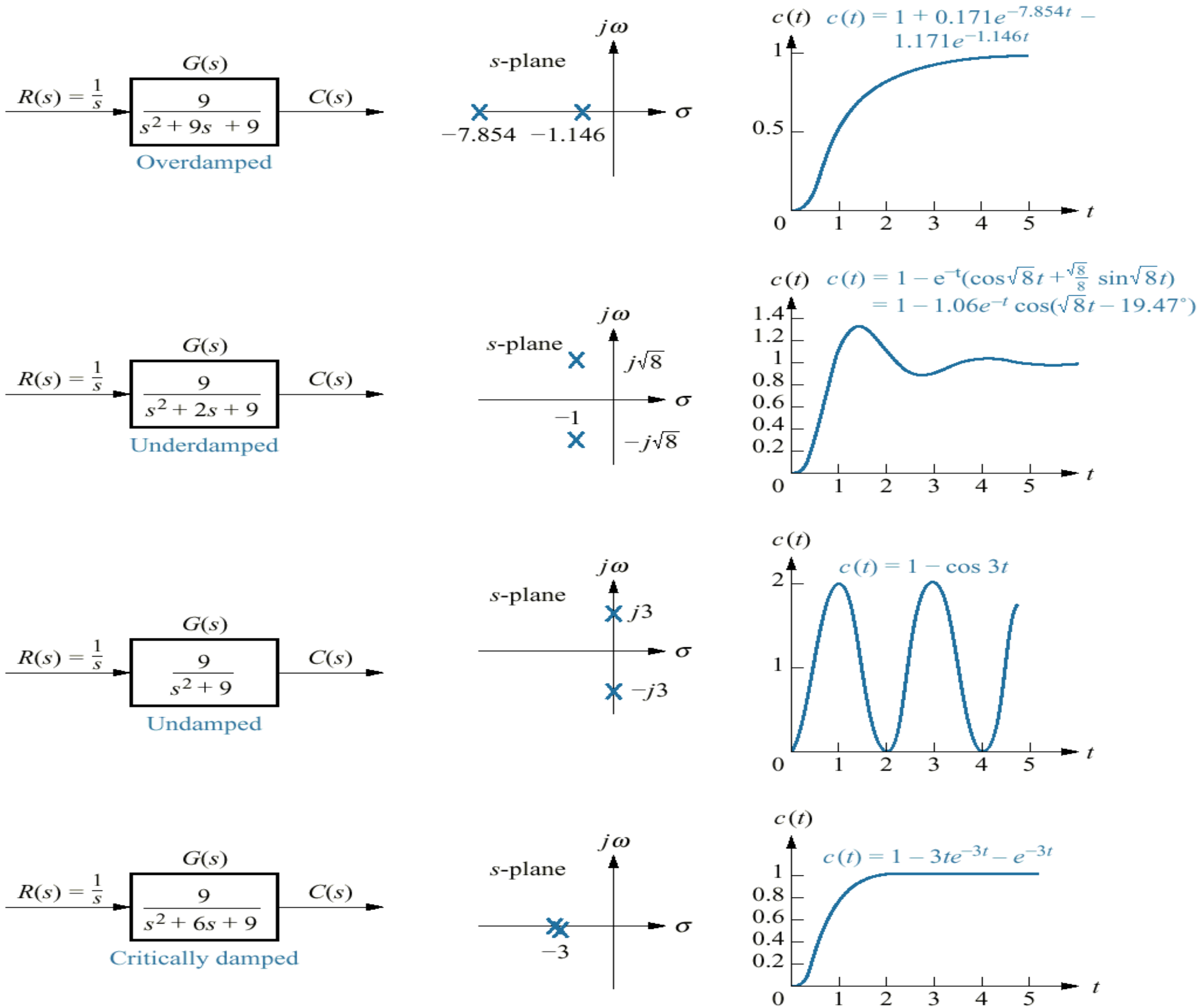
$$MP\% = 100e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$



For given *OS%*, the damping ratio can be solved from the OS% equation;

$$\varsigma = \frac{-\ln(\%MP/100)}{\sqrt{\pi^2 + \ln^2(\%MP/100)}}$$

Examples:


G(s)
R(s) = 1/s
9 / (s^2 + 9s + 9)
C(s)
Overdamped
jω
s-plane
−7.854 −1.146
σ
c(t) c(t) = 1 + 0.171e^{−7.854t} − 1.171e^{−1.146t}
1
0.5
0 1 2 3 4 5 t
G(s)
R(s) = 1/s
9 / (s^2 + 2s + 9)
C(s)
Underdamped
jω
s-plane
j√8
−1
−j√8
σ
c(t) c(t) = 1 − e^{−t}(cos√8t + √8/8 sin√8t)
= 1 − 1.06e^{−t} cos(√8t − 19.47°)
1.4 1.2 1 0.8 0.6 0.4 0.2
0 1 2 3 4 5 t
G(s)
R(s) = 1/s
9 / (s^2 + 9)
C(s)
Undamped
jω
s-plane
j3
−j3
σ
c(t)
c(t) = 1 − cos 3t
2
1
0 1 2 3 4 5 t
G(s)
R(s) = 1/s
9 / (s^2 + 6s + 9)
C(s)
Critically damped
jω
s-plane
−3
σ
c(t)
c(t) = 1 − 3te^{−3t} − e^{−3t}
1 0.8 0.6 0.4 0.2
0 1 2 3 4 5 t

IV. Higher order system:

Assume the following TF:

$$\frac{C(s)}{R(s)} = \frac{(s+z_1)(s+z_2)\cdots\cdots(s+z_{m-1})(s+z_m)}{(s+p_1)(s+p_2)\cdots\cdots(s+p_{n-1})(s+p_n)}$$

$$= \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}(s^2+2\xi_k\omega_{n_k}s+\omega_{n_k}{}^2)}$$

For step input and using partial fraction,

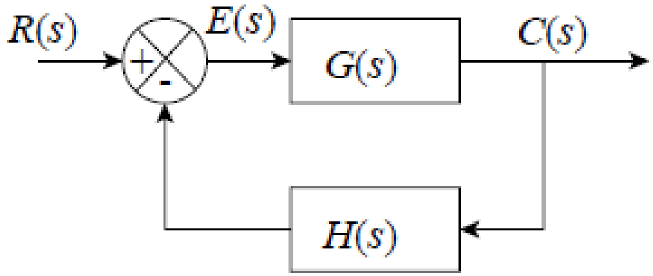
$$C(s) = \frac{a}{s} + \sum_{i=1}^{q}\frac{A_i}{s+\sigma_i} + \sum_{k=1}^{r}\frac{B_k s + D_k}{s^2 + 2\alpha_k s + \gamma_k{}^2}$$

Observations:

- The response of a higher order system is a combination of responses of first and second order systems.
- The poles far away from the *j*-axis (*Im*-axis) can be ignored (fast response).
- The dominant pole is the one nearer to the *j-axis.*
- Poles and zeros that are several order (usually five order)of magnitude smaller than the dominate, Poles and zeros can be ignored.
- A pole and a zero that coincide cancel each other.
- A pole and a zero that are near to each other tend to cancel each other.
- Generally speaking, the order of the denominator is larger than that of the numerator.
- For the system to be stable, all the poles of the transfer function must be negative, that is to say must lay in the left hand side of the s-*plane.*

Steady-state Error Analysis

For the feedback system shown in block diagram below, the transfer function is given by:



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The system error is given by:

$$\begin{aligned} E(s) &= R(s) - C(s)H(s) \\ &= \left[1 - \frac{G(s)H(s)}{1+G(s)H(s)}\right]R(s) \\ &= \frac{1}{1+G(s)H(s)}R(s) \end{aligned}$$

This last expression shows that the loop gain *G(s)H(s)* determine the amount and nature of the steady state error of a system.

The loop gain *G(s)H(s) can be expressed in the general form;*

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)(s+z_3)\cdots\cdots(s+z_m)}{s^N(s+p_1)(s+p_2)(s+p_3)\cdots\cdots(s+p_n)}$$

$$= \frac{K\prod_{i=1}^{i=m}(s+z_i)}{s^N \prod_{j=1}^{i=n}(s+p_j)}$$

The error in this case would be given by:

$$E(s) = \frac{1}{1+G(s)H(s)} R(s)$$

$$= \frac{s^N \prod_{j=1}^{i=n}(s+p_j)}{s^N \prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} R(s)$$

The steady state error is calculated as follows:

$$e_{ss} = \lim_{s\to 0}\left[s \frac{s^N \prod_{j=1}^{i=n}(s+p_j)}{s^N \prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} R(s) \right]$$

- When the standard test signals of a step (*A/s), a ramp (A/s2), and an acceleration (A/s3) are* used, the Laplace operator "*s" in the input test signal denominator will cancel or reduce* from the power of "*s" in the numerator of the expression above.*

- The power of "*s" (the poles of the G(s)H(s) located on the origin of s-plane), i.e. N, determines the* steady state error response of the system when subjected to standard test signals, and is called the **"type number" of the system.**

- For *N = 0, the system is a type zero, for N = 1, the system is a type one, and so on.*

Type Zero System:

The steady state error for a step input; *A/s is given by*

$$e_{ss} = \lim_{s\to 0}\left[s \frac{\prod_{j=1}^{i=n}(s+p_j)}{\prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} \frac{A}{s}\right]$$

$$= \frac{\prod_{j=1}^{i=n} p_j}{\prod_{j=1}^{i=n} p_j + K\prod_{i=1}^{i=m} z_i} A$$

$$= \frac{A}{1+K_p} \qquad \text{for } K_p = \frac{K\prod_{i=1}^{i=m} z_i}{\prod_{j=1}^{i=n} p_j}$$

Position error constant

$$K_p = \lim_{s\to 0} G(s)H(s)$$

The steady state error for a ramp input; *A/s2 is given by*

$$e_{ss} = \lim_{s\to 0}\left[s\,\frac{\prod_{j=1}^{i=n}(s+p_j)}{\prod_{j=1}^{i=n}(s+p_j)+K\prod_{i=1}^{i=m}(s+z_i)}\,\frac{A}{s^2}\right]$$

$$= \infty$$

Type One System:

The steady state error for a step input; *A/s is given by*

$$e_{ss} = \lim_{s\to 0}\left[s\,\frac{s\prod_{j=1}^{i=n}(s+p_j)}{s\prod_{j=1}^{i=n}(s+p_j)+K\prod_{i=1}^{i=m}(s+z_i)}\,\frac{A}{s}\right]$$

$$= 0$$

The steady state error for a ramp input; *A/s2 is given by*

$$e_{ss} = \lim_{s\to 0}\left[s \frac{s\prod_{j=1}^{i=n}(s+p_j)}{s\prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} \frac{A}{s^2}\right]$$

$$= \frac{\prod_{j=1}^{i=n} p_j}{K\prod_{i=1}^{i=m} z_i} A$$

$$= \frac{A}{K_v} \qquad \text{for } K_v = \frac{K\prod_{i=1}^{i=m} z_i}{\prod_{j=1}^{i=n} p_j}$$

Velocity error constant

$$K_v = \lim_{s\to 0} sG(s)H(s)$$

The steady state error for an acceleration input; *A/s3 is given by*

$$e_{ss} = \lim_{s\to 0}\left[s \frac{s\prod_{j=1}^{i=n}(s+p_j)}{s\prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} \frac{A}{s^3}\right]$$

$$= \infty$$

Type Two System:

The steady state error for a step input; *A/s is given by*

$$e_{ss} = \lim_{s\to 0}\left[s \frac{s^2 \prod_{j=1}^{i=n}(s+p_j)}{s^2 \prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} \frac{A}{s}\right]$$

$$= 0$$

The steady state error for an acceleration input; *A/s3* is given by

$$e_{ss} = \lim_{s\to 0}\left[s \frac{s^2 \prod_{j=1}^{i=n}(s+p_j)}{s^2 \prod_{j=1}^{i=n}(s+p_j) + K\prod_{i=1}^{i=m}(s+z_i)} \frac{A}{s^3}\right]$$

$$= \frac{\prod_{j=1}^{i=n} p_j}{K\prod_{i=1}^{i=m} z_i} A$$

$$= \frac{A}{K_a} \qquad \text{for } K_a = \frac{K\prod_{i=1}^{i=m} z_i}{\prod_{j=1}^{i=n} p_j}$$

Acceleration error constant

$$K_a = \lim_{s\to 0} s^2 G(s)H(s)$$

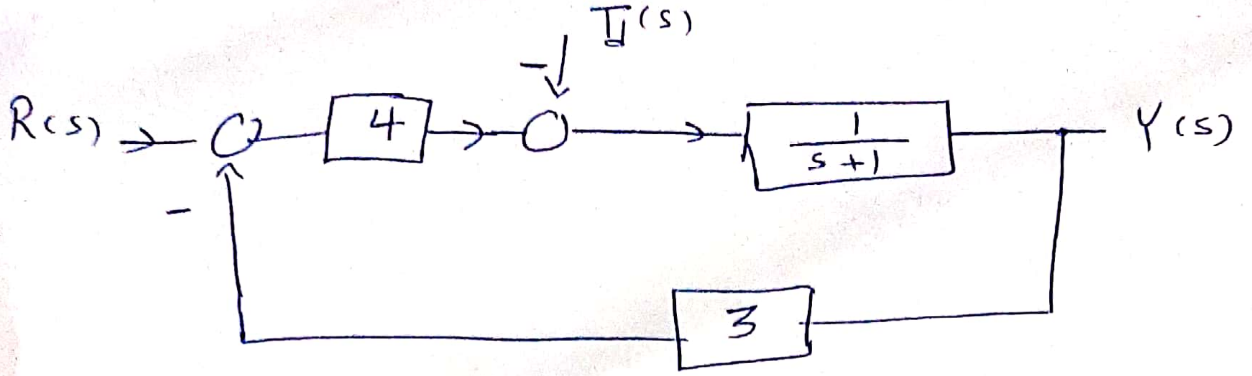
In summary;

$$\text{For } K_p = K_v = K_a = \frac{K \prod_{i=1}^{i=m} z_i}{\prod_{j=1}^{i=n} p_j}$$

TYPE	STEP INPUT $r(t)=A$, $R(s)=A/s$	RAMP INPUT $r(t)=At$, $R(s)=A/s^2$	ACCELERATION INPUT $r(t)=At^2$, $R(s)=A/s^3$
0	$\frac{A}{1+K_p}$	∞	∞
1	0	$\frac{A}{K_v}$	∞
2	0	0	$\frac{A}{K_a}$

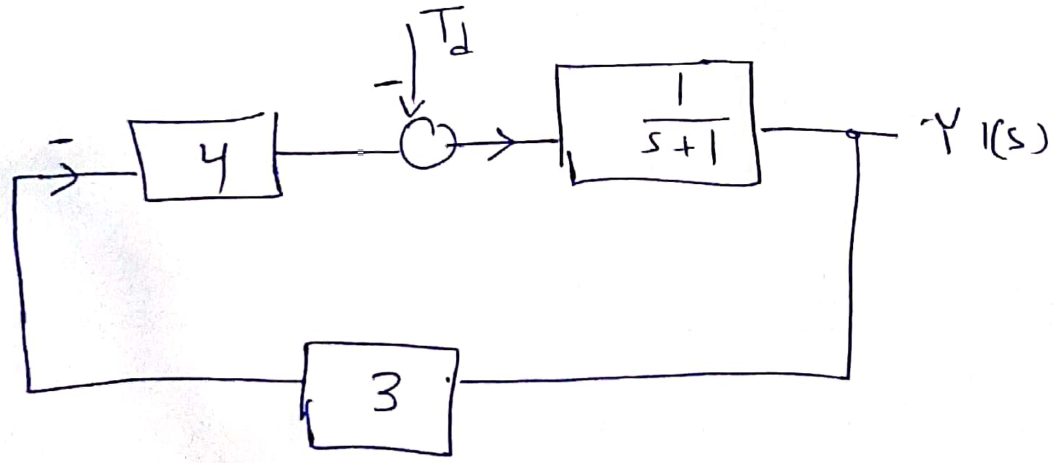
Examples !!

Ex: Find the steady-state error due to disturbance.



Solution: due to disturbance $R(s) = 0$

the block diagram when $R(s) = 0$ will be



→ forward path is $\frac{1}{s+1}$

→ feedback signal is $-3 * 4$

→ $T_d(s)$ is negative → $-T_d(s)$

$$Y(s) = \frac{\frac{1}{s+1}}{1 + \frac{1}{s+1}(12)} \left(-T_d(s)\right)$$

$$Y(s) = \frac{1}{(s+1)+12} \left(-T_d(s)\right)$$

So ∴ $E(s)\Big|_{\text{due to } T_d} = R(s) - Y(s)$

but $R(s) = 0$

$$\Rightarrow E(s) = 0 - Y(s)$$

$$= -\left(\frac{-T_d(s)}{s+1+12}\right)$$

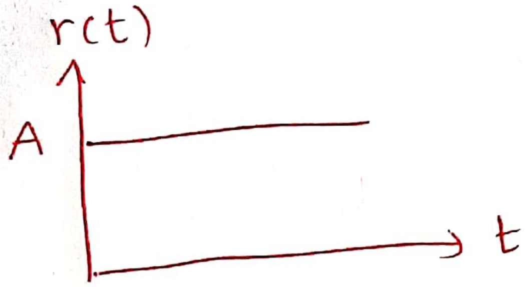
$$E(s) = + \frac{T_d(s)}{s+13}$$

chapter 5: The performance of feedback control system.

→ Test input signals for the time response of control system.

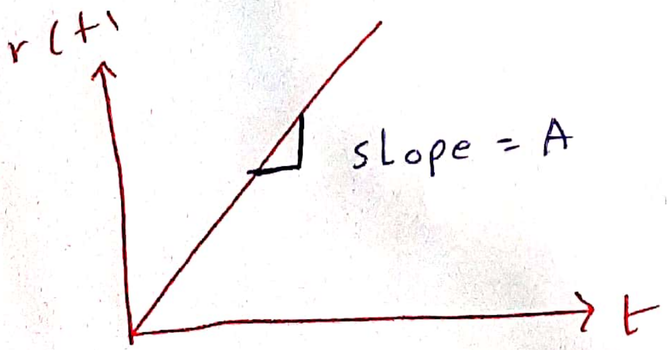
① step input

$R(t) = A \Rightarrow R(s) = \frac{A}{s}$



$$r(t) = \begin{cases} A, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

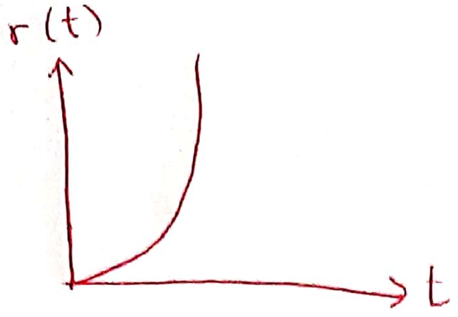
② Ramp input



$$r(t) = \begin{cases} At, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$R(s) = \frac{A}{s^2}$

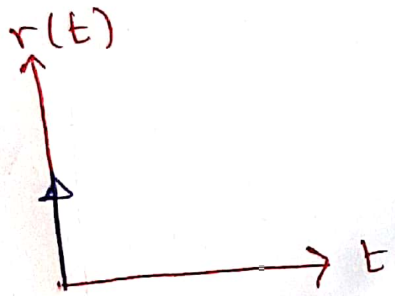
③ parabolic input



$$r(t) = \begin{cases} A t^2 & , t \geq 0 \\ 0 & , t < 0 \end{cases}$$

$$R(s) = \frac{2A}{s^3}$$

④ Unit-impulse input



unit → 1

→ for a unit-impulse input always $R(s) = 1$

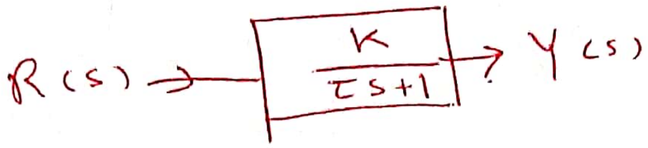
→ remember table 2.3

→ order System Response:

① Zero-order system.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K}{s^0} = K$$

(2) First-order System



$$T(s) = \frac{Y(s)}{R(s)} = \frac{K}{\tau s + 1}$$ ... standard form of first-order transfer function

where : K is the gain of the system

τ is the time constant of the system.

DC - gain $\quad K = \frac{y_{ss}}{R_{ss}}$ ... (1)

$$y(\tau) = 0.63\, y_{ss} \quad \text{--- (2)}$$

→ for a unit- step input $R(s) = \frac{1}{s}$

$$y(t) = K(1 - e^{-t/\tau}) \quad \text{... (3)}$$

note: → For first - order system

$$\frac{Y(s)}{R(s)} = \frac{K}{\tau s+1}$$

wher $R(s) = \frac{1}{s} \Rightarrow y(s) = \frac{1}{s}\left(\frac{k}{\tau s+1}\right)$

take Laplace inverse of $y(s)$

$$y(t) = k(1 - e^{-t/\tau})$$

$$y(s) = \frac{A}{s} + \frac{B}{s+\frac{1}{\tau}} = \frac{k/\tau}{s(s+\frac{1}{\tau})}$$

$$y(t) = A + B e^{-t/\tau}$$

when $s = 0 \Rightarrow A(s+\frac{1}{\tau}) + Bs = \frac{k}{\tau}$

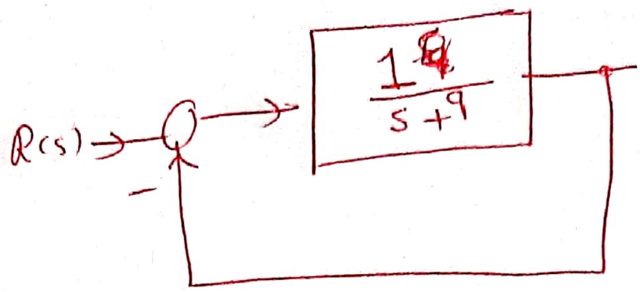
$$\boxed{A = k}$$

$$s = \frac{-1}{\tau} \Rightarrow B = -k$$

$$\Rightarrow y(t) = K - k e^{-t/\tau}$$

$$\boxed{y(t) = k(1 - e^{-t/\tau})}$$

EX 1: For a unit-step input, Find $y(t)$ for system below.



$$\frac{Y(s)}{R(s)} = \frac{\frac{1}{s+9}}{1+\left(\frac{1}{s+9}\right)(1)} = \frac{1}{s+10} = \frac{1}{s+10}$$

standard form $\frac{k}{\tau s+1}$

$$\Rightarrow \frac{\frac{1}{10}}{\frac{s}{10}+\frac{10}{10}} = \frac{0.1}{0.1s+1}$$

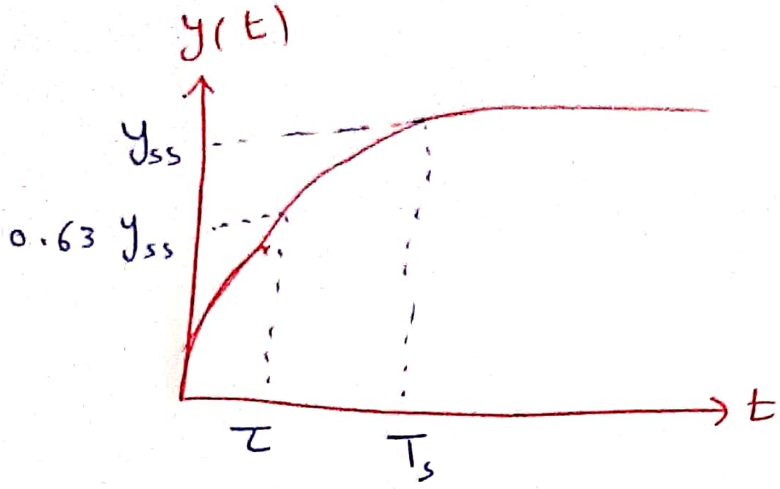
$\Rightarrow \tau = 0.1$ second

$K = 0.1$

$$\therefore y(t) = k(1 - e^{-t/\tau})$$

$$= 0.1(1 - e^{-t/0.1})$$

→ plot first – order response:

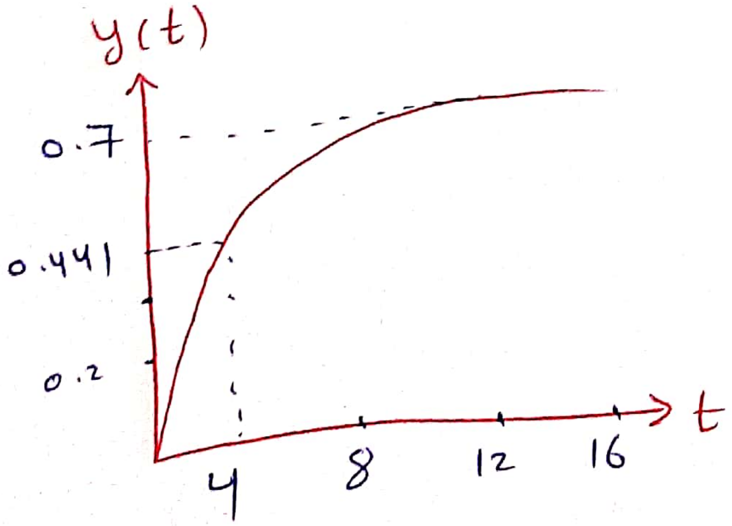


T_s = settling time

$= 4\tau$

T_s = is the time required to reach the steady - state value of the response.

Ex2:- Find $y(t)$ and T_s for the following response and $R(s) = \frac{1}{s}$



Solution:—

$y_{ss} = 0.7$ from plote

$R_{ss} = \lim_{s \to 0} s R(s)$

$= \lim_{s \to 0} s \frac{1}{s} = 1$

then, $k = \frac{y_{ss}}{R_{ss}} = \frac{0.7}{1}$

Ex2: Continue ....

$y(\tau) = 0.63\, y_{ss} = 0.63(0.7) = 0.441$

From plot when $y(t) = 0.441$

$t = 4$ seconds $= \tau$

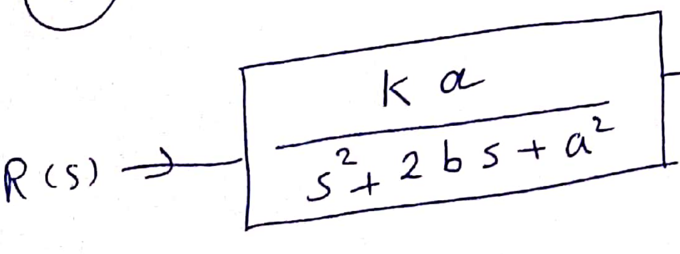
$\Rightarrow y(t) = k(1 - e^{-t/\tau})$

$$y(t) = 0.7(1 - e^{-t/4})$$

$T_s = 4\tau = 4(4) = 16$ seconds.

* * * *

(3) Second-order System:-



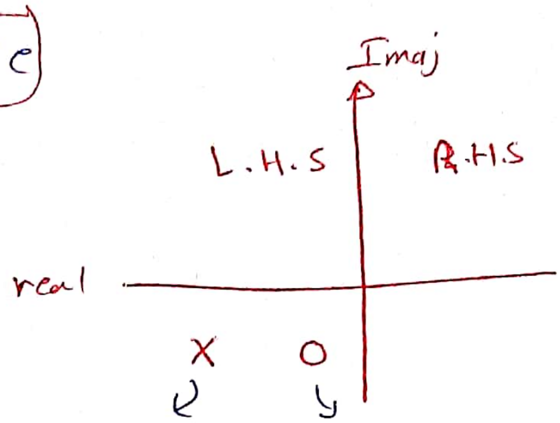
or $\dfrac{k\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

ω_n = natural frequency

ζ = damping ratio

* stabitiy is an important concept.

① The system is stable if all **Poles** are on the Left hand side of **s-plane**



Transfer function أصفار المقام لل ← Pole

zero

↓

أصفار البسط لل Transfer function

* * *

L.H.S = Left hand side

R.H.S = Right hand side

* * *

* * * → damping ration can be determined depnding on s-plane

→ A second-order system can be set into one of the following four cases:

[1] Undamped system ($\zeta = 0$)

ζ : damping ratio $= \frac{c}{2 m \omega_n}$

where c = damping coefficient

m = mass

ω_n = natural frequency

→ For undamped 2^{nd} - order system:

$$T(s) = \frac{K \omega_n^2}{s^2 + \omega_n^2} , \zeta = 0$$

← بعد تعويض قيمة ζ صفر في القانون العام للـ 2^{nd} - order

$$s^2 + \omega_n^2 = 0 \Rightarrow S_{1,2} = \mp j \omega_n$$

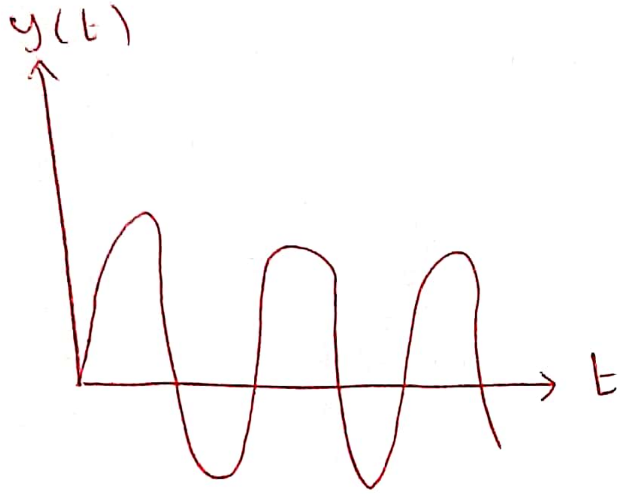
imaginary roots

⇒ y(t) will be sine or cosine

For a unit-step input ($R(s) = \frac{1}{s}$)

$$\frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + \omega_n^2}$$

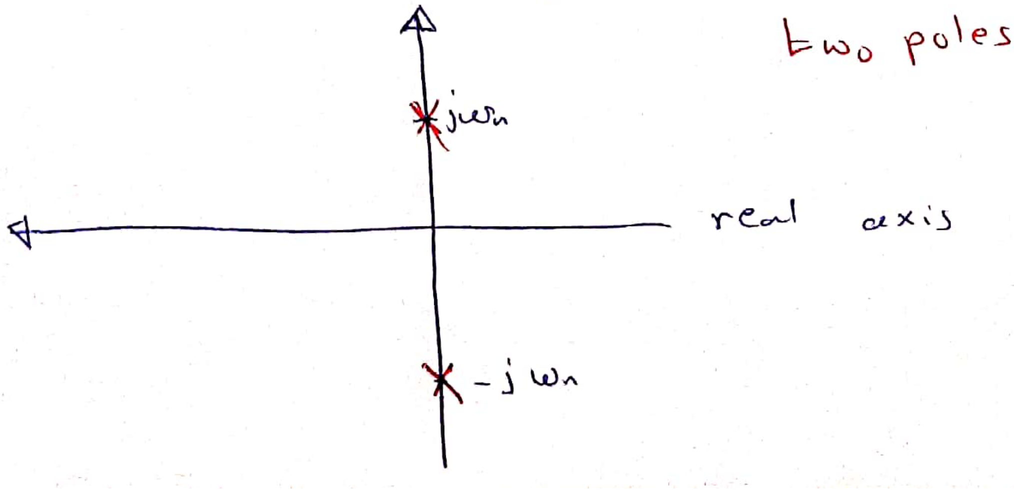
$$y(t) = 1 - \cos(\omega_n t)$$



← y(t) Bounded Input Bounded Output

(stable system)

→ s-plane



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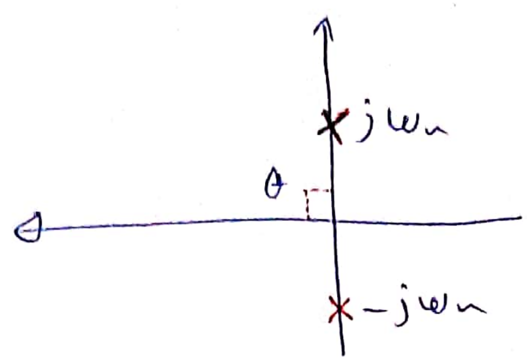
How???

For undamped system



$\zeta = \cos\theta$

$\zeta = \cos(90°)$

$\boxed{\zeta = 0}$

→ θ is measured from negative real axis

2 underdamped - 2nd order system

$\boxed{0 < \zeta < 1}$

$$T(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

⇐ تحليل العبارة التربيعية باستخدام القانون العام

$$s_{1,2} = -\zeta\omega_n \mp j\omega_n\sqrt{1-\zeta^2}$$

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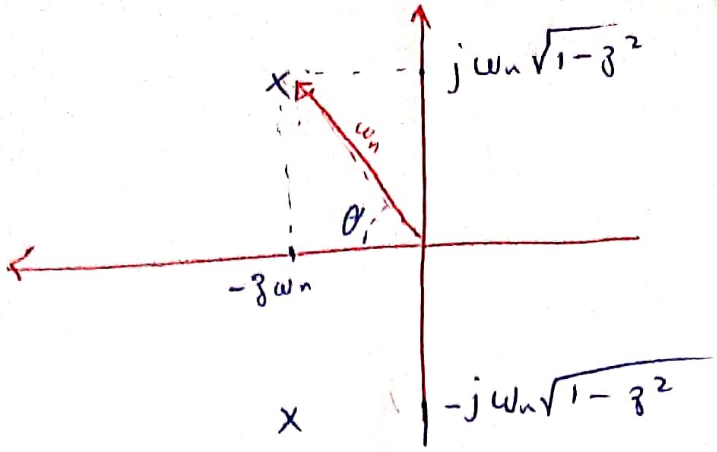
L.H.S R.H.S



$\Rightarrow \zeta = \cos\theta$

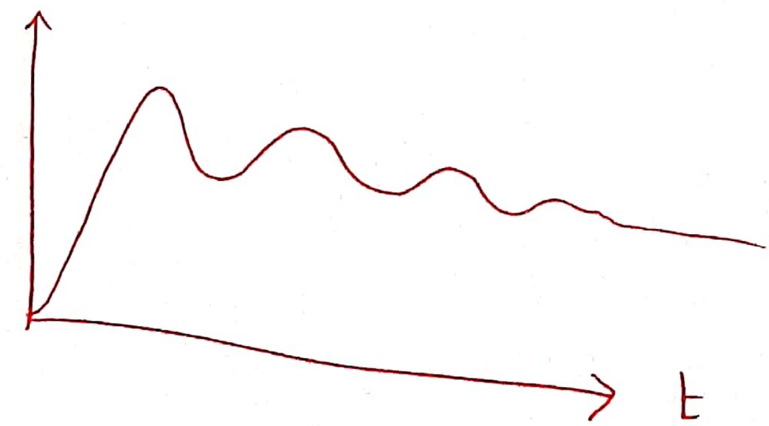
$0 < \theta < 90^\circ$

$\Rightarrow 0 < \zeta < 1$

for $R(s) = \frac{1}{s}$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \cos^{-1}(\zeta))$$

ω_d = damped frequency $= \omega_n\sqrt{1-\zeta^2}$

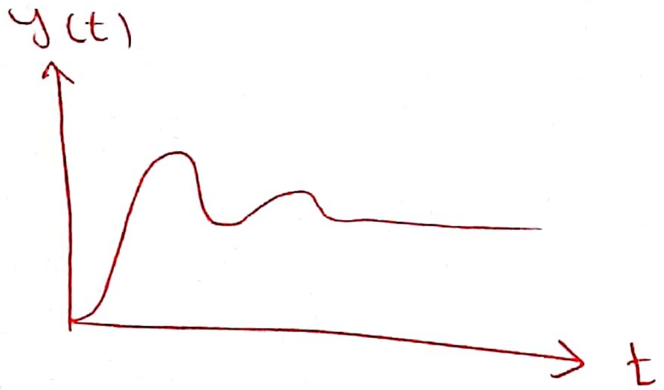


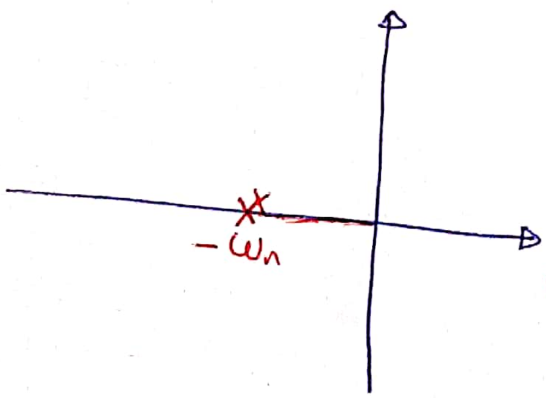
3 critically damped ($\zeta = 1$)

$$T(s) = \frac{k\,\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$S_{1,2} = -\omega_n$ (repeated and real roots)

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$





$\zeta = \cos\theta$

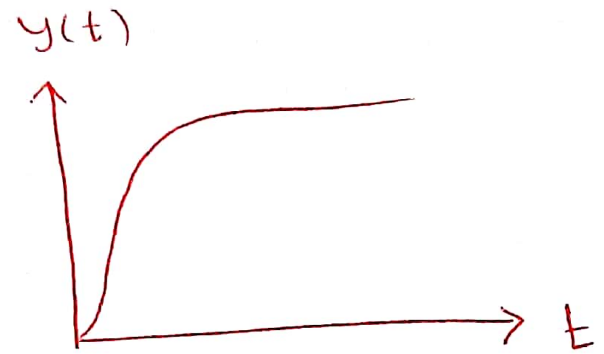
$\zeta = \cos(0)$

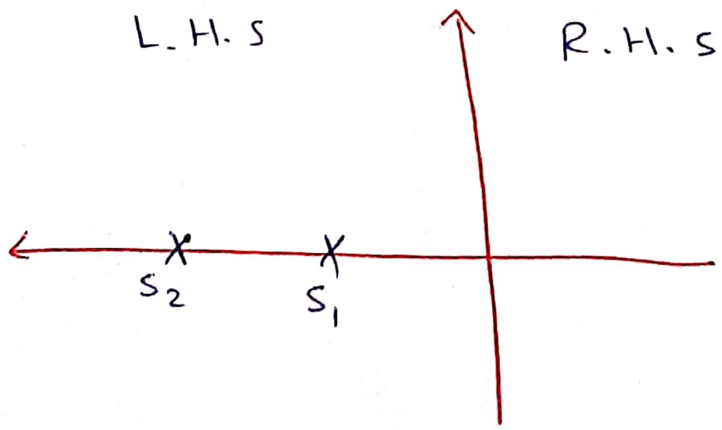
$\boxed{\zeta = 1}$

4 overdamped system ($\zeta > 1$)

$$S_{1,2} = -\zeta \omega_n \mp \omega_n \sqrt{\zeta^2 - 1}$$

$$y(t) = 1 + \frac{\omega_n}{\sqrt{\zeta^2 - 1}} \left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right]$$





Ex 1: For a unit-step input, Find

(1) $y(t) = ??$

[a] $T(s) = \dfrac{9}{s^2 + \underset{= \omega_n^2}{9}}$

$\omega_n^2 = 9 \Rightarrow \omega_n = +3$ rad/s

$\omega_n = -3$ X canceled

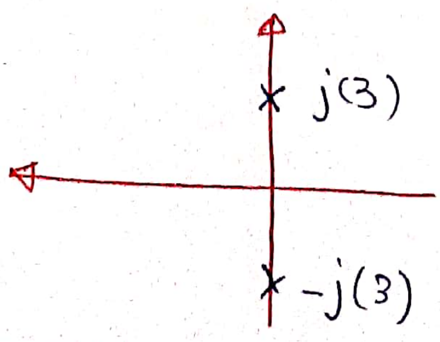
$K \omega_n^2 = 9$

$K = 1$

S ملاحظ ← $2\zeta\omega_n = 0$

$\zeta = 0$ → undamped ~~...~~ response

$\Rightarrow$ $y(t) = 1 - \cos 3t$



⟹ Marginally stable

b $T(s) = \dfrac{9}{s^2 + 2s + 9}$

$\omega_n = \sqrt{9} = 3$ rad/s

$2\zeta\omega_n = 2$

$2\zeta(3) = 2 \Rightarrow \boxed{\zeta = \frac{1}{3} < 1}$

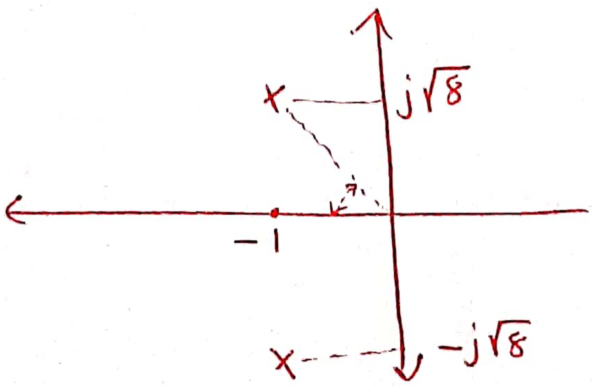
underdamped response

$K\omega_n^2 = 9 \Rightarrow \boxed{K = 1}$

$y(t) = 1 - 1.06\, e^{-t} \cos(\sqrt{8}\, t - 19.47)$

— OR —

$y(t) = 1 - 1.06\, e^{-t} \sin(\sqrt{8}\, t - 70.5)$



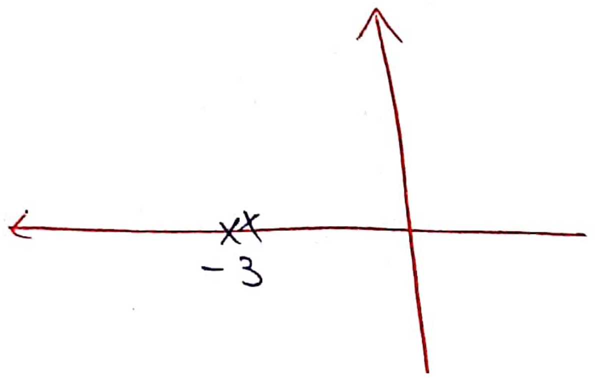
[c] $T(s) = \dfrac{9}{s^2 + 6s + 9}$

$\omega_r = 3$ rad/s and $2\zeta\omega_n = 6$

$2\zeta(3) = 6$

critically damped ← $\boxed{\zeta = 1}$

$\Rightarrow \boxed{y(t) = 1 - e^{-3t} - 3te^{-3t}}$



[d] $T(s) = \dfrac{9}{s^2 + 9s + 9}$

$\omega_r = 3$ rad/s and $2\zeta\omega_n = 9$

$2(3)\zeta = 9$

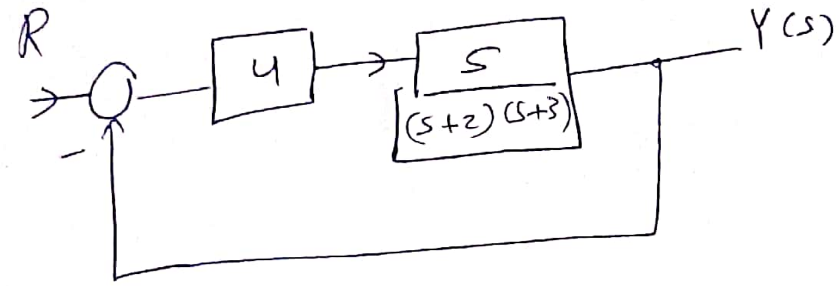
overdamped response ← $\zeta = 1.5 > 1$

$$y(t) = 1 + \frac{3}{\sqrt{1.5^2 - 1}} \left[\frac{e^{-s_1}}{s_1} - \frac{e^{-s_2}}{s_2} \right]$$

$$s_{1,2} = -(1.5)(3) \mp 3\sqrt{1.5^2 - 1}$$

* * *

Ex2 :



Find $y(t) = ?$ for a unit-step input

Sol:-

$$T(s) = \frac{\frac{20}{s^2 + 5s + 6}}{1 + \frac{20}{s^2 + 5s + 6}} = \frac{20}{s^2 + 5s + 26}$$

$\omega_n = \sqrt{26}$ and $2\zeta\omega_n = 5$

$2\zeta(\sqrt{26}) = 5$

$\zeta < 1 \Rightarrow$ underdamped

Ex3: Find system response ??

$$T(s) = \frac{4}{2s^2 + 8s + 8}$$

should be 1 ← ($2s^2$)

معامل s^2 ← 1

⇒ نقسم المعادلة على 2

ليصبح معامل $s^2 = 1$

$$\Rightarrow \quad T(s) = \frac{2}{s^2 + 4s + 4}$$

$\omega_n = \sqrt{4} = 2$ rad/s

$2\zeta\omega_n = 4 \Rightarrow 2(2)\zeta = 4$

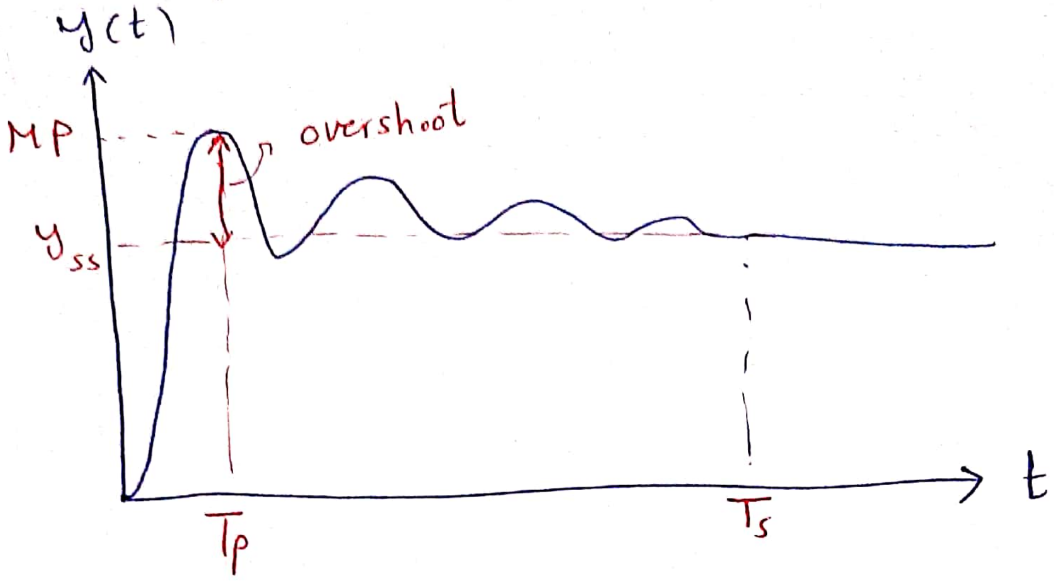
critically damped ← $\boxed{\zeta = 1}$

$k\,\omega_n^2 = 2 \Rightarrow k(4) = 2$

$\boxed{k = \frac{1}{2}}$

important ← $k = \dfrac{y_{ss}}{R_{ss}}$

* The performance of Underdamped Response



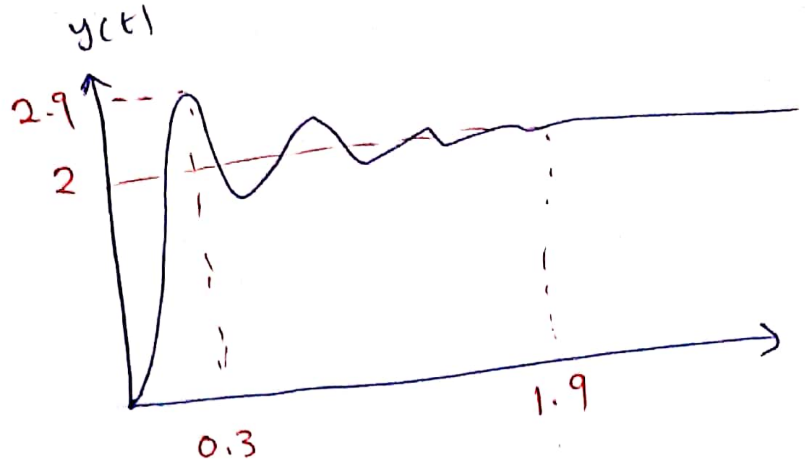
Settling time $= T_s = \dfrac{4}{\zeta \omega_n}$ using 2% criterion

$T_s = \dfrac{3}{\zeta \omega_n}$ Using 5% criterion

$T_p = \dfrac{\pi}{\omega_n \sqrt{1-\zeta^2}}$ peak time

$$MP\% = \frac{y(T_p) - y(\infty)}{y(\infty)} \times 100\% = 100\, e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Ex1: For a unit-step input, Find $T(s) = ?$



Using 2% criterion

① $\zeta = ??$

$$M_P = \frac{y(T_p) - y(\infty)}{y(\infty)} \times 100\% = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$= \frac{2.9 - 2}{2} \times 100\% = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$\Rightarrow$ $\zeta = 0.246$

$0 < \zeta < 1 \rightarrow$ underdamped

② $T_p = 0.3$ seconds

(3) $T_s = \frac{4}{\zeta \omega_n} = 1.9 = \frac{4}{0.246(\omega_n)}$

$$\boxed{\omega_n = 7.87 \text{ rad/s}}$$

(4) $T(s) = \frac{k\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$k = \frac{y_{ss}}{R_{ss}} = \frac{2}{1} = 2$

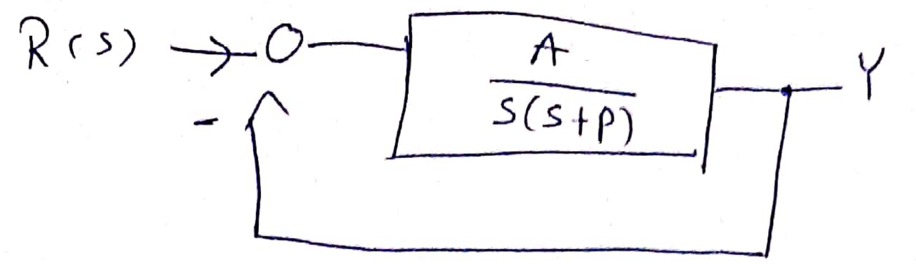
$y_{ss} \rightarrow$ from plot $= 2$

$R_{ss} \rightarrow$ (unit - step input $= \frac{1}{s}$)

$R_{ss} = \lim_{s \to 0} s\,R(s) = \lim_{s \to 0} s\left(\frac{1}{s}\right)$

$= 1$

Ex2:



Select A and P where

$T_s < 4$ sec Using 2% criterion

$\zeta = 0.707$

Sol:

$$\frac{Y(s)}{R(s)} = \frac{\frac{A}{s(s+P)}}{1 + \frac{A}{s(s+P)}}$$

$$T(s) = \frac{A}{s^2 + Ps + A}$$

$$\omega_n^2 = A \quad \ldots (1)$$

$$2\zeta\omega_n = P \quad \ldots (2)$$

$$T_s < 4 \Rightarrow \frac{4}{0.707\,\omega_n} < 4$$

$$1 < 0.707\,\omega_n$$

$$\frac{1}{0.707} < \omega_n$$

$$\boxed{1.414 < \omega_n}$$

$$\Rightarrow \boxed{A = 2} \quad \text{from eq (1)}$$

also,

$$2(0.707)(1.414) = P$$

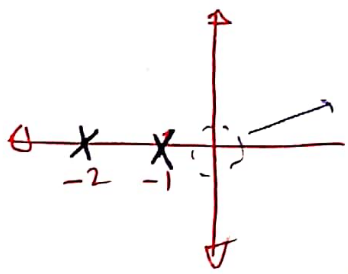
$$\boxed{2 = P} \quad \text{from eq (2)}$$

→ Steady-state error analysis :-

System type:

① Type zero system

e.g: $\frac{1}{(s+1)(s+2)}$



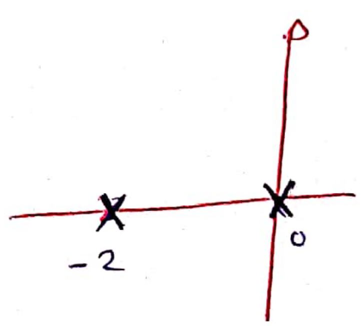
s = 0 ← لايوجد

② Type one - system.

e.g: $\frac{1}{s(s+2)}$

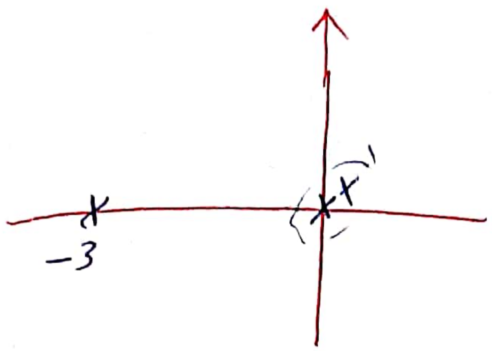
s = 0 عند pole يوجد ←

← type 1



③ Type Two System

e.g.: $\dfrac{1}{s^2(s+3)}$



Two poles يوجد at $s=0$

Type	input Step A/s	Ramp A/s^2	Acceleraton A/s^3
$N=0$	$\dfrac{A}{1+K_p}$	∞	∞
$N=1$	0	A/K_v	∞
$N=2$	0	0	$\dfrac{A}{K_a}$

note: step input $\Rightarrow r(t) = A t^{0}$

when $N=0 \Rightarrow e_{ss} = \dfrac{A}{1+K_p}$

but if $r(t) = A\,t^{0}$

and $N = 1$

$\Rightarrow 0 < 1 \Rightarrow e_{ss} = 0$

also

$N = 2 > t^{0} \Rightarrow e_{ss} = 0$

e.g : if $r(t) = A\,t^{1} \rightarrow$ Ramp

and $N = 0 \Rightarrow N < t^{1}$

$0 < 1$

$\Rightarrow e_{ss} = \infty$

but $N = 2$

$2 > t^{1}$

$e_{ss} = 0$

Remember

$$e_{ss} = \lim_{s \to 0} s\,E(s)$$

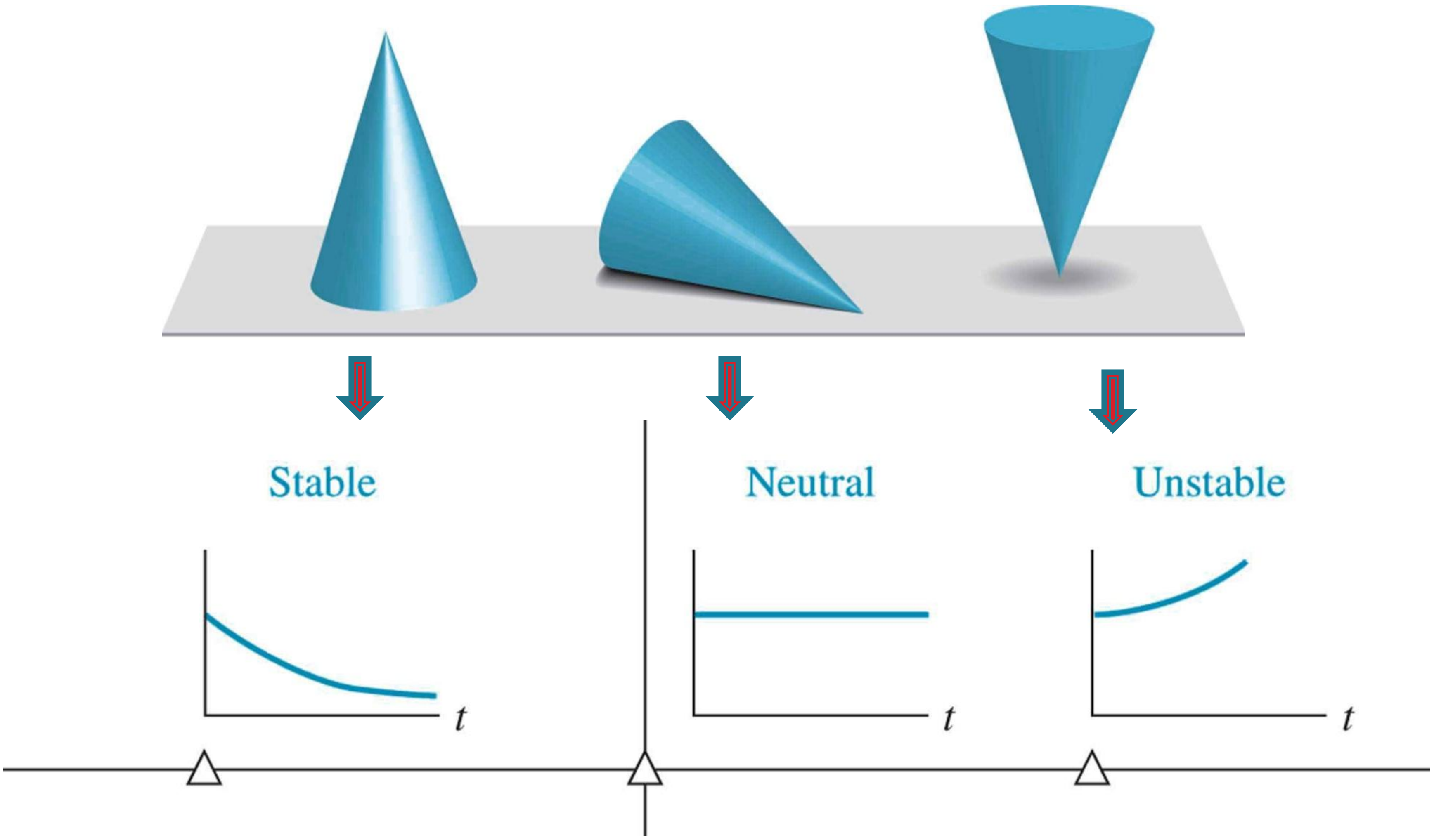
and

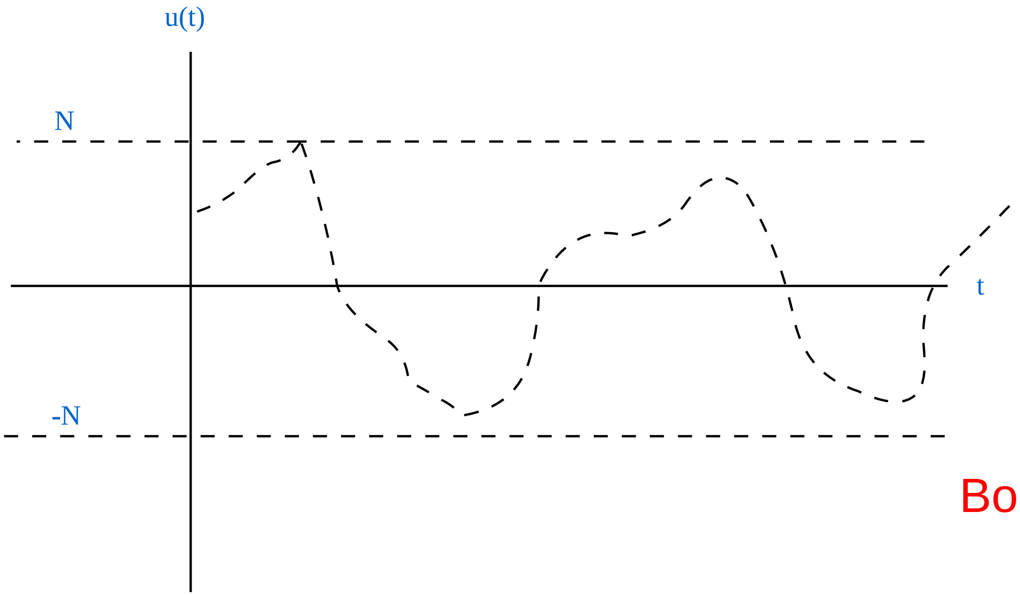
$$E(s) = R(s) - Y(s)$$

Ch.6 The Stability of linear Feedback Systems

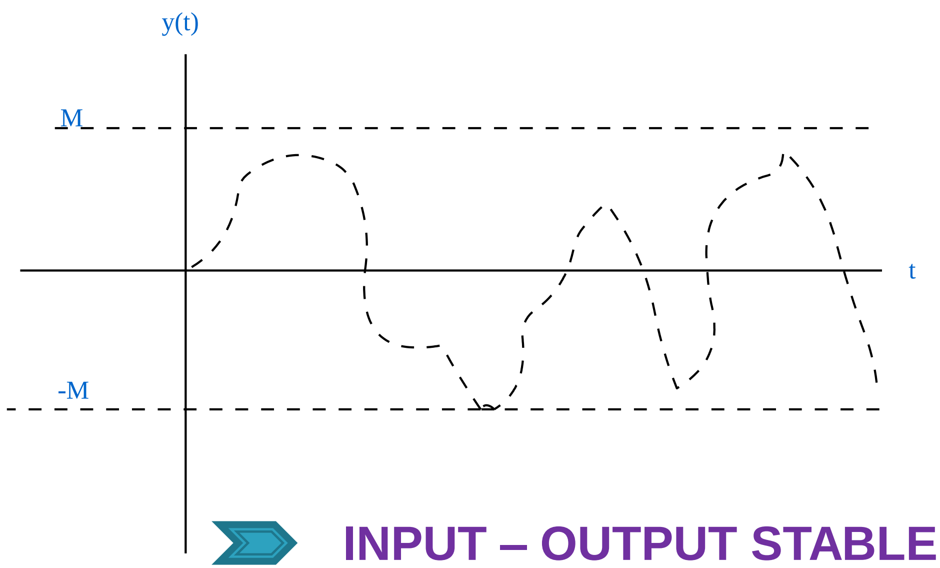
- Stability of closed-loop feedback systems is central to control system design.
- A stable system should exhibit a bounded output if the corresponding input is bounded. This is known as bounded-input-bounded-output (BIBO) stability.
- The stability of a feedback system is directly related to the location of the roots of the characteristic equation of the system transfer function.
- The Routh-Hurwitz method is introduced as a useful tool for assessing system stability.
- The technique allows us to compute the number of roots of the characteristic equation in the right half plane without actually computing the values of the roots.

A stable system is a dynamic system with a bounded response to a bounded input.





Bounded input signal

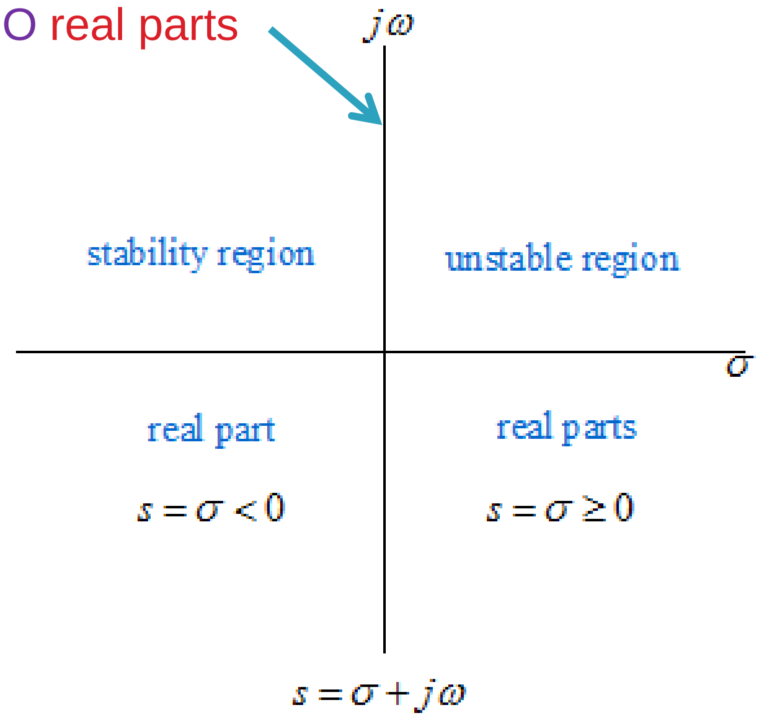


Bounded output signal

- **INPUT – OUTPUT STABLE if and only if every bounded input produces a bounded output**
- **A system is said to be unstable if it's not BIBO stable**

Input/output stability is characterized by the location of the poles of the system closed loop transfer function (i.e. roots of the characteristic equation).

Undammped critically (marginally) stable
System with poles of ZERO real parts



ROUTH – HURWITZ STABILITY CRITERION

- Here we present a method for investigating the stability of high order systems without having to obtain a complete time response or determining the precise position of the poles in s-plane.

- Assume the following general closed loop transfer function of a system :

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{\alpha_0 s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n}$$

- The denominator (Δ) can always be expressed as a polynomial of *nth order as follows:*

$$\Delta = \alpha_0 s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n$$

- If the roots of Δ are positive, that means that the system is unstable. ➡ In this case the polynomial Δ will have alternating signs

- If the roots of Δ are negative, that means that the system is stable and this case the polynomial will have all positive signs.

- If all the coefficients of the polynomial are positive, this does not mean that the system is stable

- Now, the following conditions are **necessary** (but not sufficient) for the system to be stable:

  1- All the coefficients $\alpha_o, \alpha_1, \alpha_2, \alpha_3 \ldots, \alpha_n$ have the same sign.

  2- No coefficient is zero.

➡ **Build the Routh-Hurwitz array from Δ**

Order the coefficient polynomial $a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$ as shown below:

$$
\begin{array}{c|ccccc}
s^n & a_0 & a_2 & a_4 & \bullet \\
s^{n-1} & a_1 & a_3 & a_5 & \bullet \\
s^{n-2} & b_1 & b_2 & b_3 & \bullet \\
s^{n-3} & c_1 & c_2 & c_3 & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
$$

The terms b_n, c_n, ... are given by the expressions shown below:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \qquad c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}, \qquad c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

- Routh-Hurwitz stability criterion states that the number of roots of the characteristic equation that have positive real roots is equal to the number of changes in sign of the first column of the ordered array.
- Now it is sufficient condition for the system to be stable if there is no changes in sign of the first column of the Routh-Hurwitz array.

Examples !!

chapter 6: The stability of Linear feedback System

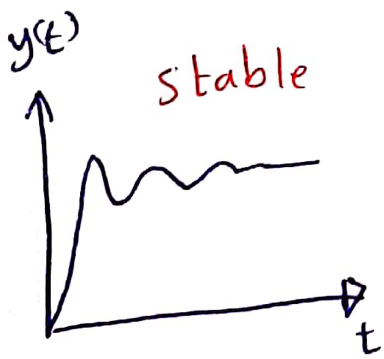
→ Stability : Bounded input Bounded output.

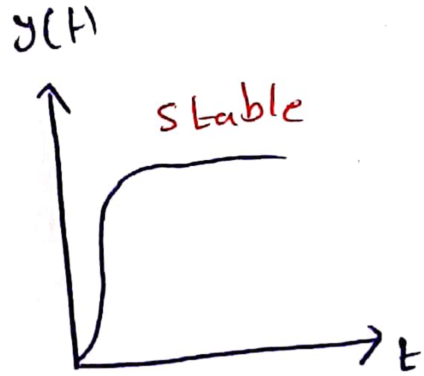
① stable: all poles on left hand side of s-plane

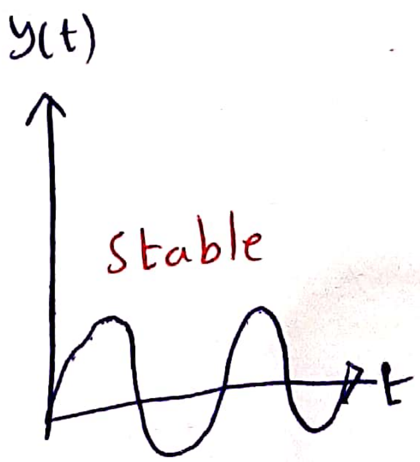
② unstable: one or more from the poles on the right hand side of the s-plane

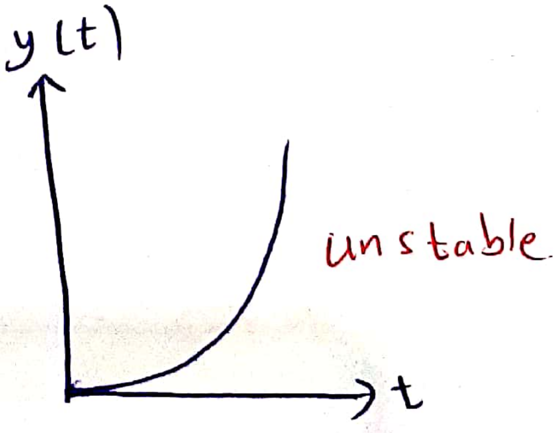
③ Maginally stable: There is poles on $\mp j\omega$ axis

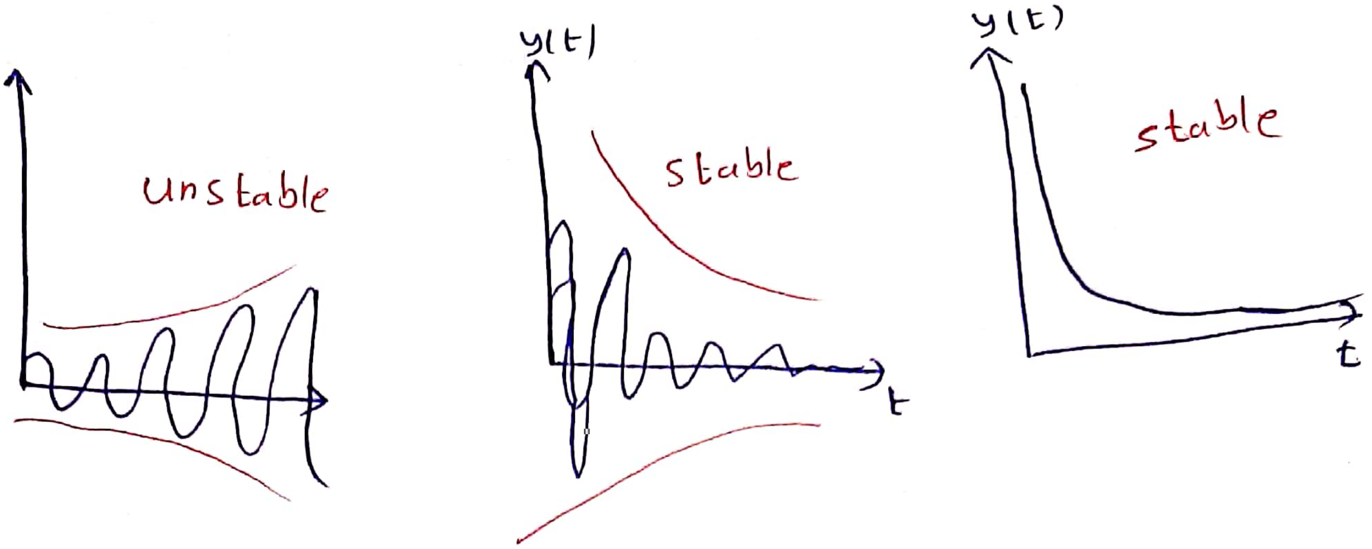
e.g:



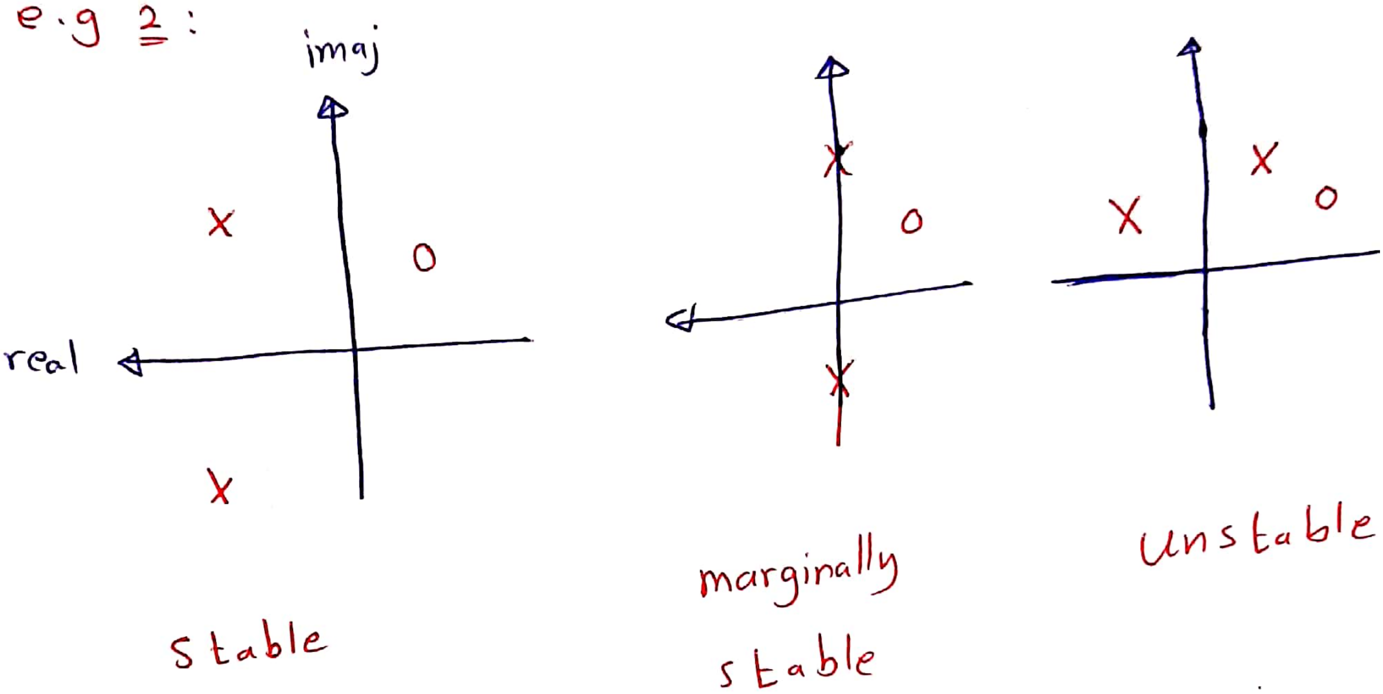








e.g 2:



* Routh-Hurwitz stability criterion:

→ for $T(s) = \frac{Z(s)}{P(s)} = \frac{\text{zeros}}{\text{poles}}$

$$p(s) = a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + \cdots + a_N$$

s^N	a_0	a_2	a_4 ......
s^{N-1}	a_1	a_3	a_5 .....
s^{N-2}	b_1	b_2	b_3 ....
⋮	c_1	c_2	c_3 ----
⋮	⋮	⋮	⋮
s^0	a_N	0	0

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

ex1: $P(s) = s^3 + 2s^2 + 4s + K$

s^3	1	4	0
s^2	2	K	0
s^1	$\frac{8-K}{2}$	0	0
s^0	K	0	0

$$b_1 = \frac{2*4 - 1*K}{2} = \frac{8-K}{2}$$

$$b_2 = \frac{2*0 - 1*0}{2} = 0$$

$$c_1 = \frac{\left(\frac{8-K}{2}\right)(K) - 2*0}{\frac{8-K}{2}}$$

$$= K$$

to be stable → all of the first coloum ~~must be~~ should have the same sign and doesn't equal to zero.

⇒
$1 > 0$
$2 > 0$
$\frac{8-K}{2} > 0 \Rightarrow K < 8$
$K > 0$

$0 < K < 8$

the system is stable in this range

Ex2: $P(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

s^5	1	2	11	0
s^4	2	4	10	0
s^3	~~0~~ ϵ	6	0	0
s^2	$-\frac{12}{\epsilon}$	10	0	0
s^1	6	0	0	0
s^0	10	0	0	0

$$b_1 = \frac{2*2 - 4*1}{2} = 0$$

$$b_2 = \frac{2*11 - 10*1}{2} = 6$$

in the third row, the first element is zero but not all row is zero

$\Rightarrow$ replace the zero with positive small value ϵ

$$\therefore c_1 = \frac{4*\epsilon - 12}{\epsilon} = -\frac{12}{\epsilon}$$

(zero: $4*\epsilon$)

$$c_2 = \frac{10*\epsilon - 2*0}{\epsilon} = \frac{10\epsilon}{\epsilon}$$

(zero: $2*0$)

$= 10$

$$d_1 = \frac{-\frac{12}{\epsilon}(6) - 10\epsilon}{-\frac{12}{\epsilon}}$$

(zero: 10ϵ)

$= 6$

$$e_1 = \frac{6*10 - \frac{-12}{\epsilon}(0)}{6}$$

$= 10$

$\therefore$ The first coloum has different sign (+ and −)

so, the system is unstable

The system has two sign changes

→ There are two poles on the R.H.S of the s-plane

* * * * *

Ex 3: $p(s) = s^4 + s^3 + s^2 + s + K$

s^4	1	1	K
s^3	1	1	0
s^2	~~0~~ ϵ	K	0
s^1	$-\frac{K}{\epsilon}$	0	0
s^0	K	0	0

$$b_1 = \frac{1\times1 - 1\times1}{1} = 0$$

$$b_2 = \frac{1\times k - 1\times 0}{1} = k$$

$$c_1 = \frac{1\times\epsilon - 1\times k}{\epsilon} = \frac{-k}{\epsilon}$$

(zero)

$$d_1 = \frac{-\frac{k}{\epsilon}(k) - \epsilon\times 0}{-\frac{k}{\epsilon}} = k$$

(zero)

→ There are two sign changes

∴ " " " poles on the R.H.S

and the system unstable for all values of K

$\frac{-k}{\epsilon} > 0 \Rightarrow$ no

$k > 0$

$\Rightarrow$ what is the rang of k that make the system marginall stable?

System is unstable for all values of k

Ex 4: $P(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$

s^5	1	2	1	0
s^4	1	2	1	0
s^3				
s^2				
s^1				
s^0				

$b_1 = \frac{2\times1 - 1\times2}{1} = 0$

$b_2 = \frac{1\times1 - 1\times1}{1} = 0$

~~The first row~~

all elements in the third row are zero

$\Rightarrow$

all values of k

Ex 4: $P(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$ / Fifth-order

s^5	1	2	1	0
s^4	1	2	1	0
s^3	~~0~~ 4	~~0~~ 4	0	0
s^2	1	1	0	0
s^1	~~0~~ 2	~~0~~ 0	0	0
s^0	1	0	0	0

$$b_1 = \frac{2\times1 - 1\times2}{1} = 0$$

$$b_2 = \frac{1\times1 - 1\times1}{1} = 0$$

~~The first row~~

all elements in the third row are zero

$\Rightarrow$

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$s^4 + 2s^2 + 1$

$4s^3 + 4s$

$c_1 = \frac{4*2 - 1*4}{4} = 1$, $c_2 = \frac{4*1 - 1*0}{4} = 1$

$d_1 = \frac{4*1 - 1\times4}{1} = 0$, $d_2 = 0$

$\Rightarrow s^2 + 1$

$2s$

$e_1 = \frac{2*1 - 1\times0}{2} = 1$

The system is stable.

Ex5: $P(s) = s^4 + 2s^3 - 2s - 1$ / Fourth-order

s^4	1	0	-1
s^3	2	-2	0
s^2	1	-1	0
s^1	~~0~~ 2	0	0
s^0	-1	0	0

$s^2 - 1$

$2s$

$$b_1 = \frac{2\times 0 - 1\times -2}{2} = 1$$

$$b_2 = \frac{2(-1) - (1\times 0)}{2} = -1$$

$$c_1 = \frac{1(-2) - (2(-1))}{1}$$

$$c_1 = 0$$

$$c_2 = \frac{1\times 0 - 2(0)}{1} = 0$$

$$d_1 = \frac{2(-1) - (1\times 0)}{2}$$

$$= -1$$

The system is unstable

one sign change (unstable roots)

1 pole on R.H.S of s-plane

3 poles " L.H.S " "

Ex 6: $P(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$

s^5	1	4	3	0
s^4	1	24	63	0
s^3	-20	-60	0	0
s^2	21	63	0	0
s^1	~~0~~ 42	0	0	0
s^0	63 (e_1)	0	0	0

The system is unstable

There are tow sign changes ⇒

There are two unstable roots

$$b_1 = \frac{1\times4 - 1\times24}{1} = -20 \quad , \quad b_2 = \frac{1\times3 - 1\times63}{1} = -60$$

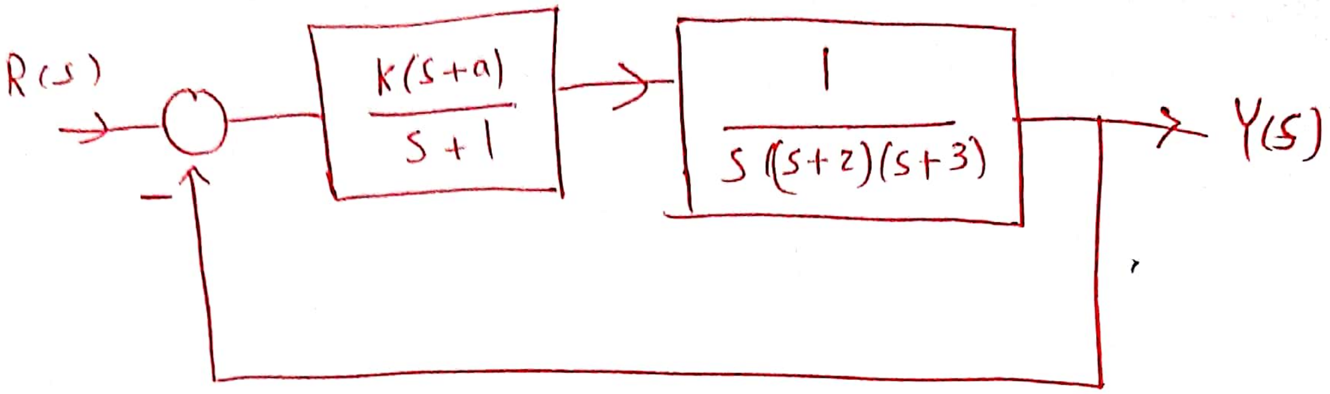
$$c_1 = \frac{-20(24) - (1 * -60)}{-20} = 21 \quad , \quad c_2 = \frac{-20(63) - 0}{-20} = 63$$

$$d_1 = \frac{21(-60) - (-20)(63)}{21} = 0 \quad , \quad d_2 = 0$$

$\therefore \; 21s^2 + 63 \Rightarrow 42s$

$$e_1 = \frac{42 * 63 - 21 * 0}{42} = 63$$

Ex 7: Determine the range of K and a for the system is stable?



$$\frac{Y(s)}{R(s)} = \frac{\dfrac{K(s+a)}{s(s+2)(s+3)(s+1)}}{1 + \dfrac{K(s+a)}{s(s+1)(s+2)(s+3)}}$$

$$\frac{Y(s)}{R(s)} = \frac{K(s+a)}{s(s+1)(s+2)(s+3) + K(s+a)}$$

The system has four poles and one zero

$$Z(s) = K(s+a)$$

$$P(s) = s^4 + 6s^3 + 11s^2 + (K+6)s + Ka$$

s^4	1	11	$K \cdot a$
s^3	6	$K+6$	0
s^2	$\frac{60-K}{6}$	$K \cdot a$	0
s^1	c_1	0	0
s^0	$K \cdot a$	0	0

$$c_1 = \frac{\left(\frac{60-K}{6}\right)(K+6) - 6Ka}{\frac{60-K}{6}}$$

* * *

To be stable →

- $\frac{60-K}{6} > 0$
- $c_1 > 0$

P1:

$$\frac{Y(s)}{R(s)} = \frac{\dfrac{K}{s^2(s+P)}}{1+\dfrac{K}{s^2(s+P)}} = \frac{K}{s^2(s+P)+K}$$

$\Rightarrow P(s) = s^3 + Ps^2 + K$

s^3	1	0	0
s^2	P	K	0
s^1	$-K$	0	0
s^0	K	0	0

$\Rightarrow$ $P>0$, $-K>0$, $K>0$?

The system is unstable for all values of K

* type two system
* third-order system.

Ph Ph

P_2 :

(a) $$\frac{Y(s)}{R(s)} = \frac{K}{(s+5)(s+2)^2 + K}$$

(b) system type (zero), third-order

$$P(s) = s^3 + 9s^2 + 24s + 20 + K$$

s^3	1	24	0
s^2	9	20+K	0
s^1	$\frac{196-K}{9}$	0	0
s^0	20+K		

$$\frac{196-K}{9} > 0$$

$$20+K > 0$$

$\therefore$ $K > -20$ and $K < 196$

$$0 < K < 196$$

(c) $R(s) = \frac{R}{s}$ $\Rightarrow$ $e_{ss} = \lim_{s \to 0} s\,E(s)$

$$= \lim_{s \to 0} s \frac{R}{s} \left(\frac{(s+5)(s+2)^2}{(s+5)(s+2)^2 + K} \right)$$

$$= \frac{20R}{20+K}$$

(d) Ramp $\Rightarrow$ $R(s) = \frac{R}{s^2}$ $\Rightarrow$ $e_{ss} = \infty$

P3 : $P(s) = s^6 + s^5 + 5s^4 + s^3 + 2s^2 - 2s - 8$

s^6	1	5	2	-8
s^5	1	1	-2	0
s^4	4	4	-8	0
s^3	16	8	0	0
s^2	2	-8	0	0
s^1	72	0	0	0
s^0	-8	0	0	0

s^4 row → $4s^4 + 4s^2 - 8$

s^3 row → $16s^3 + 8s$

The system is <u>unstable</u>

P4

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{(s+1)(s+2)}}{1 + \dfrac{K}{(s+1)(s+2)(s+3)}}$$

$$\frac{C(s)}{R(s)} = \frac{K(s+3)}{(s+1)(s+2)(s+3) + K}$$

$$P(s) = s^3 + 6s^2 + 11s + 6 + K$$

s^3	1	11	0
s^2	6	$6+K$	0
s^1	$\frac{60-K}{6}$	0	0
s^0	K	0	0

$K > 0$

$\frac{60-K}{6} > 0$

$K < 60$

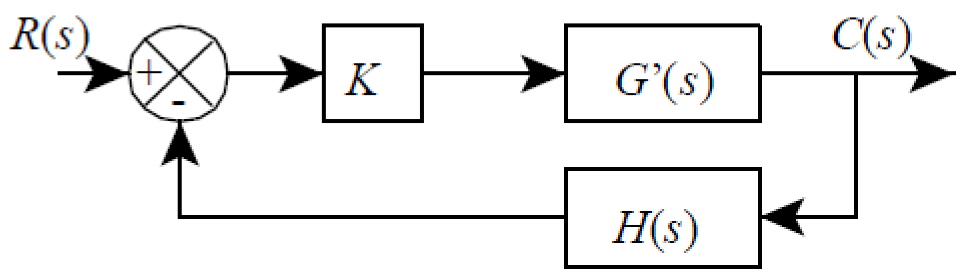
$0 < K < 60$

Ch.7 The Root Locus Method

- So far in the studies of control systems the role of the characteristic equation polynomial in determining the behavior of the system has been highlighted.

- The roots of that polynomial are the poles of the control system, and their locations in the complex *s-plane reveal information about the stability and the performance of the system.*

- Consider the system shown below:



The transfer function of the system is given by:

$$\frac{C(s)}{R(s)} = \frac{K\,G'(s)}{1+K\,G'(s)H(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The loop gain of the system can be expressed as a numerator polynomial over a denominator polynomial as follows:

$$G(s)H(s) = K\frac{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots\cdots + a_{n-1}s + a_n}{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots\cdots + b_{n-1}s + b_n}$$

The characteristic equation is then given by:

$$1 + G(s)H(s) = 0 = 1 + K\frac{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots\cdots + a_{n-1}s + a_n}{b_0 s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots\cdots + b_{n-1}s + b_n}$$

As an example consider that the system in the previous figure have the following loop gain:

$$G(s)H(s) = \frac{K}{(s+1)(s+2)} = \frac{K}{s^2+3s+2}$$

The characteristic equation for the system is given by:

$$\Delta = 1 + G(s)H(s) = 1 + \frac{K}{s^2+3s+2} = \frac{s^2+3s+(2+K)}{s^2+3s+2} = 0$$

The numerator is in fact the denominator of the closed loop system TF. The roots of this polynomial are the closed loop poles of the system.

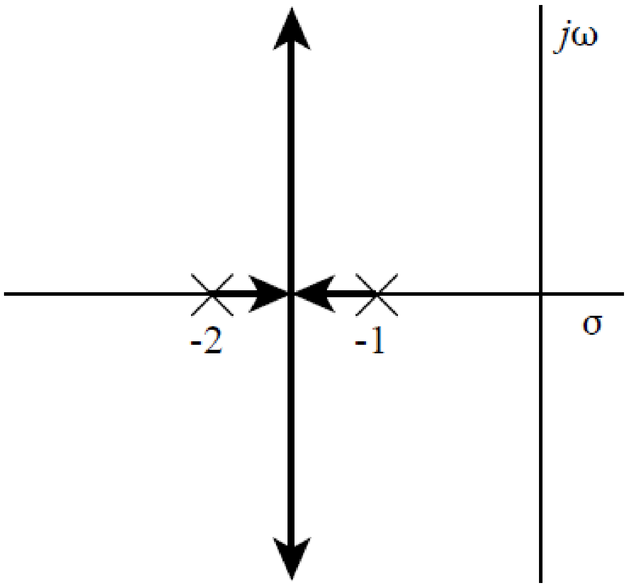
As *K* varies between zero and infinity, the closed loop poles of the system are changing and can be calculated as follows:

K	s_1	s_2
0	-2	-1
0.25	- 1.5	- 1.5
0.5	- 1.5 -j 0.5	- 1.5 + j 0.5
1	- 1.5 - j 0.866	- 1.5 + j 0.866
2	- 1.5 -j 1.3229	- 1.5 + j 1.3229
4	- 1.5 -j 1.9365	- 1.5 + j 1.9365
10	- 1.5 -j 3.1225	- 1.5 + j 3.1225
100	- 1.5 -j 9.9875	- 1.5 + j 9.9875
1000	- 1.5 - j 31.6188	- 1.5 + j 31.6188

$$\Delta = s^2 + 3s + (2 + K) = 0$$

The plot of system poles as they move throughout the s-domain when *K* varies between zero and infinity are as shown below:

The root locus is nothing but the path of the system closed loop poles as the gain of the system *K* varies between zero and infinity.



Since the root locus represent the path of the roots of the characteristic equation as the gain varies from zero to infinity, it follows that every point on the root locus must satisfy the characteristic equation, namely;

$$\Delta = 1 + G(s)H(s) = 0$$

Rearranging the characteristic equation gives;

$$G(s)H(s) = -1$$

Both *G(s)* and *H(s)* are complex quantities. The above equation can hence be cast in polar or vector form as follows:

$$\begin{aligned} G(s)H(s) &= \left|G(s)H(s)\right| \angle G(s)H(s) \\ &= \left|1\right| \angle -1 \end{aligned}$$

This last relationship specifies the conditions that must prevail for any point on the root locus.

⇨ I. Magnitude Condition

II. Angle Condition

I. Magnitude Condition:

$$|G(s)H(s)| = 1$$

$$\frac{K|(s+z_1)||(s+z_2)|\cdots\cdots|(s+z_m)|}{|(s+p_1)||(s+p_2)|\cdots\cdots|(s+p_n)|} = 1$$

For a given point on Root locii,

s=a+jb

$$\therefore \left.\frac{|(s+p_1)||(s+p_2)|\cdots\cdots|(s+p_n)|}{|(s+z_1)||(s+z_2)|\cdots\cdots|(s+z_m)|}\right|_{s=a+jb} = K$$

II. Angle Condition:

$$\angle G(s)H(s) = -1$$

$$= \pm 180^\circ (2k+1) \qquad \text{For } k = 0, 1, 2, \cdots$$

This equation can be rewritten as:

$$\angle G(s)H(s) = \angle s+z_1 + \angle s+z_2 + \angle s+z_3 + \cdots\cdots + \angle s+z_m - \angle s+p_1 - \angle s+p_2 - \cdots\cdots - \angle s+p_n$$
$$= \pm 180^\circ (2k+1)$$

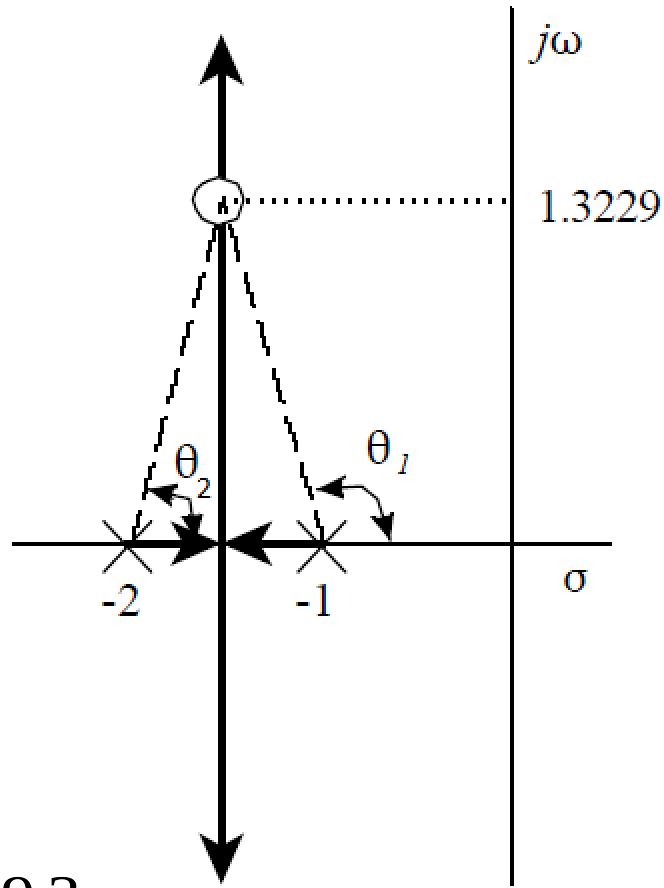
For a given point on Root locii,

s=a+jb

For the system analyzed before and shown below at the point given by -1.5 ± *j 1.3229, the* angles of the vectors from that point to the two poles are calculated as follows:

$$\theta_1 = \tan^{-1}\frac{1.3229}{-1.5+1} = 110.70446^\circ$$

$$\theta_2 = \tan^{-1}\frac{1.3229}{-1.5+2} = 69.29554$$



Angle Condition,

$$\angle G(s)H(s) = \angle zeros - \angle poles$$
$$= 0 - \angle s+1 - \angle s+2$$
$$= -\theta_1 - \theta_2 = -110.7 - 69.3$$
$$= -180$$

Magnitude Condition,

$$\left|G(s)H(s)\right|_{-1.5+j1.3229} = 1$$

$$\left|\frac{K(s+z_1)(s+z_2)\ldots\ldots}{(s+p_1)(s+p_2)\ldots\ldots\ldots}\right| = 1$$

$$\therefore K = \left|(s+1)(s+2)\right|_{-1.5+j1.3229}$$

$$= 2$$

The point -1.5 ± *j 1.3229* satisfied the both conditions since it is on root locus

Before starting the steps of sketching Root locus, we have to know

1. **The Start and End Points of a Root Locus**
2. **Number of segments (branches) of root locus**
3. **Location of root locus segments on real axis**

The characteristic equation can be rearranged as follows:

$$(s+p_1)(s+p_2)\cdots\cdots(s+p_n) = K(s+z_1)(s+z_2)\cdots\cdots(s+z_m)$$

When $K = 0$, the last equation becomes;

$$(s+p_1)(s+p_2)\cdots\cdots(s+p_n) = 0$$

This indicates that the root locii starts at the poles of the system when $K = 0$.

The characteristic equation can also be arranged as follows:

$$\frac{(s+p_1)(s+p_2)\cdots\cdots(s+p_n)}{K} = (s+z_1)(s+z_2)\cdots\cdots(s+z_m)$$

When *K is at infinity* the above equation becomes;

$$0 = (s+z_1)(s+z_2)\cdots\cdots(s+z_m)$$

This indicates that the root locii ends at the zeros of the system when $K = \infty$

Root locii starts at the system poles (when $K = 0$ *) and ends at the system zeros* (when $K = \infty$ *).*

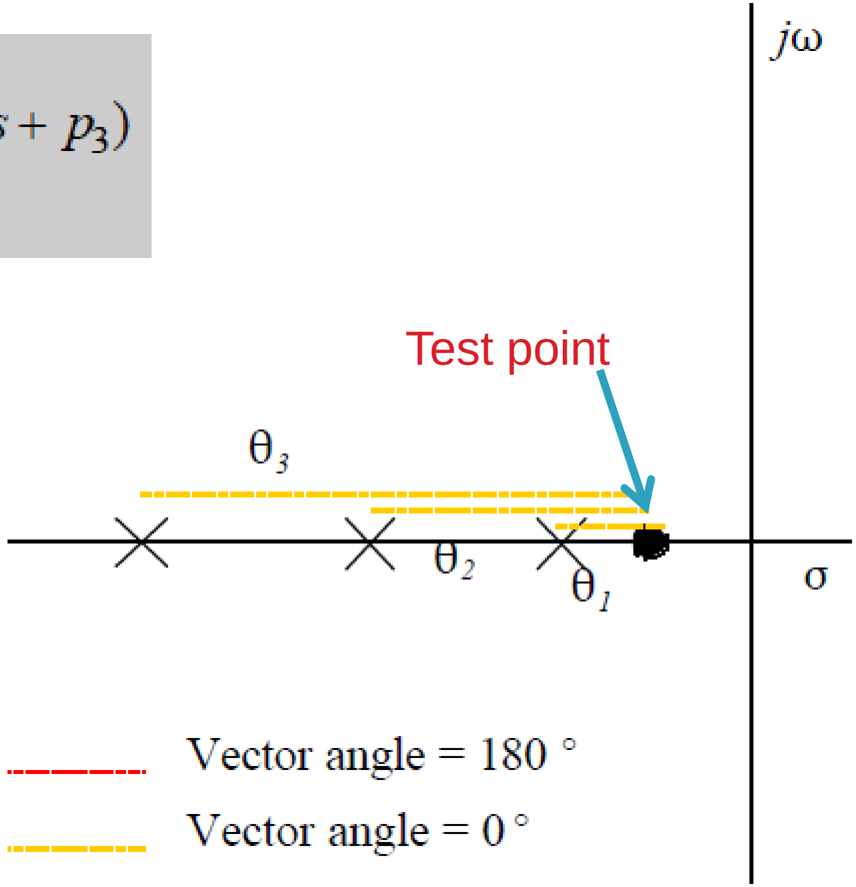
- Since root locii must start at poles and end at zeros, then it is fair to assume that the number of segments of root locii is equal to the number of poles.

- If the number of poles in the loop gain equation is larger than the number of zeros, this means that a number of root locii segments equal to the number of poles and zeros will be ending at infinity.

- Since poles and zeros that occur off the real axis must be in complex conjugate pairs, it follows that complex portions of the root locii always occur as complex conjugate portions. These portions appear as mirror images of one another with the parting line being the real axis.

- Root locii segments on the real axis can be found by applying the angle criteria as follows:

$$\angle(s+z_1)-\angle(s+p_1)-\angle(s+p_2)-\angle(s+p_3)$$
$$-\left(\theta_1+\theta_2+\theta_3\right)=0^\circ$$

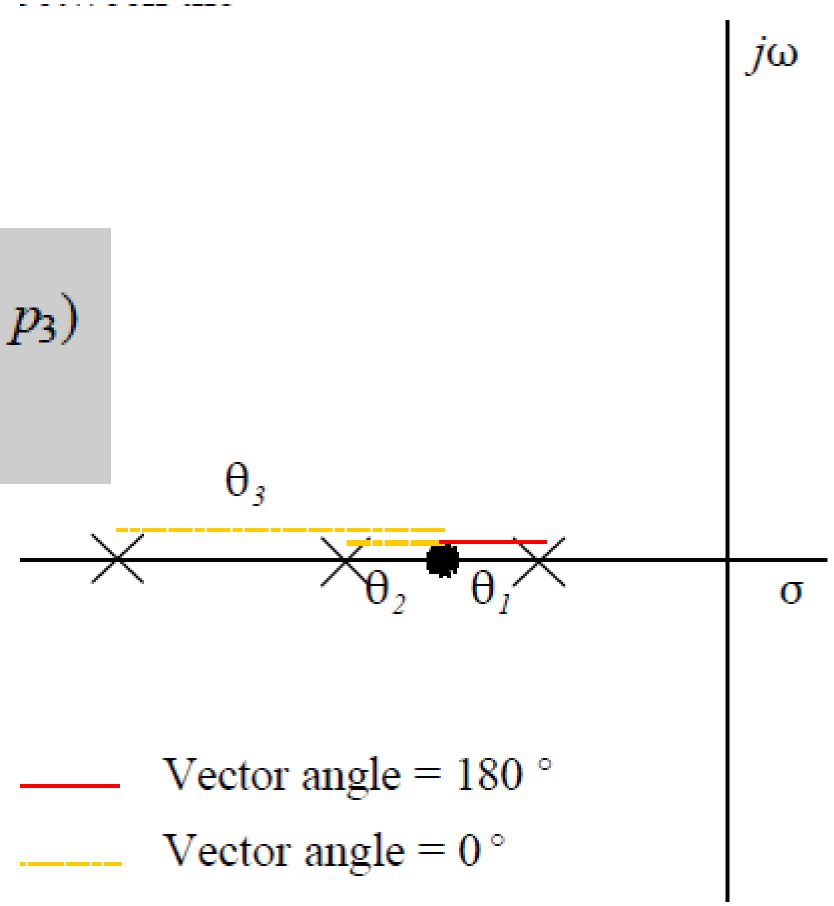


This indicates that the assumed point violates the angle criteria and hence cannot be on a segment of the root locus.

Now, let us move the test point as follows:

Applying the angle criteria gives:

$$\angle(s+z_1) - \angle(s+p_1) - \angle(s+p_2) - \angle(s+p_3)$$
$$-\left(\theta_1 + \theta_2 + \theta_3\right) = -180^\circ$$



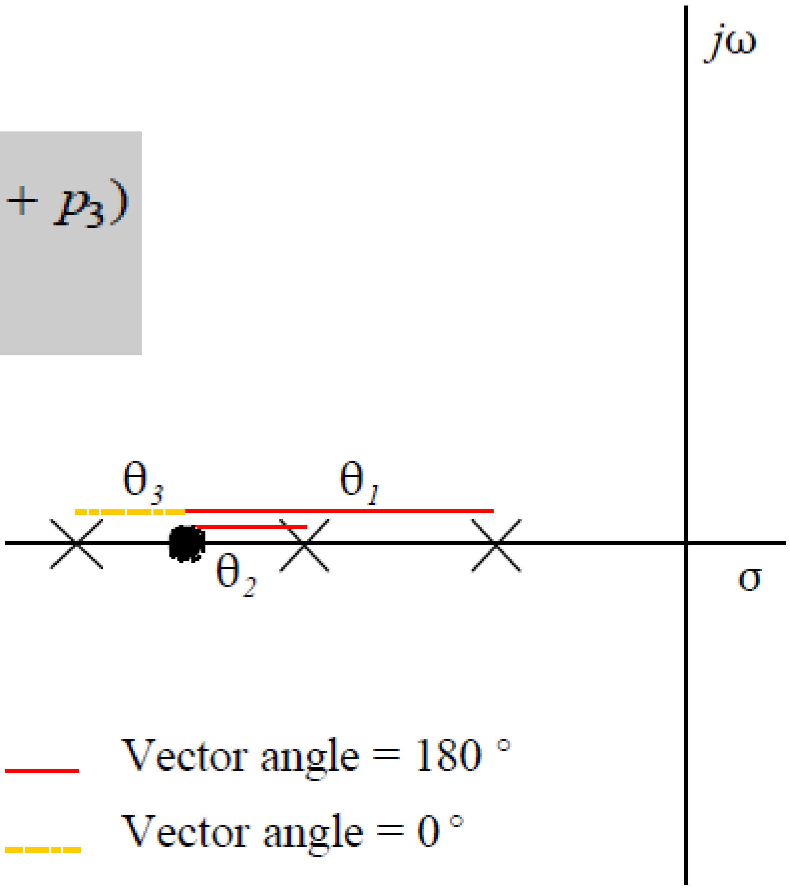
This indicates that the assumed point conforms to the angle criteria and hence is on a segment of the root locus.

Now, let us move the test point as follows:

Applying the angle criteria gives:

$$\angle(s+z_1) - \angle(s+p_1) - \angle(s+p_2) - \angle(s+p_3)$$
$$-\left(\theta_1 + \theta_2 + \theta_3\right) = -360^\circ$$

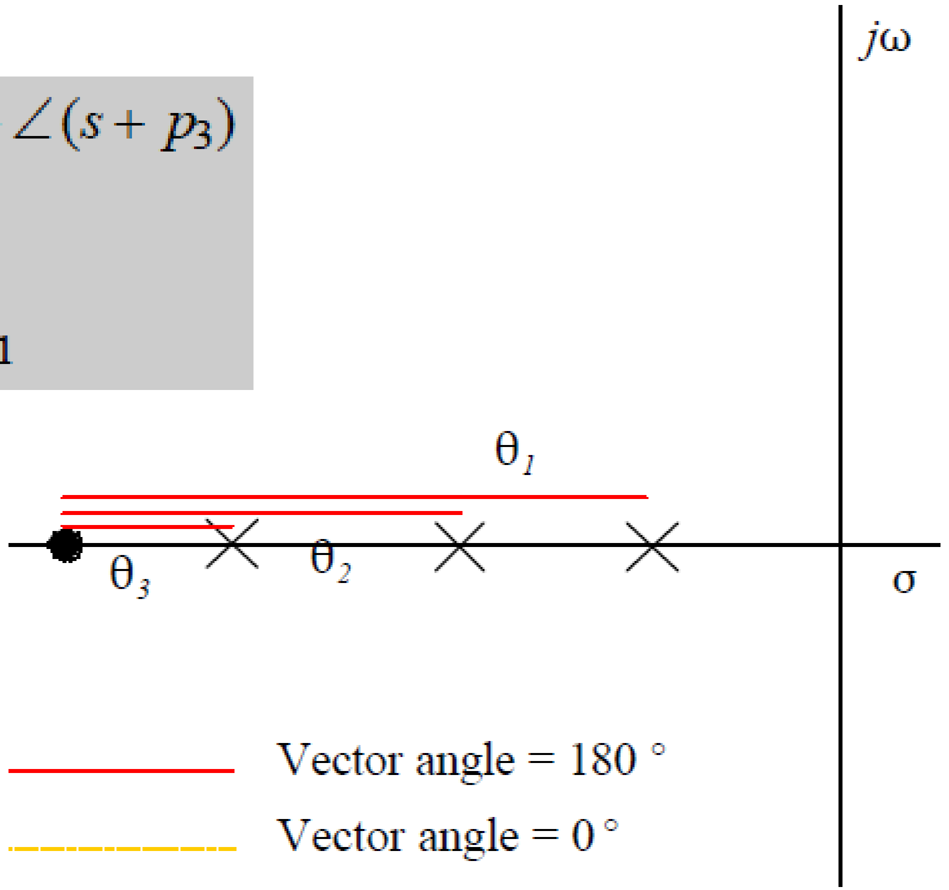
This indicates that the assumed point violates the angle criteria and hence cannot be on a segment of the root locus.



Now, let us move the test point as follows:

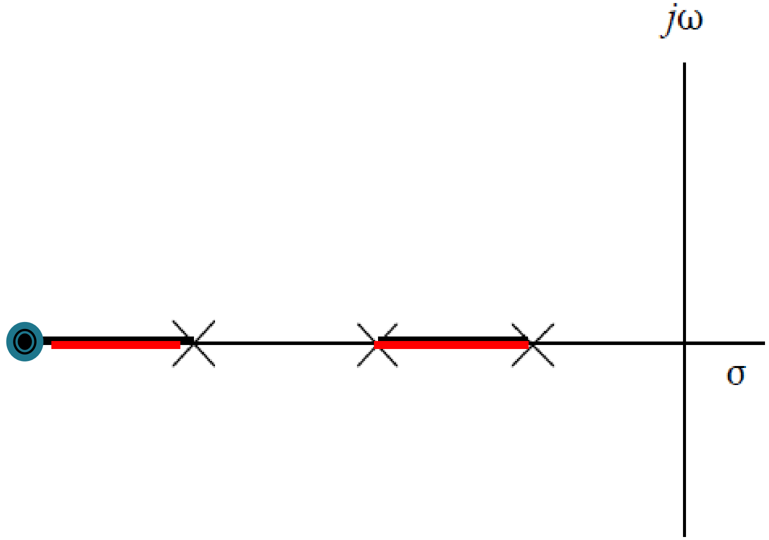
Applying the angle criteria gives:

$$\angle(s+z_1)-\angle(s+p_1)-\angle(s+p_2)-\angle(s+p_3)$$
$$-\left(\theta_1+\theta_2+\theta_3\right)=-540^\circ$$
$$=-180^\circ\,(2k+1)\big|_{k=1}$$



This indicates that the assumed point conforms to the angle criteria and hence is on a segment of the root locus.

Root locii on the real axis will occur to the left of odd number of poles and zeros .



Steps of Constructing Root Locus of a System:

1- Write the characteristic equation of the system in the following standard form

$$\Delta = 1 + K\frac{(s+z_1)(s+z_2).....(s+z_m)}{(s+p_1)(s+p_2).....(s+p_n)} = 0$$

Where *K* might be a controller gain (or system gain) and is the parameter of interest

2- Locate all poles $p_1\ ,\ p_2\ ,\ldots..p_n$ and zeros $z_1\ ,\ z_2\ ,\ldots..z_m$ in s-plane.

3- Determine the root locus segments on the real axis

4- Determine the asymptotes of the root locus:

number of asymptotes = *n-m*

intersection of asymptotes with real axis $\sigma_a = \dfrac{\sum poles - \sum zeros}{n-m}$

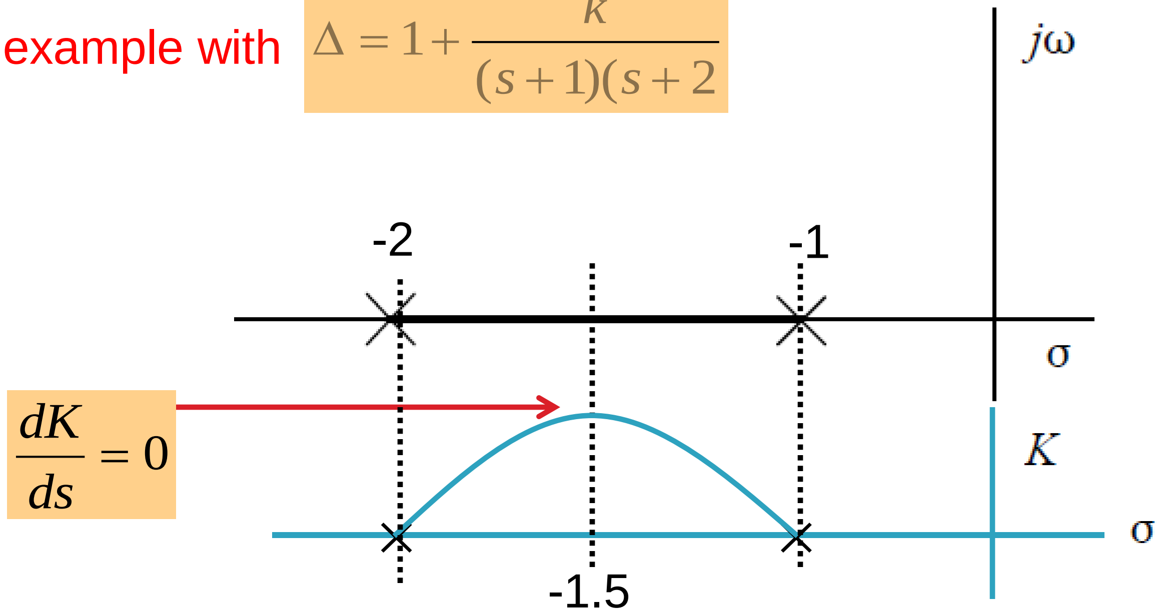
Angles of asymptotes $\alpha_a = \dfrac{\pm 180(2k+1)}{n-m},\quad k = n-m-1$

5- Find the break away / in points if any

Rearrange the characteristic equation then find the break away/in points that should result from

$$\frac{dK}{ds} = 0$$ Why ???!!

Back to our example with $\Delta = 1 + \dfrac{k}{(s+1)(s+2)}$

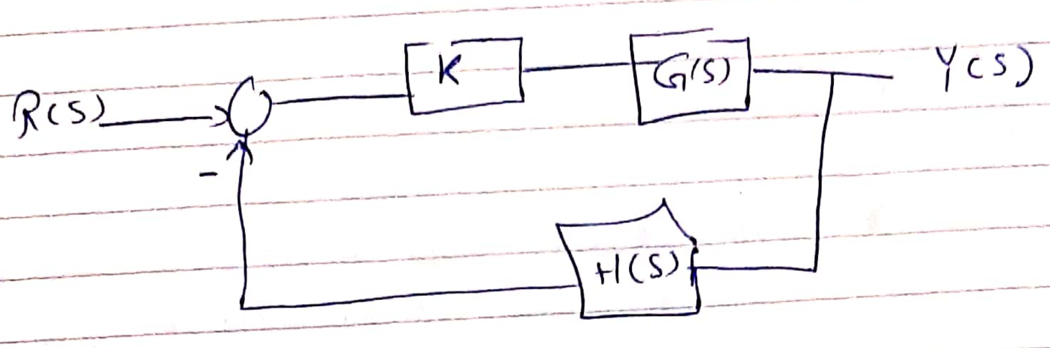


6- Find the points of intersection with *Im*-axis by applying Routh-Hurwitz criteria.

7- Determine the departure angle (arrival angle) of there is complex poles (complex zeros) by applying the angle condition

8- Calculate the desired gain *K* that corresponds to a particular desired closed loop poles by applying the magnitude condition.

Chapter 7:- The root Locus method.



$$T.F = \frac{K\,G(S)}{1 + K\,G(S)\,H(S)}$$

$$1 + K\,G(S)\,H(S) = 0$$

$$\text{Loop gain} = G(S)\,H(S) = \frac{K\,a_0 S^n + a_1 S^{n-1} + a_2 S^{n-2} + \dots + a_n}{b_0 S^n + b_1 S^{n-1} + b_2 S^{n-2} + \dots + b_n}$$

$$0 = 1 + K \text{———}$$

e.g.: $\frac{K}{(S+1)(S+2)} \Rightarrow G(S)\,H(S)$

$$1 + \frac{K}{(S+1)(S+2)} = 0 \Rightarrow (S+1)(S+2) + K = 0$$

Let $K = 0 \Rightarrow S_{1,2} = -1, -2$

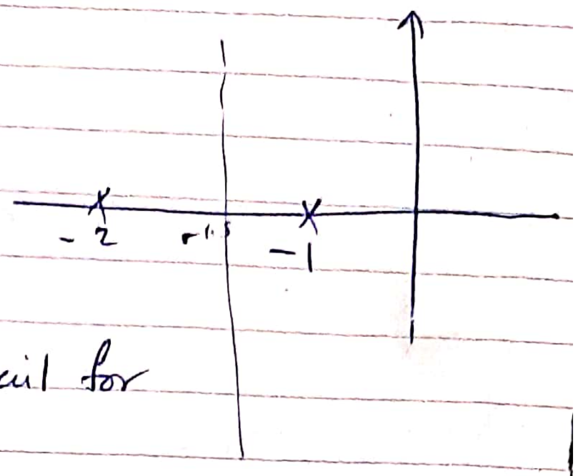
$K = 0.25 \Rightarrow S_{1,2} = -1.5$

$K = 0.5 \Rightarrow S_{1,2} = -1.5 \mp 0.5j$

$K = 1 \Rightarrow S_{1,2} = -1.5 \mp 0.866j$

⋮

$K = 1000$



* The conditions that must prevail for any point on the root locus

① Magnitude condition

$$\left| \frac{K\,|s+z_1|\,|s+z_2| \cdots |s+z_m|}{|s+p_1|\,|s+p_2| \cdots |s+p_n|} \right| = 1$$

$s = \rho$

② Angle condition

$$\angle G(s)H(s) = \mp 180^\circ (2k+1) \quad , k = 0, 1, 2 \ldots$$

$$\angle (s+z_1) + \ldots \angle s+z_m - \angle s+p_1 \ldots\ldots - \angle s+p_n$$

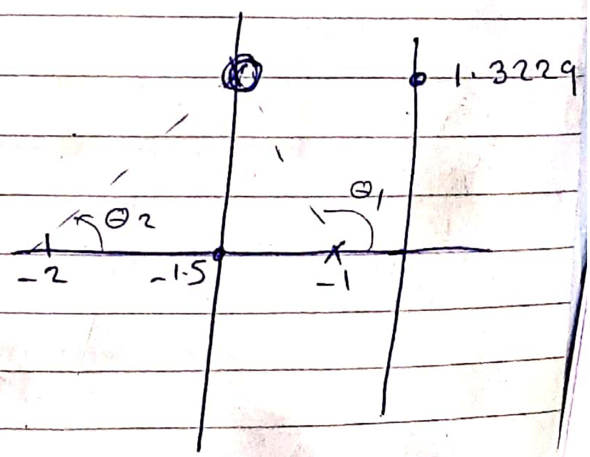
$$= \mp 180 (2k+1)$$

according to the previous example

$s = -1.5 \mp j\,1.3229$

⇒ angle condition

$$\angle s+z - \angle (s+p_1) - \angle s+p_2$$
$$= \mp 180 (2k+1)$$



0 -

$\theta_1 = \tan^{-1}\left(\frac{1.3229}{-0.5}\right) = 110.7^\circ$

$\theta_2 = \tan^{-1}\left(\frac{1.3229}{+0.5}\right) = 69.29^\circ$

$0 - \theta_1 - \theta_2 \Rightarrow 0 - 110.7 - 69.29 = -180$ ✓

⇒ Magnitude condition

$$\left| \frac{K(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)} \right| = 1 \Rightarrow K = \left|(s+p_1)(s+p_2)\right|$$

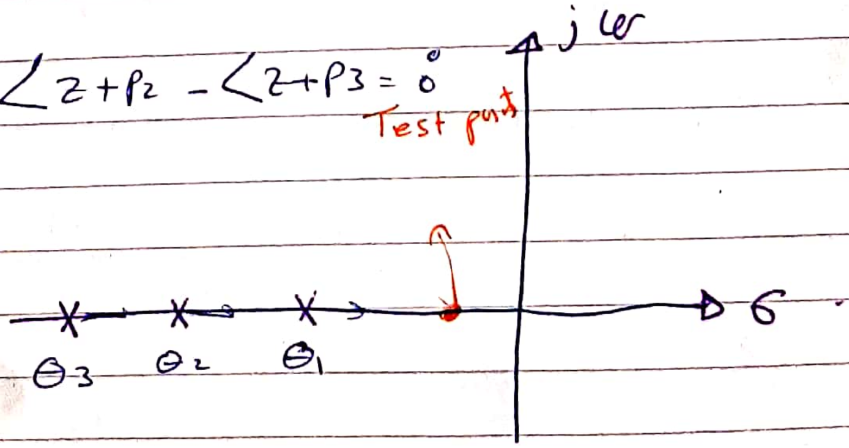
$$K = \left|(-1.5 + 1.3229j + 1)(-1.5 + 1.3229j + 2)\right| =$$

① Root locus starts at the system poles when $k = 0$ and ends at the system zeros when $k = \infty$

② number of segments (branches) is equal to the number of poles.

$\angle s+z_1 - \angle z+p_1 - \angle z+p_2 - \angle z+p_3 = 0^\circ$

$-\theta_1 - \theta_2 - \theta_3 = 0$



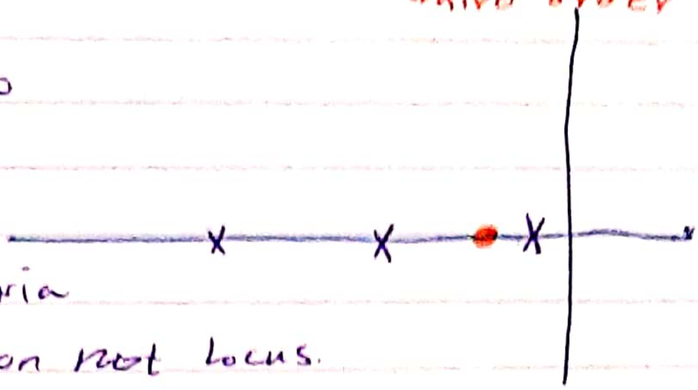
~~θ0 = -180~~ .

this point violates the angle criteria and cannot be on segment of root locus.

type zero
third order

$\theta_1 = 180$

$\theta_2 = 0$, $\theta_3 = 0$



$\Rightarrow -180$ ✓

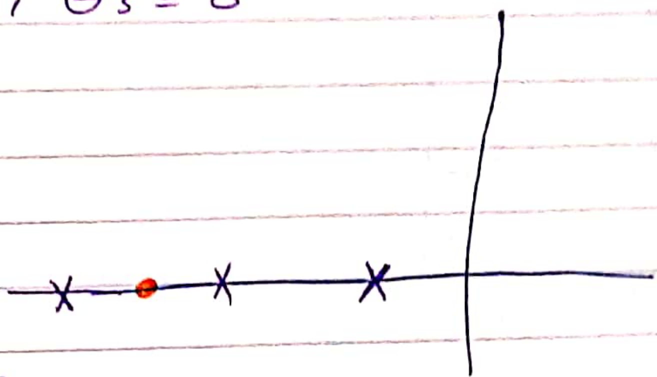
Confirms angle criteria

k. is a segment on root locus.

$\theta_1 = 180$, $\theta_2 = 180$, $\theta_3 = 0$

$\Rightarrow -360$ ✗

لا تحقق الشرط

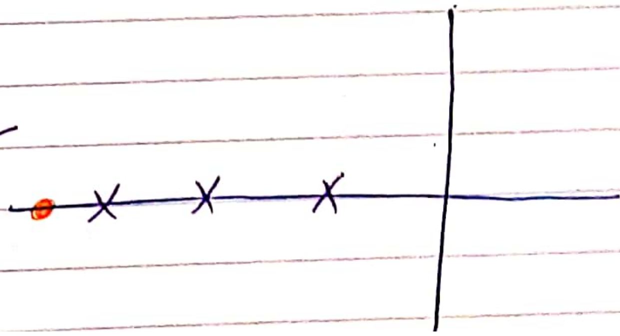


$\theta_1 = -180 = \theta_2 = \theta_3$

-540

$\Rightarrow \mp 180(2k+1)$

$k = 1$ ✓



Ex 1 :- Draw the root Locus

$$1 + KGH = 0 = 1 + \frac{k}{(s+1)(s+2)}$$

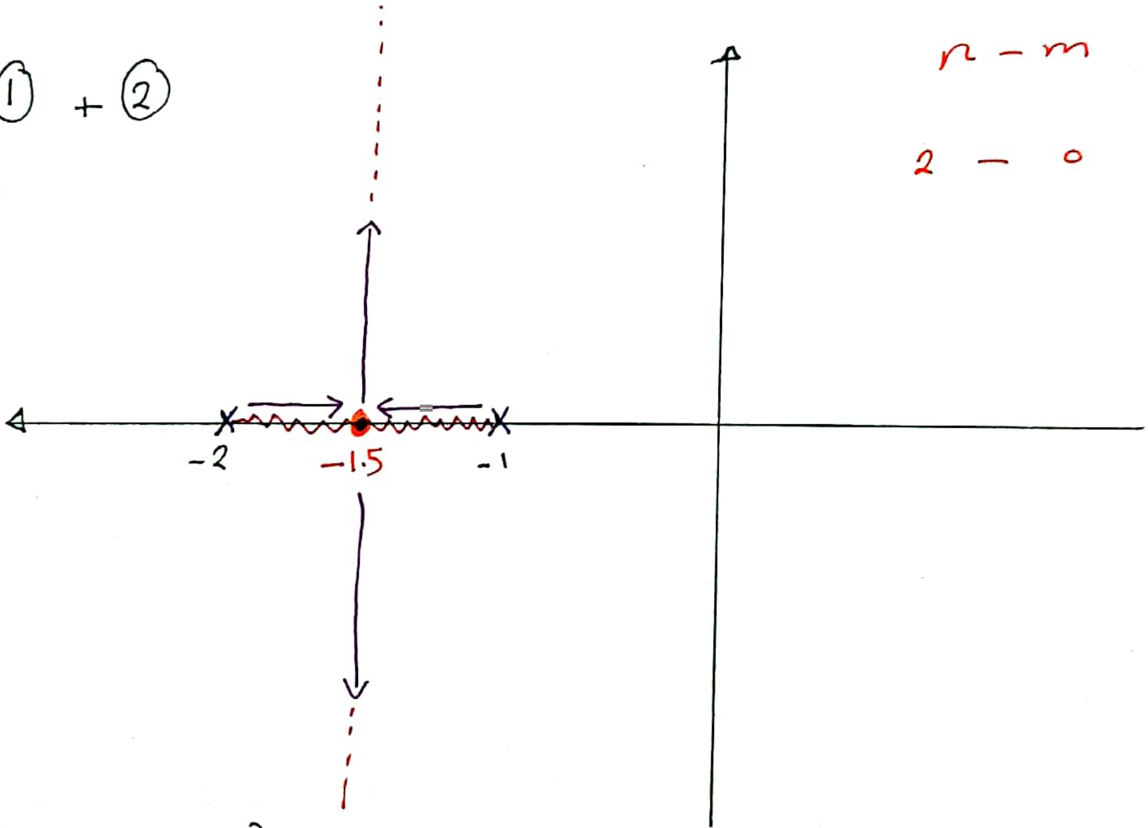
$$s^2 + 3s + 2 + k = 0$$

\# of asymptotes

$n - m$

$2 - 0 = 2$

① + ②



③ $\sigma_a = \frac{-3 - 0}{2 - 0} = -1.5$

$k = n - m - 1 = 2 - 0 - 1 \Rightarrow$

$$\theta_0 = \frac{-180(2(0)+1)}{2-0} = 90^\circ$$

$$\theta_1 = \frac{-180(2(1)+1)}{2-0} = 270^\circ$$

(4) break-awy point $\frac{dk}{ds} = 0$

$$-k \leftarrow$$

$$0 = s^2 + 3s + 2 + k \Rightarrow \frac{dk}{ds} = -(2s + 3) = 0$$

$$s = \frac{-3}{2}$$

angle of departure لا يوجد

$0 < K < \infty \rightarrow$ system is stable.

K at $s = -1$ is zero

∴ system order is second-order

∴ type = zero type.

Ex 2: Draw the root Locus for the system shown:

$$1 + \frac{K(s+1)}{s(s+2)(s+4)^2}$$

① $s(s+2)(s+4)^2 + k(s+1) = 0$

The system has one zero at $s = -1$

and Four poles at $s = 0$

$s = -2$

$s = -4, -4$ +3j repeated roots

③ # of asymptotes: $n - m = 4 - 1 = 3$

$$\sigma_a = \frac{(0 - 2 - 4 - 4) - (-1)}{4 - 1} = -3$$

$$k = n - m - 1 = 4 - 1 - 1 = 2$$

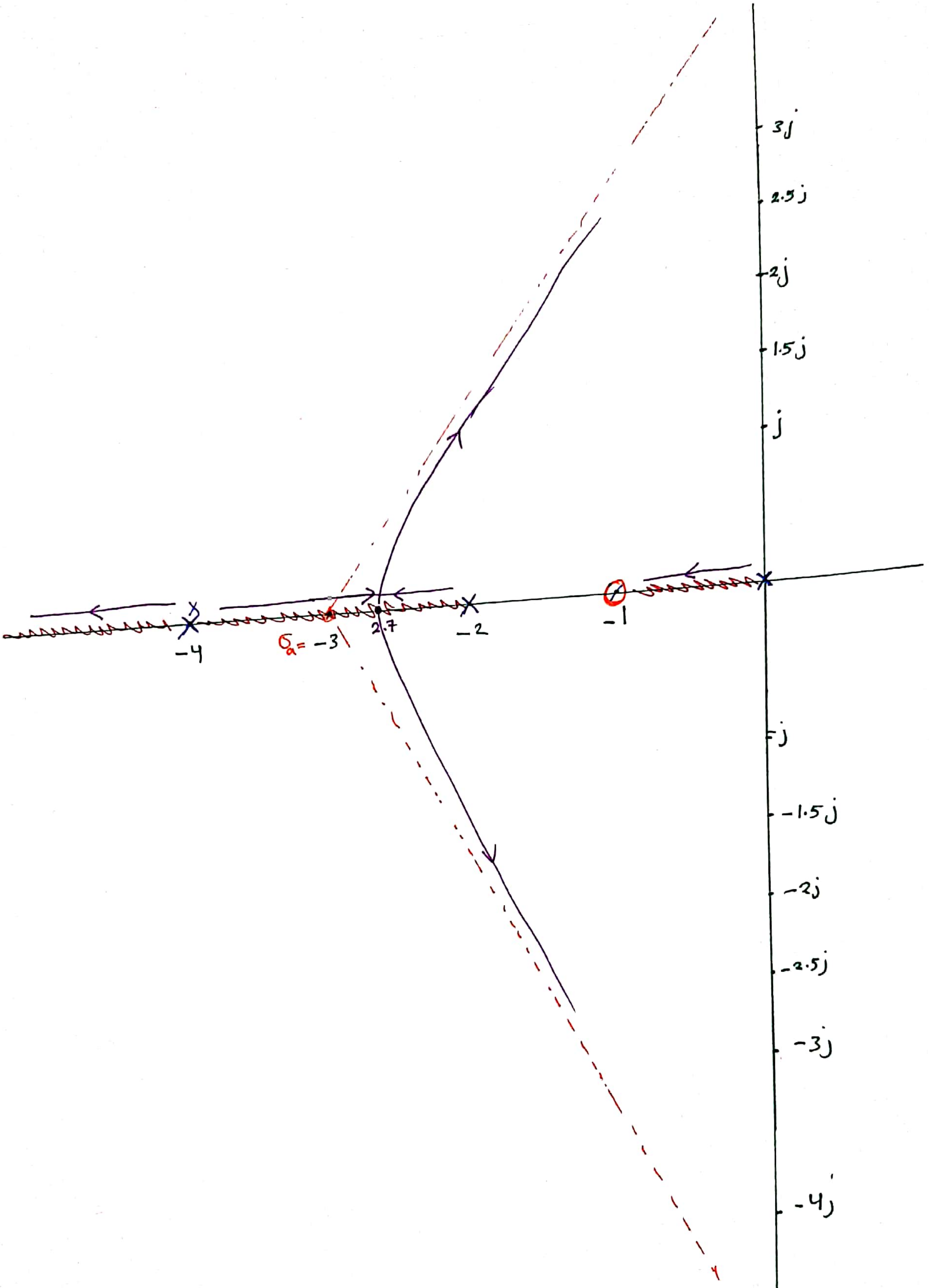
$\therefore$ at $k = 0 \Rightarrow \theta_0 = \frac{-180\,(2(0)+1)}{4-1} = 60^\circ$

at $k = 1 \Rightarrow \theta_1 = \frac{-180\,(2(1)+1)}{4-1} = -180^\circ$

at $k = 2 \Rightarrow \theta_2 = \frac{-180\,(2(2)+1)}{4-1} = 300^\circ$

④ * break-away point : $\frac{dK}{ds} = 0$

$s \cong 2.7$



$\theta = 60^\circ \Rightarrow \tan\theta = \frac{y}{3} = \tan 60^\circ$

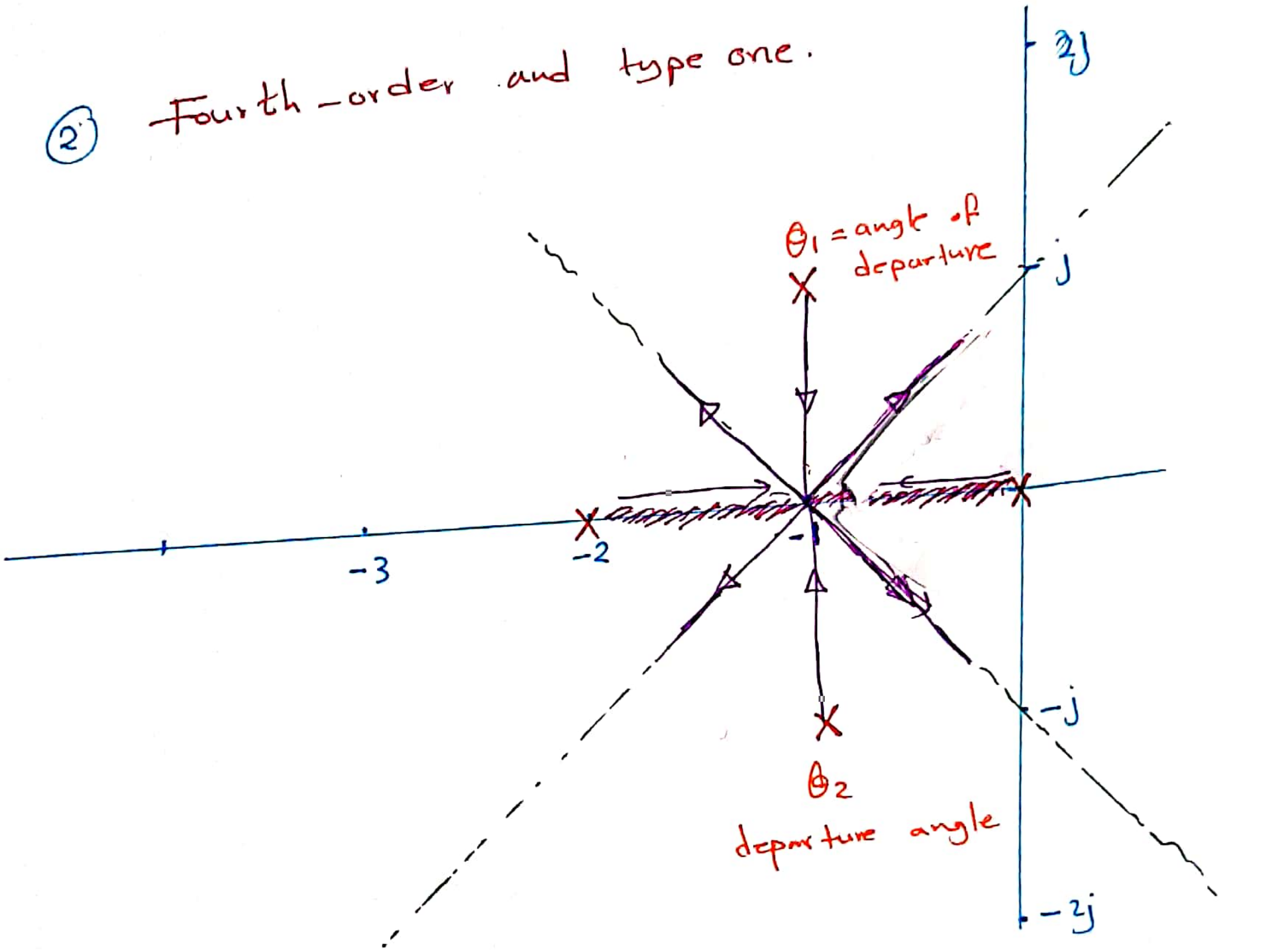
Imajenary axis مع نقطة التقاطع asymptote $\quad y = (1.732) * 3 = 5.19$

Ex 3: Draw the root locus.

note
$s = 0$
$s = -2$
$s = -1 - j$
$s = -1 + j$

① $1 + KGH = 1 + \frac{K}{s(s+2)((s+1)^2+1)}$

② Fourth-order and type one.



④ $n - m = \#$ of asymptotes

$4 - 0 = 4$

$\sigma_a = \frac{(0 - 2 - 1 + j - 1 - j) - (0)}{4 - 0} = -1$

$K = n - m - 1 = 4 - 0 - 1 = 3$

$\alpha_0 = \frac{180(1)}{4} = 45^\circ$

$\alpha_1 = \frac{180(3)}{4} = 135^\circ$

$\alpha_2 = \frac{180(5)}{4} = 225^\circ$

$\alpha_3 = \dfrac{180(7)}{4} = 315^\circ$

⑤ break-away point $\dfrac{dK}{ds} = 0$

$(s^2+2s)(s^2+2s+2) \pm K = 0$

$s^4 + 4s^3 + 6s^2 + 4s + k = 0$

$4s^3 + 12s^2 + 12s + 4 = -\dfrac{dk}{ds} = 0$

$s^3 + 3s^2 + 3s + 1 = \dfrac{dk}{ds} = 0$

$s^2 = -1$

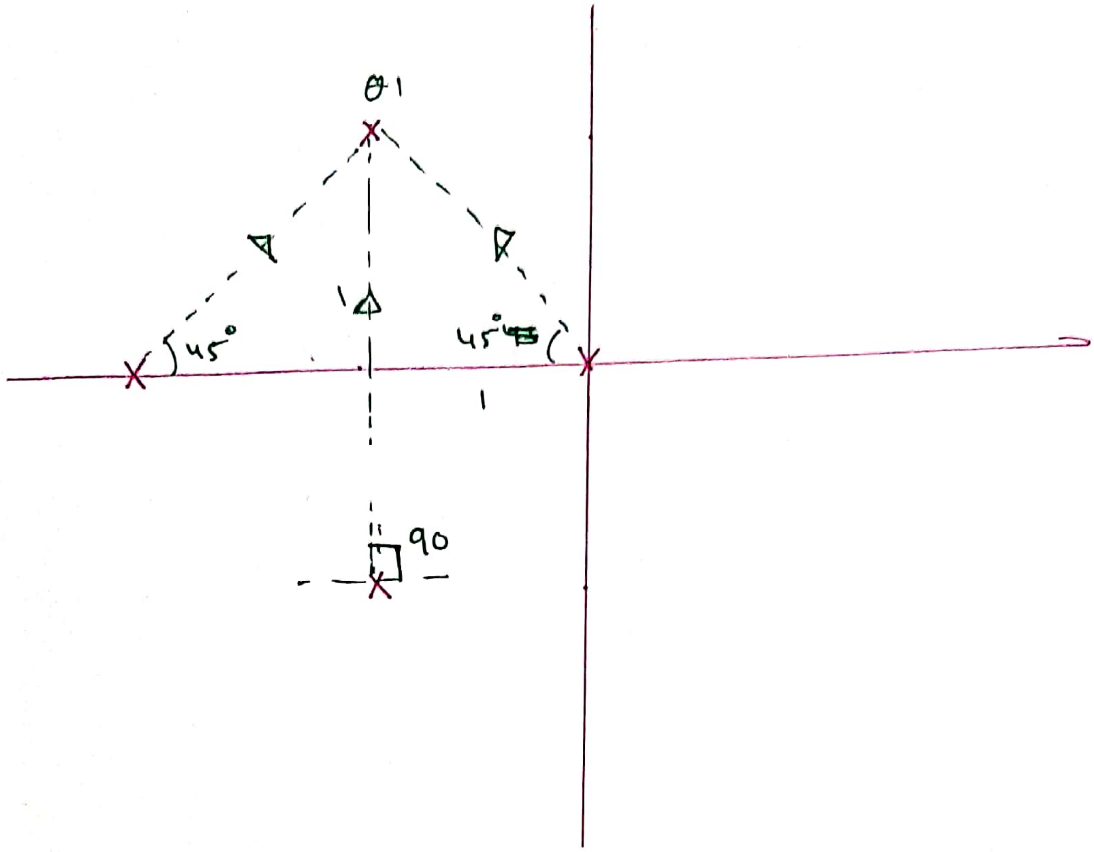
s^4	1	6	K
s^3	4	4	0
s^2	5	$\frac{4K}{4}$	0
s^1	$\frac{20-4K}{5}$	0	0 →
s^0	K		

when $\dfrac{20-4K}{4} = 0 \Rightarrow$ The system is marginally stable ✓

$\Rightarrow K = 5$ at $5s^2 + K = 0$

$5s^2 + 5 = 0 \Rightarrow s^2 = -1$

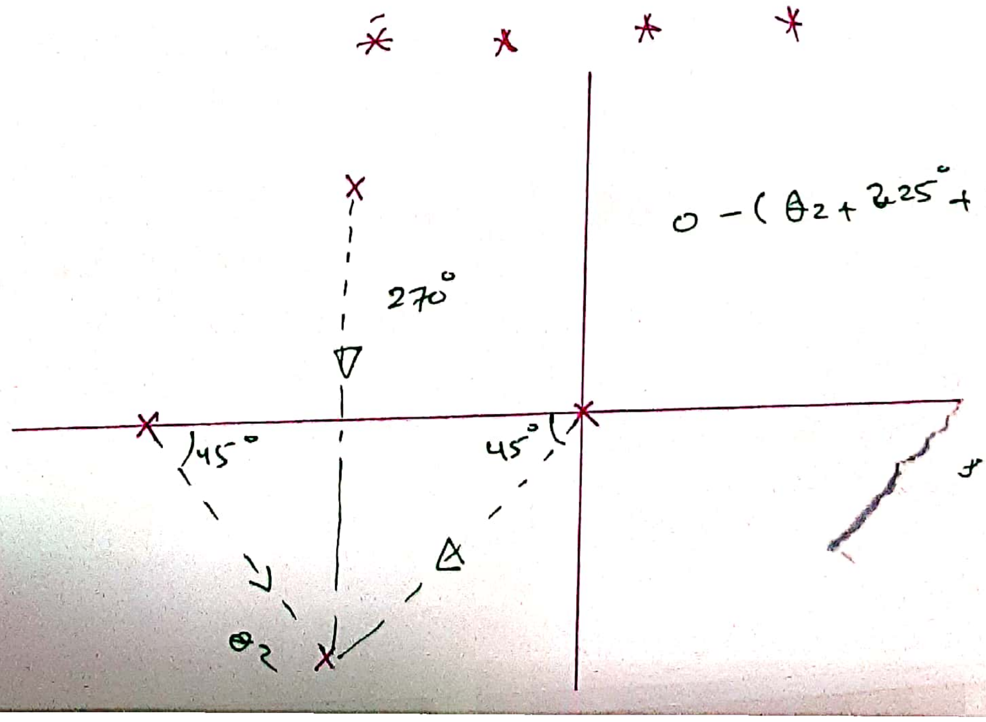
θ_1 and θ_2 → angle of departure.



$\Sigma \angle \text{zeros} - \Sigma \angle \text{poles} = -180(2K+1)$

$0 - (\theta_1 + 135° + 45° + 90) = -180$

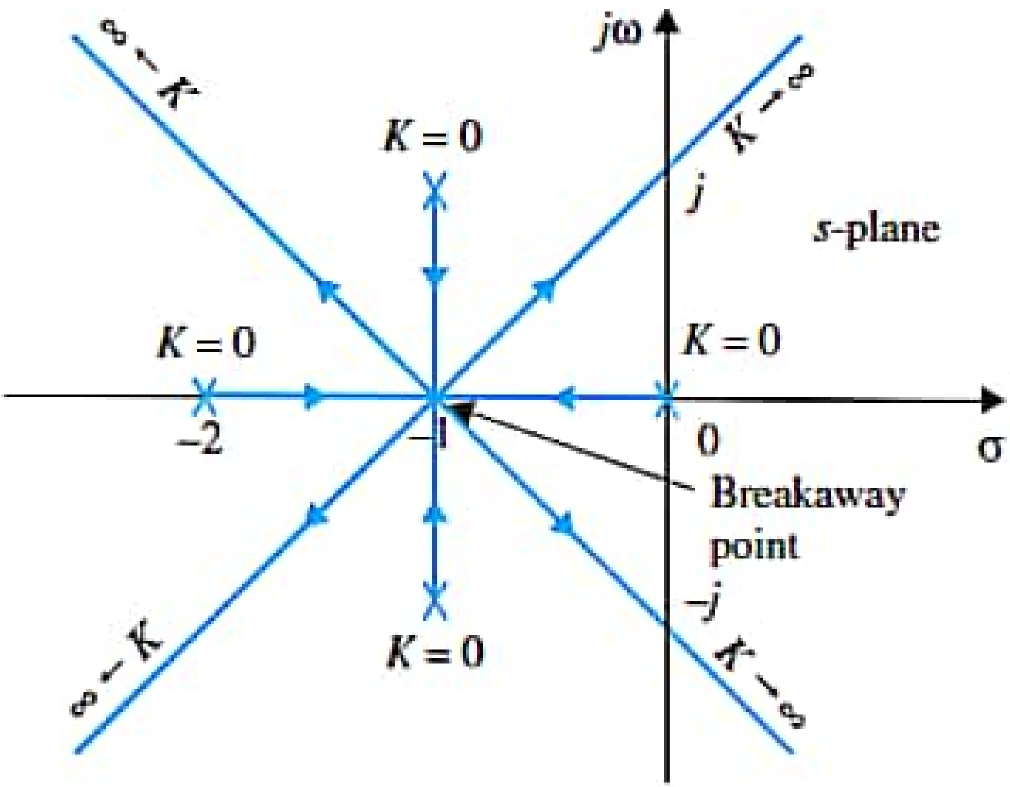
$\theta_1 = -90° = 270°$



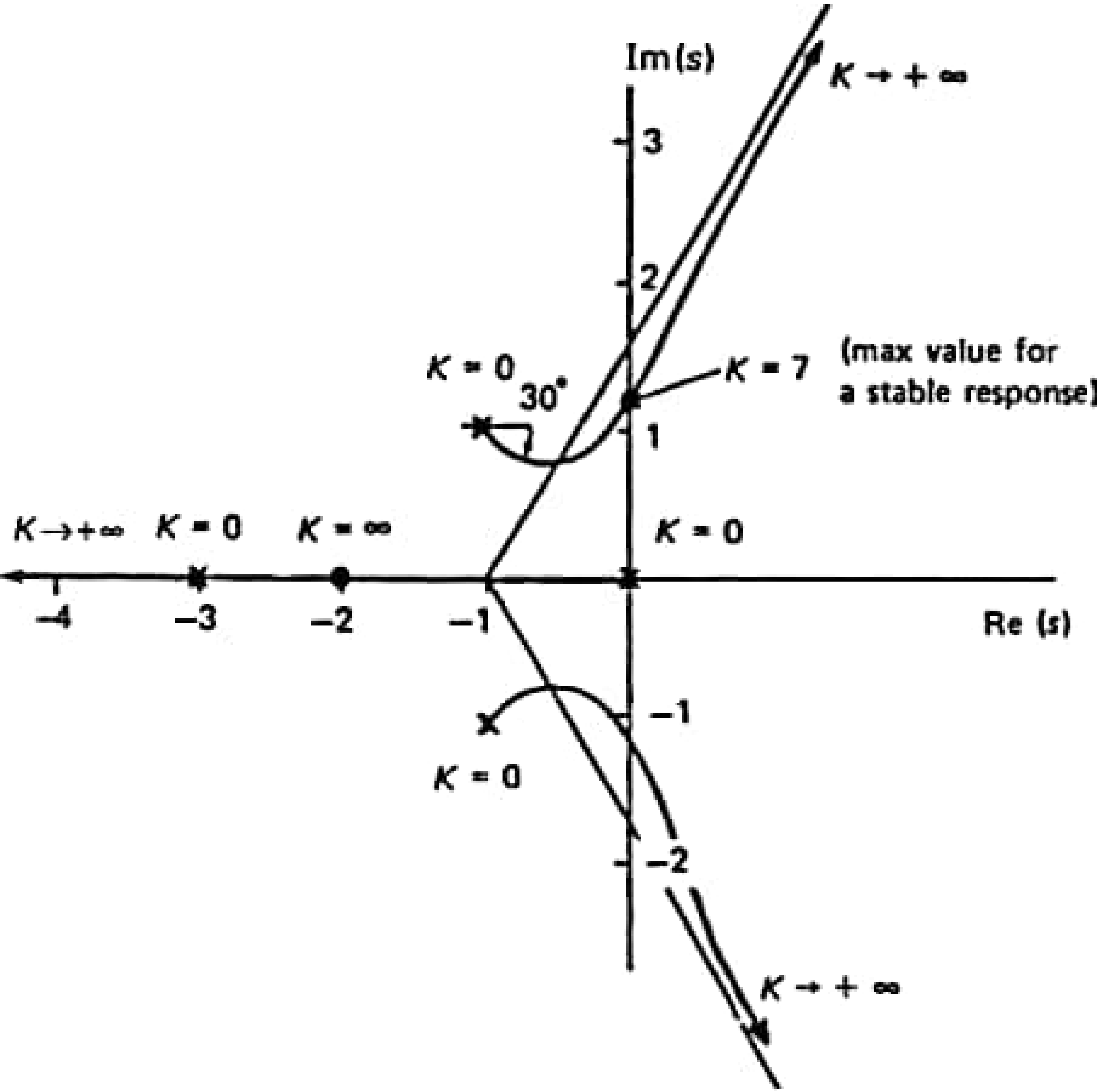
$0 - (\theta_2 + 225° + 315° + 270) = -180$

$\theta_2 = -270°$

$= +90°$



(c)



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EXAMPLE PROBLEMS AND SOLUTIONS

A-6-1. Sketch the root loci for the system shown in Figure 6–39(a). (The gain K is assumed to be positive.) Observe that for small or large values of K the system is overdamped and for medium values of K it is underdamped.

Solution. The procedure for plotting the root loci is as follows:

1. Locate the open-loop poles and zeros on the complex plane. Root loci exist on the negative real axis between 0 and -1 and between -2 and -3.
2. The number of open-loop poles and that of finite zeros are the same. This means that there are no asymptotes in the complex region of the s plane.

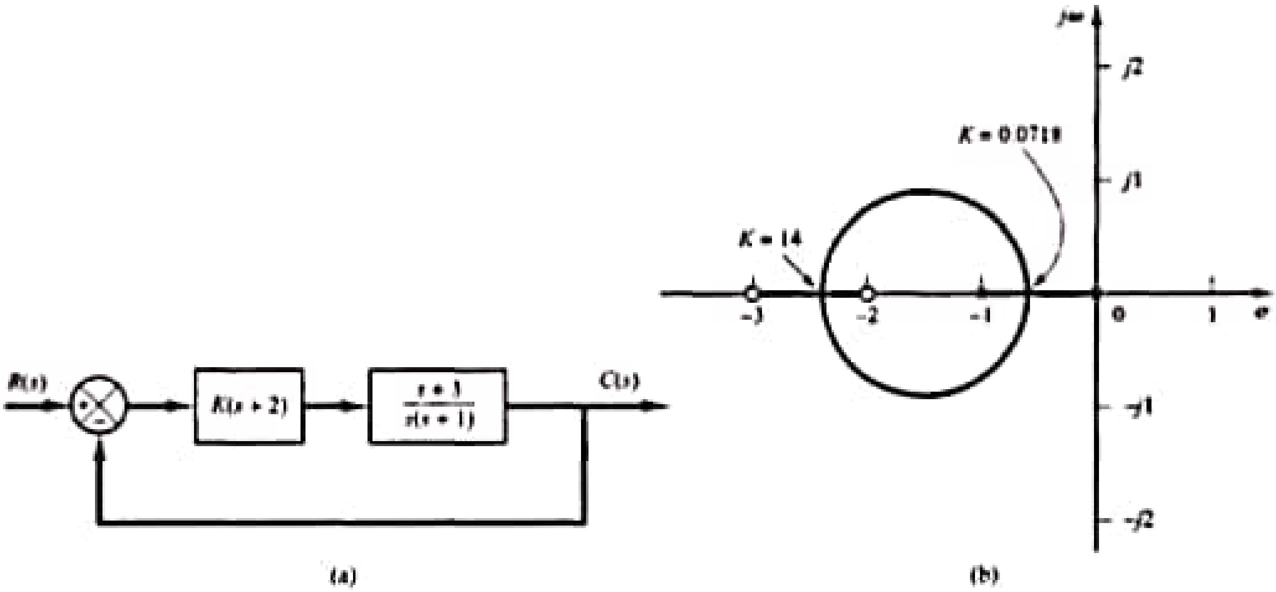


Figure 6–39
(a) Control system; (b) root-locus plot.

Ex4: Draw a root locus for the characteristic equation shown.

① $1 + \frac{K}{s^4 + 12s^3 + 64s^2 + 128s}$

② no zeros, 4 poles at

$s = 0$

$s = -4$

$s = -4 + j4$

$s = -4 - j4$

System type one

Fourth order

③ # of segments on s-plane

④ # of asymptotes $n - m = 4 - 0 = 4$

$\sigma_a = \frac{(0 - 4 - 4 + j4 - 4 - j4) - (0)}{4 - 0} = -3$

$K = n - m - 1 = 4 - 0 - 1 = 3$

$k = 0 \Rightarrow \theta_0 = \frac{180(1)}{4} = 45^\circ$

$K = 1 \Rightarrow \theta_1 = \frac{180(3)}{4} = 135^\circ$

$K = 2 \Rightarrow \theta_2 = \frac{180(5)}{4} = 225^\circ$

$K = 3 \Rightarrow \theta_3 = \frac{180(7)}{4} = 315^\circ$

(5) break-away point $\frac{dK}{ds} = 0 \Rightarrow S = -1.5$

(6) Routh-Herwitz , $p(s) = s^4 + 12s^3 + 64s^2 + 128s + K = 0$

find zero row on Routh-array

zero-row → marginally stable

and there are poles on $j\omega$-axis

s^4	1	64	K
s^3	12	128	0
s^2	53.3	K	0
s^1	c_1	0	0
s^0	K	0	0

$\Rightarrow$ if $c_1 = 0 \Rightarrow$ we have zero row

$$c_1 = \frac{(53.3)(128) - (12)(K)}{53.3} = 0$$

$$\Rightarrow \quad c_1 = \frac{6826.67 - 12K}{53.33} = 0$$

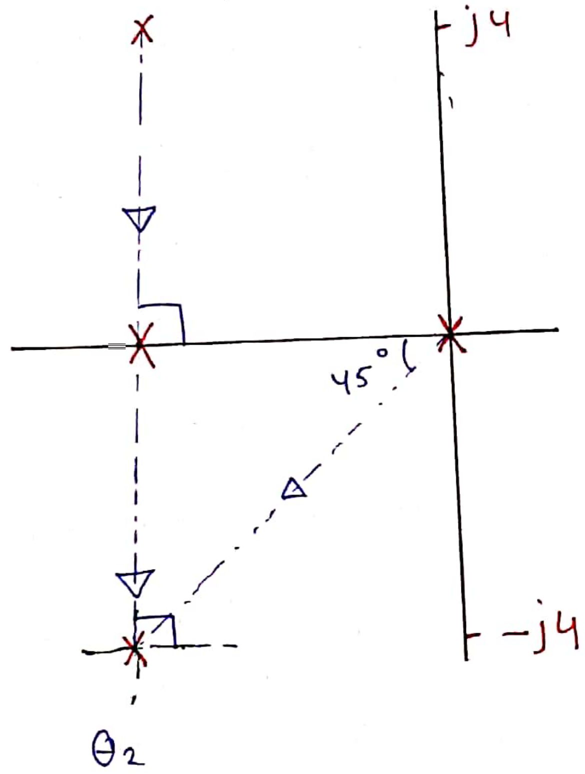
الخطوة رقم (8) (8) $\Rightarrow$ $K = 568.89$

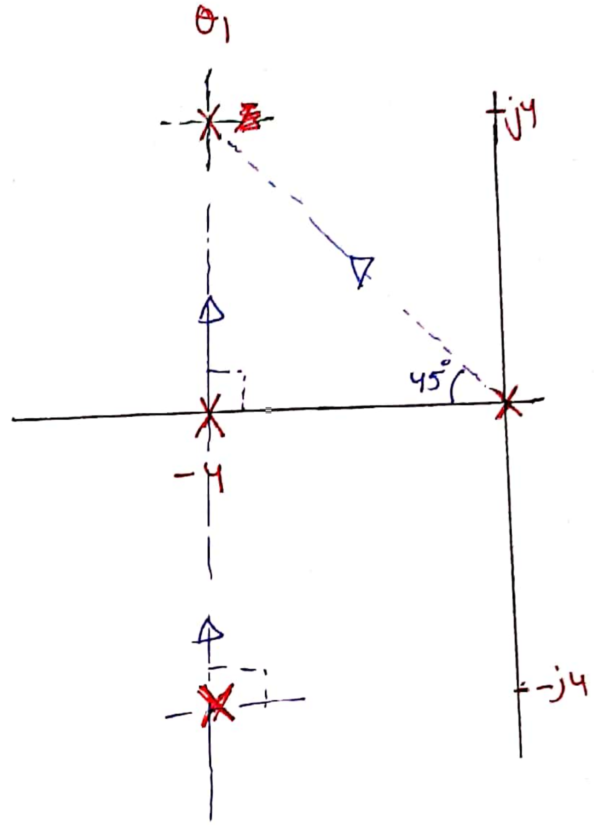
$$53.3\, s^2 + K = 0$$

$$53.3\, s^2 + 568.89 = 0 \Rightarrow s_{1,2} = \mp j3.266$$

The system is stable $0 < K < 568.89$

(7) departure angle.





by applying angle criterion

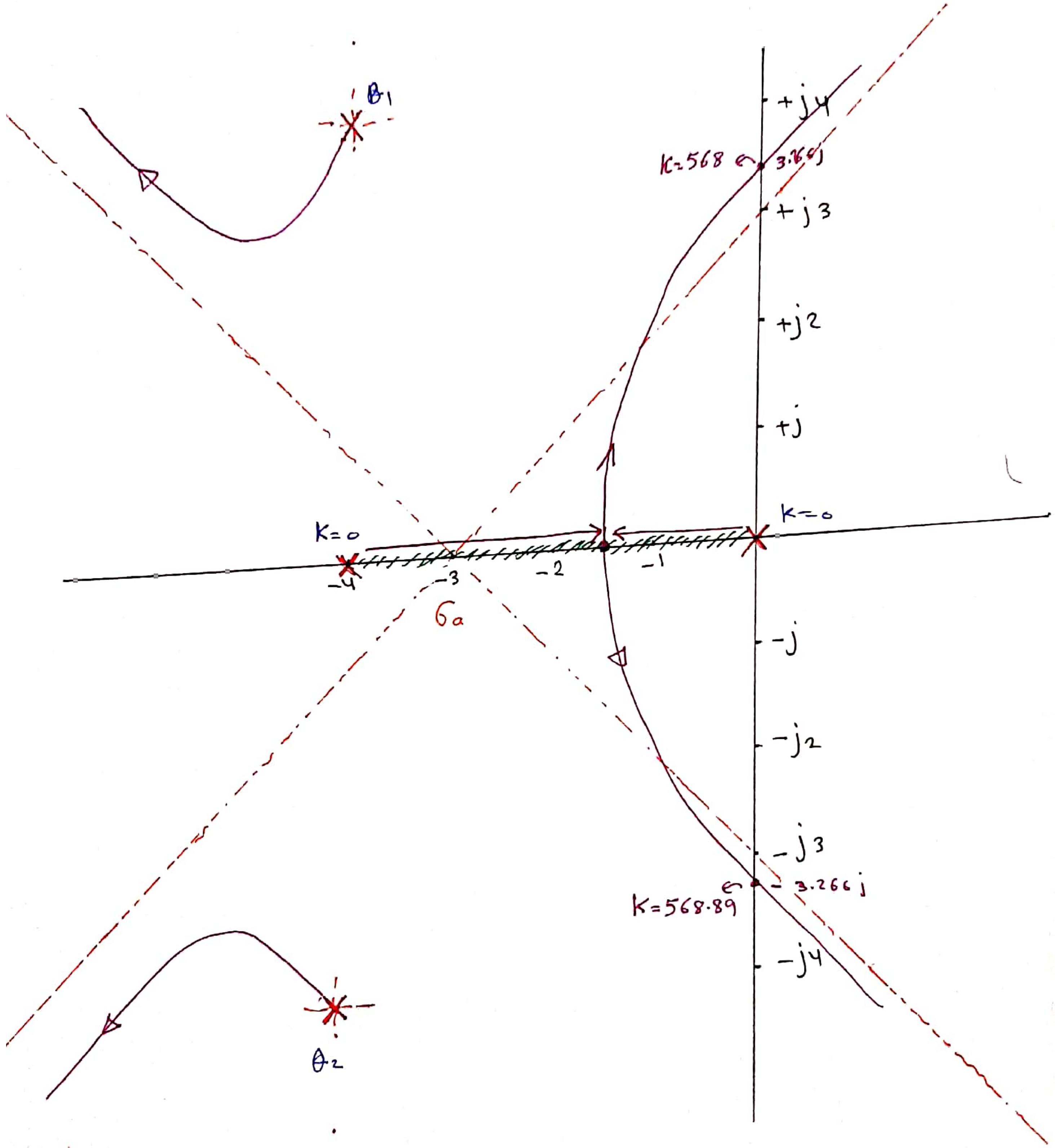
$$0 - (\theta_2 + 225^\circ + 270^\circ + 270^\circ) = -180(2K+1)$$

$$\boxed{\theta_2 = 135^\circ}$$

also,

$$0 - (\theta_1 + 135^\circ + 90^\circ + 90^\circ) = -180(2K+1)$$

$$\boxed{\theta_1 = 225^\circ}$$

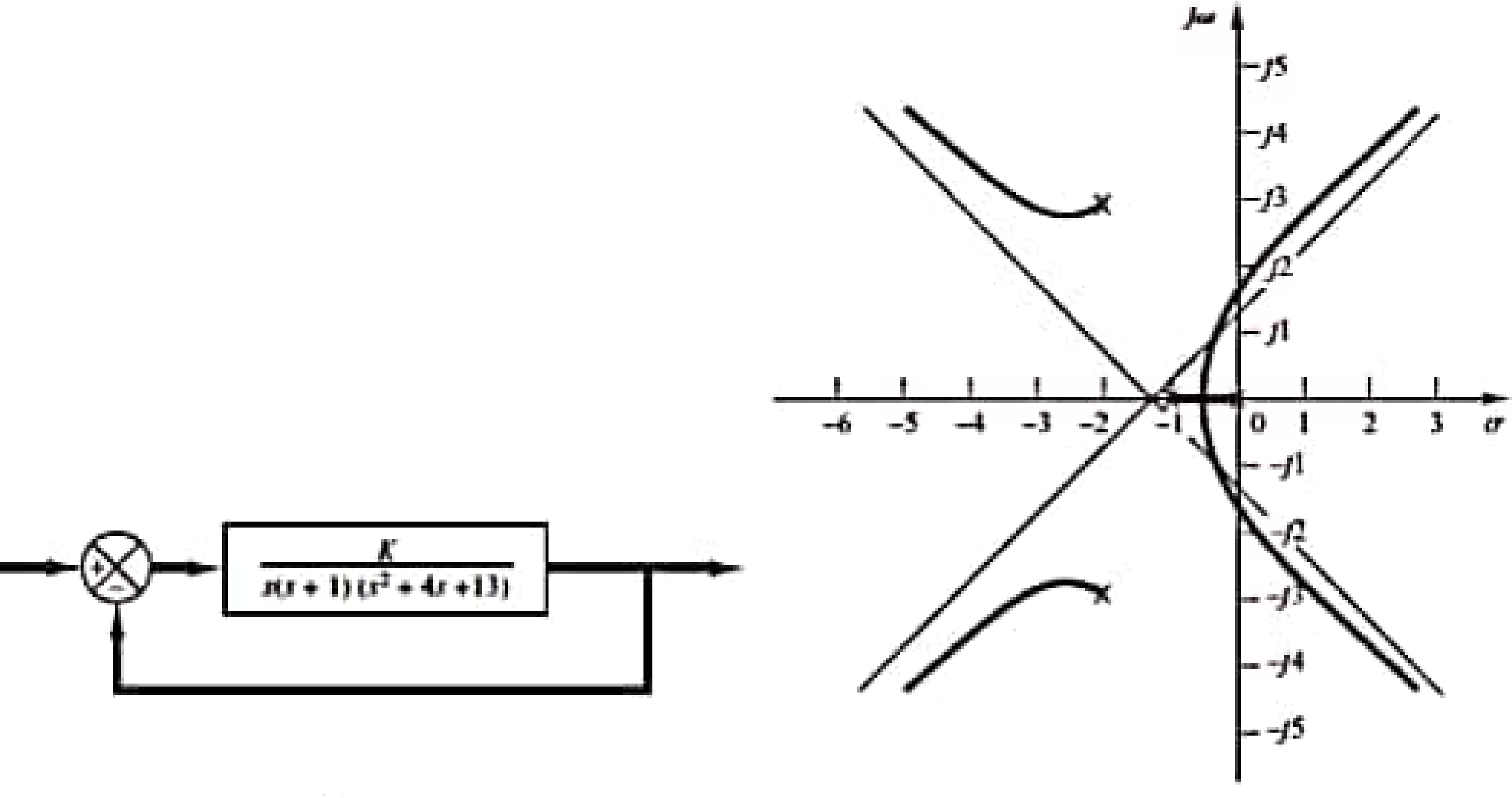


θ_1
θ_2
+j4
+j3
+j2
+j
−j
−j2
−j3
−j4
K=568
3.266j
K=568.89
−3.266j
K=o
K=o
−4
−3
−2
−1
σ_a

A–6–6. Sketch the root loci for the system shown in Figure 6–68(a).

Solution. The open-loop poles are located at $s = 0$, $s = -1$, $s = -2 + j3$, and $s = -2 - j3$. A root locus exists on the real axis between points $s = 0$ and $s = -1$. The angles of the asymptotes are found as follows:

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ(2k + 1)}{4} = 45^\circ, -45^\circ, 135^\circ, -135^\circ$$



(a) (b)

Figure 6–68
(a) Control system; (b) root-locus plot.

Examples for Construction of Root Locus

Here in this section, we will see some examples that will help you to understand how the root locus is drawn for a system to check its stability.

Example1: Suppose we have given the transfer function of the closed system as:

$$G(s)H(s) = \frac{K}{s(s+5)(s+10)}$$

We have to construct the root locus for this system and predict the stability of the same.

Firstly, writing the characteristic equation of the above system,

$$s(s+5)(s+10) = 0$$

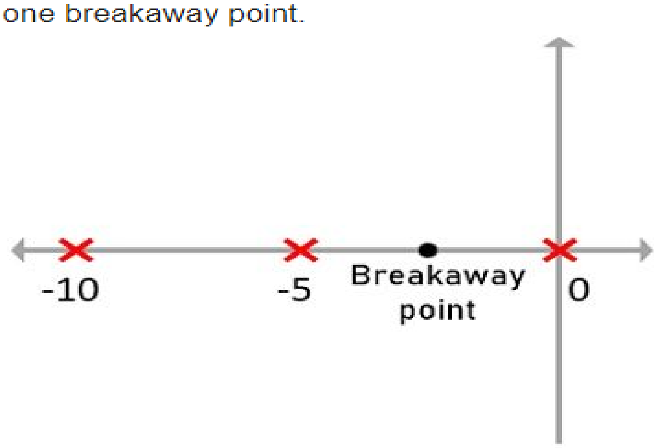
So, from the above equation, we get, s = 0, -5 and -10.

Thus, P = 3, Z = 0 and since P > Z therefore, the number of branches will be equal to the number of poles.

So, N = P = 3

Thus, under this condition, the branches will start from the locations of 0, -5 and -10 in the s-plane and will approach infinity.

Initially, with general prediction, we can say that the point -5 on the real axis has an odd sum of total number of poles and zeros to the right of it. Thus, in between 0 and -5 there will be one breakaway point.



Now, let us calculate the angle of asymptotes with the formula given below:

$$\theta = \frac{(2q + 1)180°}{P - Z}$$

: *q lies between 0 to P-Z-1*

So, in this case, θ will be calculated for q = 0, 1 and 2.

$$\theta_1 = \frac{180°}{3} = 60°$$

$$\theta_2 = \frac{3 * 180°}{3} = 180°$$

$$\theta_3 = \frac{5 * 180°}{3} = 300°$$

So, these three are the angle possessed by asymptotes approaching infinity.

Now, let us check where the centroid lies on the real axis by using the formula given below:

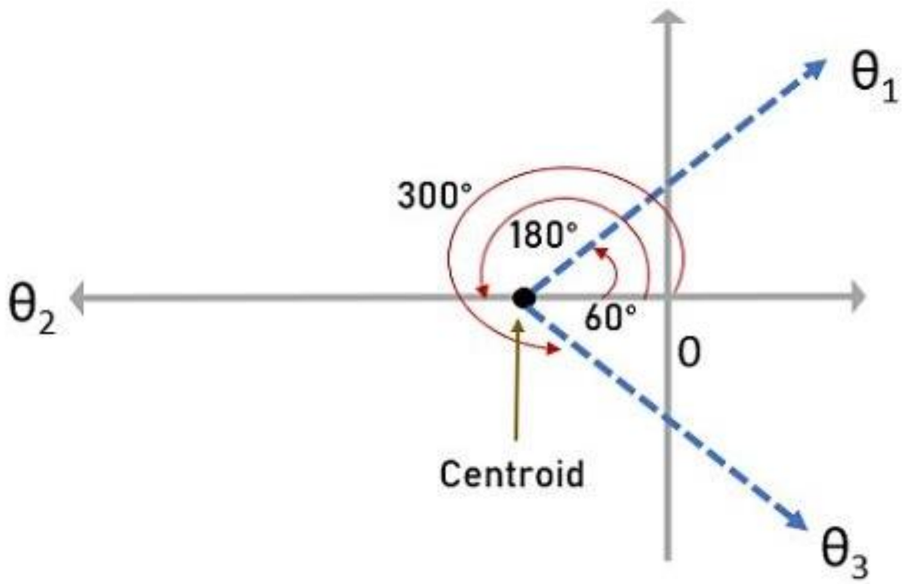
$$\sigma = \frac{\text{sum of real part of poles} - \text{sum of real part of zeros}}{P-Z}$$

$$\sigma = \frac{0-5-10-0}{3}$$

$$\sigma = \frac{-15}{3}$$

$$\sigma = -5$$

The figure below represents a rough sketch of the plot that is obtained by the above analysis



Earlier we have predicted that one breakaway point will be present in the section between points 0 and -5. So, now using the method to determine the breakaway point we will check the validity of the breakaway point.

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+5)(s+10)} = 0$$

$$s(s+5)(s+10) + K = 0$$

$$s^3 + 15s^2 + 50s + K = 0$$

$$K = -s^3 - 15s^2 - 50s$$

In this method, roots obtained on differentiating K with respect to s and equating it to 0, will be the breakaway point.

Therefore,

$$\frac{dK}{ds} = 0$$

$$\frac{d(-s^3 - 15s^2 - 50s)}{ds} = 0$$

$$-3s^2 - 30s - 50 = 0$$

Or

$$3s^2 + 30s + 50 = 0$$

Thus, on solving, roots obtained will be **-2.113** and **-7.88**.

As the root -7.88 falls beyond the predicted section for the breakaway point thus s = -2.113 is the valid breakaway point.

Further, we can get the value of K on substituting the value of s= -2.113 in the equation,

$$K = (-2.113)^3 - 15(-2.113)^2 - 50(-2.113)$$

Therefore, on solving,

$$K = 48.11$$

Here, K obtained is a positive value, hence, **s = -2.113** is valid.

Now, we have to check the at what point the root locus intersects with the imaginary axis. Thus, for this routh array is used.

Here a proper method is used where the characteristic equation is used and routh array in terms of K is formed.

$$s^3 + 15s^2 + 50s + K = 0$$

Thus, the Routh's Array:

$$\begin{array}{c|cc} s^3 & 1 & 50 \\ s^2 & 15 & K \\ s^1 & \frac{750-K}{15} & 0 \\ s^0 & K & \end{array}$$

Now, to find K_{mar}, which is the value of K from one of the rows of routh's array as row of zeros, except the row s^0.

Considering, row s^1, 750 – K = 0

Thus, **K_m = 750**

Further, with the help of coefficients of the row which is present above the row of zero, an auxiliary equation A(s) = 0 is constructed. In this case,

$$A(s) = 15s^2 + K = 0$$

So, substituting the value of Km in the above equation, we will get,

$$15s^2 + 750 = 0$$

$$s^2 = -\frac{750}{15} = -50$$

Therefore,

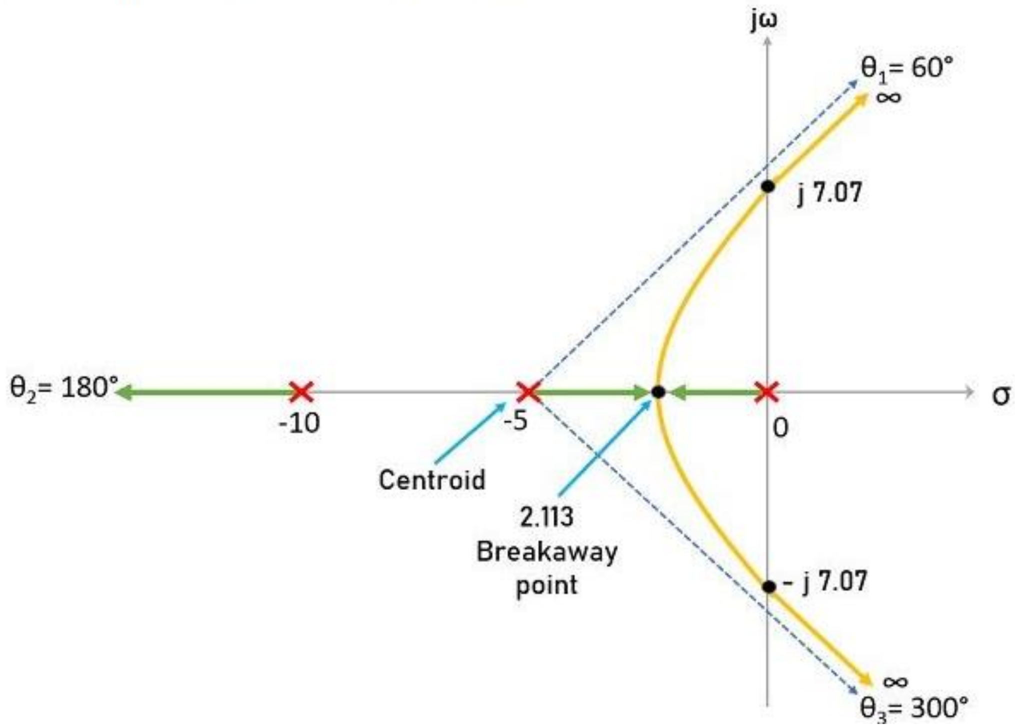
$$s = \pm j\sqrt{50}$$

$$s = + j\ 7.07, -j\ 7.07$$

Thus, these are the intersection points of the root locus with the imaginary axis.

Also, as the poles are not complex thus angles of departure not needed. Hence, at the breakaway point, the root locus breaks at ± 90°.

So, the complete root locus is given below:



From the above sketch, the stability of the system can be analyzed that for **K between 0 to 750** the system is **completely stable** as the complete root locus lies in the left half of s-plane. While at **K = 750**, the system is **marginally stable**.

While, for **K between 750 to** ∞, the system is **unstable** as the dominant roots proceed towards the right half of s-plane.

Example2: Consider that for the system with transfer function given below we have to sketch the root locus and predict its stability.

$$G(s)H(s) = \frac{K}{s(s^2 + 2s + 2)}$$

The characteristic equation provides the poles and zeros. So, writing the characteristic equation for the above system:

$$s(s^2 + 2s + 2) = 0$$

Thus, s = 0,

$$s = \frac{-2 \pm \sqrt{4-8}}{2}$$

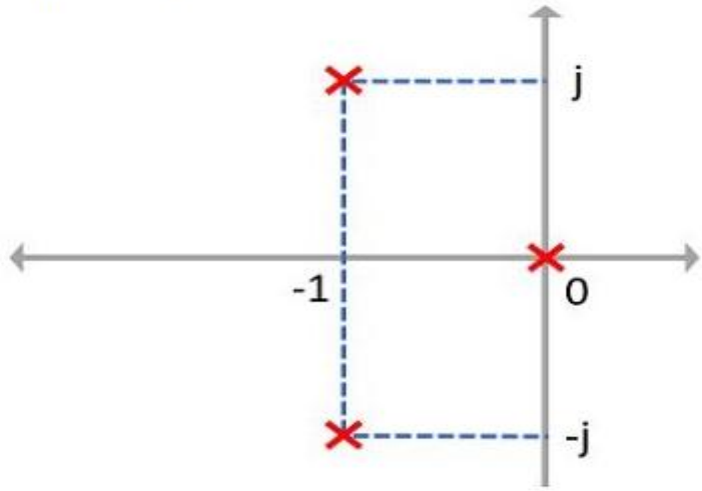
On solving

s = – 1 + j and -1 – j

Thus, here

P = 3, Z = 0 and as P > Z, so rule wise N = P = 3

The pole-zero plot is given below:



Here, it is clear that branch originating from s = 0 approaches -∞. And general predictions clear that there is no breakaway point here.

Angle of asymptotes

Since

$$\theta = \frac{(2q + 1)180°}{P - Z}$$

: q lies between 0 to P-Z-1

So, here, θ will be calculated for q = 0, 1 and 2.

$$\theta_1 = \frac{180°}{3} = 60°$$

$$\theta_2 = \frac{3 * 180°}{3} = 180°$$

$$\theta_3 = \frac{5 * 180°}{3} = 300°$$

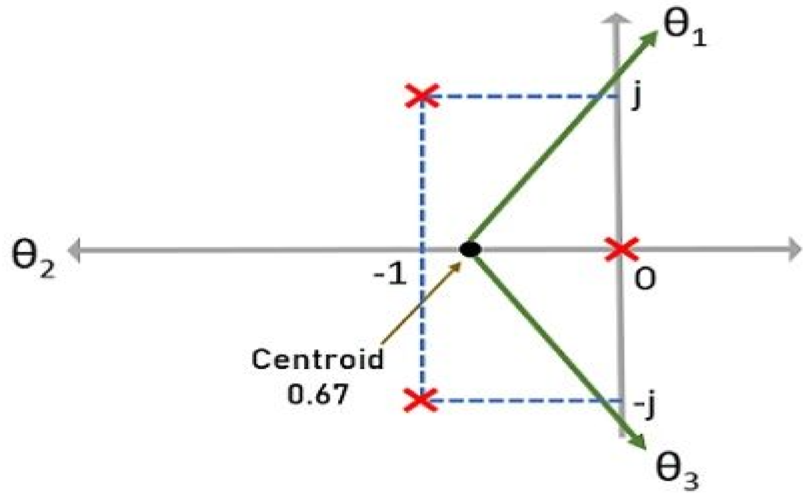
Now, centroid:

$$\sigma = \frac{\text{sum of real part of poles} - \text{sum of real part of zeros}}{P-Z}$$

$$\sigma = \frac{0-1-1-0}{3}$$

$$\sigma = \frac{-2}{3} = \ -0.67$$

So, with the help of the above analysis, the sketch of s-plane is given below:



With angle θ2, one branch from s = 0 approaches to infinity and with θ1 and θ3, branches starting from -1+j and -1-j respectively, approach infinity.

Now, let us check for the breakaway point.

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s^2+2s+2)} = 0$$

$$s^3 + 2s^2 + 2s + K = 0$$

$$K = -s^3 - 2s^2 - 2s$$

So, on differentiating,

$$\frac{dK}{ds} = 0$$

$$\frac{d(-s^3 \ - \ 2s^2 \ - \ 2s)}{ds} = 0$$

$$-3s^2 - 4s - 4 = 0$$

Or

$$3s^2 + 4s + 4 = 0$$

Breakaway points are calculated as:

$$s = \frac{-4 \pm \sqrt{16-24}}{6}$$

Therefore,

$$\mathbf{s = -0.67 \pm j\,0.47}$$

Now, as here we are having complex conjugates, thus, checking the validity of these points as breakaway point by using angle condition.

Testing, s = – 0.67 + j 0.47

$$\angle G(s)H(s) = \pm\,(2q + 1)180°$$

$$: q = 0, 1, 2$$

$$G(s)H(s) = \frac{K}{s(s+1-j)(s+1+j)}$$

As it is not an odd multiple of 180° thus, this point is not present on the root locus, hence there is no breakaway point here.

Further, checking for the intersection with the imaginary axis.

$$s^3 + 2s^2 + 2s + K = 0$$

Routh's array

s^3	1	2
s^2	2	K
s^1	$\frac{4-K}{2}$	0
s^0	-K	

The value of K_m = +4 makes $s^1 = 0$

Thus,

$$A(s) = 2s^2 + K = 0$$

Substituting $K_m = 4$, we get,

$$2s^2 + 4 = 0$$

$$s = \pm j\,1.414$$

Now, calculating angle of departure

At complex pole, **– 1 + j**

$$\varphi_d = 180° - \varphi$$

So, here $\varphi_{P1} = 135°$ and $\varphi_{P2} = 90°$

$$\varphi_d = 180° - (135° + 90°)$$

$$\varphi_d = -45°$$

Therefore, at the complex pole, **– 1 – j**,

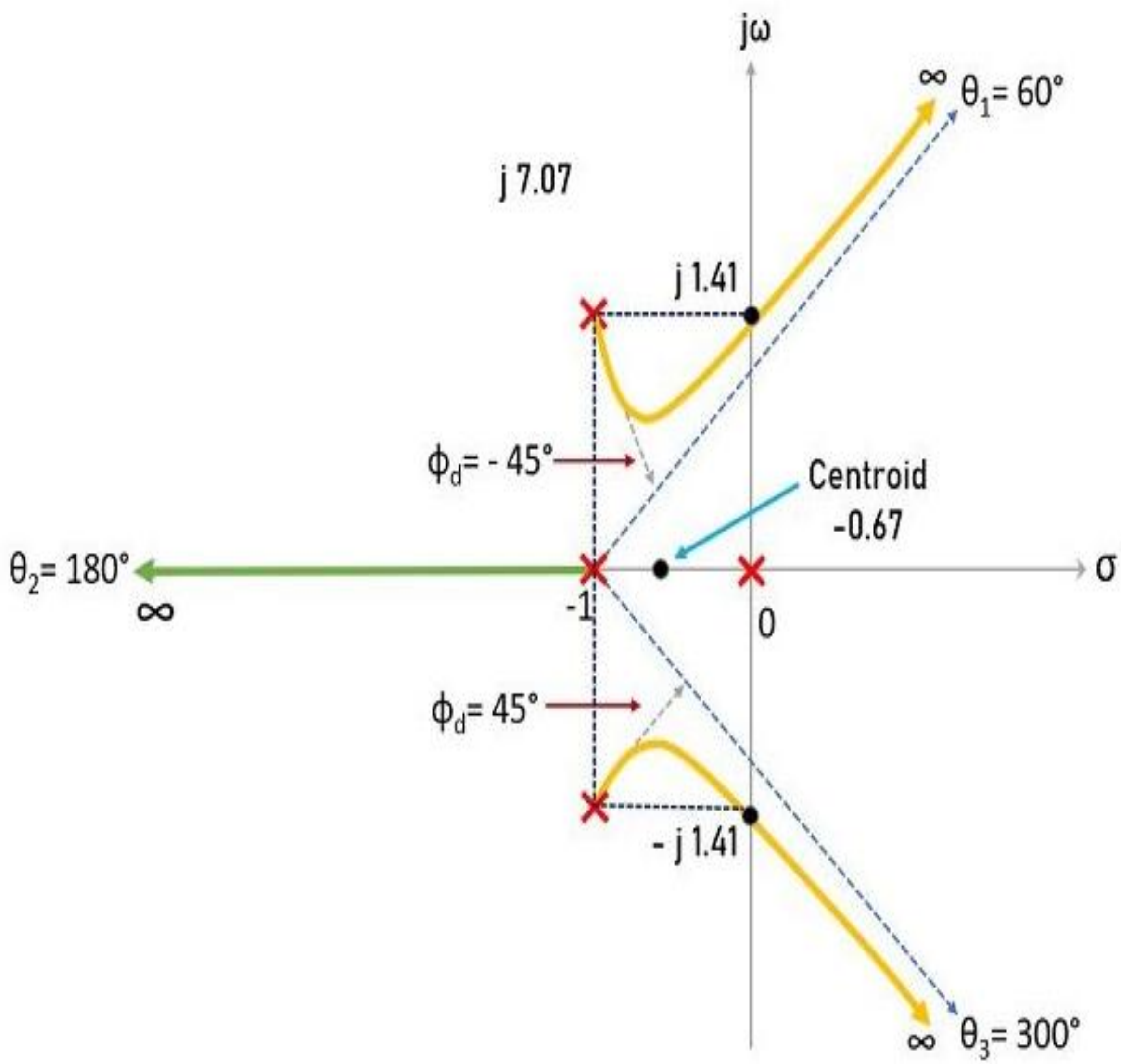
$$\varphi_d = +45°$$

So, with the above-determined values and parameters, the complete root locus sketch obtained is given below:

Now talking about stability, for **K between 0 to 4**, the roots are present on the left half of s-plane, representing a **completely stable system**.

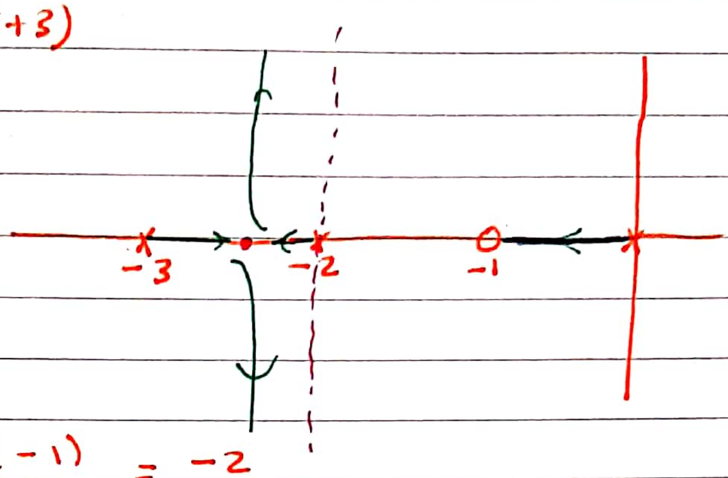
K = +4 makes the system **marginally stable** due to the presence of dominant roots on the imaginary axis. While for **K > 4**, the system becomes **unstable** as dominant roots lie in the right half of s-plane.

Thus, in this way by plotting the root locus, the stability of the system can be determined.



type one
Third order

Ex: $1 + \frac{K(s+1)}{s(s+2)(s+3)}$



$n - m = 3 - 1 = 2$

$\sigma_A = \frac{(0 - 2 - 3) - (-1)}{3 - 1} = -2$

~~$K = n - m - 1$~~ $= 3 - 1 - 1 = 1$

$\alpha_0 = \frac{180(1)}{2} = 90^\circ$

$\alpha_1 = \frac{180(3)}{2} = 270^\circ$

$\rightarrow \quad s(s+2)(s+3) + K(s+1) = 0 \Rightarrow (s^2 + 2s)(s+3) = -K(s+1)$

$-K = \frac{s^3 + 3s^2 + 2s^2 + 6s}{s+1} = \frac{s^3 + 5s^2 + 6s}{s+1} = 0$

$-\frac{\partial K}{\partial s} = 0 \Rightarrow 3s^2 + 10s + 6 = 0$

$s = -2.46..$ breakaway point

Scanned by CamScanner

→ PID Controller: propitinal - Integral - Derivative Controller

P K_p: Decrease the steady-state error

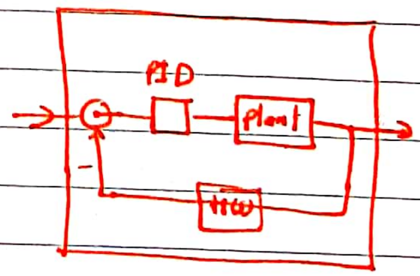
I K_I: Eliminate the steady-state error (add a pole in the origin $s=0$)

D K_D: Decrease the overshoot, and settling time
- the system become more stable.
or makes the system more stable

$$T(s)\Big|_{PID} = K_p + \frac{K_I}{s} + K_D s$$

$$= \frac{K_D s^2 + K_p s + K_I}{s}$$

PI, PD, PID



→ $T(s)\Big|_{PI} = K_p + \frac{K_I}{s}$

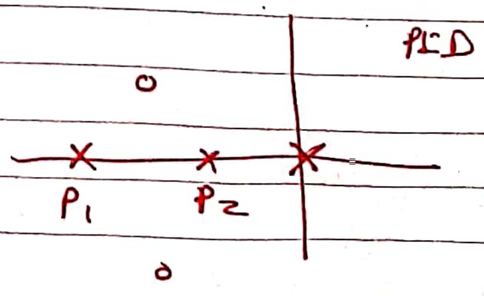
→ $T(s)\Big|_{PD} = K_p + K_D s$

$$\frac{1}{(s+p_1)(s+p_2)}$$

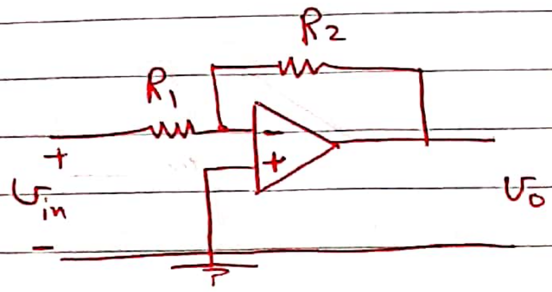
K_p: propotional gain

K_D = Derivative gain

K_I = integral gain

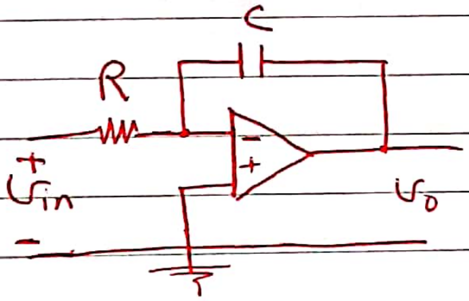


* PID – Controllers:



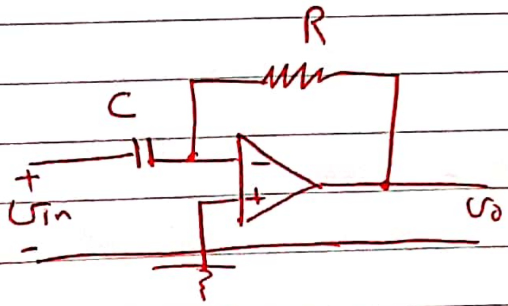
→ propotional Controller (P)

$$\frac{V_o(s)}{V_{in}(s)} = -\frac{R_2}{R_1}$$



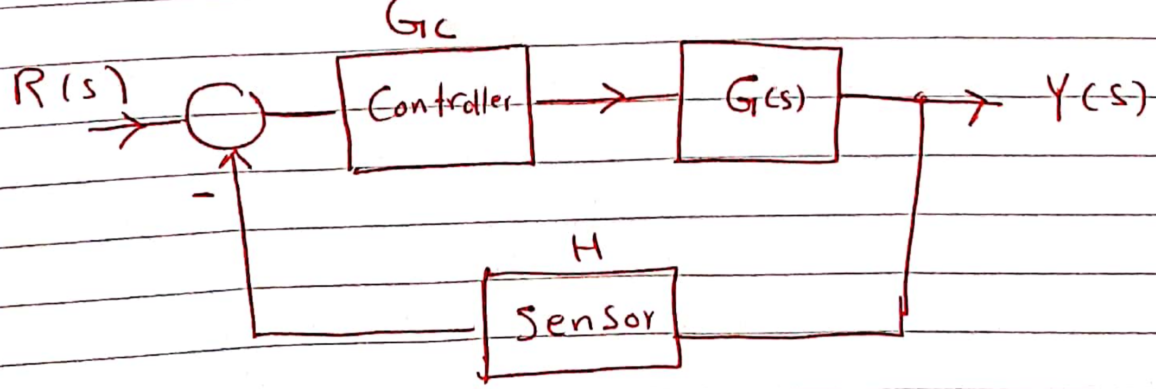
→ Integral Controller (I)

$$\frac{V_o(s)}{V_{in}(s)} = -\frac{1}{RCs}$$



→ Derivative Controller (D)

$$\frac{V_o(s)}{V_{in}(s)} = -RCs$$



$$\frac{Y(s)}{R(s)} = \frac{G_c\, G}{1 + G_c\, G\, H}$$

* * * G_c = Controller transfer function

G_c for PID $= K_P + \frac{K_I}{S} + K_D S$

$$= \frac{K_D S^2 + K_P S + K_I}{S}$$

PID added one pole at $S=0$
and Two zeros depending on
K_D, K_P, K_I values

K_P = propotional gain

K_I = Integral gain

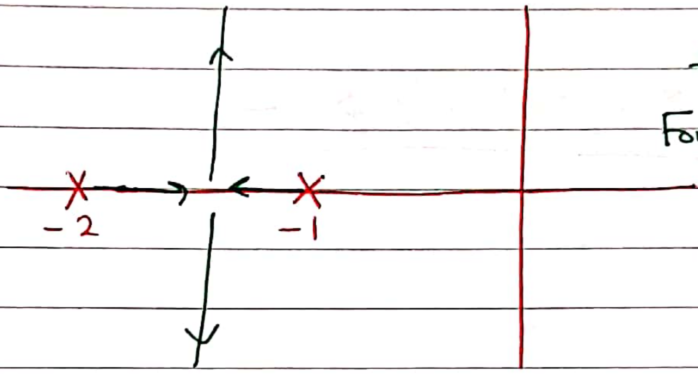
K_D = Derivative gain

Table 1: Effect of increasing parameter independently

Parameter	Rise Time	Overshoot	Settling Time	Steady-State Error	Stability
K_p	Decrease	Increase	Small Change	Decrease	Degrade
K_i	Decrease	Increase	Increase	Eliminate	Degrade
K_d	Minor Change	Decrease	Decrease	No Effect	Improve if K_d small

Ex: $T(s) = \dfrac{1}{(s+1)(s+2)}$

before



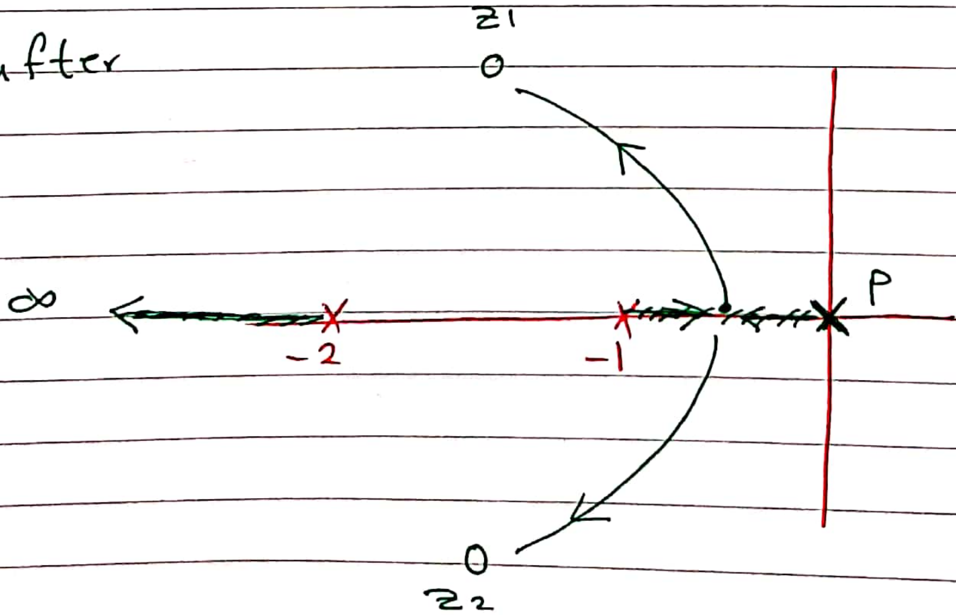
type zero

For a unit-step input

$e_{ss} = \dfrac{A}{1+K_p}$

add PID

after



type one

for a unit-step input →

$e_{ss} = 0$

↓

adding one pole at $s = 0$

→ eliminate the steady state error