

معادلات تفاضلية عادية 1

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للطالبة المبدعة
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إرادة - ثقة - تغيير

Subject Date No.

بسم الله الرحمن الرحيم

chapter 1 :- Differential Equation (DE)

Basic concepts

Def :- Equations that contain derivatives of a dependent variable respect

المعادلة التفاضلية هي معادلة فيها مشتقة بين :

For examples

$y'' - y = x^2$ Here $y = y(x)$

x : independent variable

y : dependent variable

unknown is $y = y(x)$ ← المجهول هو y والتي عبارة عن اقتران

$y' = \frac{dx}{dy}$

مشتقة جزئية

The equation

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3x \Rightarrow u = u(x, y)$ $\frac{\partial u}{\partial x}$ $\frac{\partial u}{\partial y}$

x, y independent

u dependent

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* classifications of DEs :-

الرتبة

1] order : The order of a DE is the order the highest derivative apper on the depandent vasible with secpect to the independint vasigbles.

رتبة المعادلة التفاضلية هي = رتبة اعلى مشتقة فيها

For example the DE

1] $y'' - y = x^3$ has order 2

2] $(y')^2 + y^4 = \sin x$ has order 1

3] $y^{(4)} + y''' = x^{10}$ order 4

$y''' = y^{(4)}$

2] Linear and Non linear Equation :-

A DE is called linear if it is linear of y and all it's derivative $(y', y'', y''', y^{(4)}, \ldots)$.

If the DE not linear, we call it NON linear

في تكملة ت

ي يكون انه كل المعادلة التفاضلية خطية لها الـ y وكل مشتقاتها في المعادلة يكونوا مرفوعين للقوة واحد وشي مضروبين بنفسهم وبعض

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1] $y'' + 3y = x^3$ (Linear) لأن الـ y و مشتقاتها مرفوعين لقوة واحد

2] $y' + (y^2) = 3x$ NON Linear 1st order DE

3] $yy'' + y' = \ln x$ NON Linear

↳ ممنوع وحدة منهم

مشتقاتها يكونوا مضروبين ببعض ممكن المجموع يبقا بس

4] $y'' + x^4 y = 3$ Linear DE

3] ordinary and partial DEs:-

ordinary Differential Equation ODE

If all derivative in the dependent variable are ordinary derivative, the DE is called ODE

$y = y(x)$ y dipendint on only one indipendint variable.

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* $y'' + y' - 2y = x$ ODE 2nd order linear ODE

* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is a PDE

* solutions of DEs :-

In general an ODE of order (n) can be written as :-

$$y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)})$$

↳ اكبر مشتقة داخل

مستحيل يكون بدون

$n+1$

* $\phi(x)$ is a solution to this ODE if $\phi, \phi', \phi'', \ldots, \phi^{(n)}$ exist and ϕ satisfies the ODE

$$\phi^{(n)} = f(\phi, \phi', \phi'', \ldots, \phi^{(n-1)})$$

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Example :-

$$y'' + y = 0$$

ادعي (ادعاء)

Claim: $y(x) = \cos x$ is a solution

verify :- $y(x) = \cos x$

$y' = -\sin x$

$y'' = -\cos x$

$y'' + y = 0$

$-\cos x + \cos x = 0$ ✓ عوضت بالمعادلة

so $y(x) = \cos x$ is a solution the DE

بس

الحل بالامثلة التفاضلية هو ايجاد الاقترانات التي لما اعوضه فيها بمعطيات المعادلة يحقق الحل.

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* Example : $y' + y = 0$

claim : $y(x) = e^{-x}$ is a solution

$y' = -e^{-x}$

$y' + y = -e^{-x} + e^{-x}$

$= 0$ ← صفر → صفر

$y = \varphi(x)$ Explicit form

$\Phi(x, y) = C$ Implicit form

* Example : Show that $y - \ln y = x^2 + 1$ difines an implicit solution to the DE

$$y' = \frac{2xy}{y-1}$$

$y - \ln y = x^2 + 1$

$$y' - \frac{y'}{y} = 2x$$

$$\frac{y'y - y'}{y} = 2x \Rightarrow y'(y-1) = 2xy$$

$$y' = \frac{2xy}{y-1}$$

so y satifies $y - 1$ the DE and so a solution ✱

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ممكن قيمة او اكثر

*Example: Find the value(s) of r such that $y(x) = e^{rx}$ is a solution to the DE $y'' + 5y' + 6y = 0$

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$y'' + 5y' + 6y = r^2 e^{rx} + 5re^{rx} + 6e^{rx} = 0$

$e^{rx}(r^2 + 5r + 6) = 0$

$e^{rx} \neq 0$, $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2$ $r = -3$

so $y(x) = e^{-2x}$, $y(x) = e^{-3x}$ are solution #

* Initial value problems

A DE $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$

along with initial conditions IC's

$$y(x_0) = \alpha_0$$
$$y'(x_0) = \alpha_1$$
$$y''(x_0) = \alpha_2$$
$$y^{(n-1)}(x_0) = \alpha_{(n-1)}$$

is called initial value problem IVP

$y' + 2y'' = x^2$, $y(2) = 3$

DE ⇓ , IC ⇓

⇒ IVP

* DE $\left[y'' - 2y' + 2y = 0 \right]$

2 IC's $\left[y(0) = 1 \right.$, $\left. y'(0) = -3 \right]$ ⇒ IVP

لازم يكون
نفس النقطة

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* Example :- $y(x) = e^x$ is a solution to the IVP $y' - y = 0$, $y(0) = 1$

$y(x) = e^x$

$y' = e^x$

1] $y' - y = e^x - e^x$

$= 0$ ✓

2] $y(x) = e^x$

$y(0) = e^0 = 1$ ✓

* Example :- give that $y(x) = C_1 e^x + C_2 e^{-x}$ is a solution to the DE : $y'' - y = 0$ for any choice of C_1, C_2. find C_1, C_2 so that the IC, satisfied $y(0) = 2 / y'(0) = 1$.

$y(x) = C_1 e^x + C_2 e^{-x}$ ⇒ genesal solution

$y(x) = C_1 e^x + C_2 e^{-x}$

$y' = C_1 e^x - C_2 e^{-x}$

$y'' = C_1 e^x + C_2 e^{-x}$

$y'' - y = C_1 e^x + C_2 e^{-x} - C_1 e^x - C_2 e^{-x}$

$= 0$ ✓

$y(0) = C_1 e^0 + C_2 e^0$

$\boxed{2 = C_1 + C_2}$

$y'(0) = C_1 e^0 - C_2 e^0$

$\boxed{1 = C_1 - C_2}$

$C_1 + C_2 = 2$

$C_1 - C_2 = 1$

$C_1 = \frac{3}{2} \Rightarrow C_2 = \frac{1}{2}$

$y(x) = \frac{3}{2} e^x + \frac{1}{2} e^{-x}$ → pacticulas solation.

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* First order ODE

The general form of a first order ODE is

$$y' = f(x, y)$$

A linear order ODE has the form

$$a_1(x)\, y' + a_2(x)\, y = a_3(x)$$

For example :-

$y' + y^2 = \sin x$ nonlinear 1st order ODE

$y' + x^2 y = x^{10}$ linear 1st order ODE

* The general form of a first order ODE is:

$y' = f(x, y)$ → Existance

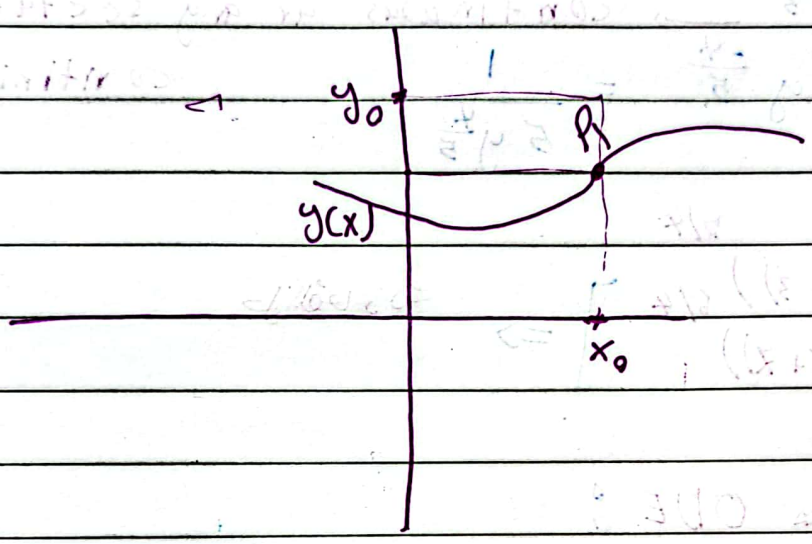
$y' = f(x, y)$ → Uniqueness

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* Theorem: consider the IVP $y' = f(x,y)$, $y(x_0) = y_0$

let R be a rectangle contins (x_0, y_0) (a)

If $f(x,y)$ continuous in R, Run the IVP has a solution in an open intival $((x_0 - s), (x_0 + s))$



b] If in addition $\frac{\partial f}{\partial y}$ continous on R have the IVP has a unique solutions on a n interval $(x_0 - s, x_0 + s)$

* For Example : $y' - 2xy = x^3$ → $f(x,y) = x^3 + 2xy$

$y' = x^3 + 2xy$

$\Downarrow$

$y' = f(x,y)$

$\frac{\partial f}{\partial y} = 2x$

* For example : ~~$y' + 2xy = x^2$~~

~~$y' = x^3 + 2xy$~~

~~$\frac{\partial f}{\partial y} = 2x$~~

$y' = y^{\frac{1}{5}}$, $y(0) = 0$

Here $f(x,y) = y^{\frac{1}{5}}$ ⟶ continuous on any rectangel containig $(0,0)$.

$$\frac{\partial f}{\partial y} = \frac{1}{5} y^{\frac{-4}{5}} = \frac{1}{5 y^{\frac{4}{5}}}$$

$$\left.\begin{aligned} y_1(x) &= \left(\tfrac{4}{5}(x+3)\right)^{5/4} \\ y_2(x) &= -\left(\tfrac{4}{5}(x+2)\right)^{5/4} \end{aligned}\right] \Rightarrow$$ حل للمعادلة

* linear First order ODE :

→ The general Form of a first order ODE :

$$[a_1(x)\cdot y' + a_2(x)\, y = a_3(x)]$$

Divide by $a_1(x)$ to get

$$y' + \frac{a_2(x)}{a_1(x)} y = \frac{a_3(x)}{a_1(x)} \Rightarrow y' + p(x) y = g(x) \quad \hookrightarrow (1)$$

This is called the standard Form of a first order linear ODE, Let $\mu(x)$ be a function to be determined lat

* Multiply Equation (1) by $\mu(x)$

$$\mu(x) y' + \mu(x) p(x) y = \mu(x) g(x)$$

~~$\mu(x) y' + \mu(x) p(x) y = \mu(x) g(x)$~~

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This is equivelent to :

$$\mu(x)y' + \mu'(x)y - \mu'(x)y + \mu(x)p(x)y = \mu(x)g(x)$$

$$(\mu(x)y)' + (\mu(x)p(x) - \mu'(x)).y = \mu(x)g(x)$$

* we choose $\mu(x)$ such that :

$$\mu(x)p(x) - \mu'(x) = 0 \Rightarrow \frac{\mu(x)p(x)}{\mu(x)} = \frac{\mu'(x)}{\mu(x)}$$

$$\int p(x)dx = \int \frac{\mu'(x)}{\mu(x)}dx$$

$$\ln \mu(x) = \int p(x)dx \Rightarrow \mu(x) = e^{\int p(x)dx}$$

* choosing $\Rightarrow \mu(x) = e^{\int p(x)dx}$

reducen the tqst equation to

$$(\mu(x)y)' = \mu(x)g(x)$$

$$\mu(x)y = \int \mu(x)g(x).dx + C$$

$$y(x) = \frac{\int \mu(x)g(x)dx + C}{\mu(x)}$$

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* solving 1st order linear ODE

1] make sure the DE in standard from

$$y' + p(x)y = g(x)$$

2] Determine $p(x)$ and $g(x)$

3] compute the integrating factor $\mu_w = e^{\int p(x)dx}$

4] The genral solution is

$$y(x) = \frac{\int \mu_w g(x)\,dx + C}{\mu(x)}$$

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Ex: Solve the DE, $xy' - 2y = x^2$

This is a first order linear ODE, To put the DE in standerd Form, we divide by x

$$y' - \frac{2}{x}y = x \Rightarrow p(x) = \frac{-2}{x}, \quad g(x) = x$$

we compute $\mu(x)$:

$$\mu(x) = e^{\int \frac{-2}{x} dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} \Rightarrow \underline{\mu(x) = x^{-2}}$$

* Now, The general solution:

$$y(x) = \frac{\int \mu(x)\, g(x)\, dx + C}{\mu(x)}$$

$$= \frac{\int x^{-2} \cdot x \, dx + C}{x^{-2}}$$

$$= x^2(\ln x + C) = x^2 \ln x + x^2 C$$

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* solving 1st osder linear ODE

1] make suse the DE in standard fsom

$$y' + p(x)y = g(x)$$

2] Determine $p(x)$ and $g(x)$

3] compute the integrating factor $\mu_w = e^{\int p(x)dx}$

4] The genrq' solution is

$$y(x) = \frac{\int \mu_w g(x)\,dx + C}{\mu(x)}$$

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* problem :- solve the DE : $(x^2+1)y' + xy = x$

we first put this 1st order ODE in standard from :-

1- اول شي بدنا نخلي معامل $y' = 1$

$$(x^2+1)y' + xy = x$$

$$y' + \frac{xy}{x^2+1} = \frac{x}{x^2+1}$$

$p(x)$ $\qquad$ $g(x)$

* The integrating factor is *

$$M(x) = e^{\int \frac{2x}{x^2+1} dx} = e^{\frac{1}{2}\int \frac{2x}{x^2+1}} = e^{\frac{1}{2}\ln(x^2+1)}$$

$$= (x^2+1)^{\frac{1}{2}} \; **$$

* The general solution *

$$y(x) = \int \frac{(x^2+1)^{\frac{1}{2}} \cdot \frac{x}{x^2+1} dx + C}{(x^2+1)^{\frac{1}{2}}}$$

خليها ذمتي I

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$$* \; I = \int \frac{(x^2+1)^{\frac{1}{2}} \; x}{x^2+1} \, dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$$I = \int (x+1)^{\frac{-1}{2}} x \, dx$$

$$= \int u^{\frac{-1}{2}} \times \frac{du}{2x} = u^{\frac{1}{2}} + C = (x^2+1)^{\frac{1}{2}} + C$$

$$y_{(x)} = \frac{(x^2+1)^{\frac{1}{2}} + C}{(x^2+1)^{\frac{1}{2}}} \; *$$

problem : solve the IVP $xy' + 2y = \sin x$, $y(\frac{\pi}{2}) = 1$

* The DE in standard form is

$$y' + \frac{2}{x} y = \frac{\sin x}{x}$$

$p(x)$ ← 1 ; $\frac{\sin x}{x}$ ↳ $g(x)$

* The integrating factor is

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$$

* The general solution is

$$y(x) = \frac{\int x^2 \frac{\sin x}{x} \, dx + C}{x^2}$$

I

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$$I = \int x \sin x \, dx$$

$u = x \longrightarrow du = 1\,dx$

$dv = \sin x\,dx \Longrightarrow v = -\cos x$

$$= -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + C$$

$$y(x) = \frac{-x\cos x + \sin x + C}{x^2}$$

* using the IC $y\left(\frac{\pi}{2}\right) = 1$

$$\frac{-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + C}{\frac{\pi^2}{4}} = 1$$

$$1 + C = \frac{\pi^2}{4} \Rightarrow C = \frac{\pi^2}{4} - 1$$

The solution to the IVP is

$$y(x) = \frac{-x\cos x + \sin x + \frac{\pi^2}{4} - 1}{x^2} \quad \#$$

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* problem : solve the DE by regarding y as the independent variable rather than x

$$(1+2xy)\frac{dy}{dx} = 1+y^2$$ Linear في x لأنها بقلبها

$$\frac{dx}{dy} = \frac{1+2xy}{1+y^2}$$

$$\frac{dx}{dy} = \frac{1}{1+y^2} + \frac{2y}{1+y^2}x$$

$$\frac{dx}{dy} - \frac{2y}{1+y^2}x = \frac{1}{1+y^2} \Rightarrow \text{Linear in } x$$

$$x' \qquad P(y)x = g(y)$$

$$\mu(y) = e^{\int \frac{-2y}{1+y^2}dy} = e^{-\ln(1+y^2)} = (1+y^2)^{-1}$$

$$x(y) = \int \frac{(1+y^2)^{-1} \times \frac{1}{1+y^2} + C}{\frac{1}{1+y^2}}$$

بالتكامل

$$x(y) = \frac{\int \frac{1}{(1+y^2)^2}dy + C}{\frac{1}{1+y^2}}$$ لا تنسى تشوفي التكملة

* problem 2 : Find the Longest interval in which the solution of the IVP is certain to exist :

$$x y' + \tan x\, y = \sin x$$

$$y(\pi) = 0$$

* بدنا نطلع y' لحالها (موضوع قانون)

$$y' + \tan x = \frac{\sin x}{x}$$

* بطلع الفترة اللي بتتوفر بالنسبة لـ y و y'

$$y' = \boxed{\frac{\sin x}{x} - \frac{\sin x\, y}{x \cos x}}$$

$f(x, y)$

لـ اول شي خليت الـ y' لحالها

$\frac{\partial f}{\partial y}$ بكون دائما متغيرات

لازم f و f' بالنسبة لـ y يكونوا متصلات

$$f(x,y) = \frac{\sin x}{x} - \frac{\sin x}{x \cos x} y$$

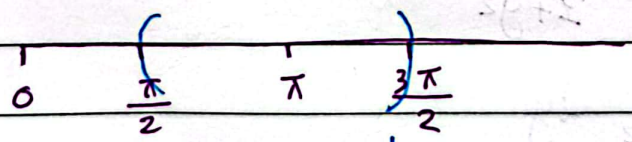
غير متصل عند اصفار المقام

$$\frac{\partial f}{\partial y} = \frac{-\sin x}{x \cos x}$$

غير متصل عند اصفار المقام

$x = 0 \;/\; \cos x = 0$

$x = \pm\frac{\pi}{2} \;/\; x = \pm\frac{3\pi}{2} \;/\; \ldots$



لـ هاي الفترة متصل عند $f(x,y)$ والـ $\frac{\partial f}{\partial y}$

وموجود بينتمي الـ π

$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

* ليش ما اخدنا الفترة من الـ $\left(0, \frac{3\pi}{2}\right)$ ؟؟؟

* Bernoulli Equations ☺

The first order ODE of the form

$$y' + p(x)y = g(x)\,y^n \qquad n \neq 0, 1$$

(non linear ODE)

is called Bernoulli equations

- To solve Bernoulli Equation we use the substitution

$$v = y^{1-n}$$

Differentiate $v' = (1-n)\,y^{-n}y'$

multiply Bernoulli Equation by $(1-n)\,y^{-n}$

$$(1-n)y^{-n}y' + (1-n)p(x)y^{-n}y = (1-n)g(x)y^{-n}$$

$$v' + (1-n)p(x)v = (1-n)g(x)$$

linear

This a linear ODE of v solve it to find v, then use $v = y^{1-n}$ to find y.

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Bernoulli Equations :

$$y' + p(x)\,y = g(x)\,y^n \quad , \quad n \neq 0, 1$$

let $v = y^{1-n}$

$$v' = (1-n)\,y^{-n}\,y'$$

$$(1-n)\,y^{-n}y' + (1-n)\,p(x)\,y$$

* Ex: solve the DE

$$y' - \frac{3}{x}\,y = x^2\sqrt{y}$$

$$y' - \frac{3}{x}y = x^2 (y)^{\frac{1}{2}}$$

This is a bernoulli equations with $n = 1/2$

let $v = y^{1-n} \Rightarrow y = y^{1-\frac{1}{2}} = y^{1/2}$

$$v = y^{1/2}$$

$$v' = \frac{1}{2} y^{-\frac{1}{2}}\,y'$$

$\frac{1}{2}$ ← اضرب الطرفين بالمقلوب

y' بعد

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$$\frac{1}{2} y^{-1/2} y' - \frac{3}{2x} y^{\frac{1}{2}} = \frac{1}{2} x^2$$

$$v' - \frac{3}{2x} v = \frac{1}{2} x^2$$

This is lenear eq,uqtion

Here $p(x) = \frac{-3}{2x}$, $g(x) = \frac{1}{2} x^2$

$$\mu(x) = e^{\int \frac{-3}{2x} dx} = e^{\frac{-3}{n} \ln x} = x^{\frac{-3}{2}}$$

$$v(x) = \frac{\int x^{\frac{-3}{2}} \frac{1}{2} x^2 dx + c}{x^{\frac{-3}{2}}}$$

$$= \frac{\frac{1}{2} \int x^{\frac{1}{2}} dx + c}{x^{-\frac{3}{2}}} = x^3 + C x^{\frac{3}{2}}$$

But $\Rightarrow v = y^{\frac{1}{2}}$

$$y^{\frac{1}{2}} = \frac{1}{3} x^3 + C x^{\frac{3}{2}}$$

implicit solution of the DE

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قابل للفصل

*: separable equations :

A first order DE is called separable if it can be written as $h(y)\,dy = g(x)\,dx$

$y' = \frac{dy}{dx}$ ⇒ differentials form

المشتقة هي الميل

التعبير في y على التعبير في x في الطرف الأيمن

إذا قدرت تفصل الـ y و الـ x كل واحد بجهة يبقى المعادلة separable

Ex : $y' = x\,Ln(y^{2x}) + 8x^2$

$\frac{dy}{dx} = 2x^2 Ln\,y + 8x^2$

$\frac{dy}{dx} = 2x^2 (Ln\,y + 4)$

$\frac{dy}{(Ln\,y+4)} = 2x^2\,dx$ The DE is separable

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* $y' - 2y = x^2$ مش كل معادلة بقدر افصل
لازم y على لازم x فيها

$\frac{dy}{dx} = 2y + x^2$ ← not separable

* To solve a separable equation just integrate both side to get an implisit solution.

• Example: solve the DE

$$y' = \frac{x^2+1}{y^3+2}$$

$$\frac{dy}{dx} = \frac{x^2+1}{y^3+2}$$

$(y^3+2)\,dy = (x^2+1)\,dx$ This DE separable

Integrate both said

$$\int (y^3+2)\,dy = \int (x^2+1)\,dx$$

$\frac{y^4}{4} + 2y = \frac{x^3}{3} + x + C$ This is implisit solution the DE

Example 8 solve the ~~DE~~ IVP

$$y' = \frac{x^2y - 4y}{x+2}, \quad y(0) = 2$$

$$\frac{dy}{dx} = \frac{y(x^2-4)}{x+2}$$

$$y' - \frac{y(x^2-4)}{x+2} = 0$$

Lenear / seperable

$$\frac{dy}{dx} = \frac{y(x^2-4)}{x+2}$$ This is seperable equation To solve it, integrate the left-hand with respect to y and the right-hand side with respect to x

$$\frac{1}{y}\,dy = \frac{(x+2)(x-2)}{x+2}\,dx$$

$$\int \frac{1}{y}\,dy = \int (x-2)\,dx$$

$$\ln y = \frac{x^2}{2} - 2x + C$$

$$\ln(2) = 0 - 0 + C$$

$$\boxed{\ln 2 = C}$$

$$\ln y = \frac{x^2}{2} - 2x + \ln y \Rightarrow$$ implicit solution to the IVP.

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Ex:- solve the DE

$$y' = e^{x+2y}$$

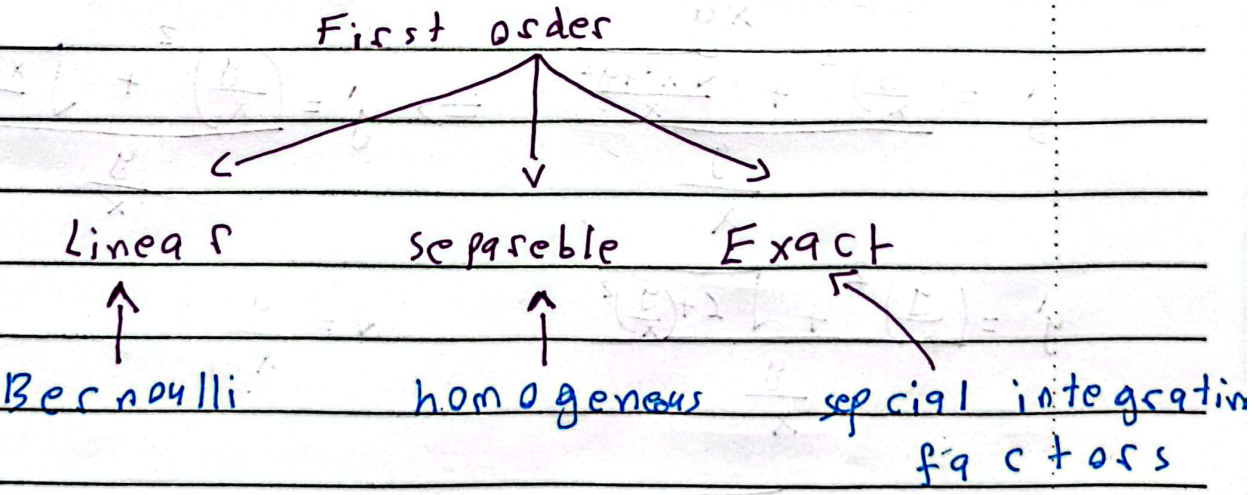
$$\frac{dy}{dx} = e^x \cdot e^{2y}$$

$$\frac{1}{e^{2y}}\,dy = e^x\,dx$$

$$\int e^{-2y}\,dy = \int e^x\,dx$$

$$\frac{e^{-2y}}{-2} = e^x + C$$

بسم الله الرحمن الرحيم 22/Oct/2023



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* Homogeneous Equations

If the DE $y' = f(x,y)$ can be written in the form $y' = G\left(\frac{y}{x}\right)$ then the DE is called homogeneous

* Examples

1] $y' = \frac{x-3y}{2x+y} = \frac{1-\frac{3y}{x}}{2+\frac{y}{x}} = \frac{1-3\left(\frac{y}{x}\right)}{2+\left(\frac{y}{x}\right)}$

let $v = \frac{y}{x}$

$y' = \frac{1-3v}{2+v} = G(v) = G\left(\frac{x}{x}\right)$

2] $y' = \frac{y^2 + x\sqrt{x^2+y^2}}{xy}$ x^2 we pull.

$y' = \frac{\left(\frac{y}{x}\right)^2 + \frac{\sqrt{x^2+y^2}}{x}}{\frac{y}{x}} \Rightarrow y' = \frac{\left(\frac{y}{x}\right)^2 + \sqrt{\frac{x^2+y^2}{x^2}}}{\frac{y}{x}}$

$y' = \frac{\left(\frac{y}{x}\right)^2 + \sqrt{1+\left(\frac{y}{x}\right)^2}}{\frac{y}{x}}$ $v = \frac{y}{x}$

$y' = \frac{v^2 + \sqrt{1+v^2}}{v}$

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* Considered the homogeneous DE

$$y' = G\left(\frac{y}{x}\right)$$

To solve it, use the substution

$v = \frac{y}{x} \Rightarrow y = xv$ Differentiate to get

$y' = x v' + v(1)$

$\frac{dy}{dx}$

Substituting in the DE

$$y' = G\left(\frac{y}{x}\right)$$

$$x v' + v = G(v)$$

$$x \frac{dv}{dx} = G(v) - v$$

v'

$$\frac{1}{G(v)-v}\, dv = \frac{1}{x}\, dx$$ a separable DE

* solve it for v, then substitute $v = \frac{y}{x}$

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* Example: solve the DE

$$y' = \frac{y^2 - 3x^2}{xy - x^3 y^{-1}} \quad / \div x^2$$

$$y' = \frac{(\frac{y}{x})^2 - 3}{\frac{y}{x} - \frac{x}{y}} \Rightarrow y' = \frac{(\frac{y}{x})^2 - 3}{\frac{y}{x} - (\frac{y}{x})^{-1}} = G(\frac{y}{x})$$

This is homogeneas DE we use the substitation

Let $v = \frac{y}{x} \rightarrow y = vx$

$y' = (x)(v') + v$

$$y' = \frac{v^2 - 3}{v - (v)^{-1}}$$

$$xv' + v = \frac{v^2 - 3}{v - (v)^{-1}}$$

$$x\frac{dv}{dx} = \frac{v^2 - 3}{v - (v)^{-1}} - v$$

$$x\frac{dv}{dx} = \frac{v^2 - 3 - v^2 + 1}{v - v^{-1}}$$

$$x\frac{dv}{dx} = \frac{-2}{v - v^{-1}} \cdot \frac{xv}{v}$$

$$x\frac{dv}{dx} = \frac{-2v}{v^2 - 1} \Rightarrow \frac{v^2 - 1}{-2v}\,dv = \frac{1}{x}\,dx$$

وهكذا بتكمل بس

$$x\frac{dv}{dx} = \frac{v^2-3}{v-v^{-1}} - v = \frac{(v^2-3)-(v^2-1)}{v-v^{-1}}$$

$$\frac{x\,dv}{dx} \cdot \frac{-2}{v-v^{-1}} = \frac{-2v}{v^2-1}$$

$$\int \frac{v^2-1}{v}\,dv = \frac{-2}{x}\,dx$$

$$\int \left(v-\frac{1}{v}\right)dv = -2\ln x + c$$

$$\frac{v^2}{2} - \ln v = -2\ln x + c$$

$$\text{so } \frac{y^2}{2x^2} - \ln\left(\frac{y}{x}\right) = -2\ln x + c$$

in a implicit solution ✻

Subject

Example: $\frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x}$

$\frac{dy}{dx} = \frac{y\left(\ln\left(\frac{y}{x}\right) + 1\right)}{x}$

$\frac{dy}{dx} = \frac{y}{x}\left(\ln\left(\frac{y}{x}\right) + 1\right)$ This is a homogeneous DE

$v = \frac{y}{x}$

$y = vx \longrightarrow y' = (v) + (x)(v')$

$xv' + v = v(\ln v + 1)$

$xv' + v = v\ln v + v$

$xv' = v\ln v$

$x\frac{dv}{dx} = v\ln v \Rightarrow \int \frac{1}{x}\,dx = \int \frac{1}{v\ln v}\,dv$

وكاملين

لو حد او عايزو باجزاء

عند هيك تقوريلن

$\ln(\ln(\frac{y}{x})) = \ln x + C$

الجواب

* Exact Equations *

Any 1st order ODE

$$y' = f(x, y)$$

Can be written in the differentials form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

* For example:

$$y' = \frac{2x^2y + 3x - 2}{xy^3 + e^{xy}}$$

$$\frac{dy}{dx} = \frac{2x^2y + 3x - 2}{xy^3 + e^{xy}}$$

$$(xy^3 + e^{xy})\,dy = (2x^2y + 3x - 2)\,dx$$

$$(xy^3 + e^{xy})\,dy + (2 - 2x^2y - 3x)\,dx = 0$$

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* Example: consider the DE

$$(2xy + y^2)\,dx + (2 + y)\,dy = 0$$

$$\Downarrow$$

$$\frac{d}{dx}(x^2 + xy^2)\,dx + \frac{d}{dy}(x^2 + xy)\,dy = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$f(x,y)$ $\qquad$ $f(x,y)$

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0 \Rightarrow dF = 0$$

this is

$F(x,y) = C$ is a solution to the DE ✱

$x^2 + xy^2 = C$ implicit solution ✱

Subject * بسم الله الرحمن الرحيم * Date 24/Oct/2023 No.

* Exact Equations *

$y' = f(x,y)$

can be written in differentials form

$M(x,y)\,dx + N(x,y)\,dy = 0$

Example of an Exact Equation

$(2x+y^2)\,dx + (2xy)\,dy = 0 \Rightarrow \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$

$M(x,y)$ $N(x,y)$

$dF = 0$

~~$F(x,y)$ such that~~

$F(x,y) = x^2 + xy^2$ such that

$\frac{\partial F}{\partial x} = 2x + y^2$

$\frac{\partial F}{\partial y} = 2xy$

* This give $F(x,y) = C$ * implicit solution

* The DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is called Exact if there is a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$.

* Theorem *

The DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Example :

$(2xy)$
$(2x + y^2)\,dx + (2xy)\,dy = 0$

Here, $M(x,y) = 2x + y^2$ / $N(x,y) = 2xy$

$\frac{\partial M}{\partial y} = 2y$ / $\frac{dN}{\partial x} = 2y$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the DE is Exact

If the DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is Exact. then these is a function $F(x,y)$ such that

$\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$

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If an find such $F(x,y)$, then $F(x,y) = C$ is an implicit to the DE.

Example: consider the DE

$$(2x + y^2)\,dx + (2xy)\,dy = 0$$

Here, $M(x,y) = 2x + y^2$ / $N(x,y) = 2 + y$

Check the Exactness?

$$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial x} = 2y$$

since $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the DE is Exact

Thus, these is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) = 2x + y^2$$

and

$$\frac{\partial F}{\partial y} = N(x,y) = 2xy$$

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we need to find such $F(x,y)$

using that $\frac{\partial F}{\partial x} = 2x + y^2$

$$F = \int (2x + y^2)\, dx$$

$$F = x^2 + xy^2 + h(y)$$

اي شي بدلالة y يعتبر ثابت

To determine $F(x,y)$ we need to find $h(y)$

using that $\frac{\partial F}{\partial y} = 2xy$

$$2xy + h'(y) = 2xy$$

$$h'(y) = 0$$

$$h(y) = C_1$$

$$F(x,y) = x^2 + xy^2 + h(y)$$

اشتقينا بالنسبة لـ y

$$F(x,y) = x^2 + xy^2 + C_1$$

$$F(x,y) = C$$

$$x^2 + xy^2 = C_2$$

is an implicit solution

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* Example : solve the DE

$$(y e^{xy}\tan x + e^{xy}\sec^2 x)\,dx + (x e^{xy}\tan x)\,dy = 0$$

$$M(x,y) = y e^{xy}\tan x + e^{xy}\sec^2 x$$

$$N(x,y) = x e^{xy}\tan x$$

* chick the Exactness ?

$$\frac{\partial M}{\partial y} = \tan x\, e^{xy}(1+xy) + x\sec^2 x\, e^{xy}$$

$$\frac{\partial N}{\partial x} = \tan x\, e^{xy}(1+xy) + x\sec^2 x\, e^{xy}$$

since $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the DE is Exact

Then these is a function $F(x,y)$ satisfie

$$\frac{\partial F}{\partial x} = M(x,y) = y e^{xy}\tan x + e^{xy}\sec^2 x$$

$$\frac{\partial F}{\partial y} = N(x,y) = x e^{xy}\tan x$$

بنكامل واحده واحده اسهل

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* To find $F(x,y)$, we start with

- $\frac{\partial F}{\partial y} = xe^{xy}\tan x$

$$F(x,y) = \int xe^{xy}\tan x\, dy$$

$$F(x,y) = x\tan x\,\frac{e^{xy}}{x} + h(x)$$

$$F(x,y) = \tan x\, e^{xy} + h(x) \longrightarrow \frac{\partial F}{\partial x} = e^{xy}\sec^2 x + ye^{xy}\tan x + h'(x)$$

* To determine $F(x,y)$ we use

- $\frac{\partial F}{\partial x} = ye^{xy}\tan x + e^{xy}\sec^2 x$

$$e^{xy}\sec^2 x + ye^{xy}\tan x + h'(x) = ye^{xy}\tan x + e^{xy}\sec^2 x$$

$\therefore$ Dengan $h'(x) = 0$

$$h(x) = C_1$$

Then $F(x,y) = e^{xy}\tan x + C_1$

$$F(x,y) = C$$

$$e^{xy}\tan x = C_2$$

→ implicit solution

* Example : Find the most general function $M(x,y)$ so that the DE is Exact

$$M(x,y)\,dx + (\sin x \cos y - xy - e^{-y})\,dy = 0$$

since the DE is exact, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \cos x \cos y - y$$

$$M(x,y) = \int (\cos x \cos y - y)\,dy$$

$$M(x,y) = \cos x \sin y - \frac{y^2}{2} + h(x)$$

The most general function

لانا لحد دلوقتي ال h(x)

Subject بسم الله الرحمن الرحيم Date 26/oct/2023 No.

* Special Integrating Factors :

* Consider the DE

$$\underbrace{(2y - 6x)}_{M(x,y)}dx + \underbrace{(3x - 4x^2y^{-1})}_{N(x,y)}dy = 0$$

check Exactness :

$$\frac{\partial M}{\partial y} = 2 \quad , \quad \frac{\partial N}{\partial x} = 3 - 8y^{-1}x$$

since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the DE is not exact.

. multiply the DE by the function $\mu = xy^2$

$$(2xy^3 - 6x^2y^2)dx + (3x^2y^2 - 4x^3y)dy = 0$$

$M(x,y)$ $N(x,y)$

$$\frac{\partial M}{\partial y} = 6xy^2 - 12x^2y \quad , \quad \frac{\partial N}{\partial x} = 6xy^2 - 12x^2y$$

After multiplying the DE by $\mu = xy^2$ The DE becomes exact.

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*consider the Non Exact DE

$M(x,y)\,dx + N(x,y)\,dy = 0$

A function $M(x,y)$ is called an integrating factor to the DE if when we multiply the DE by $M(x,y)$ the DE become Exact.

*من كلمة integrating ... نقدر نقول طالبة بعد ما نضيف الـ factor

* Let $M(x,y)$ be an integrating factor to $M(x,y)\,dx + N(x,y)\,dy = 0$

Then

$M(x,y)\,M(x,y)\,dx + M(x,y)\,N(x,y)\,dy = 0$

is Exact

$\frac{\partial}{\partial y}(MM) = \frac{\partial}{\partial x}(MN)$

((باستخدام قاعدة الضرب

$M\frac{\partial M}{\partial y} + M\frac{\partial M}{\partial y}M = M\frac{\partial N}{\partial x} + M\frac{\partial M}{\partial x}N$

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* Let's assume that $M = M(x)$

$$M\frac{\partial M}{\partial y} = M\frac{\partial N}{\partial x} + N\frac{\partial M}{\partial x},$$

$$N\frac{\partial M}{\partial x} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)M$$

$$\frac{M'}{M} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

* If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ function of x only

$$M(x) = e^{\int \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right) dx}$$

* one the other side, assume that $M = M(y)$

$$M\frac{\partial M}{\partial y} + \frac{\partial M}{\partial y}M = M\frac{\partial N}{\partial x}$$

$$\frac{M'}{M} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$ → فرض میکنیم که فقط بر حسب y باشد

If $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ depends on y only then

$$M(y) = e^{\int \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right) dy}$$

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* Example 0 Find an integrating Factor to the DE

$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)dx + (3x^2y^2 + 4y)dy = 0$

we first tray to find $M = M(x)$

$$\frac{\partial M}{\partial y} = (6xy^2 - 6x^3y^2 - 8xy)$$

$$\frac{\partial N}{\partial x} = (6xy^2)$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{6xy^2 - 6x^3y^2 - 8xy - 6xy^2}{3x^2y^2 + 4y}$$

$$= \frac{-2xy(3x^2y + 4)}{y(3x^2y + 4)}$$

$M(x) = -2x$ depends on x only

So, $M(x) = e^{\int -2x\,dx} = e^{-x^2}$ is an I

Integrating Factor to the DE

* Example. 28 Find an integrating factor to the DE

$$(2xy^3)\,dx + (3x^2y^2 + x^2y^3 + 1)\,dy = 0$$

$$\frac{\partial M}{\partial y} = 6xy^2$$

$$\frac{\partial N}{\partial x} = 6xy^2 + 2xy^3$$

* we try $\mu = \mu(x)$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{6xy^2 - 6xy^2 - 2xy^3}{3x^2y^2 + x^2y^3 + 1}$$

does not depend on x only

* we next try $\mu = \mu(y)$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{6xy^2 + 2xy^3 - 6xy^2}{2xy^3}$$

$$\mu(y) = 1 \checkmark$$

depend on y only

$$\mu(y) = e^{\int 1\,dy} = e^{y}$$ *

is an integrating factor to the DE *

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* Example 8 solve the DE. by finding an appropriate integrating factor.

$$(y+1)\,dx + (2x - e^y)\,dy = 0$$

$$M(x,y) = y+1 \;,\; \frac{\partial M}{\partial y} = 1$$

$$N(x,y) = 2x - e^y \;,\; \frac{\partial N}{\partial x} = 2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ The DE is not exact

- $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ = ما بتزبط

- $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \dfrac{2-1}{y+1} = \dfrac{1}{y+1}$ ← depend on y only *

$$M(y) = e^{\int \frac{1}{y+1}\,dy} = e^{\ln|y+1|} = y+1$$

* multiplying the DE by $M(y)$

$$(y^2+2y+1)\,dx + (2xy + 2x - ye^y - e^y)\,dy = 0$$

This DE is Exact, There is a function $F(x,y)$ satifies $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$

we stact using

$$\frac{\partial F}{\partial x} = y^2 + 2y + 1$$

$$F(x,y) = y^2x + 2xy + x + h(y)$$

To determine $h(y)$, we use

$$\frac{\partial F}{\partial y} = 2xy + 2x - ye^y - e^y$$

$$2xy + 2x + h'(y) = 2xy + 2x - ye^y - e^y$$

$$h'(y) = -ye^y - e^y$$

$$h'(y) = -e^y(y+1)$$

$$h(y) = -\int (y+1)e^y \, dy$$

$$= -\left((y+1)e^y - e^y\right)$$

$$h(y) = e^y - (y+1)e^y$$

$$= -ye^y$$

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So, $F(x,y) = xy^2 + 2xy + x - ye^y$

Then, $F(x,y) = C$

$$xy^2 + 2xy + x - ye^y = C$$

is an implicit solution to the DE ✻✻

Subject بسم الله الرحمن الرحيم Date 5/NOV/2023 No.

*** Example°** Consider the DE

$$3x^2y'' + 11xy' - 3y = 0 \qquad x > 0$$

1] Find a pair of linearly independent solution of the form $y(x) = x^r$

2] write the general solution to the DE.

$y(x) = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$y(x)$ is a solution then it satisfies the DE

$$3x^2 r(r-1)x^{r-2} + 11xrx^{r-1} - 3x^r = 0$$

$$3r(r-1)x^r + 11rx^r - 3x^r = 0$$

$$x^r(3r^2 - 8r - 3) = 0$$

Let $x^r \neq 0$ since $x > 0$

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this give that

$$3r^2 + 8r - 3 = 0$$

a b c

$$\Delta = b^2 - 4ac$$
$$= 64 - 4(3)(-3) = 100 > 0$$

$$r = \frac{-8 \pm \sqrt{100}}{2(3)} = \frac{-4 \pm 5}{3}$$

$$r_1 = \frac{1}{3}, \quad r_2 = -3$$

So, $y_1(x) = x^{\frac{1}{3}}$, $y_2(x) = x^{-3}$

$$W\left[x^{1/3}, x^{-3}\right] = \begin{vmatrix} x^{1/3} & x^{-3} \\ \frac{1}{3}x^{-2/3} & -3x^{-4} \end{vmatrix}$$

$$= -3x^{-1} - \frac{1}{3}x^{-1}$$

$$= \left(-3 - \frac{1}{3}\right)x^{-1} \neq 0 \quad x > 0$$

y_1, y_2 are the lineasly Independent solution.

The general solution to the DE is

$$y(x) = C_1 x^{1/3} + C_2 x^{-3}$$

* Reduction of order

Let $y_1(x)$ be a known solution to the DE

$$y'' + p(x)y' + q(x)y = 0$$

we need a ~~second~~ second linearly independant solution $y_2(x)$ to get the general solution

Reduction of order ~~needed~~ we that can help to find such $y_2(x)$

Reduction of Order we that assume

$$y_2(x) = v(x) y_1(x)$$

To find $v(x)$, we use that $y_2(x)$ is a solution

$$y_2(x) = v(x)\,y_1(x)$$

$$y_2' = v\,y_1' + v'\,y_1$$

$$y_2'' = v\,y_1'' + v'y_1' + v'y_1' + v''y_1$$

$$y_2'' + p(x)\,y_2' + g(x)\,y_2 = 0$$

$$v\,y_1'' + 2v'y_1' + v''y_1 + p(x)\left(v\,y_1' + v'\,y_1\right) + g(x)\,v\,y_1 = 0$$

$$v\left(y_1'' + p(x)\,y_1' + q(x)\,y_1\right) + y_1 v'' + \left(2y_1' + p(x)\,y_1\right)v' = 0$$

$$= 0$$

$$y_1\,v'' + \left(2y_1' + p(x)\,y_1\right)v' = 0$$

$$\frac{v''}{v'} = \frac{-2y_1' + p(x)\,y_1}{y_1}$$

$$\int \frac{v''}{v'}\,dv = -\int \left(\frac{2y_1'}{y_1} + p(x)\right)dx$$

البسط مشتقة المقام

$$\ln v' = -\left(2\ln y_1 + \int p(x)\,dx\right)$$

$$v' = e^{-2\ln y_1 - \int p(x)\,dx}$$

$$= e^{-2\ln y_1} \cdot e^{-\int p(x)\,dx}$$

$$v' = \frac{e^{-\int p(x)\,dx}}{y_1^2}$$

$$v = \int \left(\frac{e^{-\int p(x)\,dx}}{y_1^2} \right) dx$$

y_1 a given solution to $y'' + p(x)y' + g(x)y = 0$

to find a 2nd linearly independent solution, we assume

$$y_2(x) = v(x)\,y_1(x)$$

$$v(x) = \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

Example 9: Given that $y_1(x) = x^2$ is a solution to

$$x^2y'' - 3xy' + 4y = 0, \quad x > 0$$

Use Reduction of order we that to find a second linearly independent solution.

we first write the DE in the standard form by dividing by x^2

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

Here $p(x) = \frac{-3}{x}$

Reduction of order we that assume

$$y(x) = v(x)\,x^2$$

where

$$v(x) = \int \frac{e^{-\int p(x)\,dx}}{y_2}\,dx = \int \frac{e^{\int \frac{3}{x}\,dx}}{x^4}\,dx$$

$$= \int \frac{x^3}{x^4}\,dx = \int \frac{1}{x}\,dx = \ln x$$

So, $y(x) = x^2 \ln x$

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*** Example:** Given that $y_1(x) = e^x$ is a ~~sol~~ solution to

$$x y'' - (1+x) y' + y = 0 \quad , x > 0$$

write the general solution

✓ To write the general solution, we that a second ~~solution~~ linearly independent solution $y_2(x)$

The DE in standard form is

$$y'' - \left(\frac{1+x}{x}\right) y' + \frac{1}{x} y = 0$$

$$\text{So, } p(x) = -\frac{1+x}{x} = -\left(1 + \frac{1}{x}\right)$$

we well employ Reduction of order we that

$$y_2(x) = v(x) e^x$$

$$\text{whose } v(x) := \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx \Rightarrow \int \frac{e^{\int (1+\frac{1}{x})\,dx}}{e^{2x}}\,dx$$

$$= \int \frac{e^{x+\ln x}}{e^{2x}}\,dx = \int \frac{e^x e^{\ln x}}{e^{2x}}\,dx$$

$$= \int x e^{-x}\,dx = -x e^{-x} - e^{-x} = -(x+1) e^{-x} \quad **$$

Subject بسم الله الرحمن الرحيم Date 7/NOV/2023 No.

This is a quadratic equation in r

Called the **auxiliary equation** or

characteristic equation

To solve the auxiliary equation, we first compute the discriminant

$$\Delta = b^2 - 4ac$$

we have 3 cases:

case $\Delta > 0$: we have two distinct real roots

$$r = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$r_1 = \frac{-b + \sqrt{\Delta}}{2a}, \quad r_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

∴ So, $y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$

$$W(y_1, y_2) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

$$= r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x}$$

$$= (r_2 - r_1) e^{(r_1+r_2)x} \neq 0$$

$\{ e^{r_1 x}, e^{r_2 x} \}$ is a fundamental set of solutions

So, the general solution is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

Case $\Delta = 0$, we have repeated roots

$$r_1 = r_2 = \frac{-b}{2a} \Rightarrow y_1(x) = e^{\frac{-b}{2a} x}$$

we need a 2nd linearly independent solution Reduction of order we that is to be used to find $y_2(x)$

$$y_2(x) = v(x)\, e^{\frac{-b}{2a}x}$$

$$ay'' + by' + cy = 0$$

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$$

$$P(x) = \frac{b}{a}$$

$$v(x) = \int \frac{e^{-\int \frac{b}{a}dx}}{e^{\frac{-b}{a}x}}\, dx$$

$$= \int \frac{e^{\frac{-b}{a}x}}{e^{\frac{-b}{a}x}}\, dx = x$$

$$\text{So, } y_2(x) = x\, e^{\frac{-b}{2a}x}$$

So, the general solution is

$$y(x) = C_1 e^{\frac{-b}{2a}x} + C_2\, x\, e^{\frac{-b}{2a}x}$$

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Case $\Delta < 0$: The roots of the auxiliary equation are

$$r = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$\mathbb{R}$ $\qquad i = \sqrt{-1}$

$i = \sqrt{-1}$ or $i = -1$

complex numbers

$$\mathbb{C} = \{ a + bi \;,\; a, b \in \mathbb{R} \}$$

summary

$$ay'' + by' + cy = 0$$

we assume $y(x) = e^{rx}$

then r satisfies the auxiliary equation

$$ar^2 + br + c = 0$$

$y_1(x) = e^{\alpha x} \cos \beta x$

$y_2(x) = e^{\alpha x} \sin \beta x$

$y_1(x), y_2(x)$ are linearly independent

so, $y(x) = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$

* solve the auxiliary equation for r

Two different real root r_1, r_2

$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}$

Repeated root $r_1 = r_2 = 2$

$y_1(x) = e^{2x}, \quad y_2(x) = x e^{2x}$

comple root $r = \alpha \pm \beta i$

$y_1 = e^{\alpha x} \cos \beta x, \quad y_2(x) = e^{\alpha x} \sin \beta x$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-b \pm \sqrt{(-1)(-\Delta)}}{2a}$$

$$= \frac{-b \pm \sqrt{-\Delta}\sqrt{-1}}{2a} = \frac{-b \pm \sqrt{-\Delta}\, i}{2a}$$

$$= \frac{-b}{2a} + \frac{\sqrt{-\Delta}}{2a} i = \alpha \pm \beta i \quad *$$

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* To solve $ay'' + by' + cy = 0$ solutions are of the form $y(x) = e^{rx}$ r satisfies the auxiliary equation $ar^2 + br + c = 0$

Two real distinct root r_1, r_2 $y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

Repeated roots $r_1 = r_2 = t$

$y(x) = C_1 e^{tx} + C_2 e^{tx}$ complex root

$r = \alpha \pm \beta i$

$y(x) = C_1 e^{\alpha x} \cos\beta x + C_2 e^{\alpha x} \sin\beta$

* Example: solve the following ODEs:

1] $12y'' - 5y' - 2y = 0$

we seek solutions of the $y(x) = e^{rx}$ Then r satisfies the auxiliary equation $12r^2 - 5r - 2 = 0$

$a \quad b \quad c$

$\Delta = 25 - 4(12)(-2) = 121 > 0$

The auxiliary equation has two distint real root

$r = \frac{5 \pm \sqrt{121}}{2(12)}$ $\left(\frac{-b \pm \sqrt{\Delta}}{2a} \right)$

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$$r_1 = \frac{5 + \sqrt{121}}{24} = \frac{16}{24} = \frac{2}{3}$$

$$r_2 = \frac{5 - \sqrt{121}}{24} = \frac{-6}{24} = \frac{-1}{4}$$

$$y_1(x) = e^{r_1 x} = e^{\frac{2}{3}x}, \quad y_2(x) = e^{r_2 x} = e^{\frac{-1}{4}x}$$

* The general solution is $y(x) = C_1 y_1(x) + C_2 y_2(x)$

$$= C_1 e^{\frac{2}{3}x} + C_2 e^{\frac{-1}{4}x}$$

2] $y'' - 10y' + 25y = 0$

we look to solutions in the form $y(x) = e^{rx}$,

where r satisfies $r^2 - 10r + 25 = 0$

$a = 1, \ b = -10, \ c = 25$ → $(r-5)(r-5) = 0$

$$\Delta = 100 - 4(1)(25) = 100 - 100 = 0$$

Repeted root $r_1 = r_2 = \frac{-b}{2a} = \frac{10}{2(1)} = 5$

$$y_1(x) = e^{5x}, \quad y_2(x) = x e^{5x}$$

The general solution is

$$y(x) = C_1 e^{5x} + C_2 x e^{5x}$$

3] $2y'' - 3y' + 4y = 6$

The auxiliary equation $2r^2 - 3r + 4 = 0$

Here $a = 2,\ b = -3,\ c = 4$

$\Delta = 9 - 4(2)(4) = -23 < 0$ complex root

$$r = \frac{3 \pm \sqrt{-23}}{2(2)} = \frac{3 \pm \sqrt{23}\, i}{4}$$

$$= \frac{3}{4} \pm \frac{\sqrt{23}}{4} i$$

$$= \alpha \pm \beta i$$

$$y_1(x) = e^{\frac{3}{4}x} \cos\left(\frac{\sqrt{23}}{4}x\right), \quad y_2(x) = e^{\frac{3}{4}x} \sin\left(\frac{\sqrt{23}}{4}x\right)$$

The general solution is

$$y(x) = C_1 e^{\frac{3}{4}x} \cos\left(\frac{\sqrt{23}}{4}x\right) + C_2 e^{\frac{3}{4}x} \sin\left(\frac{\sqrt{23}}{4}x\right)$$

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4] $y'' + 3y' = 0$ The auxiliary equation

$r^2 + 3r = 0$

$r(r+3) = 0 \Rightarrow r_1 = 0, r_2 = -3$

$y_1(x) = e^{r_1 x} = e^{0x} = 1$

$y_2(x) = e^{r_2 x} = e^{-3x}$

$y(x) = c_1 + c_2 e^{-3x}$

5] $y'' + 9y = 0$ The auxiliary equation is

$r^2 + 9 = 0$

$y_1(x) = \cos 3x$ $\quad r^2 = -9$

$y_2(x) = \sin 3x$ $\quad r = \pm\sqrt{-9} = \pm 3i$

$y(x) = c_1 \cos 3x + c_2 \sin 3x$ $\quad = 0 \pm 3i$

$= \alpha \pm \beta i$

Example: Find the solution of the IVP

$y'' + 2y' + 4y = 0, \; y(0) = 0, \; y'(0) = 2$

solutions are in the form $y(x) = e^{rx}$, where r satisfies the auxiliary equation $r^2 + 2r + 4 = 0$

Here, $a = 1, b = 2, c = 4$

$\Delta = 4 - 4(1)(4) = -12 < 0$ complex root

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$$r = \frac{-2 \pm \sqrt{-12}}{2(1)} = \frac{-2 \pm \sqrt{12}\,i}{2}$$

$$= -1 \pm \sqrt{3}\,i$$

$$= \alpha \pm \beta i$$

$y_1(x) = e^{-x}\cos(\sqrt{3}x)$

$y_2(x) = e^{-x}\sin(\sqrt{3}x)$

The general solution is $y(x) = C_1 e^{-x}\cos(\sqrt{3}x) + C_2 e^{-x}\sin(\sqrt{3}x)$

$y(0) = 0$

$y'(0) = 2$

The IC $y(0) = 0$

$C_1 = 0 \Rightarrow y(x) = C_2 e^{-x}\sin(\sqrt{3}x)$

Now, $y'(x) = C_2\sqrt{3}e^{-x}\cos(\sqrt{3}x) - C_2 e^{-x}\sin(\sqrt{3}x)$

The IC $y'(0) = 2$

$$\sqrt{3}\,C_2 = 2 \Rightarrow C_2 = \frac{2}{\sqrt{3}}$$

So, $y(x) = \frac{2}{\sqrt{3}} e^{-x}\sin(\sqrt{3}x)$ The solution to the IVP

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Example: Solve the IVP

$$2y'' + y' - y = 0$$

$$y(1) = 1, \quad y'(1) = -2$$

The auxiliary equation $2r^2 + r - 1 = 0$

$a = 2, b = 1, c = -1$

$$\Delta = 1 - 4(2)(-1) = 9 > 0$$

$$r = \frac{-1 \pm \sqrt{9}}{2(2)}$$

$$r_1 = \frac{-1+3}{4} = \frac{1}{2}, \quad r_2 = \frac{-1-3}{4} = -1$$

$$y_1(x) = e^{r_1 x} = e^{\frac{1}{2}x}, \quad y_2(x) = e^{-x}$$

The general solution to the D.E is

$$y(x) = C_1 e^{-x} + C_2 e^{\frac{1}{2}x}$$

The I.C $y(1) = 1$ gives $C_1 e^{-1} + C_2 e^{\frac{1}{2}} = 1$

$$y'(x) = -C_1 e^{-x} + \frac{1}{2} C_2 e^{\frac{1}{2}x}$$

$$y'(1) = -2$$

$$2C_1 e^{-1} - C_2 e^{1/2} = 4$$

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$$e^{-1}C_1 + e^{1/2}C_2 = 1 \quad (+)$$

$$2e^{-1}C_1 - e^{1/2}C_2 = 4$$

$$3e^{-1}C_1 = 5$$

$$C_1 = \frac{5e}{3}$$

$$\frac{5}{3} + C_2 e^{1/2} = 1 \Rightarrow C_2 = \frac{-2}{3} e^{-1/2}$$

##

Subject بسم الله الرحمن الرحيم Date 14/NOV/2020

* Cauchy - Eules Equation

The 2nd osder linear ODE

$$ax^2y'' + bxy' + cy = 0 \quad , \quad x>0$$

is called Cauchy - Euler equation

To solve it , we assume solutions in the form

$$y(x) = x^r$$

$$y' = r x^{r-1}$$

$$y'' = r(r-1) x^{r-2}$$

substituting in the DE

$$a x^2 r(r-1) x^{r-2} + b x r x^{r-1} + c x^r = 0$$

$$a r(r-1) x^r + b r x^r + c x^r = 0$$

$$x^r [ar^2 - ar + br + c] = 0$$

$$x^r [ar^2 + (b-a)r + c] = 0$$

since $x>0$, then $x^r \neq 0$

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Thus, $ar^2 + (b-a)r + c = 0$

The quadrak equation is called the auxiliary equation.
solve The auxiliary equation.

Case 1: Two real distinct root r_1, r_2

$$y_1(x) = x^{r_1}, \quad y_2(x) = x^{r_2}$$

$$W[x^{r_1}, x^{r_2}] = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1-1} & r_2 x^{r_2-1} \end{vmatrix}$$

$$= r_2 x^{r_1+r_2-1} - r_1 x^{r_1+r_2-1}$$

$$= (r_2 - r_1) x^{r_1+r_2-1} \neq 0$$

The general solution is

$$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$$

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Case 2 Repeated root $s_1 = s_2 = \frac{-(b-a)}{2a}$

$$y_1(x) = x^{-\left(\frac{b-a}{2a}\right)}$$

Reduction of orders method is to be used to find a 2nd linearly independent solution $y_2(x)$

$$y_2(x) = v(x)$$

$$v(x) = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

$$ax^2y'' + bxy' + cy = 0$$

$$y'' + \left(\frac{b}{ax}\right)y' + \frac{c}{ax^2}y = 0$$

$P(x)$

$$= \int \frac{e^{-\int \frac{b}{ax}\,dx}}{x^{\frac{-(b-a)}{a}}}\,dx = \int \frac{x^{\frac{-b}{a}}}{x^{\frac{-b}{a}}\,x}\,dx = \ln x$$

$x^{\frac{-b}{a}+1}$

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So, $y_2(x) = \ln x \, y_1(x)$

The general solution is

$$y(x) = C_1 x^{\frac{-(b-a)}{2a}} + C_2 x^{\frac{-(b-a)}{2a}} \ln x$$

Case 3: complex root $r = \alpha \pm \beta i$

$ax^2y'' + bxy' + cy = 0$

$y_1(x) = x^{\alpha+\beta i}$, $y_2(x) = x^{\alpha-\beta i}$

are complex solutions. we seek real solutions

Euler formula

$$e^{iz} = \cos z + i \sin z$$

$$e^{i\beta x} = \cos \beta x + i \sin \beta x$$

$$e^{-i\beta x} = \cos \beta x - i \sin \beta x$$

$$e^{i\beta x} + e^{-i\beta x} = 2 \cos \beta x$$

$$e^{i\beta x} - e^{i\beta x} = 2i \sin \beta x$$

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$$y_1(x) + y_2(x) = x^{\alpha} x^{\beta i} + x^{\alpha} x^{-\beta i}$$

$$= x^{\alpha}\left(x^{\beta i} + x^{-\beta i}\right)$$

$$= e^{\alpha \ln x}\left(e^{\beta i \ln x} + e^{-\beta i \ln x}\right)$$

$$= 2e^{\alpha \ln x}\cos(\beta \ln x)$$

$$y_1(x) - y_2(x) = 2ie^{\alpha \ln x}\sin(\beta \ln x)$$

$$\frac{1}{2}(y_1 + y_2) = x^{\alpha}\cos(\beta \ln x)$$

$$\frac{1}{2i}(y_1 - y_2) = x^{\alpha}\sin(\beta \ln x)$$

seql solutions ##

To solve

$$ax^2y'' + bxy' + cy = 0 \quad , \quad x > 0$$

solution in the fosm $y(x) = x^r$

r satisfies the auxiliary equation

$$ar^2 + (b-a)r + c = 0$$

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r_1, r_2 ∴ $y(x) = C_1 x^{r_1} + C_2 x^{r_2}$

$r_1 = r_2$ ∴ $y(x) = C_1 x^{r_1} + C_2 x^{r_1} \ln x$

$r = \alpha \pm \beta i$ ∴ $y(x) = C_1 x^{\alpha} \cos(\beta \ln x) + C_2 x^{\alpha} \sin(\beta \ln x)$

* Example ∴ solve The DE

$$2x^2 y'' + 3x y' - y = 0 \quad , \quad x > 0$$

$a = 2 \quad , b = 3 \quad , c = -1$

This is Cauch-Eules equation we assume $y(x) = x^r$, where r satifies the auxiliary equation

$$ar^2 + (b-a)r + c = 0$$

$$2r^2 + r - 1 = 0$$

$\Delta = b^2 - 4ac = 1 - 4(2)(-1) = 9 > 0$

Two real distinct roots

$$r = \frac{-1 \pm \sqrt{9}}{2(2)}$$

$$r_1 = \frac{-1+3}{4} = \frac{1}{2} \Rightarrow y_1(x) = x^{r_1} = x^{\frac{1}{2}} = \sqrt{x}$$

$$r_2 = \frac{-1-3}{4} = -1 \Rightarrow y_2(x) = x^{r_2} = x^{-1} = \frac{1}{x}$$

The general solution

$$y(x) = C_1 \sqrt{x} + C_2 \frac{1}{x} \quad **$$

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* Cauchy-Eules Equations

$$ax^2y'' + bxy' + cy = 0 \quad , \quad x > 0$$

$y(x) = x^r$

Auxiliary Equa $ar^2 + (b-a)r + c = 0$

r_1, r_2	$x^{r_1} \quad x^{r_2}$
$r_1 = r_2 = r$	$x^r, \; x^r \ln x$
$r = \alpha \pm \beta i$	$x^\alpha \cos(\beta \ln x), \; x^\alpha \sin(\beta \ln x)$

* Example: Solve the DE.

$$2x^2y'' + 3xy' - y = 0 \quad , \quad x > 0$$

$a = 2 \quad b = 3 \quad c = -1$

This is a Cauchy-Eules equation

Let $y(x) = x^r$, then r satisfies

$$2r^2 + r - 1 = 0$$

$a = 2, \; b = 1, \; c = -1$

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$\Delta = b^2 - 4ac = 1 - 2(2)(-5) = 9 > 0$

we have two distinct real root

$$r = \frac{-1 \pm \sqrt{9}}{2(2)}$$

$$r_1 = \frac{-1+\sqrt{9}}{4} = \frac{1}{2} \quad , \quad r_2 = \frac{-1-\sqrt{9}}{4} = -1$$

$$y_1(x) = x^{r_1} = x^{1/2} = \sqrt{x}$$
$$y_2(x) = x^{r_2} = x^{-1} = \frac{1}{x}$$

The general solution is $\therefore y(x) = c_1\sqrt{x} + c_2\frac{1}{x}$

Example: solve The Cauchy-Euler equation

$$x^2y'' - 3xy' + 4y = 0 \quad , \quad x > 0$$

$a = 1$ $b = -3$ $c = 4$

Let $y(x) = x^r$ then r satisfies

$$r^2 - 4r + 4 = 0$$
$$(r-2)^2 = 0$$
$$(r-2)(r-2) = 0$$
$$r_1 = r_2 = 2$$

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$y_1(x) = x^2$, then $y_2(x) = x^2 \ln x$

The general solution is

$y(x) = C_1 x^2 + C_2 x^2 \ln x$

* Example: Find the general solution for

$x^2 y'' - 3xy' + 6y = 0$, $x > 0$

$a = 1$ $b = -3$ $C = 6$

solutions are in the form $y(x) = x^r$

Then r must satisfy the auxiliary equation

$r^2 - 4r + 6 = 0$

$a = 1$ $b = -4$ $C = 6$

$\Delta = 16 - 4(1)(6) = -8 < 0$

we have complex root

$r = \frac{4 \pm \sqrt{-8}}{2(1)} = \frac{4 \pm \sqrt{8}\, i}{2} = \frac{4 \pm 2\sqrt{2}\, i}{2}$

$\sqrt{4 \cdot 2} = 2\sqrt{2}$

~~$2 \pm 2\sqrt{2}\, i$~~

$= 2 \pm \sqrt{2}\, i$

$= \alpha \pm \beta i$

$y_1(x) = x^{\alpha} \cos(\beta \ln x) = x^2 \cos(\sqrt{2} \ln x)$

$y_2(x) = x^2 \sin(\sqrt{2} \ln x)$

The general solution is

$y(x) = C_1 x^2 \cos(\sqrt{2} \ln x) + C_2 x^2 \sin(\sqrt{2} \ln x)$

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* Example : solve $x^2y'' + xy' - 9y = 0$, $x>0$

$y(1) = 2$, $y'(1) = 3$: $a = 1$, $b = 1$, $c = -9$

$y(x) = x^r$

$r^2 - 9 = 0$

$r = \pm 3$

$y(x) = C_1 x^3 + C_2 x^{-3}$

* Example : $2x^2y'' - 3y = 0$

$ax^2y'' + bxy' + cy = 0$

$a = 2$, $b = 0$, $c = -3$

$y(x) = x^r$, r satisfies

$2r^2 - 2r - 3 = 0$

$\Delta = 4 - 4(2)(-3) = 16 > 0$

$r_1 = \frac{3}{2}$, $r_2 = \frac{-1}{2}$

$y_1(x) = x^{3/2}$, $y_2(x) = x^{-1/2}$

The general solution is : $y(x) = C_1 x^{3/2} + C_2 x^{-1/2}$

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Subject بسم الله الرحمن الرحيم Date 19/11/2023 No.

* Nonhomogeneous Equations:

Consider the nonhomogeneous equation

$$y'' + p(x)\,y' + q(x)\,y = g(x)$$

where $g(x) \neq 0$

* The corresponding homogeneous equation result when setting $g(x) = 0$,

$$y'' + p(x)\,y' + q(x)\,y = 0$$

This the last equation has solution

$$y_h(x) = c_1 y_1 + c_2 y_2$$

y_h or y_c

Any solution to the nonhomogeneous equation is called a particular solution $y_p(x)$

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Theorem: If $y_1(x)$ and $y_2(x)$ are solutions to $y'' + p(x)y' + q(x)y = g(x)$

Then $y_1(x) - y_2(x)$ solution to

$$y'' + p(x)y' + q(x)y = 0$$

* إذا عندي حلين للمعادلة non-h وطرح يعطيني حل للمعادلة h

* Theorem: If y_1, y_2 are solutions to

$$y'' + p(x)y' + q(x)y = 0$$

and $y_p(x)$ is a particular solution to

$$y'' + p(x)y' + q(x)y = g(x)$$

Then $y(x) = c_1y_1 + c_2y_2 + y_p(x)$ is the general solution to $y'' + p(x)y' + q(x)y = g(x)$

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proof: Let $\phi(x)$ be any solution to

$$y'' + p(x)y' + q(x)y = g(x)$$

$\phi(x)$, $y_p(x)$ are solution to the non-h so $\phi(x) - y_p(x)$ is a solution to the corresponding h-equation.

$$\phi(x) - y_p(x) = c_1 y_1 + c_2 y_2$$

$$\phi(x) = c_1 y_1 + c_2 y_2 + y_p(x) \quad \ast\ast$$

$\ast$ Summary: To solve $y'' + p(x)y' + q(x)y = g(x)$

step 1: solve the corresponding homogeneous equation $y'' + p(x)y' + q(x)y = 0$

$$y_h(x) = c_1 y_1 + c_2 y_2$$

step 2: Try to find a particular solution $y_p(x)$ to the nonhomogeneous equation

step 3: The general solution to the non-h equati is $y(x) = y_h(x) + y_p(x)$

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* To Find a particular solution we use one of the two methods:

1] method of undetermined coefficients.

2] method of variation of parameters.

* Method of undetermined coefficients:

This method is use to find a particular solution to the non-h equation $y'' + p(x)y' + q(x)y = g(x)$ x ثوابت

$\Rightarrow$ لازم كل معاملاتها يكونوا $ay'' + by' + cy = g(x)$

$y_p(x) = Ax + B \Longrightarrow$ إذا ما قدرت اطلع المعاملات لازم اكرف انه الفرض اللي انا فرضته خاطئ

The method works when $g(x)$ is: exponential $\Rightarrow e^{ax}$

polynomial $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

sine $\Rightarrow \sin(bx)$

cosine $\Rightarrow \cos(bx)$

or addition or multiplication of such functions

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$g(x) = 10\, e^{7x}$ ✓

$g(x) = \sin 3x \pm \cos 2x$ ✓

↳ جمع او طرح

ما يجوز

$g(x) = \tan x = \frac{\sin x}{\cos x}$ ✗

↳ ✗ القسمة

$g(x) = \frac{1}{x+3} = (x+3)^{-3}$ ↳ ✗ لازم الاس عدد صحيح موجب

polynomial ⇓

$g(x) = e^{3x} + 10x + \sin 3x$ ✓

$g(x) = x^2 e^{-3x} \sin 4x$ ✓

$g(x) = e^{3(x^2)}$ ✗

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$g(x)$	$y_p(x)$
$A e^{ax}$	$B e^{ax}$
$q_n x^n + q_{n-1} x^{n-1} + \dots + q_1 x + q_0$	$A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$
$A \sin bx$ $A \cos bx$	$A_1 \cos bx + A_2 \sin(bx)$ Ex: $\sin 3x \longrightarrow A \sin 3x + B \cos 3x$
$e^{ax} p_n(x)$	$e^{ax}(A_n x^n + A_{n-1} x^{n-1} + \dots + A x_1 + A_0)$
$e^{ax} \sin bx$ $e^{ax} \cos bx$	$e^{ax}(A_1 \sin bx + A_2 \cos bx)$
$p_n(x) \sin bx$ $p_n(x) \cos bx$	$(A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0) \sin bx$ (+) $(B_n x^n + B_{n-1} x^{n-1} + \dots + B_1 x + B_0) \cos bx$
$e^{ax} p_n(x) \sin bx$ $e^{ax} p_n(x) \cos bx$	$e^{ax}[(A_n x^n + A_n x_1^{n-1} + \dots + A_1 x + A_0) \sin bx +$ $(B_n x^n + B_{n-1} x^{n-1} + \dots + B_1 x + B_0) \cos bx]$

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$g(x)$	$y_p(x)$
e^{ax}	$x^s(Ae^{ax})$
$P_n(x)$	$x^s(A_nx^n + A_{n-1}x^{n-1} + \dots + A_1x + A_0)$
$\cos bx$ $\sin bx$	$x^s(A\cos(bx) + B\sin(bx))$
$e^{ax}\cos bx$ $e^{ax}\sin bx$	$x^s(e^{ax}[A\cos(bx) + B\sin(bx)])$
$e^{ax}P_n(x)$	$x^s(e^{ax}[A_nx^n + A_{n-1}x^{n-1} + \dots + A_1x + A_0])$
$P_n(x)\cos(bx)$ $P_n(x)\sin(bx)$	$x^s([A_nx^n + A_{n-1}x^{n-1} + \dots + A_1x + A_0]\cos(bx) +$ $[B_nx^n + B_{n-1}x^{n-1} + \dots + B_1x + B_0]\sin(bx))$
$e^{ax}P_n(x)\cos(bx)$ $e^{ax}P_n(x)\sin(bx)$	$x^s e^{ax}((A_nx^n + A_{n-1}x^{n-1} + \dots + A_1x + A_0)\cos bx +$ $(B_nx^n + B_{n-1}x^{n-1} + \dots + B_1x + B_0)\sin bx$

Subject Date No.

$$ay'' + by' + cy = g(x)$$

corresponding homogeneous equation

$$ay'' + by' + cy = 0$$

$y_p(x)$

general solution

$$y(x) = y_h(x) + y_p(x)$$

$$y_h(x) = c_1 y_1 + c_2 y_2$$

* $$y'' + p(x)y' + q(x)y = g(x)$$

$$g(x) = g_1 + g_2 + \dots + g_n$$

$$y_p(x) = y_{p_1} + y_{p_2} + \dots + y_{p_n}$$

Subject Date No.

* Determine the form of a particular solution of the give DE. Do not evaluate the coefficients.

1] $y'' - 2y' - 3y = 3e^{2x}$

The corresponding homogenous equation.

$y'' - 2y' - 3y = 0$

The auxliary equation is $r^2 - 2r - 3 = 0$

$(r - 3)(r + 1) = 0$

$r_1 = 3, r_2 = -1$

$y_1 = e^{3x}, y_2 = e^{-x}$

Here $g(x) = 3e^{2x}$ so the first guess of $y_p(x)$ is $y_p(x) = Ae^{2x}$

e^{2x} is not similar to y_1 or y_2 then $y_p(x) = Ae^{2x}$

2] $y'' - y' = 3x^2 - 5$

The corresponding homogeneous equation

$y'' - y' = 0$

has the auxiliary equation

$r^2 - r = 0$

$r(r-1) = 0$

$r_1 = 0, \; r_2 = 1$

$y_1(x) = e^{r_1 x} = e^{0x} = 1$

$y_2(x) = e^{r_2 x} = e^{x}$

Here $g(x) = 3x^2 - 5$ so, the first guess of $y_p(x)$ is

$y_p(x) = A_2 x^2 + A_1 x + A_0$

But the 3rd term is similar to $y_1(x)$ so,

$y_p(x) = A_2 x^3 + A_1 x^2 + A_0 x$

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3] $y'' - y = e^{2x}(x^2 + x + 1)$

The corresponding homogeneous equation

$y'' - y = 0$

has the auxiliary equation $r^2 - 1 = 0$

$r_1 = 1, r_2 = -1$

So, $y_1(x) = e^x$, $y_2(x) = e^{-x}$

Here, $g(x) = e^{2x}(x^2 + x + 1)$

An initial guess to $y_p(x)$ is

$y_p(x) = e^{2x}(A_2x^2 + A_1x + A_0)$

$= A_2x^2e^{2x} + A_1xe^{2x} + A_0e^{2x}$

Now of the 3 term in $y_p(x)$ is similar to y_1 or y_2 so $y_p(x) = e^{2x}(A_2x^2 + A_1x + A_0)$

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4] $y'' + 4y = x \sin 2x$

The corresponding homogeneous equation

$y'' + 4y = 0$

has the auxiliary equation

$r^2 + 4 = 0$

$r^2 = -4$

$r = \pm \sqrt{-4} = \pm 2i = 0 \pm 2i$

$= \alpha \pm 2i$

$y_1(x) = e^{\alpha x} \cos(\beta x) = \cos 2x$

$y_2(x) = \sin 2x$

Here, $g(x) = x \sin 2x$

An initial guess of $y_p(x)$ is $y_p(x) = A \cos 2x + B \sin 2x$

$\cos 2x$ is similar to y_1

$y_p(x) = x(A \cos 2x + B \sin 2x)$

Subject Date No.

5] $y'' - 3y' - 4y = 3e^{-x} + 2\sin x - 8e^{x}\cos 2x$

The corresponding homogeneous equation

$y'' - 3y' - 4y = 0$,

has an auxiliary equation $r^2 - 3r - 4 = 0$

$(r - 4)(r + 1) = 0$

$r_1 = 4$, $r_2 = -1$

$y_1(x) = e^{4x}$, $y_2(x) = e^{-x}$

Here , $g(x) = 3e^{-x} + 2\sin x - 8e^{x}\cos 2x$

$g_1(x)$ $g_2(x)$ $g_3(x)$

$g_1(x) = 3e^{-x}$, $y_{p_1}(x) = A_1 x e^{-x}$

$g_2(x) = 2\sin x$, $y_{p_2}(x) = A_2\cos x + B_3 \sin x$

$g_3(x) = -8e^{x}\cos 2x$, $y_{p_3}(x) = e^{x}(A_4\cos 2x + A_5 \sin 2x)$

So , $y_p(x) = y_{p_1} + y_{p_2} + y_{p_3}$

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6] $y'' + 2y' + 2y = 9e^{-x}(x^2-3)\sin x$

The corresponding homogeneous equation

$y'' + 2y' + 2y = 0$

has an auxiliary equation $r^2 + 2r + 2 = 0$

$\Delta = 4 - 4(1)(2) = -4 < 0$

$r = \frac{-2 \pm \sqrt{-4}}{2(1)} = -1 \pm 1i$

$= \alpha \pm \beta i$

$y_1(x) = e^{-x}\cos x \;,\; y_2(x) = e^{-x}\sin x$

Here, $g(x) = 9e^{-x}(x^2-3)\sin x$

An initial guess to the form of $y_p(x)$ is

$y_p(x) = x e^{-x}\left[(A_2x^2 + A_1x + A_0)\cos x + (B_2x^2 + B_1x + B_0)\sin x\right]$

Subject بسم الله الرحمن الرحيم Date 23/ Nov/ 2023 No.

* Example: Find the general solution of the nonhomogeneous equation $y'' - 2y' + y = 3e^x$

The corresponding homogeneous equation is

$$y'' - 2y' + y = 0$$

we seek solution in the form $y(x) = e^{rx}$, then r must satisfy the auxiliary equation

$$r^2 - 2r + 1 = 0$$

$$(r-1)(r-1) = 0$$

$r_1 = r_2 = 1$: Repeated root

$y_1(x) = e^x$, $y_2(x) = xe^x$

Thus , $y_h(x) = c_1 e^x + c_2 x e^x$

To Find a particular solution $y_p(x)$, we will use the method of Undetermined coefficients

Here, $g(x) = 3e^x$

our initial guess is $y_p(x) = Ax^2 e^x$

To find A,

$y'_p(x) = A(x^2 e^x + 2xe^x) = Ae^x(x^2 + 2x)$

$y''_p(x) = A(e^x(2x+2) + e^x(x^2+2x))$

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$y''p(x) = Ae^x(x^2+4x+2)$

Now $y_p(x)$ solution $y''_p - 2y'_p + y_p = 3e^x$

$Ae^x(x^2+4x+2) - 2Ae^x(x^2+2x) + Ax^2e^x = 3e^x$

$2Ae^x = 3e^x$

$2A = 3 \Rightarrow A = \frac{3}{2}$

so, $y_p(x) = \frac{3}{2}x^2e^x$

The general solution to the non homogeous DE is,

$y(x) = y_h(x) + y_p(x) = c_1e^x + c_2xe^x + \frac{3}{2}x^2e^x$

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* Example: Find the general solution to the DE

$$y'' + 2y' + y = x^2 + 1 - e^x$$

The corresponding homogeneous equation

$$y'' + 2y' + y = 0$$

The auxiliary equation $r^2 + 2r + 1 = 0$

$$(r+1)(r+1) = 0$$

has repeated root $r_1 = r_2 = -1$ so $y_1(x) = e^{-x}$

$y_2(x) = xe^{-x}$

and $y_h(x) = c_1 e^{-x} + c_2 x e^{-x}$

Here, $g(x) = \underbrace{x^2 + 1}_{g_1(x)} - \underbrace{e^x}_{g_2(x)} = g_1(x) + g_2(x)$

$y_{p_1}(x) = Ax^2 + Bx + C$, $y_{p_2}(x) = De^x$

So, $y_p(x) = Ax^2 + Bx + C + De^x$

Next, we will determine A, B, C, D

$y_p'(x) = 2Ax + B + De^x$

$y_p''(x) = 2A + De^x$

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$y_p(x)$ satisfies $y_p'' + 2y_p' + y_p = x^2 + 1 - e^x$

$2A + De^x + 4Ax + 2B + 2De^x + Ax^2 + Bx + C + De^x$
$= x^2 + 1 - e^x$

$Ax^2 + (4A + B)x + (2A + 2B + C) + 4De^x = x^2 + 1 - e^x$
Comparing coefficients:

x^2: $A = 1$
x: $4A + B \Rightarrow B = -4A = -4(1) = -4$
x^0: $2A + 2B + C = 1 \Rightarrow 2 - 8 + C = 1 \Rightarrow C = 7$
e^x: $4D = -1 \Rightarrow D = -1/4$

Now, $y_p(x) = x^2 - 4x + 7 - \frac{1}{4}e^x$

The general solution to the non homogenous equation is

$y(x) = y_h(x) + y_p(x)$

$= C_1 e^{-x} + C_2 x e^{-x} + x^2 - 4x + 7 - \frac{1}{4}e^x$

$y(1) = 0$, $y'(0) = 2$

$C_1 + 7 - \frac{1}{4} = 0 \Rightarrow C_1 = \frac{1}{4} - 7$

Use the method of undetermind coefficient to find the form of a pasticular solution

$$x^2y'' - 3xy' - 5y = 2x^5 \ln x + x^{-1}$$

Every Cauch - Euler equation

$$ax^2y'' + bxy' + cy = 0 \text{ , } x > 0$$

can be converted to an equation with constant coefficient Let $x = e^t$ or $t = \ln x$

$$y'' = \frac{d^2y}{dx^2} \text{ , } y' = \frac{dy}{dx}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$$

$$y'' = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{1}{x}\left(\frac{d^2y}{dt^2} \cdot \frac{1}{x}\right) - \frac{1}{x^2}\frac{dy}{dt}$$

$$= \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

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$$ax^2y'' + bxy' + cy = 0 \longrightarrow \frac{d^2y}{dt^2}$$

$$ax^2\left(\frac{1}{x^2}y'' - \frac{1}{x^2}y'\right) + bx\frac{1}{x}y' + cy = 0$$

$$ay'' - ay' + by' + cy = 0$$

$$ay'' + (b-a)y' + cy = 0$$

$$x^2y'' - 3xy' - 5y = 2x^5\ln x + x^{-1}$$

$a = 1,\ b = -3,\ c = -5$

Let $x = e^t$ or $t = \ln x$

the Cauch-Eulers equation becomes

$$y'' - 4y' - 5y = 2te^{5t} + e^{-t}$$

$$y'' - 4y' - 5y = 0$$

$$r^2 - 4r - 5 = 0$$

$$(r-5)(r+1) = 0$$

$$r_1 = 5,\ r_2 = -1$$

$y_1 = e^{5t}$, $y_2 = e^{-t}$

$y_p(t) = te^{5t}(At + B) + Cte^{-t}$

$y_p(x) = \ln x \, x^5 (A \ln x + B) + C \ln x \, x^{-1}$

Subject بسم الله الرحمن الرحيم Date 26/11/2023 No.

* variation of parameters :

To solve the nonhomogeneous DE

$$y'' + p(x)\,y' + q(x)\,y = g(x)$$

we need first to solve cosesponding homogeneous equation $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$

then we need to find a particlar solution $y_p(x)$
Then we general solution is $y(x) = y_h(x) + y_p(x)$

Method of undetermined coefficint has restrictions.

$$ay'' + by' + cy = g(x)$$

- e^{qx}
- $P_n(x)$
- $\sin qx$
- $\cos qx$

we need another general method, to find $y_p(x)$ for any equation of form

$$y'' + p(x)\,y' + q(x)\,y = g(x)$$

This method is called variation of parameters

The method assume $y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$

where y_1, y_2 are the solution to the corresponding homogeneous equation.

Now, $y_p(x) = u_1 y_1(x) + u_2(x) y_2(x)$

$y_p'(x) = u_1 y_1' + y_1 u_1' + u_2 y_2' + u_2' y_2$

Let $u_1' y_1 + u_2' y_2 = 0$

Now, $y_p'(x) = u_1 y_1' + u_2 y_2'$

$y_p''(x) = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$

substitute is $y_p'' + p(x) y_p' + q(x) y_p = g(x)$

$u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + p(x)[u_1 y_1' + u_2 y_2']$

$+ q(x)[u_1 y_1 + u_2 y_2] = g(x)$

$$u_1 \underbrace{[y_1'' + p(x)y_1' + q(x)y_1]}_{=0} + u_2[y_2'' + p(x)y_2' + q(x)y_2] + u_1'y_1' + u_2'y_2' = g(x)$$

$$u_1'y_1' + u_2'y_2' = 0$$

$$u_1'y_1 + u_2'y_2 = 0$$

solve the system of u_1', u_2'

$$u_1'(x) = \frac{-y_2(x)\,g(x)}{W[y_1, y_2]}$$

$$u_2'(x) = \frac{y_1(x)\,g(x)}{W[y_1, y_2]}$$

$$u_1(x) = -\int \frac{y_2(x)\,g(x)}{W[y_1, y_2]}\,dx$$

$$u_2(x) = \int \frac{y_1(x)\,g(x)}{W[y_1, y_2]}\,dx$$

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

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Example 10 Find the general solution to

$$y'' + y = \tan x \quad , \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

The corresponding homogeneous equation

$$y'' + y = 0$$

has an auxiliary equation

$$r^2 + 1 = 0$$

$$r^2 = -1$$

$$r = \pm\sqrt{-1} \implies r = \pm i$$

$$r = 0 \pm 1i$$

$$= \alpha \pm \beta i$$

$$y_1(x) = \cos x \quad , \quad y_2(x) = \sin x$$

so, $y_h(x) = C_1 \cos x + C_2 \sin x$

here $g(x) = \tan x$

$$W[y_1, y_2] = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

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Method of variation of parameters assume

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

$$= u_1(x) \cos x + u_2 \sin x$$

$$u_1(x) = -\int \frac{y_2(x)\, g(x)}{w[y_1, y_2]}\, dx = -\int \frac{\sin x \tan x}{1}\, dx$$

$$= -\int \frac{\sin x \sin x}{\cos x}\, dx = -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1 - \cos^2 x}{\cos x}\, dx$$

$$u_1(x) = -\int (\sec x - \cos x)\, dx$$

$$= -\ln|\sec x + \tan x| + \sin x$$

$$u_2(x) = \int \frac{y_1(x)\, g(x)}{w[y_1, y_2]}\, dx = \int \frac{\cos x \sin x}{\cos x}\, dx$$

$$= -\cos x$$

$$y_p(x) = -\cos x \ln|\sec x + \tan x| + \cos x \sin x - \cos x \sin x$$

$$= -\cos x \ln|\sec x + \tan x|$$

The general solution is

$$y(x) = y_h(x) + y_p(x)$$

$$= c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$$

* Example 2: Find the general solution to *

$$x^2y'' - 4xy' + 6y = x^3 + 1 \quad , \quad x > 0$$

The corresponding homogeneous equation

$$x^2y'' - 4xy' + 6y = 0$$

Cauchy-Eular with $a = 1, b = -4, c = 6$

solution of the form $y(x) = x^r$, r satisfies:

$$ar^2 + (b-a)r + c = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r-3)(r-2) = 0$$

$$r_1 = 3, \quad r_2 = 2$$

So

$$y_1(x) = x^3, \quad y_2(x) = x^2$$

So, $y_h(x) = C_1x^3 + C_2x^2$

Here, $g(x) \neq x^3 + 1$ إذا كان y'' معاملها واحد، $g(x)$ هو الطرف الأول الذي معاملها واحد

The DE in standard form

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = \frac{x^3+1}{x^2}$$

So, $g(x) = \dfrac{x^3+1}{x^2}$

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$$W[y_1, y_2] = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix} = 2x^4 - 3x^4 = -x^4$$

Variation of parameters method assume

$$y_p(x) = u_1(x)\, x^3 + u_2(x)\, x^2$$

$$u_1(x) = \int \frac{x^2 \cdot \frac{x^3+1}{x^2}}{+x^4}\, dx = \int \left(\frac{1}{x} + x^{-4}\right) dx$$

$$= \ln x - \frac{x^{-3}}{3}$$

$$u_2(x) = \int \frac{x^3 \cdot \frac{x^3+1}{x^2}}{-x^4}\, dx$$

$$= -\int \frac{x^3+1}{x^3}\, dx = -\int (1 + x^{-3})\, dx$$

$$= -x + \frac{x^{-2}}{2}$$

Thuse $y_p(x) = x^3 \ln x - \frac{1}{3} + -x^3 + \frac{1}{2}$

$$= x^3 \ln x - x^3 + \frac{1}{6}$$

The genesal solution ~~$y(x) = C_1$~~

$$y(x) = C_1 x^3 + C_2 x^2 + x^3 \ln x - x^3 + \frac{1}{6}$$

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* $x^2y'' - 4xy' + 6y = x^3 + 1$

$x = e^t$ / من $t = \ln x$

$y'' - 5y' + 6y = e^{3t} + (1)$

e^{3t}, e^{2t} $\Rightarrow$ $A + e^{3t} + B$

$x^2y'' - 4xy' + 6y = x^3 \ln x$

حل طرفي المعادلة على الطريقة الاولى افضل

Subject بسم الله الرحمن الرحيم Date 20/11/2023 No.

chapter 4: Higher order ODE

The DE $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$

in the general form of an ODE of order n if $n \geq 2$, the DE is called a higher order.

$$y^{(n)} + p_{(n-1)}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \dots + p_1(x) y' + p_0(x) y = g(x)$$

Linear DE of order n

If $g(x) = 0$ we call it homogeneous

If $g(x) \neq 0$ we call it nonhomogeneous

If all coefficient are constant

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

is called with constant coefficient

otherwise, we call it with variable coefficient.

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* For example :

$$y''' - 4y'' + 7y = 0$$

$$7x^4 y^{(4)} + 3x^3 y^{(3)} - 2x^2 y^{(2)} + x y' + 10 y = 0$$

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \dots + p_1(x) y' + p_0(x) y = g(x)$$

$$y(x_0) = \alpha_0 \,,\; y'(x_0) = \alpha_1 \,,\; y''(x_0) = \alpha_2 + \dots, \; y^{(n-1)}(x_0) = \alpha_{n-1}$$

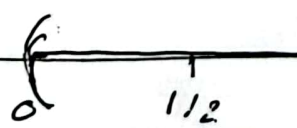
If $p_{(n-1)}(x), \dots, p_1(x), p_0(x), g(x)$ are continous on an open interval I containing x_0, then the DE has a unique solution on I.

Example : Find the longest interval in which the IVP $y''' - \ln x\, y'' + \frac{1}{x-1} y' + 10 y = \sin x$

$y(\frac{1}{2}) = 0$, $y'(\frac{1}{2}) = 2$, $y''(\frac{1}{2}) = 1$

is guranted to have a unique solution

$\ln x \Rightarrow x > 0$, $\frac{1}{x-1} \Rightarrow x \neq 1$



$I = (0, 1)$

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* Def° The functions $f_1, f_2, \ldots, f_n$ are said to be linearly dependant on I if there exist constant $c_1, c_2, \ldots, c_n$ not all zero such that $c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = 0$

If these are not linearly dependant we say they are linearly independant.

If $y_1, y_2, \ldots, y_n$ are n linearly independant solution to $y^{(n)} + p_{n-1}(x) y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = 0$

these $\{y_1, y_2 \ldots, y_3\}$ form a fundemantal set of solution. This means that, the general solution is $y(x) = c_1 y_1 + c_2 y_2 + c_3 y_3 + \ldots + c_n y_n$

* Homogenouose Equation with constant coefficients

$a_n y^{(n)} + a_{n-1} y^{n-1} + \ldots + a_1 y' + a_0 y = 0$

we seek solution of the form ~~$y(x) = x^r$~~ $y(x) = e^{rx}$

~~$y' = r x^{r-1}$, $y'' = r(r-1) x^{r-2}$, $\ldots$, $y^{(n)}$~~

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substitute in the DE $a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$

$y = re^{rx}$, $y'' = r^2 e^{rx}$, $\dots$, $y^{(n)} = r^n e^{rx}$

substitute in the DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \dots + a_1 r e^{rx} + a_0 e^{rx} = 0$$

$$e^{rx} \left[a_n r^n + a_{(n-1)} r^{n-1} + \dots + a_1 r + a_0 \right] = 0$$

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$$

and this equation is called the auxiliary equation to the DE.

Subject بسم الله الرحمن الرحيم Date 30/Nov/2023 No.

Example 1° Find the general solution to the DE

$$y''' + 3y'' - 10y = 0$$

3rd order homo linear with constant coefficient

we set in the form $y(x) = e^{sx}$, then s satisfies the aux. eq ° $s^3 - 3s^2 - 10s = 0$

$$s(s^2 - 3s - 10) = 0 \Rightarrow s(s+5)(s-2) = 0$$

$$s = 0,\ s = 2,\ s = -5$$

we have three distinct real roots.

$$y_1(x) = e^{0x} = 1,\ y_2(x) = e^{-5x},\ y_3(x) = e^{2x}$$

The general solution is $\Rightarrow y(x) = c_1 + c_2 e^{-5x} + c_3 e^{2x}$

Example 2° $y^{(4)} + 2y'' + y = 0$ 4^{th} order linear homo wit constant coefficient

The aux. eq is: $s^4 + 2s^2 + 1 = 0$

$$(s^2+1)^2 = 0$$

$$s^2 + 1 = 0 \Rightarrow s = \pm i,\ \alpha = 0,\ \beta = 1$$

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$y_1(x) = e^0 \cos x = \cos x$, $y_2(x) = \sin x$

$y_3(x) = x \cos x$, $y_4(x) = x \sin x$

repeted complex root : $\pm i$, $\pm i$

The general solution is

↳ $y(x) = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$

*Example 3 : $9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$

the aux. eq is : $9s^5 - 6s^4 + s^3 = 0$

$s^3(9s^2 - 6s + 1) = 0 \Rightarrow s^3(3s-1)^2 = 0$

$sss(3s-1)(3s-1) = 0$

$s_1 = s_2 = s_3 = 0$, $s_4 = s_5 = \frac{1}{3}$

$y_1(x) = e^{0x} = 1$, $y_2(x) = x$, $y_3(x) = x^2$, $y_4(x) = e^{1/3x}$, $y_5(x) = x e^{1/3x}$

The general solution

↳ $y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^{1/3x} + C_5 x e^{1/3x}$

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* Example 4 : $y^{(3)} + y' - 10y = 0$

the aux. eq is: $r^3 + r - 10 = 0$

By faail and trial, we found that $r = 2$ is a root

So, $r - 2$ is a factor of $r^3 + r - 10 = 0$

$r^3 + r - 10 = (r-2)(\quad)$

← ناتج القسمة الطويلة او التركيبية

$r^3 + r - 10 = (r-2)(r^2 + 2r + 5) = 0$

$\Delta = -16 < 0$ complex root

$r = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$

$\alpha \pm \beta i$

$r_1 = 2 \;,\; r_{2,3} = -1 \pm 2i$

$$\begin{array}{r|l} r^2+2r+5 & \\ \hline r-2 & r^3+r-10 \\ - & r^3-2r^2 \\ \hline & +2r^3+r-10 \\ & 2r^2-4r \\ \hline & 5r-10 \\ & 5r-10 \\ \hline & 0 \end{array}$$

$y_1(x) = e^{2x}$, $y_2(x) = e^{-x}\cos 2x$, $y_3(x) = e^{-x}\sin 2x$

The general solution

$\therefore y(x) = c_1 e^{2x} + c_2 e^{-x}\cos 2x + c_3 e^{-x}\sin 2x$

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* Example 5: $y^{(5)} + 3y^{(4)} - 2y^{(3)} - 4y'' = 0$

The aux. eq is: $r^5 + 3r^4 - 2r^3 - 4r^2 = 0$

$r^2(r^3 + 3r^2 - 2r - 4) = 0$

$r^2 = 0 \Rightarrow r_1 = r_2 = 0$

$r^3 + 3r^2 - 2r - 4 = 0 \Rightarrow$ By trial and fail $r = -1$ is a root. So, $r+1$ is a factor of $r^3 + 3r^2 - 2r - 4$

$r^3 + 3r^2 - 2r - 4 = (r+1)(r^2 + 2r - 4) = 0$

$r + 1 \;\big|\; r^3 + 3r^2 - 2r - 4 \;\big\lceil\; r^2 + 2r - 4$

$-(r^3 + r^2)$

$2r^2 - 2r - 4$

$-(2r^2 + 2r)$

$-4r - 4$

$-(-4r - 4)$

0

$r_3 = -1$, $r^2 + 2r - 4 = 0 \Rightarrow \Delta = 20$

$r = \frac{-2 \pm \sqrt{20}}{2} = \frac{-2 \pm 2\sqrt{5}}{2}$

$= -1 \pm \sqrt{5}$

$r_4 = -1 + \sqrt{5}$, $r_5 = -1 - \sqrt{5}$

$y_1(x) = 1$, $y_2(x) = x$, $y_3(x) = e^{-x}$, $y_4(x) = e^{(-1+\sqrt{5})x}$

$y_5(x) = e^{(-1-\sqrt{5})x}$

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* The roots of the auxiliary equation of a homogeneous equation with constant coffecients are:

$0, 5, 5, 7, 7, 7, 7, 2 \pm 3i, 2 \pm 3i$

$y_1(x) = e^0 = 1, \; y_2(x) = e^{5x}, \; y_3(x) = x e^{5x}, \; y_4(x) = e^{7x}$

$y_5(x) = x e^{7x}, \; y_6(x) = x^2 e^{7x}, \; y_7(x) = x^3 e^{7x}$

$y_8(x) = e^{2x} \cos 3x, \; y_9(x) = e^{2x} \sin 3x$

$y_{10}(x) = x e^{2x} \cos 3x, \; y_{11}(x) = x e^{2x} \sin 3x$

The general solution is:

$$y(x) = c_1 + c_2 e^{5x} + c_3 x e^{5x} + c_4 e^{7x} + c_5 x e^{7x} + c_6 x^2 e^{7x}$$

$$+ c_7 x^3 e^{7x} + c_8 e^{2x} \cos 3x + c_9 e^{2x} \sin 3x$$

$$+ c_{10} x e^{2x} \cos 3x + c_{11} x e^{2x} \sin 3x$$

NOTEBOOK

Subject بسم الله الرحمن الرحيم Date 3/12/2023 No.

* Rational Root Theorem :

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

where $a_n, a_{n-1}, \dots, a_1, a_0$ are integers

The rational root for this polynomial equation, are in the form $\Rightarrow$ $\pm$ factors of a_0 / factors of a_n

Ex : Find the general solution to the DE

$$y^{(4)} + 4y''' + 6y'' + 4y' = 0$$

$$r^4 + 4r^3 + 6r^2 + 4r = 0$$

$$r(r^3 + 4r^2 + 6r + 4) = 0$$

$r = -2 \Rightarrow -8 + 16 - 12 + 4 = 0$ ✓ بالتجريب

$r = -1 \Rightarrow -1 + 4 - 6 + 4 \neq 0$

$$r(r+2)(r^2 + 2r + 2) = 0$$

$$r^3 + 4r^2 + 6r + 4 = (r+2)(r^2 + 2r + 2)$$

$r^3 + 2r^2$

$2r^2 + 6r + 4$

$2r^2 + 4r$

$2r + 4$

$2r + 4$

$$\Delta = b^2 - 4ac = 4 - 8 = -4 < 0 \quad \text{complex}$$

$$r = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm 1i = \alpha \pm \beta i$$

$$y_1(x) = e^0 = 1, \quad y_2(x) = e^{-2x}, \quad y_3(x) = e^{-x}\cos x, \quad y_4(x) = e^{-x}\sin x$$

$$y(x) = c_1 + c_2 e^{-2x} + c_3 e^{-x}\cos x + c_4 e^{-x}\sin x$$

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Ex 1 Find the general solution to the DE

$$y''' - 5y'' - y' + 5y = 0$$

$$r^3 - 5r^2 - r + 5 = 0$$

$r = 1 \Rightarrow 1 - 5 - 1 + 5 = 0$ ✓

$$(r-1)(r^2 - 4r - 5) = 0$$

$$(r-1)(r-5)(r+1) = 0$$

$r_1 = 1, r_2 = 5, r_3 = -1$

$$y_1(x) = e^x, \quad y_2(x) = e^{5x}, \quad y_3(x) = e^{-x}$$

$$y(x) = C_1 e^x + C_2 e^{5x} + C_3 e^{-x}$$

$$\begin{array}{r|l} r-1 & r^3 - 5r^2 - r + 5 \\ & -r^3 + r^2 \\ \hline & -4r^2 - r + 5 \\ & +4r^2 - 4r \\ \hline & -5r + 5 \\ & -5r + 5 \\ \hline & 0 \end{array} \qquad r^2 - 4r - 5$$

Ex 2 Find the value of r for which the function $y(x) = x^r$ is a solution of the DE

$$x^3 y''' - 2x^2 y'' - 5x y' + 5y = 0$$

$y(x) = x^r$

$y'(x) = r x^{r-1}$

$y''(x) = r(r-1) x^{r-2}$

$y'''(x) = r(r-1)(r-2) x^{r-3}$

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$$r(r-1)(r-2)x^r - 2r(r-1)x^r - 5(rx^r + x^r) = 0$$

$$x^r\left(r(r-1)(r-2) - 2r(r-1) - 5(r+1)\right) = 0$$

$$x^r\left[r^3 - 3r^2 + 2r - 2r^2 + 2r - 5r + 5\right] = 0$$

$$x^r(r^3 - 5r^2 - r + 5) = 0$$

نحله بالمثال السابق

$r_1 = 1,\ r_2 = -1,\ r_3 = 5$

$$y_1(x) = e^x,\ y_2(x) = e^{-x},\ y_3(x) = e^{5x}$$

The general solution is $\Rightarrow y(x) = c_1e^x + c_2e^{-x} + c_3e^{5x}$

Ex: The auxiliary equation of 7th order homo DE with constant coefficients is:

$$(r-1)^2(r^2+7)^2(r+6) = 0$$

$r_1 = 1,\ r_2 = 1,\ r_{34} = \pm\sqrt{7}i,\ r_{56} = 0 \pm \sqrt{7}i,\ r_7 = -6$

$$y_1(x) = e^x,\ y_2(x) = xe^x,\ y_3(x) = \cos(\sqrt{7}x),\ y_4(x) = \sin\sqrt{7}x$$

$$y_5(x) = x\cos\sqrt{7}x,\ y_6 = x\sin\sqrt{7}x,\ y_7(x) = e^{-6x}$$

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* 3rd osder homo DE with constant coefficiens has the solution e^{-x}, $\sin 3x$, $\cos 3x$, wsite the DE.

$s_1 = -1$, $s_{23} = \pm 3i$

$(s+1)(s-3i)(s-(-3i)) = 0$

$(s+1)(s^2+3si-3si+9i^2) = 0$

$(s+1)(s^2+9) = 0$

$s^3+9s+s^2+9 = 0$

$s^3+s^2+9s+9 = 0$

$y''' + y'' + 9y' + 9y = 0$ ※

Subject بسم الله الرحمن الرحيم Date 5/Dec/2023.

* Method of undetermined coefficients.

Consider the non-homo DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)$$

where $g(x) \neq 0$

To solve it we first the corresponding homogenous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

to get $y_h(x) = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$

Next we find a particular solution $y_p(x)$ to the non-home equation.

Then $y(x) = y_h(x) + y_p(x)$

We will use only the method of undetermined coefficients to find $y_p(x)$

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*Example

Determine a suitable form for $y_p(x)$ if the method of undetermined coefficient is to be used.

1] $y^{(4)} - 4y''' + 4y'' = \underbrace{3e^{2x}}_{g(x)}$

The corresponding homo equation

$y^{(4)} - 4y''' + 4y'' = 0$ has the auxiliary equation

$r^{(4)} - 4r^3 + 4r^2 = 0$

$r^2(r^2 - 4r + 4) = 0$

$r^2(r-2)(r-2) = 0$

$r_1 = 0, r_2 = 0, r_3 = 2, r_4 = 2$

$y_1(x) = e^{0x} = 1$, $y_2(x) = x$, $y_3(x) = e^{2x}$, $y_4(x) = xe^{2x}$

Now $g(x) = 3e^{2x}$

The initial guess is $y_p(x) = Ae^{2x}$

$\rightarrow y_p(x) = Ax^2e^{2x}$

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2] $y''' - 3y'' + 3y' - y = \underbrace{4e^x}_{g(x)}$

The corresponding homo equation is

$y''' - 3y'' + 3y' - y = 0$

It has the auxiliary equation

$r^3 - 3r^2 + 3r - 1 = 0$

The possible rational root are the divisors of $a_0 = -1$ which are ± 1

by trial and fail, we find that $r = 1$ is a root

So, $r^3 - 3r^2 + 3r - 1 = 0$

$= (r - 1)(r^2 - 2r + 1)$

$= (r-1)(r-1)(r-1)$

$r_1 = r_2 = r_3 = \underbrace{1}_{\equiv}$

$r^2 - 2r + 1$

$r - 1 \;\big|\; r^3 - 3r^2 + 3r - 1$

$-\ r^3 - r^2$

$-2r^2 + 3r - 1$

$-\ -2r^2 + 2r$

$r - 1$

$-\ r - 1$

0

$y_1(x) = e^x, \; y_2(x) = xe^x, \; y_3(x) = x^2 e^x$

Here $g(x) = 4e^x$

so, the initial guess $y_p(x)$ is

$y_p(x) = Ae^x$

$y_p(x) = Ax^3 e^x$

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3] $y^{(4)} + 2y'' + y = \underbrace{2\sin x - 5\cos x}_{g(x)}$

The corresponding homo equation

$y^{(4)} + 2y'' + y = 0$

has the auxiliary equation $r^4 + 2r^2 + 1 = 0$

$(r^2+1)^2 = 0$

$(r^2+1)(r^2+1) = 0$

$r = \pm i \;,\; r = \pm i$

$y_1(x) = \cos x \;,\; y_2(x) = \sin x$

$y_3(x) = x\cos x \;,\; y_4(x) = x\sin x$

Now, $y_p(x) = A\sin x + B\cos x$ ~~→~~

$y_p(x) = Ax\sin x + Bx\cos x$ ~~→~~

$y_p(x) = Ax^2\sin x + Bx^2\cos x$ ✓

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4] $y''' + 9y' = \underbrace{x \sin 3x}_{g_1(x)} + \underbrace{x^2 e^{2x}}_{g_2(x)}$

* $y''' + 9y' = 0$

* $s^3 + 9s = 0$

$s(s^2 + 9) = 0$

$s_1 = 0$, $s = \pm 3i$

$y_1(x) = 1$, $y_2(x) = \cos 3x$, $y_3(x) = \sin 3x$

$y_{p_1}(x) = [(A_1 x + A_2)\sin 3x + (A_3 + A_4)\cos 3x]\, x$

$y_{p_2}(x) = e^{2x}(A_5 x^2 + A_6 x + A_7)$

$y_p(x) = A_1 x^2 \sin 3x + A_2 x \sin 3x + A_3 x^2 \cos 3x + A_4 x \cos 3x$
$+ A_5 x^2 e^{2x} + A_6 x e^{2x} + A_7 e^{2x}$

~~Find the~~

Subject

* Find the general solution of

$$y^{(3)} + y'' = 3e^x + 4x^2$$

The corresponding homo equation

$$y^{(3)} + y'' = 0$$

has the auxiliary equation

$$r^3 + r^2 = 0$$

$$r^2(r+1) = 0$$

$$r_1 = r_2 = 0 \ , \ r_3 = -1$$

$$y_1(x) = e^{0x} = 1 \ , \ y_2(x) = x \ , \ y_3(x) = e^{-x}$$

So, $y_h(x) = C_1 + C_2 x + C_3 e^{-x}$ homo-part ال

NOW, $g(x) = \underbrace{3e^x}_{g_1(x)} + \underbrace{4x^2}_{g_2(x)}$

$$y_{p_1}(x) = A_1 e^x \ , \ y_{p_2}(x) = (A_2 x^2 + A_3 x + A_4) x^2$$

$$y_{p_2}(x) = A_2 x^4 + A_3 x^3 + A_4 x^2$$

$$y_p(x) = A_1 e^x + A_2 x^4 + A_3 x^3 + A_4 x^2$$

To Find A_1, A_2, A_3, A_4

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$y_p' = A_1 e^x + 4A_2 x^3 + 3A_3 x^2 + 2A_4 x$

$y_p'' = A_1 e^x + 12A_2 x^2 + 6A_3 x + 2A_4$

$y_p''' = A_1 e^x + 24A_2 x + 6A_3$

~~$y_p'''' = A_1 e^x + 24A_2$~~

$y''' + y'' = 3e^x + 4x^2$

$A_1 e^x + 24A_2 x + 6A_3 + A_1 e^x + 12A_2 x^2 + 6A_3 x + 2A_4 = 3e^x + 4x^2$

$2A_1 e^x + 12A_2 x^2 + (24A_2 + 6A_3)x + (6A_3 + 2A_4) = 3e^x + 4x^2$

e^x : $2A_1 = 3 \Rightarrow A_1 = \frac{3}{2}$

x^2 : $12A_2 = 4 \Rightarrow A_2 = 1/3$

x : $24A_2 + 6A_3 = 0 \Rightarrow A_3 = -4/3$

$6A_3 + 2A_4 = 0 \Rightarrow A_4 = 4$

The general solution is $y(x) = y_h(x) + y_p(x)$

Subject بسم الله الرحمن الرحيم Date 7/12/2023 No.

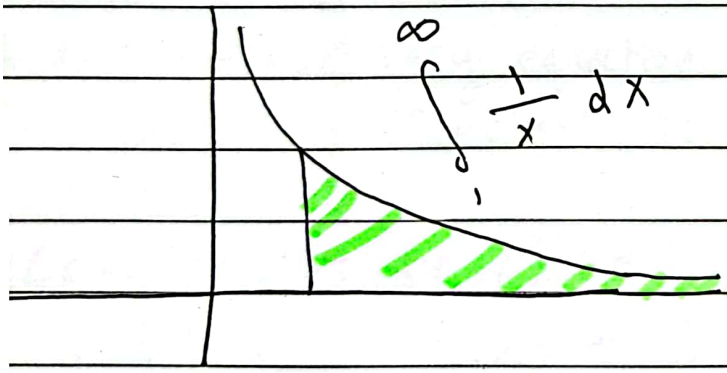
Chapter 6 : laplace Transform

Given a function $f(t)$, $t \geq 0$, the laplace transform of $f(t)$ is denoted by $\mathcal{L}[f(t)] = \mathcal{L}[f(t)]$

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$\int_a^b f(x)\, dx$ This is an improper integral

$$\mathcal{L}[f(t)] = \lim_{a \to \infty} \int_0^a e^{-st} f(t)\, dt$$



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Example : Find $\mathcal{L}[e^{at}]$

$$\mathcal{L}[e^{2t}] = \int_0^\infty e^{-st} e^{2t}\, dt$$

$$= \int_0^\infty e^{-(s-2)t}\, dx$$

$$= \lim_{a\to\infty} \int_0^a e^{-(s-2)t}\, dx$$

$$= \lim_{a\to\infty} \frac{e^{-(s-2)t}}{-(s-2)} \Bigg]_0^a$$

$$= \lim_{a\to\infty} \frac{e^{-(s-2)a} - 1}{-(s-2)}, \quad s>2$$

$$= \frac{1}{s-2}, \quad s>2$$

$\mathcal{L}$: $e^{2t} \longrightarrow \frac{1}{s-2}$

$$\mathcal{L}[f(t)] = \lim_{a\to\infty} \int_0^a e^{-st} f(t)\, dt$$

exist → converge

d.n.e → diverge

$\mathcal{L}$: $f(t) \longrightarrow F(s)$

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Example: Find $\mathcal{L}[t]$

$$\mathcal{L}[t] = \int_0^\infty e^{-st}\, t\, dt \quad \lim_{a\to\infty} \frac{-a}{s e^{as}}$$

$$= \lim_{a\to\infty} \int_0^a t\, e^{-st}\, dt \quad \frac{-1}{as e^{as}}$$

$$\lim_{a\to\infty} \left(t\frac{e^{-st}}{-s} - \int \frac{e^{-st}}{-s}\, t\, dt \right) \Bigg]_0^a$$

$$= \lim_{a\to\infty} \left(\frac{-t\, e^{-st}}{s} - \frac{e^{-st}}{s^2} \right) \Bigg]_0^a$$

$$= \lim_{a\to\infty} \left(\frac{-a\, e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) + \frac{1}{s^2}$$

$$= \frac{1}{s^2}, \quad s > 0$$

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Example: Let $f(t) = \begin{cases} 2, & 0 \le t < 1 \\ 4, & t \ge 1 \end{cases}$

Find $\mathcal{L}[f(t)]$

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 2e^{-st}\,dt + \int_1^\infty 4e^{-st}\,dt$$

$$= \lim_{a\to\infty}$$

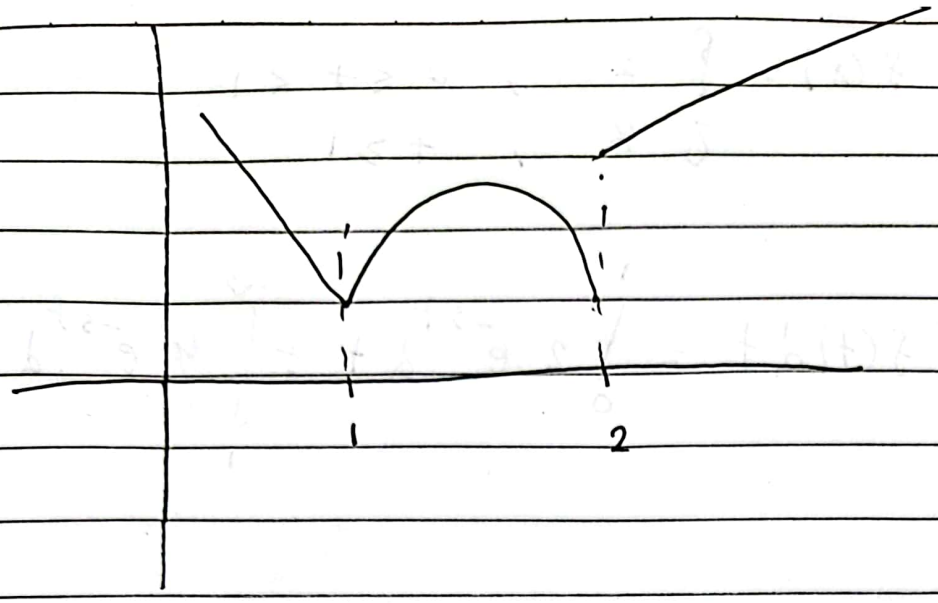
$$= 2\left[\frac{e^{-st}}{-1}\right]_0^1 + \lim_{a\to\infty}\int_1^a 4e^{-st}\,dt$$

$$= \frac{-2}{s}\left(e^{-s} - 1\right) + \lim_{a\to\infty}\left[\frac{4}{-s}e^{-st}\right]_1^a$$

$$= \frac{-2}{s}\left(e^{-1} - 1\right) + \lim_{a\to\infty}\frac{-4}{s}\left(e^{-as} - e^{-s}\right)$$

$$= \frac{-2}{s}\left(e^{-1} - 1\right) + \frac{4}{s}e^{-s}$$

$$= \frac{-2}{s}e^{-s} + \frac{2}{s}, \quad s > 0$$

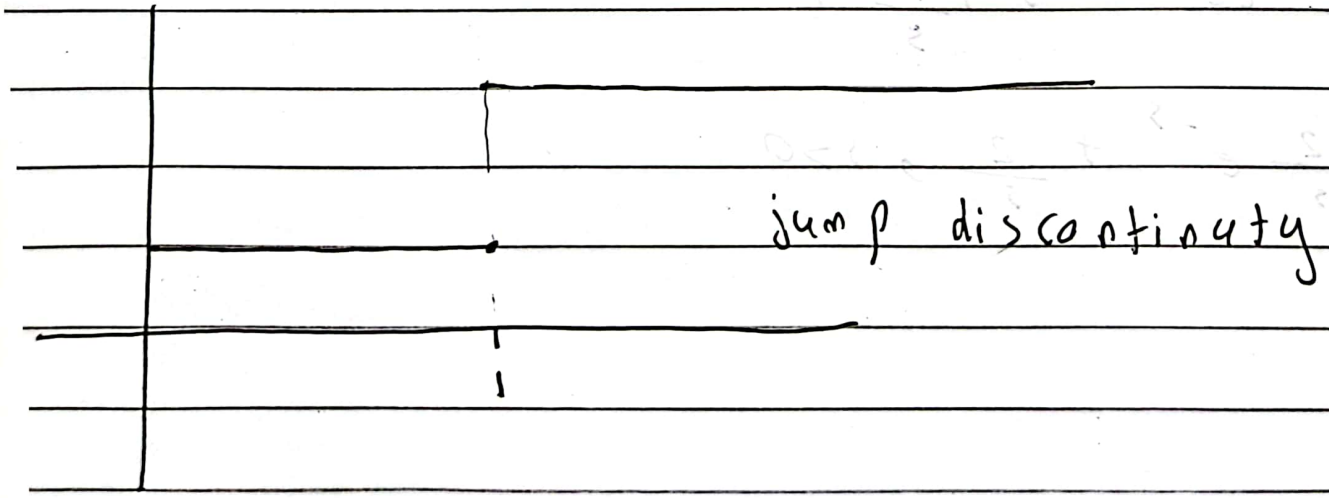


Theorem: suppose that

1. $f(t)$ is piecewise continuous on the interval $0 \leq t \leq A$

2. $|f(t)| \leq K e^{at}$ when $t \geq M$

Then $\mathcal{L}[f(t)]$ exist for $s > a$



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$f(x) \longrightarrow \mathcal{L}[f(t)]$

* $c \longrightarrow \frac{c}{s}$
* $e^{at} \longrightarrow \frac{1}{s-a}$
* $t^n \longrightarrow \frac{n!}{s^{n+1}}$
* $\sin at \longrightarrow \frac{a}{s^2+a^2}$
* $\cos at \longrightarrow \frac{s}{s^2+a^2}$
* $\sinh at \longrightarrow \frac{a}{s^2-a^2}$
* $\cosh at \longrightarrow \frac{s}{s^2-a^2}$

* Example ⁰ Evaluate

1] $\mathcal{L}[3] = \frac{3}{s}$

2] $\mathcal{L}[1] = \frac{1}{s}$

3] $\mathcal{L}[e^{2t}] = \frac{1}{s-2}$

4] $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$

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5] $L[t^2] = \frac{2!}{s^3} = \frac{2}{s^3}$

6] $L[t] = \frac{1!}{s^2} = \frac{1}{s^2}$

7] $L[t^4] = \frac{4!}{s^5} = \frac{24}{s^5}$

8] $L[\sin 3t] = \frac{3}{s^2+9}$

9] $L[\sin t] = \frac{1}{s^2+1}$

10] $L[\cos 3t] = \frac{s}{s^2+9}$

11] $L[\cos t] = \frac{s}{s^2+1}$

12] $L[\sin \sqrt{10}t] = \frac{\sqrt{10}}{s^2+10}$

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*Properties of Laplace Transform:

1] Linearity of the transform

$$L[f(t) \pm g(t)] = L[f(t)] \pm L[g(t)]$$

$$L[cf(t)] = cL[f(t)]$$

$$L[c_1 f(t) \pm c_2 g(t)] = c_1 L[f(t)] + c_2 L[g(t)]$$

*Example: $L[2t^3 + 4t^2 + 10]$

$$= 2L[t^3] + 4L[t^2] + L[10]$$

$$= 2\frac{3!}{s^4} + 4\frac{2!}{s^3} + \frac{10}{s}$$

$$= \frac{12}{s^4} + \frac{8}{s^3} + \frac{10}{s}$$

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* Example 2 : $\mathcal{L}[e^{3t} - t + 1]$

$$= \mathcal{L}[e^{3t}] - \mathcal{L}[t] + \mathcal{L}[1]$$

$$= \frac{1}{s-3} - \frac{1}{s^2} + \frac{1}{s}$$

* Example 3 : $\mathcal{L}[\sinh 3t]$

$$= \mathcal{L}\left[\frac{e^{3t} - e^{-3t}}{2}\right]$$

$$= \frac{1}{2}\left(\mathcal{L}[e^{3t}] - \mathcal{L}[e^{-3t}]\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s+3}\right)$$

$$= \frac{1}{2}\,\frac{s+3-(s-3)}{s^2-9}$$

$$= \frac{3}{s^2+9}$$

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* Example 4 : $\mathcal{L}[\sin^2 t] = \mathcal{L}\left[\frac{1}{2}(1-\cos 2t)\right]$

$= \frac{1}{2}\left(\mathcal{L}[1] - \mathcal{L}[\cos 2t]\right)$

$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$

* $\mathcal{L}[\cos^2 t] = \mathcal{L}\left[\frac{1}{2}(1+\cos 2t)\right]$

2] shifting property

If $\mathcal{L}[f(t)] = F(s)$ Then

$\mathcal{L}[e^{at} f(t)] = F(s-a)$

e^{at}

Example 1 : $\mathcal{L}[e^{2t}\sin 3t] = \frac{3}{(s-2)^2+9}$

Example 2 : $\mathcal{L}[e^{-t}\cos 2t] = \frac{s+1}{(s+1)^2+4}$

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* Example 3 : $L[t^2 e^{-3t}] = \frac{2}{(s+3)^3}$

* Example 4 : $L[e^{2t} \cos^2 4t]$

$$= \frac{1}{2}\left(\frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 64}\right)$$

$$= L[\cos^2(4t)] = L\left[\frac{1}{2}(1+\cos 8t)\right]$$

$$= \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+64}\right)$$

3] Multiplication by t^n

If $L[f(t)] = F(s)$ Then

$$L[t^n f(t)] = (-1)^n F^{(n)}(s)$$

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Example 1 : Evaluate $L[t\cos 2t]$

$$= (-1)^1 \left(\frac{s}{s^2+4}\right)'$$

$$= -1 \frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

Example 2 : $L[t^2 \sin t]$

$$= (-1)^2 \left(\frac{1}{s^2+1}\right)''$$

$$= \left(\frac{-2s}{(s^2+1)^2}\right)'$$

$$= -2 \left(\frac{(s^2+1)^2(1) - 2s(s^2+1)(2s)}{(s^2+1)^4}\right)$$

$$= \frac{6s^2-2}{(s^2+1)^3}$$

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* Example 3 : Evaluate $\mathcal{L}[t \sin^2 t]$

$$\mathcal{L}[\sin^2 t] = \mathcal{L}\left[\frac{1}{2}(1-\cos 2t)\right]$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= (-1)^1 \left(\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right)'$$

$$= \frac{-1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)'$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2} - \frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right)$$

Example 4 : $\mathcal{L}[t \sin 5t] = (-1)^1 \left(\frac{5}{s^2+25}\right)'$

$$= -5 \quad \frac{-2s}{(s^2+25)^2}$$

$$= \frac{10s}{(s^2+25)^2}$$

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* Example 5 : Evaluate $\mathcal{L}[t e^{2t} \sin 5t]$

$$= \mathcal{L}[e^{2t}(t \sin 5t)]$$

$$= \frac{10(s-2)}{((s-2)^2+25)^2}$$

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4] If $L[f(x)] = F(s)$, Then $\lim_{s \to \infty} F(s) = 0$

e.g. $\frac{1}{s^2+10}$ $\frac{s^2}{s^3+7}$ $\frac{s^2}{s^2+1}$ ↳ X

5] Let $f(t), f'(t), \ldots, f^{(n-1)}(t)$ be continuous for $t \geq 0$, and Let $f^{n}(t)$ be piecewise continuous for $t \geq 0$. and $L[f(x)] = F(s)$

Then $L[f^{(n)}(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f(0) \ldots f^{(n-$

speciql case

- $L[f'(t)] = sF(s) - f(0)$
- $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$
- $L[f'''(t)] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

* For example a $y'' + 2y' + y = \sin t$, $y(0) = 1$, $y'(0) = 0$

solution $\Rightarrow$ $y(t)$ Let $L[y(t)] = Y(s)$

$L[y'' + 2y' + y] = L[\sin t]$

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$$L[y''] + 2L[y'] + L[y] = L[\sin t]$$

$$s^2 Y(s) - s y(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = \frac{1}{s^2+1}$$

$$\left[s^2 Y(s) - s + 2sY(s) - 2 + Y(s) = \frac{1}{s^2+1} \right]$$

$$(s^2 + 2s + 1) Y(s) = \frac{1}{s^2+1} + s + 2$$

$$Y(s) = \frac{1}{(s^2+1)(s^2+2s+1)} + \frac{s+2}{s^2+2s+1}$$

But we want $y(t) = L[Y(s)]$

$Y(s)$ is given we want $y(t)$

* Laplace Inversse :

~~If $L[y(t)]$~~ If $L[f(t)] = F(s)$, then the opesators L^{-1} called laplace inversse wosks as follows $L^{-1}[F(s)] = f(t)$

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$L^{-1}[F(s)]$ we think of this problem in this way we seek a function $f(t)$ such that $L[f(t)] = F(s)$

Example: Find Laplace inverse in the following

1] $L^{-1}\left[\frac{7}{s}\right] = 7$

2] $L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t \quad L[\sin 2t] = \frac{2}{s^2+4}$

$$\frac{1}{2} L[\sin 2t] = \frac{1}{s^2+4}$$

$$L\left[\frac{1}{2}\sin 2t\right] = \frac{1}{s^2+4}$$

3] $L^{-1}\left[\frac{1}{s+4}\right] \Rightarrow L[e^{-4t}] = \frac{1}{s+4}$

$= e^{-4t}$

4] $L^{-1}\left[\frac{1}{s^3}\right] = \frac{1}{2}t^2 \quad L[t^2] = \frac{2}{s^3}$

$$L\left[\frac{1}{2}t^2\right] = \frac{1}{s^3}$$

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5] $\mathcal{L}^{-1}\left[\frac{3}{s^4}\right] = \frac{1}{2}t^3$ $\quad \mathcal{L}[t^3] = \frac{3!}{s^4}$

$$\mathcal{L}[t^3] = \frac{6}{s^4}$$

$$\mathcal{L}\left[\frac{1}{2}t^3\right] = \frac{3}{s^4}$$

6] $\mathcal{L}^{-1}\left[\frac{s}{s^2+10}\right] = \cos\sqrt{10}\,t$

$$\mathcal{L}[\cos\sqrt{10}\,t] = \frac{s}{s^2+10}$$

7] $\mathcal{L}^{-1}\left[\frac{1}{s^2-2}\right] = \frac{1}{\sqrt{2}}\sinh\sqrt{2}\,t$

$$\mathcal{L}[\sinh\sqrt{2}\,t] = \frac{\sqrt{2}}{s^2-2}$$

$$\mathcal{L}\left[\frac{1}{\sqrt{2}}\sinh\sqrt{2}\,t\right] = \frac{1}{s^2-2}$$

8] $\mathcal{L}^{-1}\left[\frac{1}{(s-2)^3}\right] = \frac{1}{2}e^{2t}t^2$

$$\mathcal{L}[t^2] = \frac{2}{s^3}$$

$$\mathcal{L}[e^{2t}t^2] = \frac{2}{(s-2)^3}$$

$$\mathcal{L}\left[\frac{1}{2}e^{2t}t^2\right] = \frac{1}{(s-2)^3}$$

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9] $L^{-1}\left[\frac{1}{s^2+4s+6}\right] = L^{-1}\left[\frac{1}{(s+2)^2+2}\right]$

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$s^2+4s+6 = \left(\frac{\text{معامل } s}{2}\right)^2$

$s^2+4s+6 = s^2+4s+6+4-4$

$= \underbrace{s^2+4s+4}+2$

$= (s+2)^2+2$

$L[\sin\sqrt{2}t] = \frac{\sqrt{2}}{s^2+2}$

$L[e^{-2t}\sin\sqrt{2}t] = \frac{\sqrt{2}}{(s+2)^2+2}$

$L\left[\frac{1}{\sqrt{2}}e^{-2t}\sin\sqrt{2}t\right] = \frac{1}{(s+2)^2+2}$ #

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[0] $L^{-1}\left[\frac{1}{s^2-s}\right] = 2e^{1/2t}\sinh\frac{1}{2}t$

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$s^2 - s = s^2 - s + \frac{1}{4} - \frac{1}{4}$

$= \left(s - \frac{1}{2}\right)^2 - \frac{1}{4}$

$L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2 - \frac{1}{4}}\right]$

$= L\left[\sinh\frac{1}{2}t\right] = \frac{1/2}{s^2 - \frac{1}{4}}$

$L\left[e^{1/2t}\sinh\frac{1}{2}t\right] = \frac{1/2}{\left(s-\frac{1}{2}\right)^2 - \frac{1}{4}}$

$L\left[2e^{1/2t}\sinh\frac{1}{2}t\right] = \frac{1}{\left(s-\frac{1}{2}\right)^2 - \frac{1}{4}}$

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* Example : Find

1] $L^{-1}\left[\frac{1}{(s-2)^4}\right] = \frac{1}{6} e^{2t} t^3$

2] $L^{-1}\left[\frac{1}{s^2-6s}\right] = \frac{1}{3} e^{3t} \sinh 3t$

* Using partial Fraction to find $L^{-1}[f(t)]$:

To illustrate the idea, we will start with this example,

$$\left[^{-1}\left[\frac{1}{s^3-s}\right]\right.$$

$$\frac{1}{s^3-s} = \frac{1}{s(s^2-1)} = \frac{1}{s(s-1)(s+1)}$$

$$= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$= \frac{A(s^2-1) + Bs(s+1) + Cs(s-1)}{s(s-1)(s+1)}$$

$$1 = A(s^2-1) + Bs(s+1) + Cs(s-1)$$

$s=0 \Rightarrow 1 = -A \Rightarrow A=-1$ $\quad s=-1 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$

$s=1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$

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$$\frac{1}{s^3-s} = \frac{-1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1}$$

$$L^{-1}\left[\frac{1}{s^3-s}\right] = L^{-1}\left[\frac{-1}{s}\right] + L^{-1}\left[\frac{1/2}{s-1}\right] + L^{-1}\left[\frac{1/2}{s+1}\right]$$

$$= -1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}$$

* Example 8 Determine

1] $$L^{-1}\left[\frac{s^2+8s+6}{(s+1)(s^2+4)}\right]$$

$$\frac{s^2+8s+6}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$= \frac{A(s^2+4)+(Bs+C)(s+1)}{(s+1)(s^2+4)}$$

$$s^2+8s+6 = A(s^2+4)+(Bs+C)(s+1)$$

$s=-1 \Rightarrow -1 = 5A \Rightarrow A = -1/5$

$s=0 \Rightarrow 6 = -\frac{4}{5} + C \Rightarrow C = 6+\frac{4}{5} = \frac{34}{5}$

$s=1 \Rightarrow 15 = -1 + 2B + \frac{68}{5} \Rightarrow 2B = \frac{12}{5} \Rightarrow B = 6/5$

$$\mathcal{L}^{-1}\left[\frac{s^2+8s+6}{(s+1)(s^2+4)}\right] = \mathcal{L}^{-1}\left[\frac{-11/5}{s+1} + \frac{\frac{6}{5}s + \frac{34}{5}}{s^2+4}\right]$$

$$= \frac{-1}{5}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{6}{5}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + \frac{34}{5}\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{-1}{5}e^{-t} + \frac{6}{5}\cos 2t + \frac{17}{5}\sin 2t$$

2] $\mathcal{L}^{-1}\left[\frac{2s+3}{s^2+6s+1}\right]$

$$= \mathcal{L}^{-1}\left[\frac{2s+3}{(s+3)^2-8}\right]$$

$$= 2\mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2-8}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2-8}\right]$$

$$= 2e^{-3t}\cosh\sqrt{8}t - \frac{3}{\sqrt{8}}e^{-3t}\sinh\sqrt{8}t$$

PRENI

3] $\left[^{-1}\left[\frac{s}{(s+2)^2+16}\right]\right.$

$= \left[^{-1}\left[\frac{s+2}{(s+2)^2+16}\right] - 2\left[^{-1}\left[\frac{1}{(s+2)^2+16}\right]\right.\right.$

$= e^{-2t}\cos 4t - \frac{1}{2}e^{-2t}\sin 4t$

بسم الله الرحمن الرحيم

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Recall the multiplication by t^n rule

$$\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$$

$$F^{(n)}(s) = \mathcal{L}[(-1)^n t^n f(t)]$$

$$\mathcal{L}^{-1}[F^{(n)}(s)] = (-t)^n f(t)$$

* Example: $\mathcal{L}^{-1}[\underbrace{\tan^{-1}(\frac{1}{s})}_{F(s)}] = f(t)$

$$F(s) = \tan^{-1}(\frac{1}{s})$$

$$F'(s) = \frac{\frac{-1}{s^2}}{1+\frac{1}{s^2}} = \frac{-1}{s^2+1}$$

$$\mathcal{L}^{-1}[F'(s)] = -\sin t$$

$$\mathcal{L}^{-1}[F'(s)] = -t\ f(t)$$

$$-\sin t = -t\ f(t)$$

$$f(t) = \frac{\sin t}{t}$$

* Example 2 & $L^{-1}\left[\underbrace{\ln\left(\frac{s-4}{s-3}\right)}_{F(s)}\right] = f(t)$

$$F(s) = \ln\left(\frac{s-4}{s-3}\right) = \ln(s-4) - \ln(s-3)$$

$$F'(s) = \frac{1}{s-4} - \frac{1}{s-3}$$

$$L^{-1}[F'(s)] = e^{4t} - e^{3t}$$

$$L^{-1}[F'(s)] = (-t) f(t)$$

$$e^{4t} - e^{3t} = -t f(t)$$

$$f(t) = \frac{e^{3t} - e^{4t}}{t}$$

* Example 3 : $L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = f(t)$

$F(s) = \frac{1}{s^2+4} \longrightarrow L^{-1}[F(s)] = f(t) = \frac{1}{2}\sin 2t$

$F'(s) = \frac{-2s}{(s^2+4)^2}$

$L^{-1}[F'(s)] = (-t)\, f(t)$

$\left[^{-1}\left[\frac{-2s}{(s^2+4)^2}\right] = -t\,\frac{1}{2}\sin 2t\right.$

$\left[^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{1}{4}\, t \sin 2t\right.$

* $f(t) = L^{-1}[F(s)]$ *

↳ $f(t)$ = laplac inverse

للاقتران نفسه

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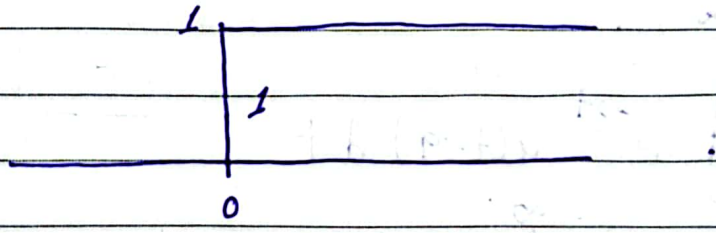
24/12/2023

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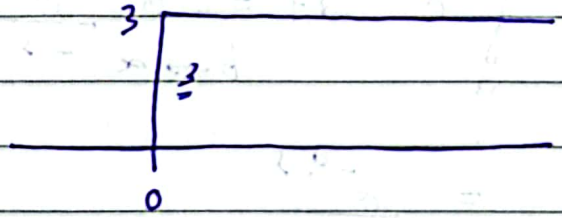
* Transforms of Discontinuous Functions :

function $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$

The unit step function



* 3 $u(t) = \begin{cases} 0 & t < 0 \\ 3 & t > 0 \end{cases}$



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We can make a shift

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

$u(t-a)$ is called the unit step function at a.

$$* \; \mathcal{L}[u(t-a)] = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st}\, u(t-a)\, dt + \int_a^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \lim_{b\to\infty} \int_a^b e^{-st}\, dt = \lim_{b\to\infty} \frac{e^{-st}}{-s}\Big]_a^b$$

$$= \lim_{b\to a} \frac{e^{-bs} - e^{-as}}{-s} = \frac{e^{-as}}{s}$$

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$$\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}, \quad a \geq 0$$

Examples: 1] $\mathcal{L}[u(t-3)] = \frac{e^{-3s}}{s}$

2] $\mathcal{L}[u(t-\pi)] = \frac{e^{-\pi s}}{s}$

* Any discontinuous function with finite number of jump discontinuous can be expressed as a combination of unit step functions.

Example: Express the given function using step functions and compute its Laplace transform

1] $$f(t) = \begin{cases} 1 & 0 < t < 1 \\ -2 & 1 < t < 2 \\ 3 & t > 2 \end{cases}$$

$f(t) = 1 - 3u(t-1) + 5u(t-2)$

$= 1 - 3(1) + 5(0) = -2$

$\mathcal{L}[f(t)] = \mathcal{L}[1] - 3\mathcal{L}[u(t-1)] + 5\mathcal{L}[u(t-2)]$

$= \frac{1}{s} - 3\frac{e^{-s}}{s} + 5\frac{e^{-2s}}{s}$

PRENK

2] $g(t) = \begin{cases} 3 & t<2 \\ 1 & 2<t<5 \\ 10 & 5<t<8 \\ x & t>8 \end{cases}$

$g(t) = 3 - 2u(t-2) + 9u(t-5) - 3u(t-8)$

$L[g(t)] = \frac{3}{s} - 2\frac{e^{-2s}}{s} + 9\frac{e^{-5s}}{s} - 3\frac{e^{-8s}}{s}$

$\lim_{s \to \infty} F(s) = 0$

* we need also to know that :

$L[g(t)\,u(t-a)] = e^{-as} L[g(t+a)]$

$L^{-1}[e^{-as} F(s)] = f(t-a)\,u(t-a)$

Example compute

1] $\mathcal{L}[t\,u(t-3)] = e^{-3s}\,\mathcal{L}[t+3]$

$= e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$

2] $\mathcal{L}[t^2 u(t-1)] = e^{-s}\,\mathcal{L}[(t+1)^2]$

$= e^{-s}\,\mathcal{L}[t^2+2t+1] = e^{-s}\left(\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}\right)$

3] $\mathcal{L}[f(t)]$ when $f(t) = \begin{cases} 1 & 0 < t < 2 \\ t+2 & t > 2 \end{cases}$

$f(t) = 1 + (t+1)\,u(t-2)$

$\mathcal{L}[f(t)] = \mathcal{L}[1] + \mathcal{L}[(t+1)\,u(t-2)]$

$= \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$

$\mathcal{L}[(t+1)\,u(t-2)] = e^{-2s}\,\mathcal{L}[t+3]$

$= e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$

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Example 1: Find $L^{-1}\left[\frac{e^{-2s}}{s^2+1}\right]$

$= \sin(t-2)\,u(t-2)$ ← الجواب النهائي

$F(s) = \frac{1}{s^2+1} \Rightarrow L^{-1}[F(s)] = \sin t$

Example 2: $L^{-1}\left[\frac{e^{-3s}}{s^4}\right]$

$F(s) = \frac{1}{s^4}$

$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^4}\right] = \frac{1}{6}t^3$

$L\left[\frac{1}{6}t^3\right] = \frac{1}{s^4}$

$= \frac{1}{6}(t-3)^3\,u(t-3)$ ← الجواب النهائي

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Recall : $\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$

$\mathcal{L}[f(t)\,u(t-a)] = e^{-as}\,\mathcal{L}[f(t+a)]$

$\mathcal{L}^{-1}[e^{-as}F(s)] = u(t-a)\,f(t-a)$

* Example : compute the Laplace transform in the following :

1] $\mathcal{L}[t^2 u(t-3)]$

$= e^{-3s}\,\mathcal{L}[(t+3)^2]$

$= e^{-3s}\,\mathcal{L}[t^2+6t+9]$

$= e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$

$t^2 u(t-3) = t^2 \begin{cases} 0 & t<3 \\ 1 & t>3 \end{cases}$

$= \begin{cases} 0 & t<3 \\ t^2 & t>3 \end{cases}$

PRENKO

2] $\mathcal{L}[f(t)]$ when $f(t) = \begin{cases} 1 & t<2 \\ t+4 & t>2 \end{cases}$

$f(t) = 1 + (t+3)\,u(t-2)$

$\mathcal{L}[f(t)] = \frac{1}{s} + \mathcal{L}[u(t-2)(t+3)]$

$= \frac{1}{s} + e^{-2s}\mathcal{L}[t+5]$

$= \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{5}{s}\right)$

3] $\mathcal{L}[e^{3t}\,u(t-4)]$

$= e^{-4s}\mathcal{L}[e^{3(t+4)}]$

$= e^{-4s}\mathcal{L}[e^{3t+12}]$

$= e^{-4s}\mathcal{L}[(e^{12})\,e^{3t}]$ (e^{12}: constant)

$= e^{12}\,e^{-4s}\mathcal{L}[e^{3t}]$

$= e^{12}\,e^{-4s}\,\frac{1}{s-3}$

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Example: compute the laplace inverse in the following

1] $\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2}\right] = u(t-3)\,f(t-3)$

$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$

2] $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2+9}\right] = u(t-2)\,f(t-2) = \frac{1}{3}u(t-2)\sin 3(t-2)$

$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+9}\right] = \frac{1}{3}\sin 3t$

$\frac{1}{3}\mathcal{L}[\sin 3t] = \frac{1}{s^2+9}$

$\mathcal{L}\left[\frac{1}{3}\sin 3t\right] = \frac{1}{s^2+9}$

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3] $\mathcal{L}^{-1}\left[\frac{s\,e^{-3s}}{s^2+4s+5}\right] = u(t-3)\,f(t-3)$

$f(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+4s+5}\right] = \mathcal{L}^{-1}\left[\frac{(s+2)-2}{(s+2)^2+1}\right]$

s^2+4s+5 اكمال المربع

$s^2+4s+4-4+5$ $\neq$

$(s+2)^2+1$

$= \mathcal{L}^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - 2\,\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+1}\right]$

$= e^{-2t}\cos t - 2e^{-2t}\sin t$

$= u(t-3)\,e^{-2(t-3)}\left(\cos(t-3) - 2\sin(t-3)\right)$

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4] $\mathcal{L}^{-1}\left[\frac{e^{-s}(s-3)}{(s+1)(s+2)}\right] = u(t-1)\,f(t-1)$

$f(t) = \mathcal{L}^{-1}\left[\frac{s-3}{(s+1)(s+2)}\right]$

↳ partial fraction

$\frac{s-3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2)+B(s+1)}{(s+1)(s+2)}$

$s-3 = A(s+2) + B(s+1)$

$s = -2 \Rightarrow -5 = -B \Rightarrow B = 5$

$s = -1 \Rightarrow -4 = A$

$\mathcal{L}^{-1}\left[\frac{-4}{s+1} + \frac{5}{s+2}\right]$

$= -4e^{-t} + 5e^{-2t}$

$= u(t-1)\,f(t-1)$

$= u(t-1)\,f\left(-4e^{-(t-1)} + 5e^{-2(t-1)}\right)$

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5] $\mathcal{L}^{-1}\left[\frac{e^{-2s} - 3e^{-4s}}{s+7}\right]$

$= \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s+7}\right] - 3\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s+7}\right]$

$= u(t-2)\, e^{-7(t-2)} - 3u(t-4)\, e^{-7(t-4)}$

$= \begin{cases} 0 & t<2 \\ e^{-7(t-2)} & t>2 \end{cases} + 3\begin{cases} 0 & t<4 \\ -3e^{-7(t-4)} & t>4 \end{cases}$

0	$e^{-7(t-2)}$	$e^{-7(t-2)} - 3e^{-7(t-4)}$
	2	4

$= \begin{cases} 0 & t<2 \\ e^{-7(t-2)} & 2<t<4 \\ e^{-7(t-2)} - 3e^{-7(t-4)} & t>4 \end{cases}$

PRENKO

31/12/2023

بسم الله الرحمن الرحيم

* solving IVPs By Laplace Transform :

consider an IVP

$$y' + ay = f(t)$$
$$y(0) = \alpha$$

Recall that

$$y' + ay = f(t)$$

Apply the laplace operation to the DE

$$L[y' + ay] = L[f(t)]$$

$$L[y'] + aL[y] = F(s)$$

$$sY(s) - y(0) + aY(s) = F(s)$$

$$(s+a)Y(s) = F(s) + y(0)$$

$$Y(s) = \frac{F(s) + y(0)}{s+a}$$

But, $y(t) = L^{-1}[Y(t)]$

$$= L^{-1}\left[\frac{F(s) + y(0)}{s+a}\right]$$

$$\mathcal{L}[y'(t)] = sY(s) - y(0)$$

$$\mathcal{L}[y''(t)] = s^2Y(s) - sy(0) - y'(0)$$

Example: Use laplace Transform to solve the IVP

$$y' + 3y = e^{2t}, \quad y(0) = -1$$

Let $\mathcal{L}[y(t)] = Y(s)$, Apply the laplace to the DE

$$\mathcal{L}[y' + 3y] = \mathcal{L}[e^{2t}]$$

$$\mathcal{L}[y'] + 3\mathcal{L}[y] = \frac{1}{s-2}$$

$$sY(s) - y(0) + 3Y(s) = \frac{1}{s-2}$$

$$sY(s) + 1 + 3Y(s) = \frac{1}{s-2}$$

$$(s+3)Y(s) = \frac{1}{s-2} - 1$$

$$Y(s) = \frac{1}{(s-2)(s+3)} - \frac{1}{s+3}$$

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$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$$= \frac{A(s+3) + B(s-2)}{(s-2)(s+3)}$$

$$1 = A(s+3) + B(s-2)$$

$s = 2 \Rightarrow 1 = 5A \Rightarrow A = \frac{1}{5}$

$s = -3 \Rightarrow 1 = -5B \Rightarrow B = -1/5$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{(s-2)(s+3)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1/5}{s-2} + \frac{-1/5}{s+3}\right] - e^{-3t}$$

$$= \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t} - e^{-3t}$$

$$y(t) = \frac{1}{5}e^{2t} - \frac{6}{5}e^{-3t}$$

$$y' + 3y = e^{2t}$$

$$M = e^{\int 3\,dt} = e^{3t}$$

$$y(t) = \frac{\int e^{5t}\,dt + C}{e^{3t}}$$

$$= \frac{e^{5t}}{5} + C = \frac{1}{5}e^{2t} - \frac{6}{5}e^{-3t}$$

$$y(0) = C + \frac{1}{5} = -1$$

$$C = -\frac{6}{5}$$

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Example 8: Use Laplace transform to solve the IVP

$$y'' - y' - 2y = 0$$

$$y(0) = 1, \quad y'(0) = 0$$

Let $\mathcal{L}[y(t)] = Y(s)$. Apply the laplace to the DE

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0]$$

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 0$$

$$s^2 Y(s) - s\,y(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = 0$$

$$s^2 Y(s) - s - sY(s) + 1 - 2Y(s) = 0$$

$$(s^2 - s - 2)\,Y(s) = s - 1$$

$$Y(s) = \frac{s-1}{s^2 - s - 2}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{s-1}{s^2-s-2}\right]$$

کسر جزئی

$$= \mathcal{L}^{-1}\left[\frac{s-1}{(s-2)(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1/3}{s-2} + \frac{2/3}{s+1}\right]$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Example 6 Use Laplace transform to solve

$$y'' - 6y' + 5y = 3e^{2t}$$
$$y(0) = 2,\ y'(0) = 3$$

Let $\mathcal{L}[y(t)] = Y(s)$ Applying the laplace operator to the given DE given

$$\mathcal{L}[y''] - 6\mathcal{L}[y'] + 5\mathcal{L}[y] = \frac{3}{s-2}$$

$$s^2Y(s) - 2s - 3 - 6(sY(s) - 2) + 5Y(s)$$
$$= \frac{3}{s-2} - 6sY(s) + 12$$

$$(s^2 - 6s + 5)Y(s) = \frac{3}{s-2} + \frac{2s-9}{1}$$

$$(s-1)(s-5)\,Y(s) = \frac{2s^2-13s+21}{s-2}$$

$$Y(s) = \frac{2s^2-13s+21}{(s-1)(s-2)(s-5)}$$

$$\frac{2s^2-13s+21}{(s-1)(s-2)(s-5)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-5}$$

$$= \frac{A(s-2)(s-5) + B(s-1)(s-5) + C(s-1)(s-2)}{(s-1)(s-2)(s-5)}$$

$$2s^2-13s+21 = A(s-2)(s-5) + B(s-1)(s-5) + C(s-1)(s-2)$$

$s=1 \Rightarrow 10 = 4A \Rightarrow A = 5/2$

$s=2 \Rightarrow 3 = -3B \Rightarrow B = -1$

$s=5 \Rightarrow 6 = 12C \Rightarrow C = 1/2$

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{2s^2-13s+21}{(s-1)(s-2)(s-5)}\right]$$

$$= L^{-1}\left[\frac{5/2}{s-1} + \frac{-1}{s-2} + \frac{1/2}{s-5}\right]$$

$$= \frac{5}{2}e^{t} - e^{2t} + \frac{1}{2}e^{5t}$$

PRENT

21/1/2024

بسم الله الرحمن الرحيم

Example: solve the IVP

$$y' + 3y = \begin{cases} 0 & 0 \le t < 2 \\ e^{-t} + 1 & t > 2 \end{cases}$$

$$y(0) = 1$$

The DE: $y' + 3y = (e^{-t} + 1)\, u(t-2)$

Apply the laplace to the DE

$$\mathcal{L}[y'] + 3\mathcal{L}[y] = \mathcal{L}[(e^{-t} + 1)\, u(t-2)]$$

$$sY(s) - 1 + 3Y(s) = e^{-2s}\mathcal{L}[e^{-(t+2)} + 1]$$

$$(s+3)Y(s) - 1 = e^{-2s}\mathcal{L}[e^{-2}e^{-t} + 1]$$

$$(s+3)Y(s) = e^{-2s}\left(\frac{e^{-2}}{s+1} + \frac{1}{s}\right) + 1$$

$$Y(s) = e^{-2s}\left(\frac{e^{-2}}{(s+1)(s+3)} + \frac{1}{s(s+3)}\right) + \frac{1}{s+3}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{e^{-2s}}{2}\, u(t-2)\left(e^{-(t+2)} - e^{-2(t-3)}\right)$$

$$+ u(t-2)\left(\frac{1}{3} - \frac{1}{3}e^{-3(t-3)}\right) + e^{-3t}$$

PRENK

$$\frac{1}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} = \frac{A(s+3)+B(s+1)}{(s+1)(s+3)}$$

$$1 = A(s+3)+B(s+1)$$

$s=-1 \Rightarrow 1 = 2A \Rightarrow A = 1/2$

$s=-3 \Rightarrow 1 = -2B \Rightarrow B = -1/2$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+3)}\right] = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{A(s+3)+Bs}{s(s+3)}$$

$$1 = A(s+3)+Bs$$

$s=0 \Rightarrow A = 1/3$

$s=-3 \Rightarrow B = -1/3$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = \frac{1}{3} - \frac{1}{3}e^{-3t}$$

Example: The current $I(t)$ in an LC series circuit is governed by

$$I''(t) + 4I(t) = g(t)$$

$$I(0) = 1, \quad I'(0) = 3$$

$$g(t) = \begin{cases} 3\sin t & 0 \le t \le 2\pi \\ 0 & t > 2\pi \end{cases}$$

$$= 3\sin t - 3\sin t\, u(t-2\pi)$$

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Apply The laplace opesalvs to the DE using

$$L[I(t)] = F(s)$$

$$I'' + 4I = 3\sin t - 3\sin t\, u(t-2\pi)$$

$$L[I''] + 4L[I] = \frac{3}{s^2+1} - 3e^{-2\pi s} L[\sin(t+2\pi)]$$

$$s^2F(s) - sI(s) - I'(0) + 4F(s) = \frac{3}{s^2+1} - \frac{3e^{-2\pi s}}{s^2+1}$$

$$s^2F(s) - s - 3 + 4F(s) = \frac{3}{s^2+1} - \frac{3e^{-2\pi s}}{s^2+1}$$

$$F(s) = \frac{3}{(s^2+1)(s^2+4)} = \frac{3e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s+3}{s^2+4}$$

$$\frac{3}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$3 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$s=0 \Rightarrow 3 = 4B + D$

$s=1 \Rightarrow 3 = 5(A+B) + 2(C+D)$

$3 = 5A + 5B + 2C + 2D$

$s=-1 \Rightarrow 3 = 5(B-A) + 2(D-C)$

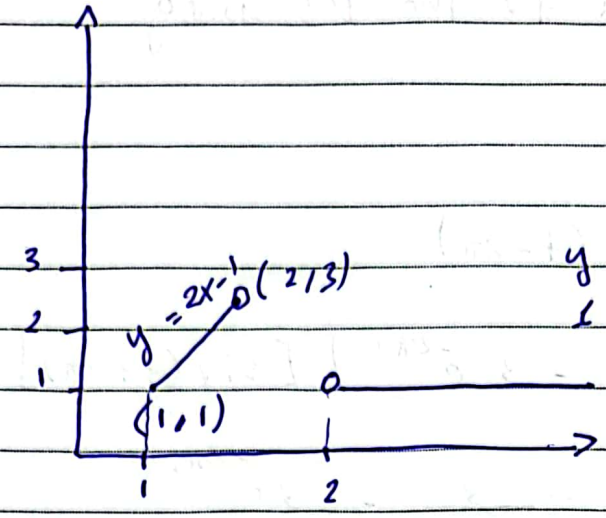
$3 = -5A + 5B - 2C + 2D$

$A=0,\ B=1,\ C=0,\ D=-1 \Rightarrow \frac{3}{(s^2+1)(s^2+4)} = \frac{1}{s^2+1} - \frac{1}{s^2+4}$

PREN

4/1/2023

بسم الله الرحمن الرحيم



$y = mx + b = 2x + b$

$1 = 2(1) + b \Rightarrow b = -1$

$$f(t) = \begin{cases} 2t-1 & 1<t<2 \\ 2 & t>2 \end{cases}$$

$= 2t-1 + (3-2t)u(t-2)$

$= L[f(t)] = 2\frac{1}{s^2} - \frac{1}{s} + e^{-2s}L[3-2(t+2)$

$= \frac{2}{s^2} - \frac{1}{s} + e^{-2s}\left(\frac{-1}{s} - \frac{2}{s^2}\right)$

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سؤال من الكتاب

Ex 20 The current $I(t)$ in an LC series circuit is governed by the initial value problem

$$I''(t) + 4I(t) = g(t)$$
$$I(0) = 1, \quad I'(0) = 3$$

where $g(t) = \begin{cases} 3\sin t, & 0 \le t \le 2\pi \\ 0, & 2\pi < t \end{cases}$

Determine the current as a function of time t

$$y'' + 4y = 3\sin t - 3\sin t\, u(t-2\pi)$$

$$s^2 Y(s) - s - 3 + 4Y(s) = \frac{3}{s^2+1} - \frac{3e^{-2\pi s}}{s^2+1}$$

$$Y(s) = \frac{3}{(s^2+1)(s^2+4)} - \frac{3e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s+3}{s^2+4}$$

$$\mathcal{L}^{-1}\left[\frac{e^{-2s} - se^{-4s}}{s+3}\right] = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s+3}\right] - 3\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s+3}\right]$$

$$= u(t-2)\,e^{-3(t-2)} - 3u(t-4)\,e^{-3(t-4)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] = te^{-2t}$$

$$\mathcal{L}[t] = \frac{1}{s^2} \rightarrow, \quad \mathcal{L}[e^{-2t}t] = \frac{1}{(s+2)^2}$$

PREN

$$\mathcal{L}[e^{-2t}] = \frac{1}{s+2}$$

$$\mathcal{L}[t e^{-2t}] = (-1)^1 \frac{-1}{(s+2)^2} = \frac{1}{(s+2)^2}$$

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right]$$

$$\mathcal{L}^{-1}\left[\ln\left(\frac{s}{s^2+1^2}\right)\right]$$

$$\mathcal{L}[t^n f(t)] = (-1)^n F(s)^{(n)}$$

$$\mathcal{L}^{-1}\left[\frac{-1/s^2}{\frac{1}{s}}\right] = \mathcal{L}^{-1}\left[\frac{-1}{s}\right]$$

$$= t e^{-2t}$$

$$\mathcal{L}^{-1}[F(s)^{(n)}] = (-1)^n t^n f(t)$$

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