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Strength of Materials Course Contents

Engineering Mechanics

Statics & Strength of Materials

Resultant of Force systems and equilibrium

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International System of Units (SI):

Length = meters (m);

Time = seconds (s);

Mass = kilograms (kg).

Force = Newtons (N) is derived from F=m*a.

Therefore a 1 kilogram mass has a weight of 9.81 Newton at the earth's surface.

Giga	X 10^9	G
Mega	X 10^6	M
Kilo	X 10^3	k
Meter		m
milli	X 10^{-3}	mm
micro	X 10^{-6}	μ
nanometer	X 10^{-9}	n
Newton		N
Pascal	Pa	N/m^2
	X 10^6Pa	MPa

Dimensional homogeneity - in an equation where you are adding terms, they must all have the same units.

Force Vectors

Scalar and Vectors

Scalar: is a quantity which has magnitude only.

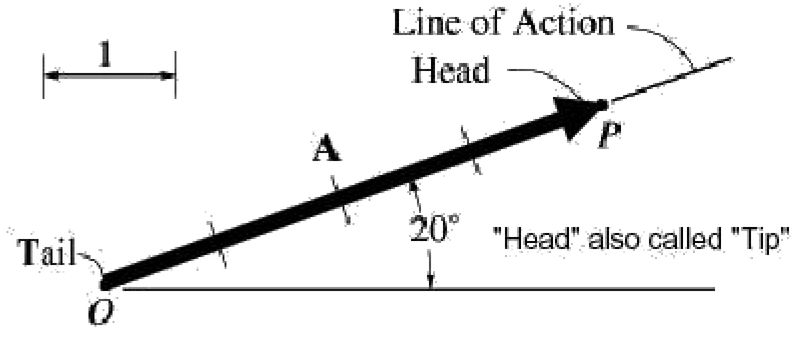
Examples of scalars: speed, distance, energy, charge, volume, mass and temperature. .

Vectors are quantities which are fully described by both a magnitude and a direction. Vectors are physical quantities.

Examples of vectors are displacement, velocity, acceleration, force and electric field

<u>Vector notation:</u>

A widely used convention is to denote a vector quantity in bold type, such as **A** ‹and that is the convention that will be used. The magnitude of a vector **A** is written as**| A.|**

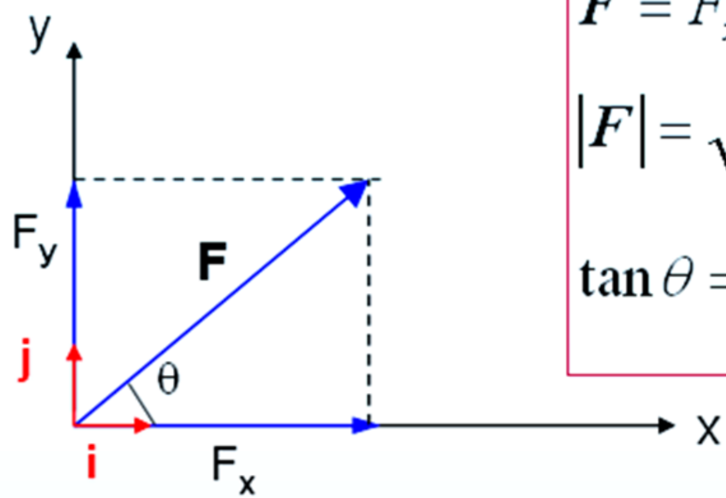


Addition of a system of coplanar forces

Cartesian Vector Notation
(2D)

$$F_x = F\cos\theta$$

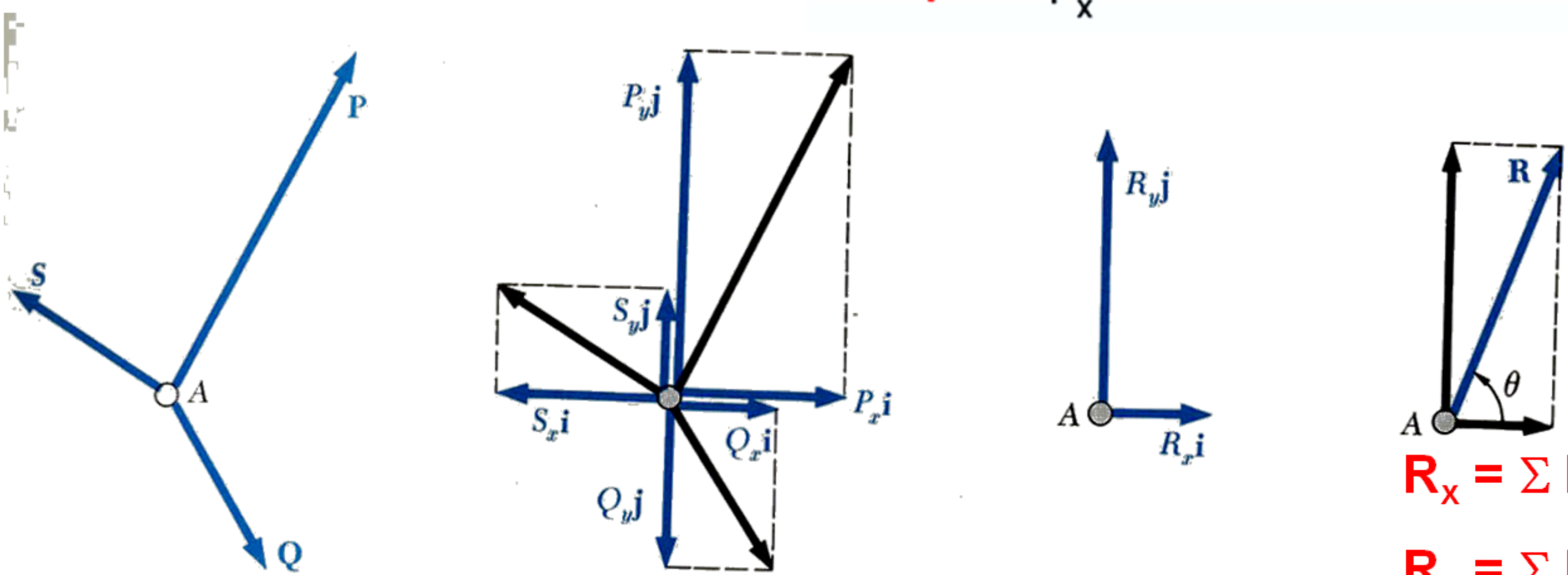
$$F_y = F\sin\theta$$



$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j}$$

$$|\boldsymbol{F}| = \sqrt{F_x^2 + F_y^2}$$

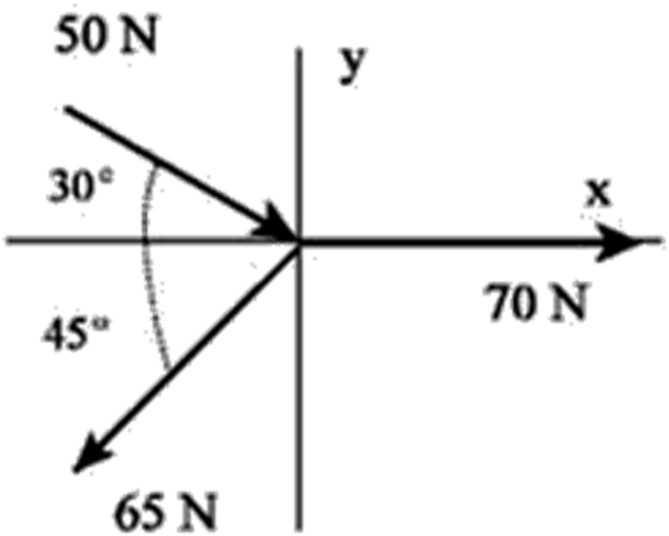
$$\tan\theta = \frac{F_y}{F_x}$$

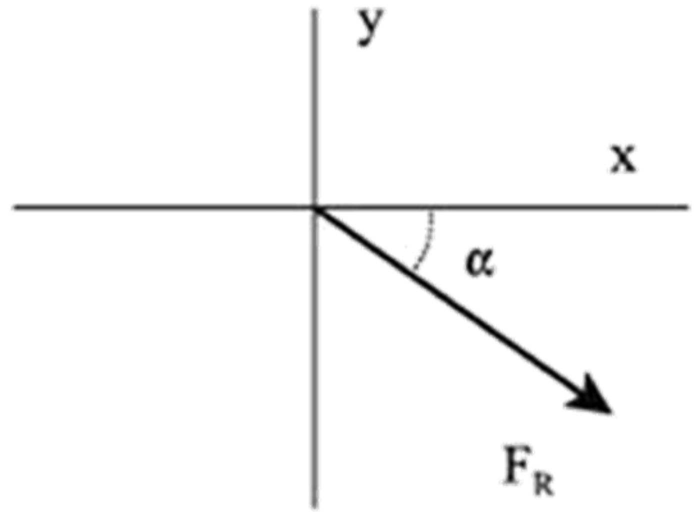


$R_x = \Sigma F_x$

$R_y = \Sigma F_y$

Determine the magnitude of the resultant force and its direction measured from the positive x axis.





$$\sum F_x: 70\,N + 50\,N\cos[30°] - 65\,N\cos[45°] = 67.3\text{ N}$$

$$\sum F_y: -50\,N\sin[30°] - 65\,N\sin[45°] = -71.0\text{ N}$$

$$\tan\alpha = \frac{71.0\,N}{67.3\,N} = 1.054 \Rightarrow \underline{\alpha = 46.5°}$$

3D *Three Dimensional Vectors*

Cartesian vector representation:

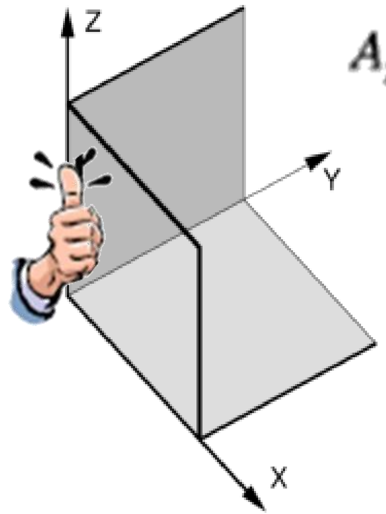
$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

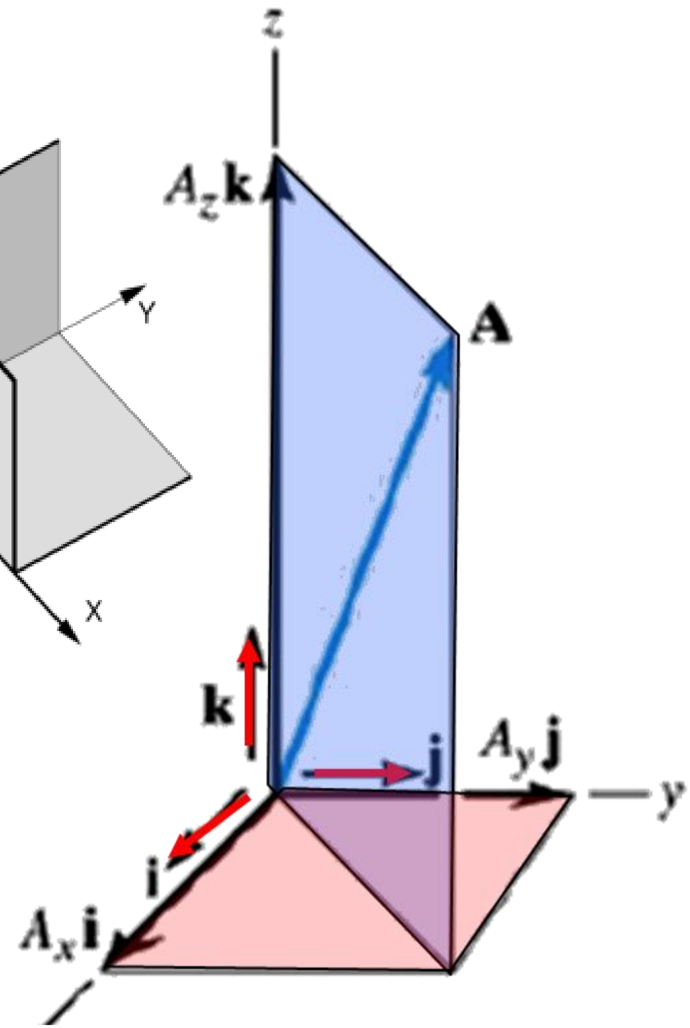
Magnitude of a Cartesian Vector.

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Direction of a Cartesian Vector

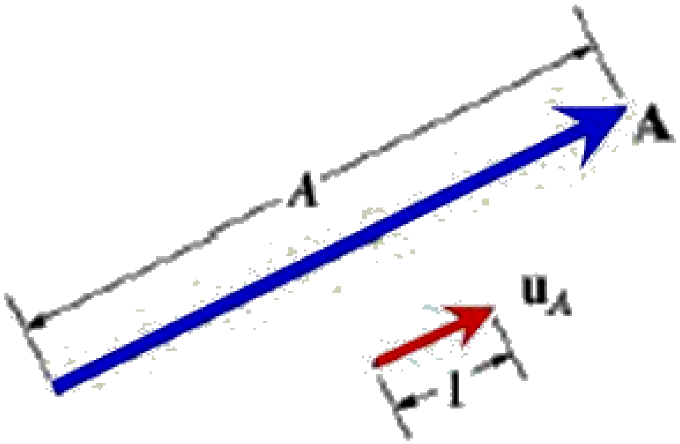
$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}$$





Unit vector Representation of a Vector

vector $\mathbf{u}_A$ is just a vector in the same direction as **A**, but with magnitude = 1,



$$\mathbf{u}_A = \bar{\mathbf{A}} / |A|$$

$\mathbf{u}_A$ is **dimensionless.** It serves only to indicate direction and sense.

Direction (orientation) of a Cartesian vector in 3D

- α = angle between **A** and *positive* x axis
- β = angle between **A** and *positive* y axis
- γ = angle between **A** and *positive* z axis

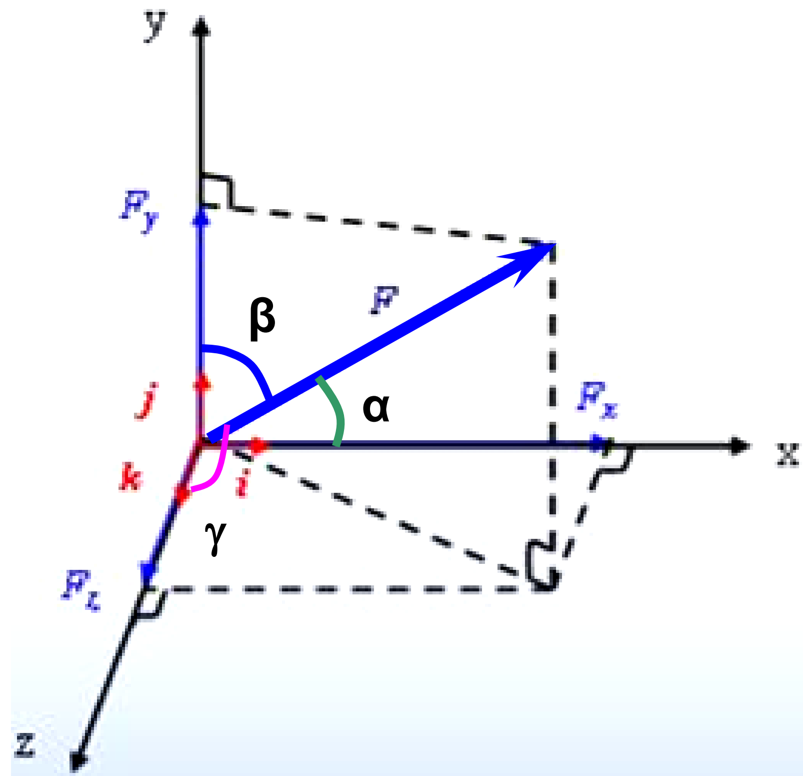
$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}$$

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}$$

$$\mathbf{U} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

$$1 = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$



Eg. Determine the magnitude and directional cosines of the vector.

$$\vec{A} = 700\,\vec{i} - 820\,\vec{j} + 900\vec{k}$$

The magnitude of the vector is

$$\vec{A} = 700\ \ \vec{i} - 820\,\vec{j} + 900\ \vec{k}$$

$$|A| = \sqrt{(700)^2 + (-820)^2 + (900)^2} = 1404.42$$

The directional cosines are

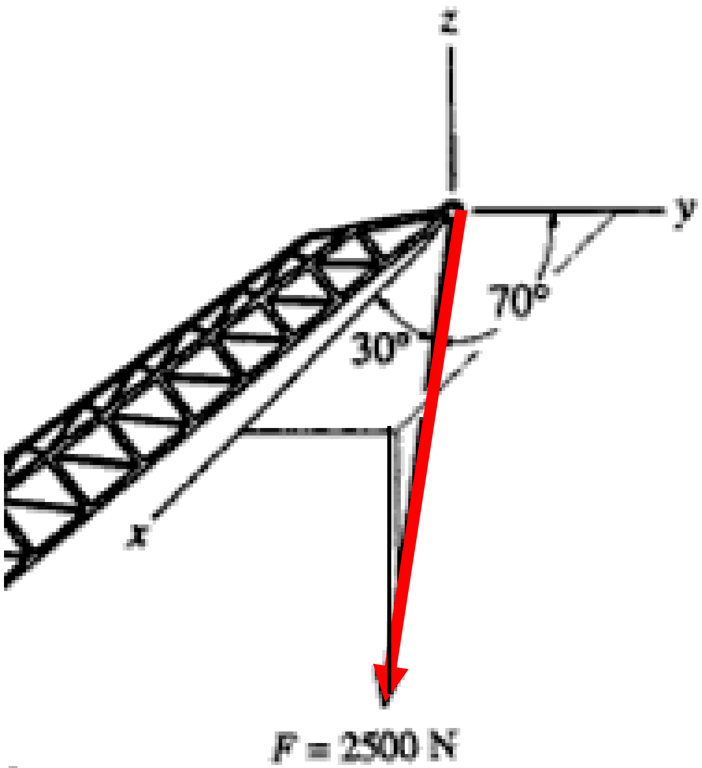
$$\begin{cases} \cos\theta_x = \dfrac{700}{1404.42} = 0.498 \Rightarrow \theta_x = 60.1^\circ \\ \cos\theta_y = \dfrac{-820}{1404.42} = -0.584 \Rightarrow \theta_y = 125.7^\circ \\ \cos\theta_z = \dfrac{900}{1404.42} = 0.641 \Rightarrow \theta_x = 50.1^\circ \end{cases}$$

Check the cosines

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

$$(0.498)^2 + (-0.584)^2 + (0.641)^2 = 1$$

Express F as Cartesian vector



$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\cos^2 30^\circ + \cos^2 70^\circ + \cos^2 \gamma = 1$$

$$\cos \gamma = \pm 0.3647$$

$$\gamma = 68.61^\circ \text{ or } 111.39^\circ$$

By inspection, $\gamma = 111.39^\circ$ since the force **F** is directed in negative octant.

$$\mathbf{F} = 2500\{\cos 30^\circ \mathbf{i} + \cos 70^\circ \mathbf{j} + \cos 111.39^\circ \mathbf{k}\}\ \text{N}$$

Adding and Subtracting 3D Cartesian Vectors

Given $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$, $\mathbf{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}$,

Addition: $\mathbf{R} = \mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$

Subtraction: $\mathbf{R}' = \mathbf{A} - \mathbf{B} = (A_x - B_x)\mathbf{i} + (A_y - B_y)\mathbf{j} + (A_z - B_z)\mathbf{k}$

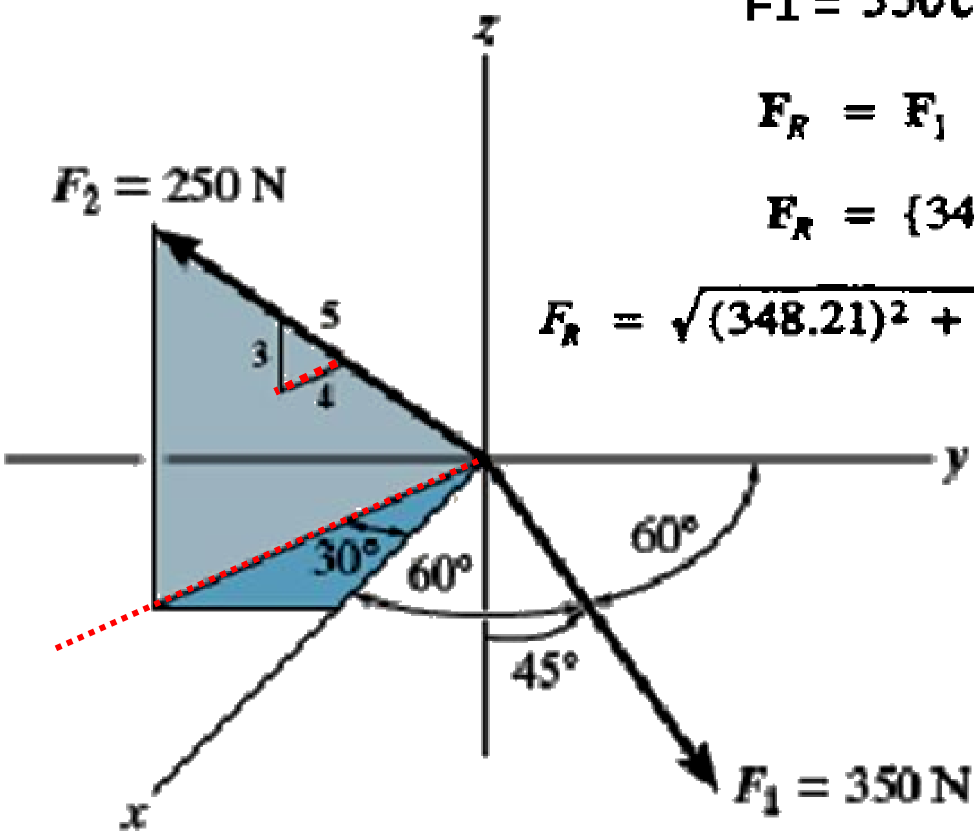
Given several vectors,

$$\mathbf{F}_R = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}$$

2-71 Determine the magnitude and coordinate direction angles of the resultant force and sketch this vector on the coordinate system.

$$F2 = 250\left(\frac{4}{5}\right)\cos 30^\circ \mathbf{i} - 250\left(\frac{4}{5}\right)\sin 30^\circ \mathbf{j} + 250\left(\frac{3}{5}\right)\mathbf{k}$$

$$F1 = 350\cos 60^\circ \mathbf{i} + 350\cos 60^\circ \mathbf{j} - 350\cos 45^\circ \mathbf{k}$$



$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$$

$$\mathbf{F}_R = \{348.21\mathbf{i} + 75.0\mathbf{j} - 97.487\mathbf{k}\}\ \text{N}$$

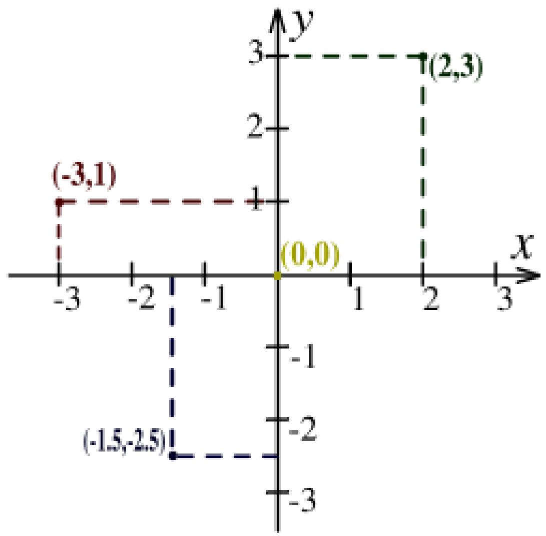
$$F_R = \sqrt{(348.21)^2 + (75.0)^2 + (97.487)^2} = 369.29\ \text{N}$$

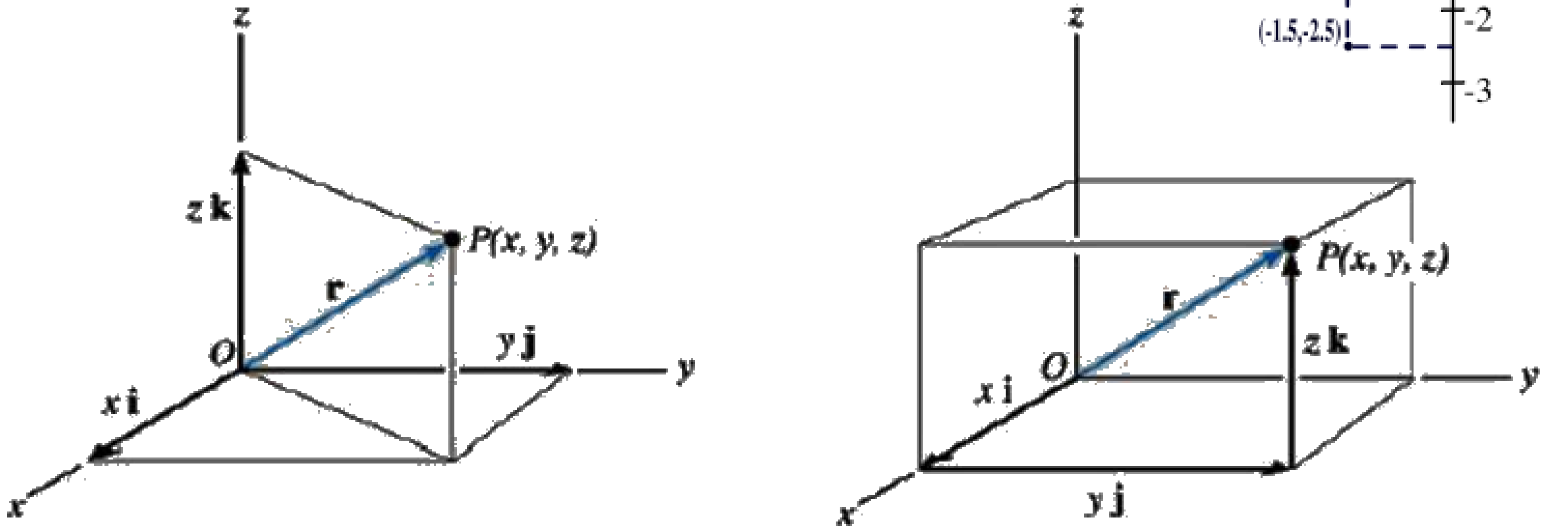
$$\alpha = \cos^{-1}\left(\frac{348.21}{369.29}\right) = 19.5^\circ$$

$$\beta = \cos^{-1}\left(\frac{75.0}{369.29}\right) = 78.3^\circ$$

$$\gamma = \cos^{-1}\left(\frac{-97.487}{369.29}\right) = 105^\circ$$

Position Vectors

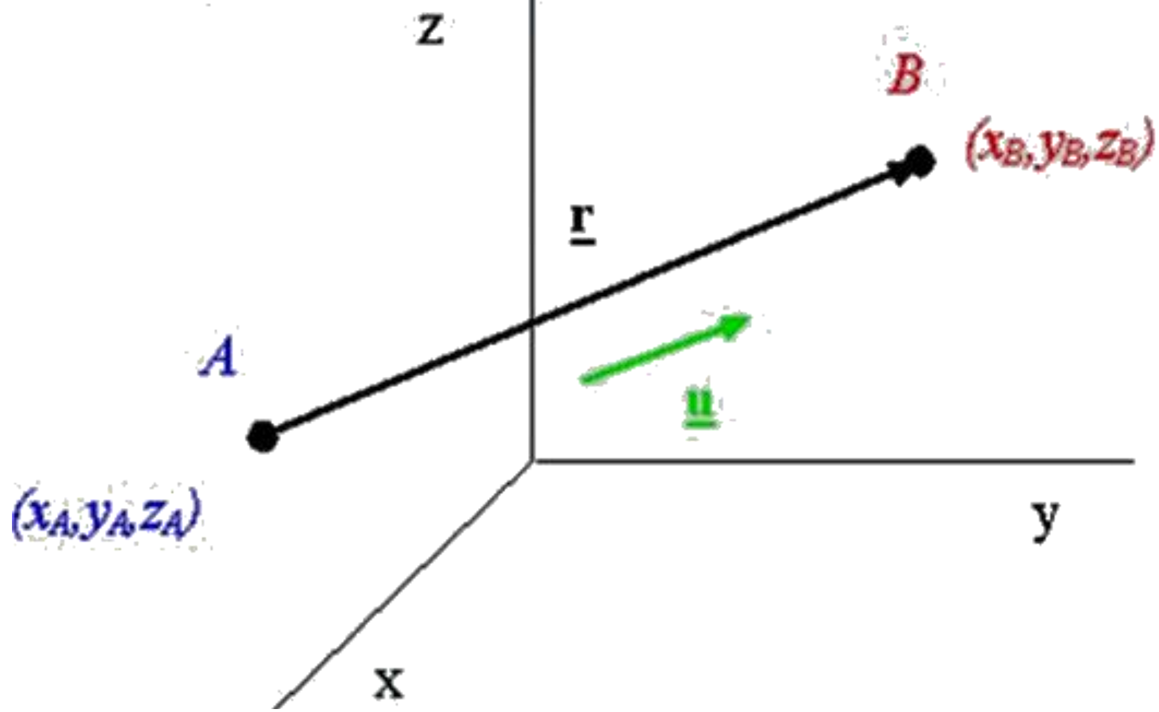




(a) (b)

Cartesian position vector from origin O to point P(x,y,z):

r = x**i** + y**j** + z**k**



$$\boldsymbol{r} = (x_B - x_A)\boldsymbol{i} + (y_B - y_A)\boldsymbol{j} + (z_B - z_A)\boldsymbol{k}$$

$$\boldsymbol{u} = \frac{\boldsymbol{r}}{r} = \frac{(x_B - x_A)\boldsymbol{i} + (y_B - y_A)\boldsymbol{j} + (z_B - z_A)\boldsymbol{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}$$

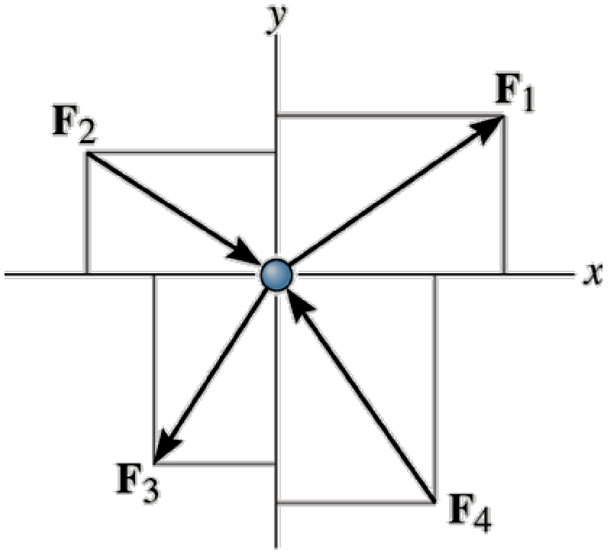
$$\boldsymbol{F} = F\frac{\boldsymbol{r}}{r} = F\boldsymbol{u}$$

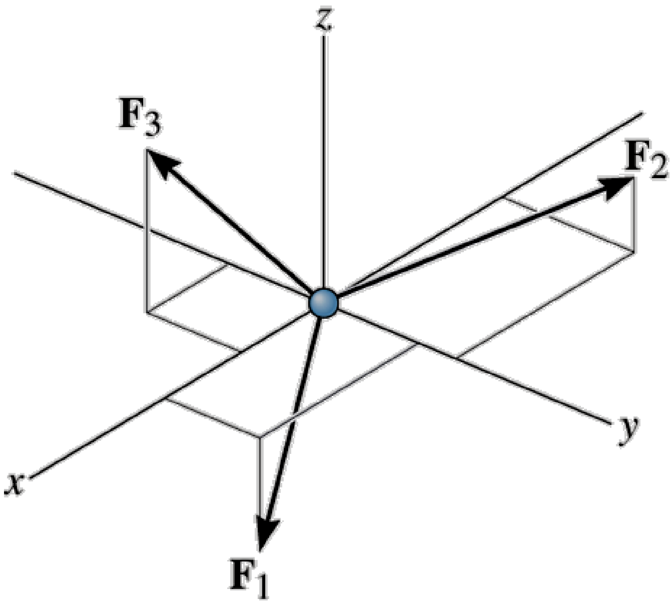
Conditions for equilibrium of a Particle

To maintain a state of equilibrium, ***The resultant force acting on a particle must be zero***.

$$\sum F = 0$$

$$\sum F_x\, i + \sum F_y\, j + \sum F_z\, k = 0$$





$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum F_y = 0$$

Equilibrium equations

The Free Body diagram

Construction of a free body diagram.

Step 1: **Isolate the body or combination of bodies are to be shown on the free-body diagram.**

Step 2: **Prepare drawing or sketch of the outline of the isolated or free body.**

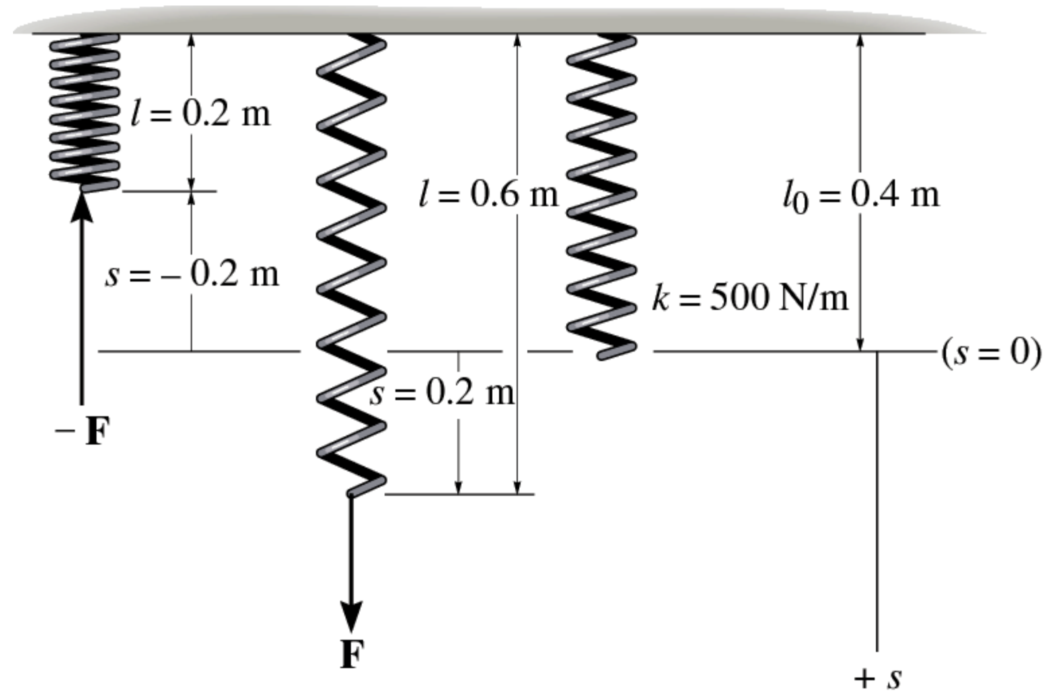
Step 3: **identify all the forces exerted by contacting or attracting bodies that were removed during isolation**

Step 4: **Choose the set of coordinate axes to be used in solving the problem and indicate their directions on the free-body diagram. Place any dimensions required for solution of the problem on the diagram.**

1. Spring

$$\mathbf{F = ks}$$

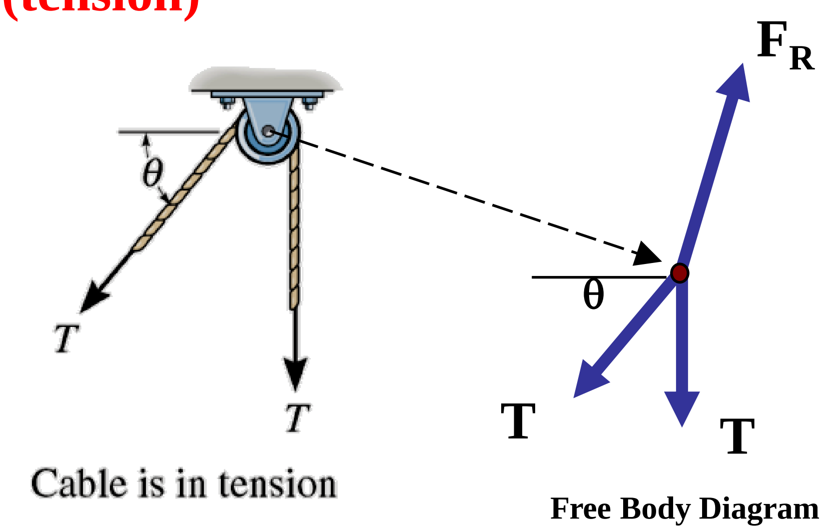
K : Spring constant = stiffness



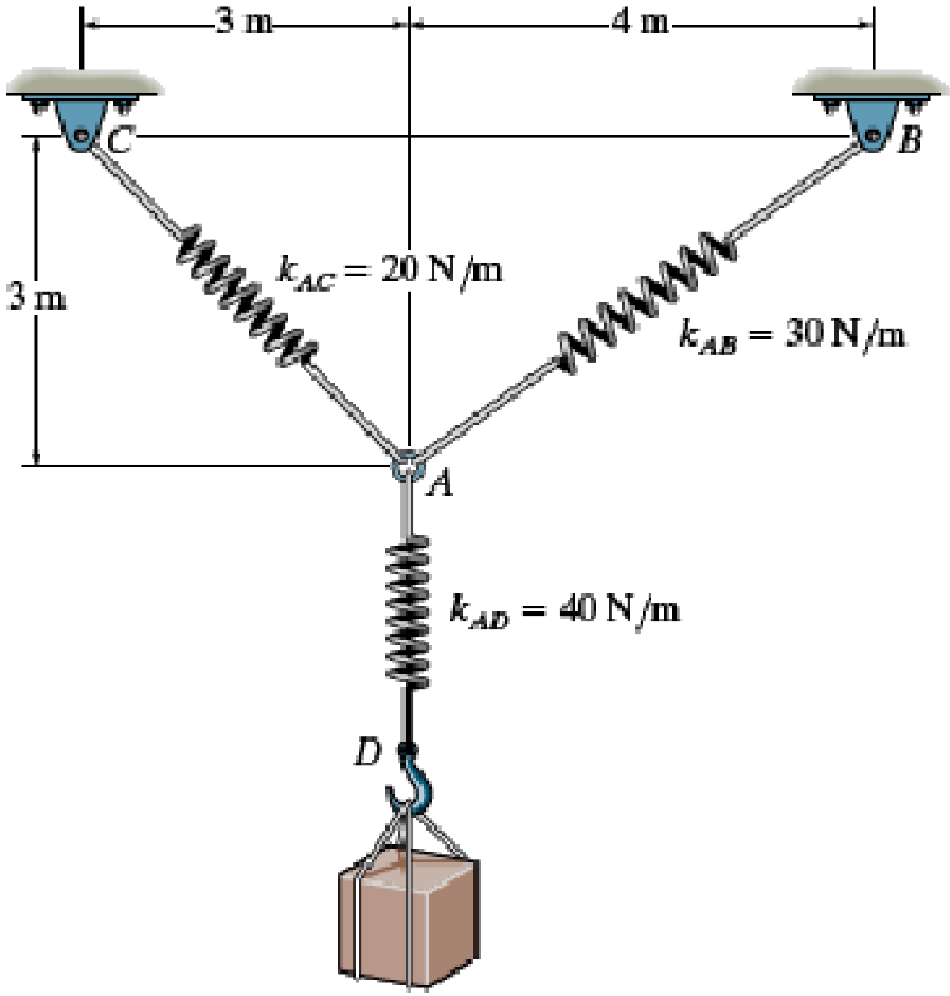
2. Cables and pulleys (tension)

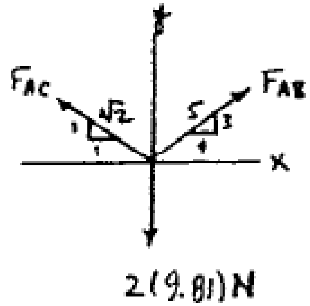
Cable forces are along the cable

Cables on frictionless pulleys have same tension force in every part of the cable



Determine the stretch in each spring for equilibrium of the 2-kg block. The springs are shown in the equilibrium position





$$F_{AD} = 2(9.81) = x_{AD}(40)$$

$$x_{AD} = 0.4905 \text{ m} \qquad \textbf{Ans}$$

$$\xrightarrow{+} \Sigma F_x = 0; \qquad F_{AB}\left(\frac{4}{5}\right) - F_{AC}\left(\frac{1}{\sqrt{2}}\right) = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad F_{AC}\left(\frac{1}{\sqrt{2}}\right) + F_{AB}\left(\frac{3}{5}\right) - 2(9.81) = 0$$

$$F_{AC} = 15.86 \text{ N}$$

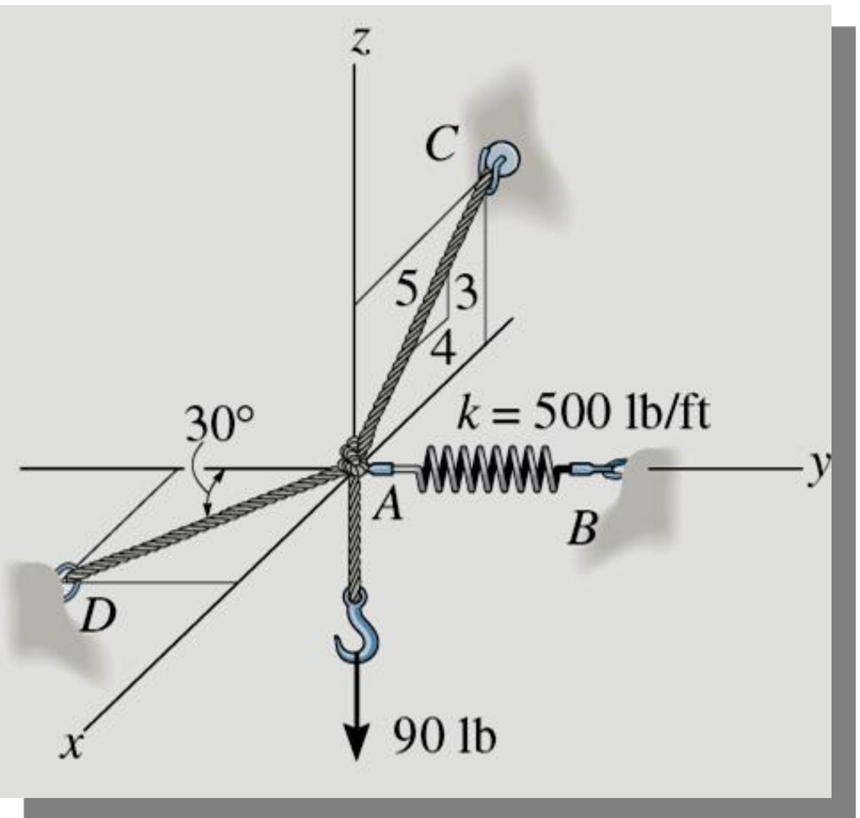
$$x_{AC} = \frac{15.86}{20} = 0.793 \text{ m} \qquad \textbf{Ans}$$

$$F_{AB} = 14.01 \text{ N}$$

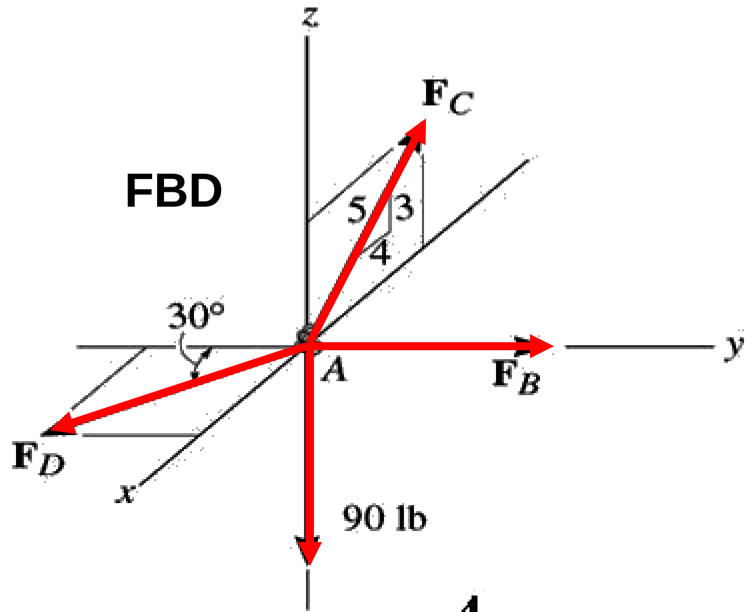
$$x_{AB} = \frac{14.01}{30} = 0.467 \text{ m} \qquad \textbf{Ans}$$

Example

A 90 lb load is suspended from the hook as shown. The load is supported by two cables and a spring with k=500 lb/ft. Determine the force in the cables and the stretch of the spring for equilibrium. Cable AD lies in the x-y plane and cable AC lies in the x-z plane.



FBD



$$\sum F_x = 0 \quad F_D \sin 30^\circ - \frac{4}{5} F_C = 0$$

$$\sum F_y = 0 \quad -F_D \cos 30^\circ + F_B = 0$$

$$\sum F_z = 0 \quad \frac{3}{5} F_C - 90\,\text{lb} = 0$$

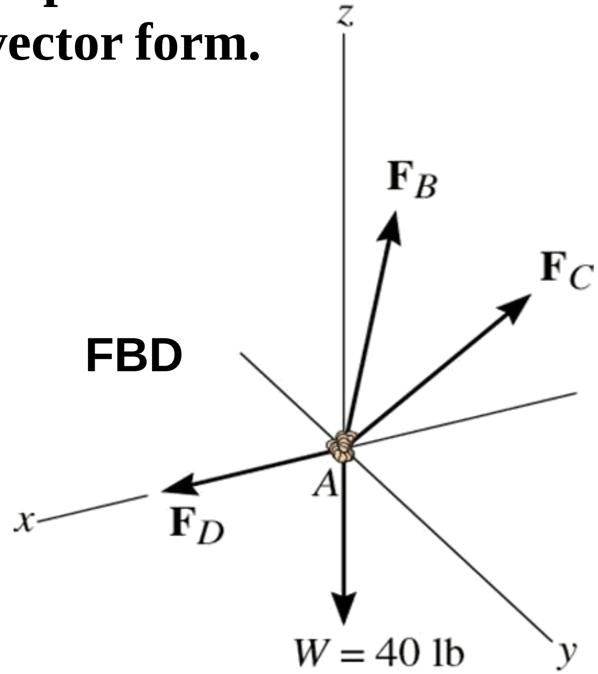
$$F_C = 150\,\text{lb}$$
$$F_D = 240\,\text{lb}$$
$$F_B = 208\,\text{lb}$$

Stretch

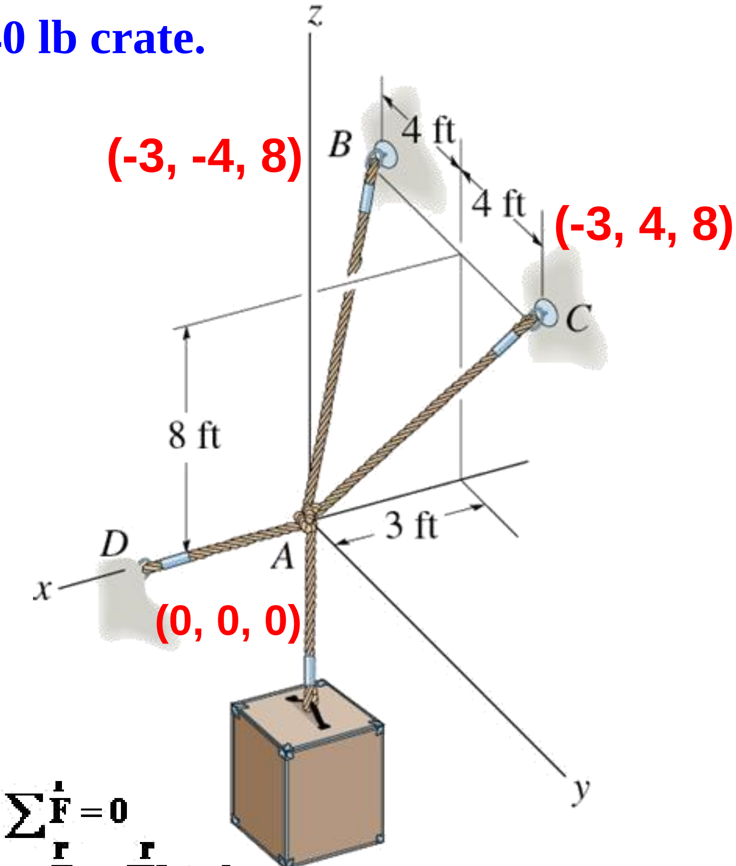
$$F_B = 208\,\text{lb}$$
$$F_B = k\, s_{AB}$$
$$208\,\text{lb} = 500 \frac{\text{lb}}{\text{ft}} s_{AB}$$
$$s_{AB} = 0.416\,\text{ft}$$

Express each force in Cartesian vector form.

Determine the force in each cable used to support the 40 lb crate.



$$\overline{F}_B = F_B \hat{u}_{AB}$$
$$\overline{F}_C = F_C \hat{u}_{AC}$$
$$\overline{F}_D = F_D \hat{u}_{AD}$$



$$\vec{F}_B = F_B\left(\frac{\vec{r}_{AB}}{r_{AB}}\right) = F_B\left[\frac{-3\,\hat{i} - 4\,\hat{j} + 8\,\hat{k}}{\sqrt{(-3)^2 + (-3)^2 + (8)^2}}\right]$$

$$\vec{F}_B = -0.318\,F_B\,\hat{i} - 0.424\,F_B\,\hat{j} + 0.848\,F_B\,\hat{k}$$

$$\vec{F}_C = F_C\left(\frac{\vec{r}_{AC}}{r_{AC}}\right) = F_C\left[\frac{-3\,\hat{i} + 4\,\hat{j} + 8\,\hat{k}}{\sqrt{(-2)^2 + (-3)^2 + (8)^2}}\right]$$

$$\vec{F}_C = -0.318\,F_C\,\hat{i} + 0.424\,F_C\,\hat{j} + 0.848\,F_C\,\hat{k}$$

$$\vec{F}_D = F_D\,\hat{i}$$

$$\vec{W} = (-40\,\hat{k})\,\text{lb}$$

$$\sum \vec{F} = 0$$

$$\vec{F}_B + \vec{F}_C + \vec{F}_D + \vec{W} = 0$$

$$-0.318\,F_B\,\hat{i} - 0.424\,F_B\,\hat{j} + 0.848\,F_B\,\hat{k}$$
$$-0.318\,F_C\,\hat{i} + 0.424\,F_C\,\hat{j} + 0.848\,F_C\,\hat{k} + F_D\,\hat{i} - 40\,\hat{k} = 0$$

$$\sum F_x = 0 \qquad -0.318\,F_B - 0.318\,F_C + F_D = 0$$

$$\sum F_y = 0 \qquad -0.424\,F_B + 0.424\,F_C = 0$$

$$\sum F_z = 0 \qquad 0.848\,F_B + 0.848\,F_C - 40 = 0$$

$$F_B = F_C = 23.6\ \text{lb}$$

$$F_D = 15.0\ \text{lb}$$

Engineering Mechanics

Statics & Strength of Materials

Force System Resultant

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Equilibrium requires the body to have No Translation and No rotation.

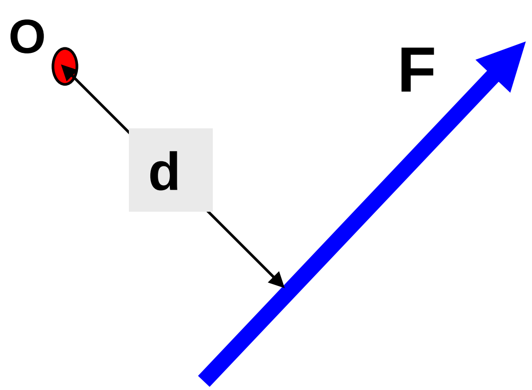
Moment of a Force - Scalar Formulation

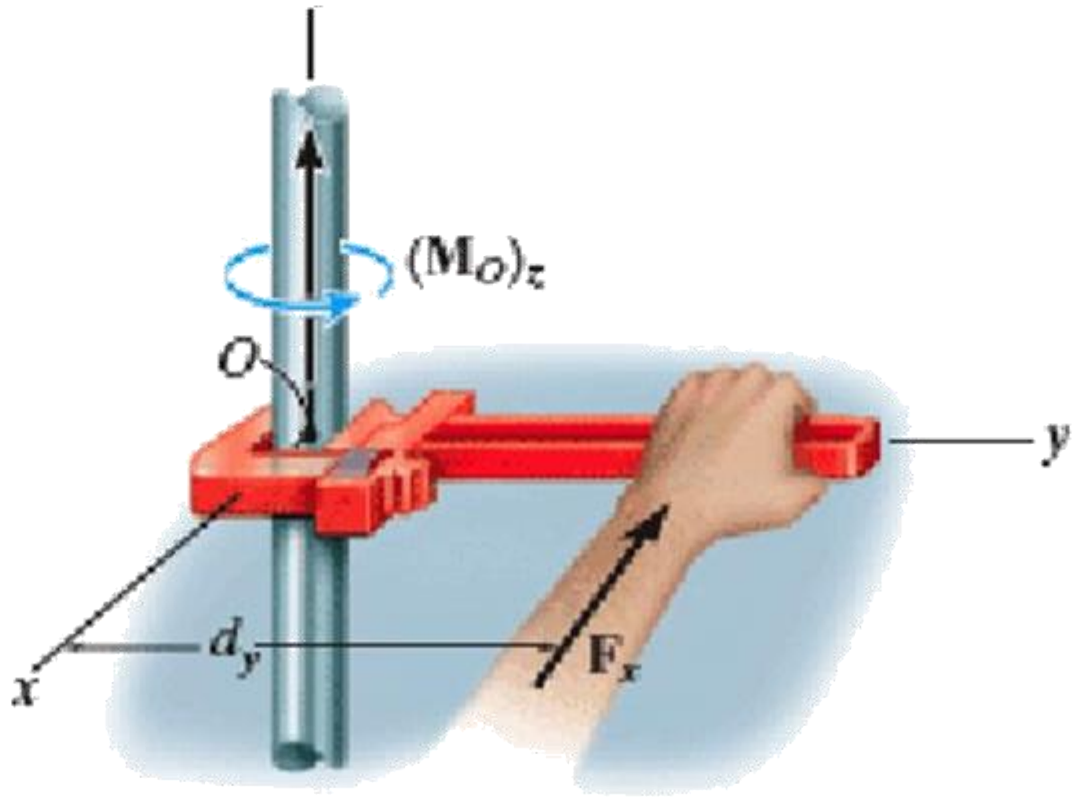
The moment of the force about a point O is the tendency of the force to rotate the object about point O.

Scalar Formulation

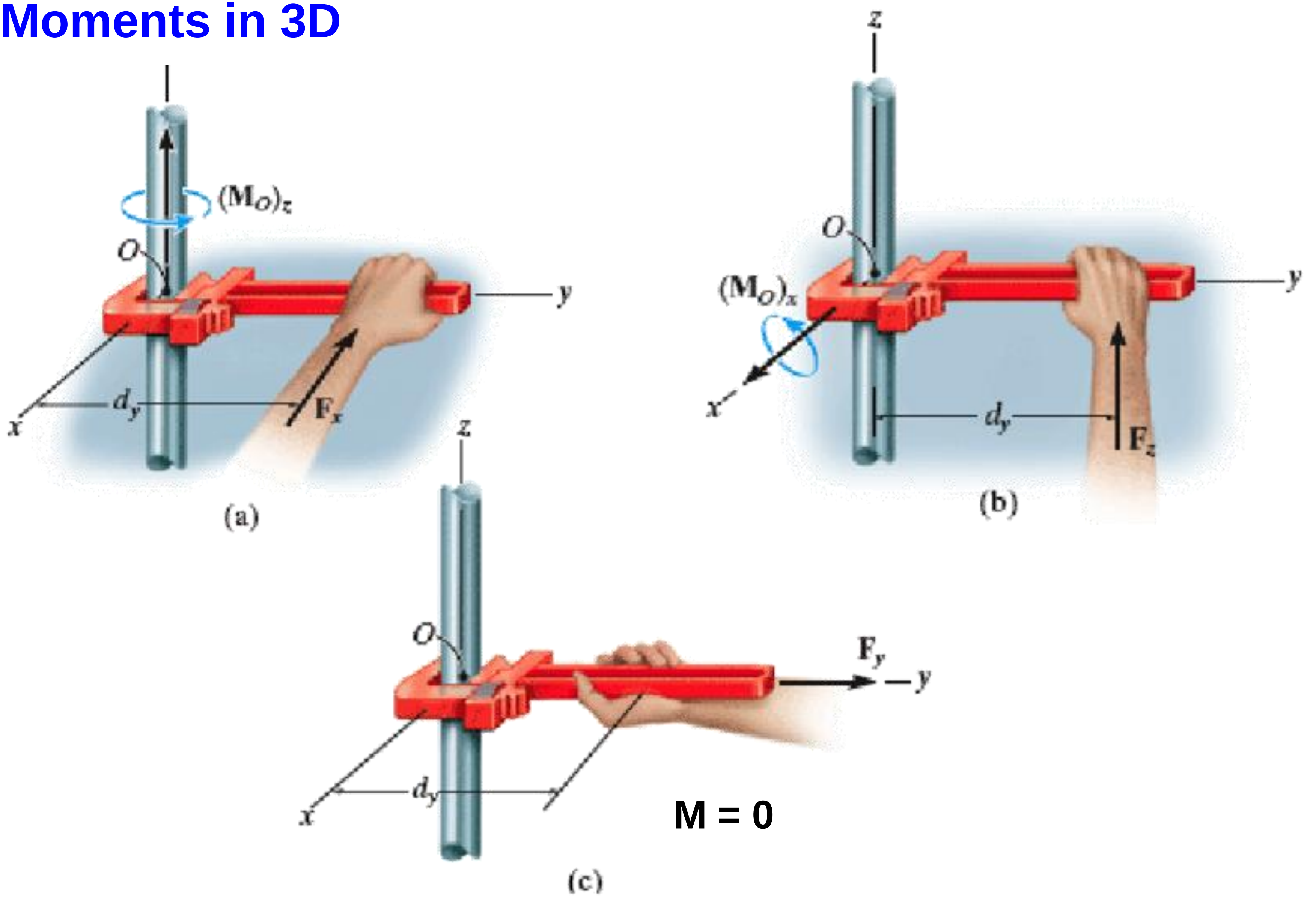
Moment = Torque = Twist

Mo = (F)(d) (force x distance)
d = "perpendicular distance"



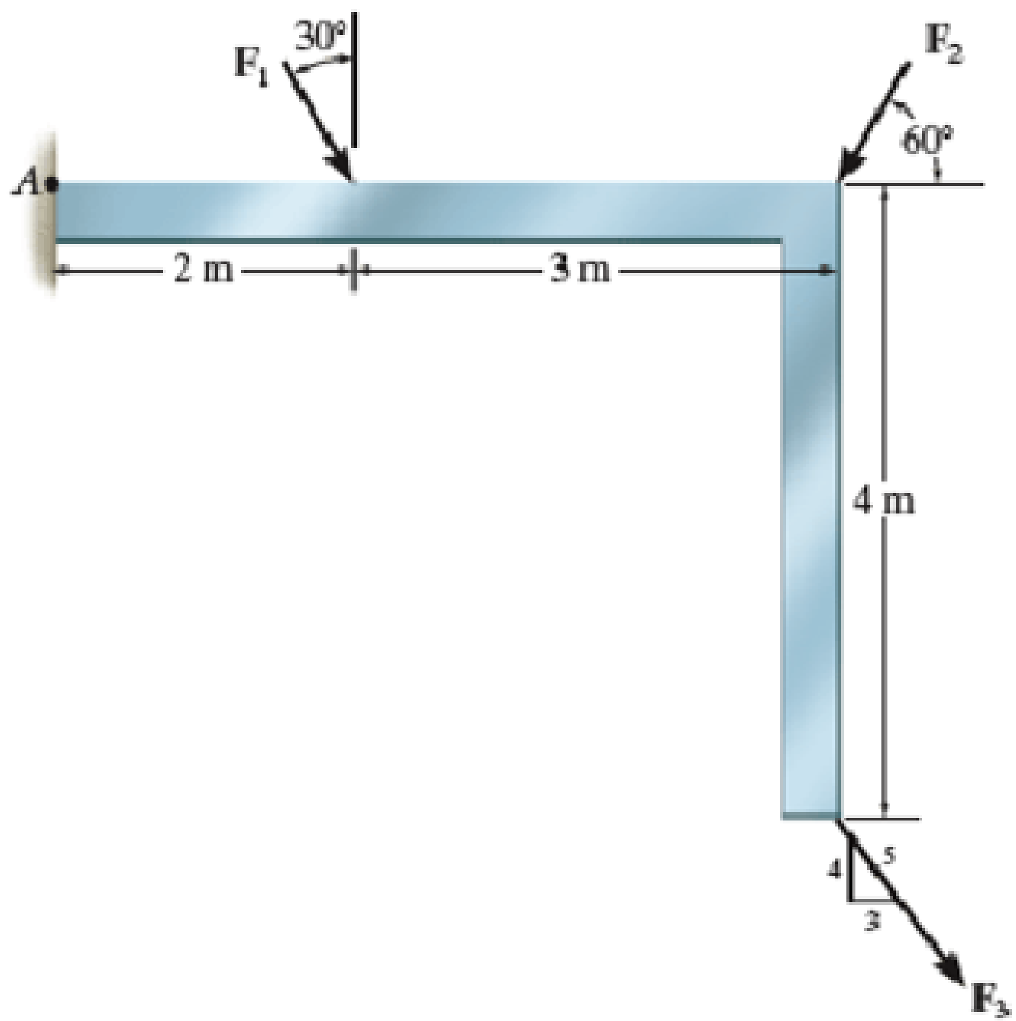


Moments in 3D

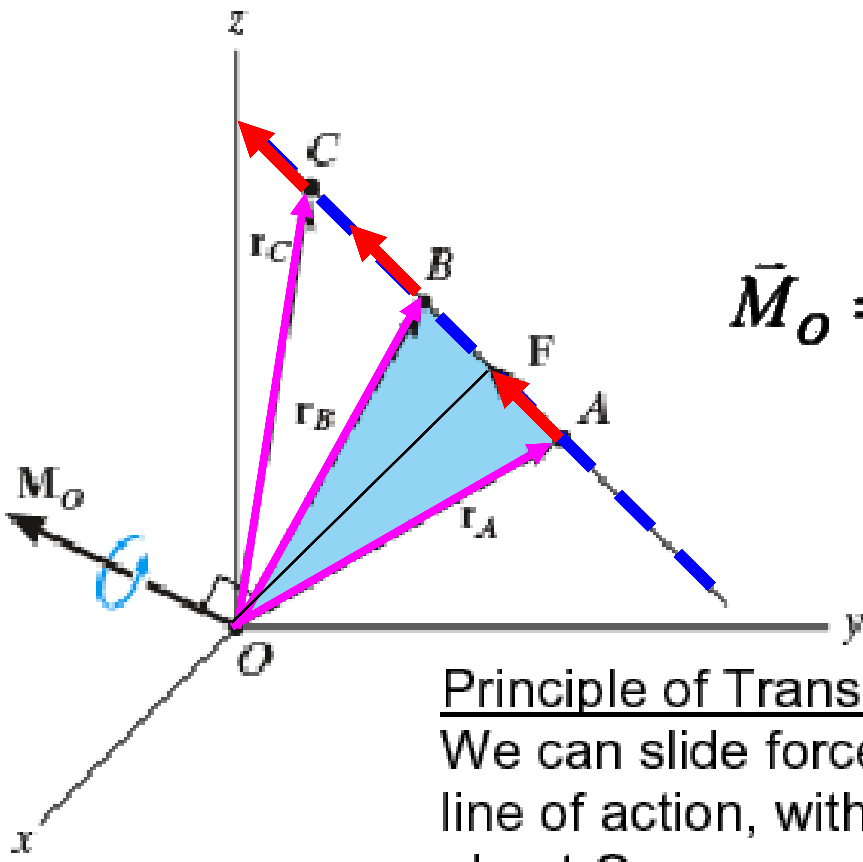


M = 0

If the resultant moment about point A is 4800 N · m clockwise, determine the magnitude of F3 if F1 = 300 N and F2 = 400 N.



Transmissibility of a force



$$\vec{M}_O = \vec{r}_A \times \vec{F} = \vec{r}_B \times \vec{F} = \vec{r}_C \times \vec{F}$$

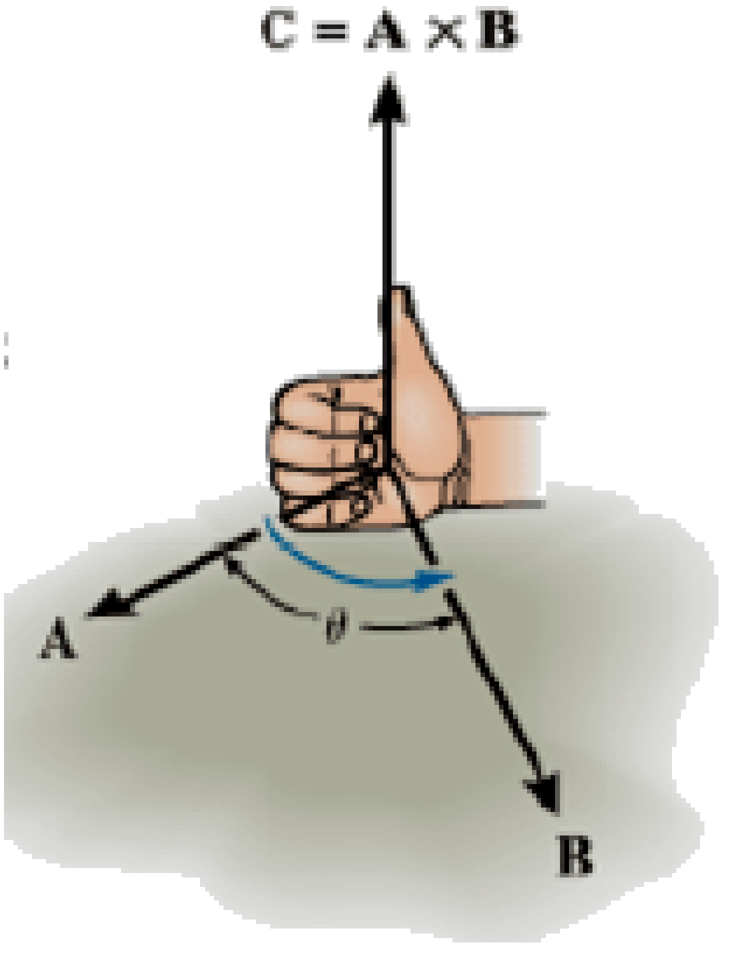
<u>Principle of Transmissibility</u>

We can slide force **F** anywhere we want along its line of action, without changing the moment of **F** about O.

Sliding the force along its line of action doesn't change the magnitude F or the perpendicular distance d

Cross product

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

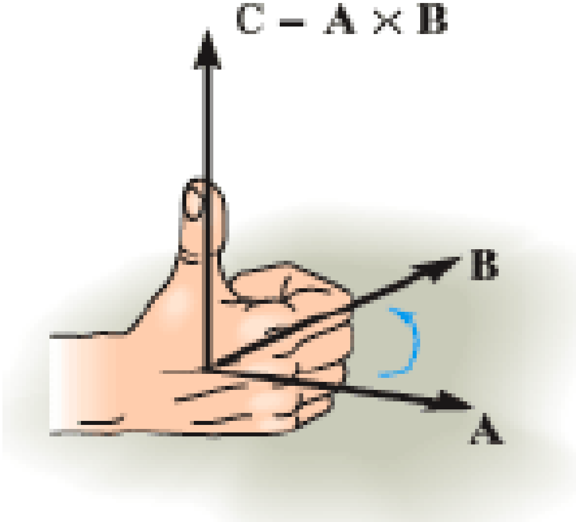


Scalar formulation

C magnitude = A B sin θ

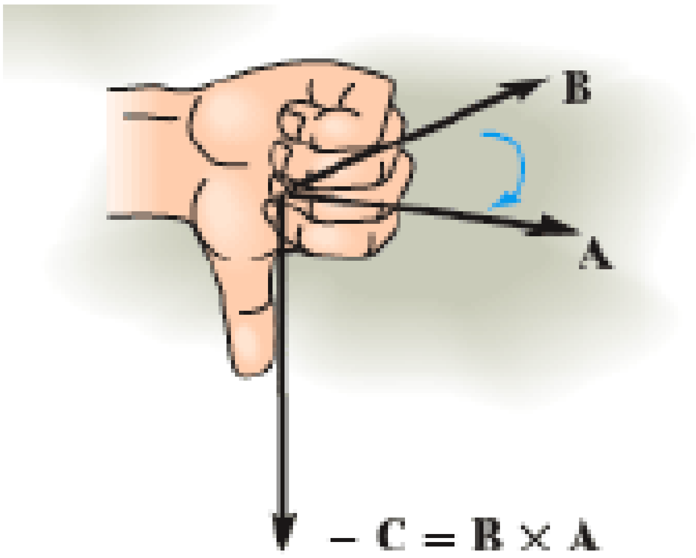
If **C** = **A** X **B,** then **C** will be perpendicular to **A** and **B**.

Commutative law - be very careful here

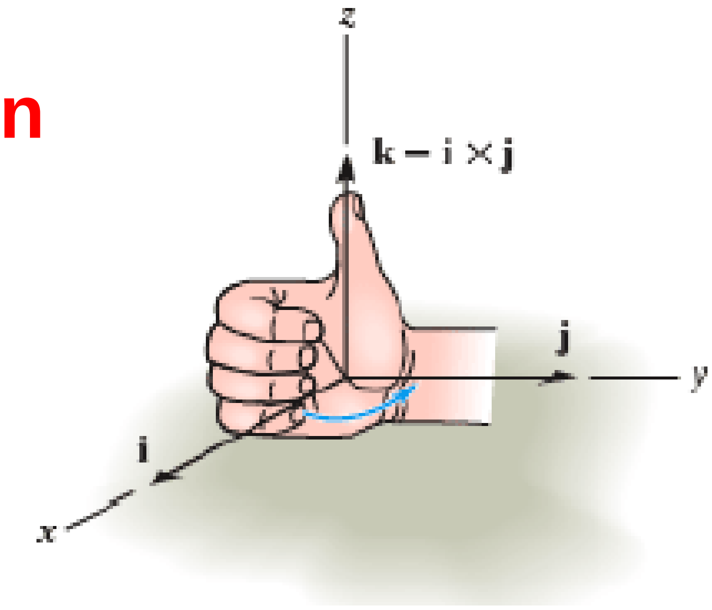


$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$



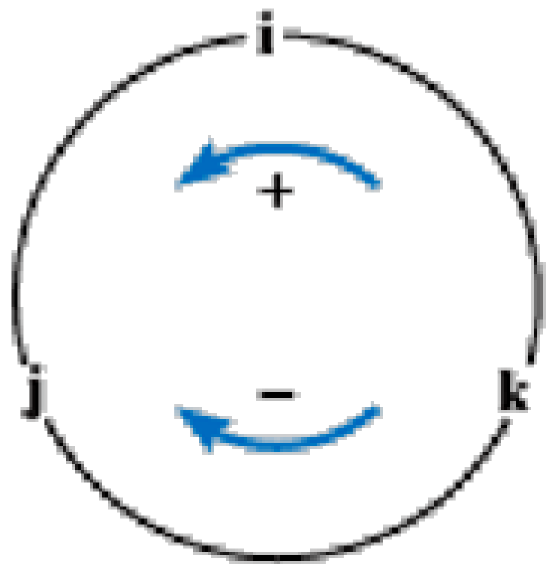
Cartesian Vector Formulation



$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \qquad \mathbf{i} \times \mathbf{i} = \mathbf{0}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{j} \times \mathbf{j} = \mathbf{0}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{k} \times \mathbf{k} = \mathbf{0}$$



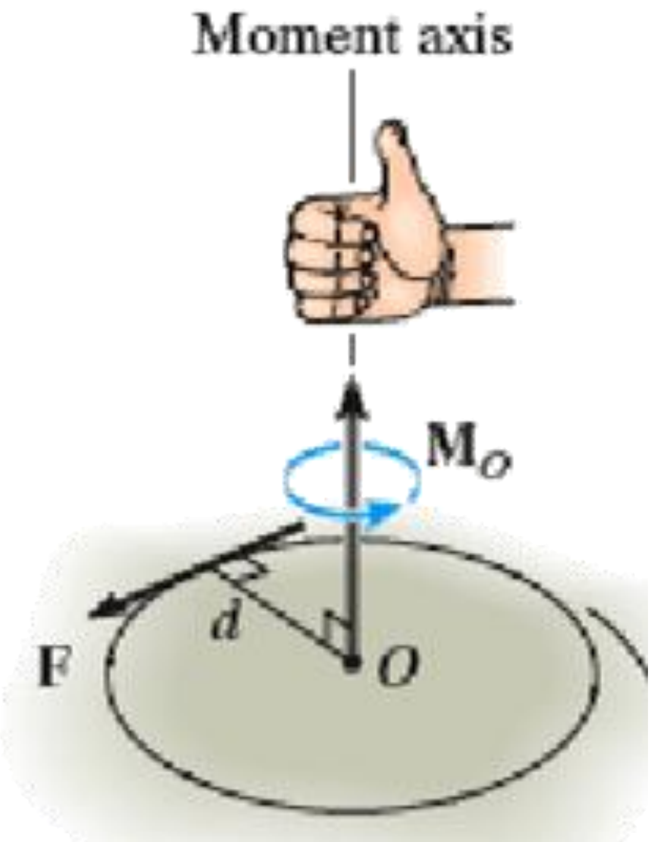
The cross product of two vectors is

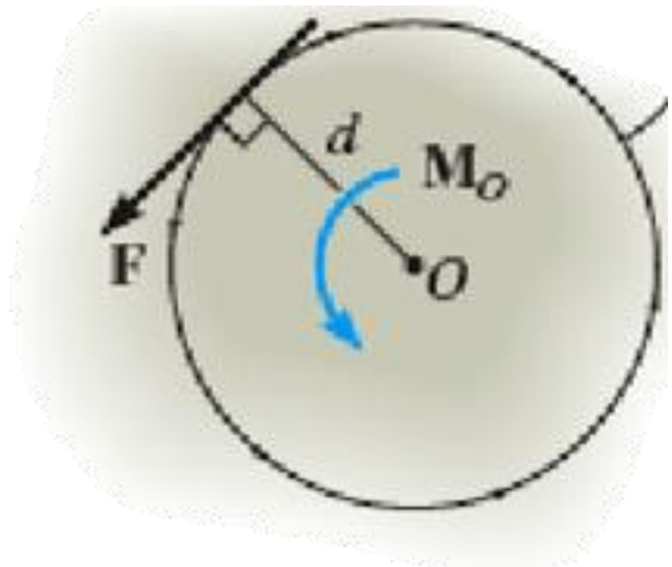
$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \vec{i} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} j + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \vec{k}$$

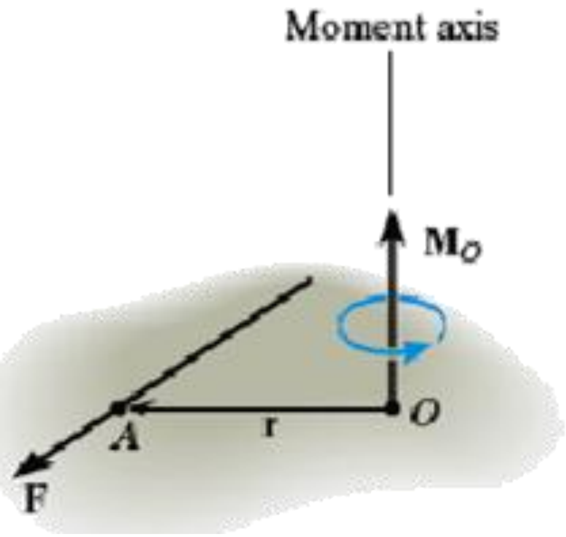
$$= \left(A_y B_z - A_z B_y \right) \vec{i} - \left(A_x B_z - A_z B_x \right) \vec{j} + \left(A_x B_y - A_y B_x \right) \vec{k}$$

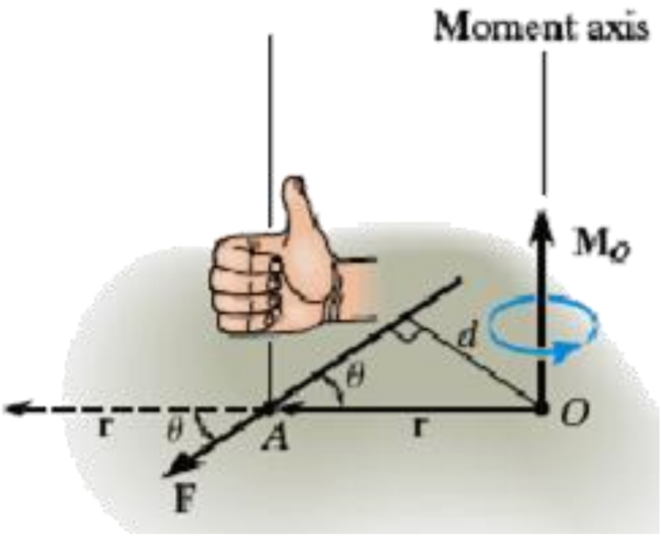
Direction of Moments



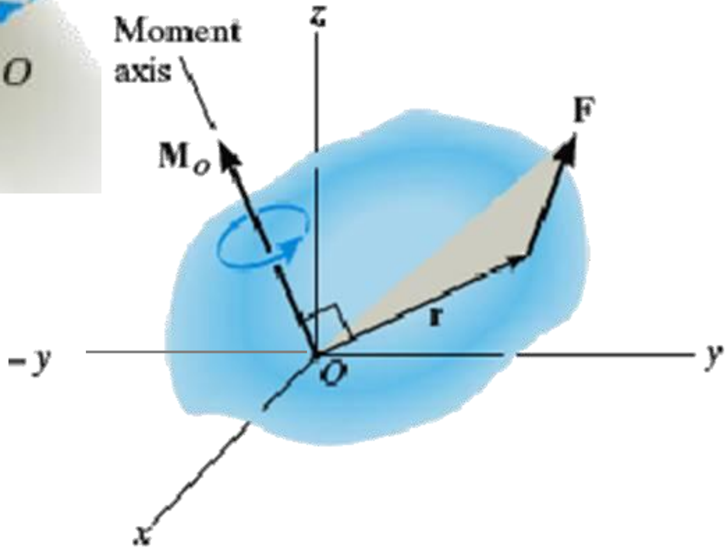


Moment of a Force - Vector Formulation





$\vec{M}_O = \vec{r} \times \vec{F}$ = r X F



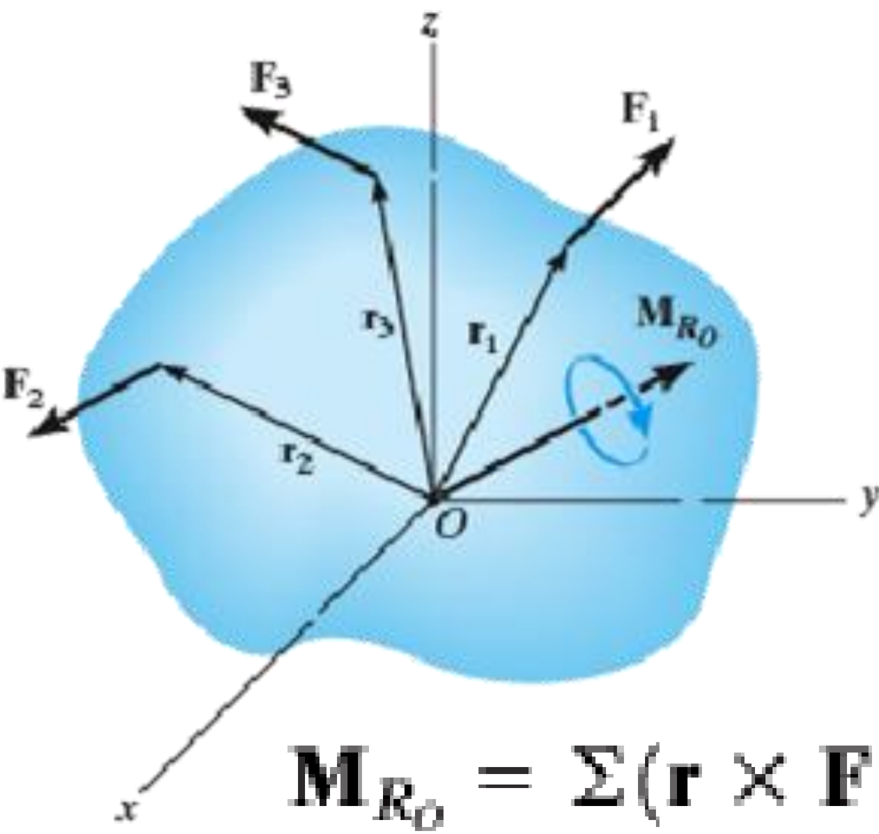
$$M_O = |r \times F| = rF\sin(\theta) = Fd = rF_{\perp}$$

$$\underline{\mathbf{F}} = F_x\underline{\mathbf{i}} + F_y\underline{\mathbf{j}} + F_z\underline{\mathbf{k}}$$

$$\underline{\mathbf{r}} = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}}$$

$$\underline{\mathbf{M}}_O = \underline{\mathbf{r}} \times \underline{\mathbf{F}} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = (yF_z - zF_y)\underline{\mathbf{i}} - (xF_z - zF_x)\underline{\mathbf{j}} + (xF_y - yF_x)\underline{\mathbf{k}}$$

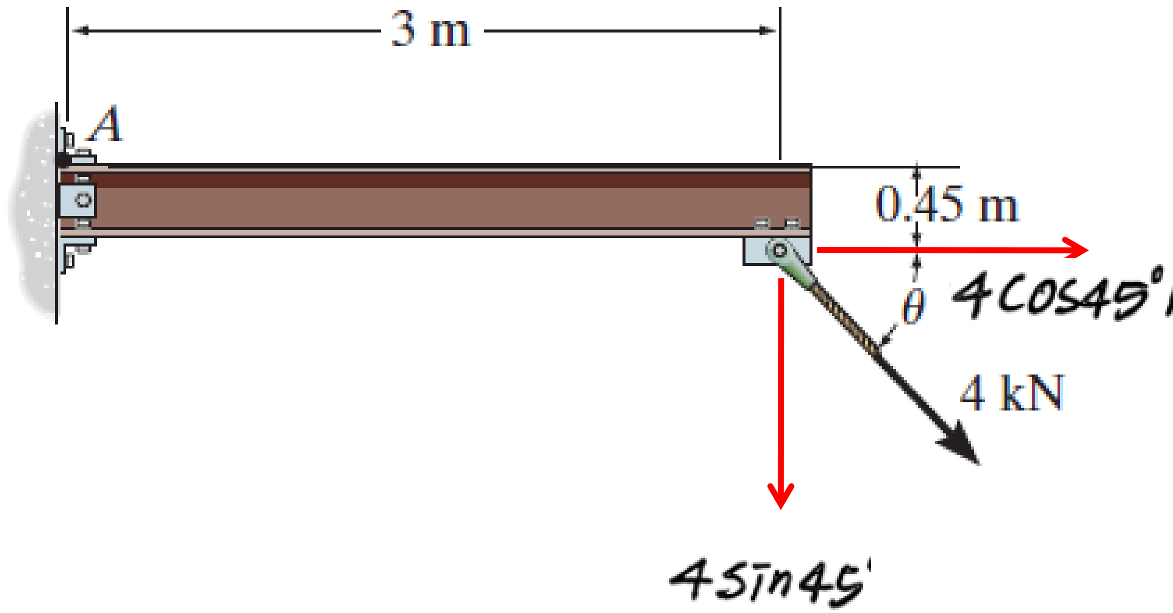
Resultant moment of a system of forces



$$\mathbf{M}_{R_O} = \Sigma(\mathbf{r} \times \mathbf{F})$$

$$\mathbf{M}_{R_O} = \mathbf{r}_1 \times \mathbf{F}_1 + \ldots + \mathbf{r}_n \times \mathbf{F}_n = \sum \mathbf{r} \times \mathbf{F}$$

If ,θ =45 determine the moment produced by the 4-kN force about point *A*.

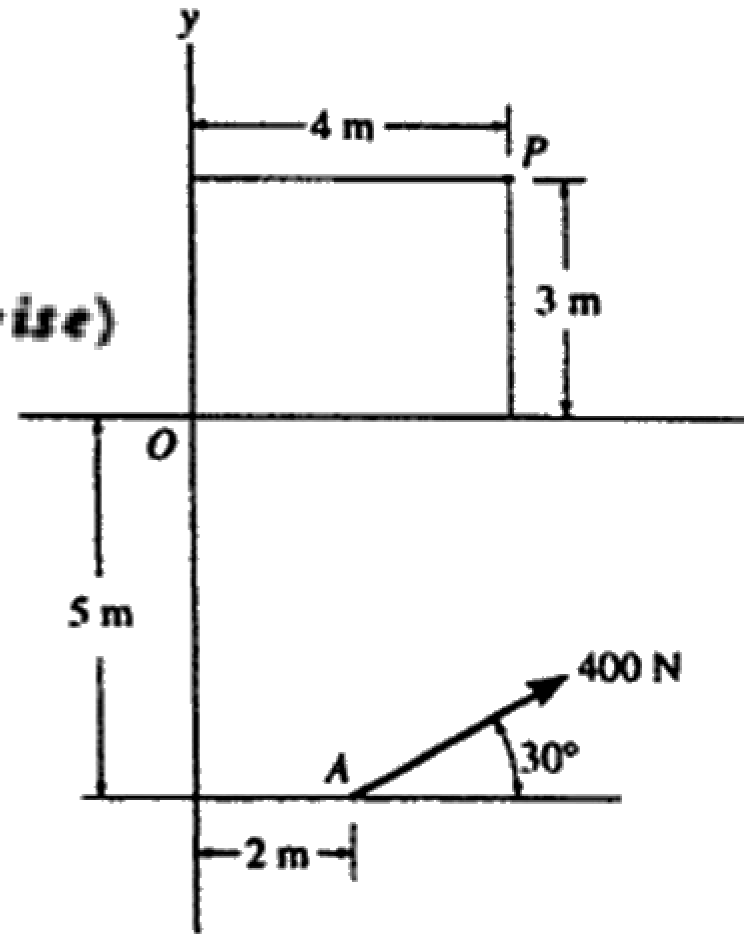


$$\circlearrowleft + M_A = 4\cos 45°(0.45) - 4\sin 45°(3)$$

$$= -7.21\ \text{kN}\cdot\text{m} = 7.21\ \text{kN}\cdot\text{m}\quad \text{(clockwise)}$$

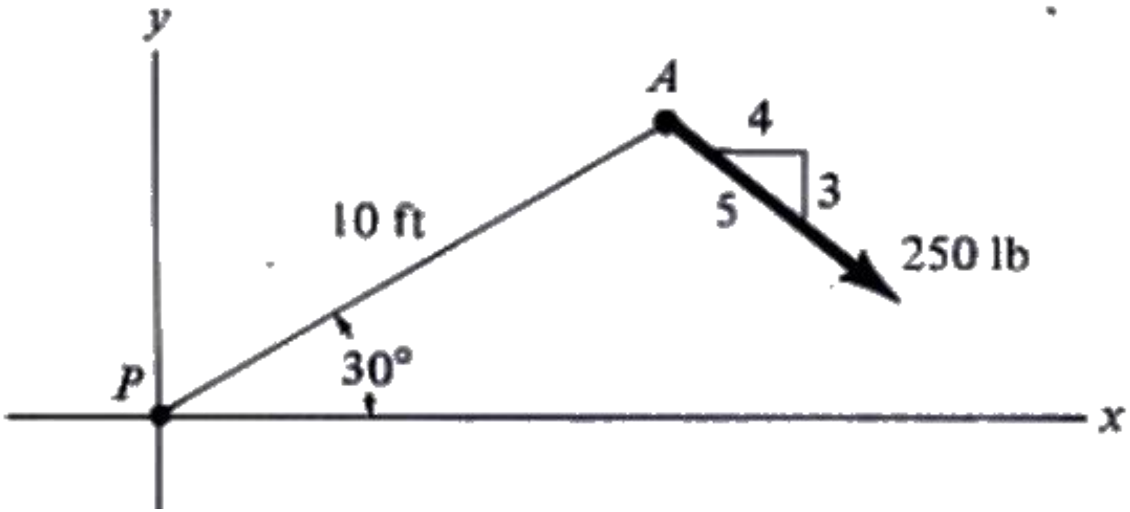
Determine the magnitude and directional sense of the moment of the force at A about point P. And O

$$\circlearrowleft + \quad M_O = 400\cos 30^\circ(5) + 400\sin 30^\circ(2)$$
$$= 2132\ \text{N}\cdot\text{m}$$
$$= 2.13\ \text{kN}\cdot\text{m} \quad (\textit{Counterclockwise})$$



$$\circlearrowleft + \quad M_P = 400\cos 30^\circ(8) - 400\sin 30^\circ(2)$$
$$= 2371\ \text{N}\cdot\text{m}$$
$$= 2.37\ \text{kN}\cdot\text{m} \quad (\textit{Counterclockwise})$$

Determine the magnitude and direction of the moment of the force at *A* about point *P*.



$$M_P = -\left(\tfrac{4}{5}250\,lb\right)\left(10\,ft\sin 30°\right) - \left(\tfrac{3}{5}250\,lb\right)\left(10\,ft\cos 30°\right)$$

$$M_P = -2300\,ft\,lb$$

The force F={6i+8j+10k} N creates a moment about point O of M_o={-14i+8j+2k} N m. If the force passes through a point having an x coordinate of 1 m, determine the y and z coordinates of the point. Also, realizing that M_o=Fd, determine the perpendicular distance d from point O to the line of action of F.

Solution:

$$-14\mathbf{i} + 8\mathbf{j} + 2\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & y & z \\ 6 & 8 & 10 \end{vmatrix}$$

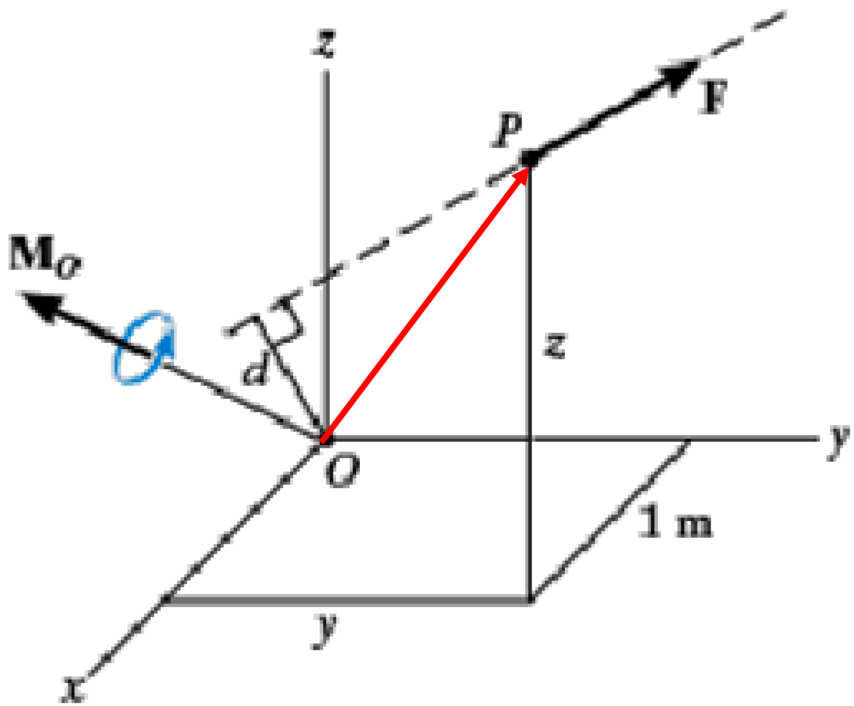
$$-14 = 10y - 8z$$

$$8 = -10 + 6z$$

$$2 = 8 - 6y$$

$y = 1$ m **Ans**

$z = 3$ m **Ans**



The pipe assembly is subjected to the 80-N force. Determine the moment of this force about point *B*.

Position Vector And Force Vector :

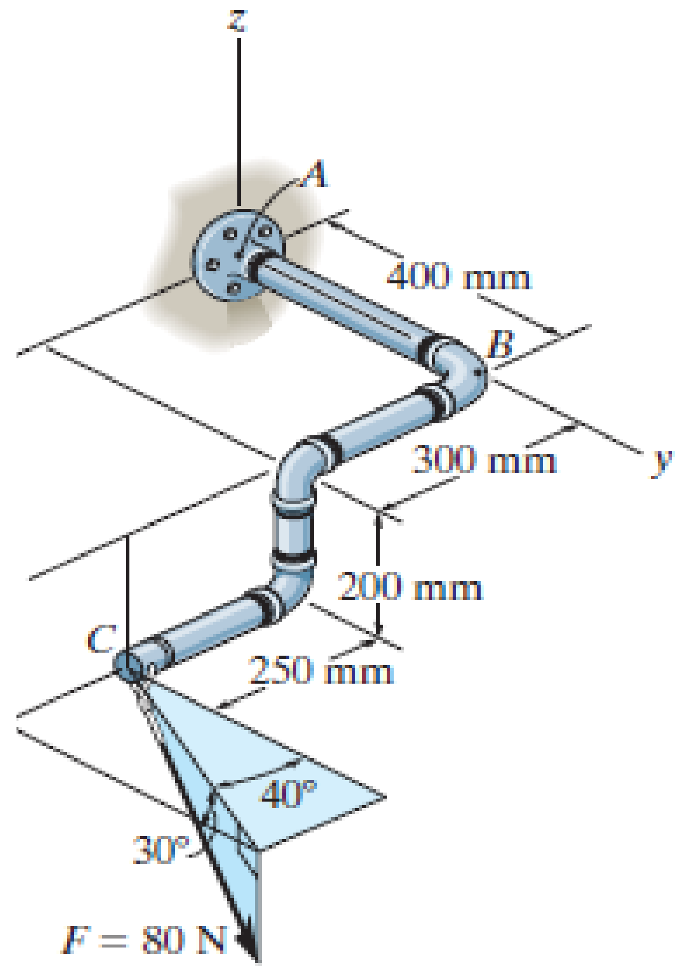
$$\mathbf{r}_{BC} = \{(0.55-0)\mathbf{i} + (0.4-0.4)\mathbf{j} + (-0.2-0)\mathbf{k}\}\ \text{m}$$
$$= \{0.55\mathbf{i} - 0.2\mathbf{k}\}\ \text{m}$$

$$\mathbf{F} = 80(\cos 30^\circ \sin 40^\circ \mathbf{i} + \cos 30^\circ \cos 40^\circ \mathbf{j} - \sin 30^\circ \mathbf{k})\ \text{N}$$
$$= \{44.53\mathbf{i} + 53.07\mathbf{j} - 40.0\mathbf{k}\}\ \text{N}$$

Moment of Force F About Point B : Applying Eq. 4−7, we have

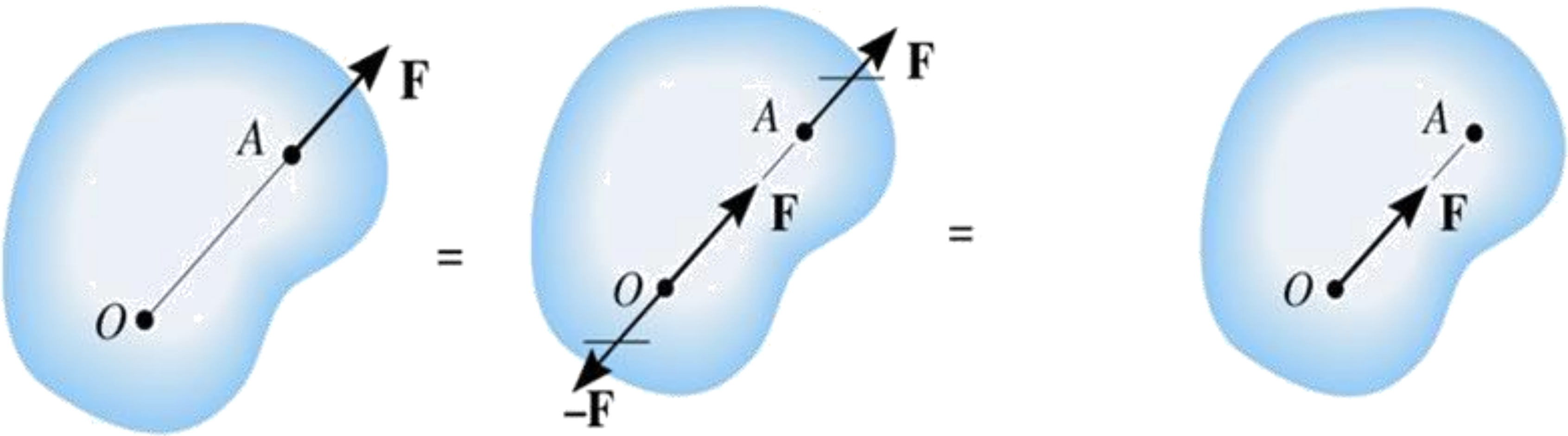
$$\mathbf{M}_B = \mathbf{r}_{BC} \times \mathbf{F}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.55 & 0 & -0.2 \\ 44.53 & 53.07 & -40.0 \end{vmatrix}$$

$$= \{10.6\mathbf{i} + 13.1\mathbf{j} + 29.2\mathbf{k}\}\ \text{N}\cdot\text{m}$$



Moment of a Force on a Rigid Body

Point O is on the line of action of the force



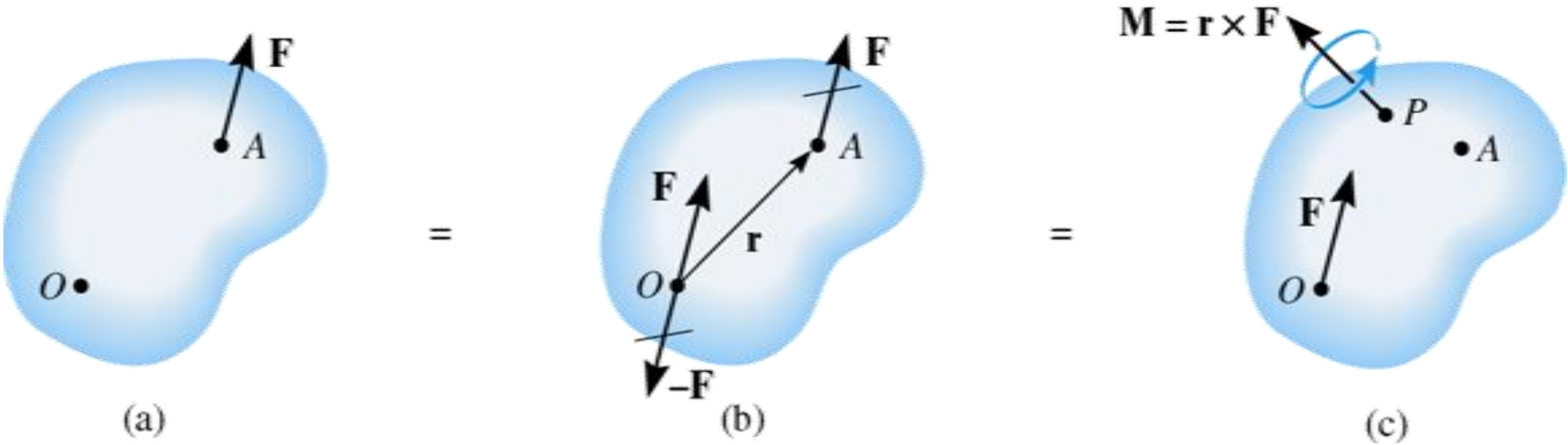
(b)

(c)

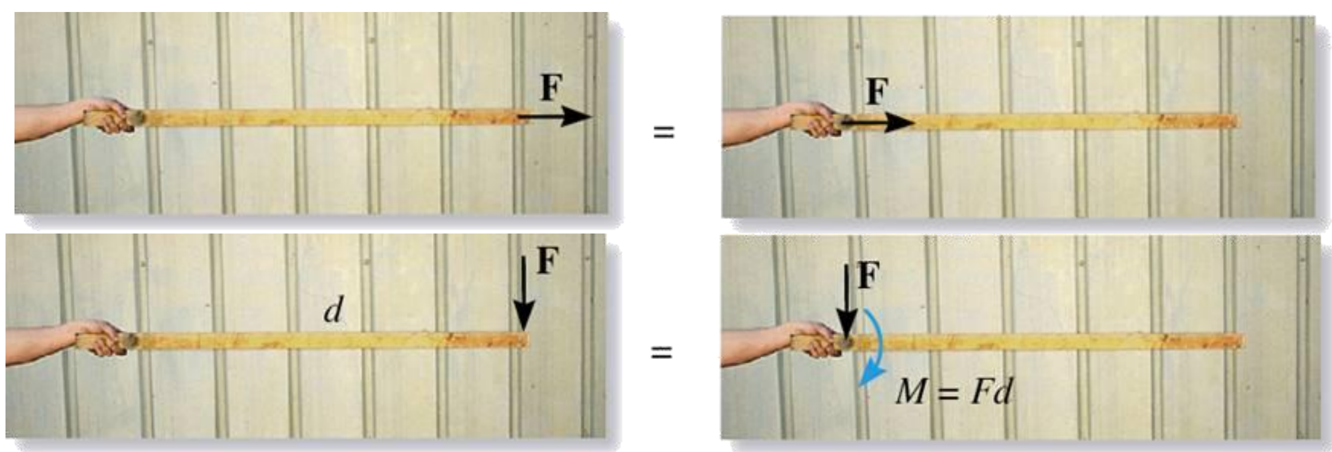
Principle of Transmissibility:

The external effect on a rigid body remains unchanged when a force acting at a given point on the body, is applied to another point lying on the line of action of the force.

Point O Not on Line of Action of Force



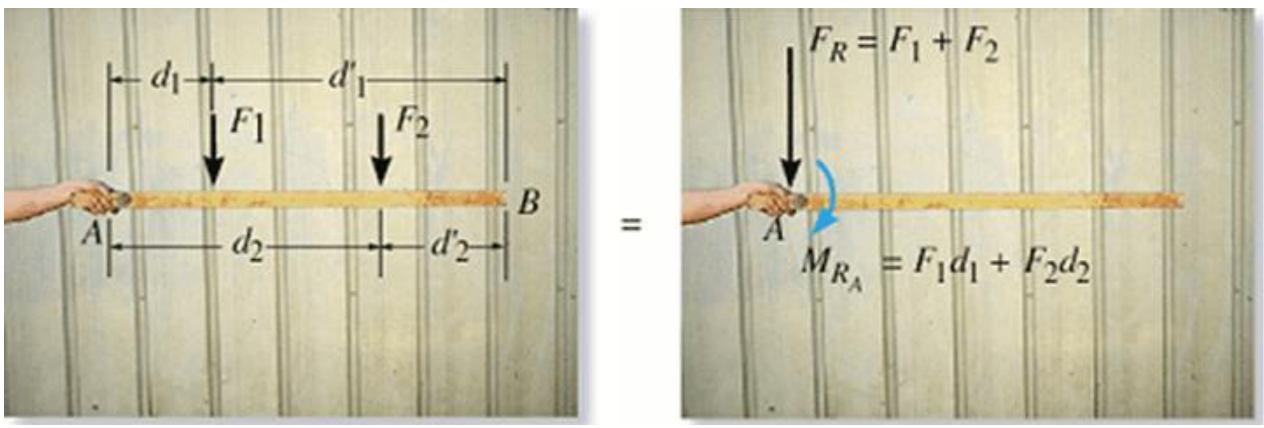
Equivalent to a force in the same direction and a moment of magnitude = r x F

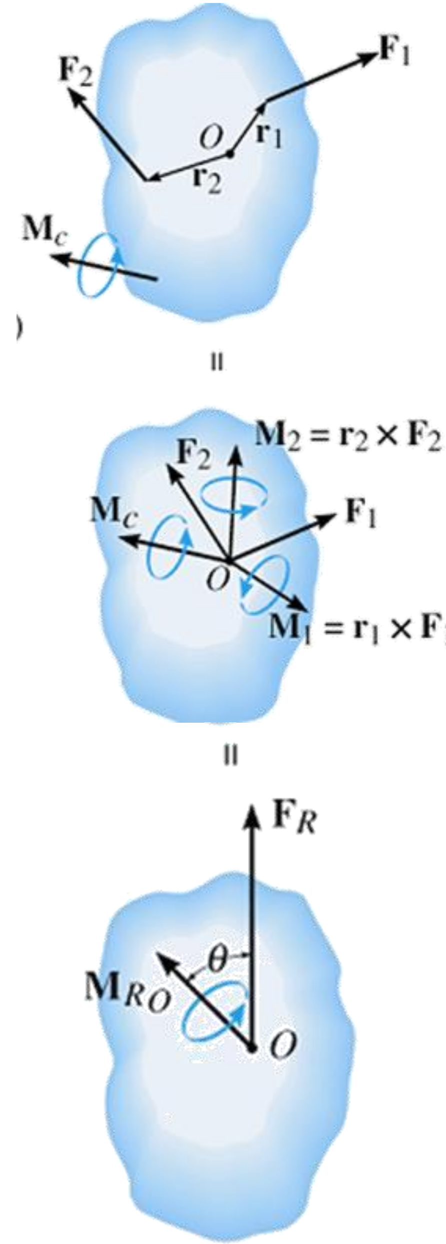


Resultant of a Force and Couple System

$$\vec{F}_R = \sum \vec{F}$$

$$\vec{M}_{R_O} = \sum \vec{M}_c + \sum \vec{M}_O$$





Resultant of a Force and Couple System – 2D

Scalar :

$$F_{R_x} = \sum F_x$$

$$F_{R_y} = \sum F_y$$

$$M_{R_O} = \sum M_c + \sum M_O$$

Eg. Replace the forces acting on the brace shown below with an equivalent resultant force and couple moment at point A.

$$F_{R_x} = \sum F_x$$

$$F_{R_x} = -100\text{ N} - 400\cos 45^{o} = -382.8\text{ N}$$

$$F_{R_x} = 382.8\text{ N} \longleftarrow$$

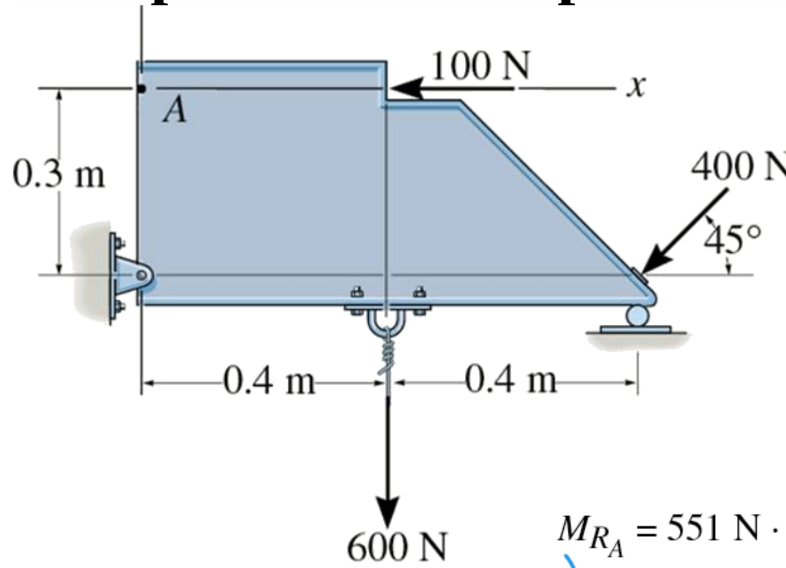
$$F_{R_y} = \sum F_y$$

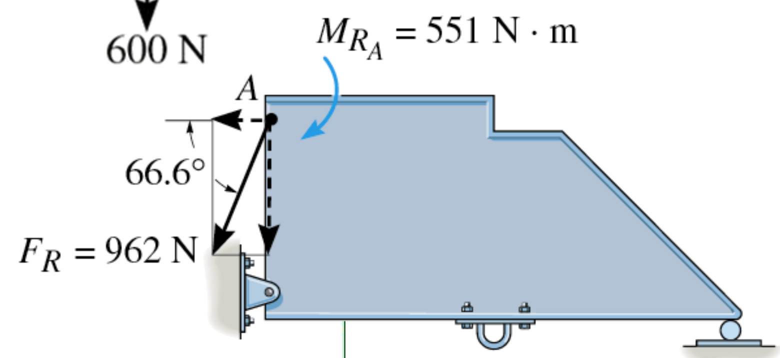
$$F_{R_y} = -600\text{ N} - 400\sin 45^{o} = -882.8\text{ N}$$

$$F_{R_y} = 882.8\text{ N}$$

$$F_R = \sqrt{(382.8)^2 + (882.8)^2} = 962\text{ N}$$

$$\theta = \tan^{-1}\left(\frac{F_{R_y}}{F_{R_y}}\right) = \tan^{-1}\left(\frac{-882.8}{-382.8}\right) = 66.6^{o}$$



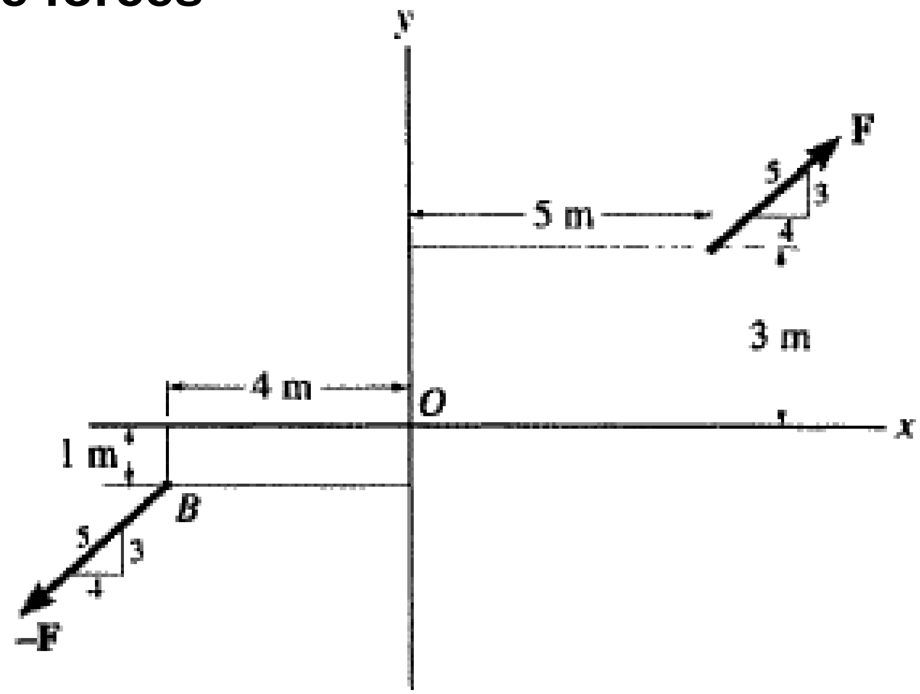


$$(+\ \text{ccw})\ M_{R_A} = \sum M_A\ (+\ \text{ccw})$$

$$M_{R_A} = (100\text{ N})(0) - (600\text{ N})(0.4\text{ m}) - (400\sin 45^{o}\text{ N})(0.8\text{ m})$$
$$-(400\sin 45^{o}\text{ N})(0.8\text{ m})$$

$$M_{R_A} = -551\text{ N}\cdot\text{m} = 551\text{ N}\cdot\text{m (cw)}$$

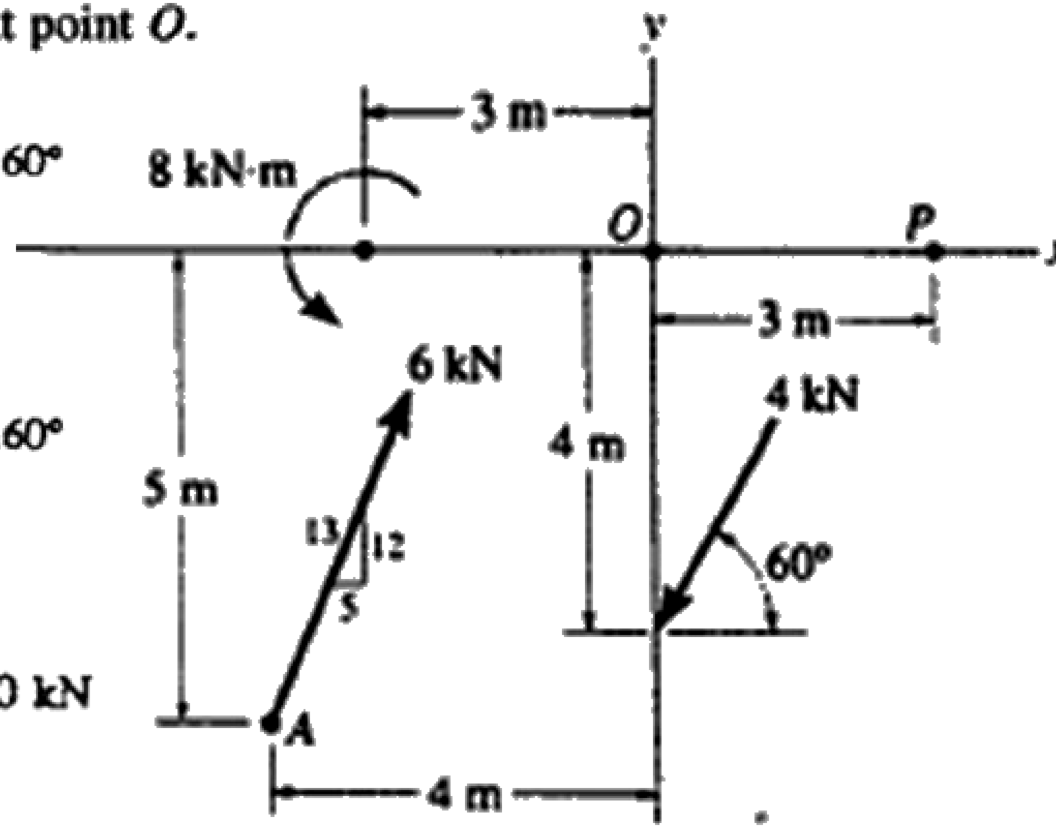
The couple moment has a magnitude of 220 N.m determine the magnitude of F of the couple forces



$$+\ 220 = -F\left(\frac{4}{5}\right)(3+1) + F\left(\frac{3}{5}\right)(4+5)$$

$$F = 100\ \mathrm{N \cdot m} \qquad (\textit{Counterclockwise})$$

Replace the force and couple system by an equivalent force and couple moment at point O.



$\xrightarrow{+} \Sigma F_{Rx} = \Sigma F_x : \quad F_{Rx} = 6(\frac{5}{13}) - 4\cos 60°$

$= 0.30769 \text{ kN}$

$+\uparrow \Sigma F_{Ry} = \Sigma F_y : \quad F_{Ry} = 6(\frac{12}{13}) - 4\sin 60°$

$= 2.0744 \text{ kN}$

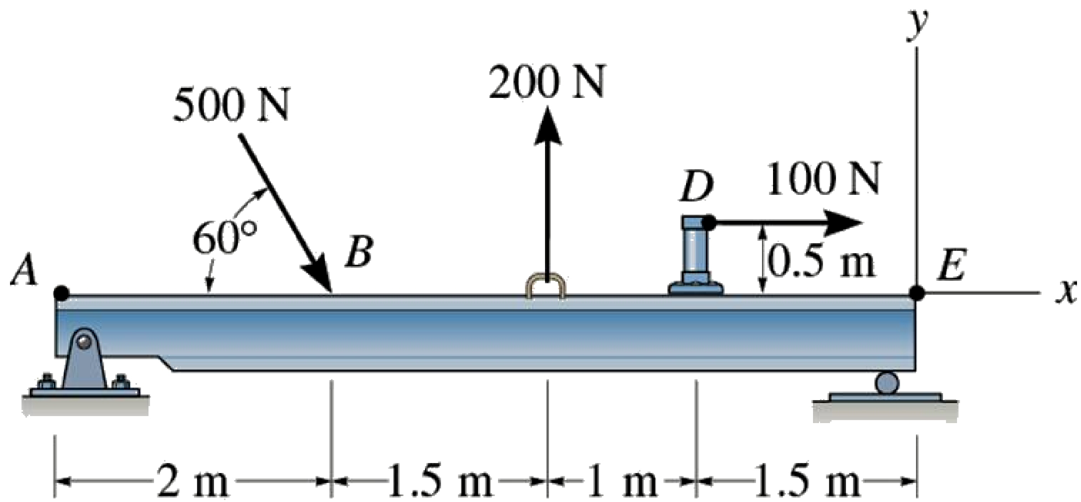
$F_R = \sqrt{(0.30769)^2 + (2.0744)^2} = 2.10 \text{ kN}$

$\theta = \tan^{-1}\left[\frac{2.0744}{0.30769}\right] = 81.6° \; \measuredangle$

$\circlearrowleft + M_O = \Sigma M_O : \quad M_O = 8 - 6(\frac{12}{13})(4) + 6(\frac{5}{13})(5) - 4\cos 60°(4)$

$M_O = -10.62 \text{ kN}\cdot\text{m} = 10.6 \text{ kN}\cdot\text{m} \; \circlearrowright$

Equivalent system at E ?



$$F_{Rx} = \sum F_x = 500 \cos 60^o \text{ N} + 100 \text{ N} = 350 \text{ N}$$

$$F_{Ry} = \sum F_y = -500 \sin 60^o \text{ N} + 200 \text{ N} = -233 \text{ N}$$

$$F_R = \sqrt{(350)^2 + (-233)^2} = 420.5 \text{ N}$$

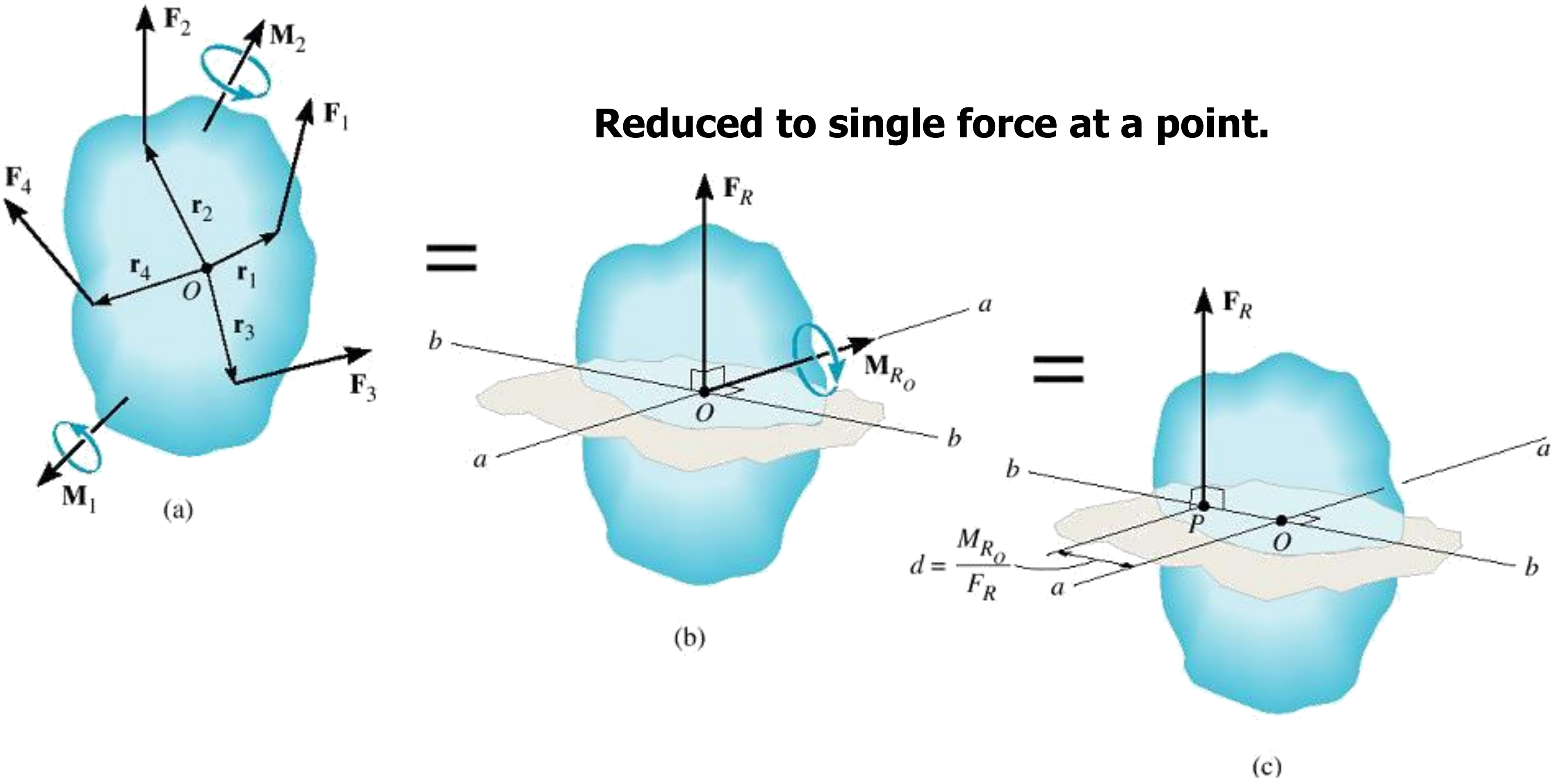
$$\theta = \tan^{-1}\left(\frac{233}{350}\right) = 33.7^o$$

$$M_{R_E} = \sum M_E \quad (+ \circlearrowleft)$$

$$= \left(500 \ \sin 60^o\right)(4) + \left(500 \ \cos 60^o\right)(0) - (100)(0.5) - (200)(2.5)$$

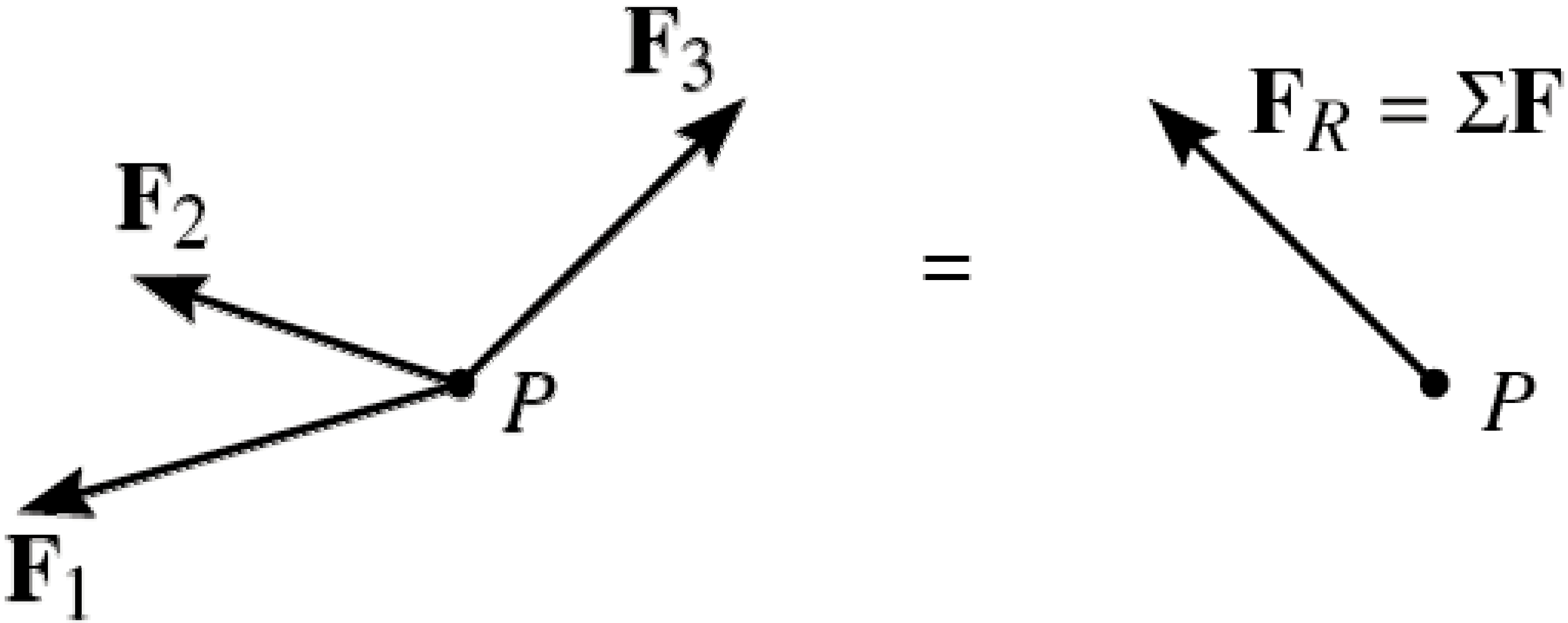
$$= 1182.1 \ \text{N} \cdot \text{m}$$

Further Reduction of a Force System

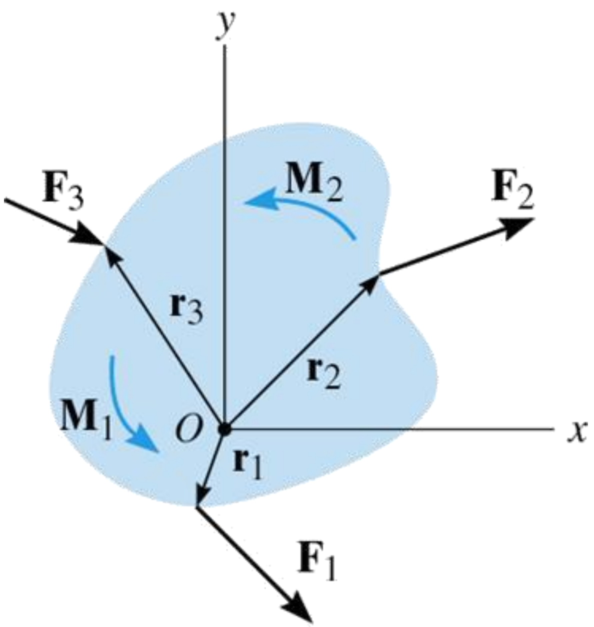


If a system of forces is either concurrent, coplanar or parallel, it can always be reduced to a single resultant force acting through a unique point .

Concurrent Force Systems



Coplanar Systems



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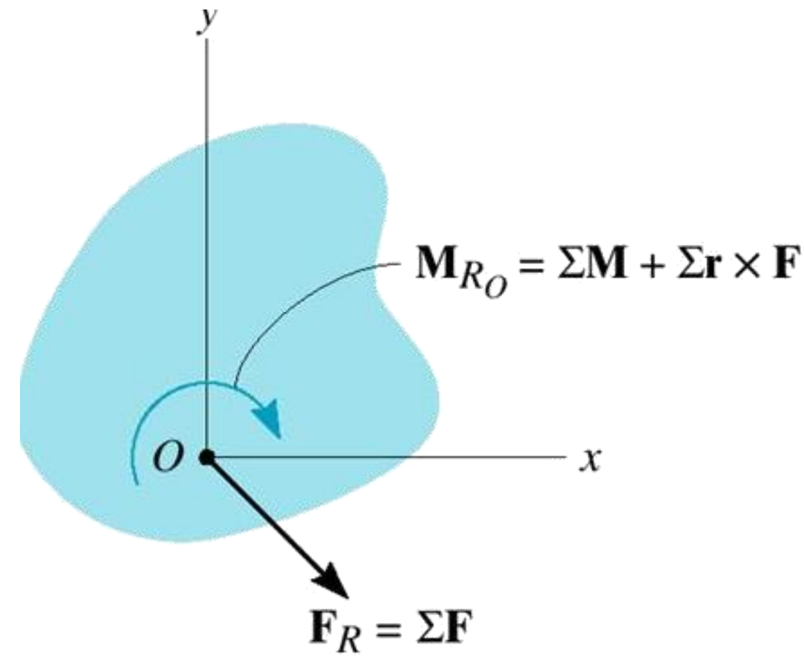

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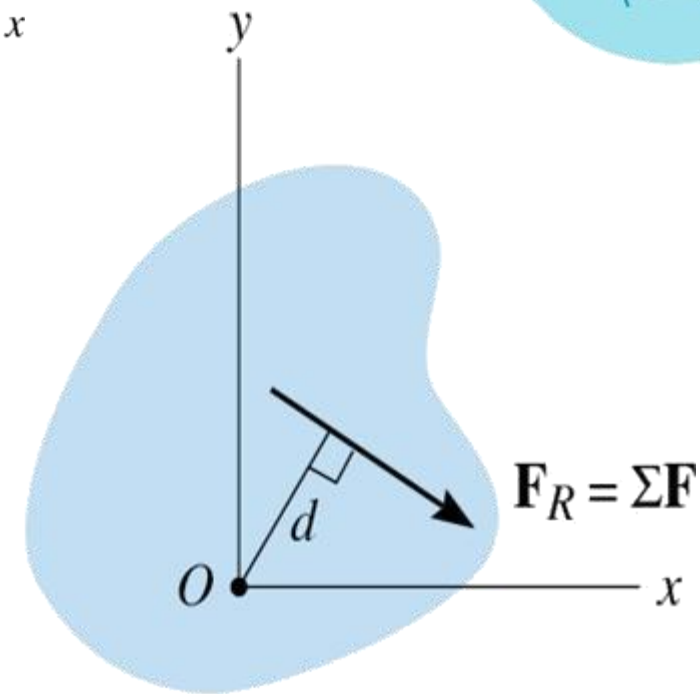
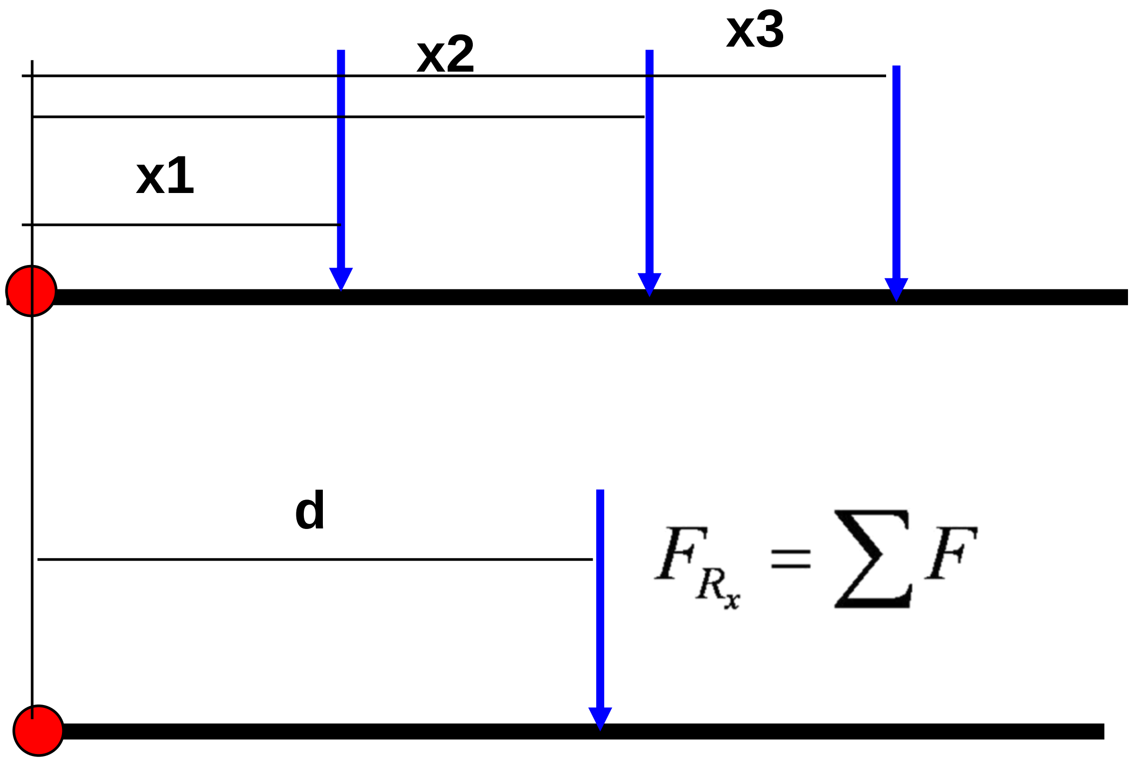
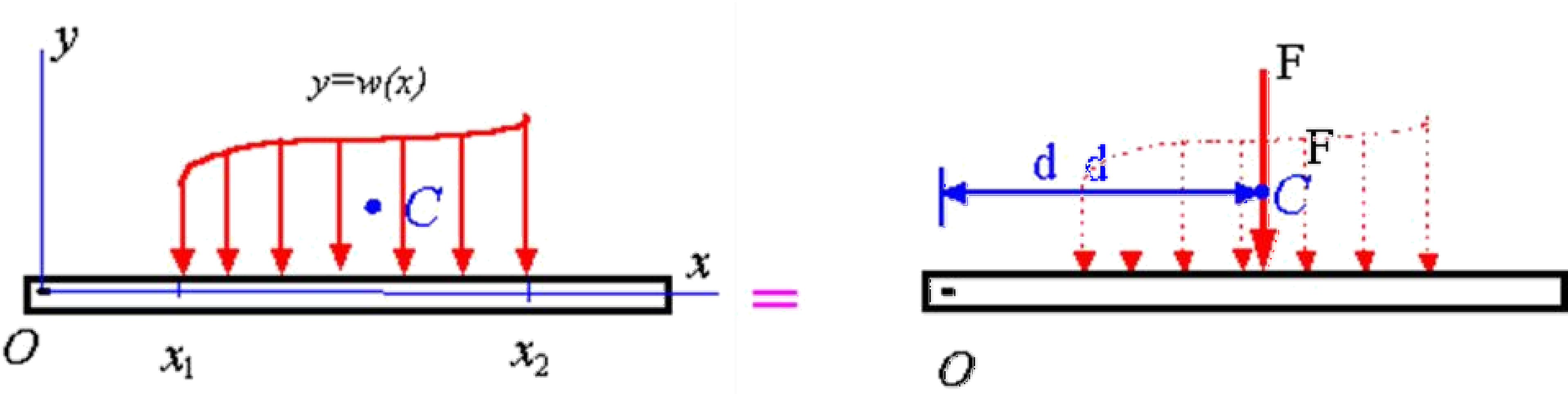




$$F_{R_x} = \sum F$$

$$d = \frac{\sum X_i \; F}{\sum F}$$

Replacing a distributed load by single resultant load:



$$(+\downarrow)\quad \sum F = F = \int_{x_1}^{x_2} dF = \int_{x_1}^{x_2} w(x)dx = area\ under\ w(x)\ curve$$

$$(+CW)\quad \sum M_O = Fd = \int_{x_1}^{x_2} xdF = \int_{x_1}^{x_2} xw(x)dx \Rightarrow d = \frac{1}{F}\int_{x_1}^{x_2} xw(x)dx$$

***F* must pass through the centroid of the area under the curve *w(x)*.**

$$F_R = \int_L w(x)\,dx = \int_A dA = A$$

Magnitude of resultant force is equal to the total area under the loading diagram

w = w(x)

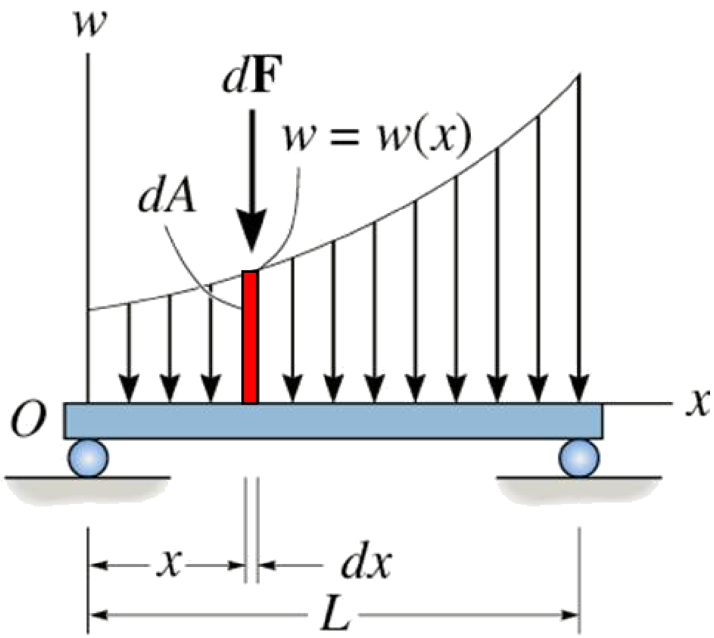
dF produces a moment

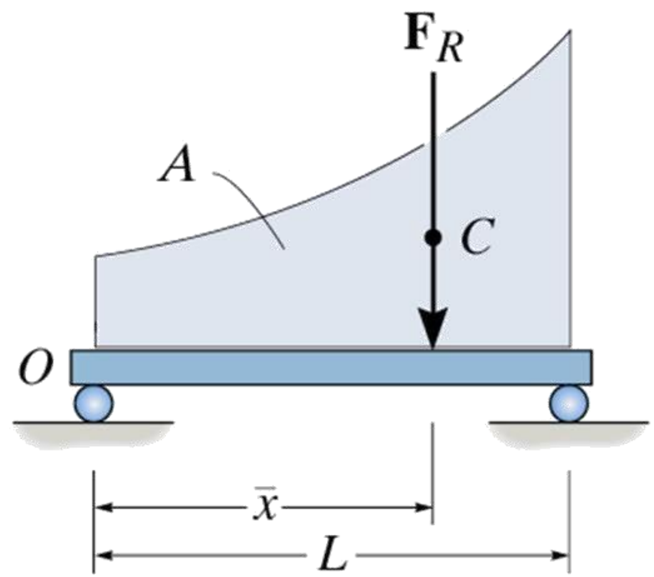
$$dM = x\,dF = x\,w(x)\,dx$$

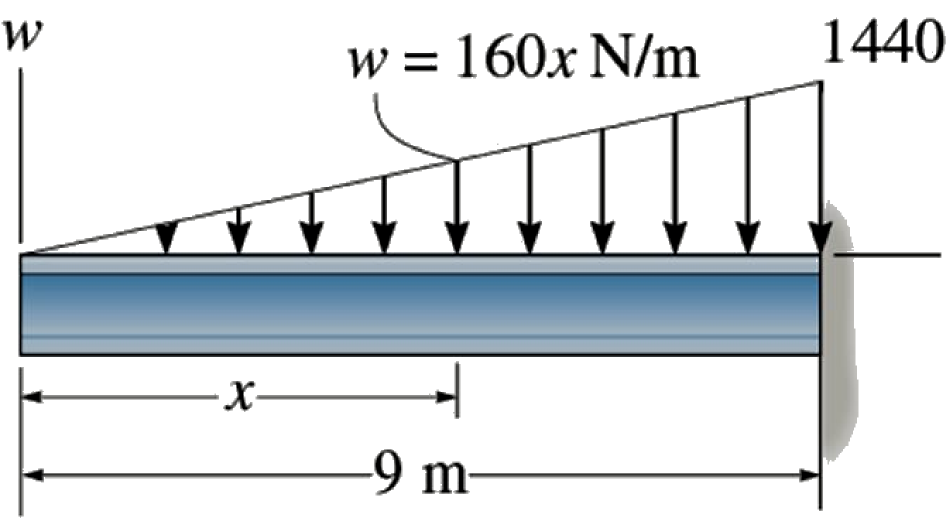
$$\bar{x}\,F_R = \int_L x\,dF = \int_L x\,w(x)\,dx = \int_A x\,dA$$

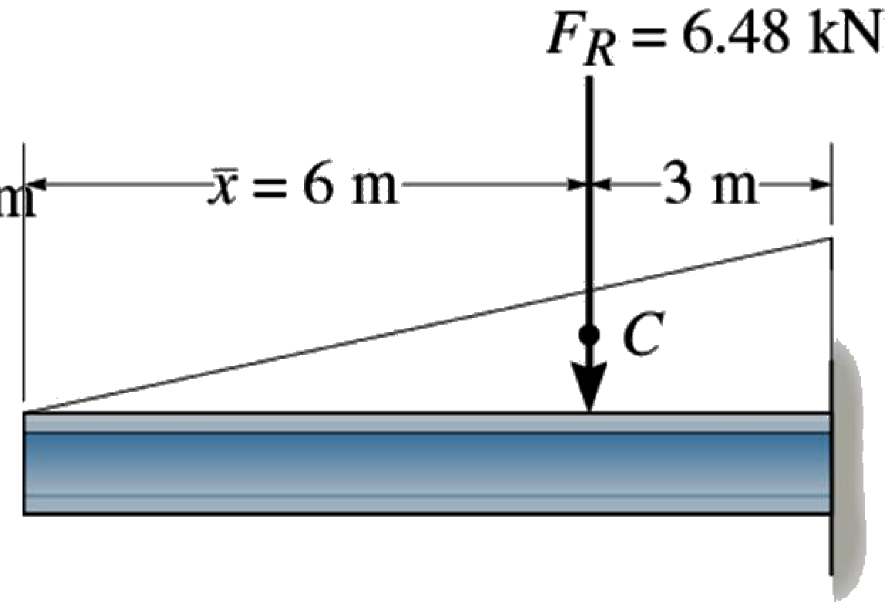
$$\bar{x} = \frac{\int_L x\,w(x)\,dx}{F_R} = \frac{\int_L x\,w(x)\,dx}{\int_L w(x)\,dx} = \frac{\int_A x\,dA}{\int_A dA}$$

$$d = \frac{\sum X_i\ F}{\sum F}$$



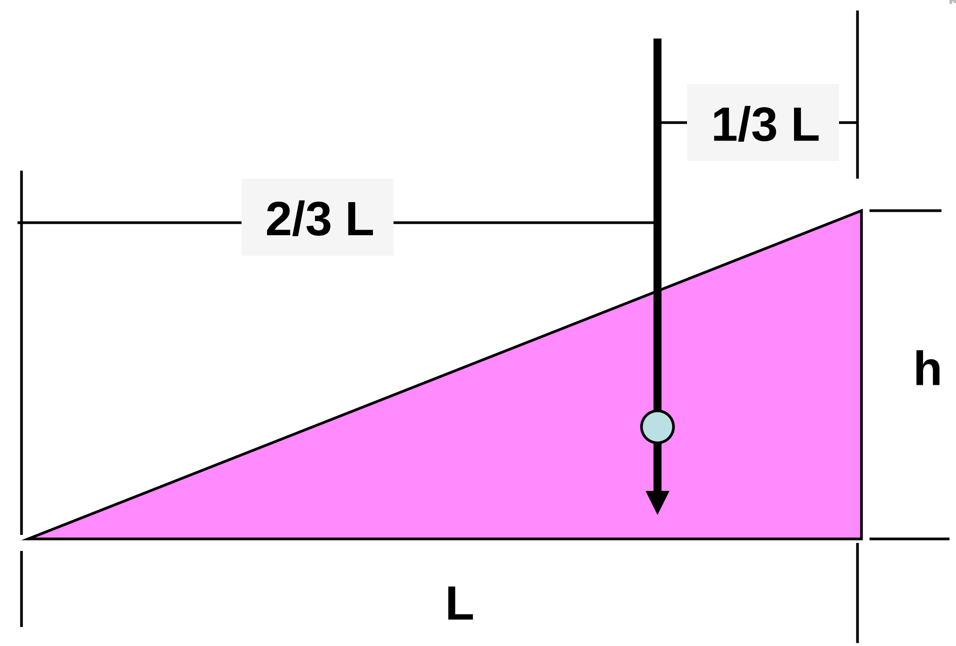




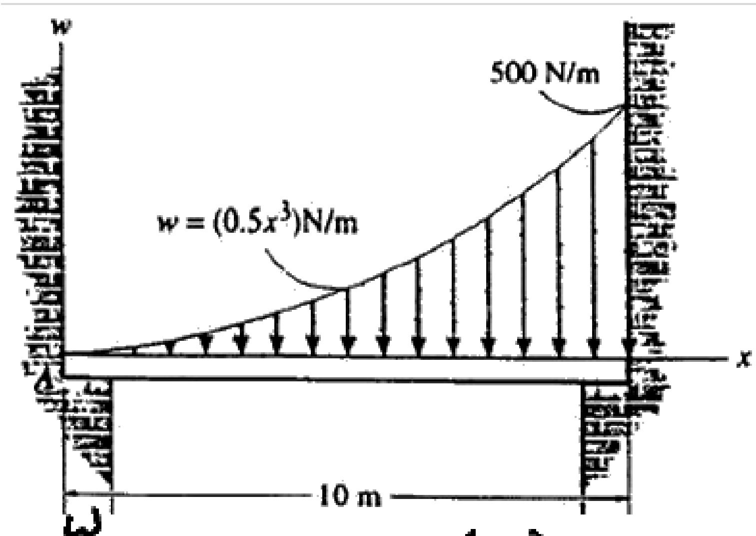


$$A = \tfrac{1}{2}(9\text{m})\left(1440\,\text{N}/\text{m}\right) = 6.48\text{ kN} = F_R$$

$$\bar{x} = \frac{\int_0^9 x(160\,x)\,dx}{\int_0^9 (160\,x)\,dx} = \frac{160\frac{x^3}{3}\Big|_0^9}{160\frac{x^2}{2}\Big|_0^9} = \frac{38880}{6480} = 6\text{ m}$$



Find the equivalent resultant force and specify the magnitude and location of the force measured from A.



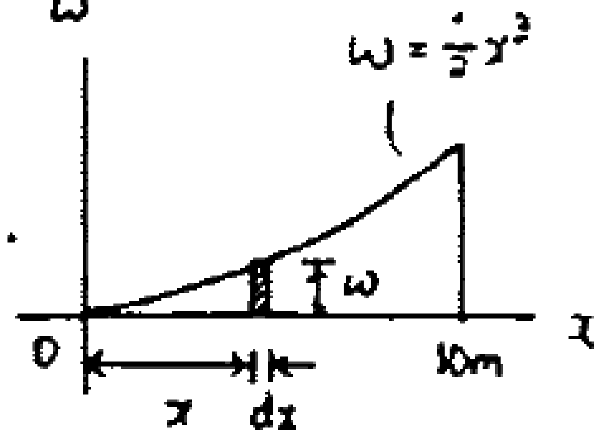
$$dA = w\,dx$$

$$F_R = \int dA = \int_0^{10} \frac{1}{2}x^3\,dx$$

$$= \left[\frac{1}{8}x^4\right]_0^{10}$$

$$= 1250 \text{ N}$$

$$F_R = 1.25 \text{ kN}$$

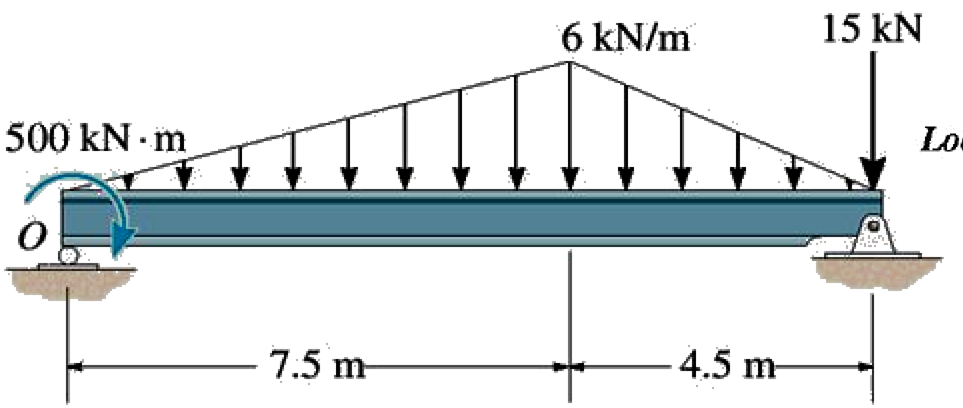


$$\int \tilde{x}\,dA = \int_0^{10} \frac{1}{2}x^4\,dx$$

$$= \left[\frac{1}{10}x^5\right]_0^{10} = 10\,000 \text{ N}\cdot\text{m} \qquad \bar{x} = \frac{10\,000}{1250} = 8.00 \text{ m}$$

Replace the loading by a simple resultant force, and specify the location of the force on the beam measured from point O.

Solution:



Equivalent Resultant Force :

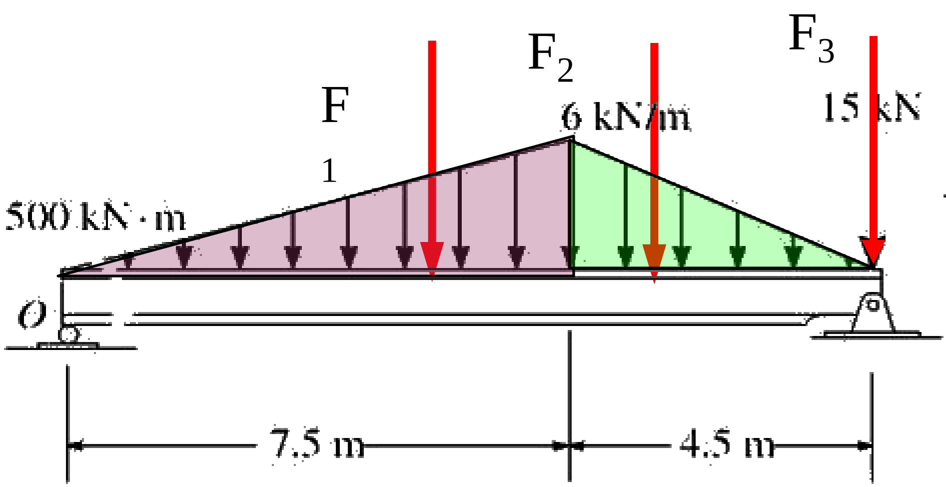
$+\uparrow F_R = \Sigma F_y; \qquad -F_R = -22.5 - 13.5 - 15$

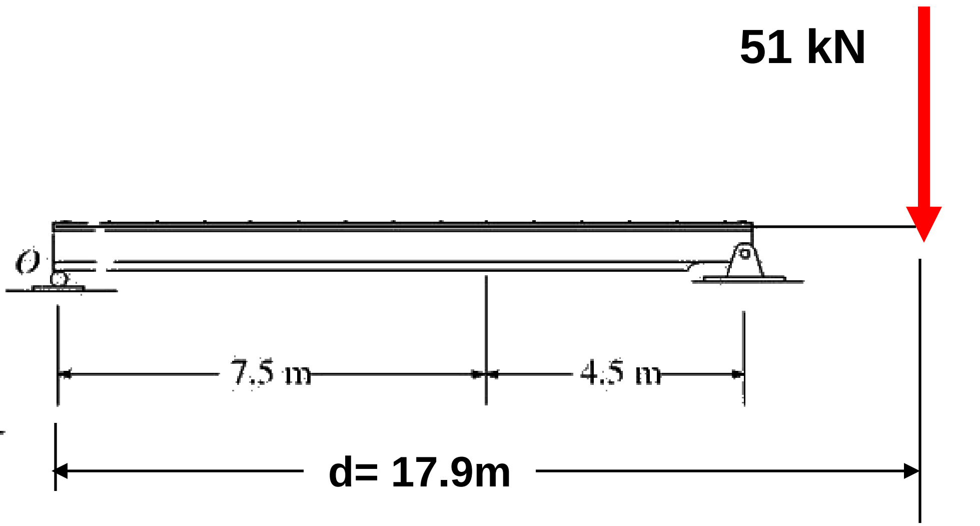
$F_R = 51.0 \text{ kN} \downarrow$

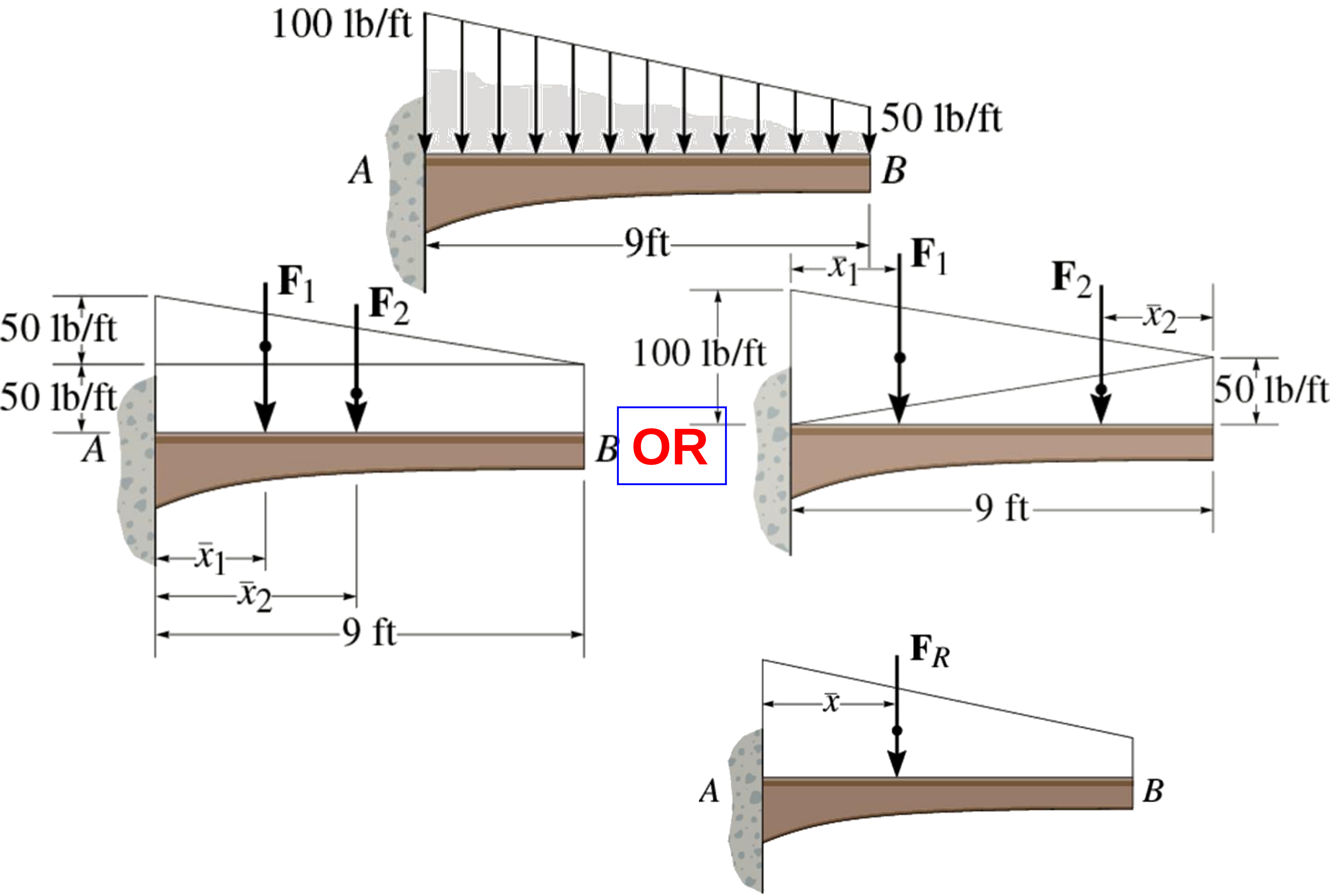
Location of Equivalent Resultant Force :

$\circlearrowleft + (M_R)_O = \Sigma M_O; \; -51.0(d) = -500 - 22.5(5) - 13.5(9) - 15(12)$

$d = 17.9 \text{ m}$

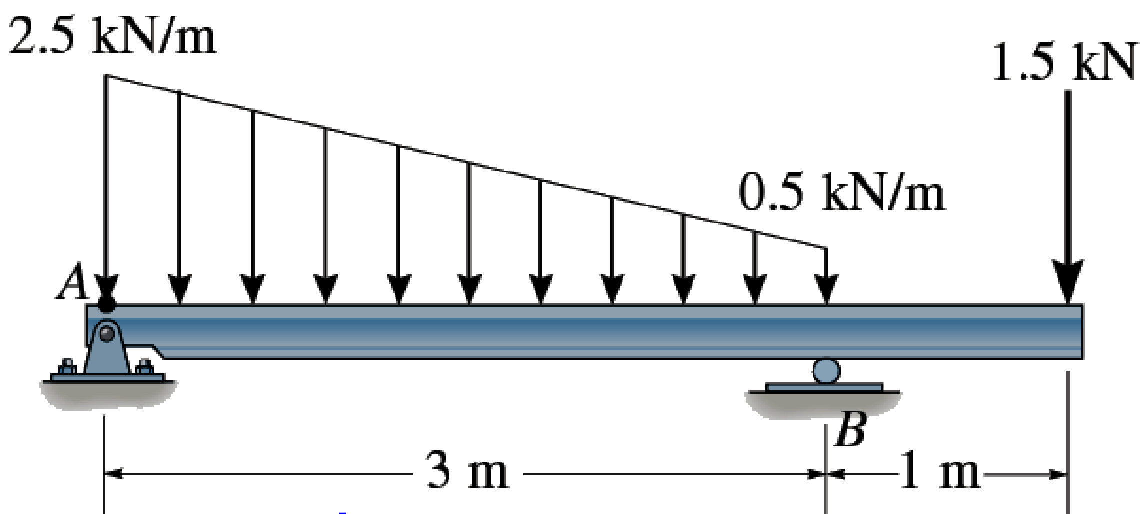


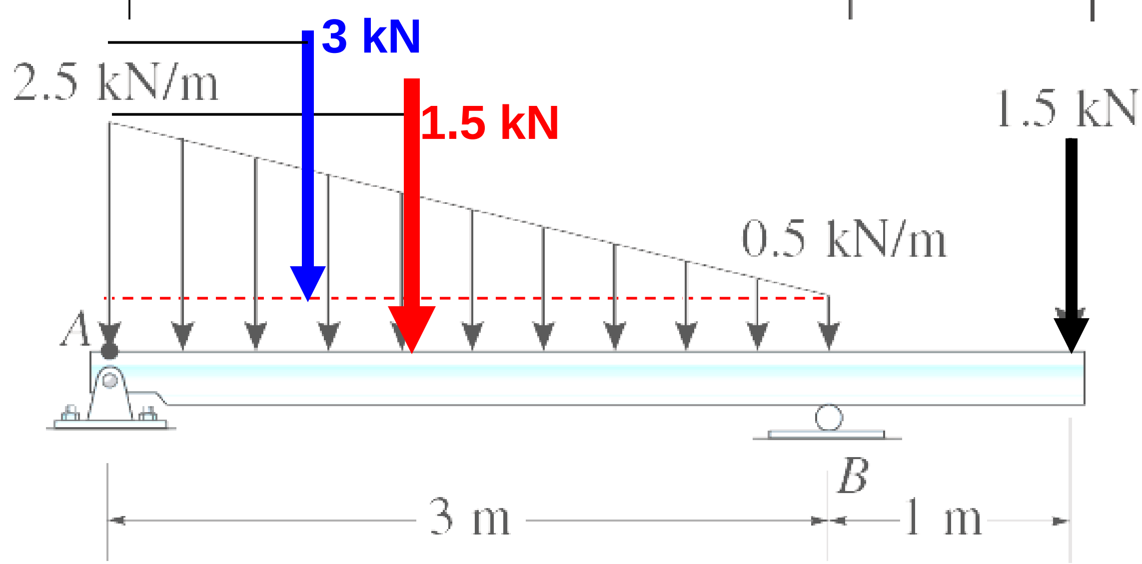




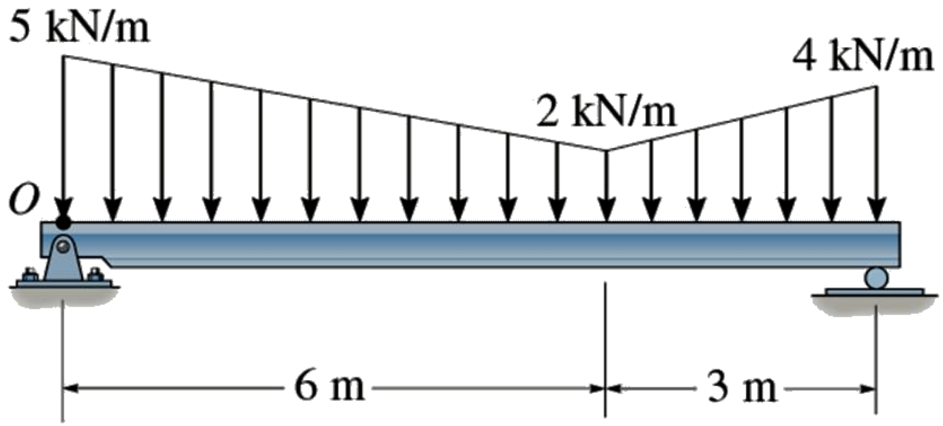
100 lb/ft
50 lb/ft
A
B
9ft
F_1
$\bar{x}_1$
F_2
$\bar{x}_2$
50 lb/ft
50 lb/ft
A
B
$\bar{x}_1$
$\bar{x}_2$
9 ft
OR
100 lb/ft
50 lb/ft
9 ft
F_R
$\bar{x}$
A
B

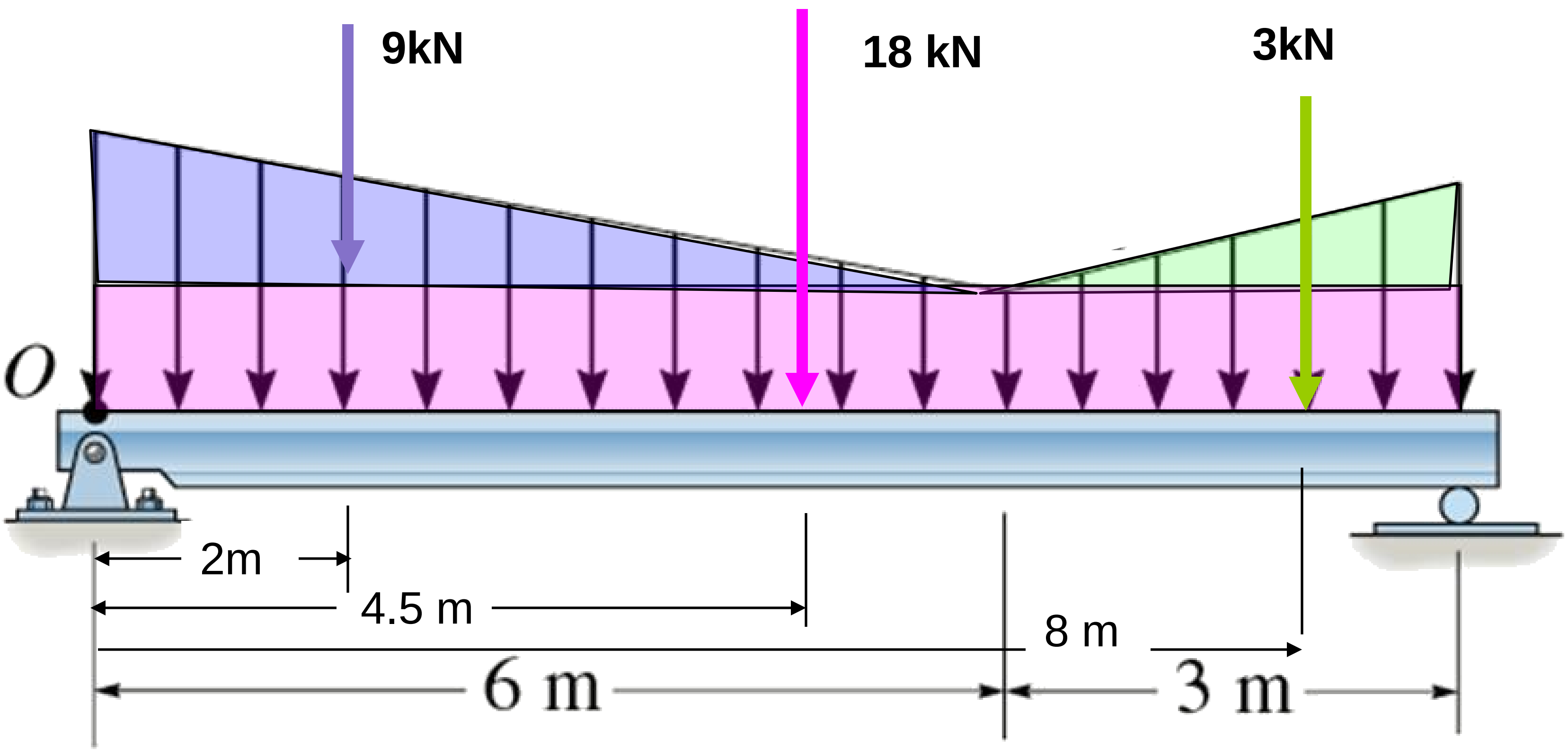
Replace the force system acting on the beam by an equivalent force and specify its location from point A.





$$D = \frac{3*1 + 1.5*1.5 + 1.5*4}{3+1.5+1.5} = 1.625 \text{ m}$$


5 kN/m
4 kN/m
2 kN/m
O
6 m
3 m


9kN
18 kN
3kN
O
2m
4.5 m
8 m
6 m
3 m

Engineering Mechanics

Statics & Strength of Materials

Equilibrium of rigid body

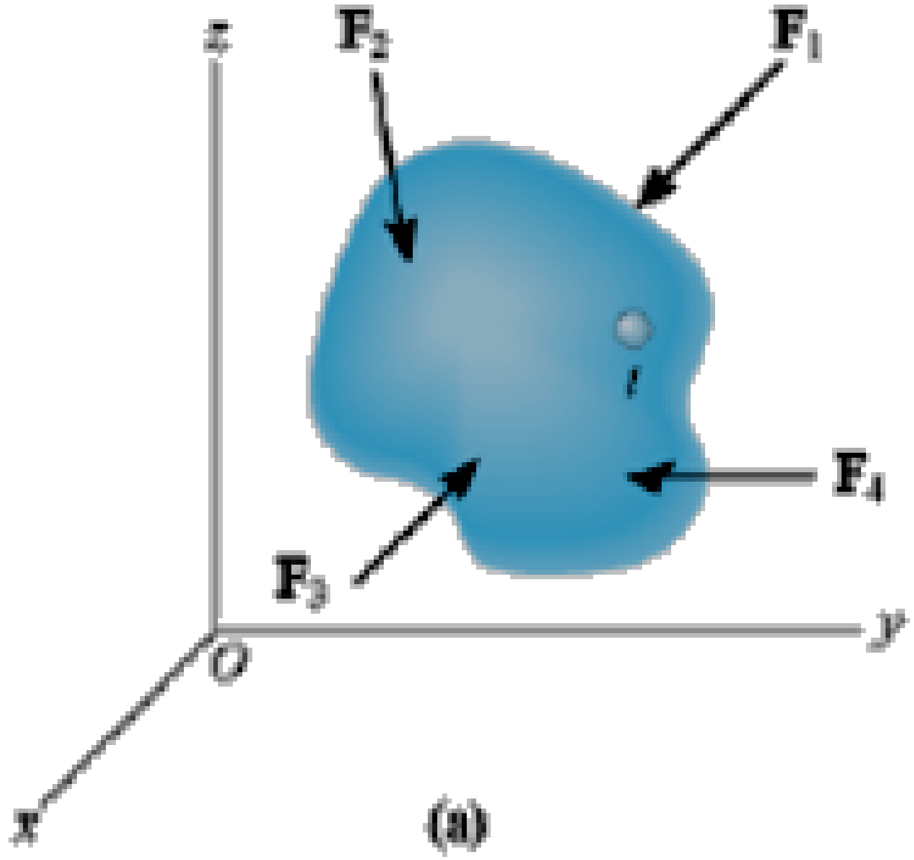
Eng. Iqbal Marie

iqbal@hu.edu.jo

5.1 Conditions for Rigid-Body Equilibrium

$$\sum \vec{F} = 0$$

$$\sum \vec{M}_O = 0$$



For equilibrium of a rigid body:
Moments (applied pure twists, and due to external forces) should sum to zero about any point.

Equilibrium in Two Dimensions

Free Body Diagrams

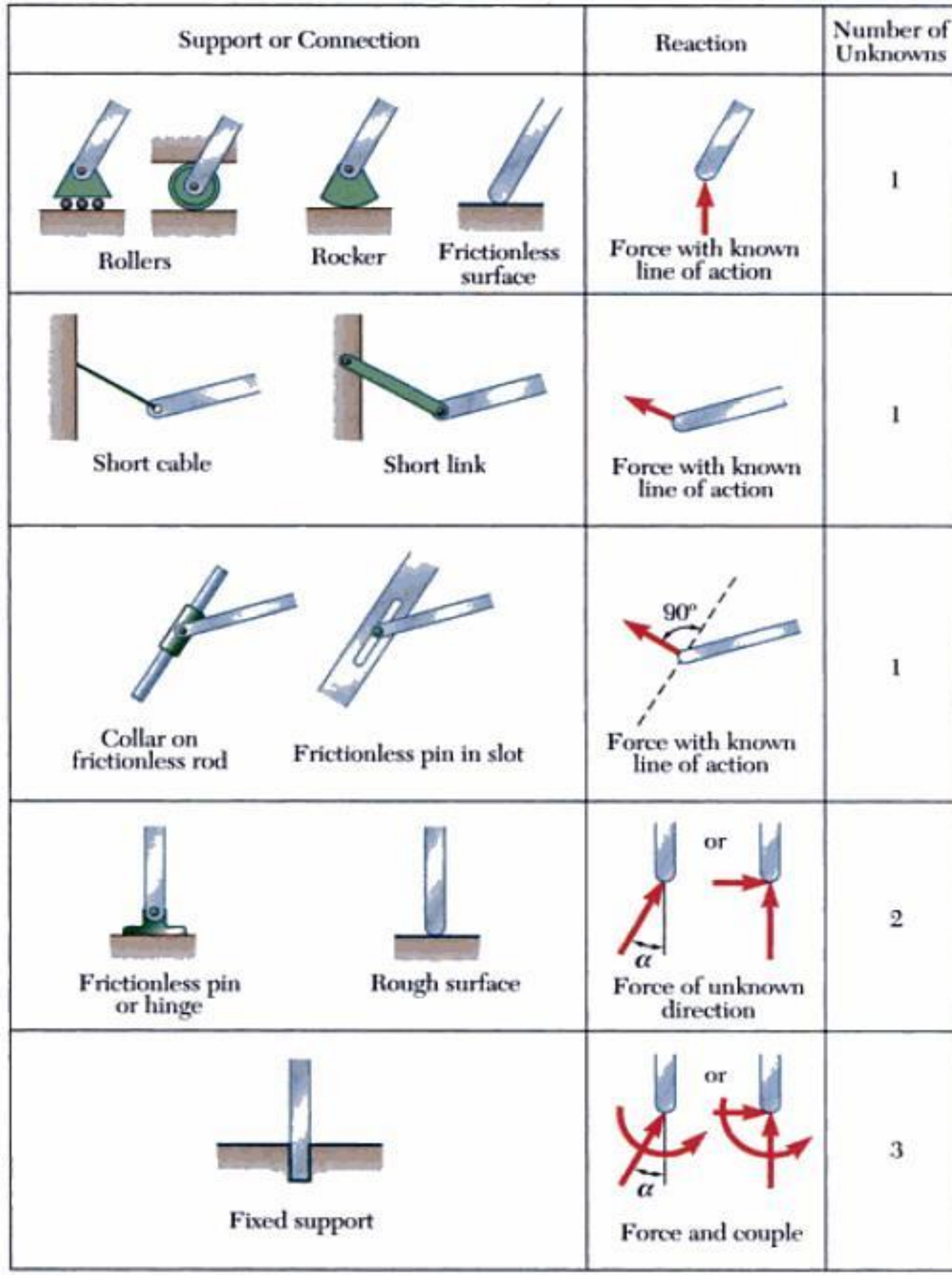
Support Reactions

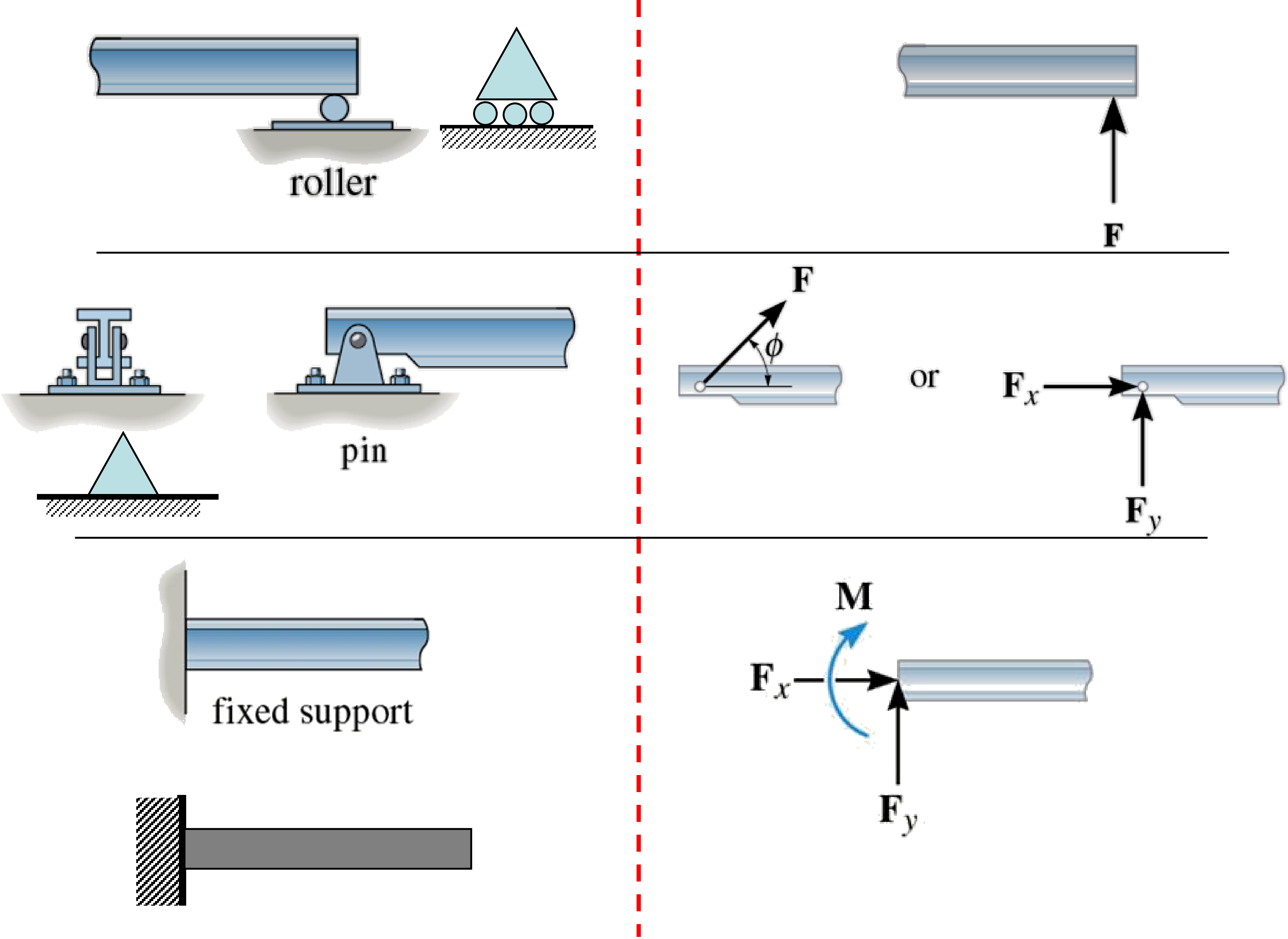
Rule:

If a support prevents the translation of a body in a given direction, then a force is developed on the body in that direction.

if rotation is prevented a couple moment is exerted on the body.

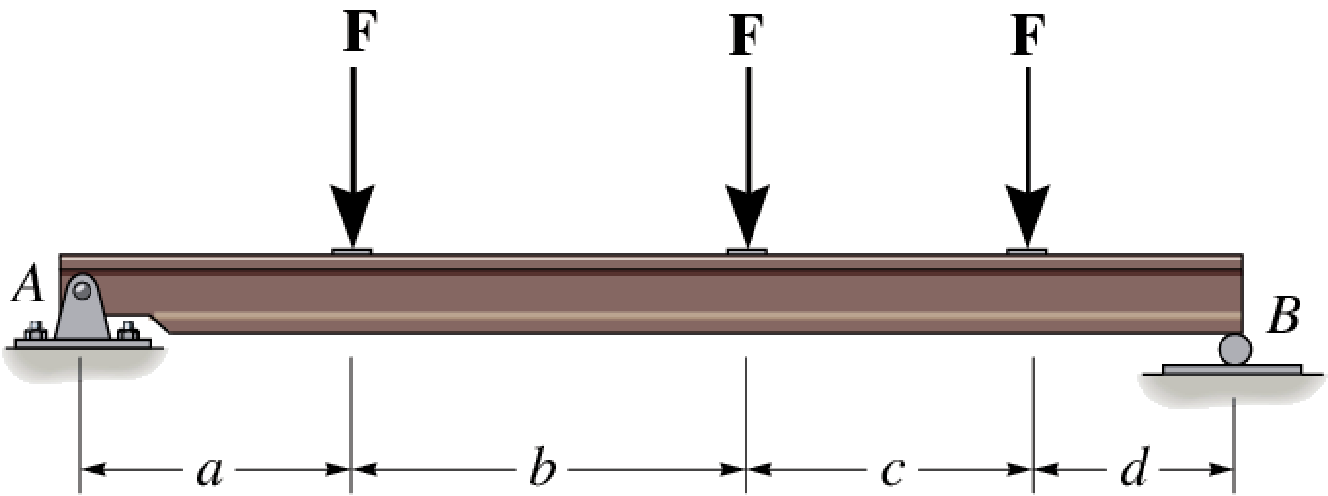
Type of supports





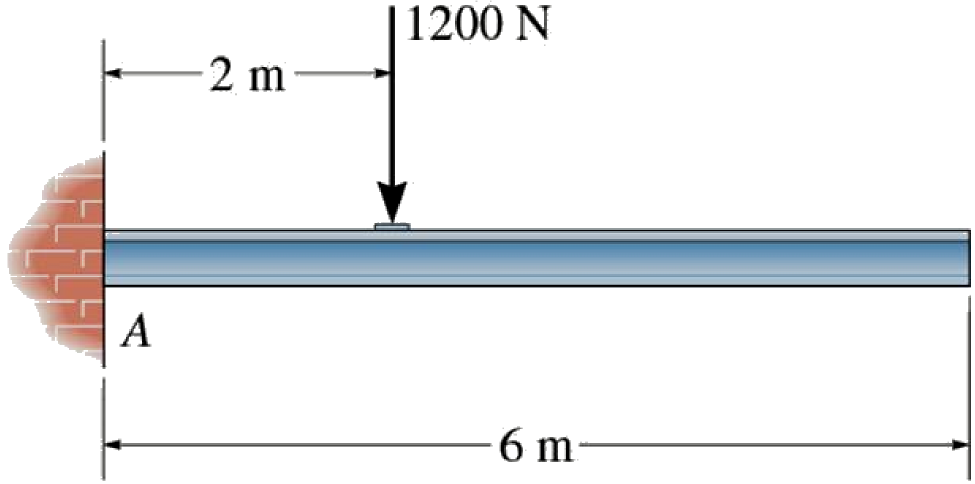
roller
F
pin
F
ϕ
or
Fx
Fy
fixed support
M
Fx
Fy

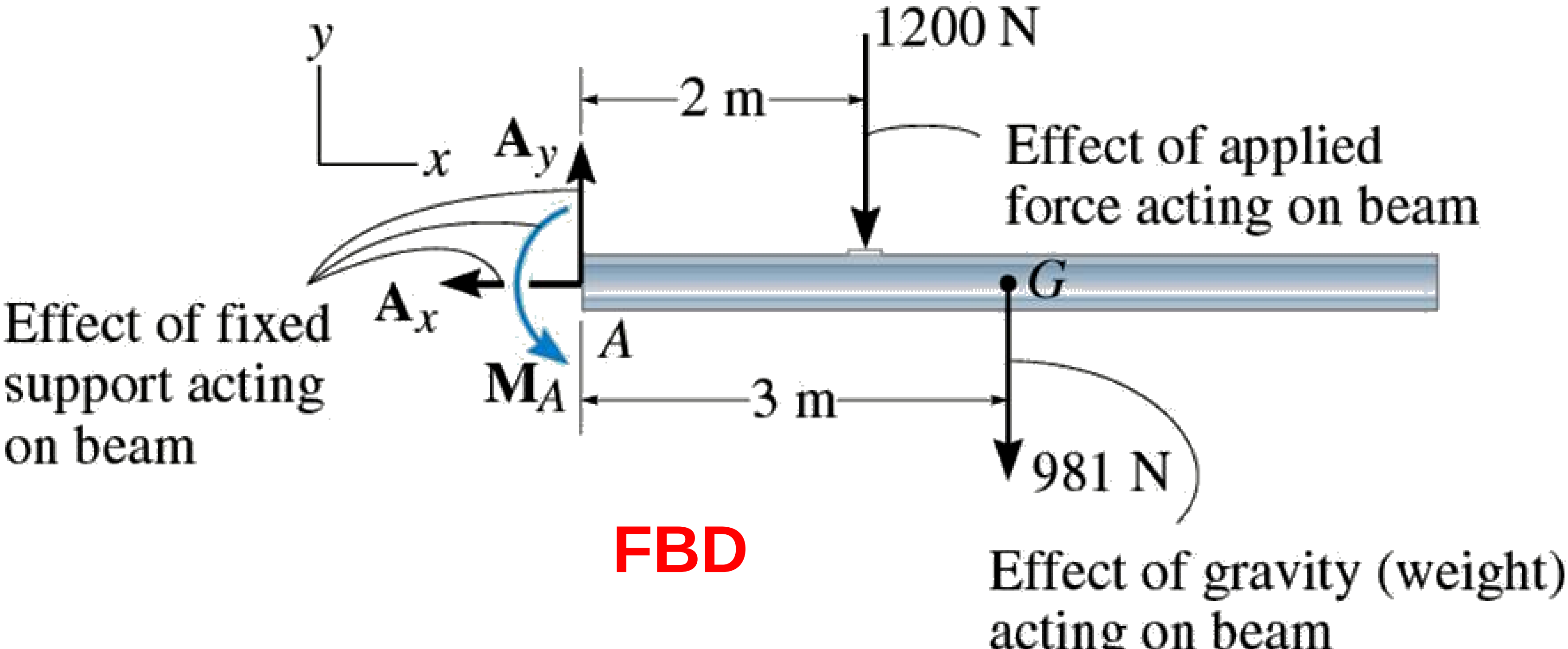
Modeling



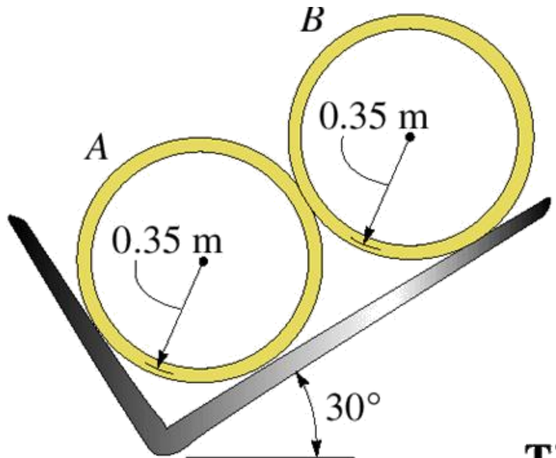
Procedure for Drawing a Free-Body Diagram

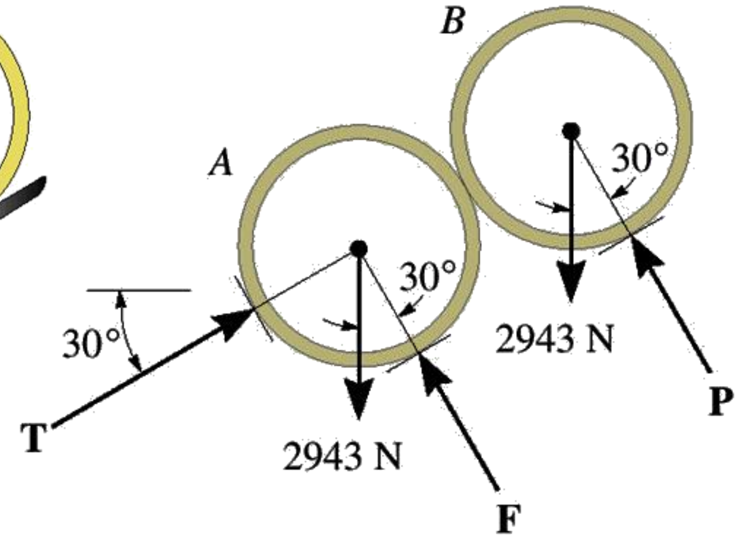
1. Select co-ordinate axes.
2. Draw outlined shape isolated or cut “free” from its constraints and connections.
3. Show all forces and moments acting on the body. Include applied loadings and reactions.
4. Identify each loading and give dimensions. Label forces and moments with proper magnitudes and directions. Label unknowns.

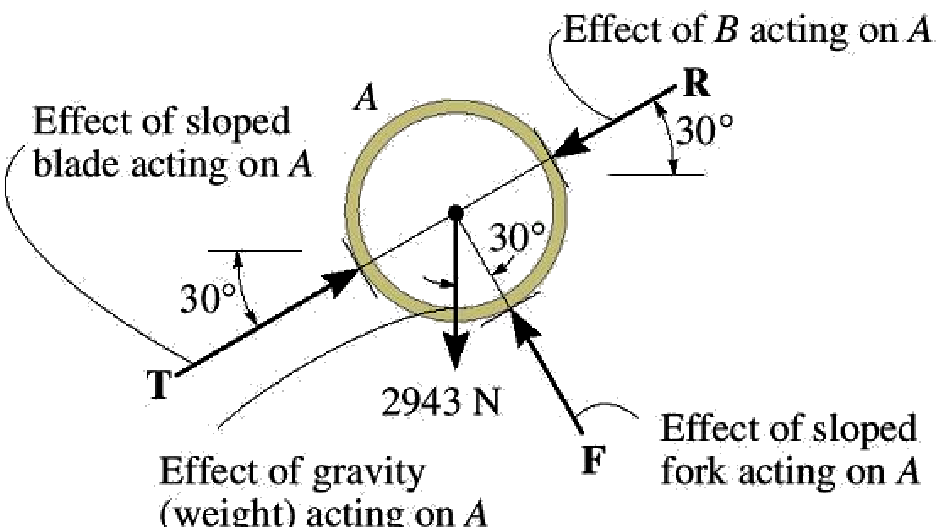

1200 N
2 m
A
6 m

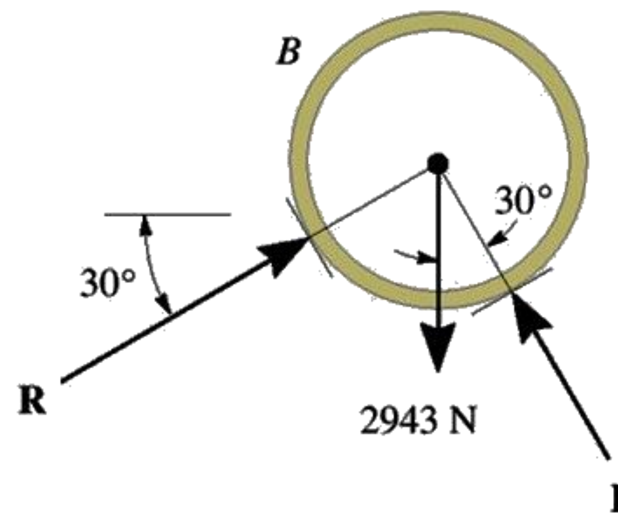

y
x
1200 N
2 m
Effect of applied force acting on beam
A_y
A_x
M_A
A
G
3 m
981 N
Effect of fixed support acting on beam
Effect of gravity (weight) acting on beam

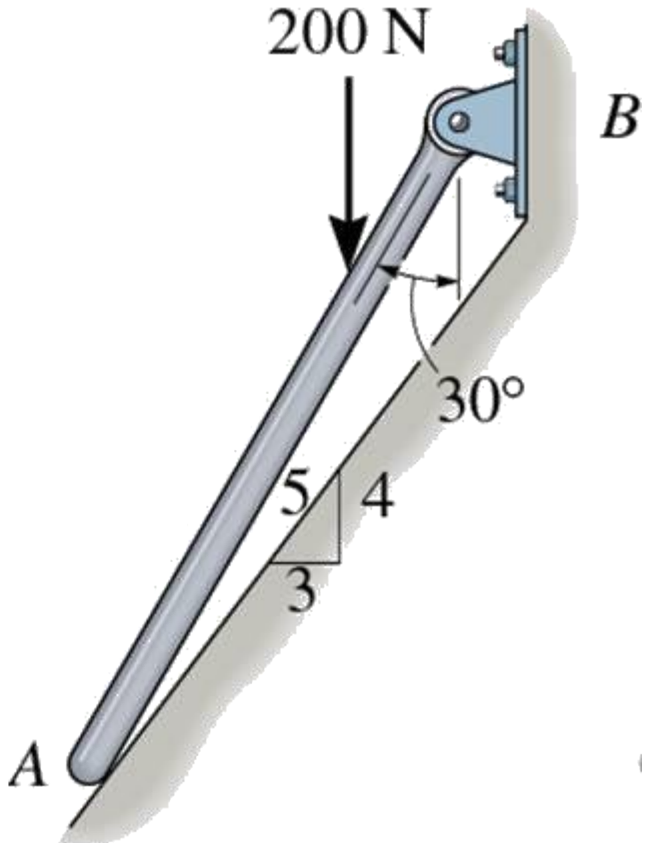
FBD

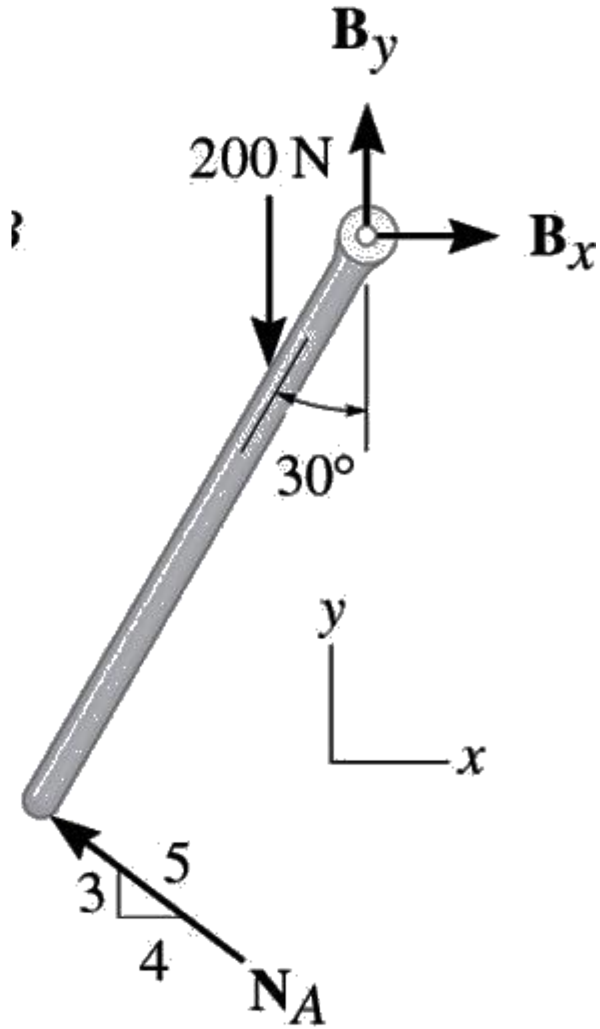

B
A
0.35 m
0.35 m
30°


B
A
30°
30°
30°
T
2943 N
2943 N
F
P

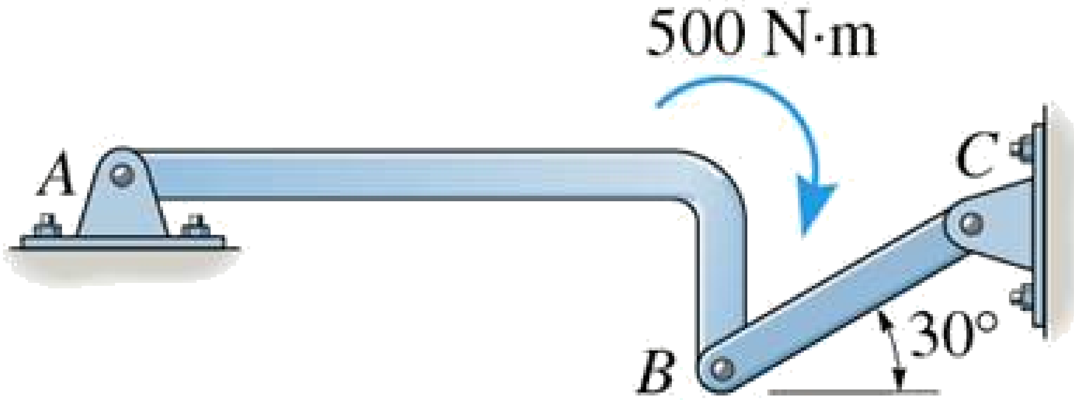

Effect of B acting on A
R
30°
Effect of sloped blade acting on A
A
30°
30°
T
2943 N
F
Effect of gravity (weight) acting on A
Effect of sloped fork acting on A

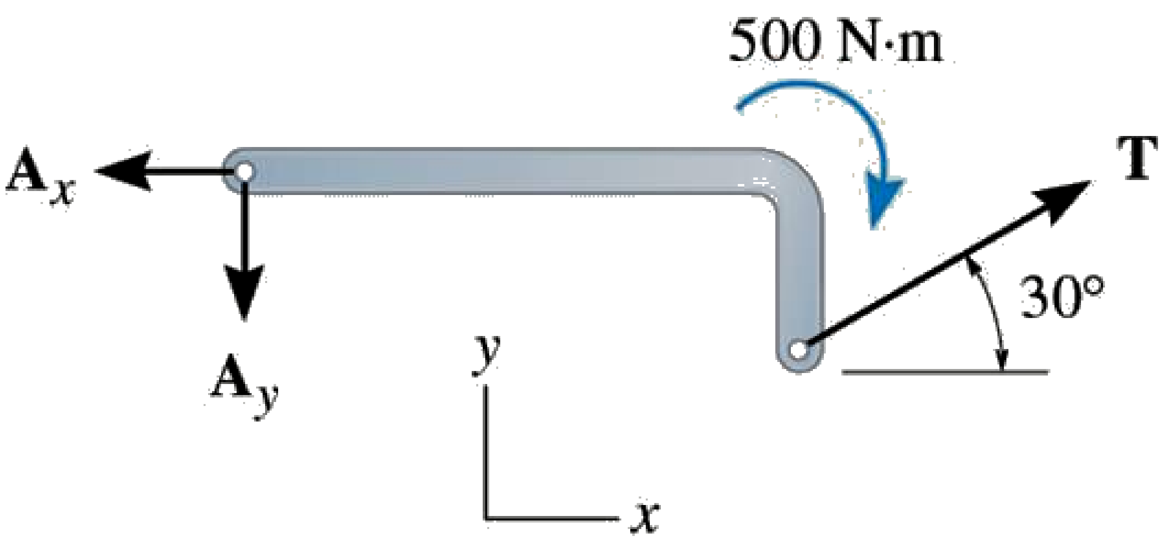

B
30°
30°
R
2943 N
P



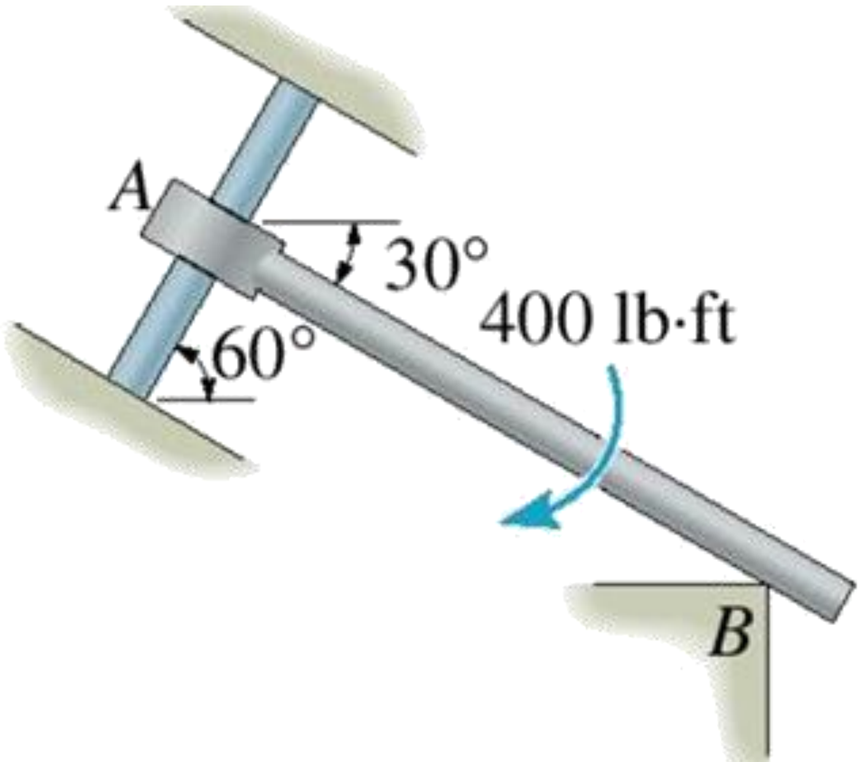


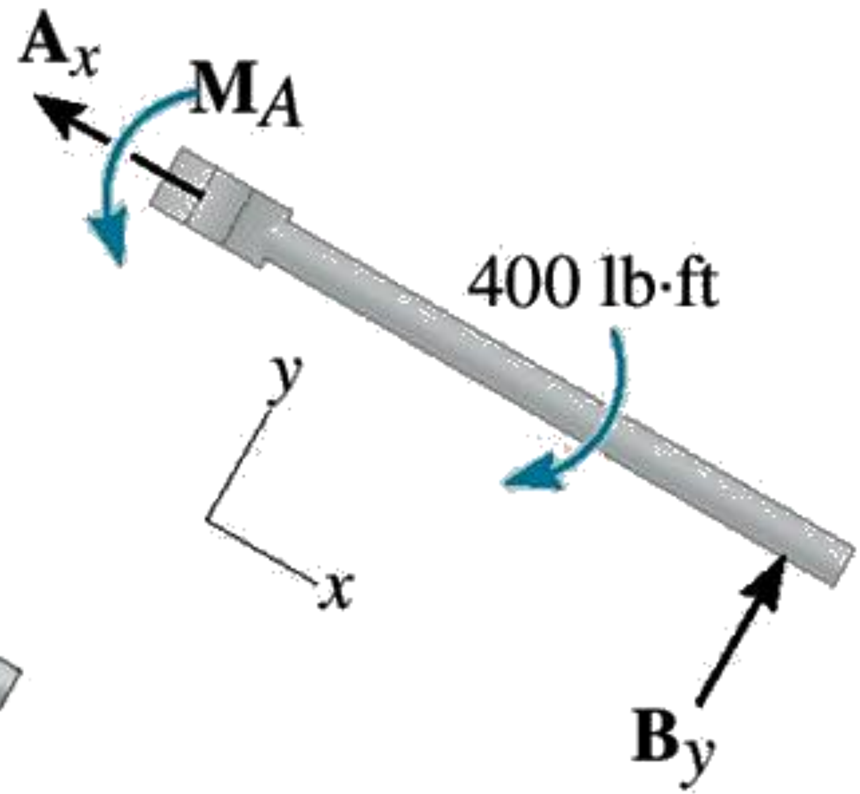
F B D



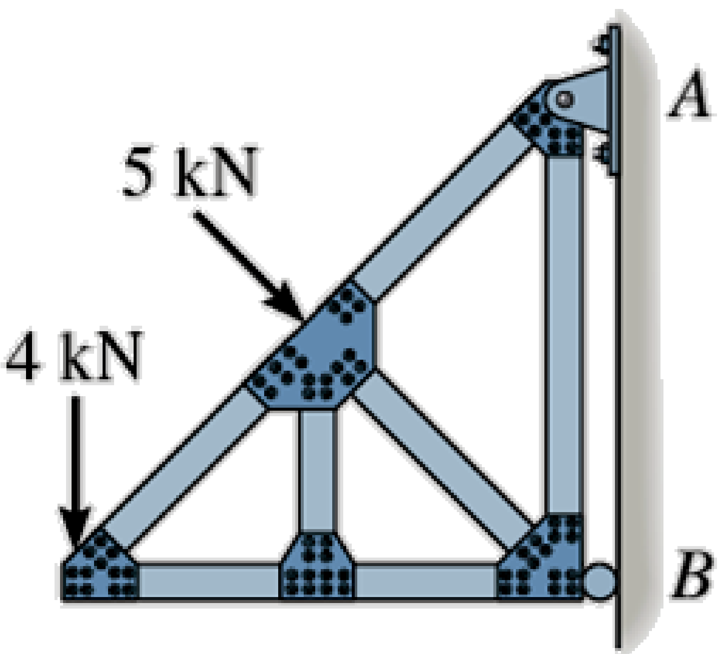


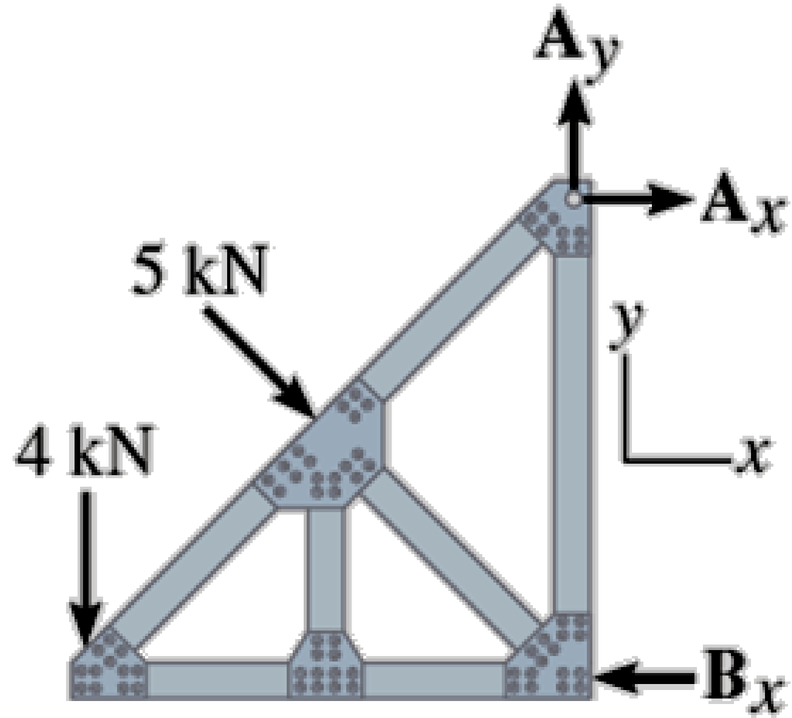
F B D



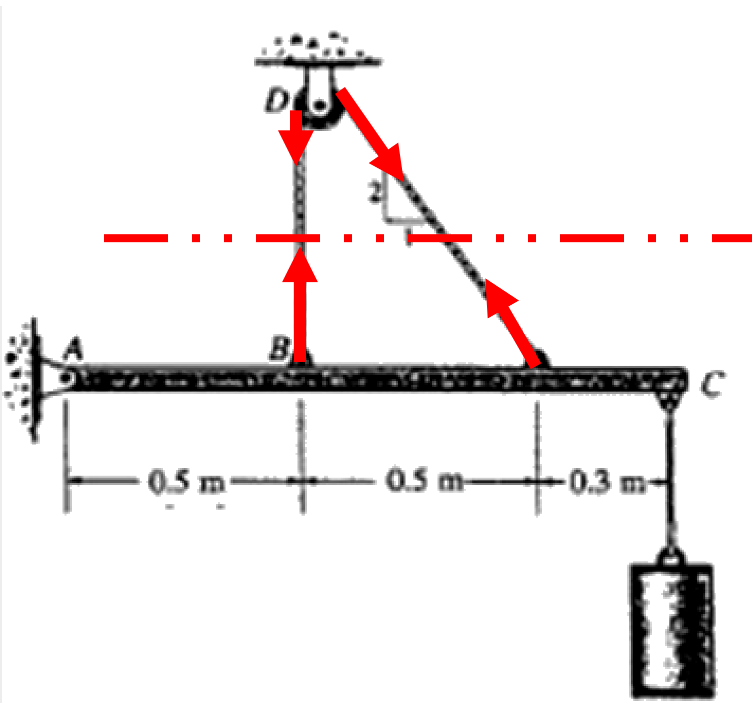


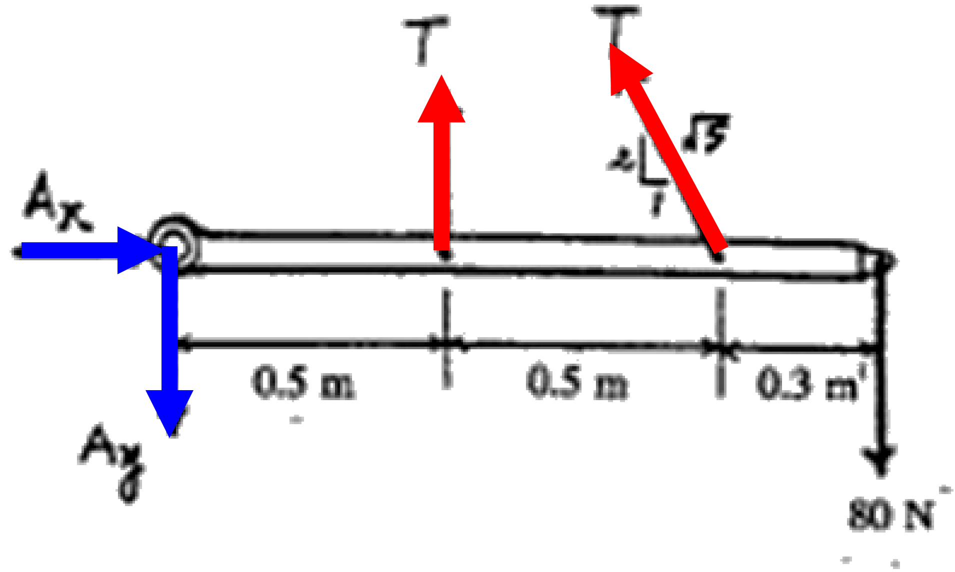
F B D





F B D





F B D

Important Points

1. **No equilibrium problem should be solved without first drawing the appropriate F.B.D.**
2. **If a support prevents translation in a direction, then it exerts a force on the body in that direction.**
3. **If a support prevents rotation of the body then it exerts a moment on the body.**
4. **Couple moments are *free vectors* and can be placed anywhere on the body.**
5. **Forces can be placed anywhere along their line of action. They are *sliding vectors*.**

Equilibrium of a Rigid Body in Two Dimensions

- **Equations of equilibrium become**

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

where *A* is any point in the plane of the structure.

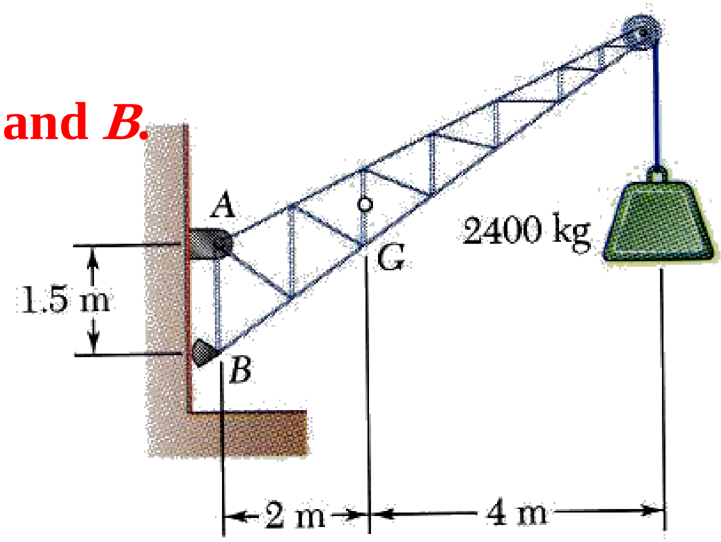
- **The 3 equations can be solved for no more than 3 unknowns.**

- **The 3 equations can not be augmented with additional equations, but they can be replaced with:**

$$\sum F_x = 0 \quad \sum M_A = 0 \quad \sum M_B = 0$$

A fixed crane has a mass of 1000 kg and is used to lift a 2400 kg crate. It is held in place by a pin at *A* and a rocker at *B*. The center of gravity of the crane is located at *G*.

Determine the components of the reactions at *A* and *B*.

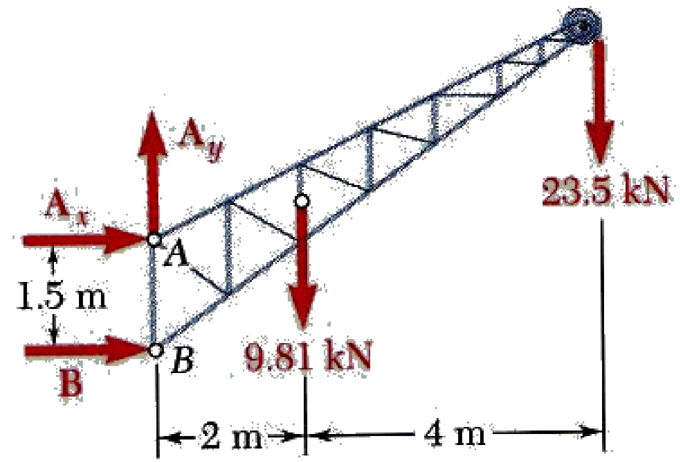


$$\sum M_A = 0: \quad +B(1.5\text{m}) - 9.81\,\text{kN}(2\text{m}) - 23.5\,\text{kN}(6\text{m}) = 0$$

$$\boxed{B = +107.1\,\text{kN}}$$

$$\sum F_x = 0: \quad A_x + B = 0$$

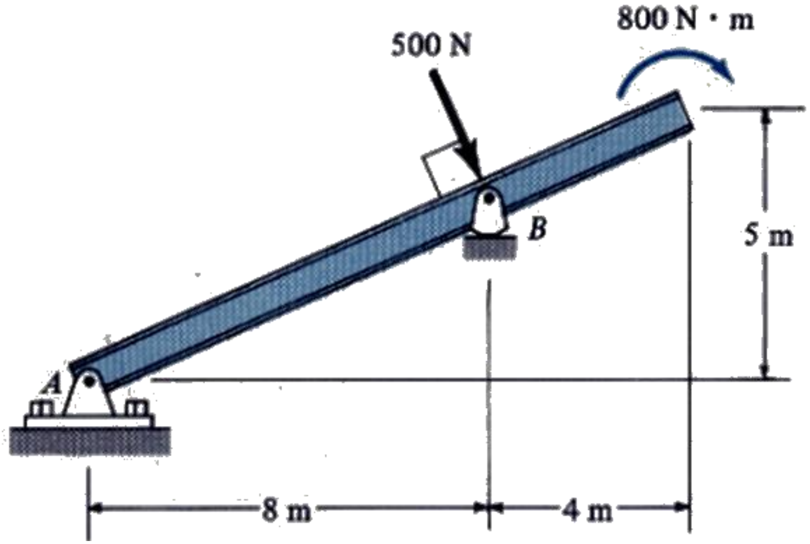
$$\boxed{A_x = -107.1\,\text{kN}}$$

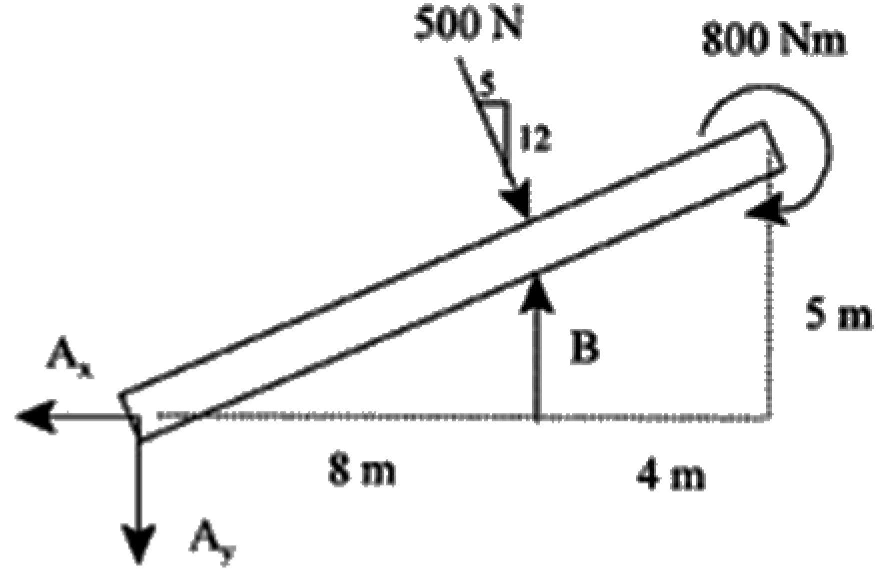


$$\sum F_y = 0: \quad A_y - 9.81\,\text{kN} - 23.5\,\text{kN} = 0$$

$$\boxed{A_y = +33.3\,\text{kN}}$$

Determine the reactions at the supports necessary for equilibrium of the beam.





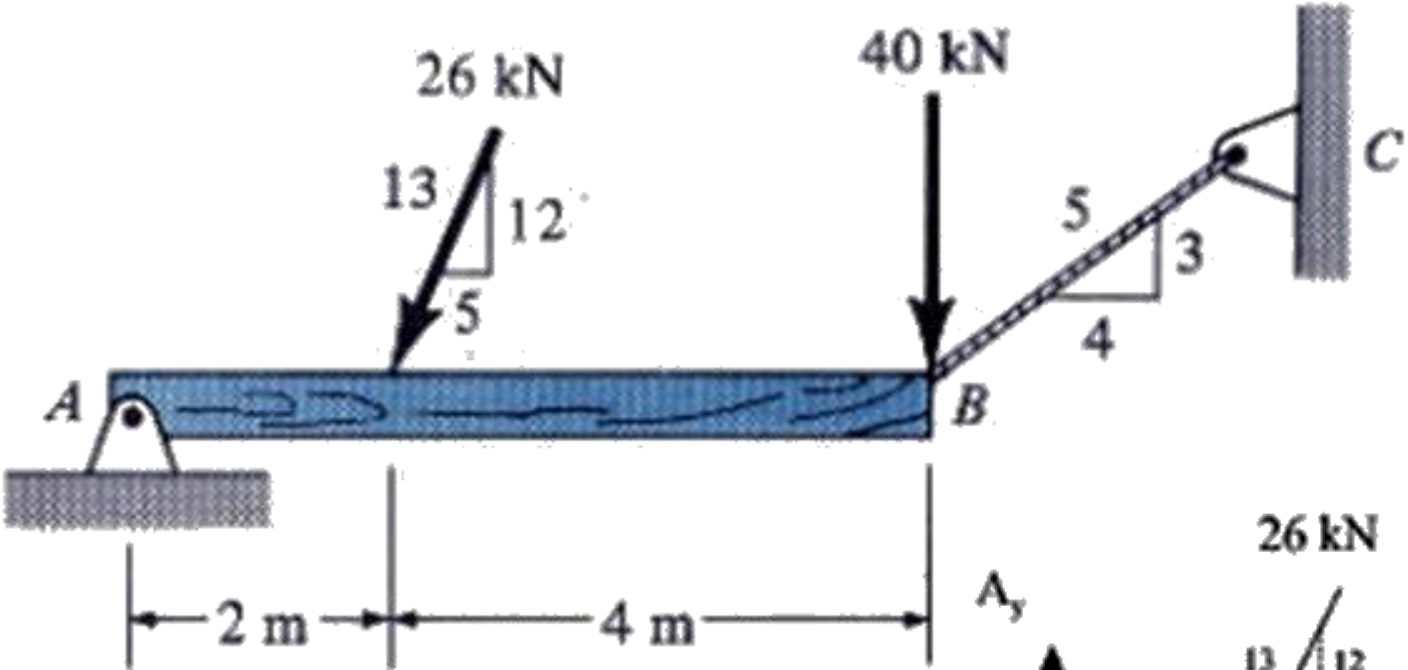
$$\sum M_A : -800\,Nm - (500\,N)\left(\tfrac{2}{3}13m\right) + B(8m) = 0$$

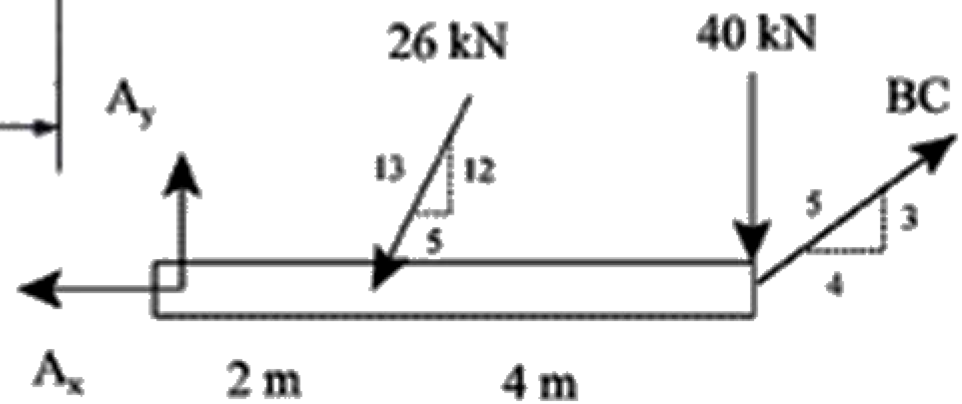
$$\sum F_x : -A_x + \tfrac{5}{13}500\,N = 0$$

$$\sum F_y : -A_y - \tfrac{12}{13}500\,N + B = 0$$

These equations can easily be solved.:

$$A_x = 192.3\,N;\quad A_y = 180.1\,N;\quad B = 642\,N$$





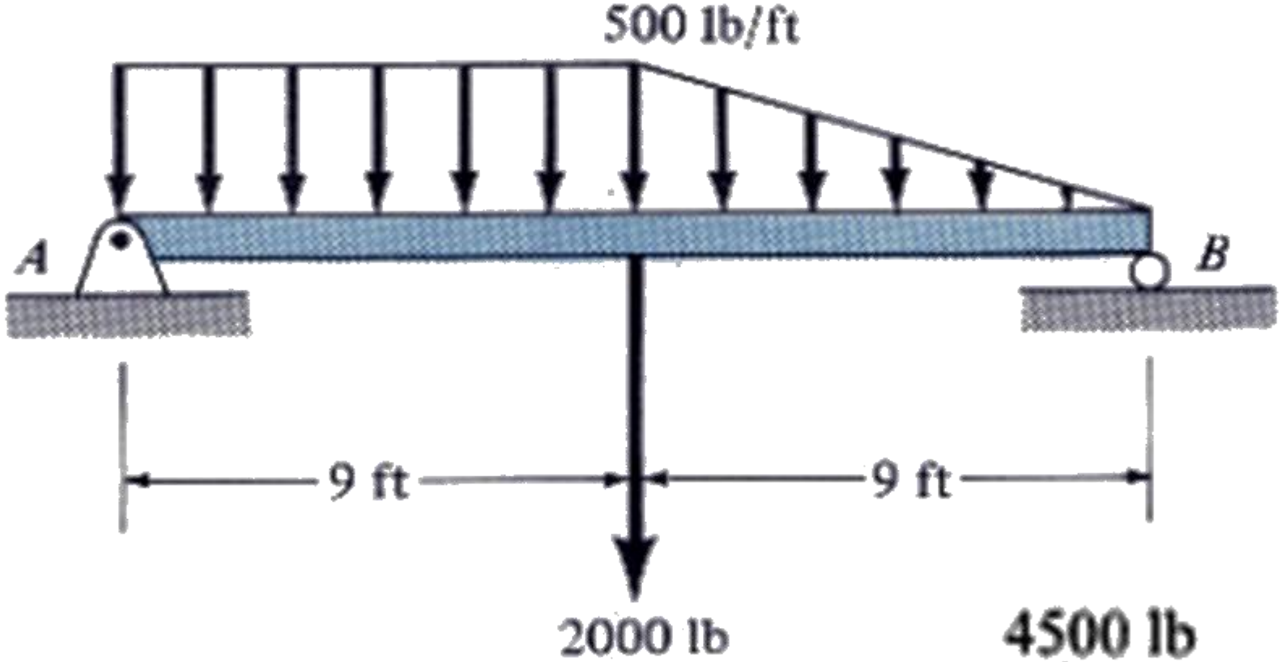
$$\sum F_x: -A_x - \tfrac{5}{13}26kN + \tfrac{4}{5}BC = 0$$

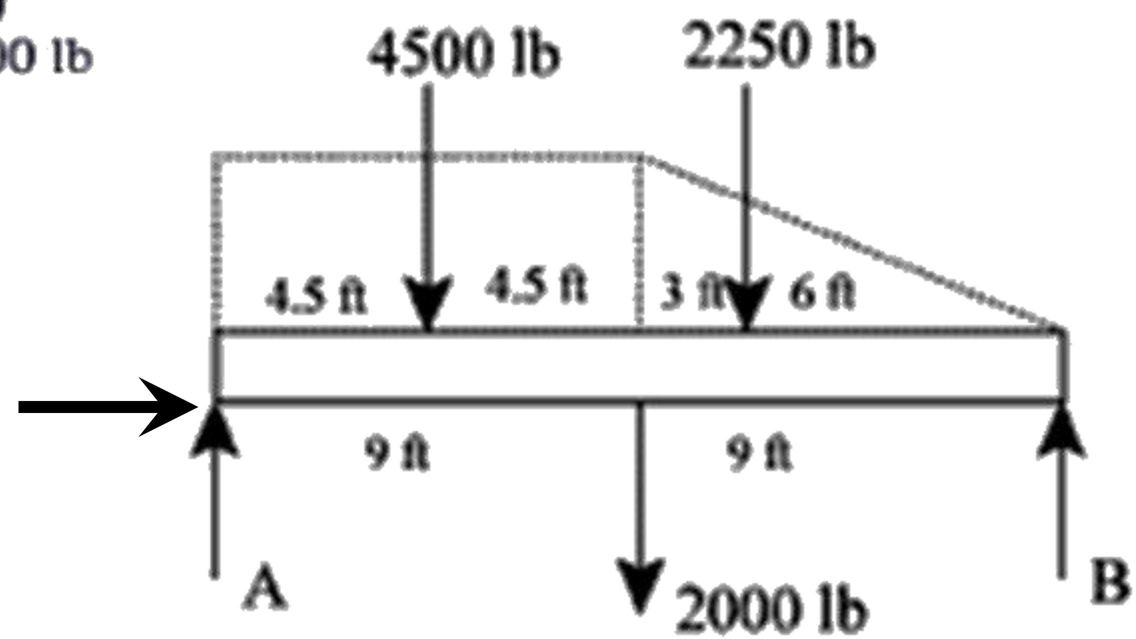
$$\sum F_y: A_y - \tfrac{12}{13}26kN - 40kN + \tfrac{3}{5}BC = 0$$

$$\sum M_A: -\left(\tfrac{12}{13}26kN\right)(2m) - (40kN)(6m) + \left(\tfrac{3}{5}BC\right)(6m) = 0$$

These equations are easily solved:

$$\boxed{A_x = 54kN;\; A_y = 16kN;\; BC = 80kN}$$

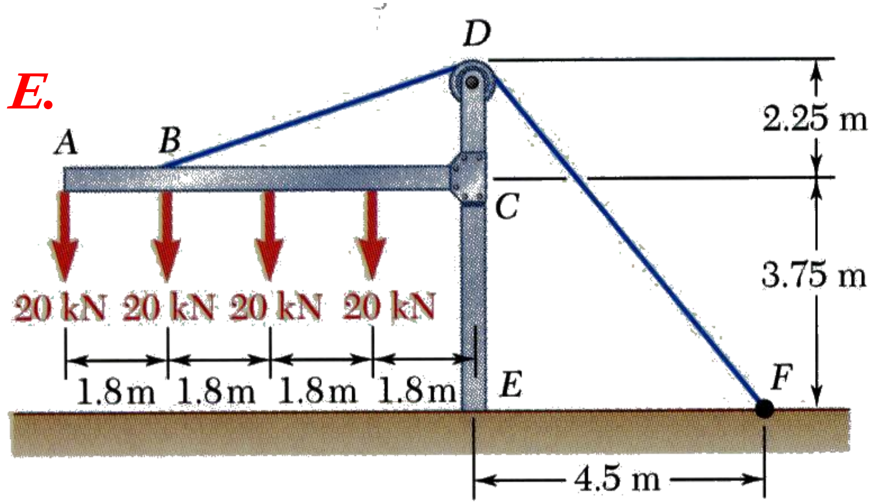




$$A = 5125\,lb, \quad B = 3625\,lb$$

The frame supports part of the roof of a small building. The tension in the cable is 150 kN.

Determine the reaction at the fixed end *E*.



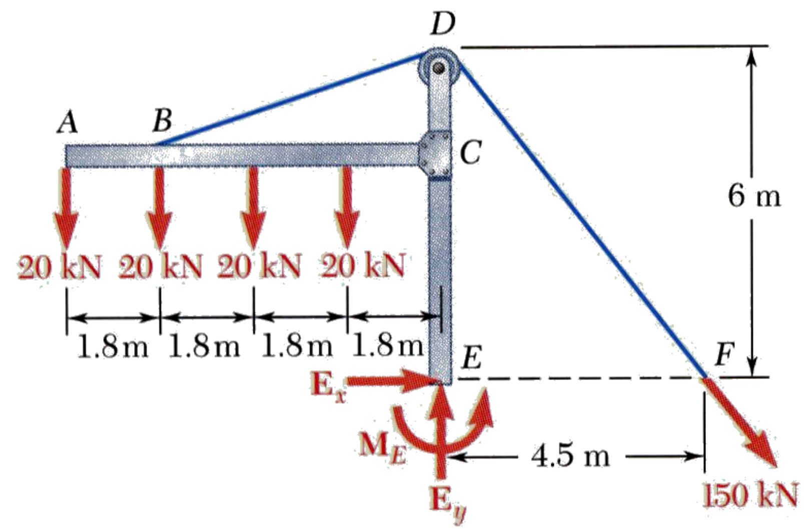
$$\sum F_x = 0: \quad E_x + \frac{4.5}{7.5}(150\,\text{kN}) = 0$$

$$\boxed{E_x = -90.0\,\text{kN}}$$

$$\sum F_y = 0: \quad E_y - 4(20\,\text{kN}) - \frac{6}{7.5}(150\,\text{kN}) = 0$$

$$\boxed{E_y = +200\,\text{kN}}$$

$$\sum M_E = 0: \quad +20\,\text{kN}(7.2\,\text{m}) + 20\,\text{kN}(5.4\,\text{m})$$
$$+20\,\text{kN}(3.6\,\text{m}) + 20\,\text{kN}(1.8\,\text{m})$$
$$-\frac{6}{7.5}(150\,\text{kN})4.5\,\text{m} + M_E = 0$$



$$\boxed{M_E = 180.0\,\text{kN}\cdot\text{m}}$$

A loading car is at rest on an inclined track. The gross weight of the car and its load is 5500 lb, and it is applied at at *G*. The cart is held in position by the cable.

Determine the tension in the cable and the reaction at each pair of wheels.

- Determine the reactions at the wheels.

$$\sum M_A = 0: \quad -(2320\,\text{lb})25\text{in.} - (4980\,\text{lb})6\text{in.} + R_2(50\text{in.}) = 0$$

$$\boxed{R_2 = 1758\,\text{lb}}$$

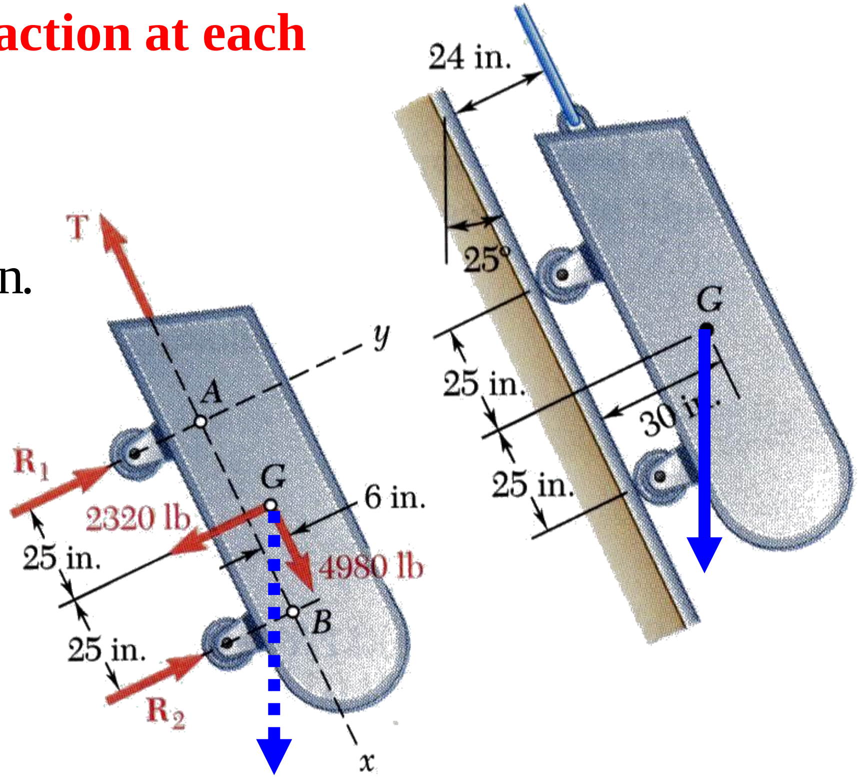
$$\sum M_B = 0: \quad +(2320\,\text{lb})25\text{in.} - (4980\,\text{lb})6\text{in.} - R_1(50\text{in.}) = 0$$

$$\boxed{R_1 = 562\,\text{lb}}$$

- Determine the cable tension.

$$\sum F_x = 0: \quad +4980\,\text{lb} - \text{T} = 0$$

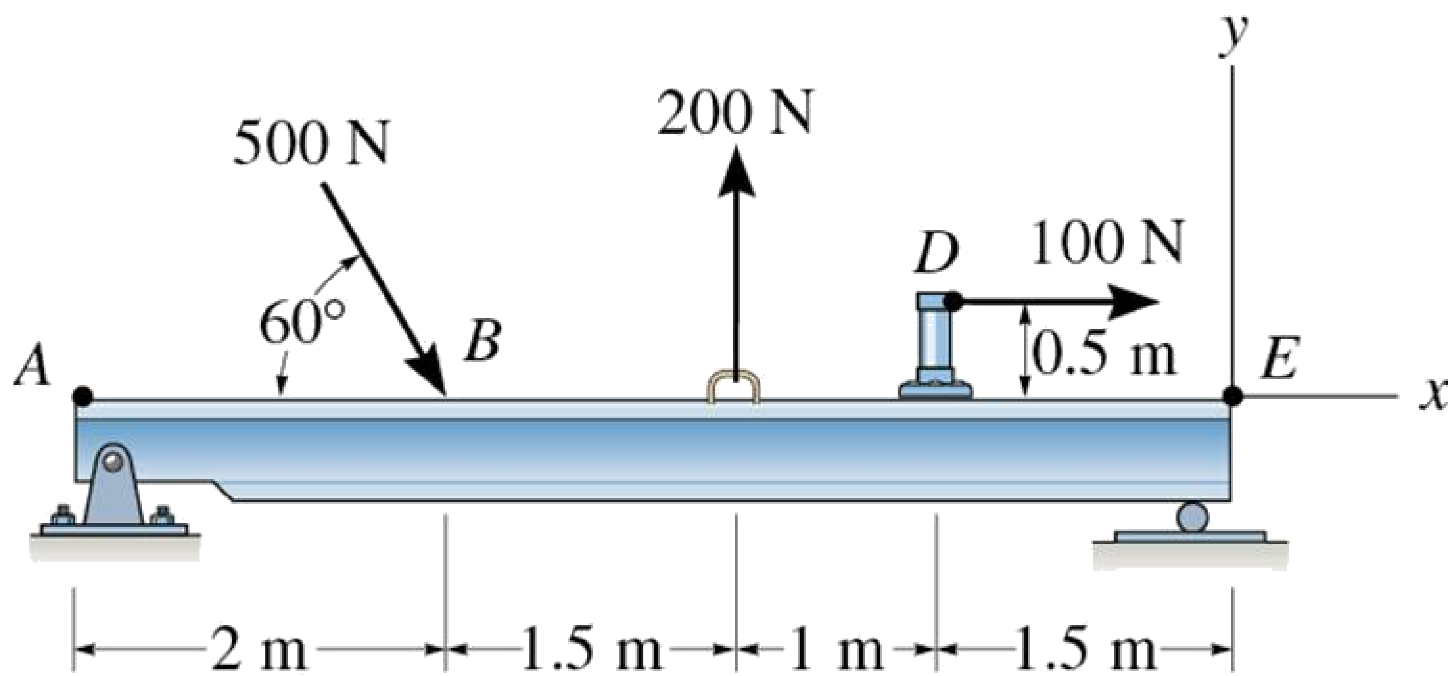
$$\boxed{T = +4980\,\text{lb}}$$



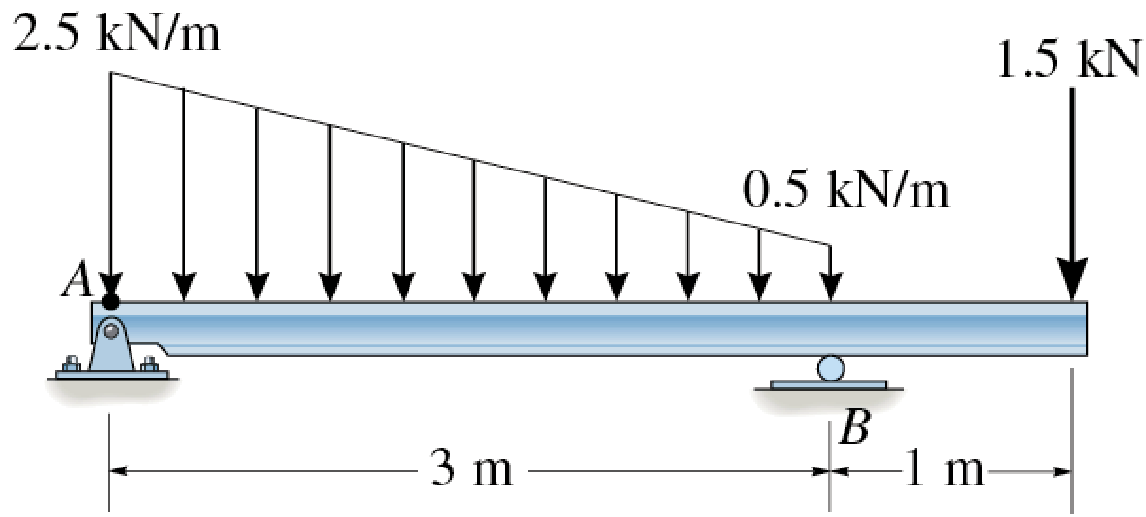
$$W_x = +(5500\,\text{lb})\cos 25^\circ = +4980\,\text{lb}$$

$$W_y = -(5500\,\text{lb})\sin 25^\circ = -2320\,\text{lb}$$

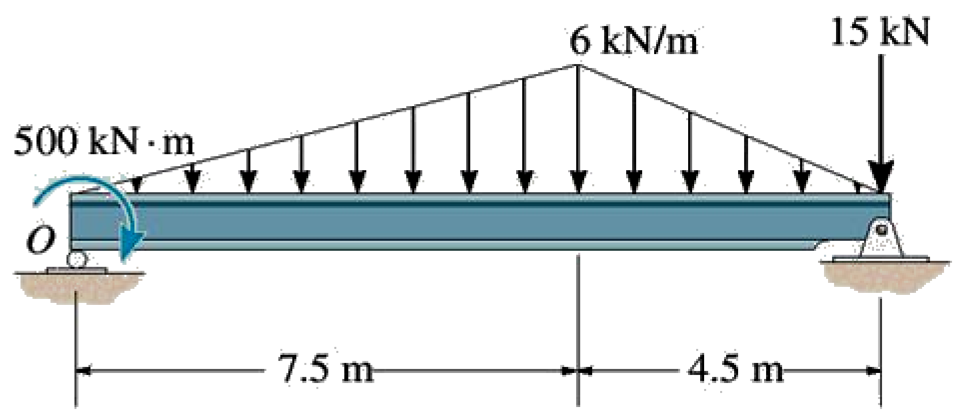
Find the reaction at A and E



Find the reactions

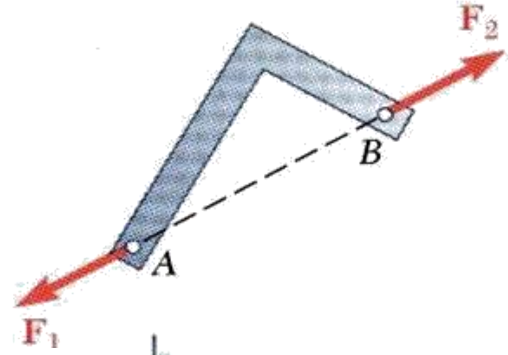


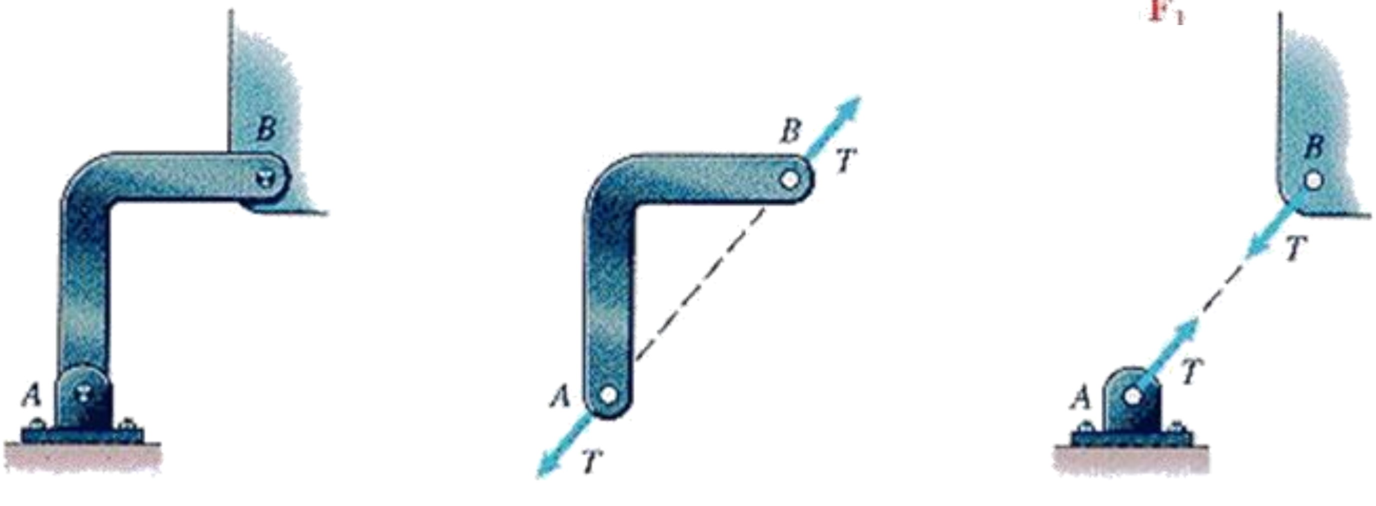
Find the reactions



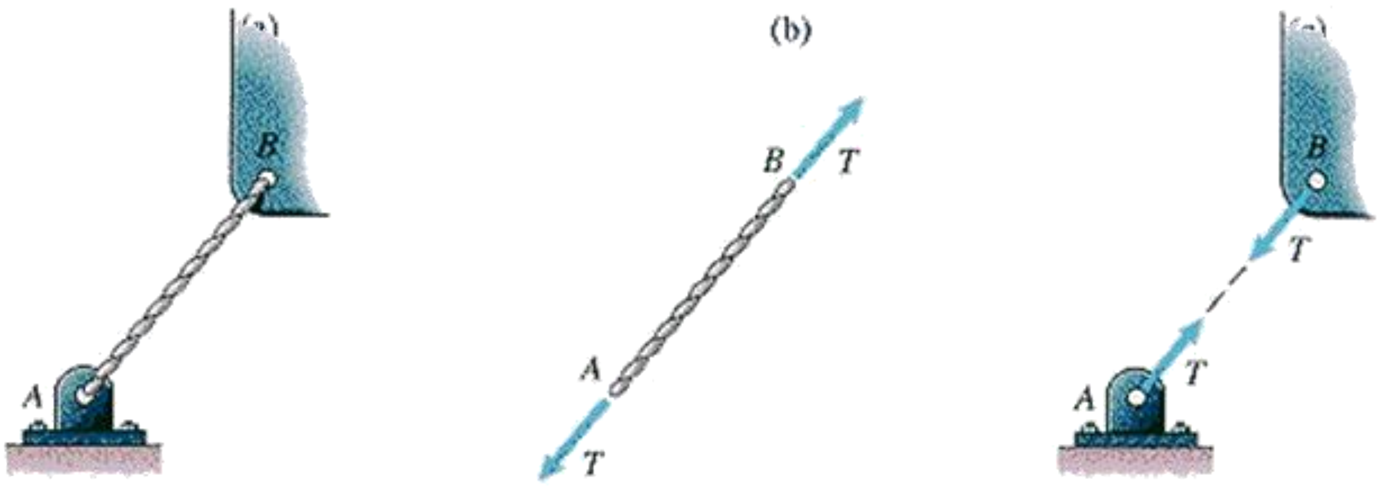
2 - Force Members

F1 **and** ***F2*** **must have equal magnitude but opposite sense.**

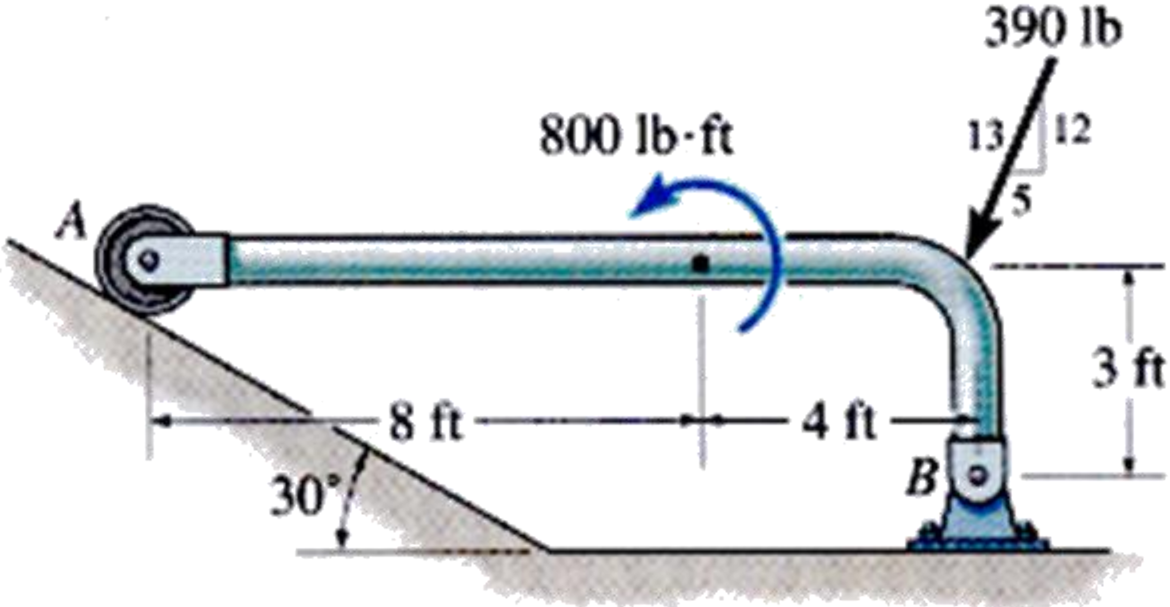


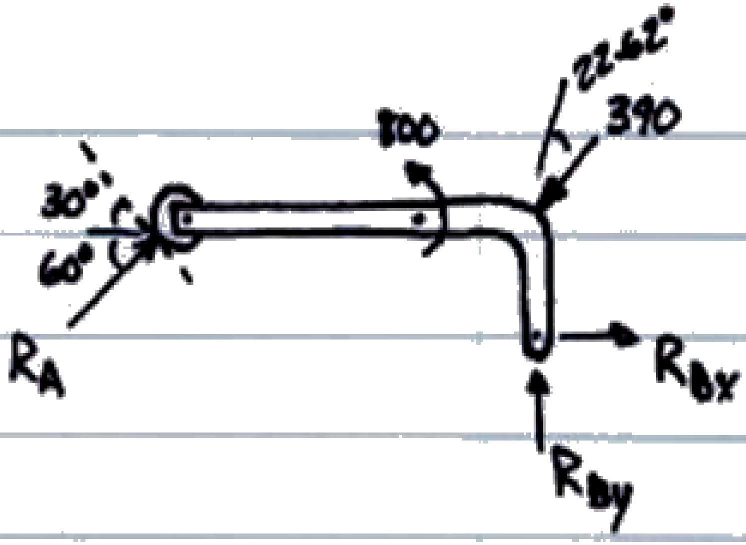


(b)



(a) (b) (c)





$$R_A \cos 60 + R_{Bx} - 390 \sin 22.62 = 0$$

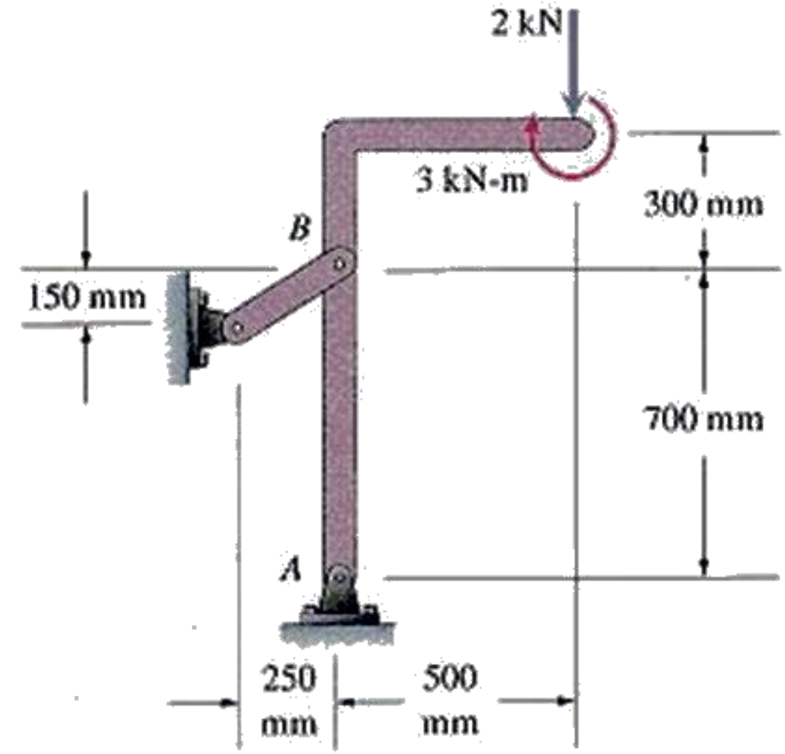
$$R_A \sin 60 + R_{By} - 390 \cos 22.62 = 0$$

$$800 + 3(390)\sin 22.62 - R_A(3\cos 60 + 12 \sin 60) = 0$$

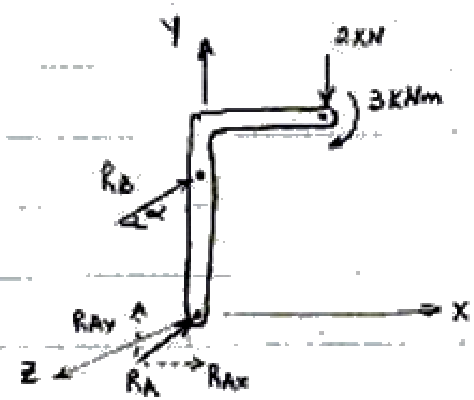
$R_A = 105.1$ lb

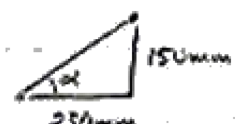
$R_{Bx} = 97.45$ lb

$R_{By} = 269.0$ lb



FBD of L-shaped bar:





$\alpha = \tan^{-1}\left(\frac{150mm}{250mm}\right)$

$= 30.96°$

$\xrightarrow{+} \Sigma F_x = 0 = R_{Ax} + R_{Bx}$

$+\uparrow \Sigma F_y = 0 = R_{Ay} + R_{By} - 2KN$

$R_{Bx} = R_B \cos\alpha$

$R_{By} = R_B \sin\alpha$

$+\circlearrowleft \Sigma M_A = 0 = -(.7m) R_{Bx} - (.5m)(2KN) - 3KNm$

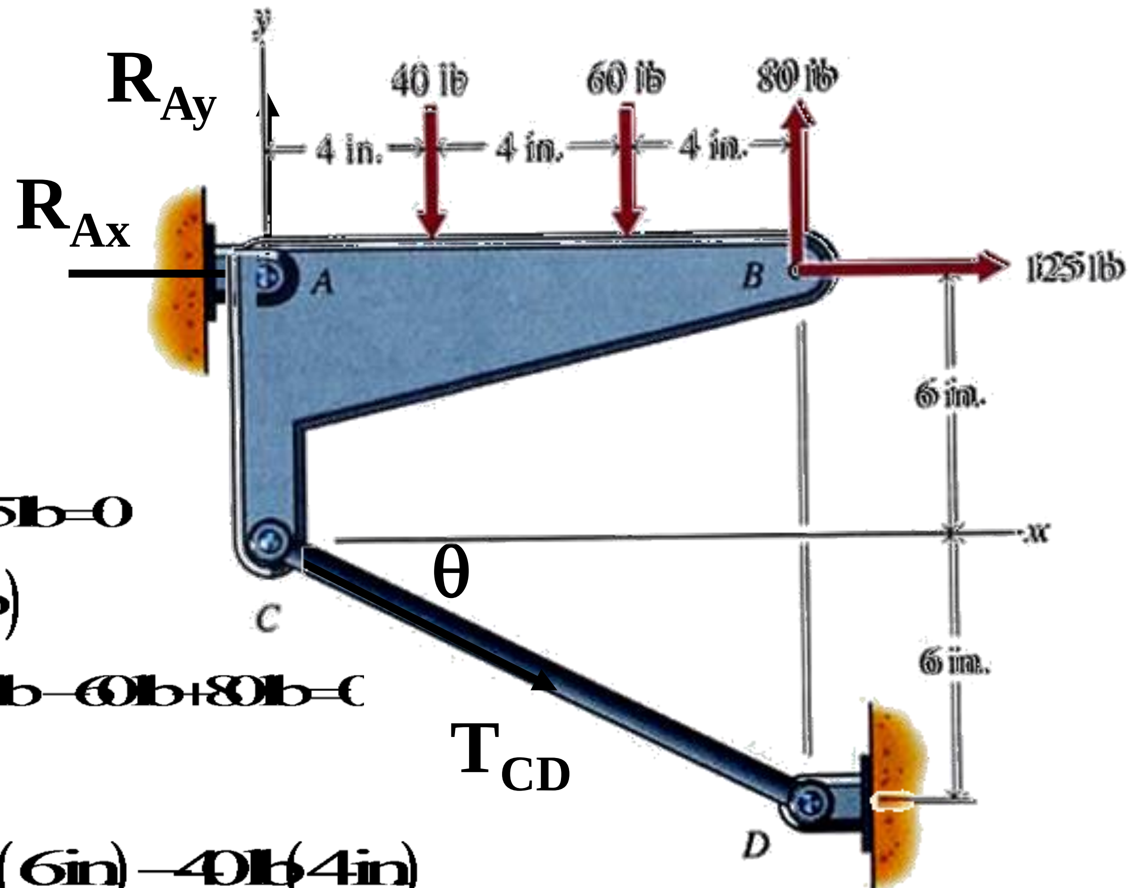
$= -(.7m) R_B \cos 30.96° - (.5m)(2KN) - 3KNm$

$R_B = -6.664\ KN$, $R_{Bx} = -5.714\ KN$, $R_{By} = -3.428\ KN$

$R_{Ax} = -R_{Bx} = 5.714 KN$, $R_{Ay} = 2KN - R_{By} = 5.428\ KN$, $R_A = 7.881\ KN$

Determine the reactions at A and the force in bar CD due to the loading.

$$\theta = \tan^{-1}\left(\frac{-6\,in}{12\,in}\right) = -26.565°$$



$$\sum F_x = R_{Ax} + T_{CD}\cos(-26.565°) + 125\,lb = 0$$

$$R_{Ax} = -\left(T_{CD}\cos(-26.565°) + 125\,lb\right)$$

$$\sum F_y = R_{Ay} + T_{CD}\sin(-26.565°) - 40\,lb - 60\,lb + 80\,lb = 0$$

$$R_{Ay} = -\left(T_{CD}\sin(-26.565°) - 20\,lb\right)$$

$$\sum M_A = 0 = T_{CD}\cos(-26.565°)(6\,in) - 40\,lb(4\,in)$$

$$-60\,lb(8\,in) + 80\,lb(12\,in)$$

$$T_{CD}\cos(-26.565°)(6\,in) = -320\,lb\cdot in$$

$$T_{CD} = -59.628\,lb$$

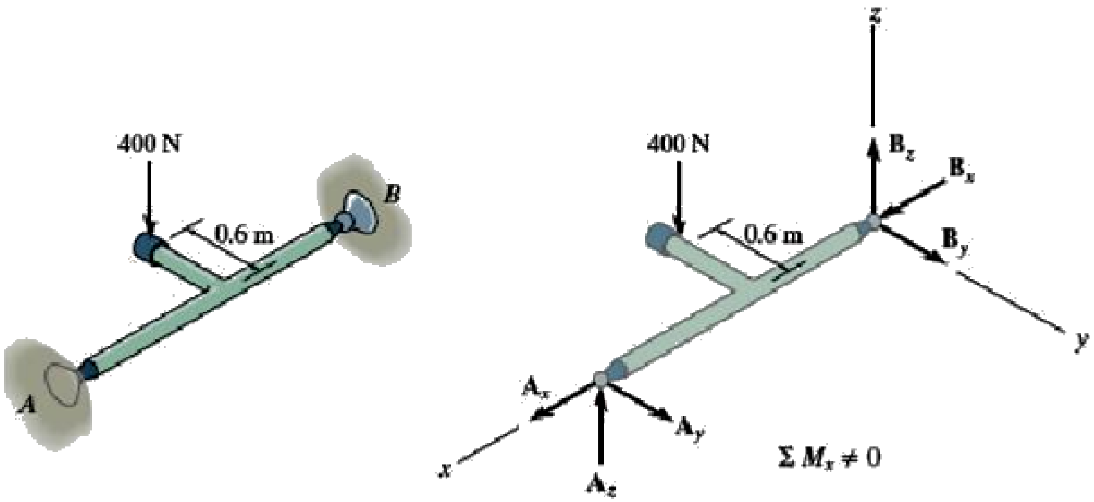
$$R_{Ax} = -\left((-59.628\,lb)\cos(-26.565°) + 125\,lb\right)$$

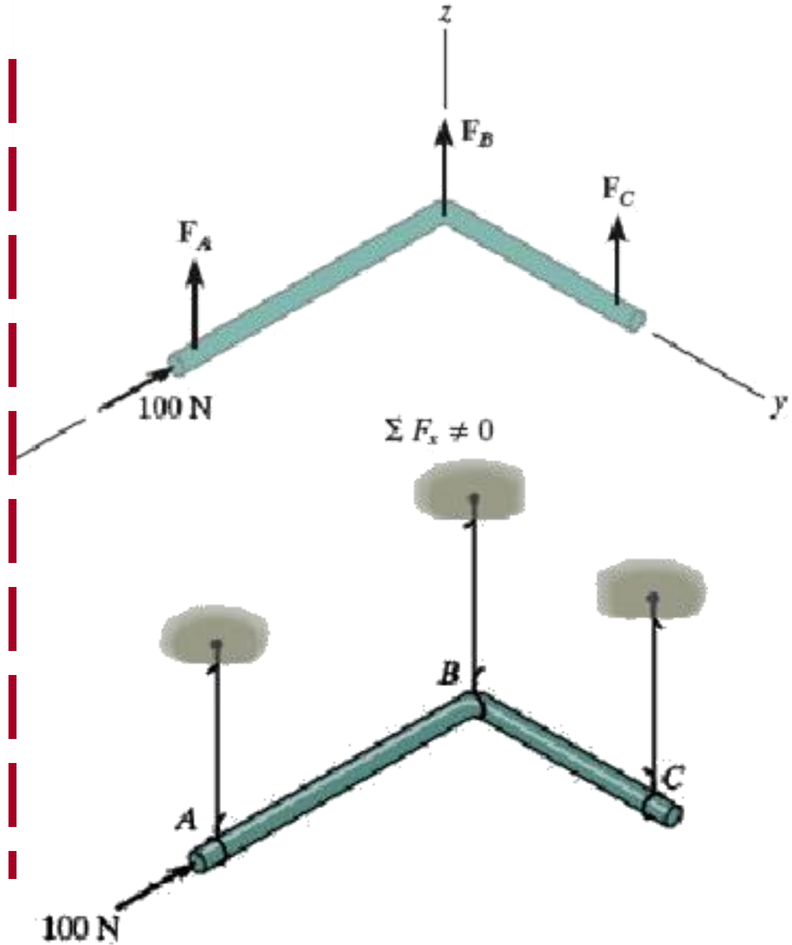
$$= -71.67\,lb$$

$$R_{Ay} = -\left((-59.628\,lb)\sin(-26.565°) - 20\,lb\right)$$

$$= -6.67\,lb$$

Improper Constraints





"**Properly constrained**" **means that**

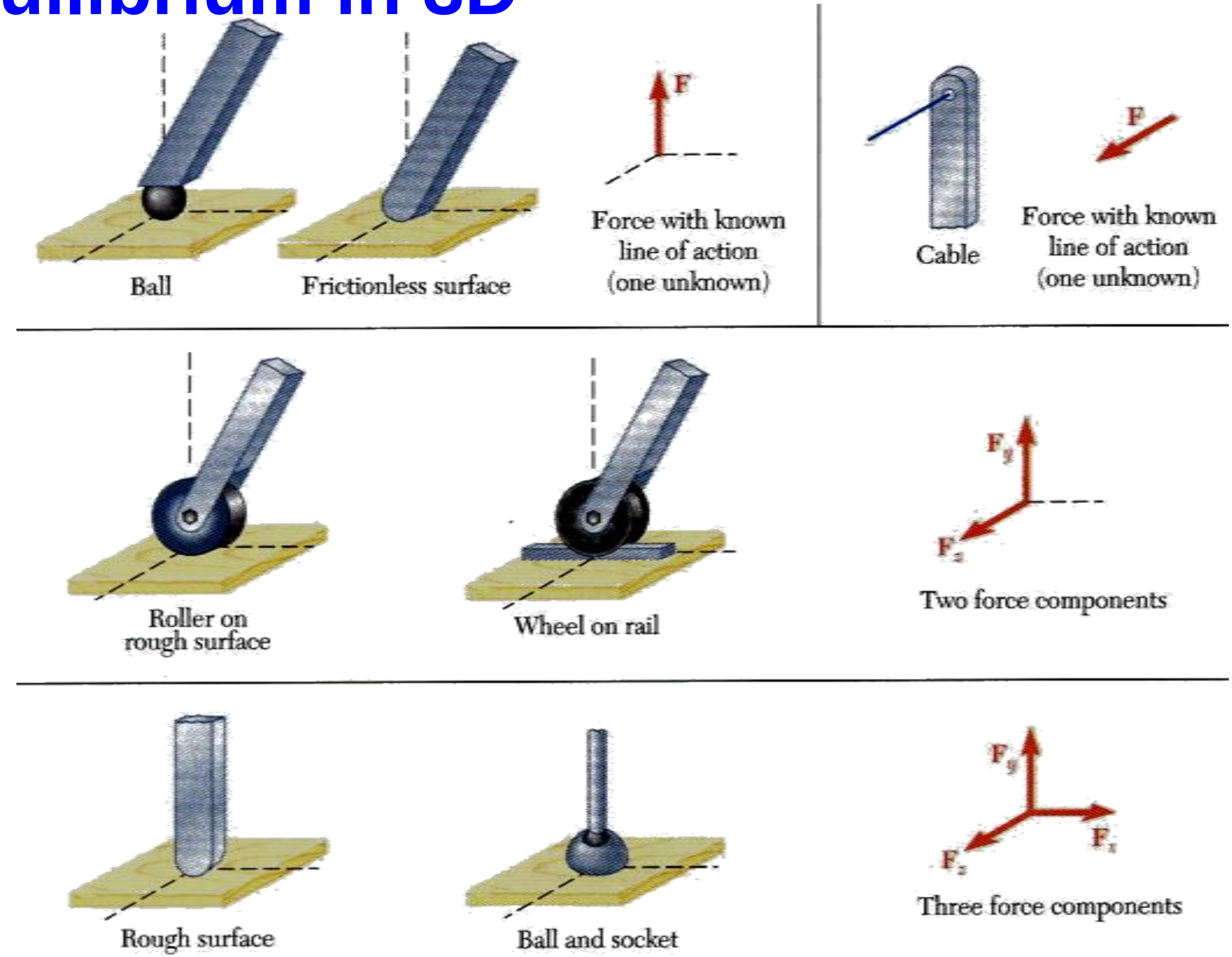
- **the supports can theoretically maintain equilibrium regardless of what forces and moments are applied**

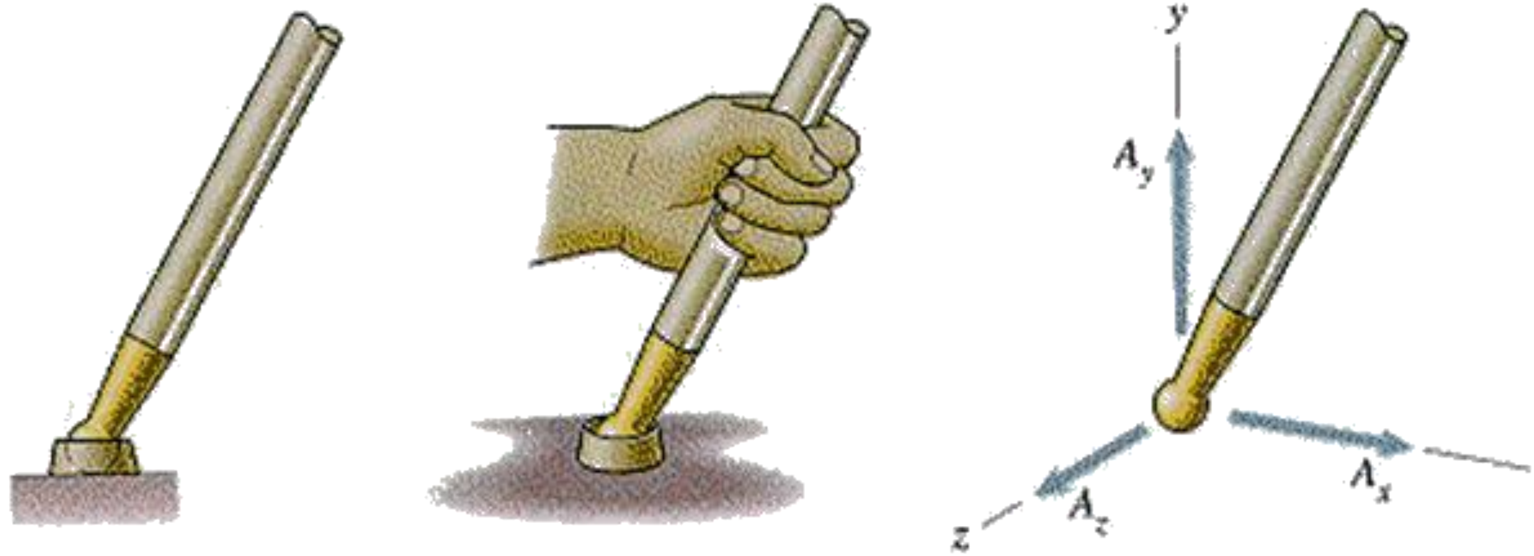
To "properly constrain" a body, if the only support reactions are forces (no moments):

- **reaction forces must not all intersect a common axis**
- **reaction forces must not all be parallel**

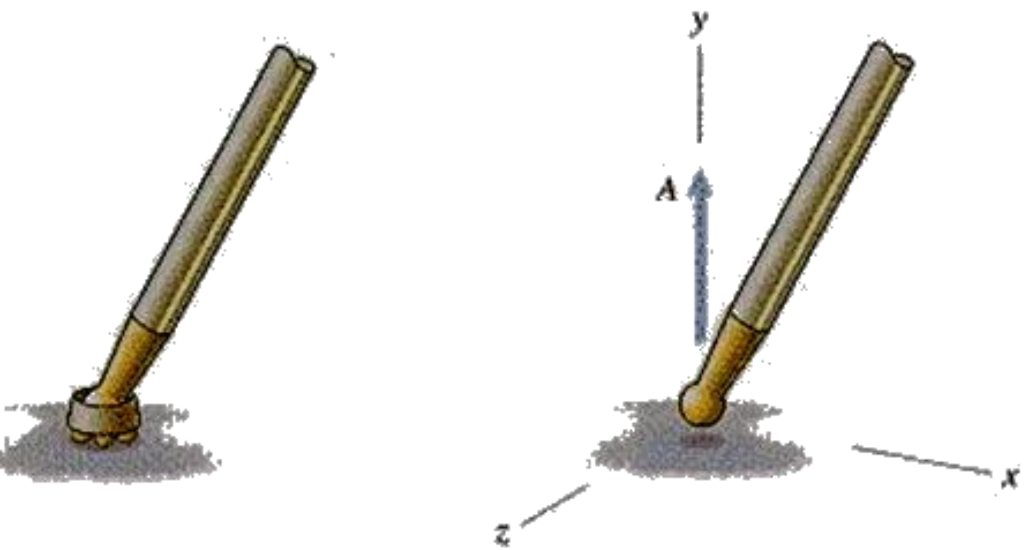
Equilibrium in 3D

Reactions



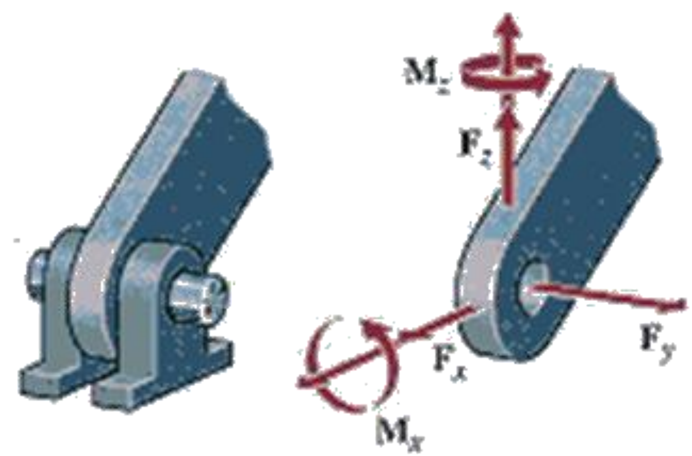


(a) (b) (c)

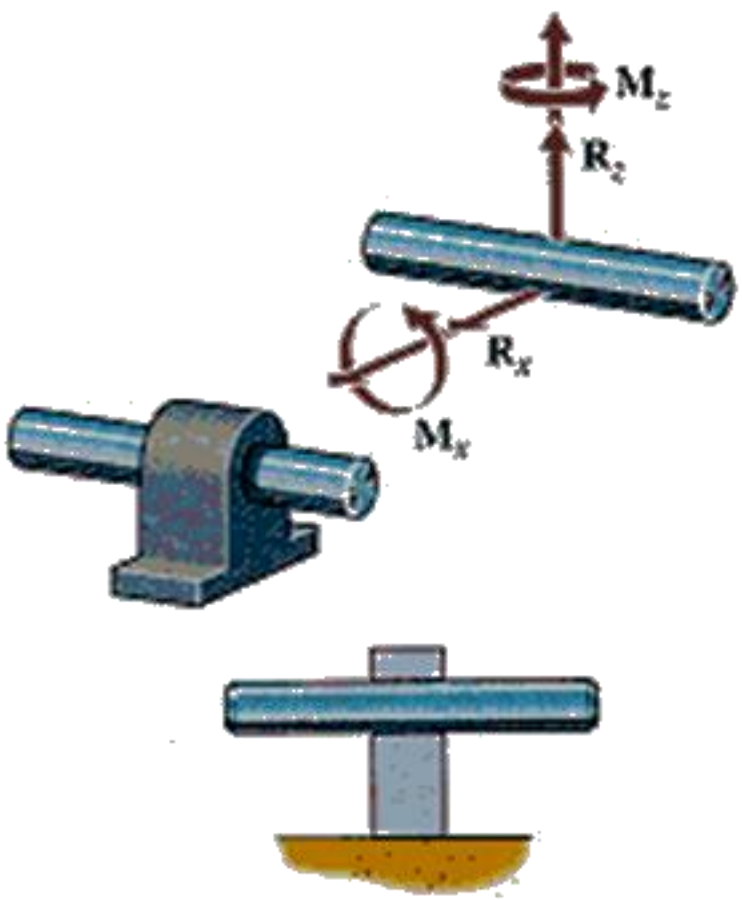


(a) (b)

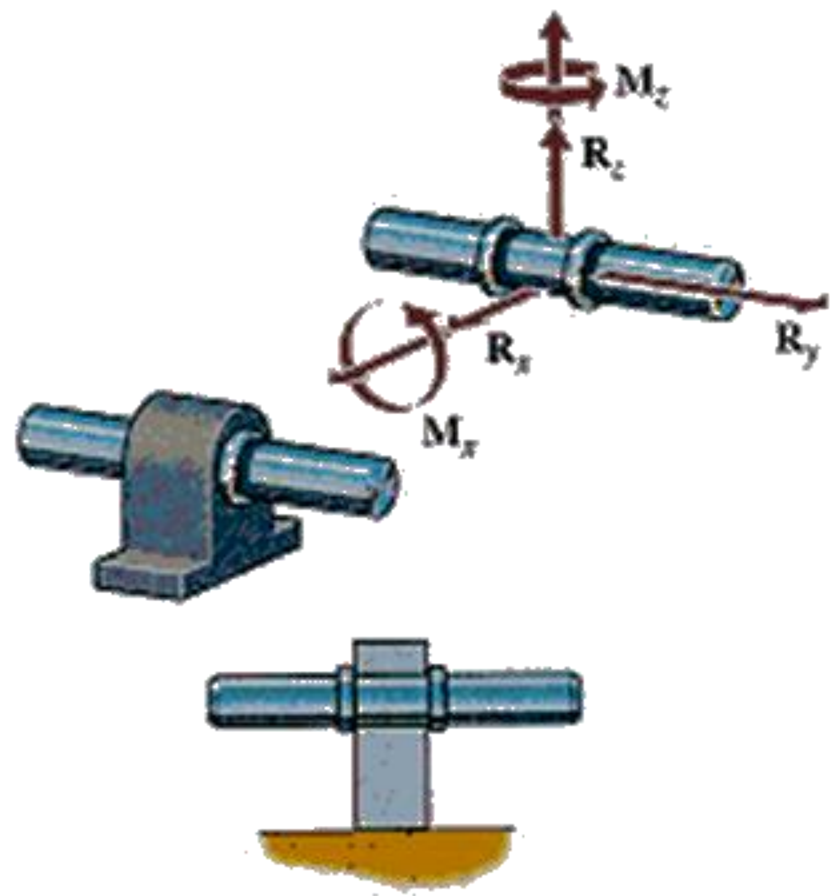
Smooth pin and bracket

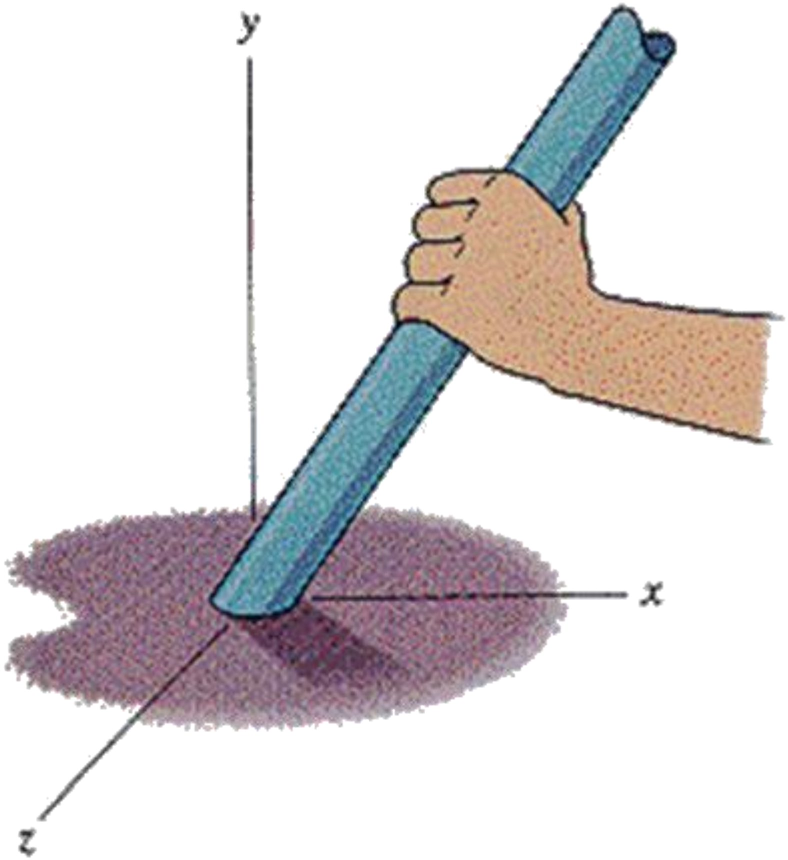


Journal bearing



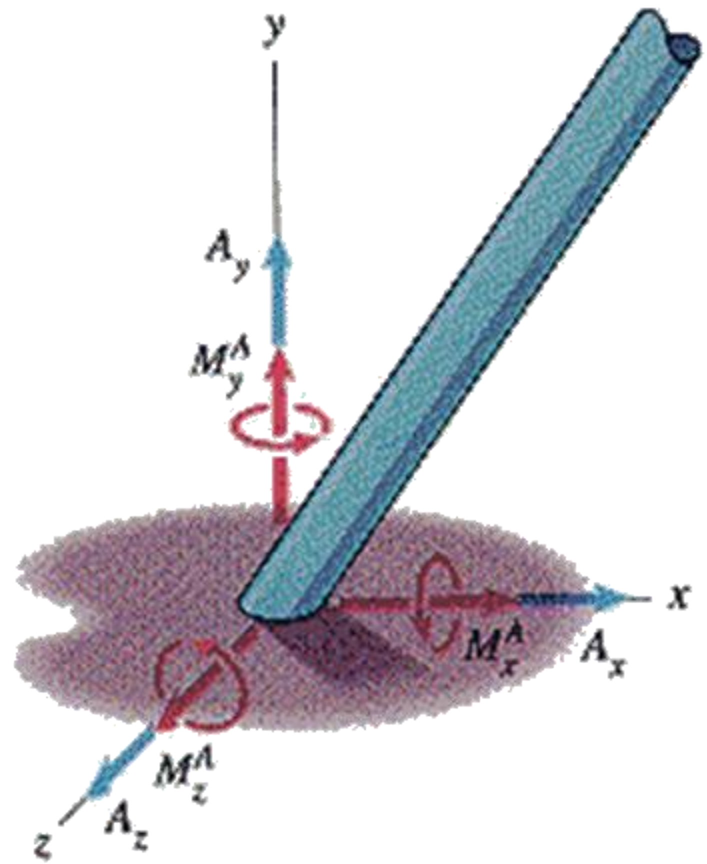
Thrust bearing



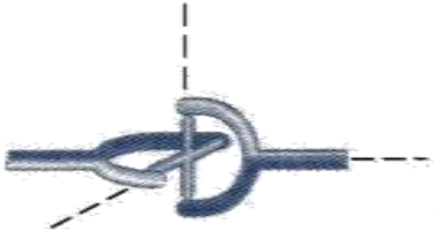


(b)

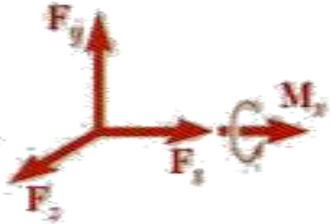
(b) Holding a supported bar.
(c) A built-in support can exert six reactions: three force components and three couple components.



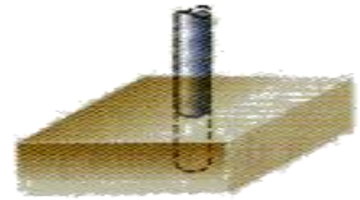
(c)



Universal joint

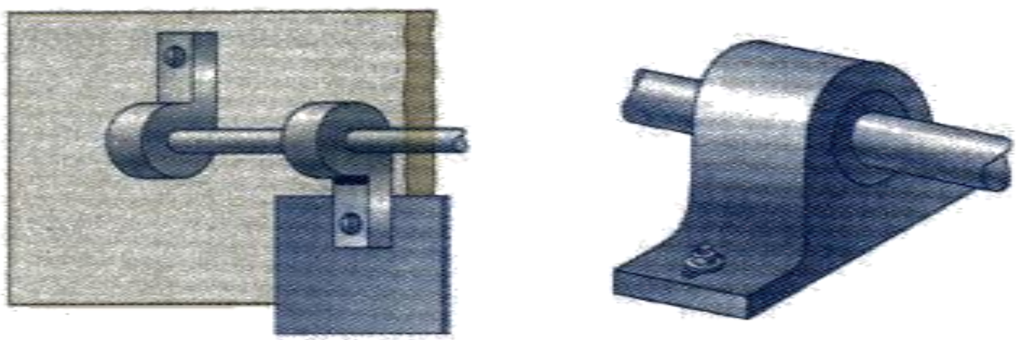


Three force components and one couple

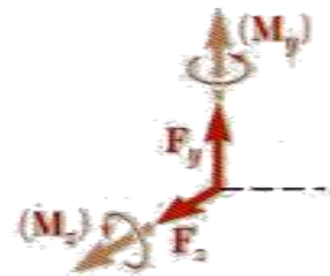


Fixed support

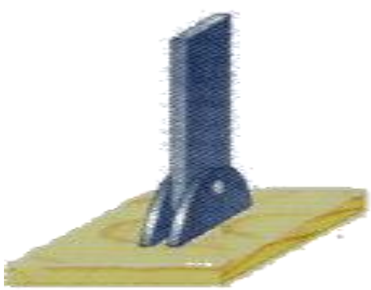
Three force components and three couples



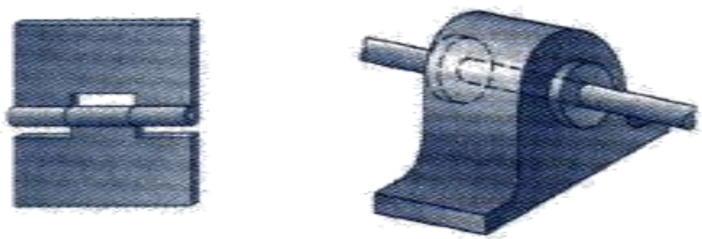
Hinge and bearing supporting radial load only



Two force components (and two couples)



Pin and bracket



Hinge and bearing supporting axial thrust and radial load

Three force components (and two couples)

Engineering Mechanics

Statics & Strength of Materials

Structural Analysis

Eng. Iqbal Marie

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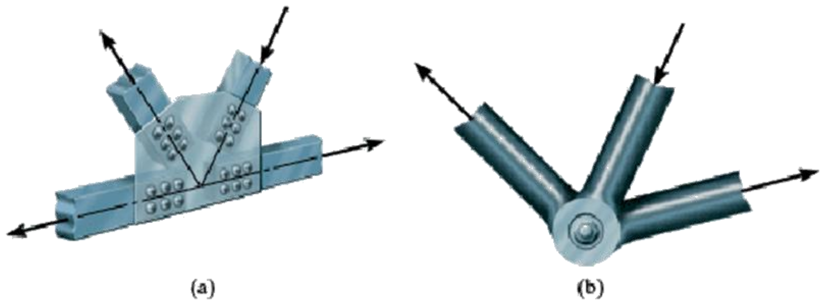
Simple Trusses

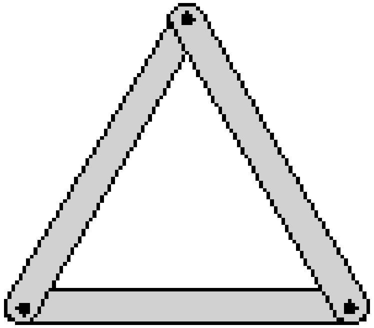
Simple Trusses truss

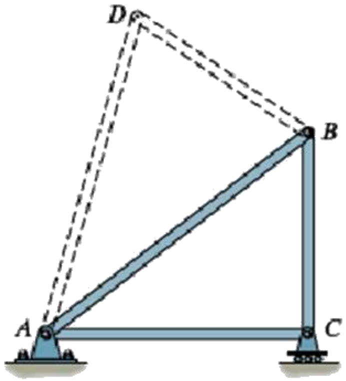
structure composed of straight, slender members joined at their endpoints

- joint connection can consist of pin through the ends of the members
- ends of members can bolted or welded to a gusset plate

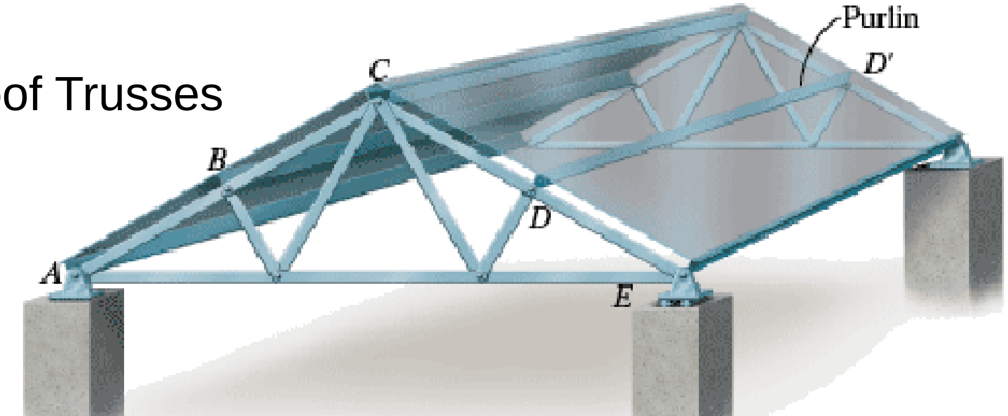


- Simplest shape that is rigid or stable is a triangle
- Trusses are generally built out of triangular elements



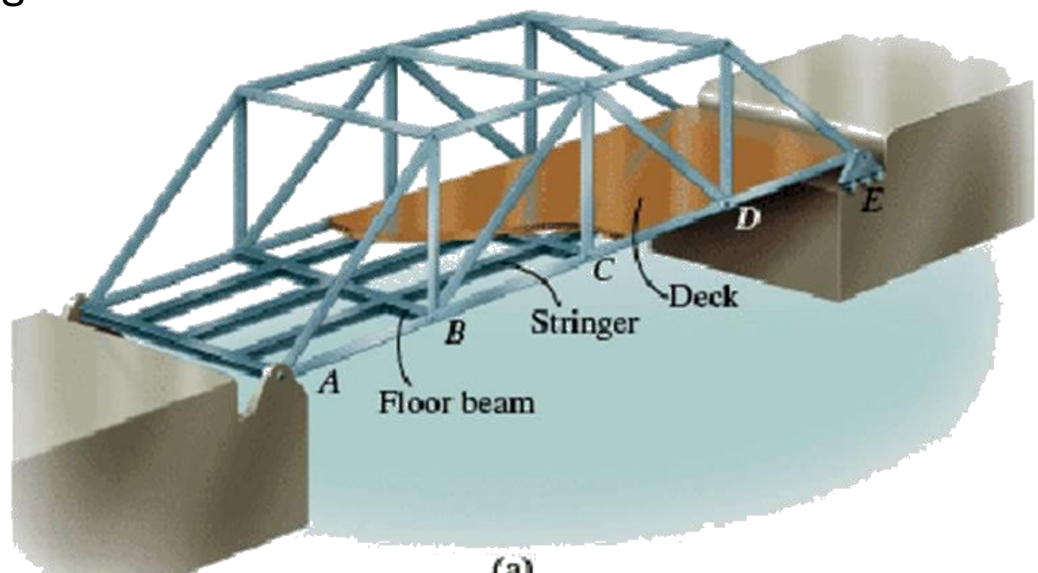


Roof Trusses

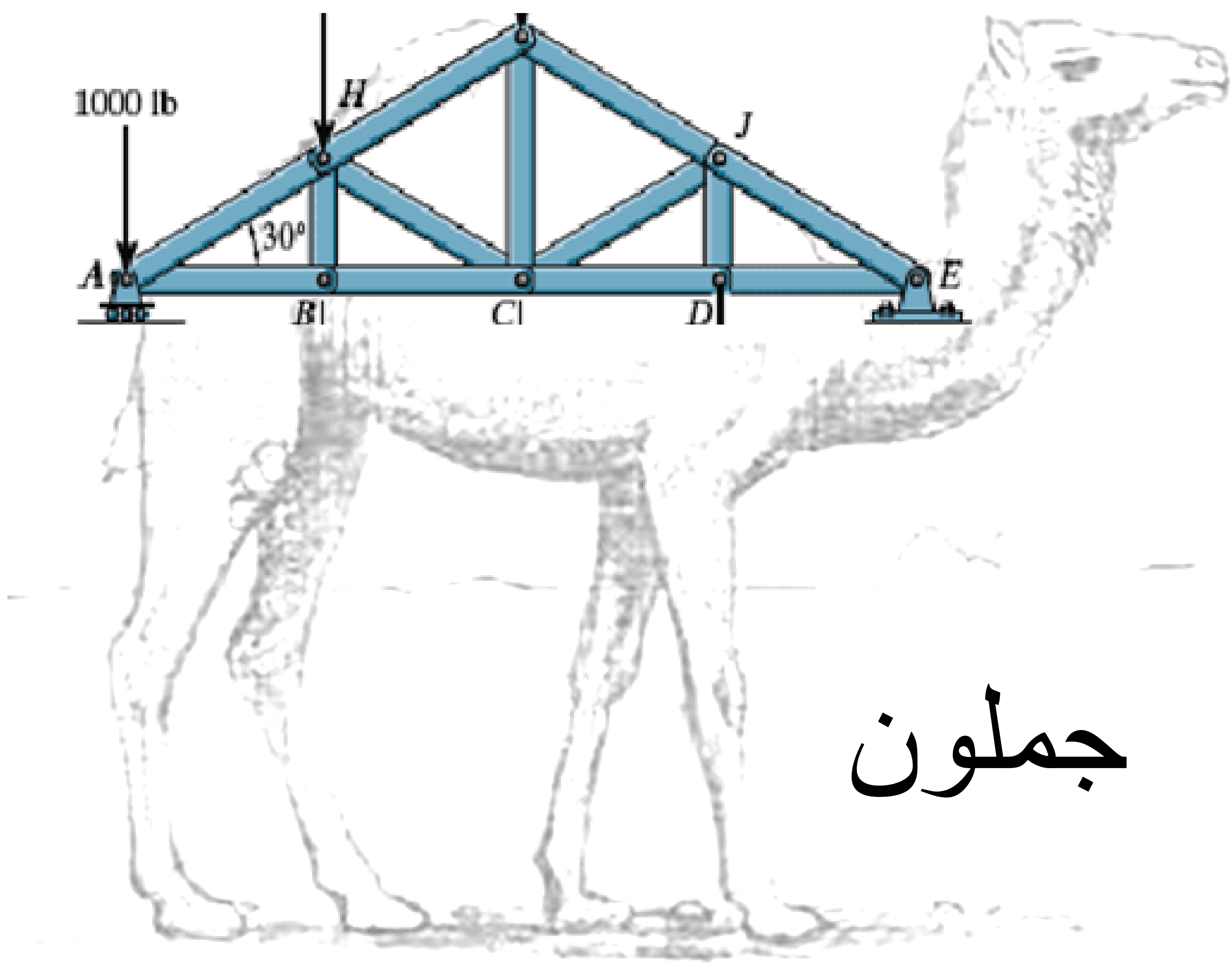


(a)

Bridge Trusses



(a)



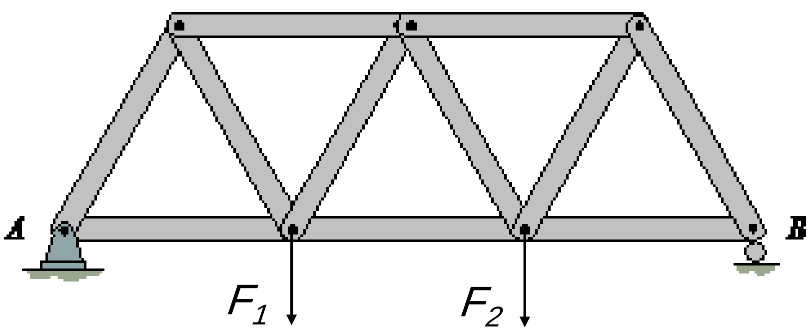
جملون

Truss analysis

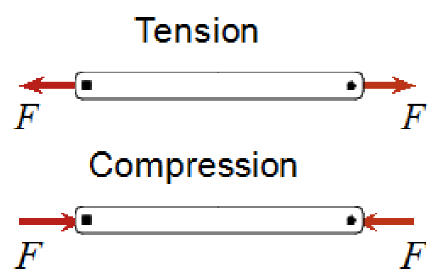
Truss analysis consists of finding forces in individual members when a truss is subject to a given loading.

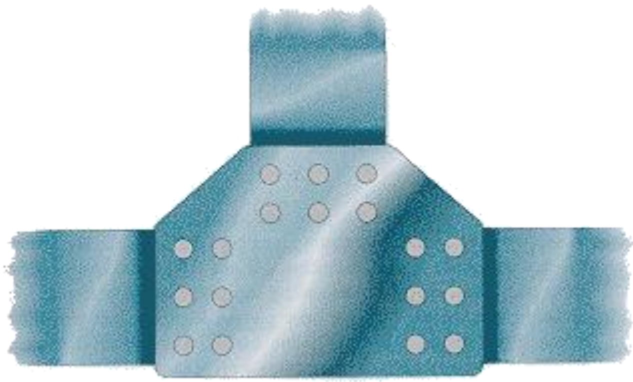
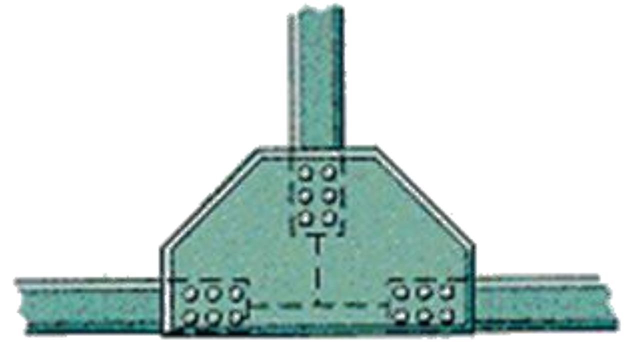
Assumptions

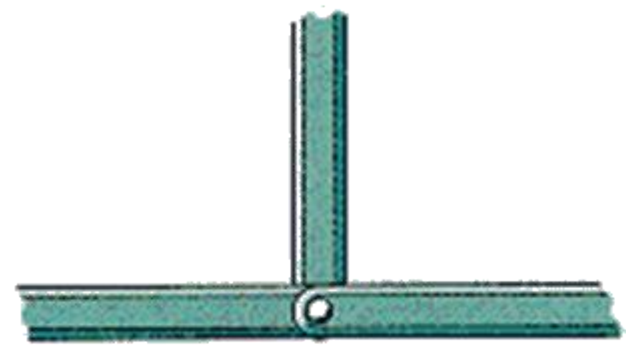
- **Trusses are composed of two force members**
- **Members are pin connected**
- **Loads are applied at joints**
  - **Weight of member usually neglected**
  - **If not neglected, typically split & applied at each joint**



- Truss members are two force members
  - Either in *tension* (T) or *compression* (C)





Truss Analysis Methods

- Two types of analysis

1. Method of joints: used when axial forces in all members are desired

2. Method of sections: used when axial forces in only a few members are desired

6.2 Method of Joints

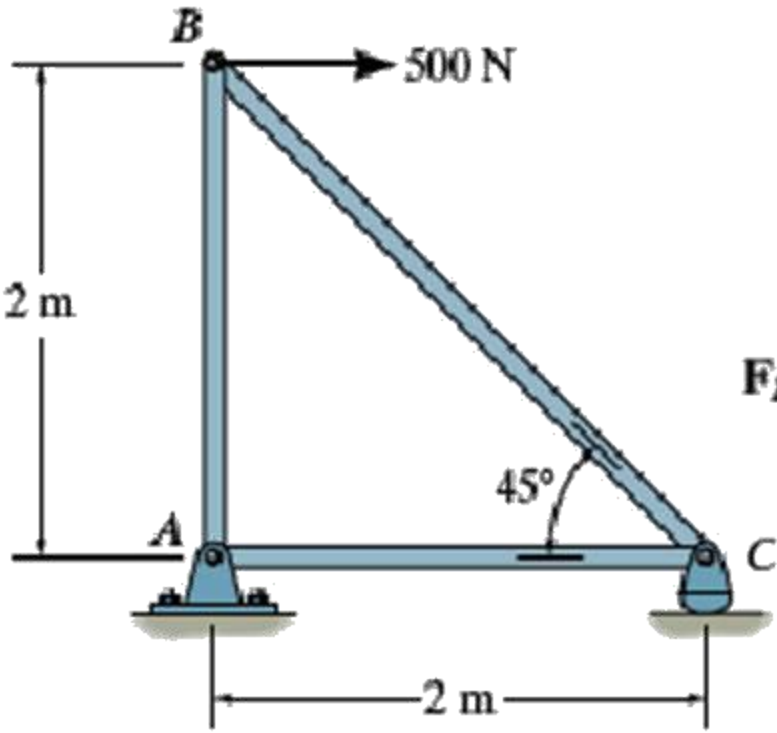
If a truss is in equilibrium, then all parts of the truss are in equilibrium. • Every joint (assume a pin joint) is in equilibrium.
• Every pin is acted on by external forces, support reactions, or forces from two-force members
• Draw FBD of each pin
• Use 2D particle equilibrium equations to solve for unknown tension or compression forces

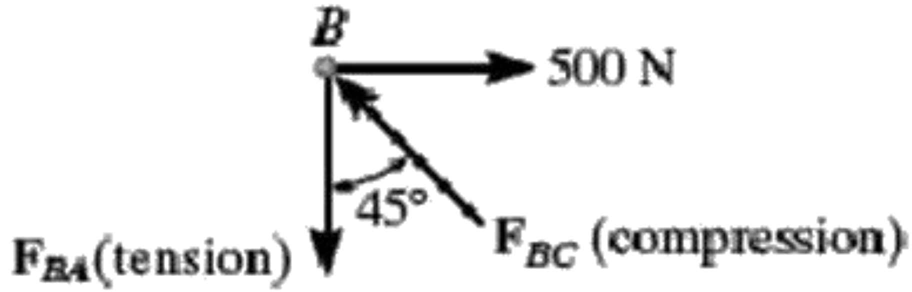
Procedure for Analysis

1. **Determine support reactions**
2. **Draw a FBD**
3. **Write equilibrium equations and solve**

2. Method of joints

- **For each joint:**
  - **Draw a FBD**
  - **Write equilibrium equations (x and y components) and solve**



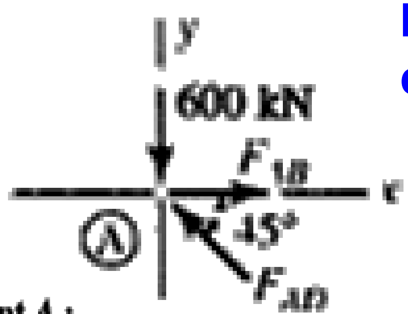


Particle equilibrium conditions:

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$

Tension vs. Compression members in compression "push" on the pin, members in tension "pull" on the pin

Find the force in each member and indicate if it is tension of compression



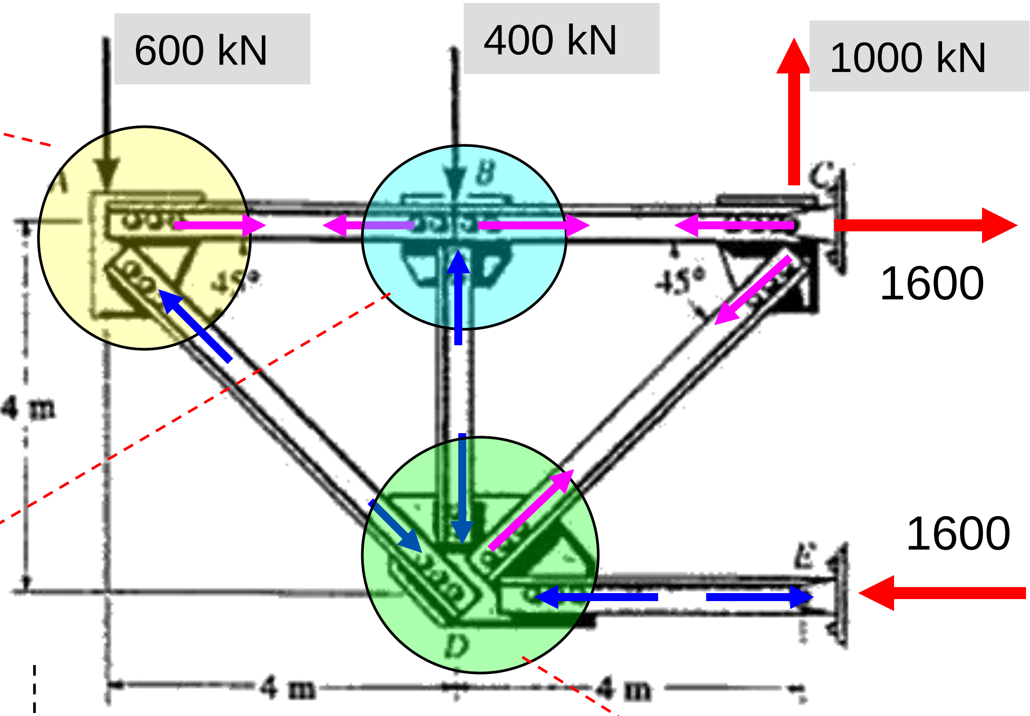
Joint *A* :

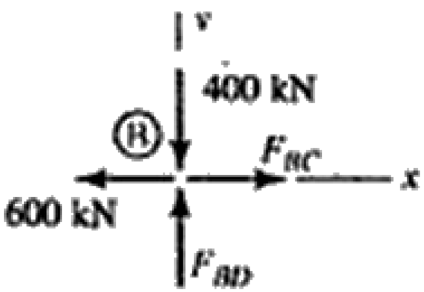
$+\uparrow \Sigma F_y = 0; \quad F_{AD} \sin 45° - 600 = 0$

$F_{AD} = 848.528 = 849 \text{ kN (C)}$

$\xrightarrow{+} \Sigma F_x = 0; \quad F_{AB} - 848.528 \cos 45° = 0$

$F_{AB} = 600 \text{ kN (T)}$





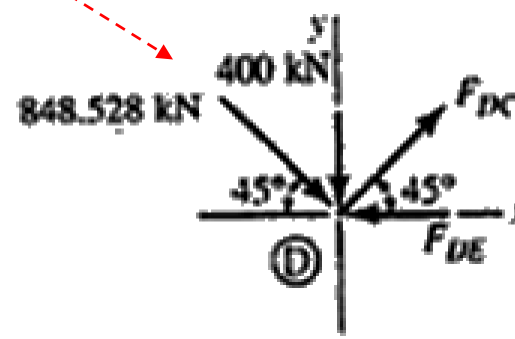
Joint *B* :

$+\uparrow \Sigma F_y = 0; \quad F_{BD} - 400 = 0$

$F_{BD} = 400 \text{ kN (C)}$

$\xrightarrow{+} \Sigma F_x = 0; \quad F_{BC} - 600 = 0$

$F_{BC} = 600 \text{ kN (T)}$



$+\uparrow \Sigma F_y = 0; \quad F_{DC} \sin 45° - 400 - 848.528 \sin 45° = 0$

$F_{DC} = 1414.214 \text{ kN} = 1.41 \text{ MN (T)}$

$\xrightarrow{+} \Sigma F_x = 0; \quad 848.528 \cos 45° + 1414.214 \cos 45° - F_{DE} = 0$

$F_{DE} = 1600 \text{ kN} = 1.60 \text{ MN (C)}$

6.3 Zero-Force Members

Why are they there?

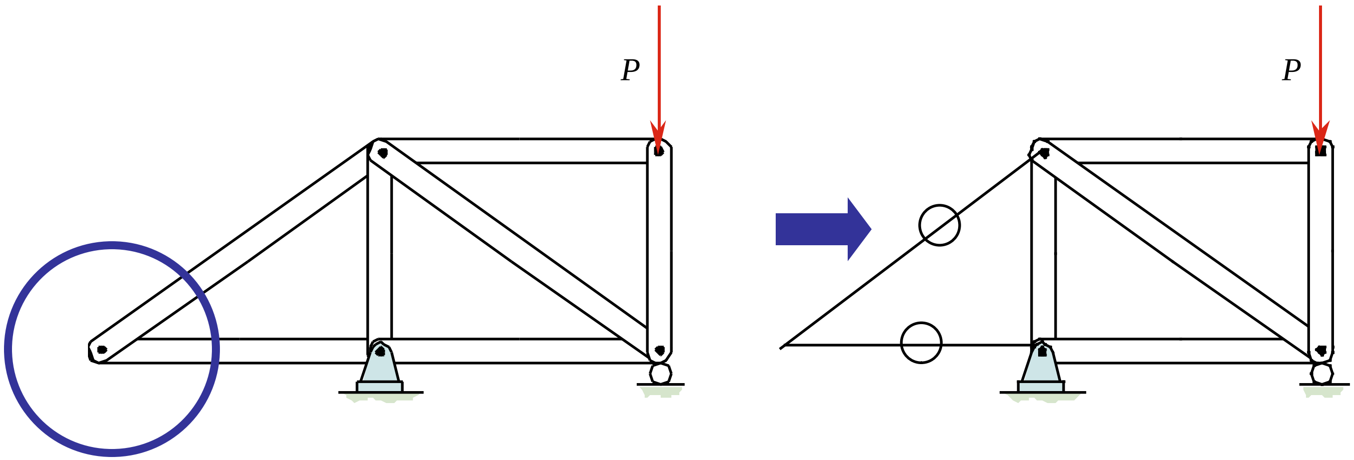
1. **sometimes zero-force members are added during construction of the truss to improve stability, and aren't removed afterwards**

2. **sometimes if the applied loading changes, they will no longer be zero-force members'**

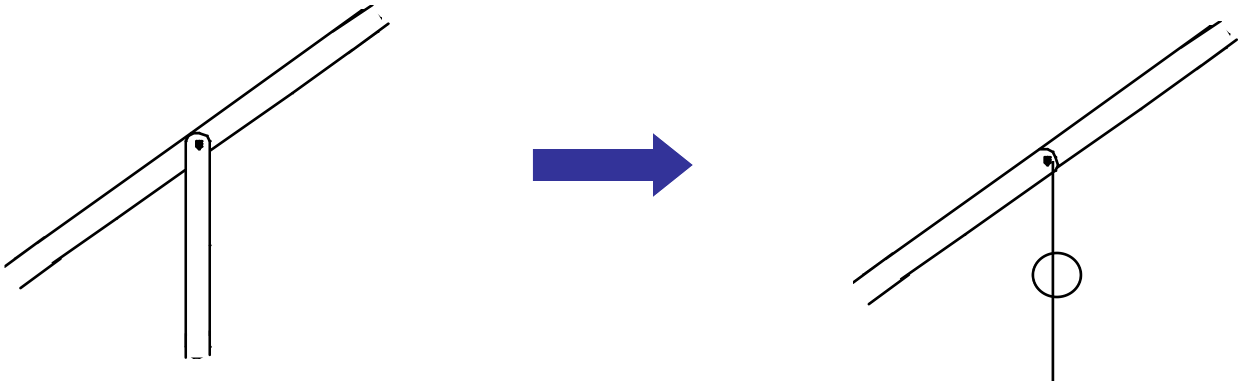
Two cases where you can tell if a member is a zero-force member.

Case 1 - pin joins two members and has no external load applied to it
Case 2 - pin joins three members, has no external load applied to it, and two of the members are collinear

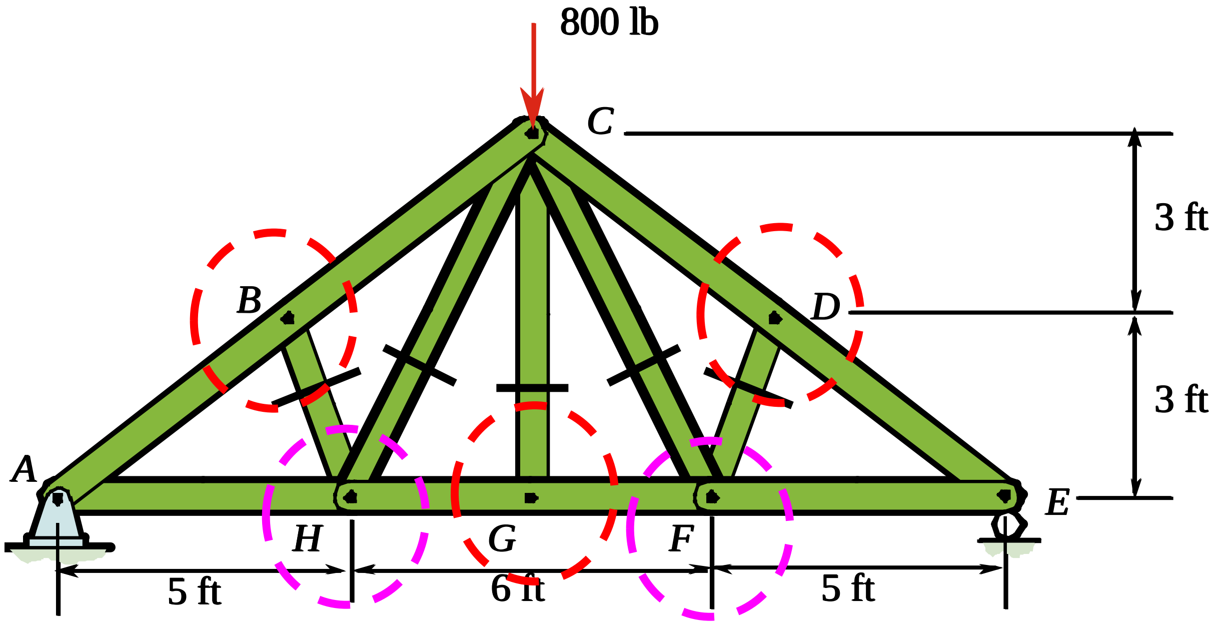
- Case 1: Two non-collinear members, no load at a pin
  - Both members are zero force members



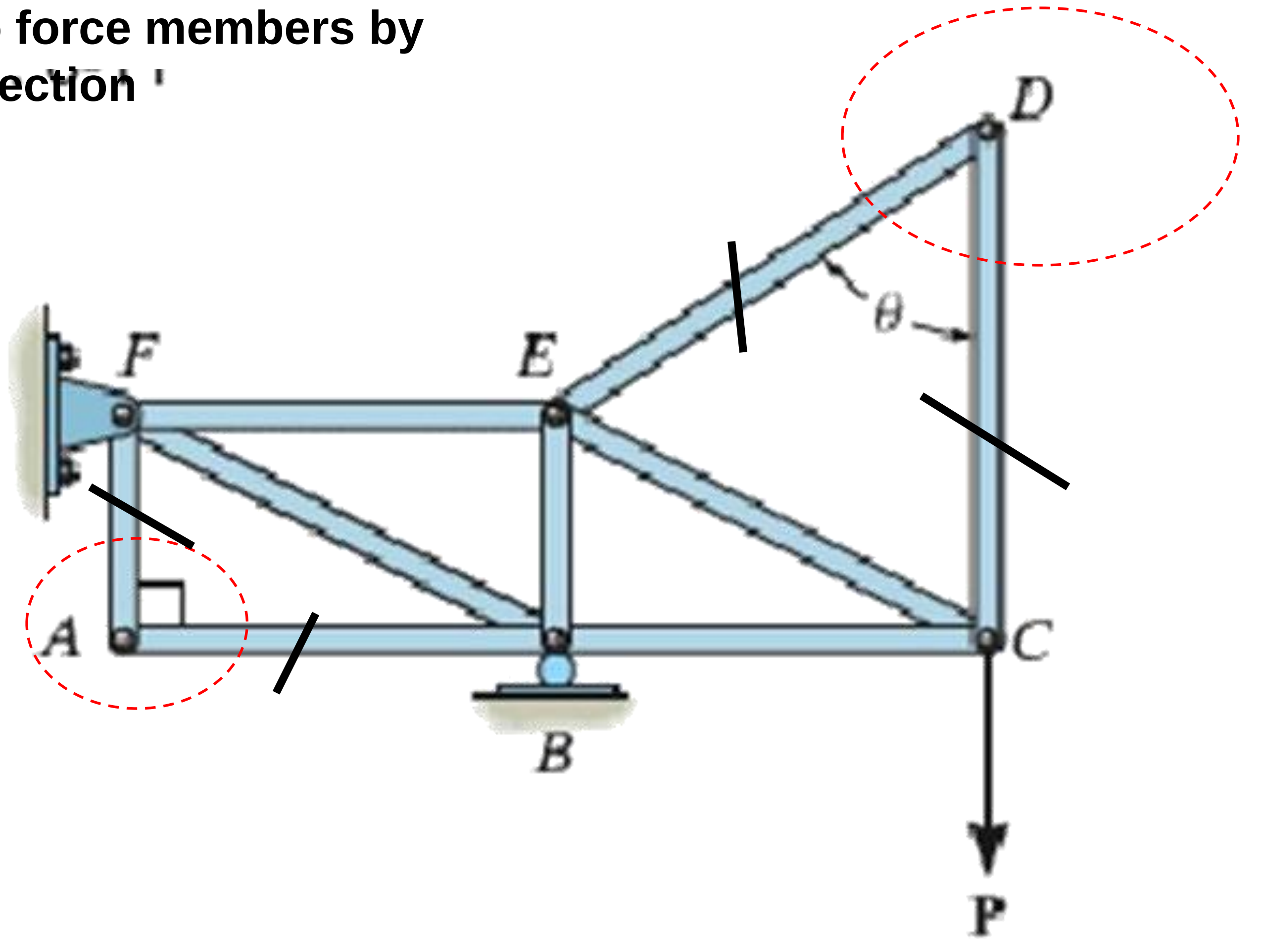
- Case 2: Three members connect at a pin, two members are collinear and no applied load
  - Non-collinear member is a zero force member



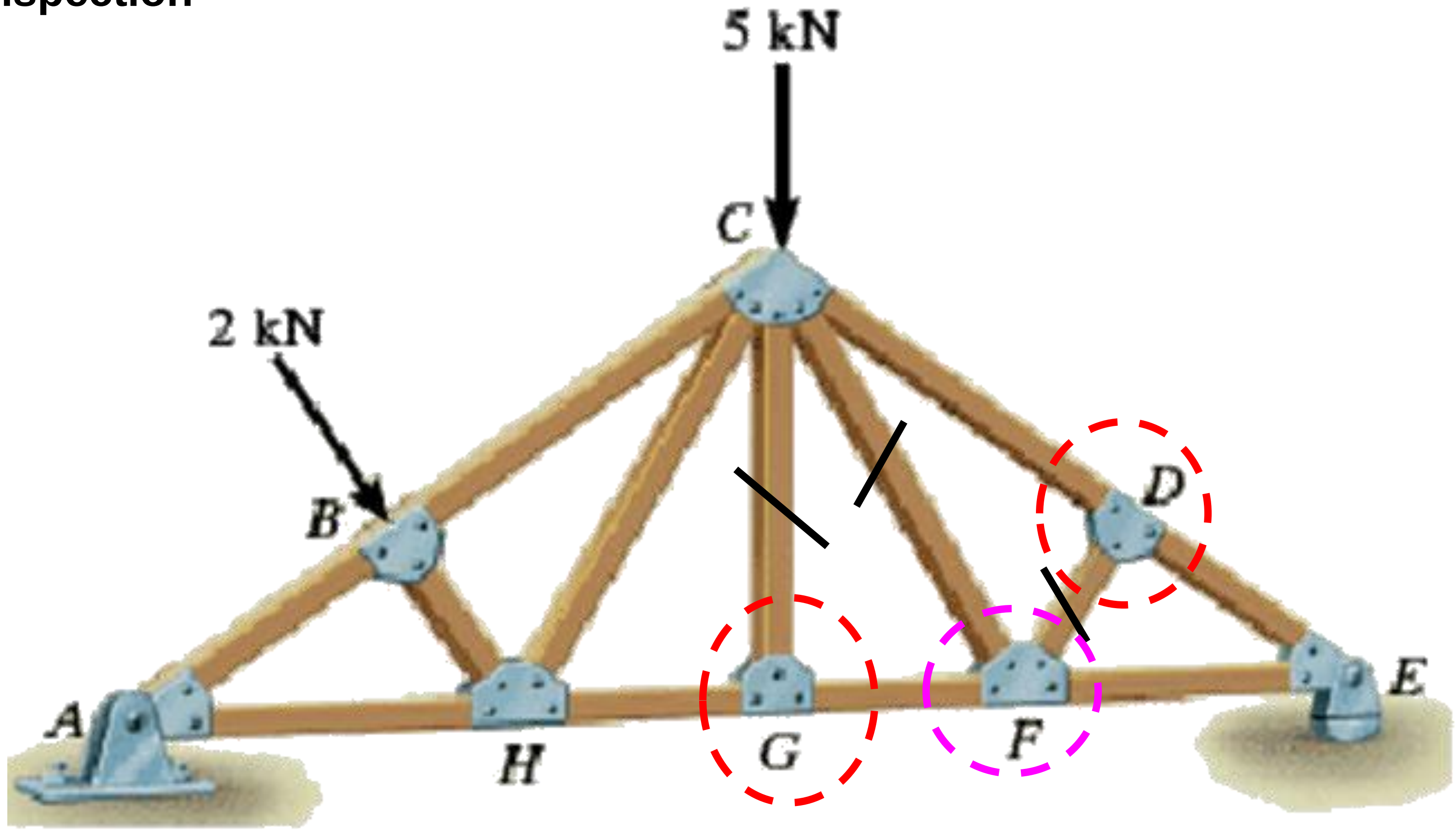
- Identify the zero force members



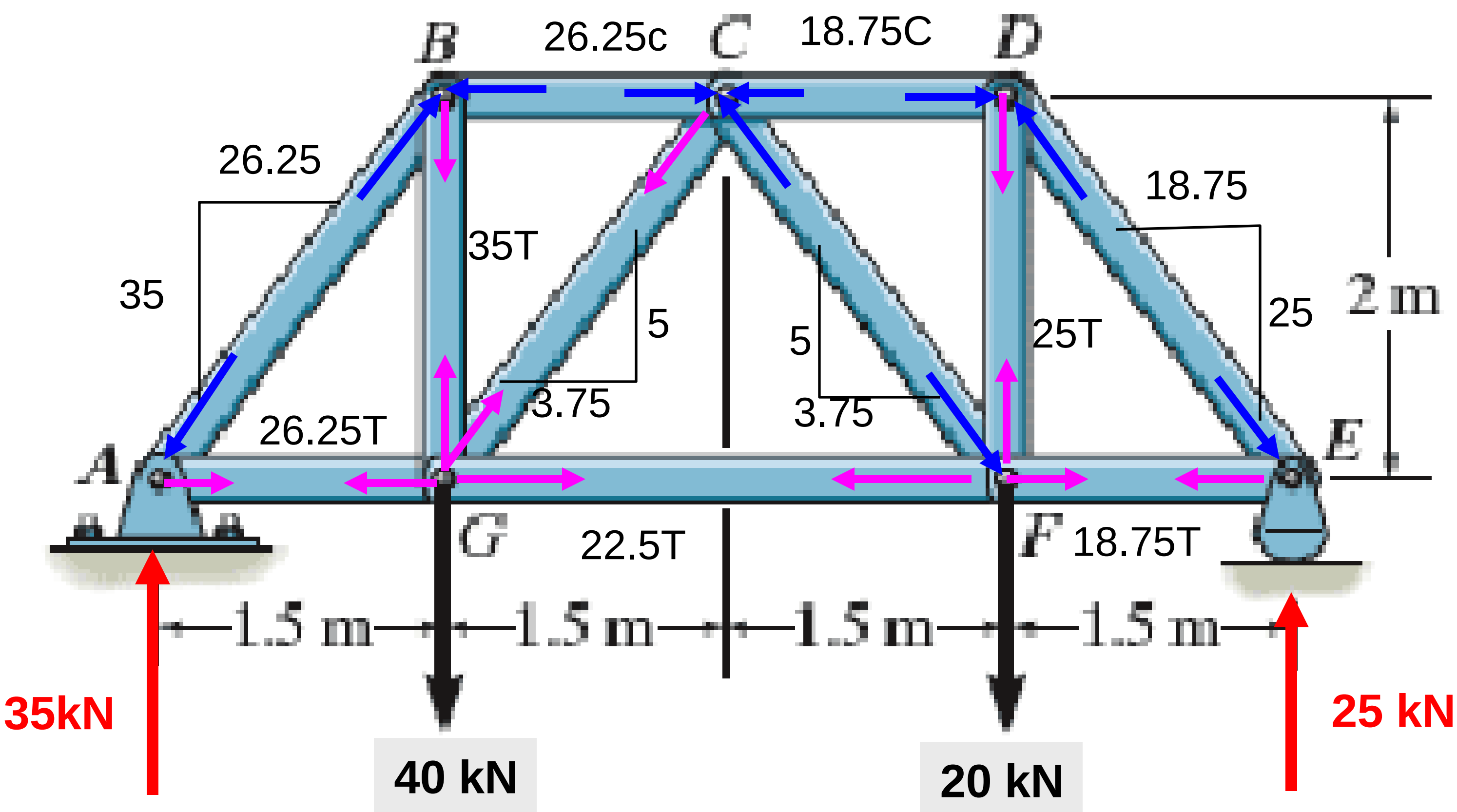
Zero force members by inspection



Zero force members by inspection

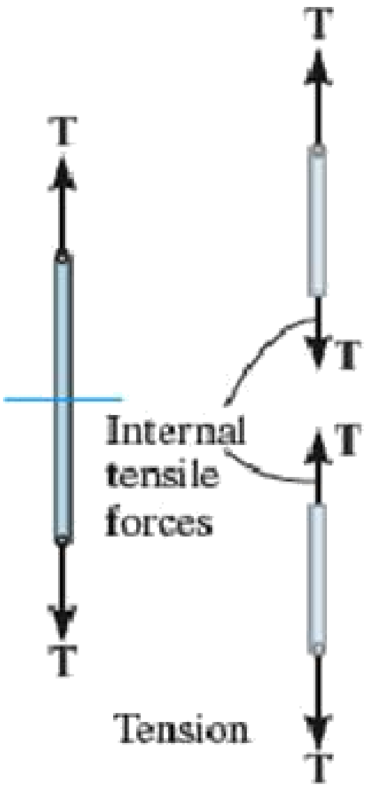


6–6. Determine the force in each member of the truss and state if the members are in tension or compression.



6.4 Method of Sections

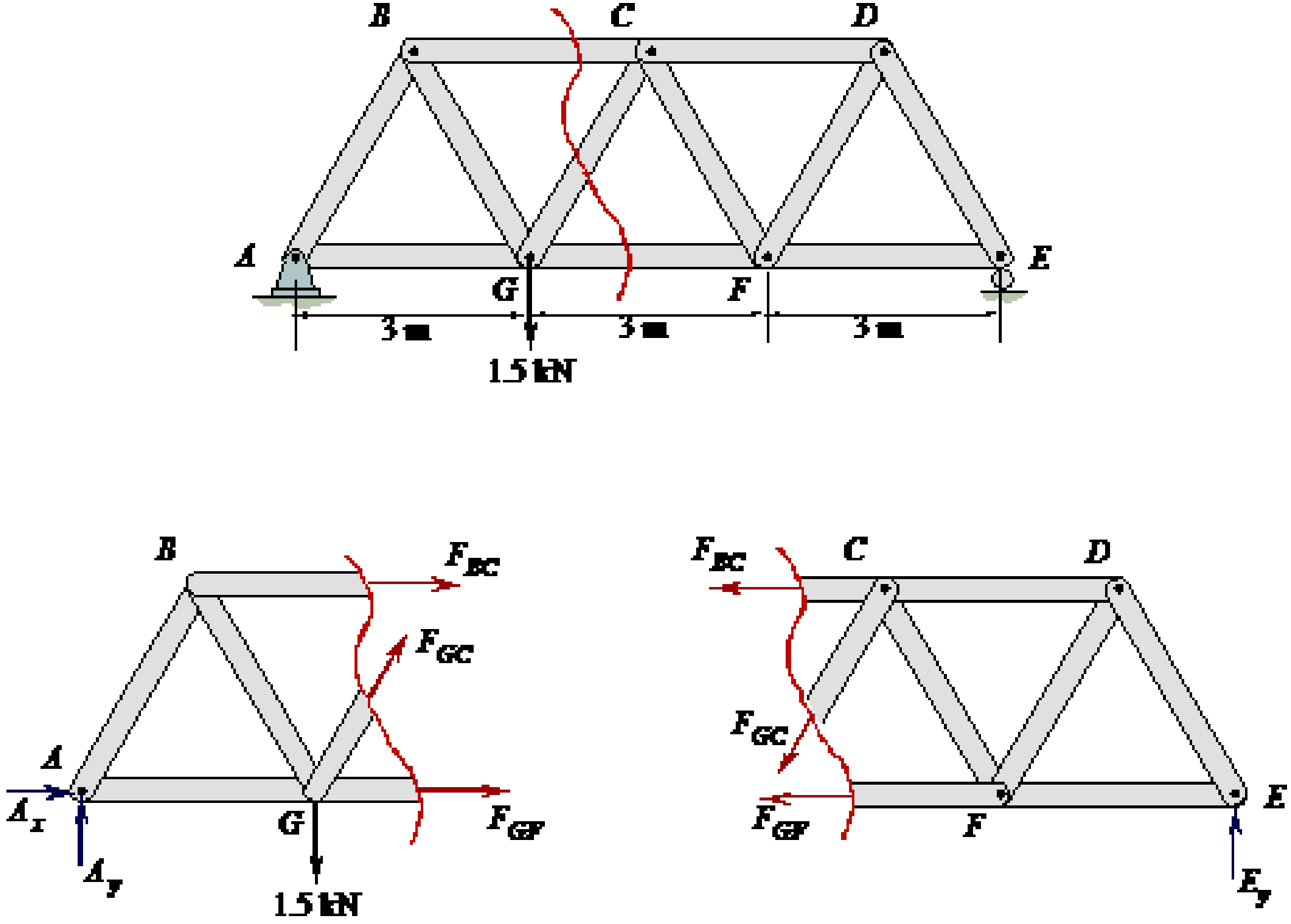
Method of Joints typically used to find forces in all members of a truss
Method of Sections typically used to find forces in a few members



Method of Sections:

1. **make a cut or "section" through the entire truss**
2. **section should divide truss into two parts**
3. **section should pass through no more than three members**
4. **for which forces are unknown (we have only 3 equilibrium equations, after all)**

- If a truss is in equilibrium, then *any sub-part* of the truss must be in equilibrium

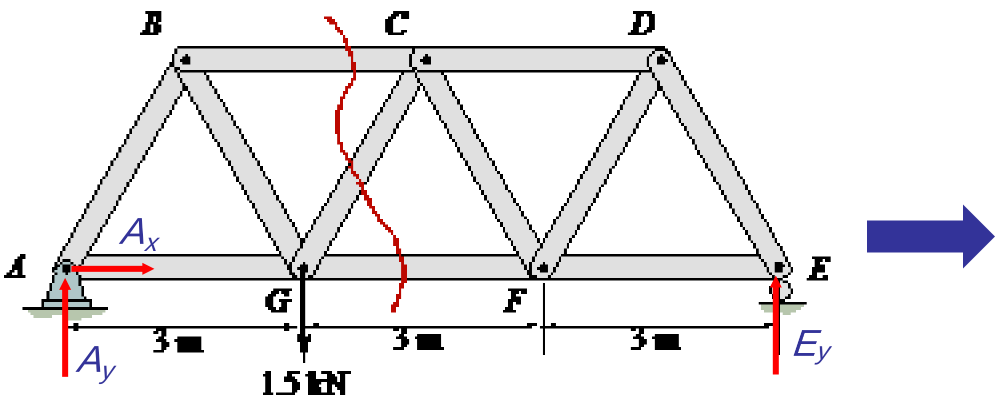


Procedure for Analysis

Determine support reactions (in general)

- **Draw a FBD**
- **Write equilibrium equations and solve**

- **Find forces in members *BC*, *GC* and *GF* and whether they are in (T) or (C)**



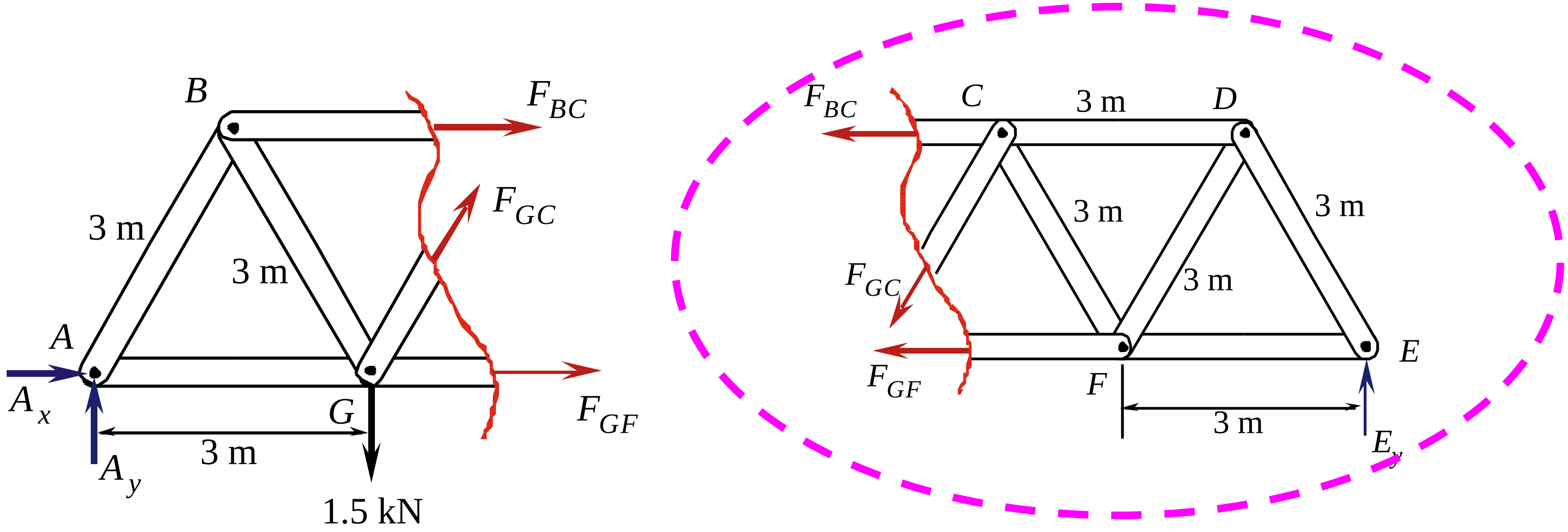
E_y = 0.5 kN↑

A_y = 1.0 kN↑

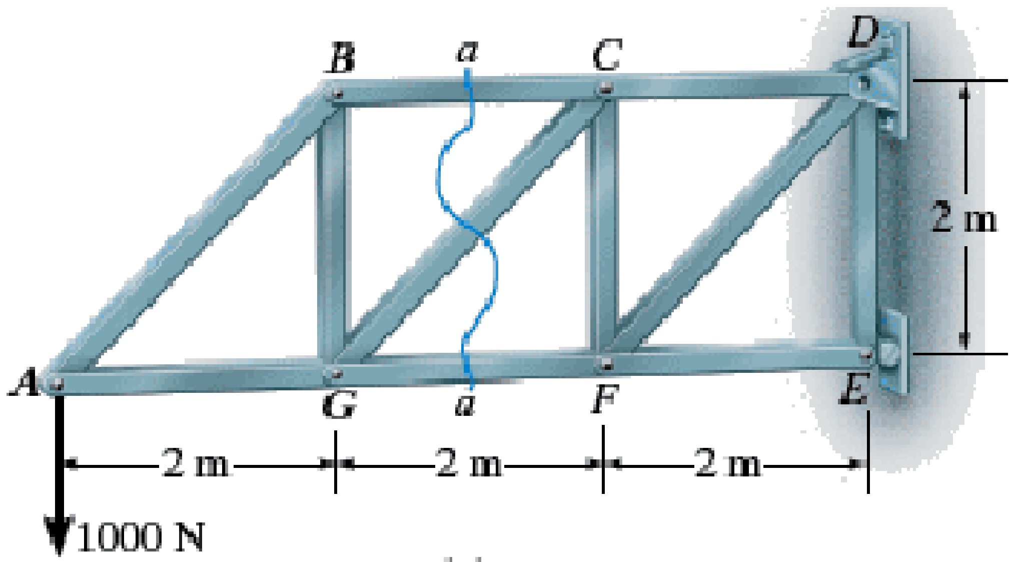
A_x = 0

Method of sections

– **Decide which part to analyze**
  - **Typically pick the one with the fewest number of forces**

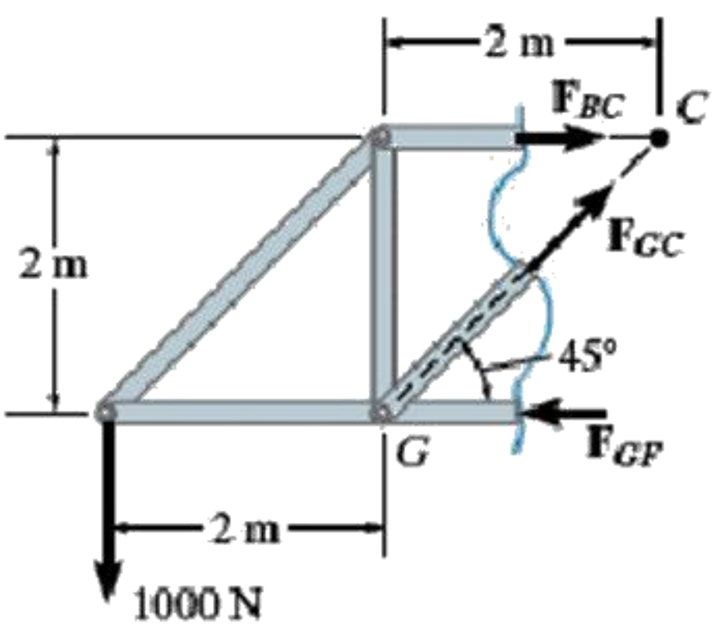


- **Draw FBD of the truss sub-part to be analyzed**
- **Write equilibrium equations to solve for unknown forces (at most three)**
- **Try to write equations with direct solutions, e.g. sum moments about a point with multiple forces acting on it**

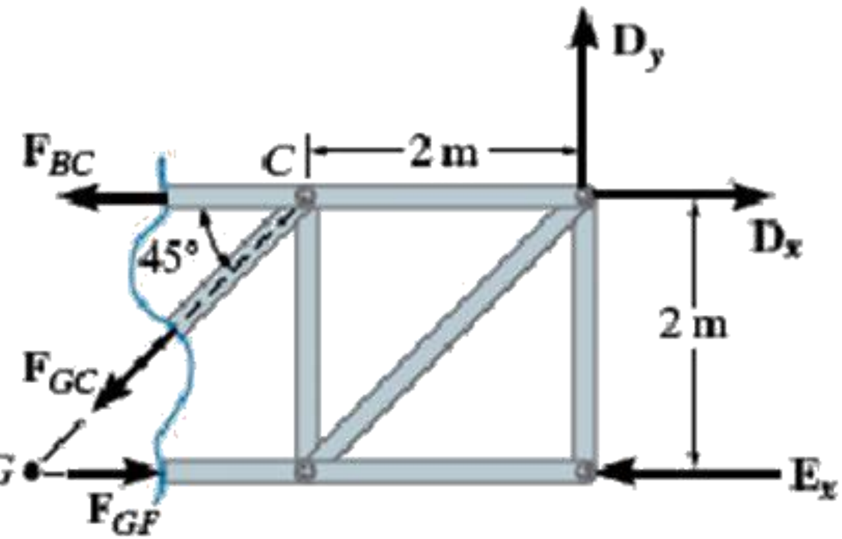


B
a
C
D
2 m
A
G
a
F
E
2 m
2 m
2 m
1000 N

(a)



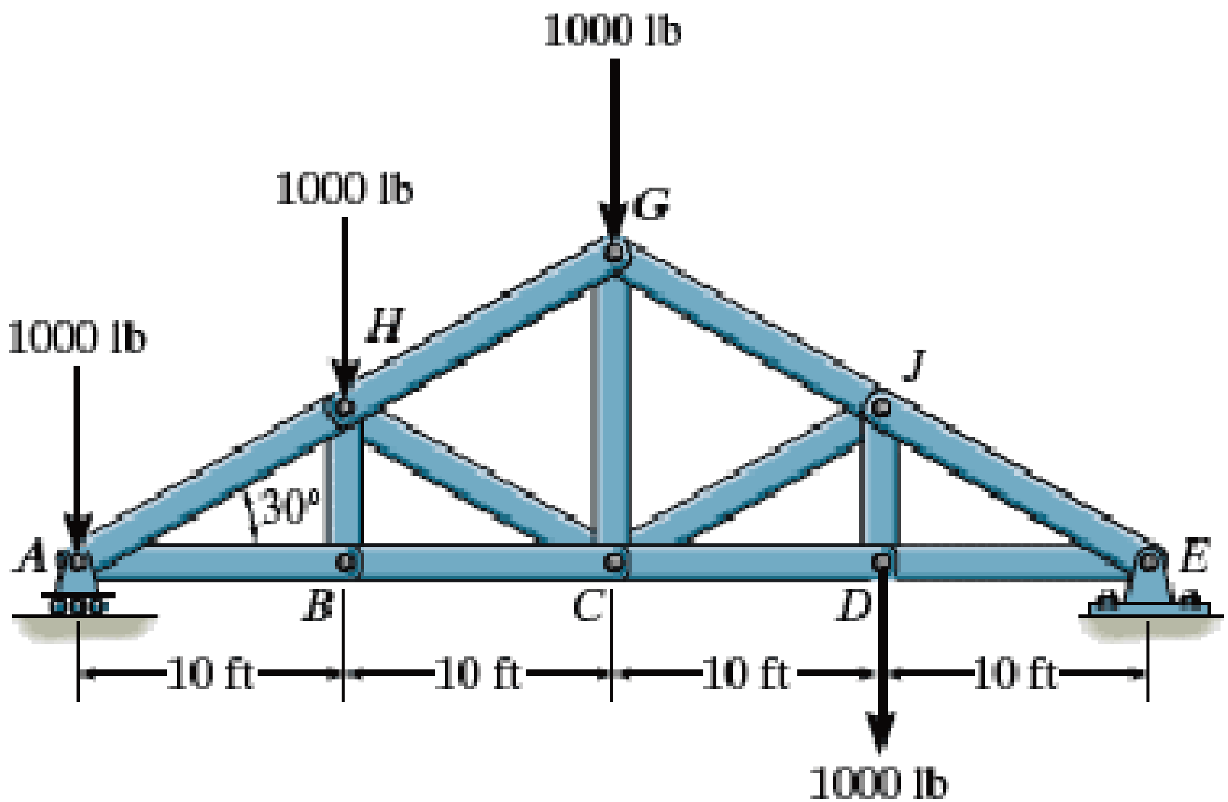
2 m
F_{BC}
C
F_{GC}
2 m
45°
F_{GF}
G
2 m
1000 N

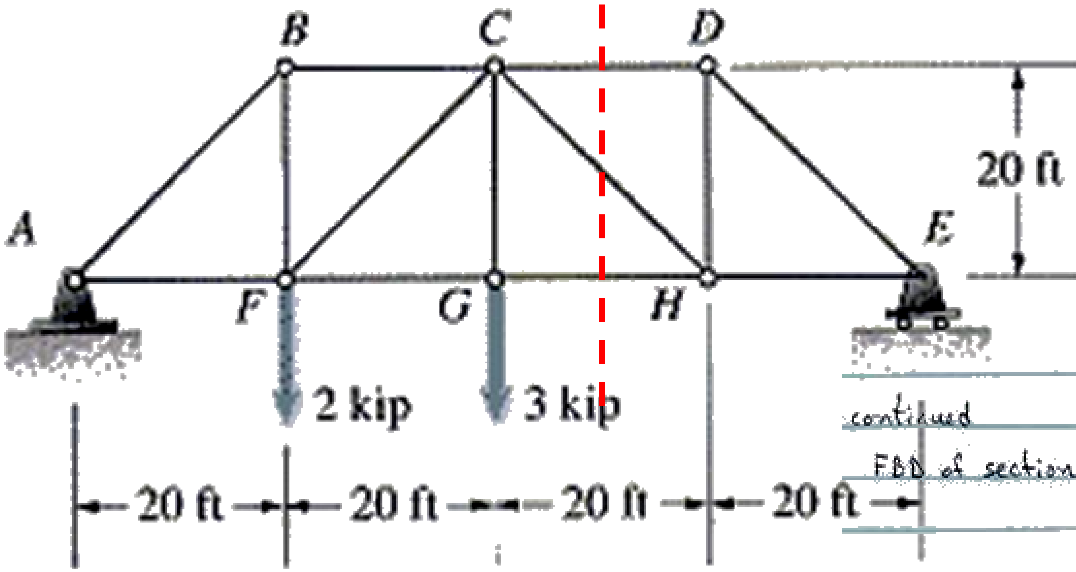


D_y
F_{BC}
C
2 m
D_x
45°
2 m
F_{GC}
G
F_{GF}
E_x

(c)

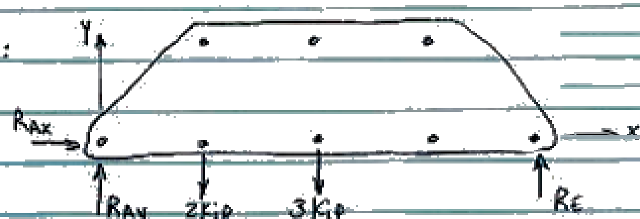
Determine the force in member GC of the truss and state if this member is in tension or compression.





Solution:

FBD of entire truss:



$\Sigma F_x = 0 = R_{Ax}$

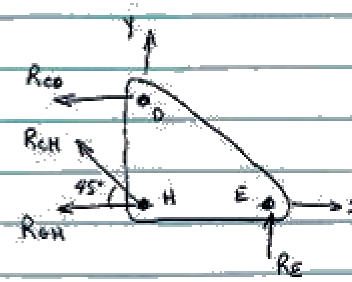
$\Sigma F_y = 0 = R_{Ay} + R_E - 2\text{kip} - 3\text{kip}$

$\circlearrowleft + \Sigma M_A = 0 = -(20')(2\text{kip}) - (40')(3\text{kip}) + (80')R_E \Rightarrow R_E = 2\text{kip}$

$\Rightarrow R_{Ay} = 3\text{ kip}$

continued

FBD of section ②:



$\Sigma F_x = 0 = -R_{CD} - R_{GH} - R_{CH}\cos 45°$

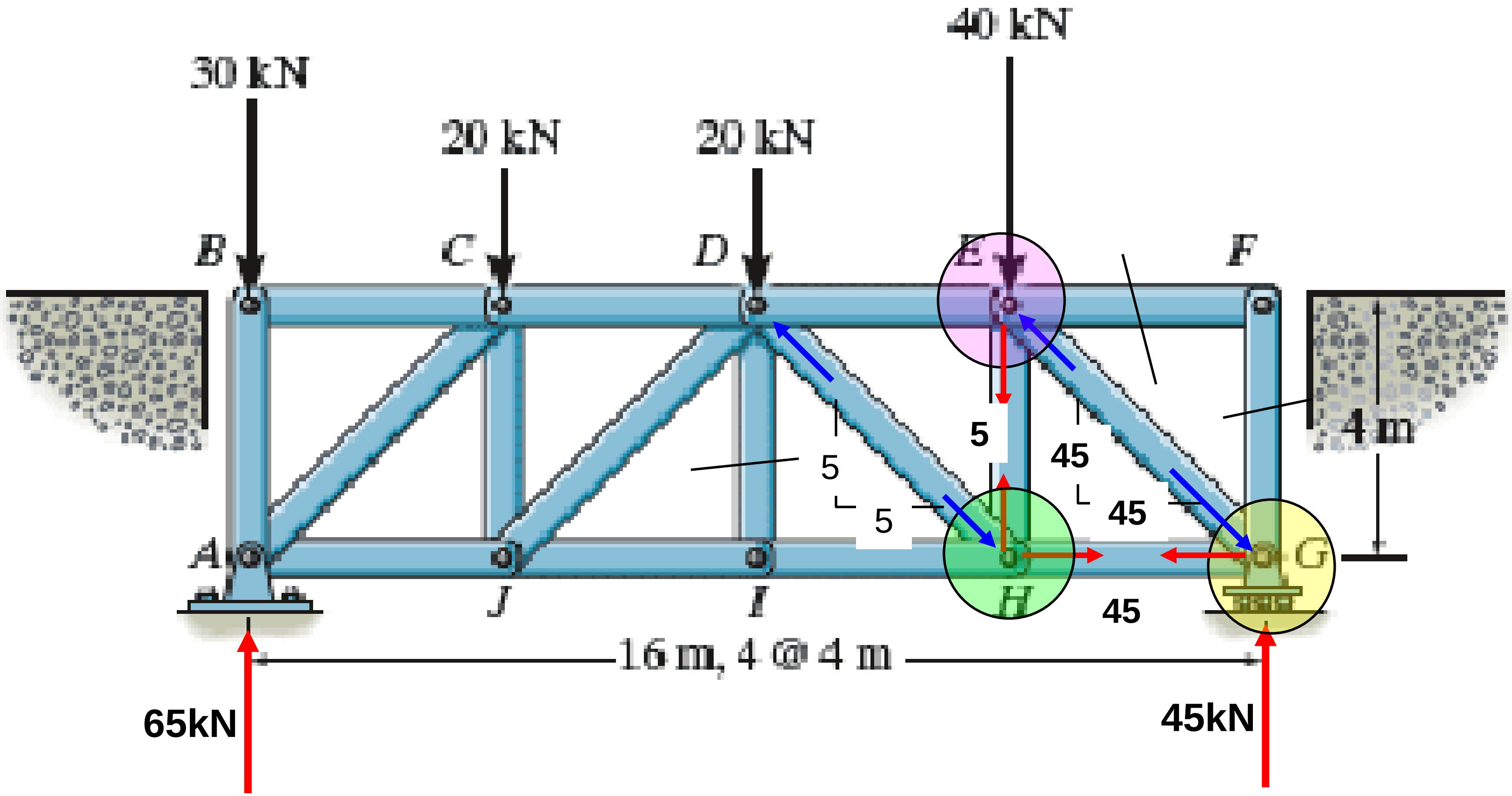
$\Sigma F_y = 0 = R_E + R_{CH}\sin 45° \Rightarrow R_{CH} = -2.828\text{ kip}$ ✓

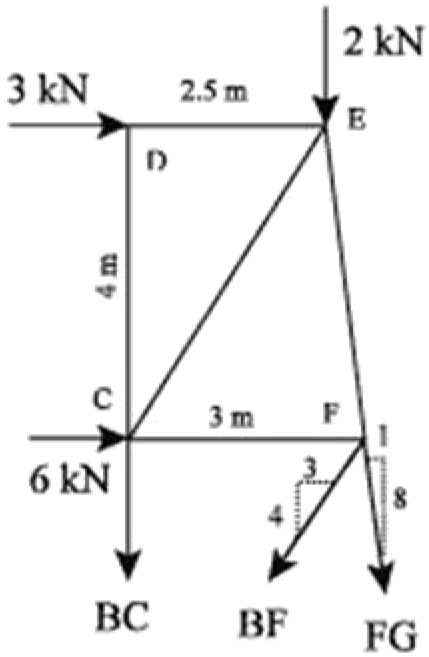
$\circlearrowleft + \Sigma M_H = 0 = (20')R_E + (20')R_{CD} \Rightarrow R_{CD} = -R_E = -2\text{ kip}$ ✓

$\Rightarrow R_{GH} = -R_{CD} - R_{CH}\cos 45° = 4\text{ kip}$ ✓

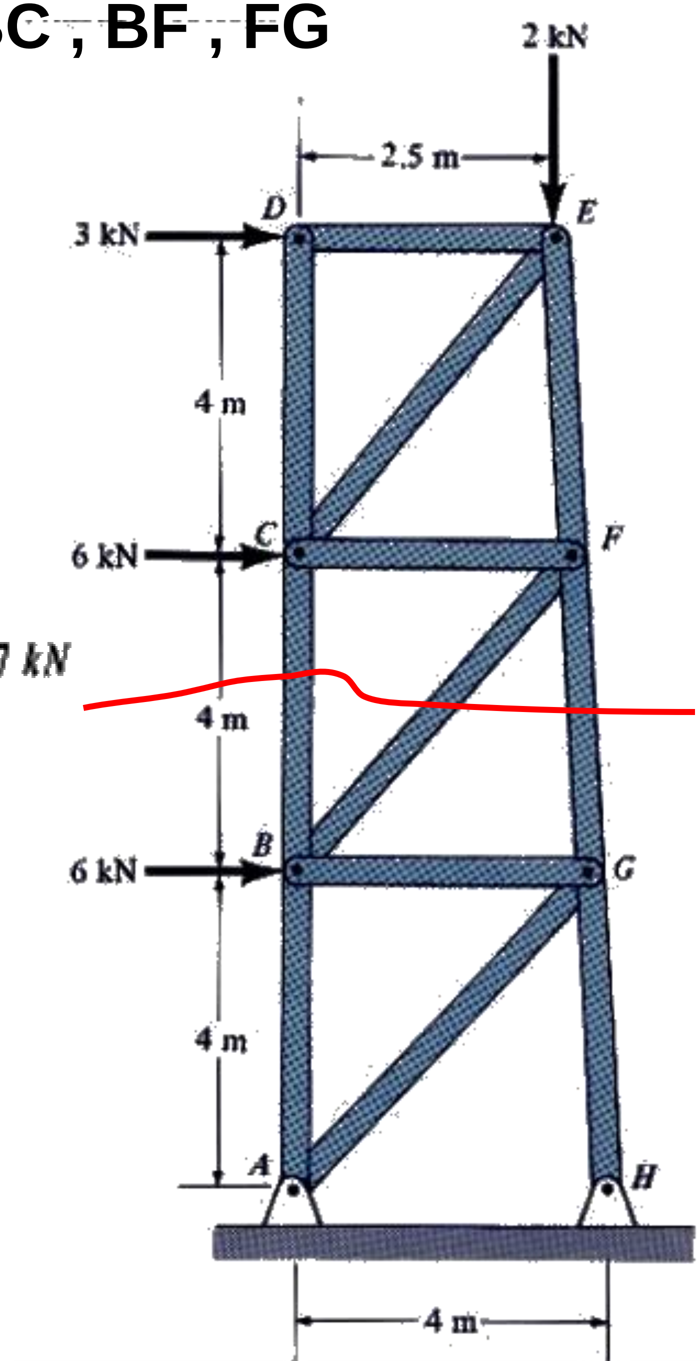
$\boxed{R_{GH} = 4\text{ kip (T)}}$

The Howe bridge truss is subjected to the loading shown. Determine the force in members DE, EH, and HG, and state if the members are in tension or compression.





BC , BF , FG



$$\sum M_F: (2kN)(0.5m) - (3kN)(4m) + BC(3m) = 0 \;\Rightarrow\; BC = 3.67\,kN$$

$$\sum M_B: -(6kN)(4m) - (3kN)(8m) - (2kN)(2.5m) - \frac{1}{\sqrt{65}}FG(4m) - \frac{8}{\sqrt{65}}FG(3m) = 0 \;\Rightarrow\; FG = -15.26\,kN$$

$$\sum M_Q: (3kN)(20m) + (6kN)(24m) - (2kN)(2.5m) - \frac{3}{5}BF(24m) - \frac{4}{5}BF(3m) = 0 \;\Rightarrow\; BF = 11.85\,kN$$

In Summary

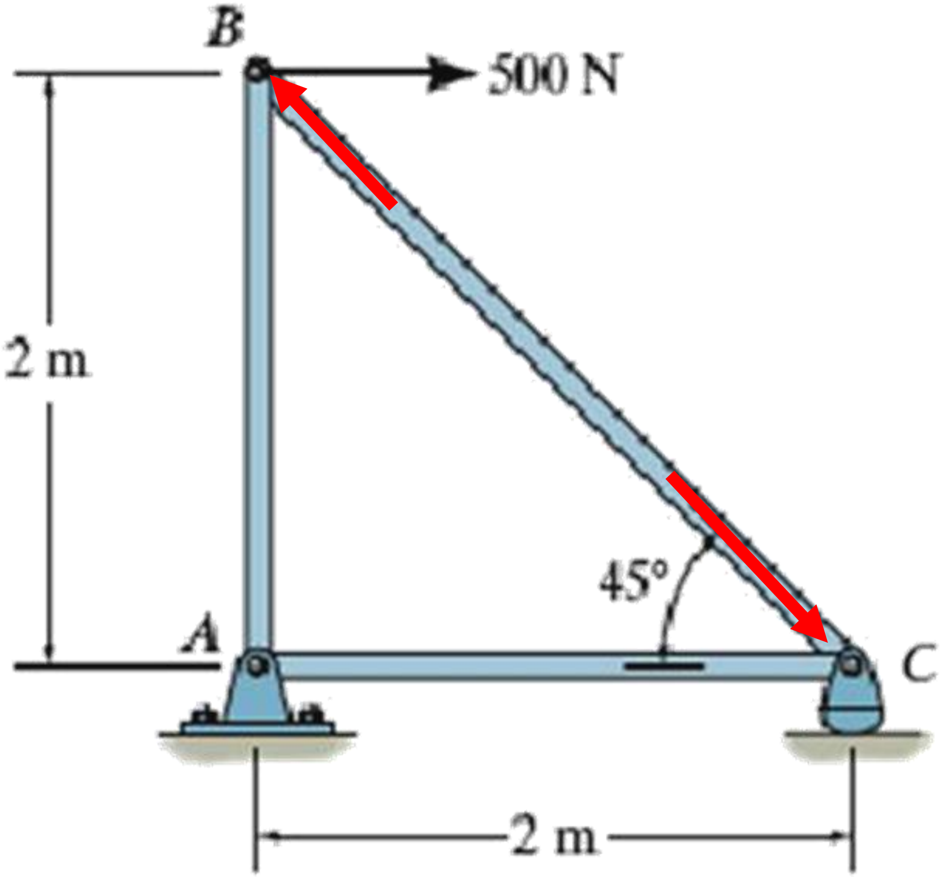
$BC = 3.67\,kN \;\; T$

$FG = 15.26\,kN \;\; C$

$BF = 11.85\,kN \;\; T$

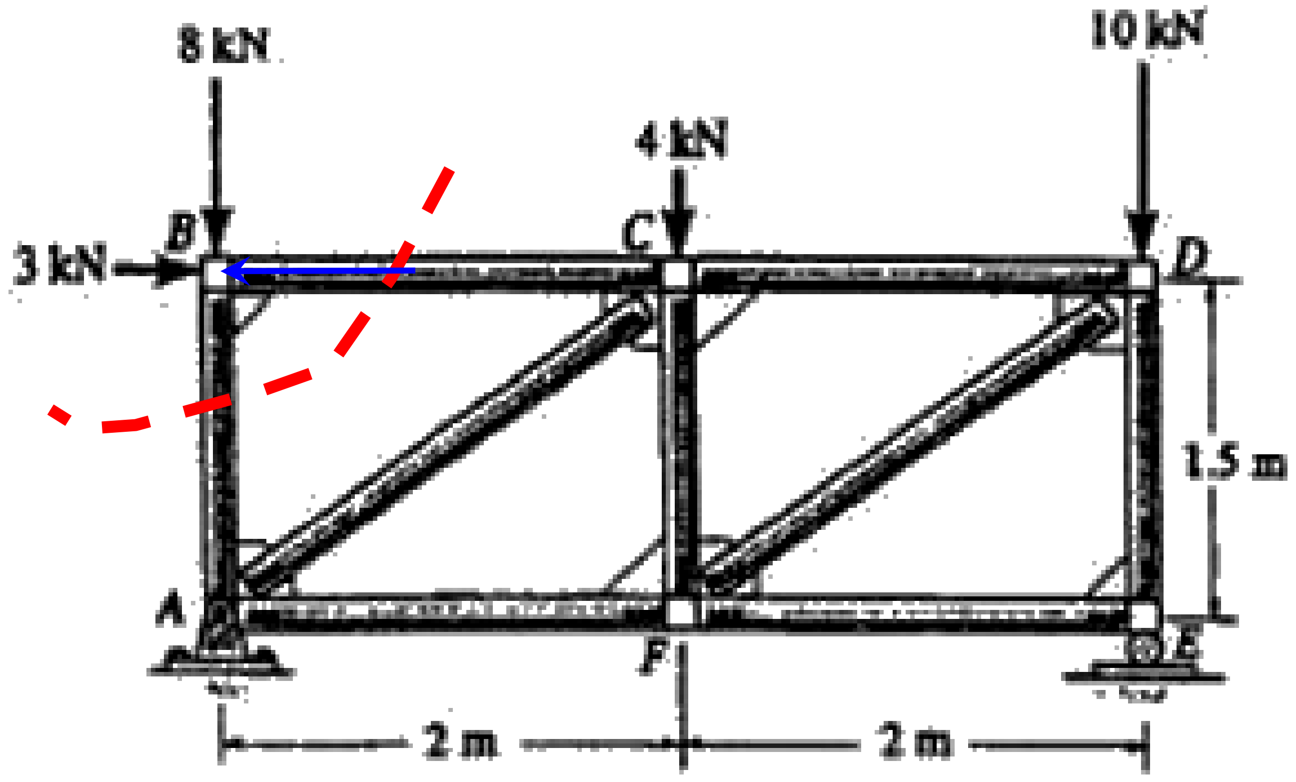
4. Find the force in member BC and indicate if it is in tension or compression

F_{BC} = ...**707.1N (C) ....** (1 point)



Find the force in member BC and indicate if it is in tension or compression

F_{BC} = …………3kN…………………… (1 point)
Tension or compression …………C………….(1 point)

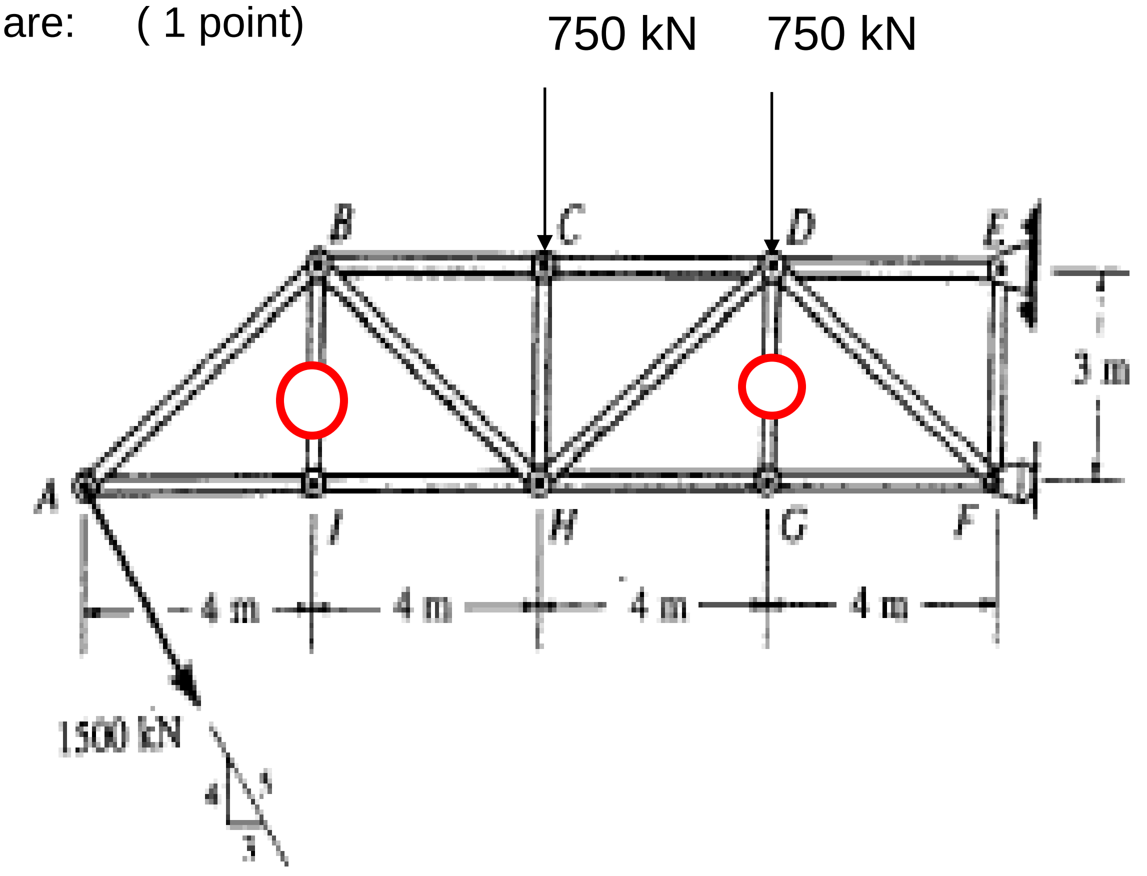


6 . Find two zero force members by inspection

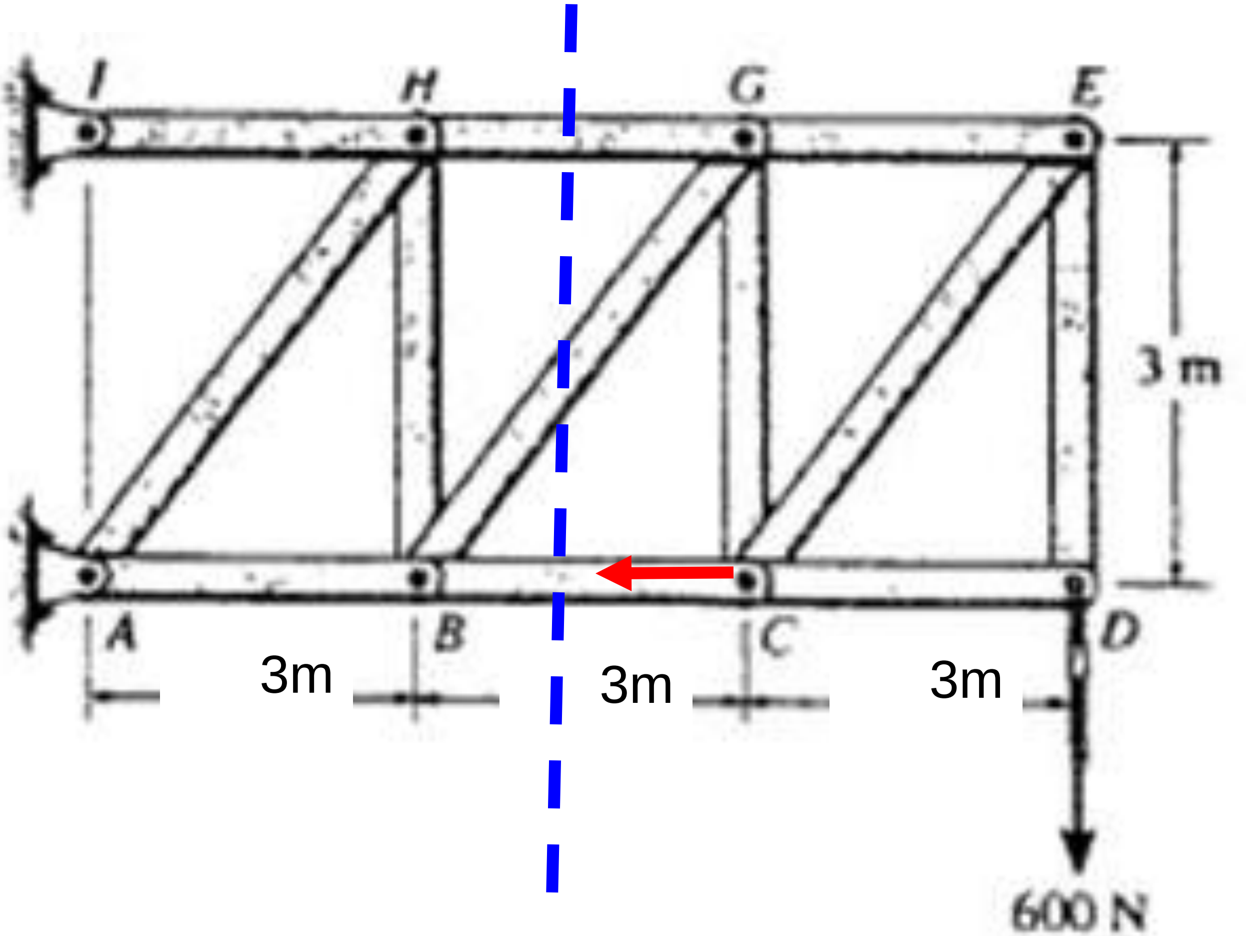
The zero force members are: (1 point)

1.……………………

2.……………………



7. Find the force in member BC



F $_{BC}$ = ……………**600N**……………… (1 point)

6.6 Frames and Machines

Frames: support loads, generally stationary
Machines: transmit or alter forces, often have moving parts

As with trusses, if a frame or machine is in static equilibrium, then every individual part of the frame or machine is in static equilibrium

- **any collection of individual parts are, together, in static equilibrium**
- **every joint in the frame or machine is in static equilibrium•**

Steps

Determine the reactions at the supports .

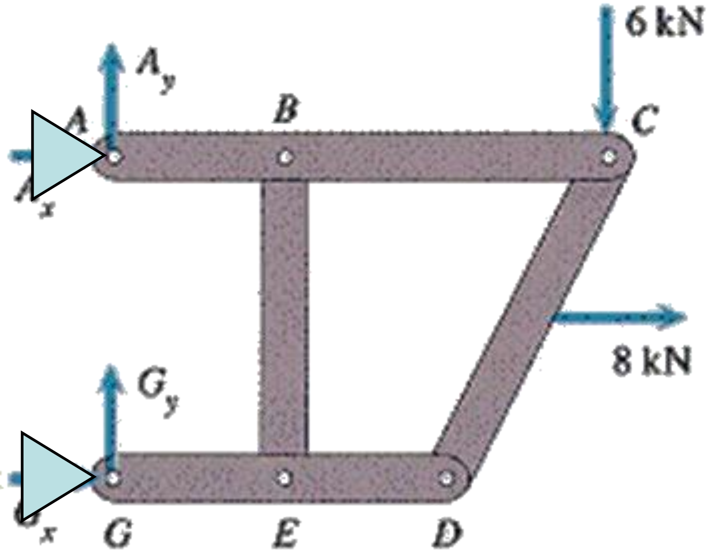
FBD ,$\Sigma F = 0$, $\Sigma M = 0$

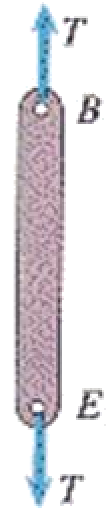
If the structure is statically indeterminate, determine as many of the reactions as possible .

Identify any 2-force members (simplify the problem)

Analyze the members .FBD, $\Sigma F = 0$, $\Sigma M = 0$

If a load is applied at a joint, place the load on only one of the members at that joint

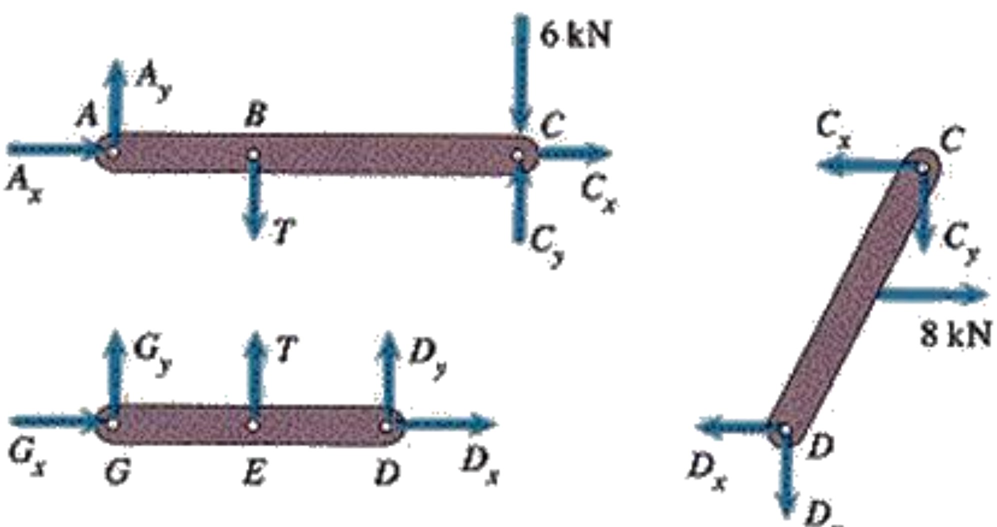


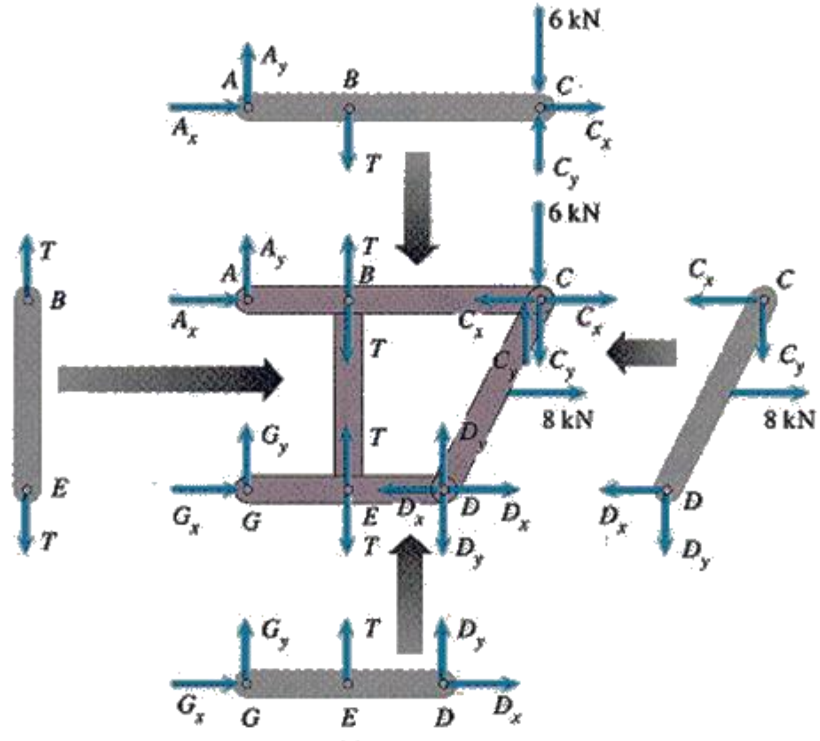


Identify any 2-force members

statically indeterminate, determine as many of the reactions as possible.

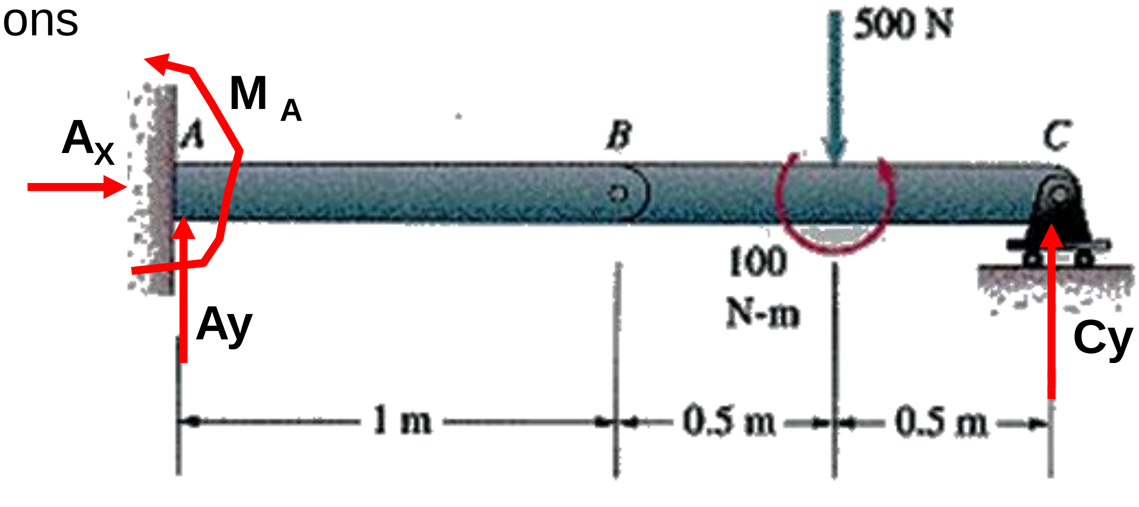
FBD, ΣF = 0 , ΣM= 0



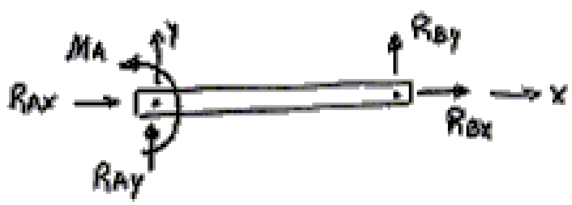


If a load is applied at a joint, place the load on only one of the members at that joint

Find the reactions



FBD of member AB:



$\Sigma F_x = 0 = R_{Ax} + R_{Bx} \Rightarrow R_{Bx} = 0$

$\Sigma F_y = 0 = R_{Ay} + R_{By}$

$\Sigma M_A = 0 = M_A + (1m) R_{By}$

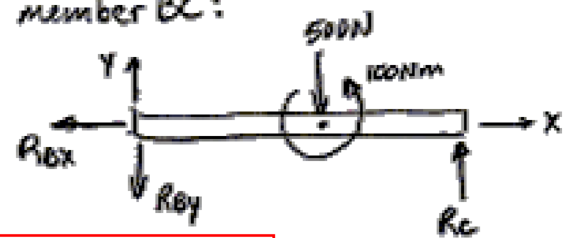
$R_C = 150N$

$R_{By} = -350N$

$M_A = 350 \, Nm$

$R_{Ay} = 350N$

FBD of member BC:

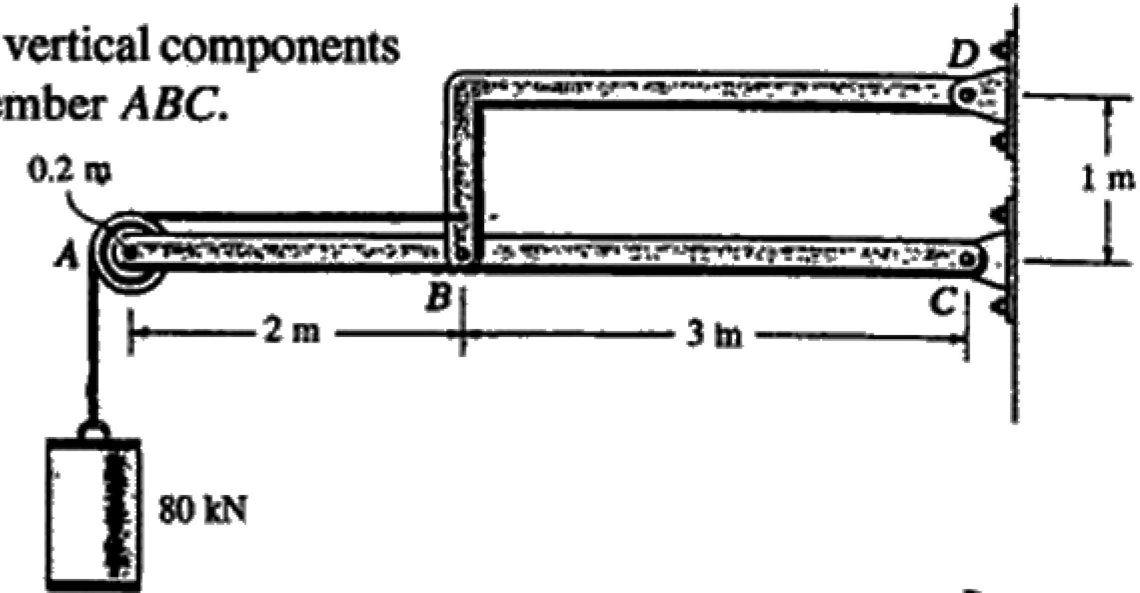


$\Sigma F_x = 0 = R_{Bx}$ ✓

$\Sigma F_y = 0 = -R_{By} - 500N + R_C$

$\Sigma M_B = 0 = -(.5m)(500N) + 100Nm + (1m) R_C \Rightarrow R_C = 150N$

6-85. Determine the horizontal and vertical components of force which the pins exert on member *ABC*.



$\xrightarrow{+} \Sigma F_x = 0;\quad A_x = 80 \text{ kN}$

$+\uparrow \Sigma F_y = 0;\quad A_y = 80 \text{ kN}$

$\circlearrowleft + \Sigma M_C = 0;\quad 80(5) - B_y(3) = 0$

$B_y = 133.3 = 133 \text{ kN}$

$\circlearrowleft + \Sigma M_D = 0;\quad -80(0.8) + 133.3(3) - B_x(1) = 0$

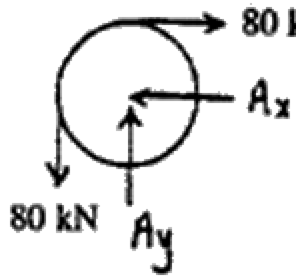
$B_x = 336 \text{ kN}$

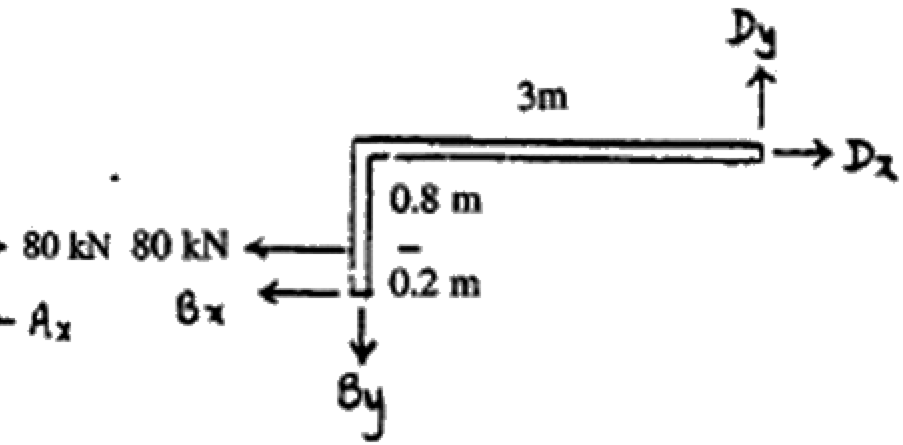
$\xrightarrow{+} \Sigma F_x = 0;\quad 80 + 336 - C_x = 0$

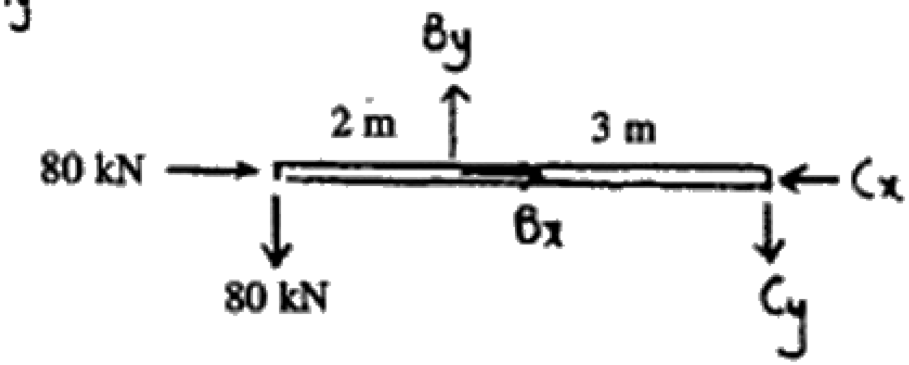
$C_x = 416 \text{ kN}$

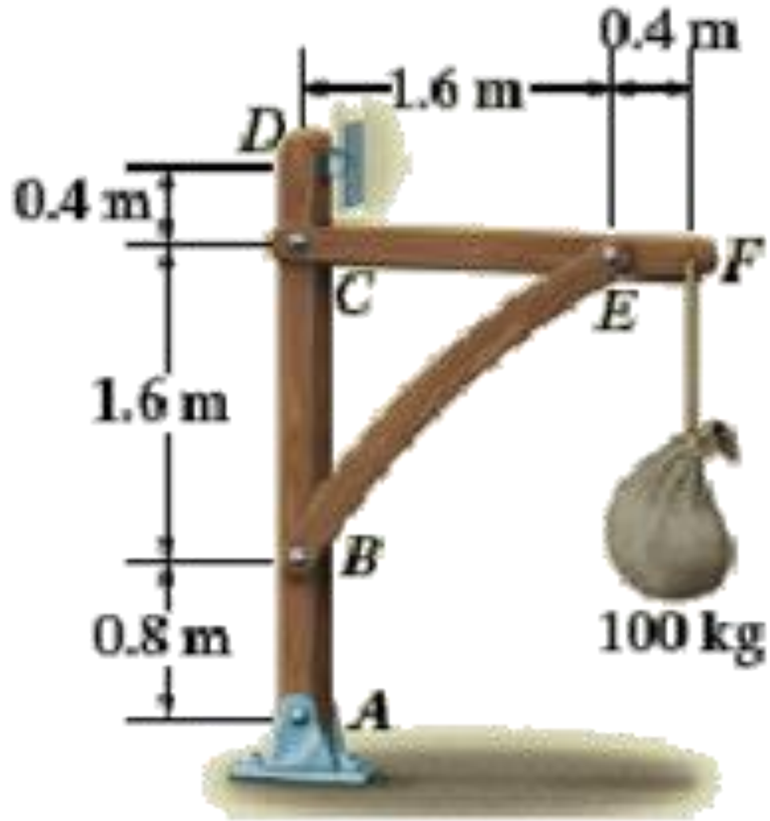
$+\uparrow \Sigma F_y = 0;\quad -80 + 133.3 - C_y = 0$

$C_y = 53.3 \text{ kN}$



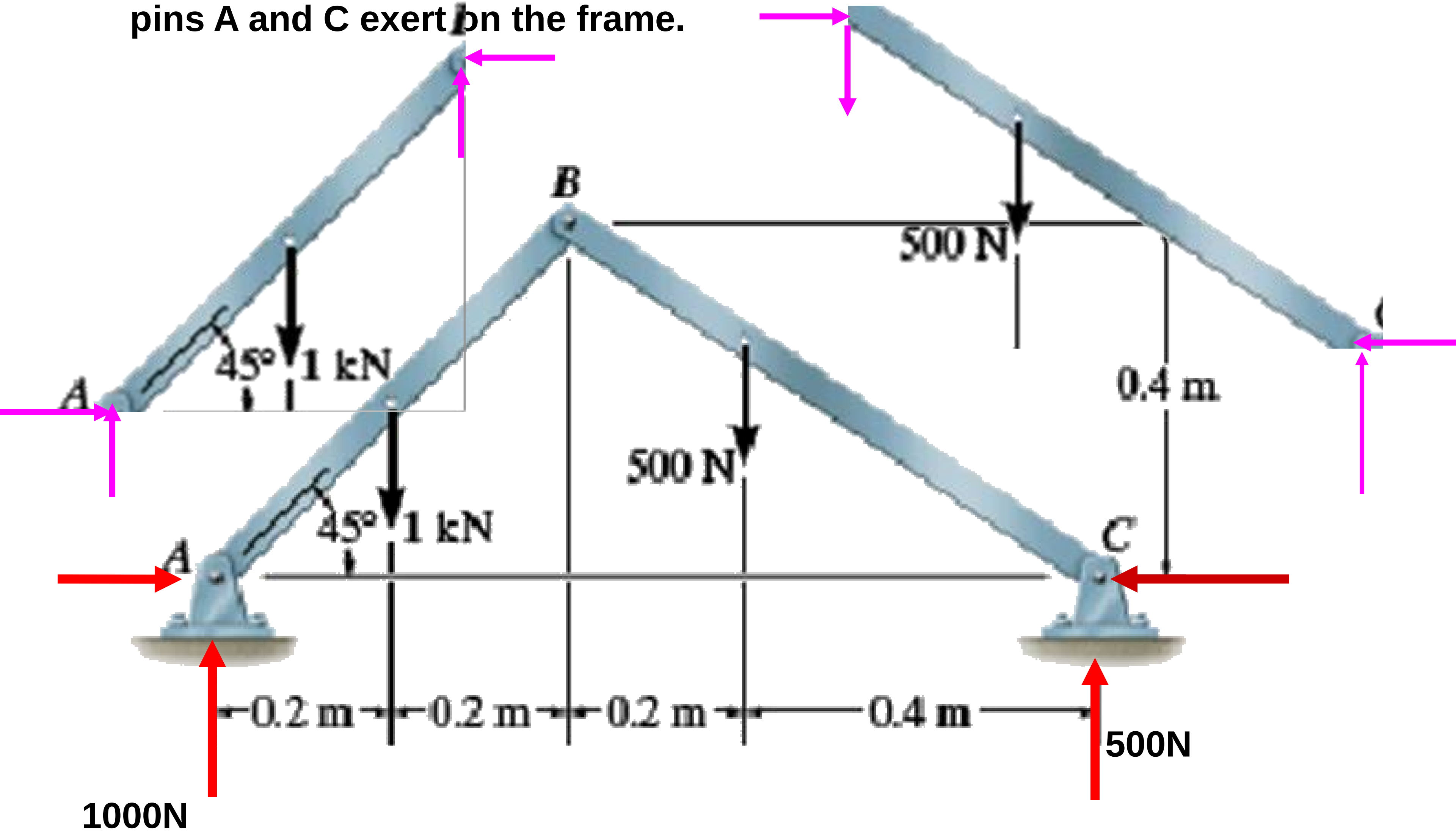




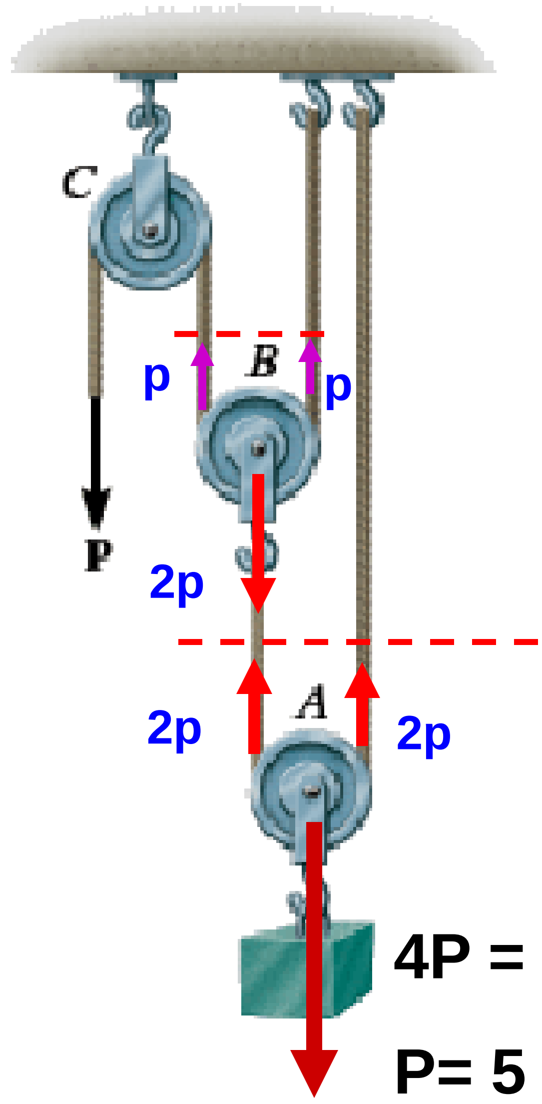


Find pin forces, and force in member BE.

6–72. Determine the horizontal and vertical components of force that pins A and C exert on the frame.



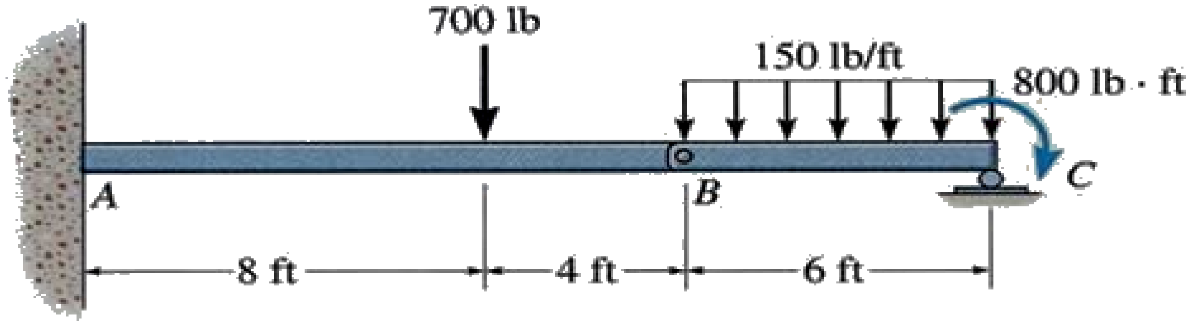
6–68. Determine the force P needed to hold the 20-lb block in equilibrium.

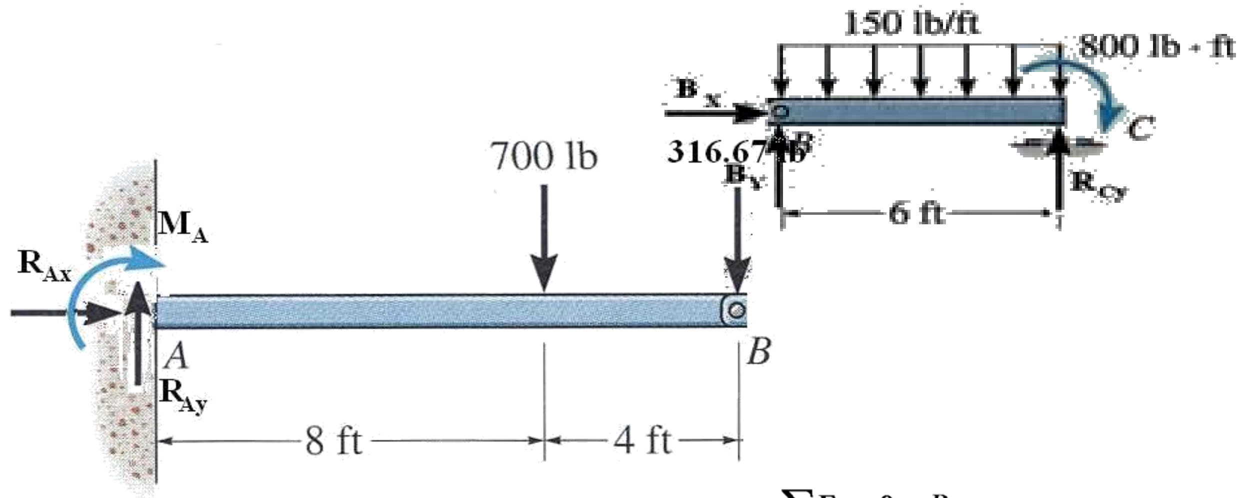


4P = 20

P= 5 lb

Find the reactions





$$\sum F_x = 0 = R_{Ax}$$

$$\sum F_y = 0 = R_{Ay} - 700\text{ lb} - 316.67\text{ lb}$$

$$\sum M_B = 0 = -M_A - 700\text{ lb}(8\text{ ft}) - 316.67\text{ lb}(12\text{ ft})$$

$$R_{Ay} = 1016.67\text{ lb}$$

$$R_{Ax} = 0\text{ lb}$$

$$M_A = -9400\text{ lb-ft}$$

$$\sum F_x = 0 = B_X$$

$$\sum F_y = 0 = R_{Cy} + B_Y - 150\text{ lb/ft}(6\text{ ft})$$

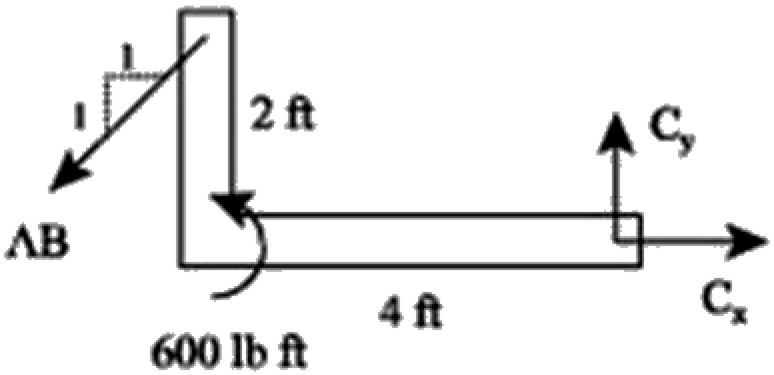
$$\sum M_B = 0 = R_{Cy}(6\text{ ft}) - 150\text{ lb/ft}(6\text{ ft})(3\text{ ft}) - 800\text{ lb-ft}$$

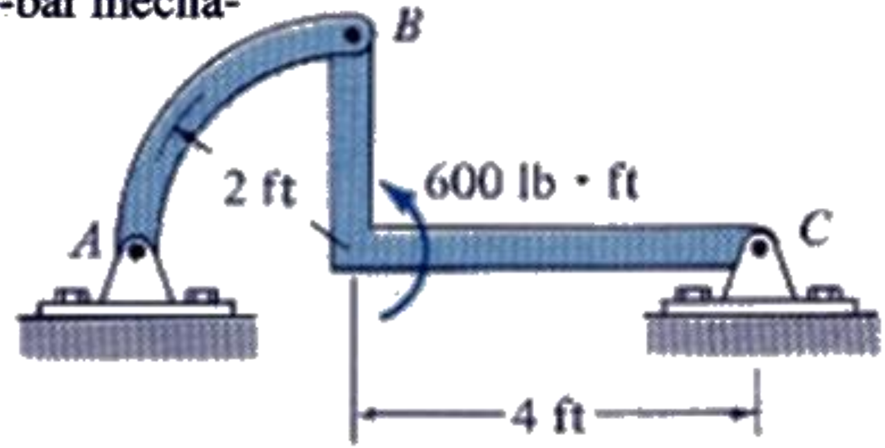
$$R_{Cy} = 583.33\text{ lb}$$

$$B_X = 0\text{ lb}$$

$$B_Y = 316.67\text{ lb}$$

Determine the horizontal and vertical components of force that the pins at *A* and *C* exert on the two-bar mechanism.





$$\sum M_C: 600\,lb\,ft + \tfrac{1}{\sqrt{2}} AB(4\,ft) + \tfrac{1}{\sqrt{2}} AB(2\,ft) = 0 \;\Rightarrow\; AB = -141.4\,lb$$

$$\sum F_x: C_x - \tfrac{1}{\sqrt{2}} AB = 0 \;\Rightarrow\; C_x = -100\,lb$$

$$\sum F_y: C_y - \tfrac{1}{\sqrt{2}} AB = 0 \;\Rightarrow\; C_y = -100\,lb$$

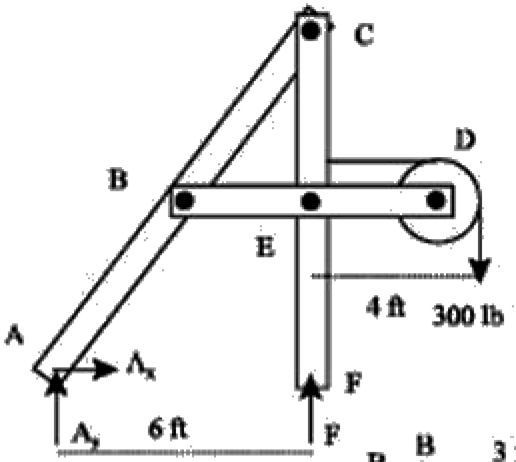
Note that the horizontal and vertical forces in the pin at A are just

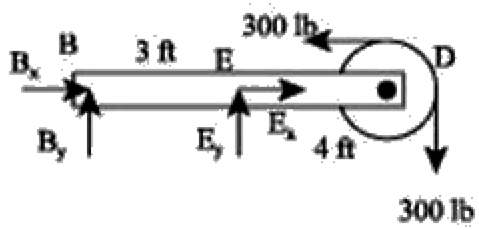
$$A_x = A_y = -\tfrac{1}{\sqrt{2}} AB = -100\,lb$$

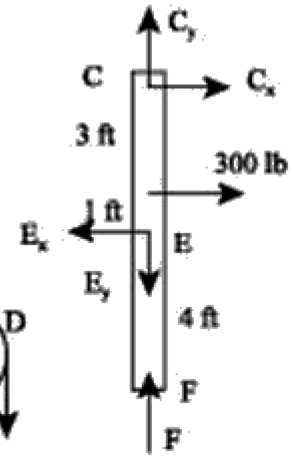
In summary

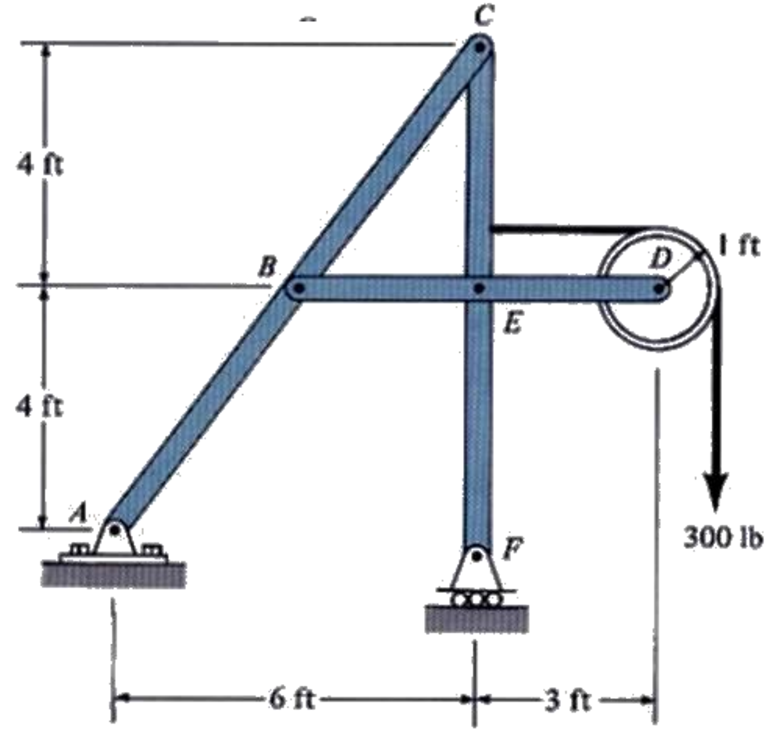
$$A_z = A_y = C_x = C_y = -100\,lb$$

Determine the horizontal and vertical components of force at C which member ABC exerts on member CEF.









$$\sum M_A: -(300\,lb)(10\,ft) + F(6\,ft) = 0 \;\Rightarrow\; F = 500\,lb$$

Now let's examine BED

$$\sum M_B: E_y(3\,ft) + (300\,lb)(1\,ft) - (300\,lb)(7\,ft) = 0 \;\Rightarrow\; E_y = 600\,lb$$

These are the two extra pieces of information that we need. Now we are ready to look at body CEF.

$$\sum M_E: -C_x(4\,ft) - (300\,lb)(1\,ft) = 0 \;\Rightarrow\; C_x = -75\,lb$$

$$\sum F_y: C_y - E_y + F = 0 \;\Rightarrow\; C_y = 100\,lb$$

$$C_x = -75\,lb$$
$$C_y = 100\,lb$$

$$\circlearrowleft + \Sigma M_A = 0; \qquad -750(2) + B_y(3) = 0$$

$$B_y = 500 \text{ N}$$

$$\circlearrowleft + \Sigma M_C = 0; \qquad -1200(1.5) - 900(1) + B_x(3) - 500(3) = 0$$

$$B_x = 1400 \text{ N}$$

$$\xrightarrow{+} \Sigma F_x = 0; \qquad -A_x + 1400 = 0$$

$$A_x = 1400 \text{ N} = 1.40 \text{ kN}$$

$$+\uparrow \Sigma F_y = 0; \qquad A_y - 750 + 500 = 0$$

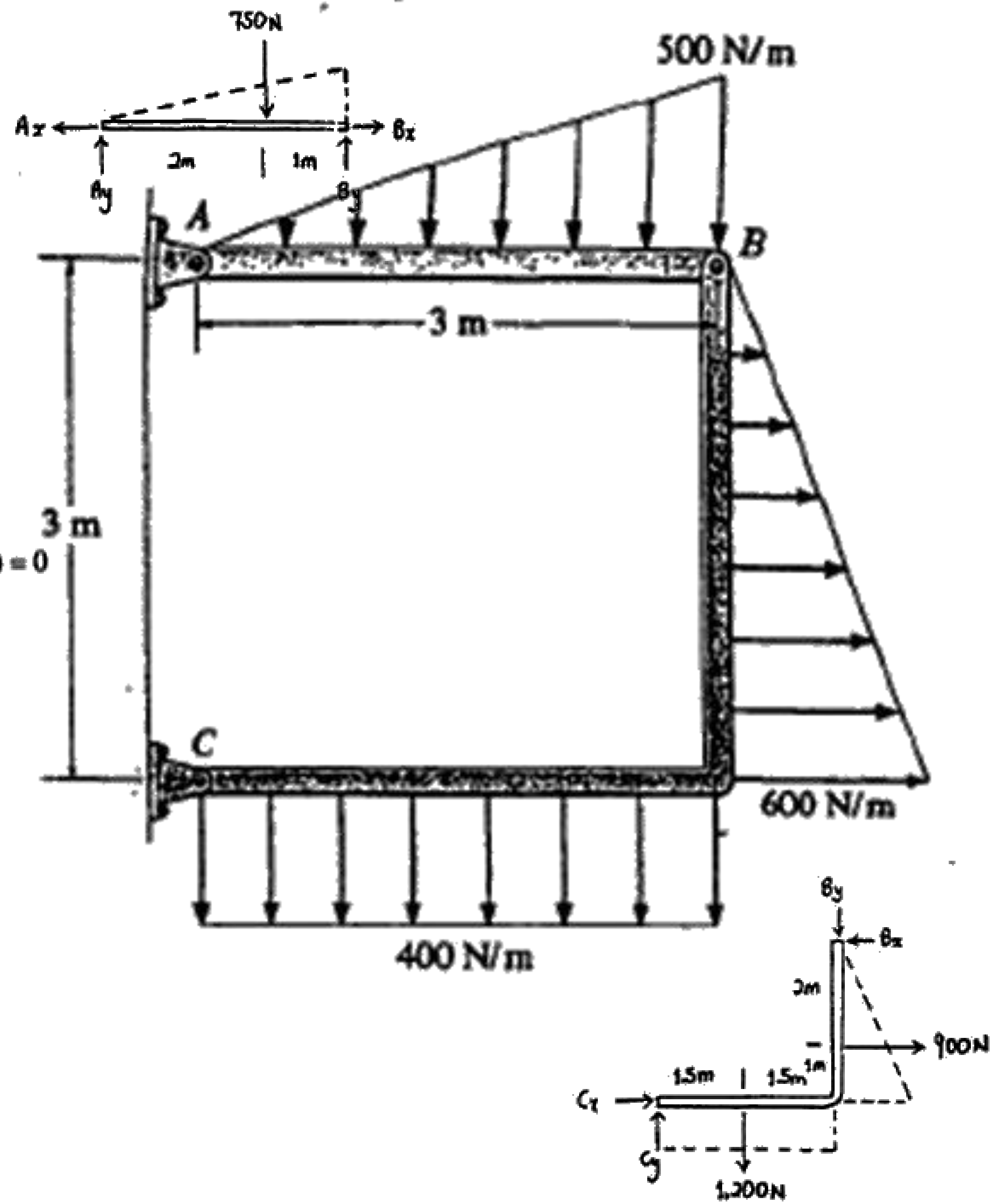
$$A_y = 250 \text{ N}$$

$$\xrightarrow{+} \Sigma F_x = 0; \qquad C_x + 900 - 1400 = 0$$

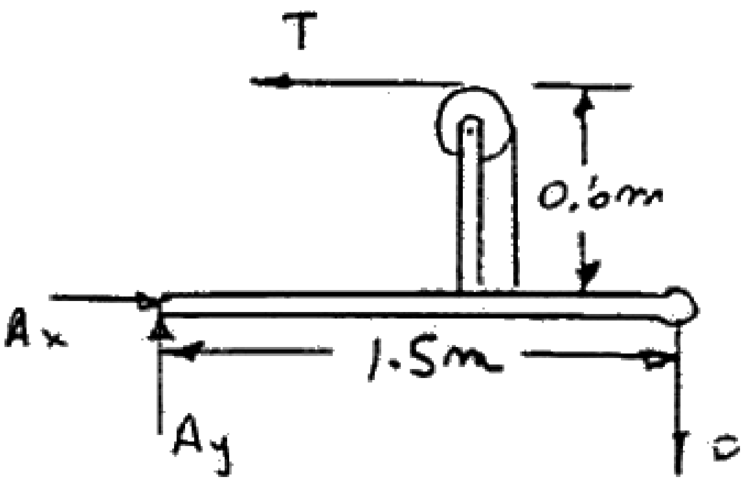
$$C_x = 500 \text{ N}$$

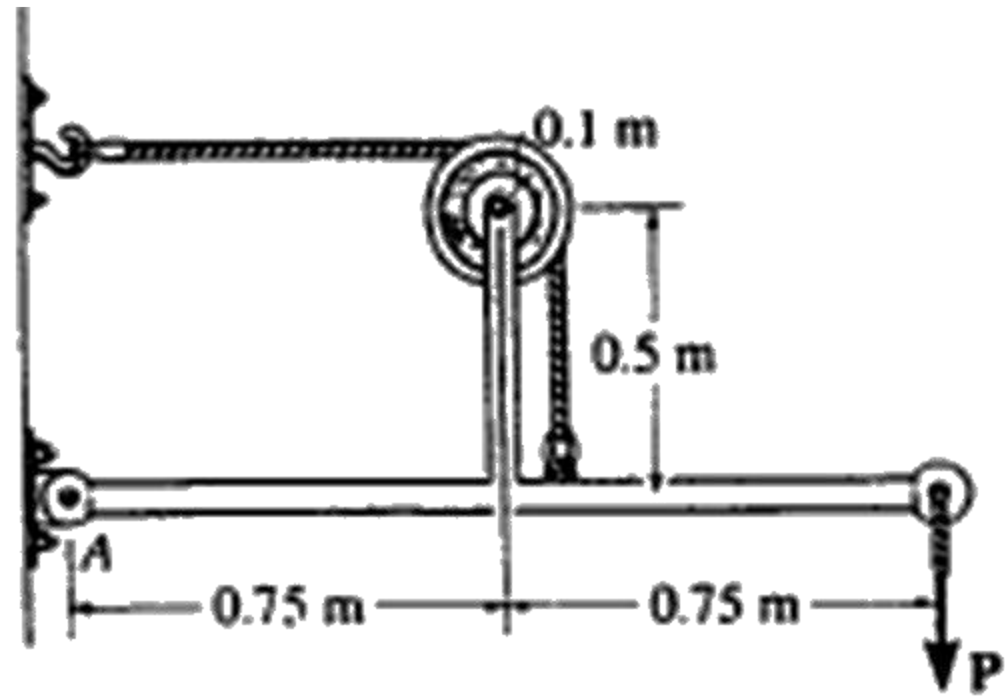
$$+\uparrow \Sigma F_y = 0; \qquad -500 - 1200 + C_y = 0$$

$$C_y = 1700 \text{ N} = 1.70 \text{ kN}$$



Determine the greatest force P that can be applied to the frame if the largest force resultant acting at A can have a magnitude of 2 kN.





$\circlearrowleft + \Sigma M_A = 0; \quad T(0.6) - P(1.5) = 0$

$\xrightarrow{+} \Sigma F_x = 0; \quad A_x - T = 0$

$+\uparrow \Sigma F_y = 0; \quad A_y - P = 0$

Thus, $\quad A_x = 2.5P, \quad A_y = P$

$$2 = \sqrt{(2.5P)^2 + (P)^2}$$

$$P = 0.743 \text{ kN} = 743 \text{ N}$$

Engineering Mechanics

Statics & Strength of Materials

Internal Forces

Eng. Iqbal Marie

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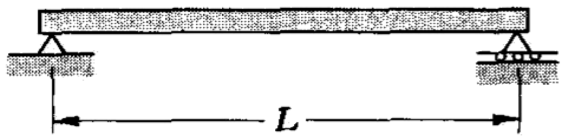
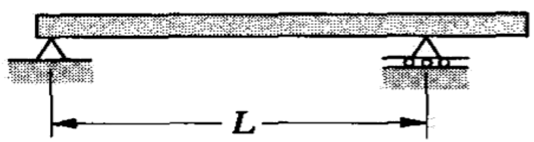
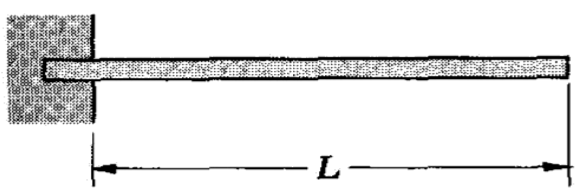
Beams – Types



(*a*) Simply supported beam

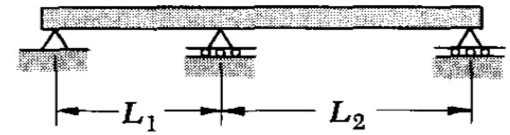


(*b*) Overhanging beam

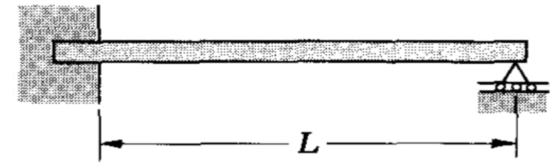


(*c*) Cantilever beam

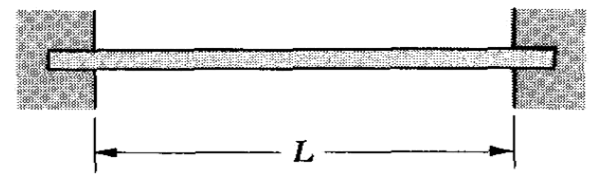
Statically Determinate Beams



(*d*) Continuous beam

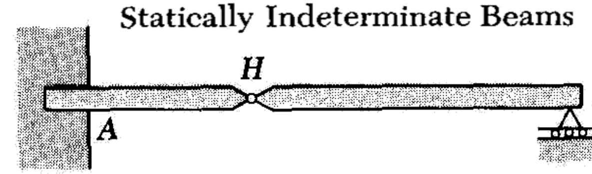


(*e*) Beam fixed at one end and simply supported at the other end

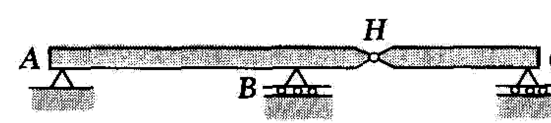


(*f*) Fixed beam

Statically Indeterminate Beams



(*a*)

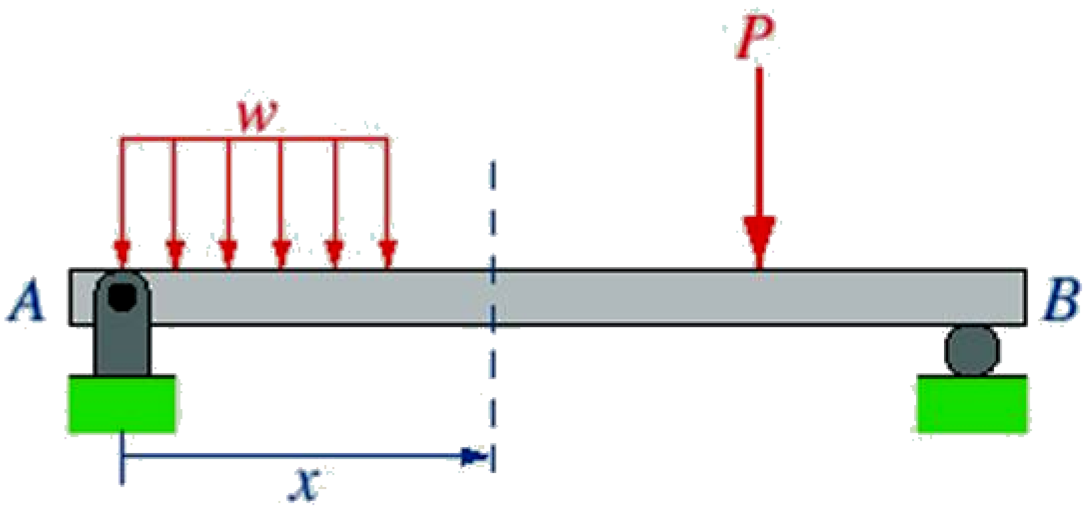


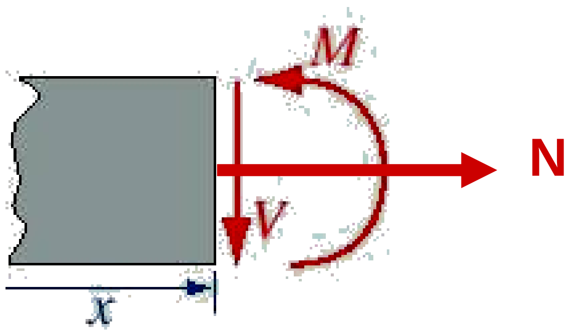
(*b*)

Combined beams

Internal Forces Developed in Structural Members

For each cross section, there is a shear force V and a bending moment M and a normal force N.

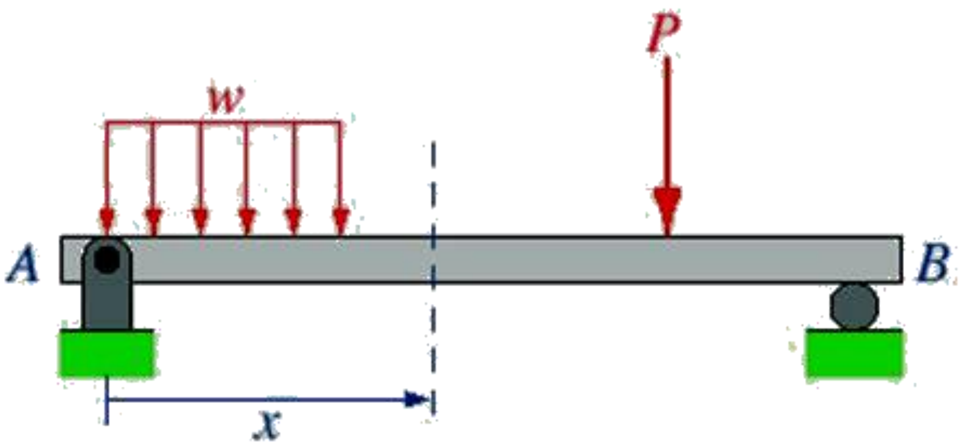


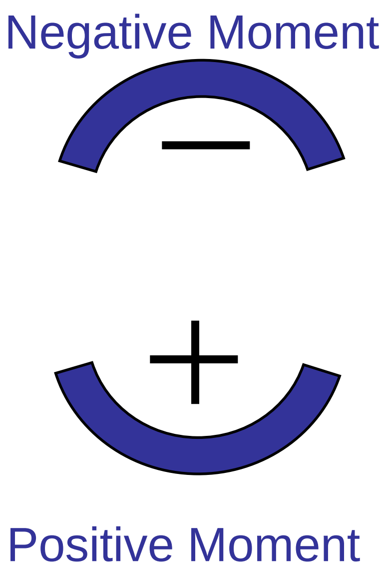


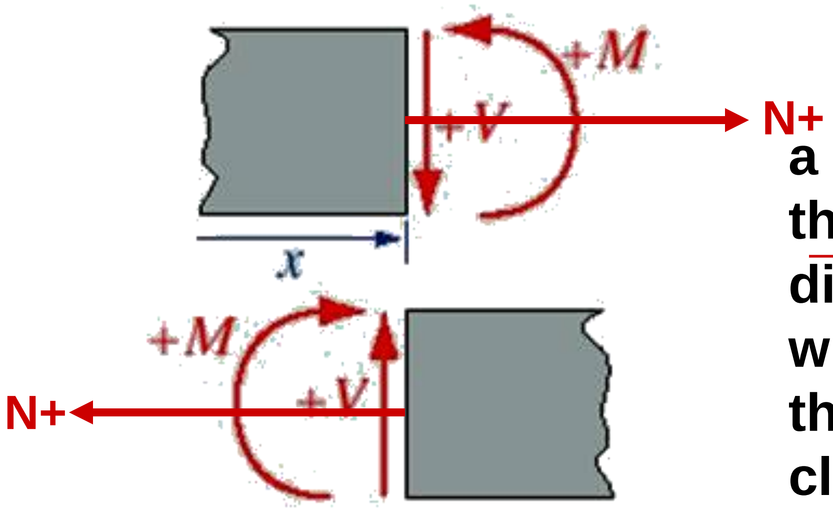
Internal Forces

- Shear Forces
- Bending Moment
- Normal Forces

Sign Convention

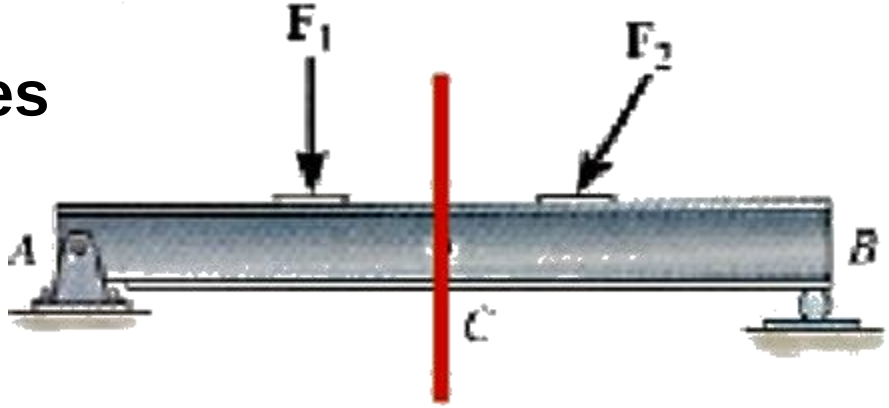


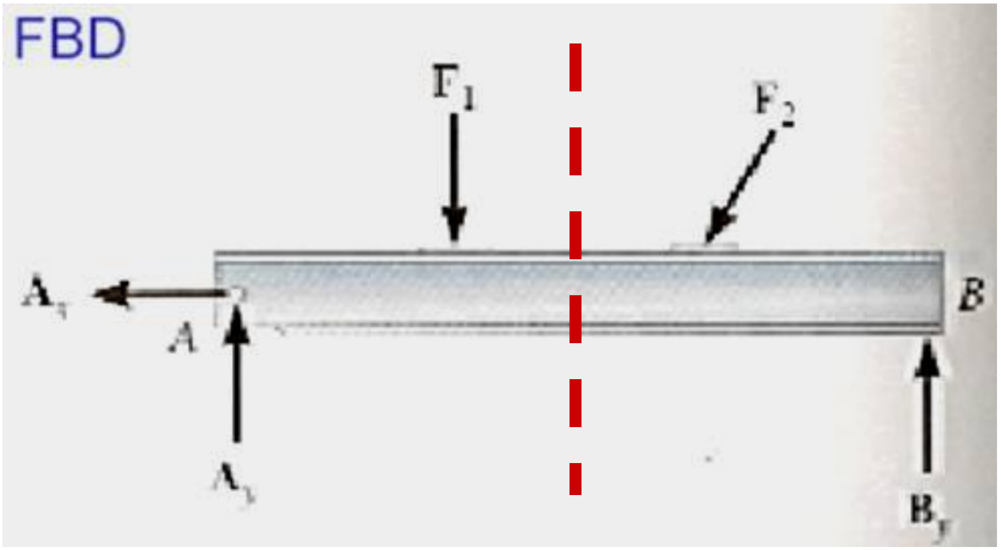


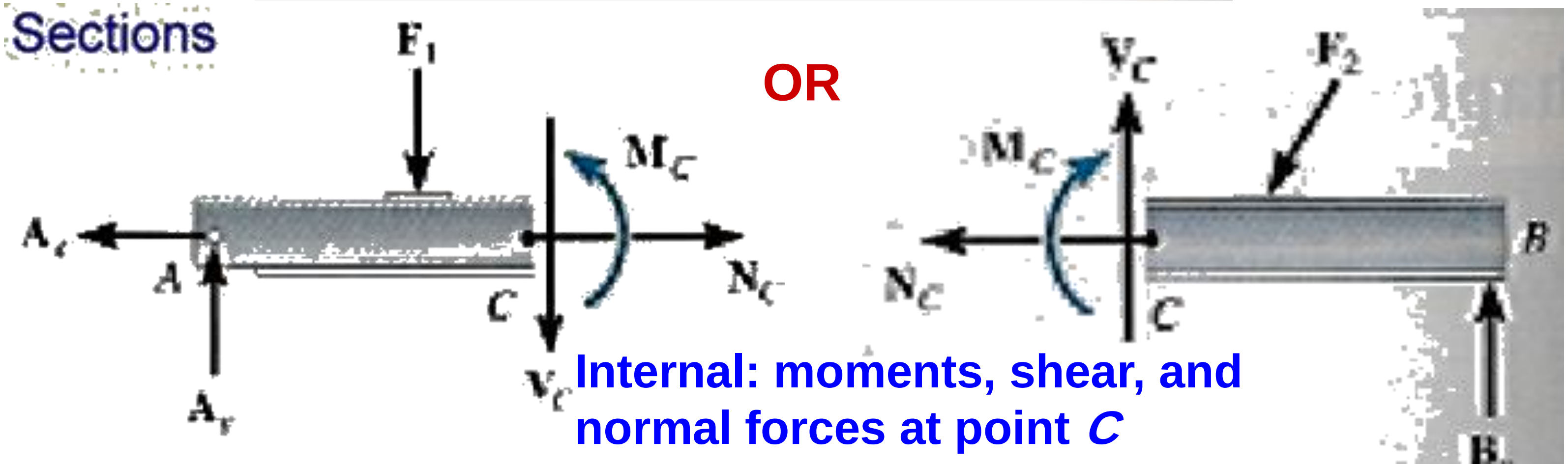


a shear force, which tends to rotate the beam fibers in the clockwise direction, is a negative shear force whereas a force that tends to rotate the beam fibers in the counter clockwise is positive.

Internal forces at point C







Internal: moments, shear, and normal forces at point *C*

General Solution Scheme

The general scheme for finding the internal set of forces is

a) Draw the free-body diagram

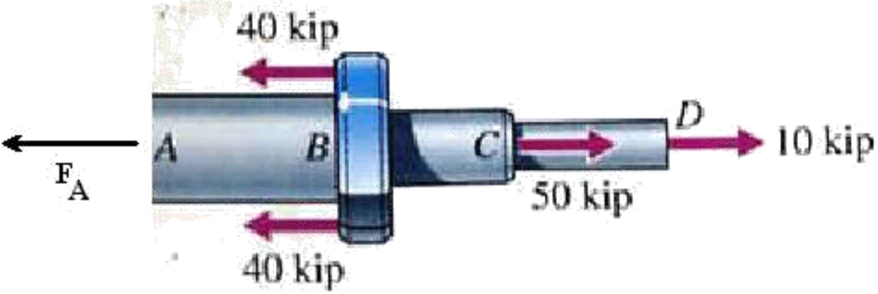
b) Determine the support reactions

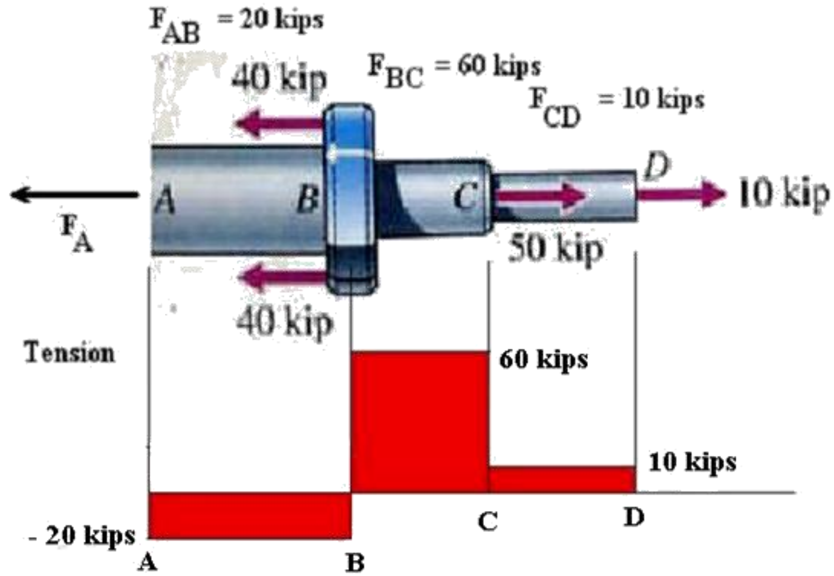
c) Apply the equations of equilibrium

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$

$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0$$

Determine the axial forces at point B and C,



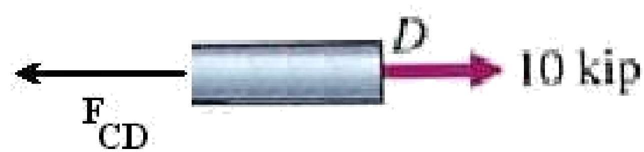


$$\sum F_x = -F_{BC} + 50 \text{ kips} + 10 \text{ kips}$$

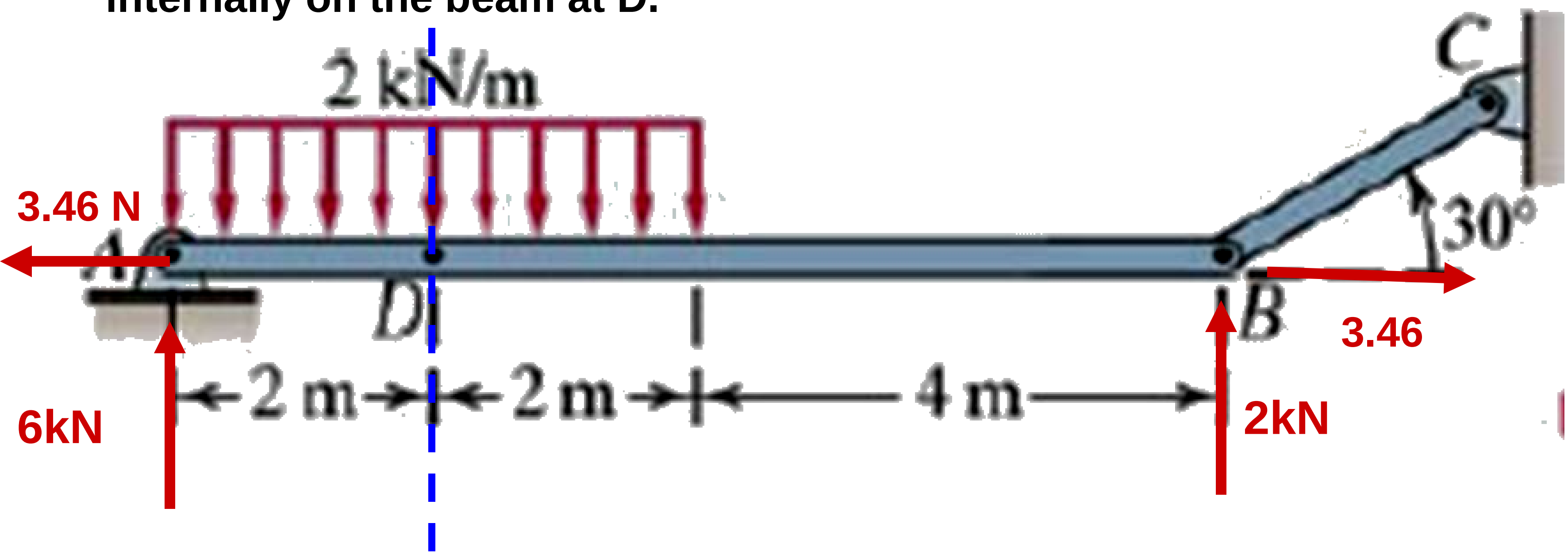
$$F_{BC} = 60 \text{ kips}$$

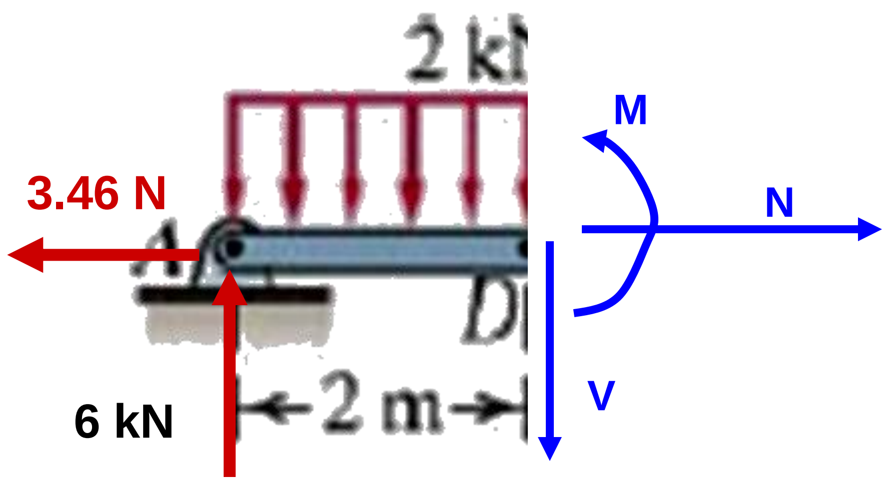
$$\sum F_x = -F_{CD} + 10 \text{ kips}$$

$$F_{CD} = 10 \text{ kips}$$



Find the axial force, shear force, and bending moment acting internally on the beam at D.



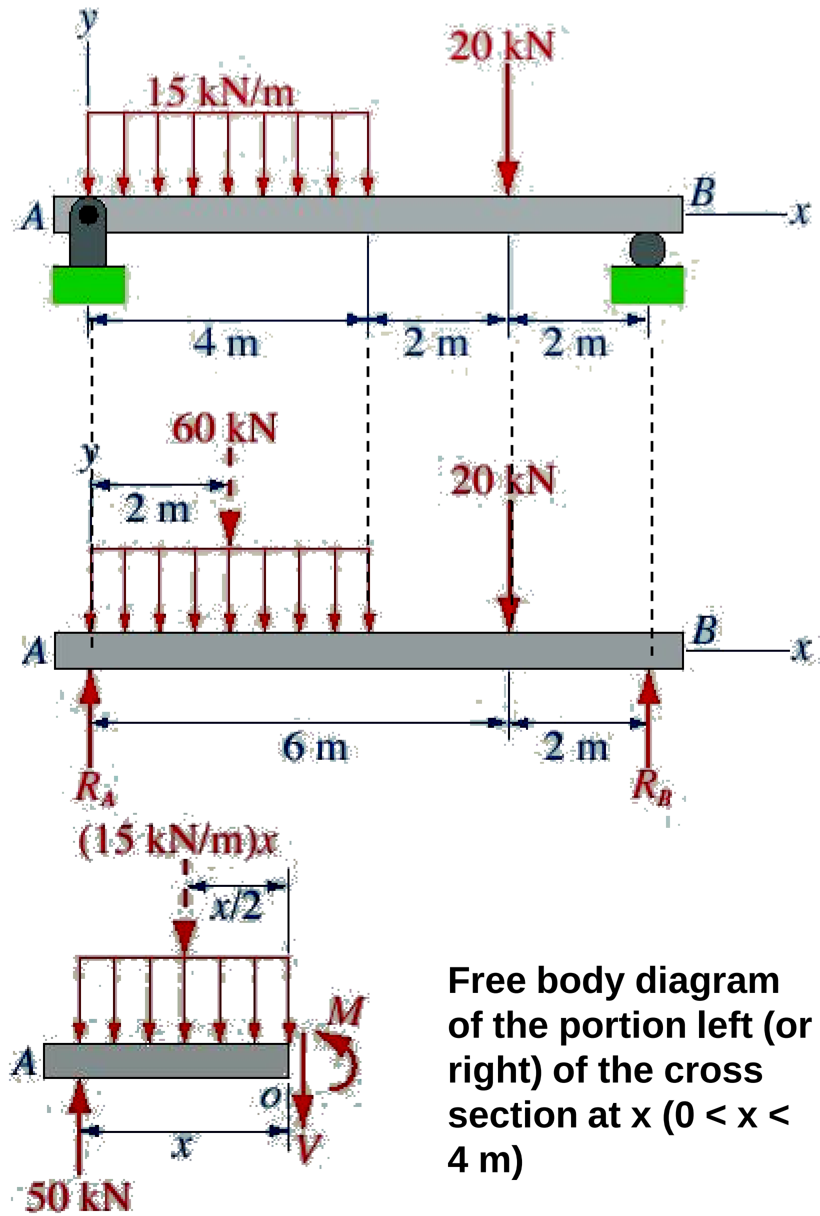


$$\sum M_A = -(2\text{ m})(60\text{ kN}) - (6\text{ m})(20\text{ kN}) + (8\text{ m})R_B = 0$$

$$\Longrightarrow \boxed{R_B = 30\text{ kN}}$$

$$\sum F_y = R_A + R_B - (60\text{ kN}) - (20\text{ kN}) = 0$$

$$\Longrightarrow \boxed{R_A = 50\text{ kN}}$$



Free body diagram of the portion left (or right) of the cross section at x (0 < x < 4 m)

Internal forces at distance x from A

$$\sum F_y = (50\text{ kN}) - \left(15\frac{\text{kN}}{\text{m}}\right)x - V = 0$$

$$\Longrightarrow V = 50 - 15x\ (\text{kN}) \quad \{x \text{ in meters}\}$$

$$\sum M_o = M - (50\text{ kN})x + \left(15\frac{\text{kN}}{\text{m}}\right)(x)\left(\frac{x}{2}\right) = 0$$

$$\Longrightarrow M = 50x - 7.5x^2\ (\text{kN}\cdot\text{m})$$

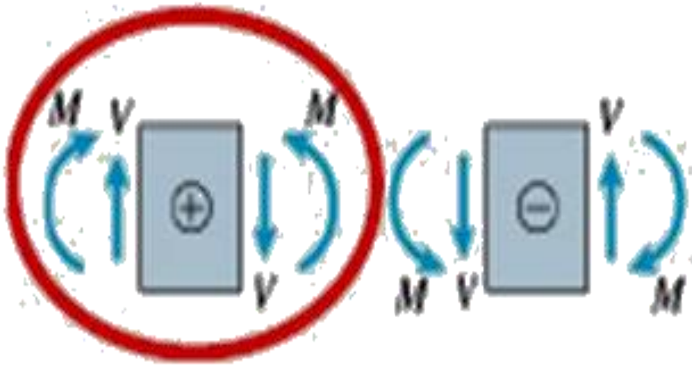
Force Diagrams

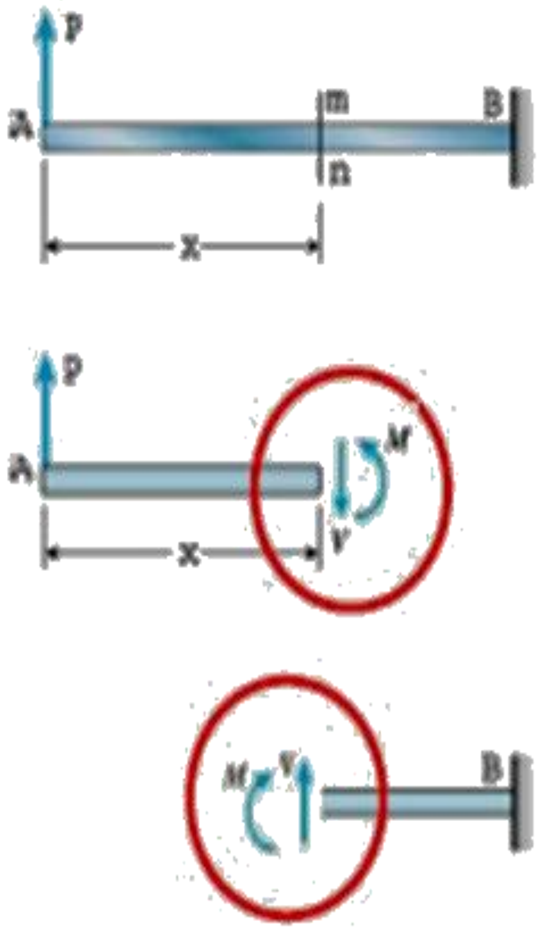
Force diagrams show the all of the internal forces acting in the member.

1) **Axial Force Diagram**

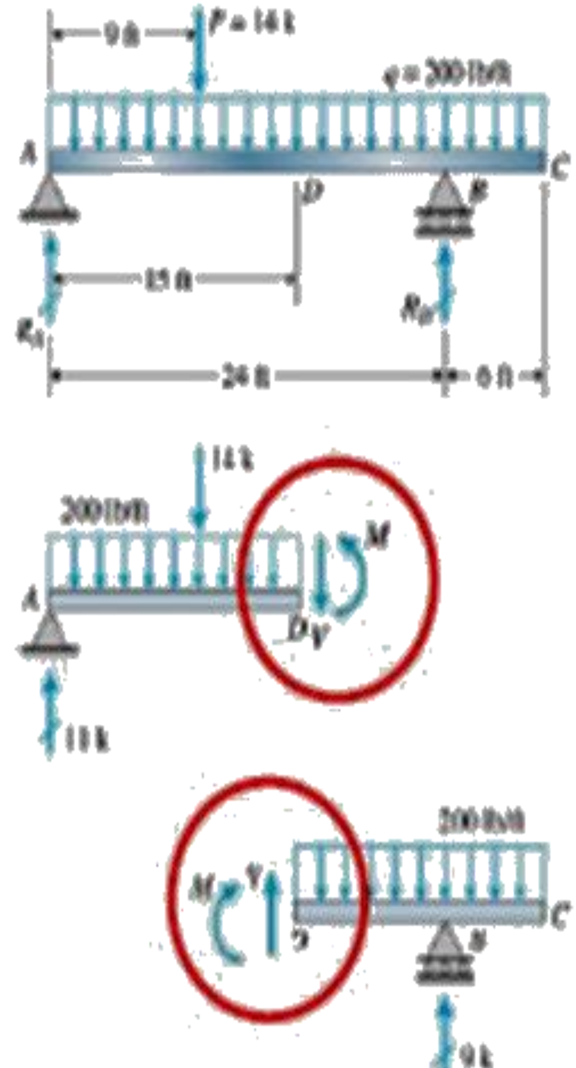
2) **Shear Diagram**

3) **Moment Diagram**





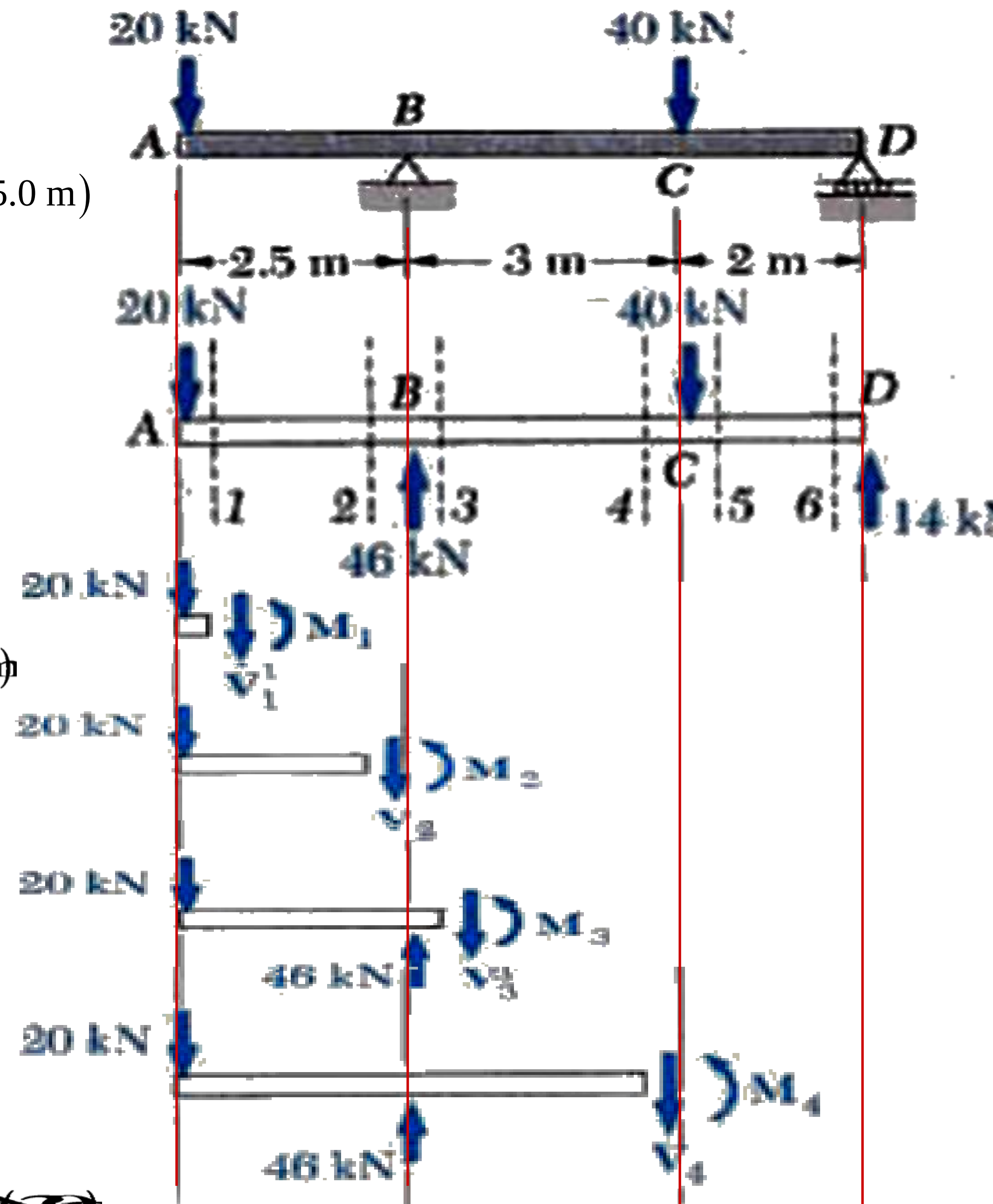
When drawing free-body diagrams for beams, always apply positive shear forces and bending moments



$$\sum F_y = 0 = -20\text{ kN} + R_{By} - 40\text{ kN} + R_D$$
$$\Rightarrow R_{By} + R_D = 60\text{ kN}$$

$$\sum M_B = 0 = 20\text{ kN}(2.5\text{ m}) - 40\text{ kN}(3.0\text{ m}) + R_D(5.0\text{ m})$$

$$\Rightarrow R_D = 14\text{ kN}$$
$$\Rightarrow R_{By} = 46\text{ kN}$$



sections 1-1.

$$\sum F_{1-1} = V_{1-1} = -20\text{kN}$$

$$\sum M_{1-1} = M_{1-1} = -20\text{kN}(x_1)$$

section 2-2

$$\sum F_{2-2} = V_{2-2} = -20\text{kN}$$

$$\sum M_{2-2} = 0 = M_{2-2} + 20\text{kN}(2.5\text{m})$$
$$\Rightarrow M_{2-2} = -50\text{kNm}$$

section at 3-3.

$$\sum F = V_{3-3} = -20\text{kN} + 46\text{kN}$$
$$\Rightarrow V_{3-3} = 26\text{kN}$$

$$\sum M_B = 0 = M_{3-3} + 20\text{kN}(2.5\text{m})$$
$$\Rightarrow M_{3-3} = -50\text{kNm}$$

section 4-4.

$$\sum F_y = V_{4-4} = -20\text{kN} + 46\text{kN}$$
$$\Rightarrow V_{4-4} = 26\text{kN}$$

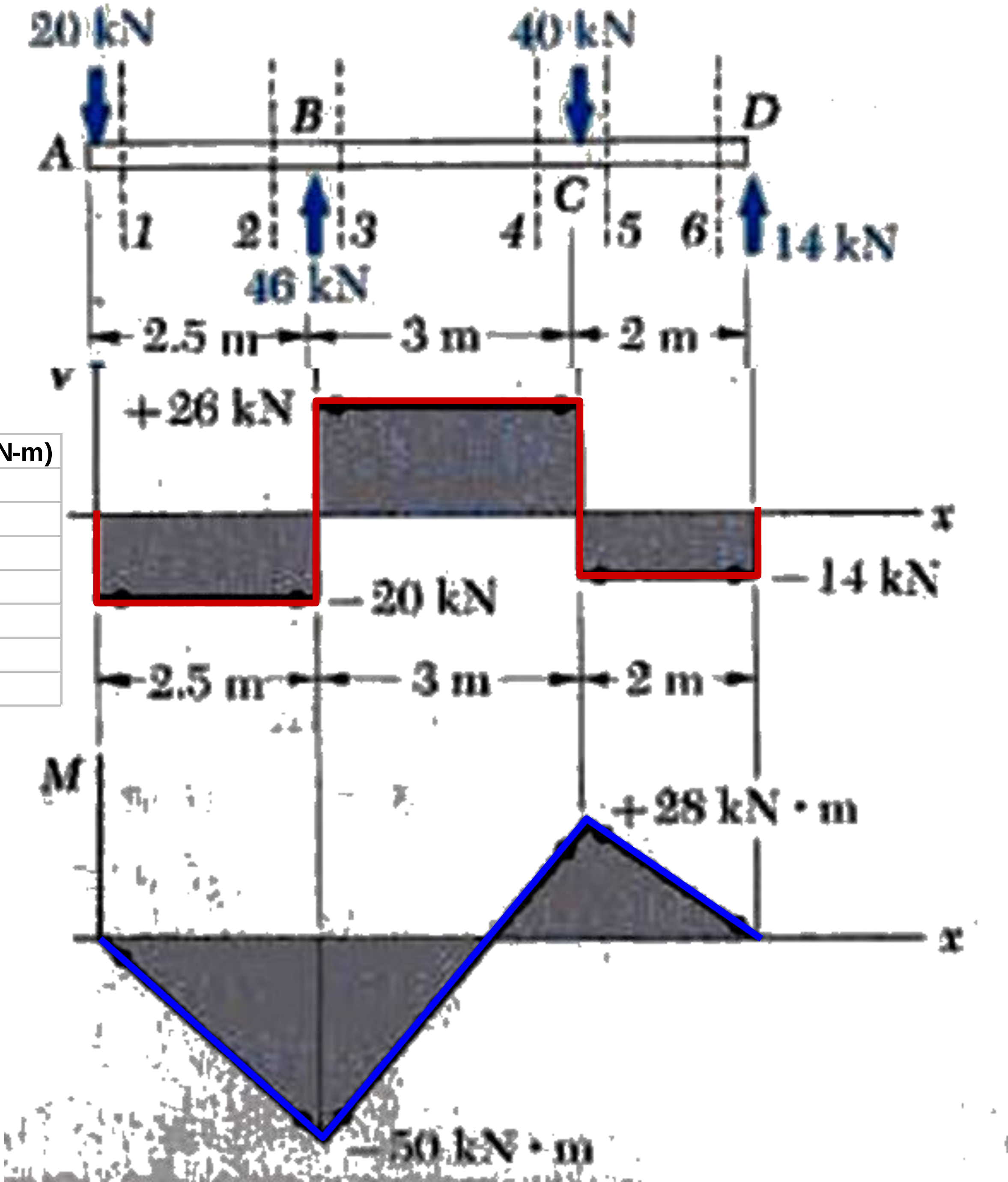
$$\sum M_{4-4} = M_{4-4} = -20\text{kN}(5.5\text{m}) + 46\text{kN}(3.0\text{m})$$
$$\Rightarrow M_{4-4} = 28\text{kNm}$$

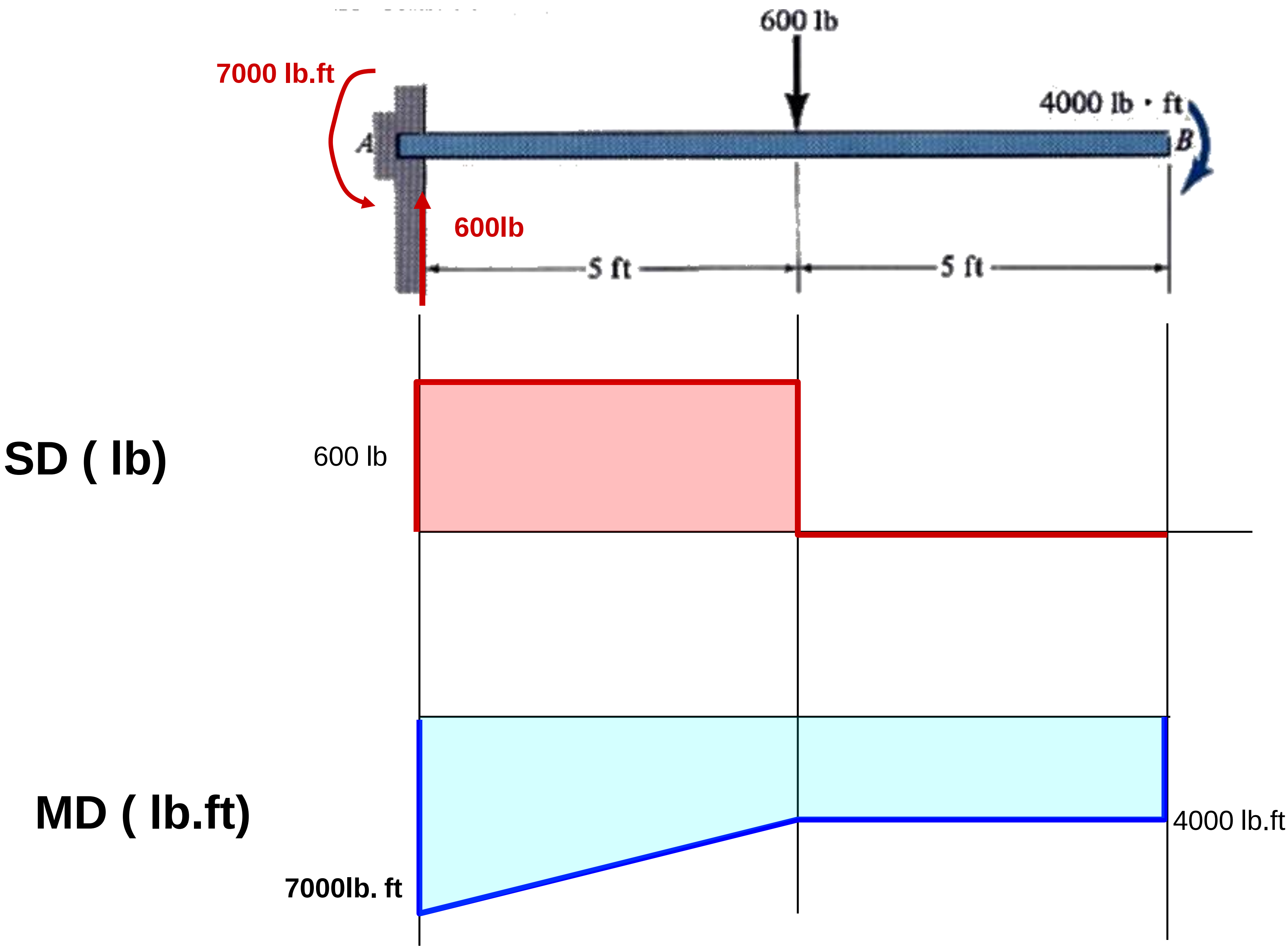
Shear and Bending moment Diagrams

In order to generate a shear and bending moment diagram one needs to

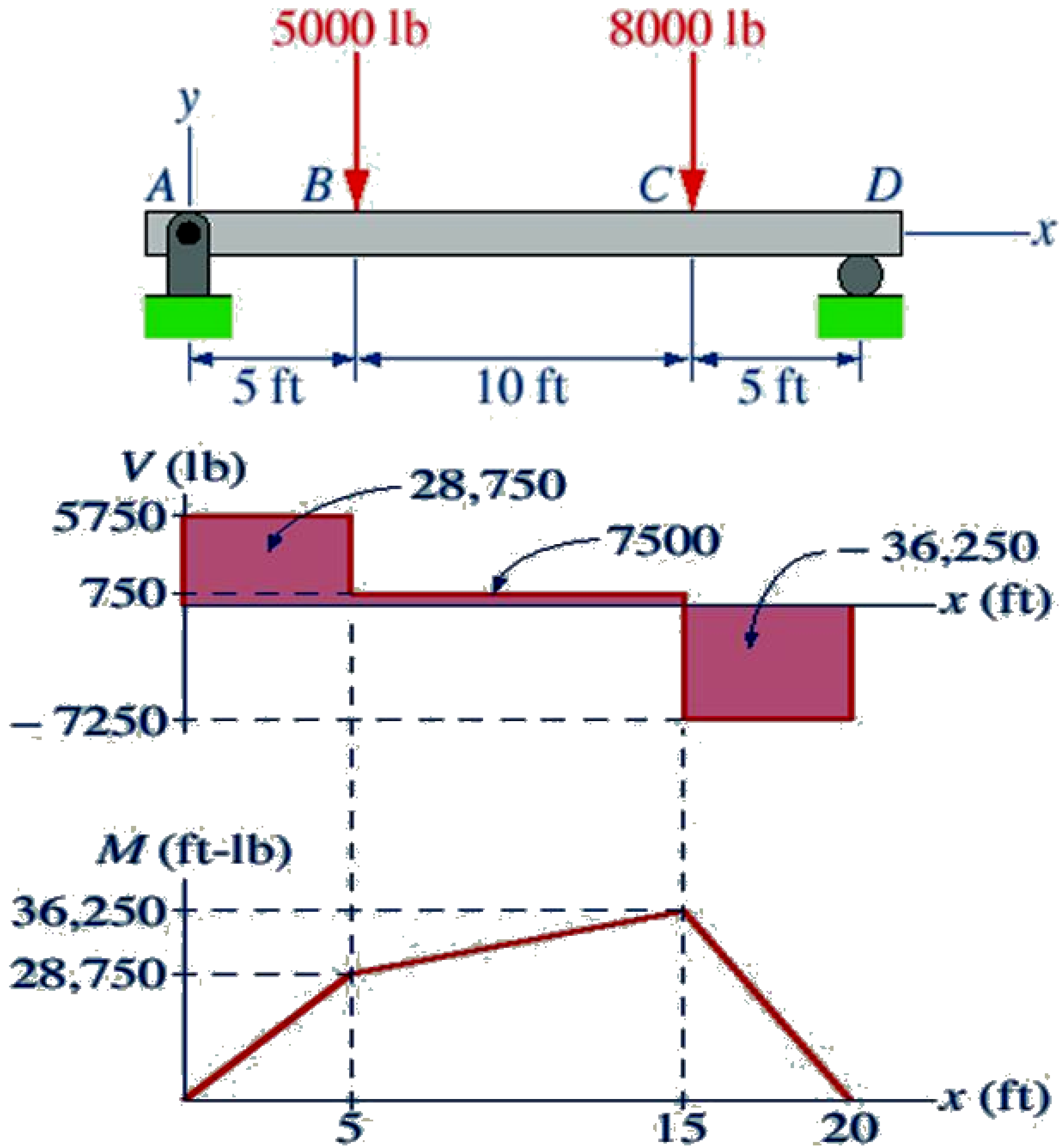
- **Draw the free-body diagram**
- **Solve for reactions**
- **Solve for the internal forces (shear, V, and bending moment, M)**

	Location (m)	Shear (kN)	Moment (kN-m)
1	0	-20	0
2	2.5	-20	-50
3	2.5	26	-50
4	5.5	26	28
5	5.5	-14	28
6	7.5	-14	0
7	7.5	0	0



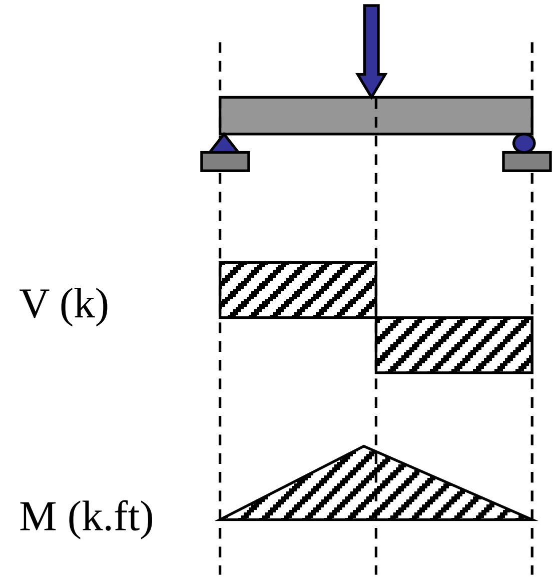


600 lb
7000 lb.ft
4000 lb · ft
A
B
600lb
5 ft
5 ft
SD (lb)
600 lb
MD (lb.ft)
4000 lb.ft
7000lb. ft



5000 lb
8000 lb
y
A
B
C
D
x
5 ft
10 ft
5 ft
V (lb)
28,750
5750
7500
– 36,250
750
x (ft)
– 7250
M (ft-lb)
36,250
28,750
x (ft)
5
15
20

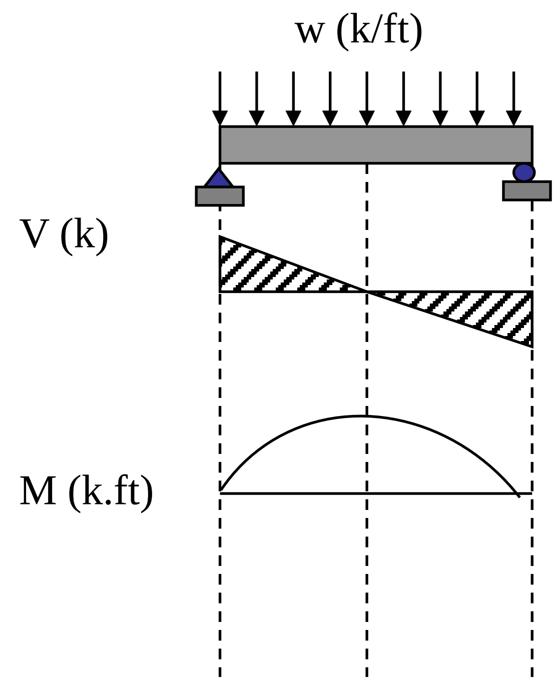
Concentrated Loads:



▪ Shear forces are consistent in magnitude. Therefore, shear diagrams are flat lines (no slope; horizontal).

▪ Moment vary linearly between concentrated loads. Therefore, moment diagrams are composed of sloping lines for concentrated loads.

Uniformly Distributed Loads:



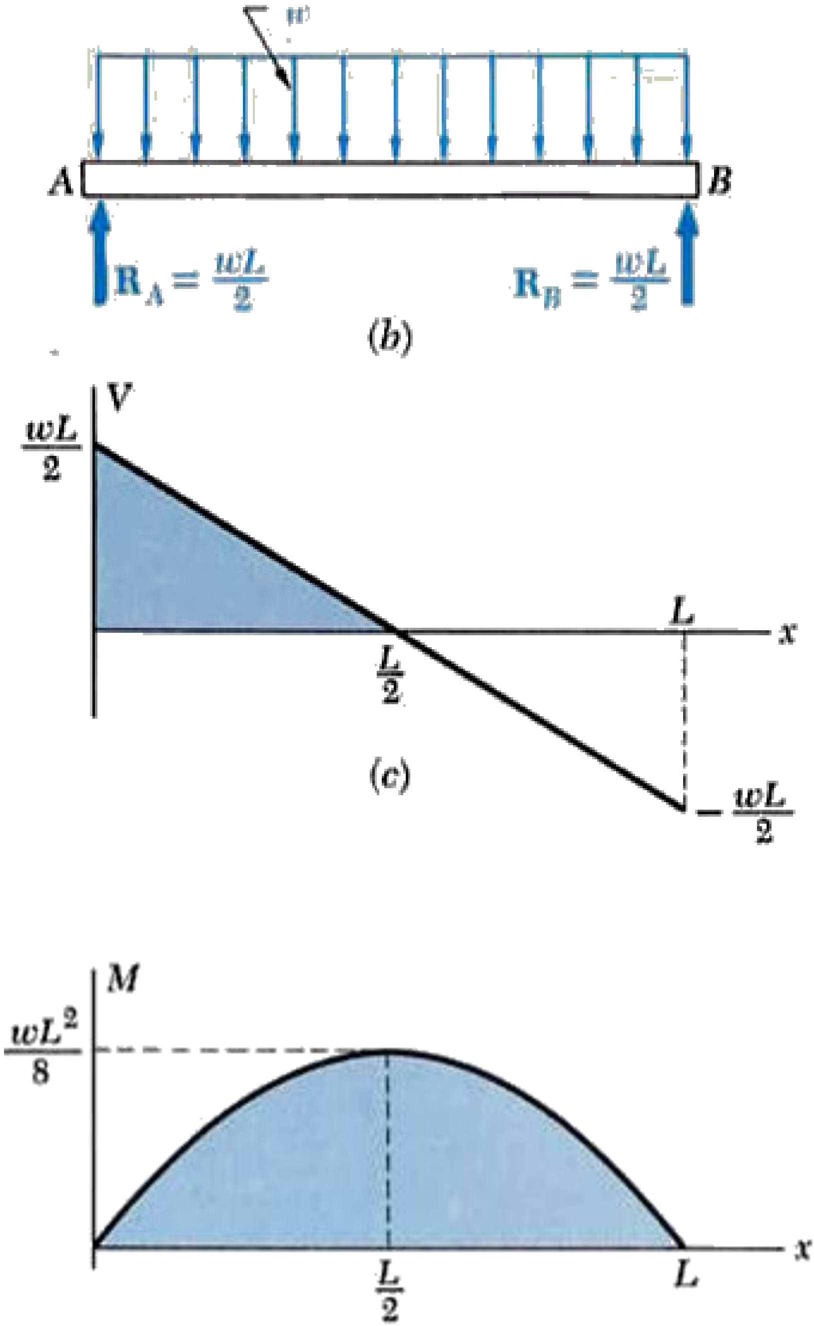
▪ UDLs produce linearly varying shear forces—shear diagrams consist of sloped lines.

▪ UDLs produce parabollically varying moments; therefore, moment diagrams are curves.

$$\frac{dM}{dx} = V$$

$$M = \frac{1}{2}\left(\frac{wL}{2}\right)\left(\frac{L}{2}\right) = \frac{wL^2}{8}$$

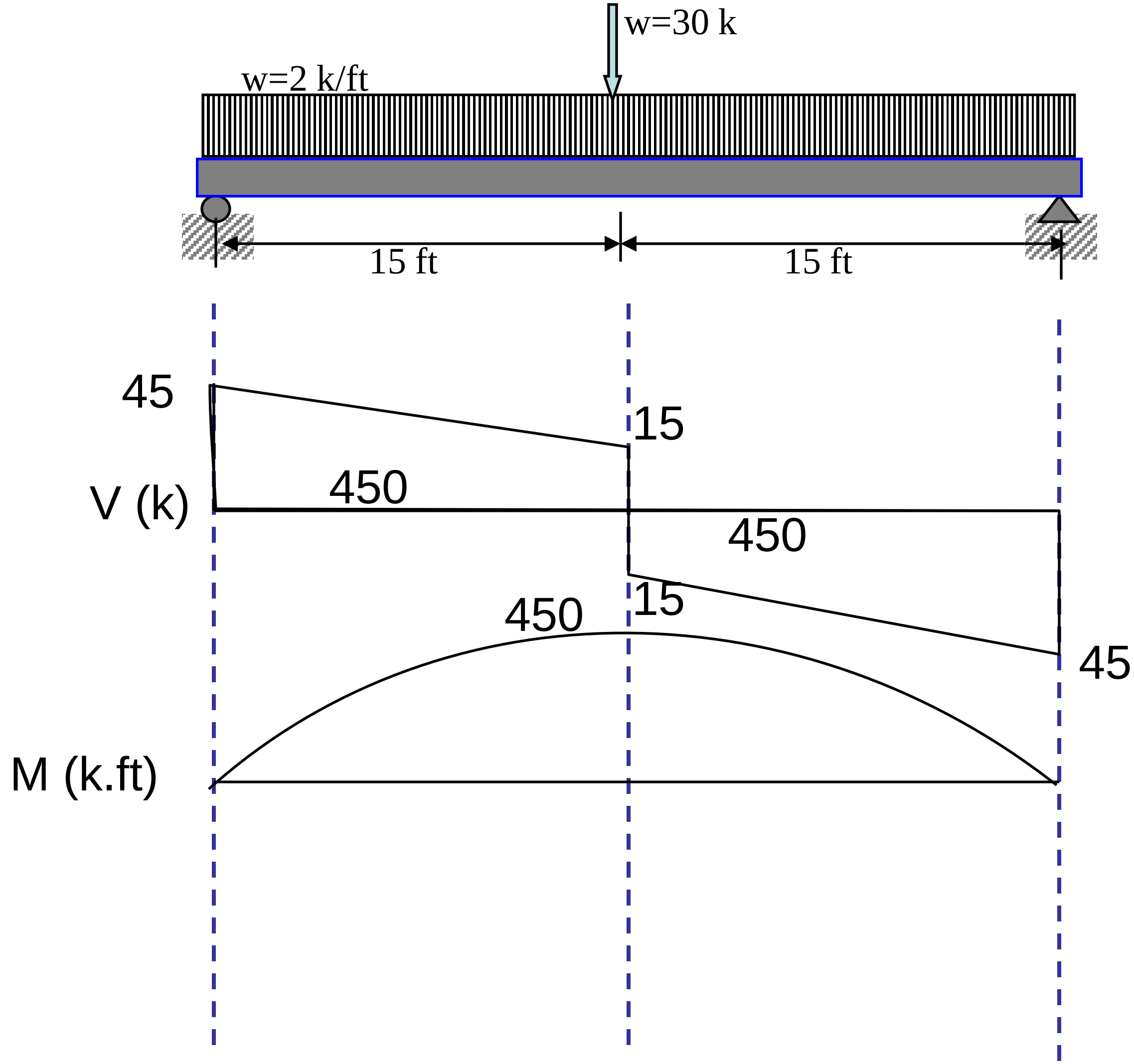
Note that the slope of the moment diagram is equal to the shear.

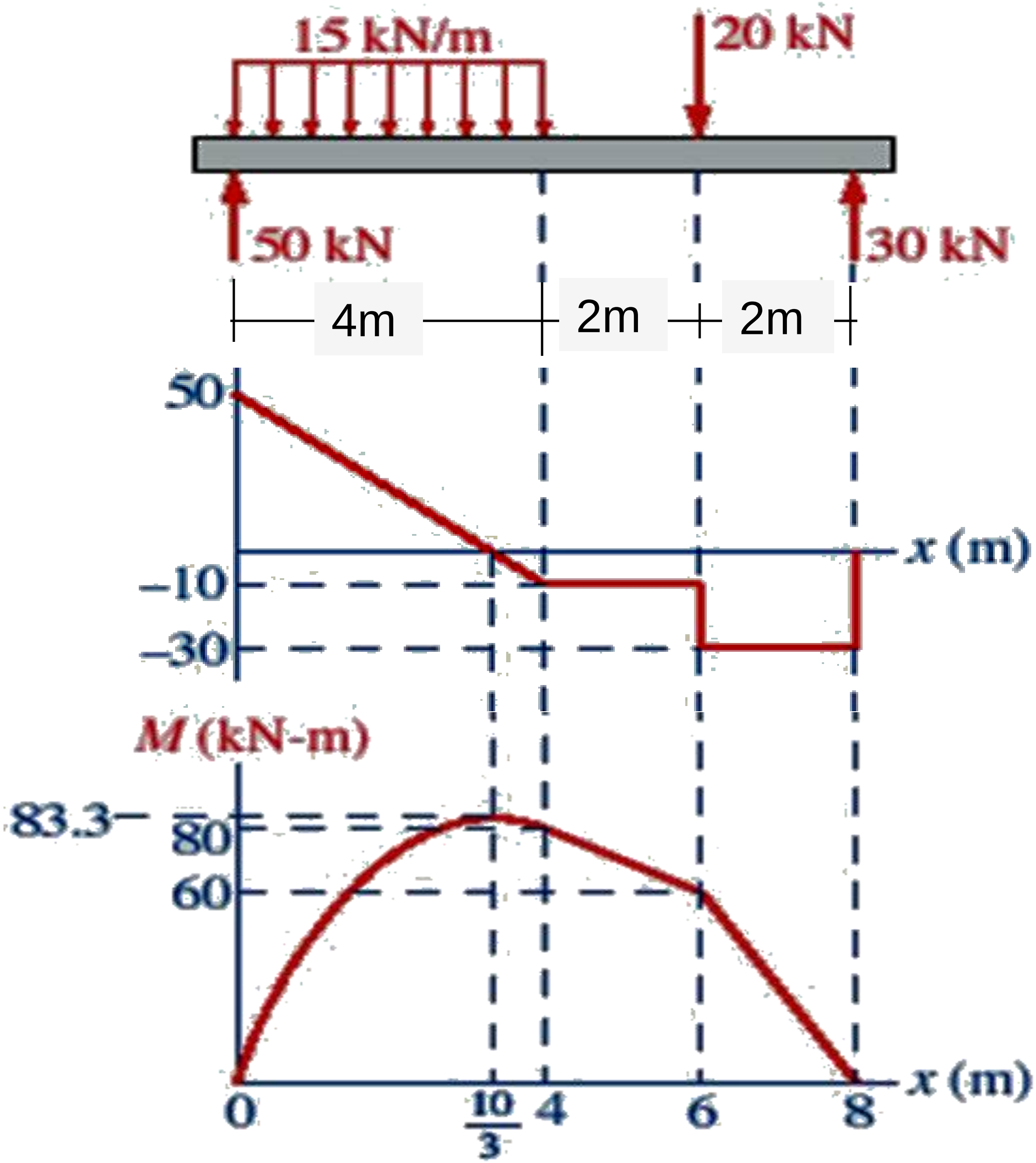


For the beam shown draw the shear and moment diagram:

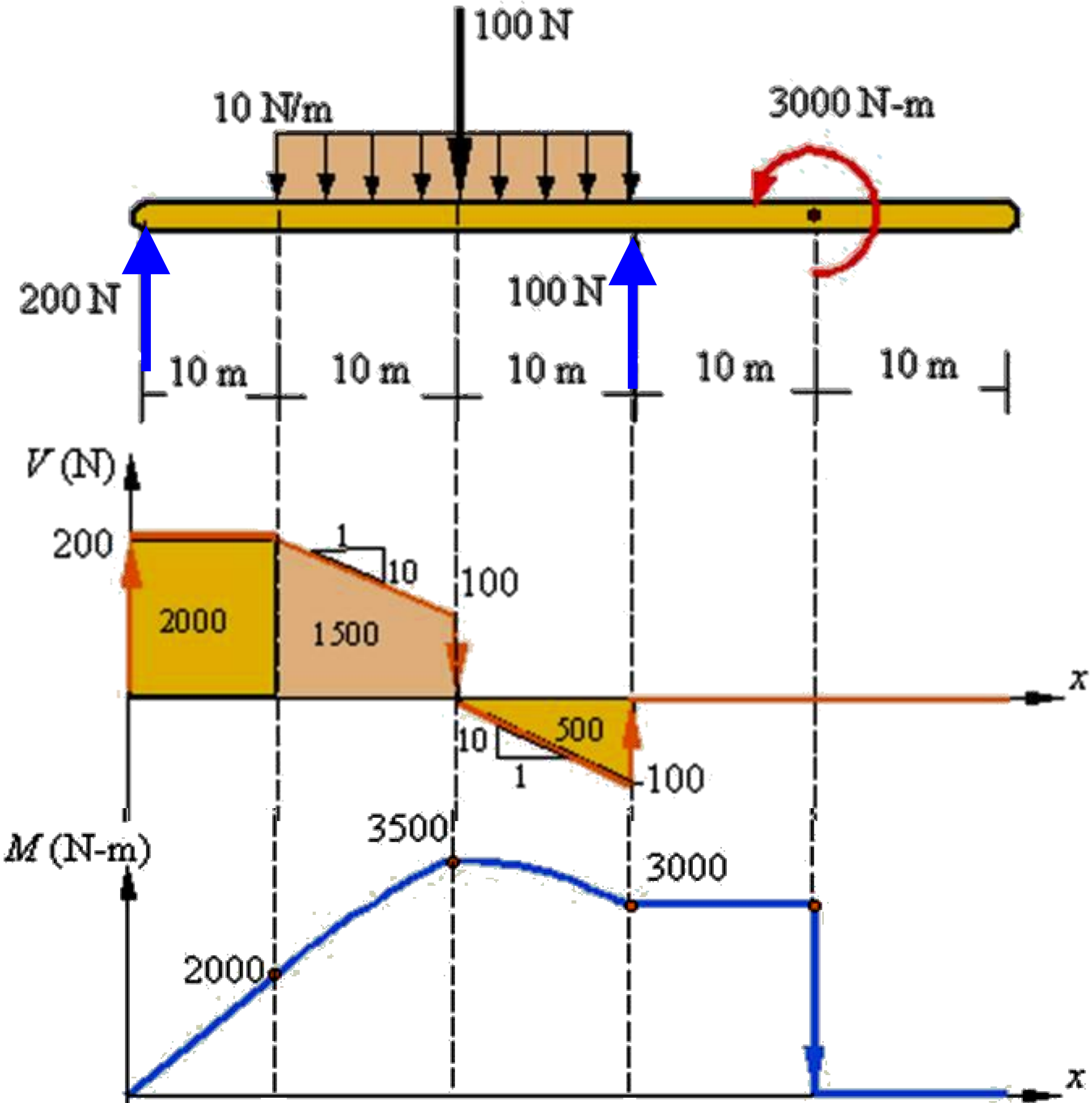
$$R_A = R_B = wl/2 + P/2$$

$$R_A = R_B = 2\times 30/2 + 30/2 = 45 lb$$





15 kN/m
20 kN
50 kN
30 kN
4m
2m
2m
50
-10
-30
x (m)
M (kN-m)
83.3
80
60
x (m)
0
10/3
4
6
8



100 N
10 N/m
3000 N-m
200 N
100 N
10 m
10 m
10 m
10 m
10 m
V (N)
200
1
10
100
2000
1500
x
500
10
1
-100
M (N-m)
3500
3000
2000
x

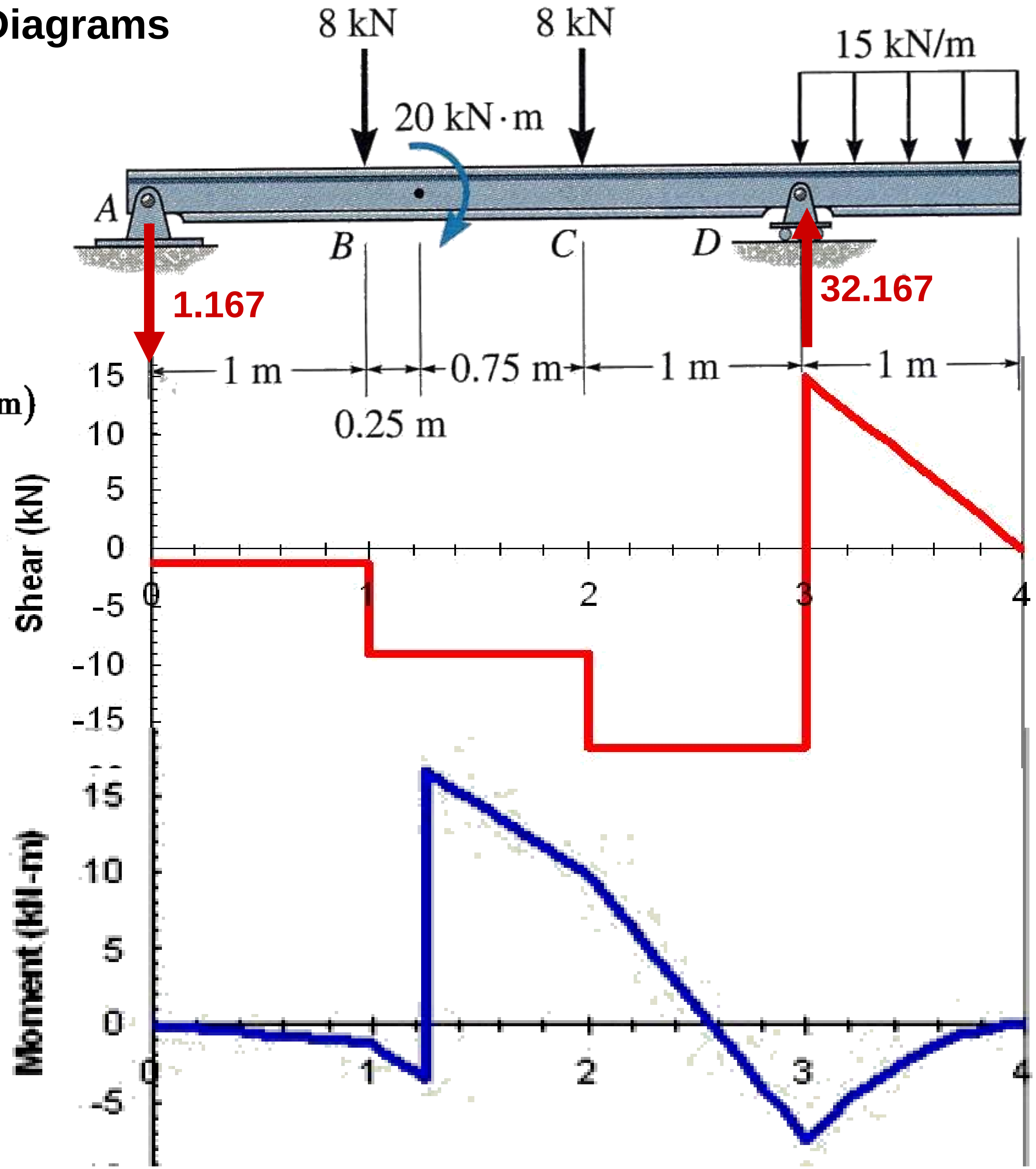
Draw shear and moment Diagrams

$$\sum F_x = 0 = R_{Ax}$$

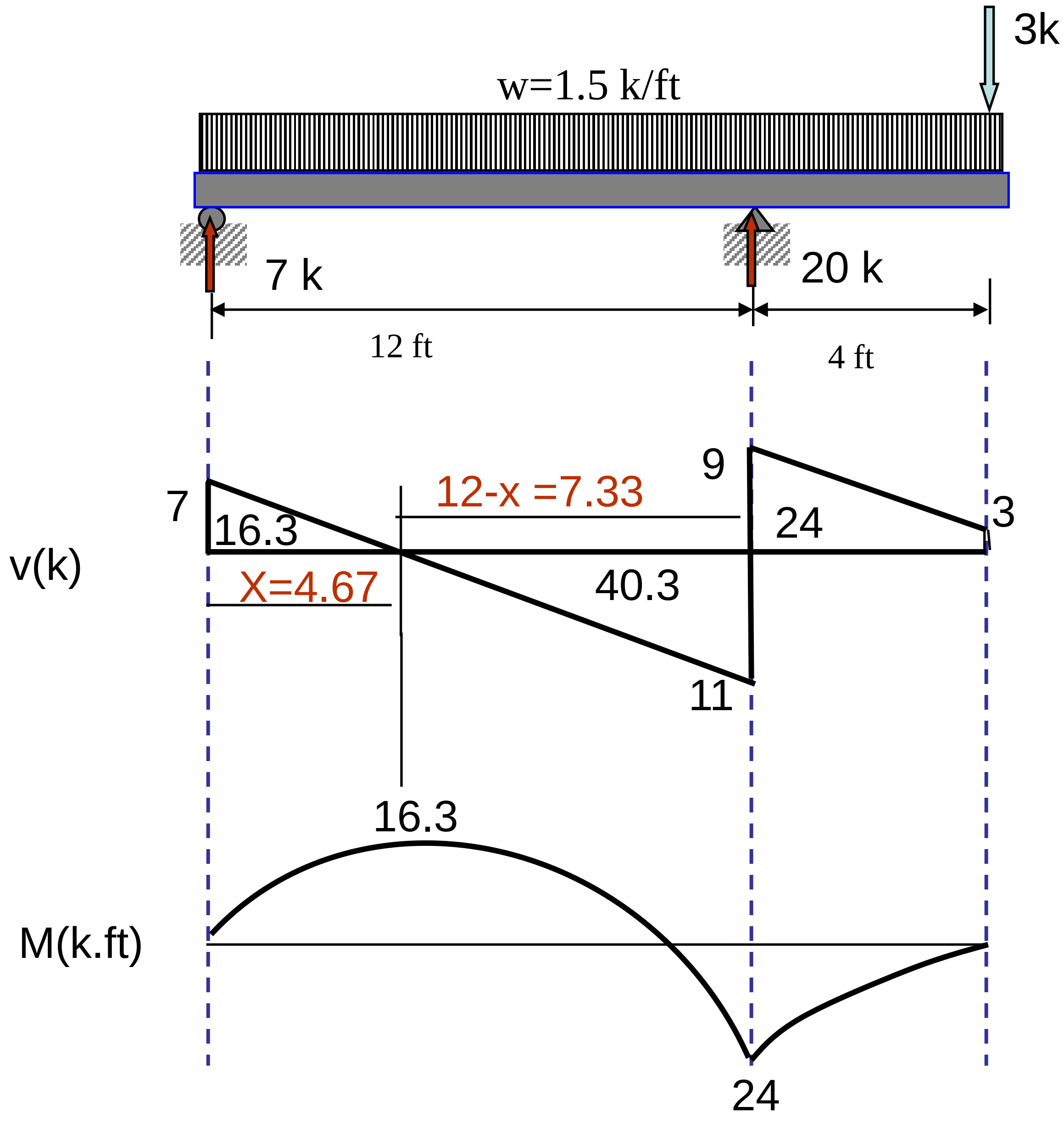
$$\sum F_y = 0 = R_{Ay} - 8\text{ kN}$$
$$-8\text{ kN} - 15\text{ kN/m}(1\text{ m}) + R_{Ey}$$

$$\sum M_A = 0 = -8\text{ kN}(1\text{ m}) - 20\text{ kN-m}$$
$$-8\text{ kN}(2\text{ m})$$
$$-15\text{ kN/m}(1\text{ m})(3.5\text{ m}) + R_{Dy}(3\text{ m})$$

$$R_{Dy} = 32.167\text{ kN}$$
$$R_{Ax} = 0\text{ kN}$$
$$R_{Ay} = -1.167\text{ kN}$$



For the beam shown here draw the shear and moment diagram:



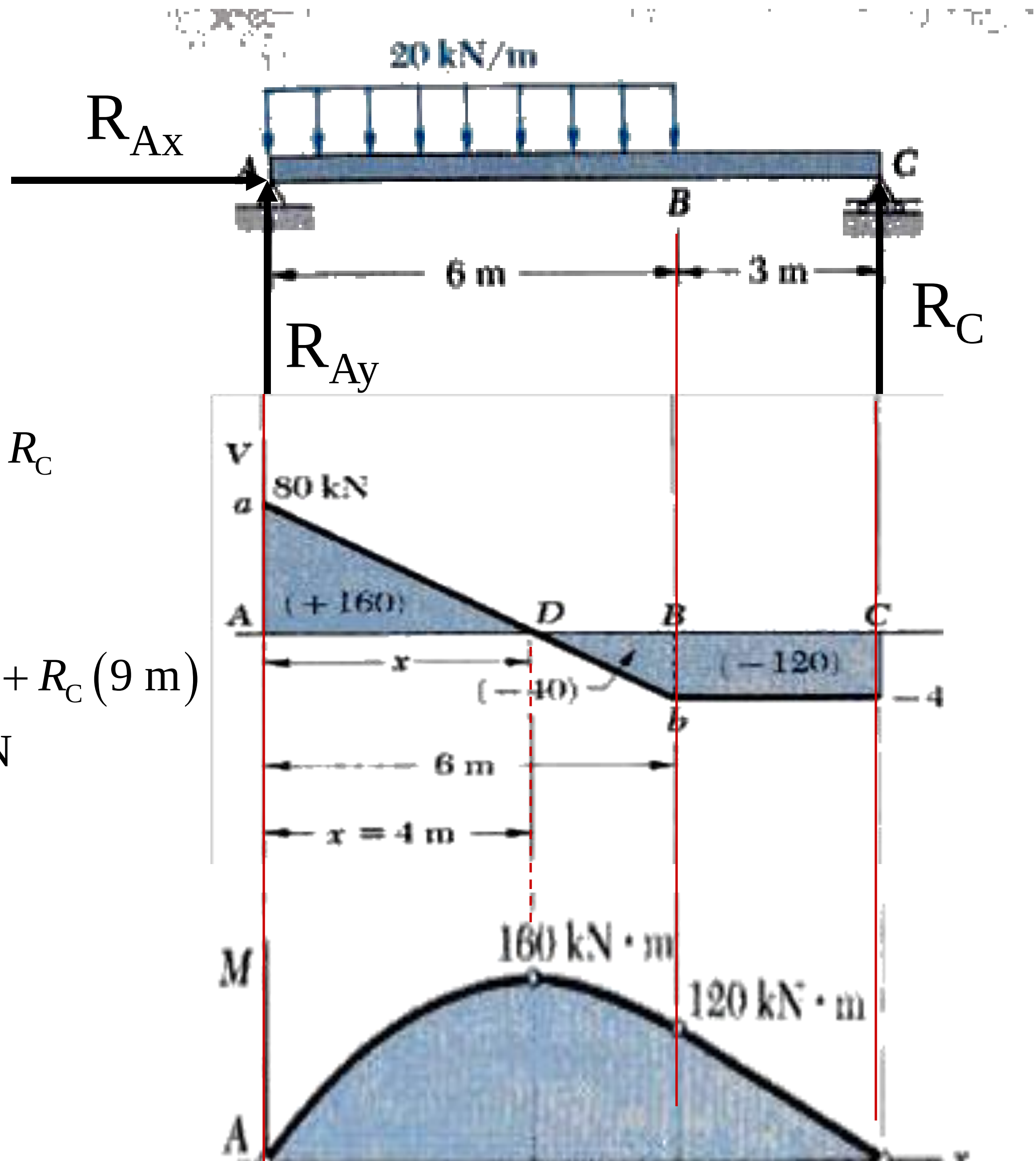
$$\sum F_x = 0 = R_{Ax}$$

$$\sum F_y = 0 = R_{Ay} - 20\ \text{kN/m}\left(6\ \text{m}\right) + R_C$$

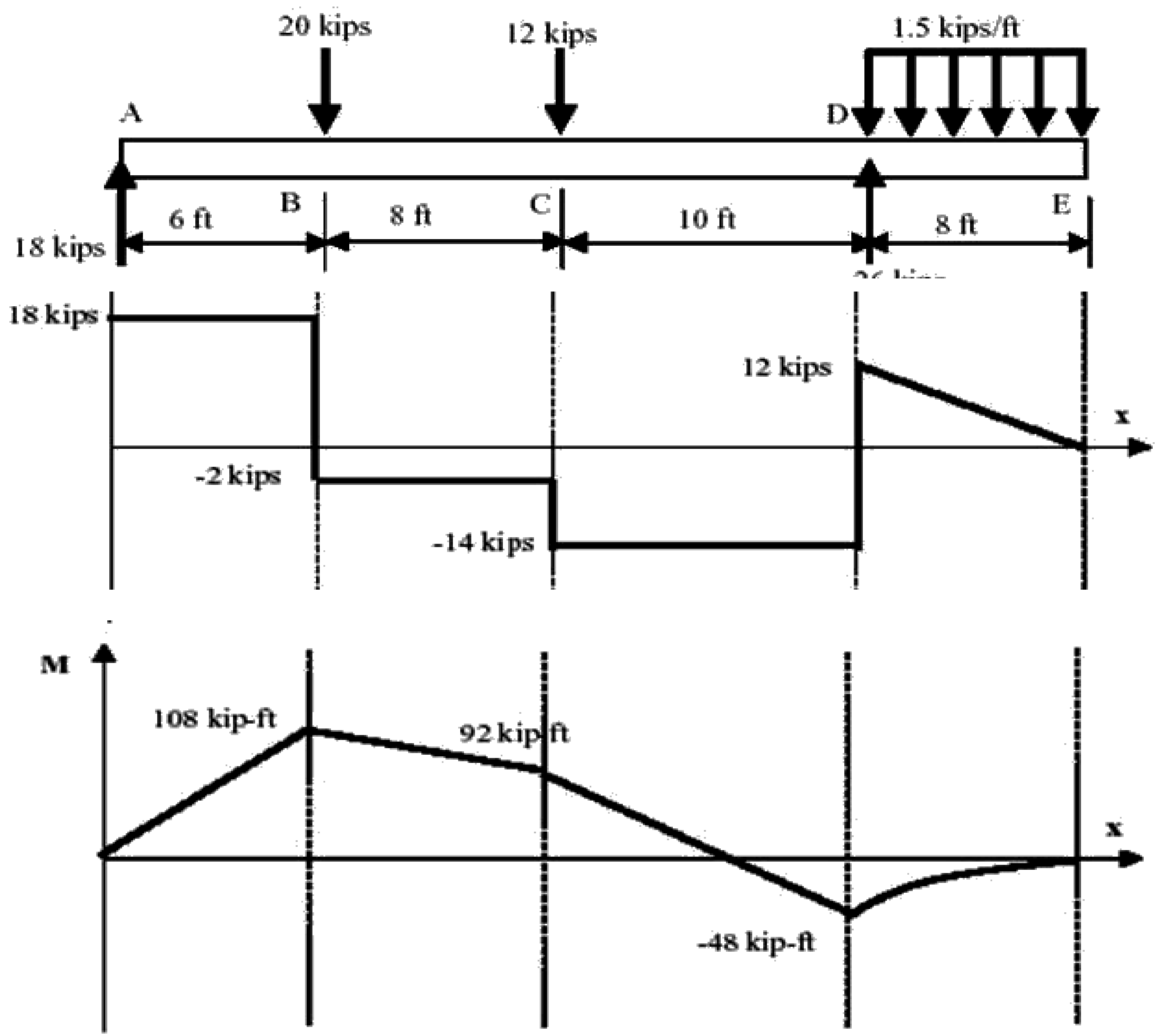
$$\Rightarrow R_{Ay} + R_C = 120\ \text{kN}$$

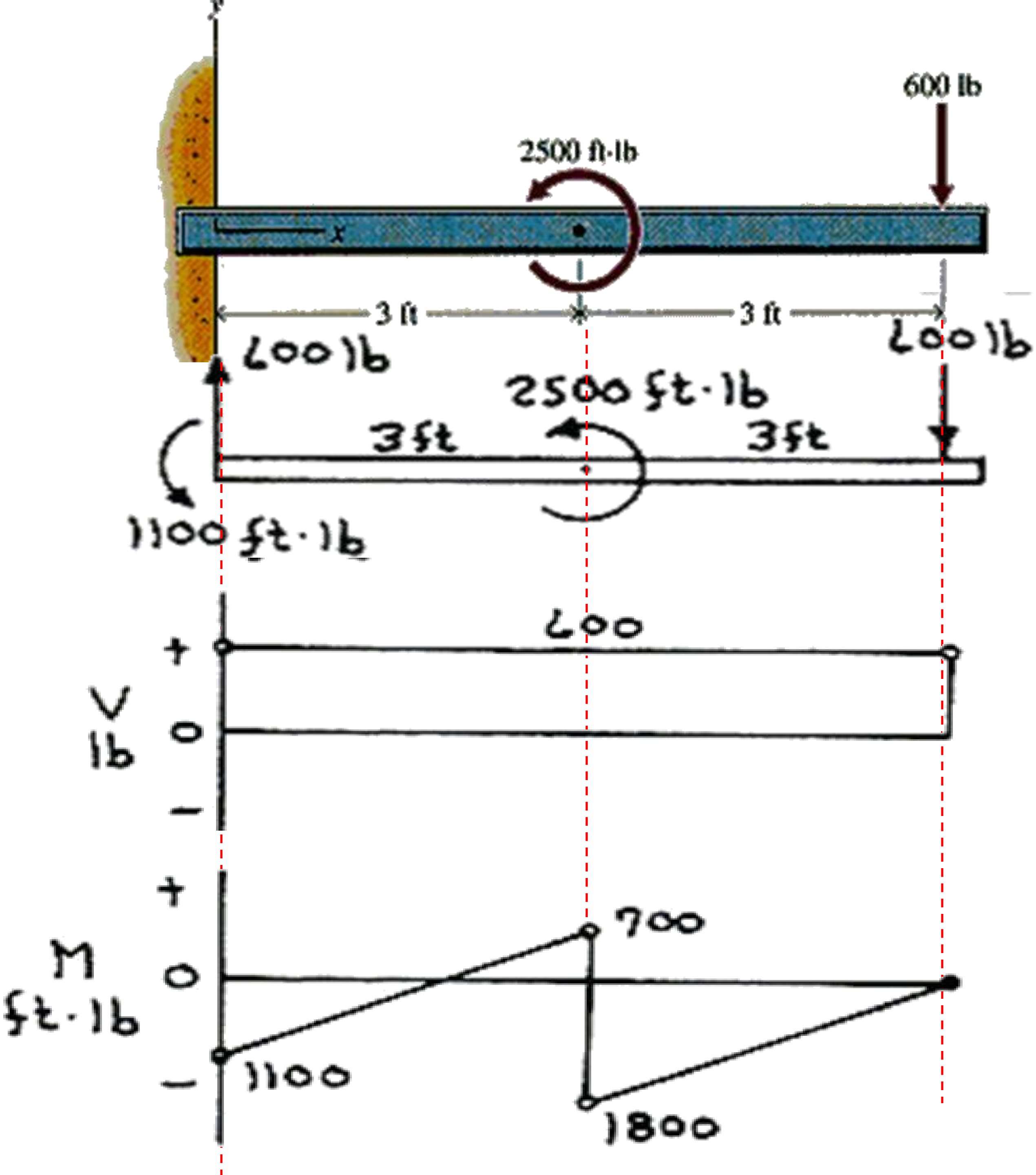
$$\sum M_A = 0 = -20\ \text{kN/m}\left(6\ \text{m}\right)\left(3\ \text{m}\right) + R_C\left(9\ \text{m}\right)$$

$$\Rightarrow R_C = 40\ \text{kN}\ \&\ R_{Ay} = 80\ \text{kN}$$



Construction of the shear force diagram





600 lb
2500 ft·lb
3 ft
3 ft
600 lb
600 lb
2500 ft·lb
3ft
3ft
1100 ft·lb
600
V
lb
+
0
−
700
M
ft·lb
+
0
−
1100
1800

$$\sum F_x = 0 = B_X$$

$$\sum F_y = 0 = R_{Cy} + B_Y - 150\ \text{lb/ft}(6\ \text{ft})$$

$$\sum M_B = 0 = R_{Cy}(6\ \text{ft}) - 150\ \text{lb/ft}(6\ \text{ft})(3\ \text{ft}) - 800\ \text{lb-ft}$$

$R_{Cy} = 583.33$ lb

$B_X = 0$ lb

$B_Y = 316.67$ lb

$$\sum F_x = 0 = R_{Ax}$$

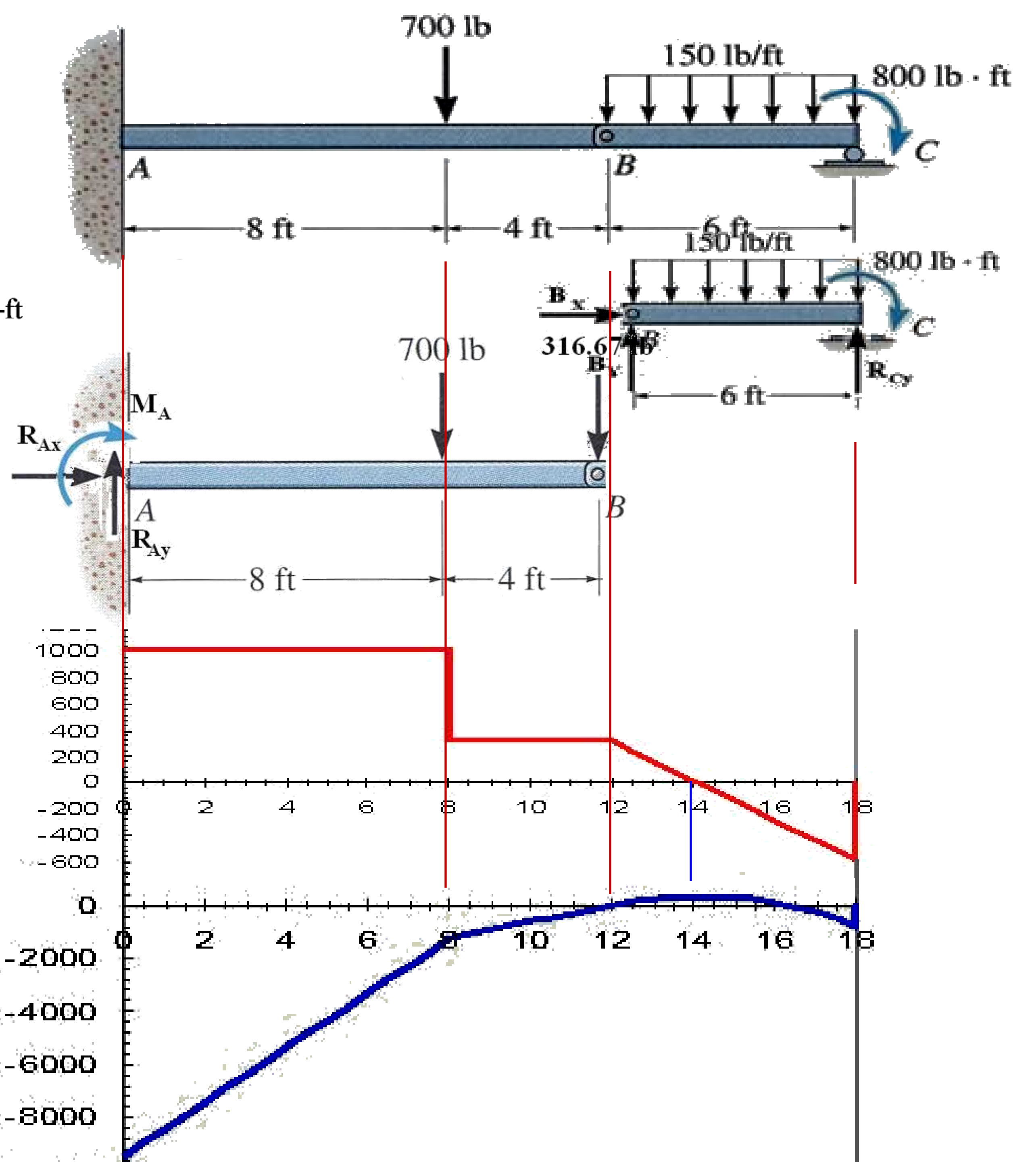
$$\sum F_y = 0 = R_{Ay} - 700\ \text{lb} - 316.67\ \text{lb}$$

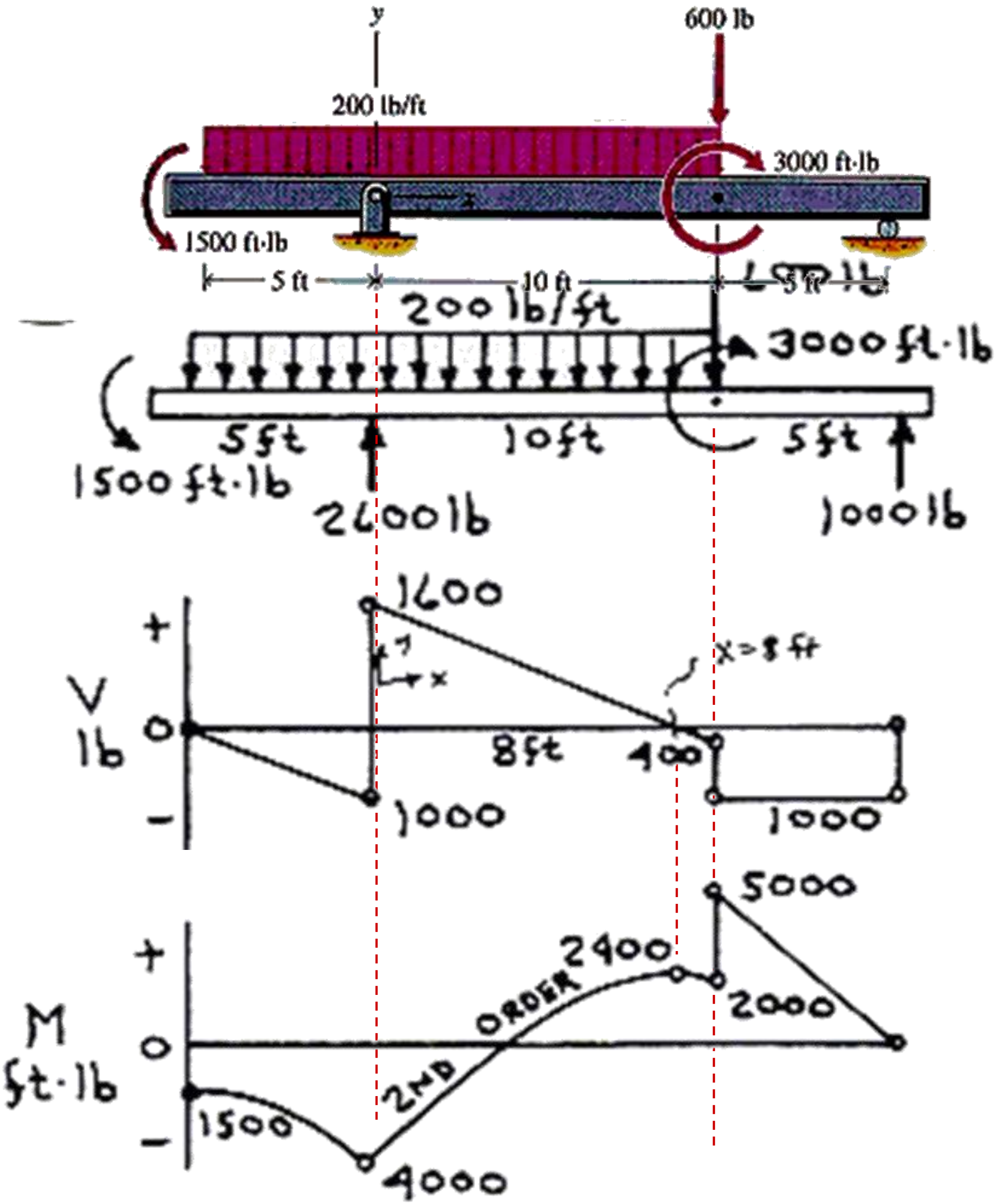
$$\sum M_B = 0 = -M_A - 700\ \text{lb}(8\ \text{ft}) - 316.67\ \text{lb}(12\ \text{ft})$$

$R_{Ay} = 1016.67$ lb

$R_{Ax} = 0$ lb

$M_A = -9400$ lb-ft





y
600 lb
200 lb/ft
3000 ft·lb
1500 ft·lb
5 ft
10 ft
5 ft
200 lb/ft
3000 ft·lb
1500 ft·lb
5ft
10ft
5ft
2600 lb
1000 lb
V
lb
1600
x = 8 ft
8ft
400
1000
1000
M
ft·lb
5000
2400
2ND ORDER
2000
1500
4000

Engineering Mechanics

Statics & Strength of Materials

Centroids of an Area

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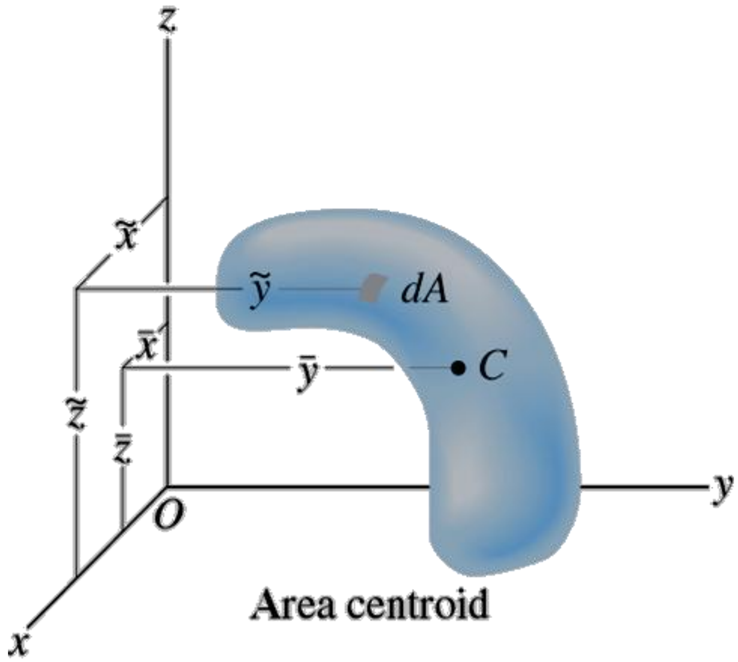
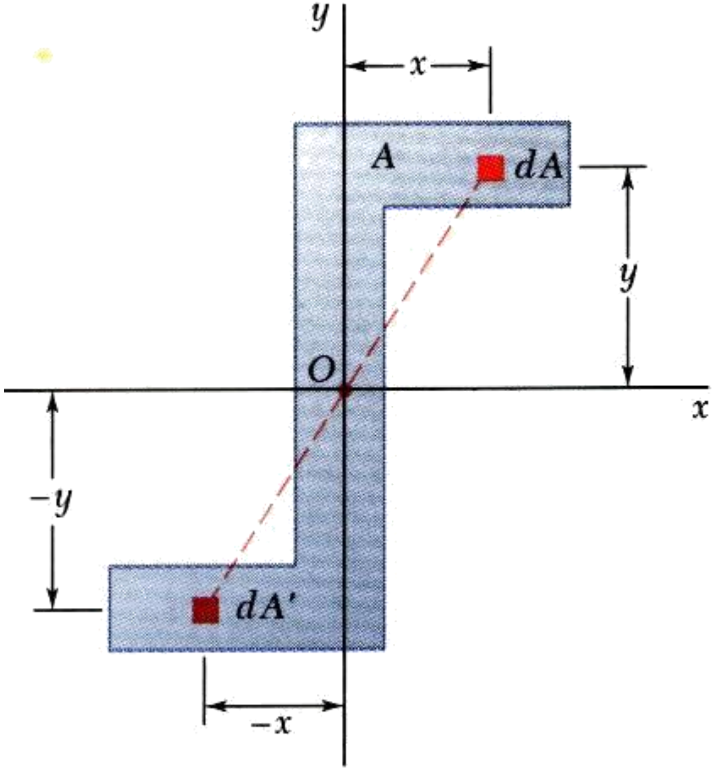
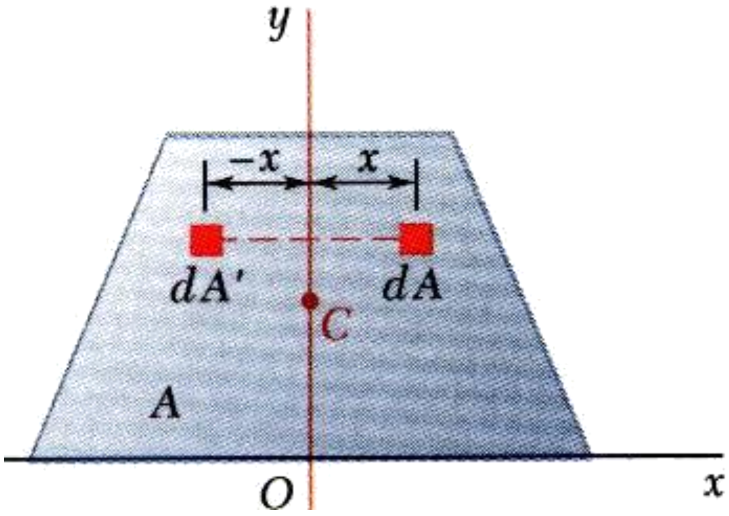
Centroid of an Area

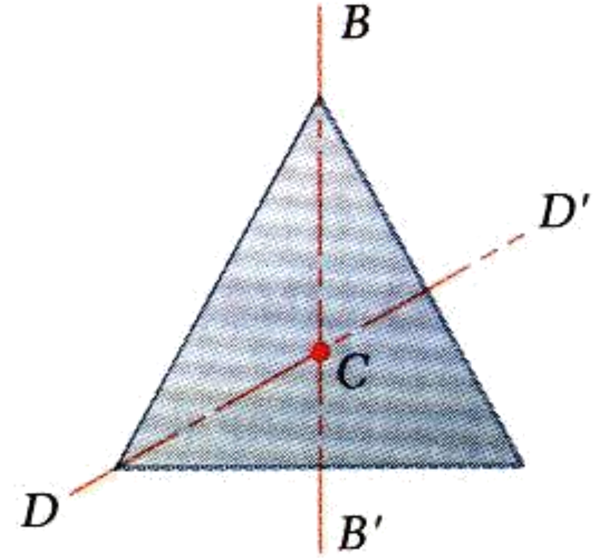
$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} \qquad \bar{z} = \frac{\int_A \tilde{z}\, dA}{\int_A dA}$$



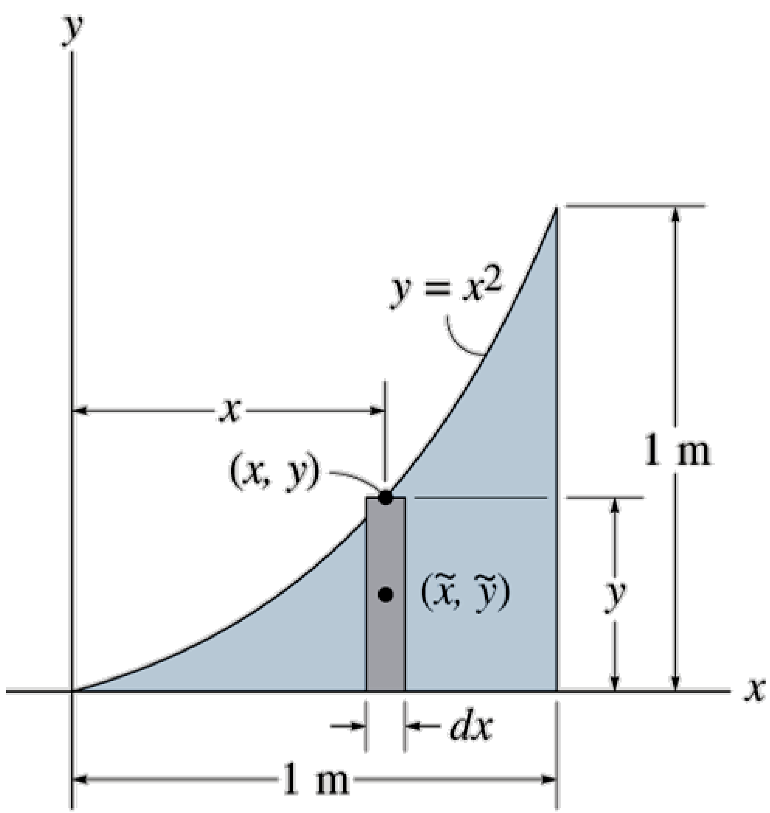
The centroid of the area coincides with the center of symmetry.

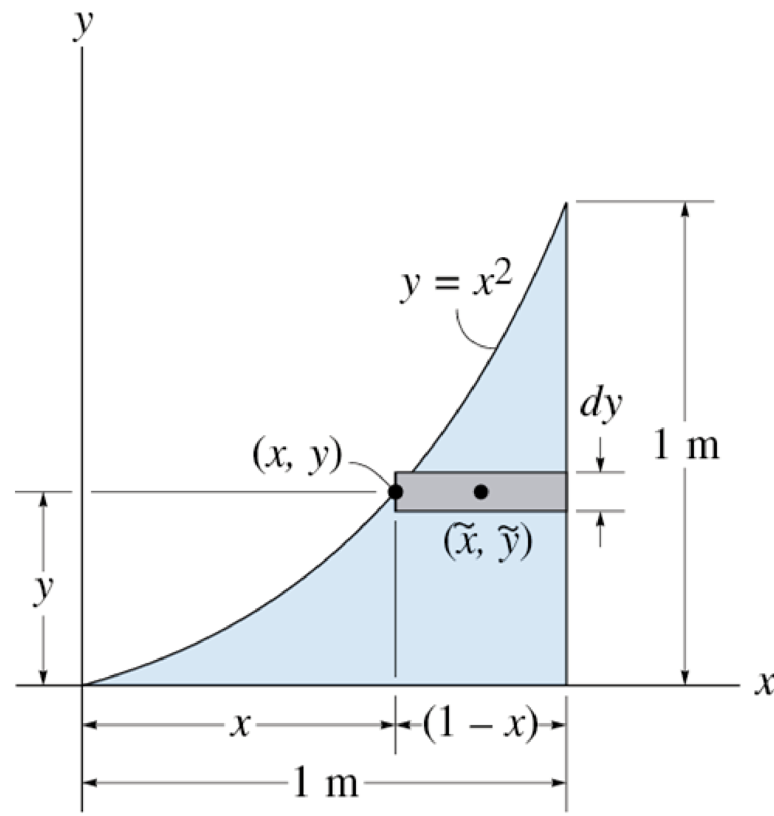






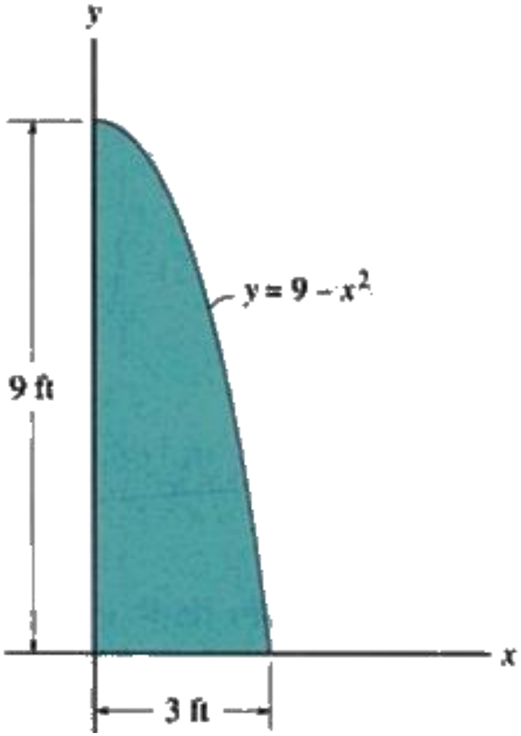
Locate the Centroid of the area shown

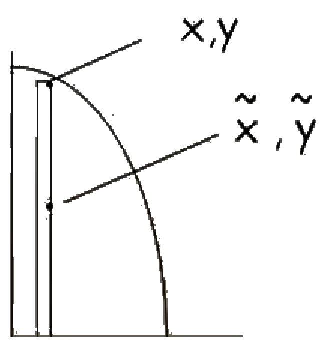




$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA}$$

Find: The centroid location ($\bar{x}$, $\bar{y}$)





1. Since y is given in terms of x, choose dA as a vertical rectangular strip.

2. $dA = y\,dx = (9 - x^2)\,dx$

3. $\tilde{x} = x$ and $\tilde{y} = y / 2$

4. $\bar{x} = (\int_A \tilde{x}\, dA) / (\int_A dA)$

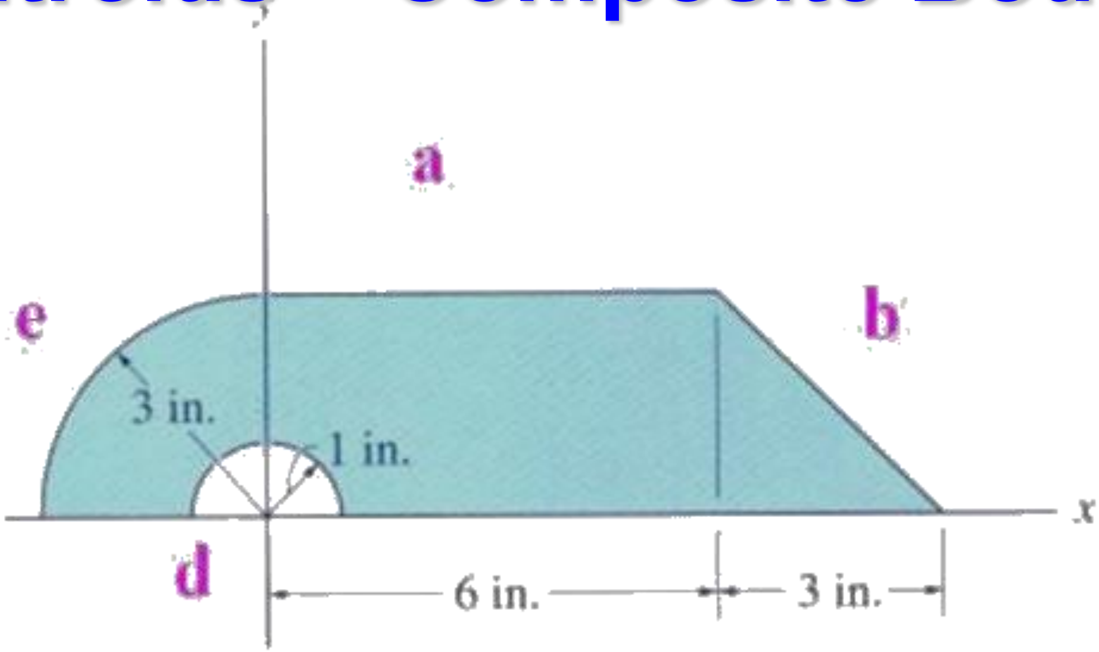
$$= \frac{\int_0^3 x\,(9 - x^2)\,dx}{\int_0^3 (9 - x^2)\,dx} = \frac{[\,9\,(x^2)/2 - (x^4)/4\,]_0^3}{[\,9x - (x^3)/3\,]_0^3}$$

$= (9 (9) / 2 - 81 / 4) / (9 (3) - (27 / 3))$

$= 1.13$ ft

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{½ \int_0^3 (9 - x^2)\,(9 - x^2)\,dx}{\int_0^3 (9 - x^2)\,dx} = 3.60 \text{ ft}$$

9.3 Centroids – Composite Bodies



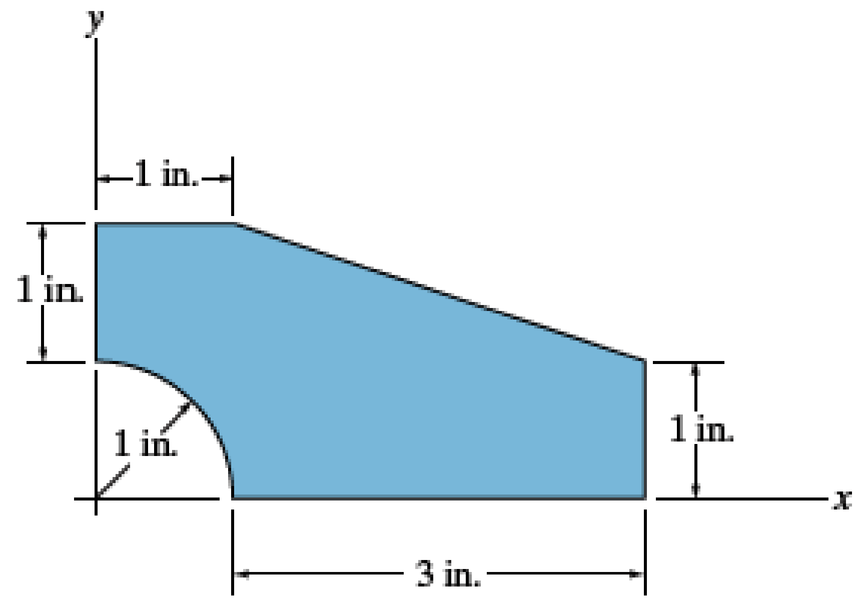
Many industrial objects can be considered as composite bodies made up of a series of connected "simpler" shaped parts or holes, like a rectangle, triangle, and semicircle.

Knowing the location of the centroid, C, or center of gravity, G, of the simpler shaped parts, we can easily determine the location of the C or G for the more complex composite body.

Composite Bodies

$$\bar{x} = \frac{1}{A_T} \sum \bar{x}_i A_i$$

$$\bar{y} = \frac{1}{A_T} \sum \bar{y}_i A_i$$

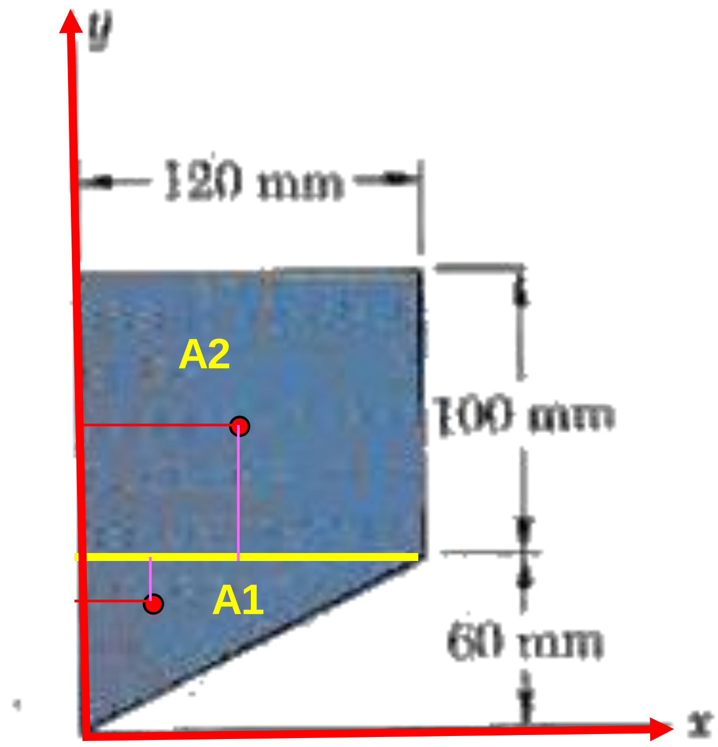


	A_i	x_i	y_i	z_i	x_iA_i	y_iA_i	z_iA_i
1							
2							
Sum	ΣA_i				$\Sigma A_i x_i$	$\Sigma A_i y_i$	$\Sigma A_i z_i$

Find the centroid of the given area

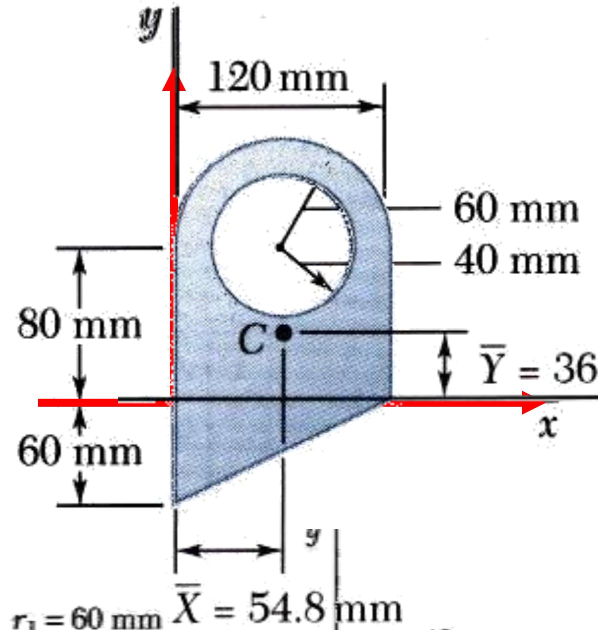
$$\bar{x} = \frac{1}{A_T} \sum \bar{x}_i A_i$$

$$\bar{y} = \frac{1}{A_T} \sum \bar{y}_i A_i$$



Body	Area(mm^2)	x (mm)	y(mm)	x*Area (mm^3)	y*Area (mm^3)
Triangle	3600	40	40	144000	144000
Square	12000	60	110	720000	1320000
Sum	15600			864000	1464000
centroid (x)	55.38	mm			
centroid (y)	93.85	mm			

For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.

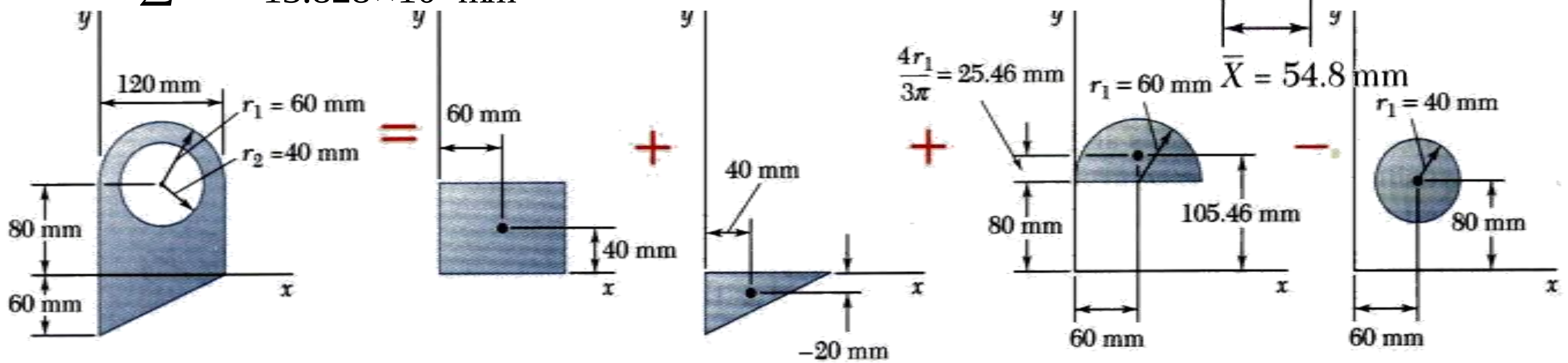


$$\bar{X} = \frac{\sum \bar{x}A}{\sum A} = \frac{+757.7\times10^3\,\text{mm}^3}{13.828\times10^3\,\text{mm}^2}$$

$$\bar{X} = 54.8\ \text{mm}$$

$$\bar{Y} = \frac{\sum \bar{y}A}{\sum A} = \frac{+506.2\times10^3\,\text{mm}^3}{13.828\times10^3\,\text{mm}^2}$$

$$\bar{Y} = 36.6\ \text{mm}$$



Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle	$(120)(80) = 9.6\times10^3$	60	40	$+576\times10^3$	$+384\times10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6\times10^3$	40	−20	$+144\times10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655\times10^3$	60	105.46	$+339.3\times10^3$	$+596.4\times10^3$
Circle	$-\pi(40)^2 = -5.027\times10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828\times10^3$			$\Sigma\bar{x}A = +757.7\times10^3$	$\Sigma\bar{y}A = +506.2\times10^3$

Engineering Mechanics

Statics & Strength of Materials

Moment of Inertia

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Moment of Inertia (MoI) (second moment of an area (m^4))

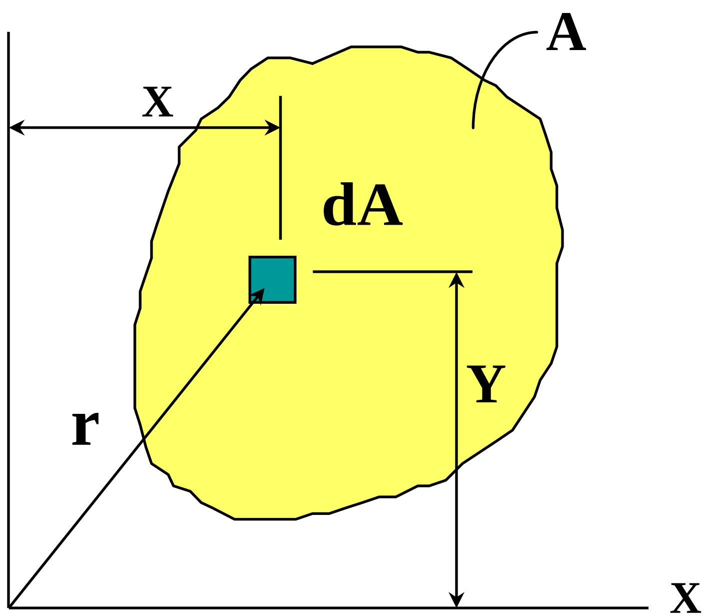
Definition of Moments of Inertia for Areas: Used in formulas for Mechanics of Materials, Fluid Mechanics, Structural Mechanics

The moment of inertia is found by integrating

$$I_x = \int_{Area} y^2 dA$$

$$I_y = \int_{Area} x^2 dA$$

$$j_o = \int_{Area} \mathbf{r}^2 dA$$



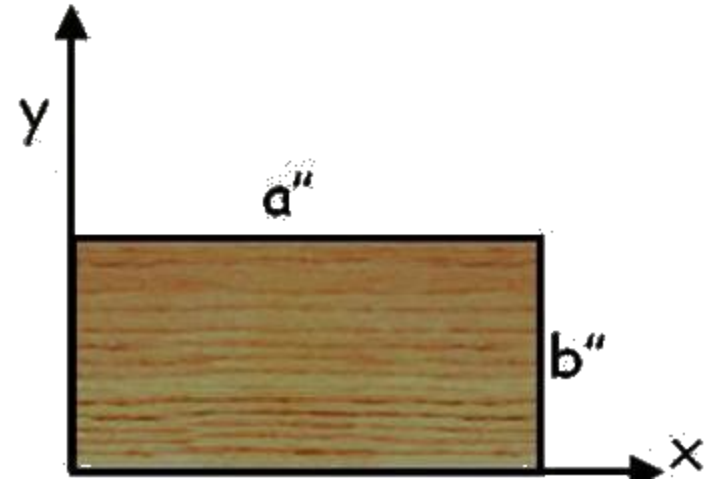
J_o is the polar moment of inertia about the pole O or Z axis

$$J_O = \int_A r^2 \, dA = \int_A (x^2 + y^2)\, dA = I_x + I_y$$

EXAMPLE

Given: The shaded area shown in the figure.

Find: The MoI of the area about the x- and y-axes.



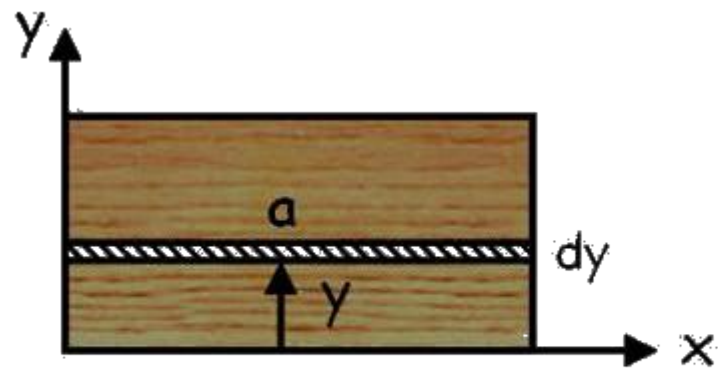
Solution

$I_x = \int y^2 \, dA$

$dA = a. \, dy$

$I_x = \int_0^b y^2 \, . \, a. \, dy$

$= [a \, . y^3/3]_0^b = ab^3/3 \quad in^4$

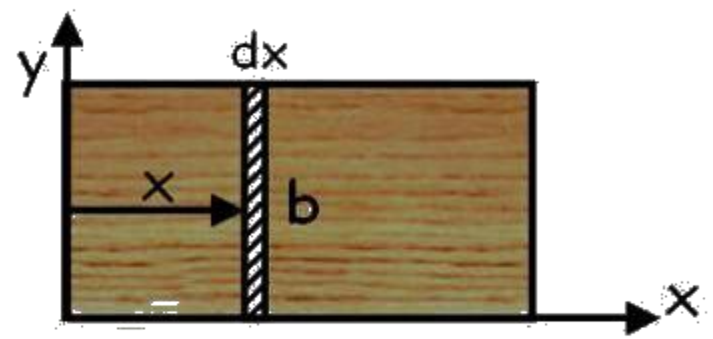


$I_y = \int x^2 \, dA$

$dA = b. \, dx$

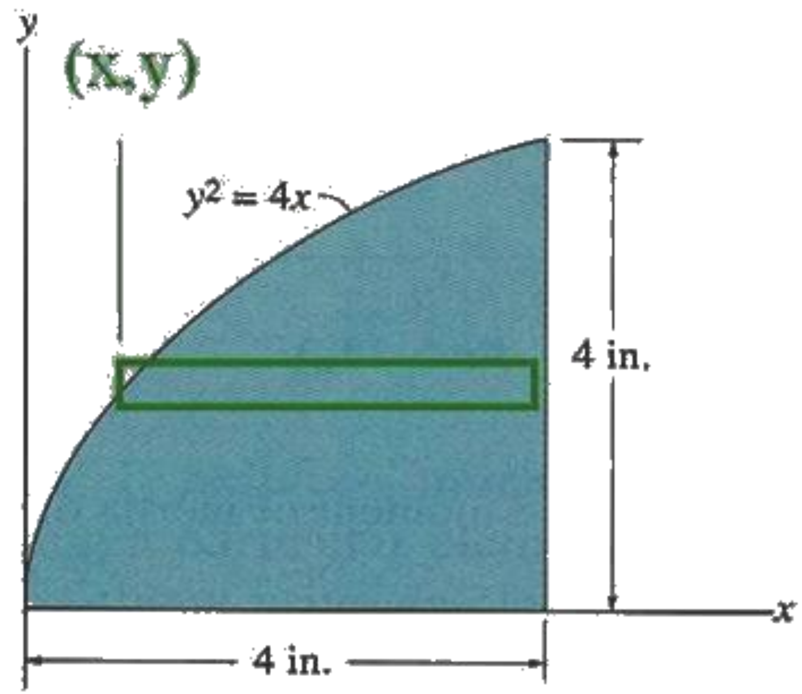
$I_y = \int_0^a x^2 \, . \, b. \, dx$

$= [b \, . x^3/3]_0^a = ba^3/3 \quad in4$



Given: The shaded area shown in the figure.

Find: The MoI of the area about the x- and y-axes.



Solution

$$I_x = \int y^2 \, dA$$

$$dA = (4 - x)\, dy = (4 - y^2/4)\, dy$$

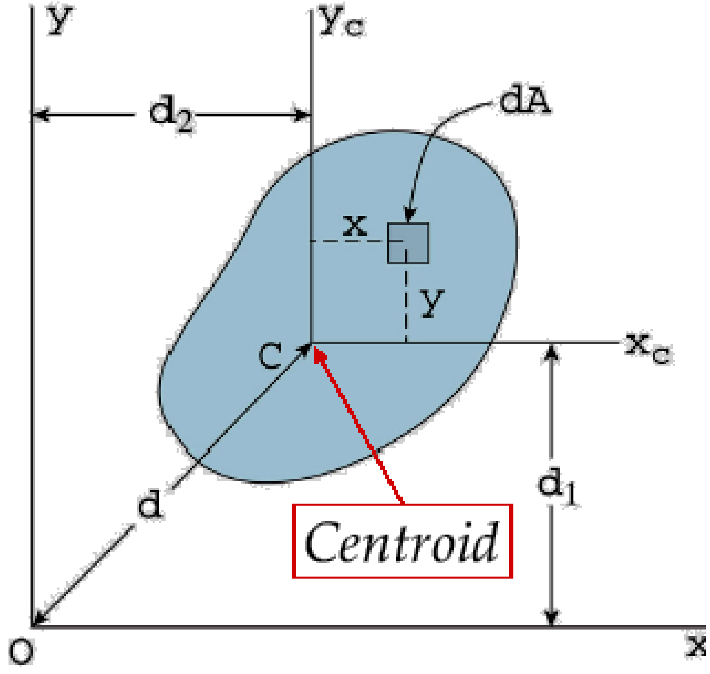
$$I_x = \int_0^4 y^2 \, (4 - y^2/4)\, dy$$

$$= \left[(4/3)\, y^3 - (1/20)\, y^5 \right]_0^4 = 34.1 \text{ in}^4$$

Moment of Inertia: ***Parallel-Axis Theorem***

$$I_x = \int_A y^2 dA, \quad I_y = \int_A x^2 dA$$

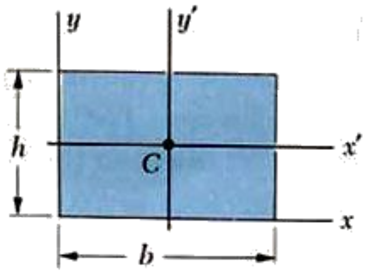
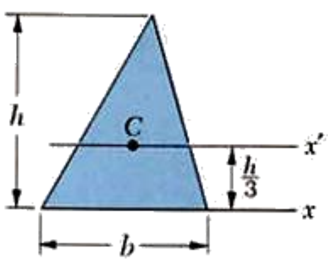
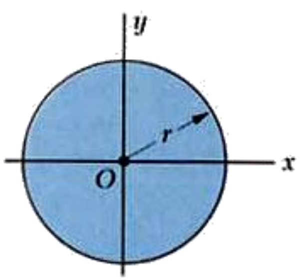
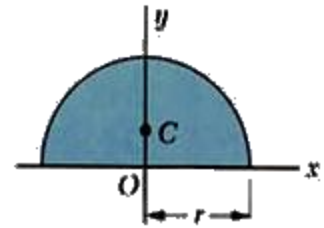
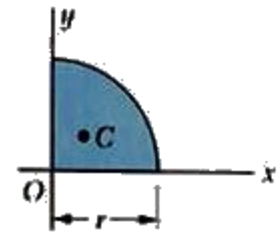
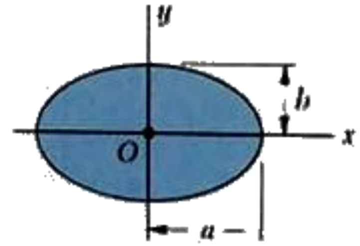
Given I_{xc}, I_{yc} (or I_x, I_y), determine I_x, I_y (or I_{xc}, I_{yc})



$$I_x = \int (y + d_1)^2 dA$$
$$= \int y^2 dA + 2d_1 \int y dA + d_1^2 \int dA$$
$$= I_{xc} + A d_1^2$$

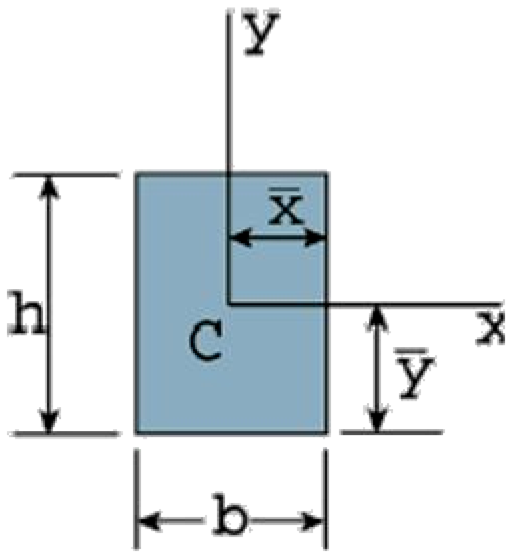
$$I_y = \int (x + d_2)^2 dA$$
$$= \int x^2 dA + 2d_2 \int x dA + d_2^2 \int dA$$
$$= I_{yc} + A d_2^2$$

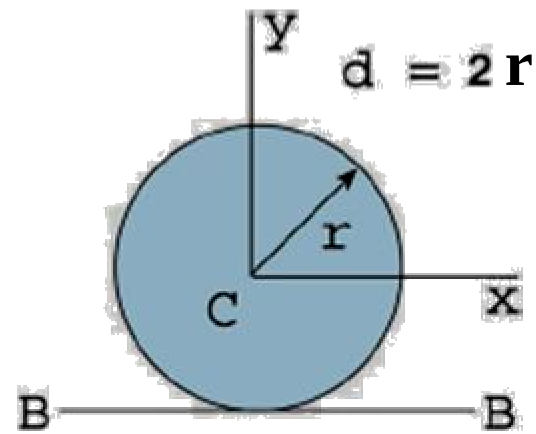
Moment of Inertia for simple shapes

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$
Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Moment of Inertia

$$I_x = \int_A y^2 dA, \quad I_y = \int_A x^2 dA$$





$$I_x = \frac{bh^3}{12}, \quad I_y = \frac{hb^3}{12}$$

$$I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

Memorize the moments of inertia of these two cross sections!

Moments of Inertia for Composite Areas

Composite area consists of group of connected simple shapes.

If MOI of parts about common axis can be determined, then MOI of the composite is algebraic sum of parts.

$$I_x = \sum_i I_x^i, \qquad I_y = \sum_i I_y^i$$

Procedure for Analysis

Composite Area Moment of Inertia about reference axis.

1.0 Composite Parts. Divide area into composite parts. Indicate perpendicular distance from centroid of parts to reference axis.

2.0 Apply Parallel Axis Theorem. Determine MOI of each part about centroidal axis parallel to reference axis. Use parallel axis theorem to calculate MOI of parts about reference axis.

3.0 Sum MOI of parts. Calculate MOI of component by summing MOI of parts. If any part is a "hole", subtract the MOI of hole in making summation.

MOI about the centroid of composite the section

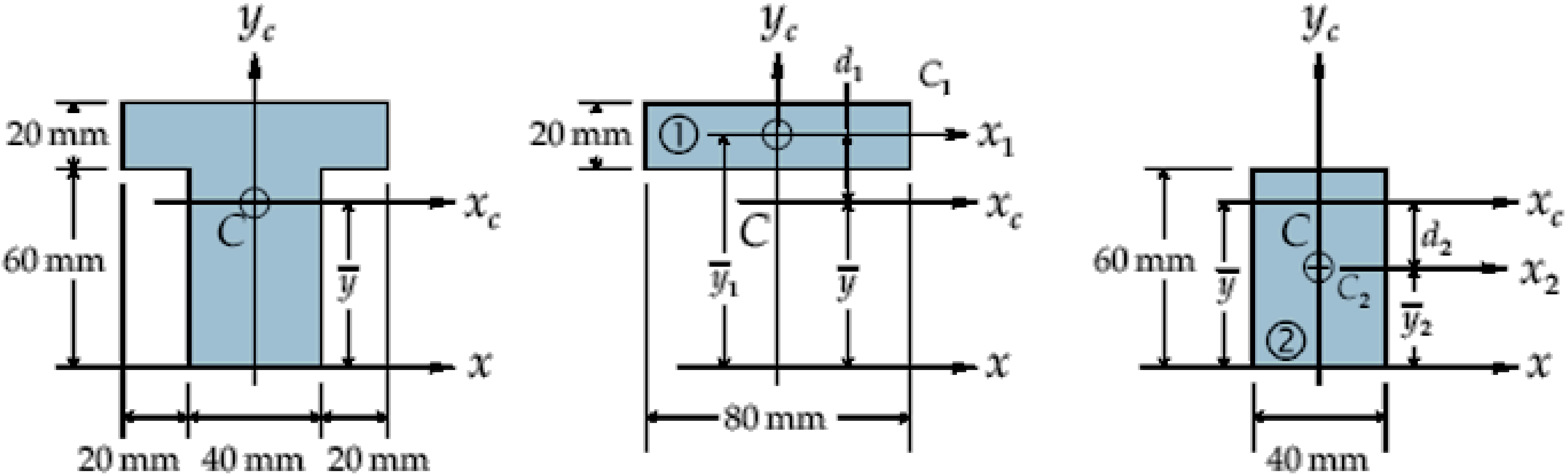
Bodies	A_i	y_i	y_i*A_i	I_i	$d_i=y_i$-ybar	$d_i^2A_i$

$$\bar{y} = \frac{\sum y_i A_i}{\sum A_i}$$

$$I_x = \sum \left(\bar{I}_x + d^2 A \right)_i$$

$$= \sum \bar{I}_{xi} + \sum \left(y_i - \bar{y} \right)^2 A_i$$

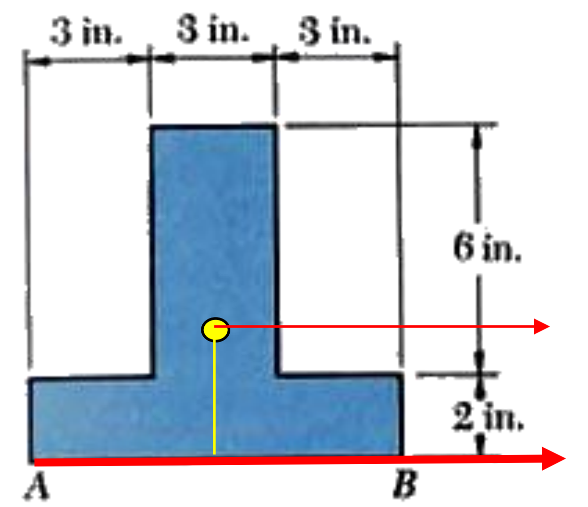
Moment of Inertia – *Composite Area*



	A_i	$\bar{y}_i$	$Q_i = A_i\bar{y}_i$	$I_{xi}^{(i)}$	$d_i = \bar{y}_i - \bar{y}$	$A_i d_i^2$	$I_x^{(i)} = I_{xi}^{(i)} + A_i d_i^2$
①	1,600	70	112,000	$I_{x1}^{(1)} = \frac{(80)(20)^3}{12} = 53{,}333$	24	921,600	974,933
②	2,400	30	72,000	$I_{x2}^{(2)} = \frac{(40)(60)^3}{12} = 720{,}000$	-16	614,400	1,334,400
$\sum$	4,000		184000				2,309,333

$$\bar{y} = \frac{\sum Q_i}{\sum A_i} = \frac{184{,}000}{4{,}000} = 46 \text{ mm}$$

Find I_x and r_x (x- axis passing through the centroid of the section



Bodies	A_i	y_i	$y_i{*}A_i$	I_i	$d_i=y_i\text{-ybar}$	$d_i^2A_i$
1	18	1	18			
2	18	5	90			
	36		108			

$$\bar{y}=\frac{\sum y_i A_i}{\sum A_i}=\frac{108\text{ in}^3}{36\text{ in}^2}=3.0\text{ in.}$$

Bodies	A_i	y_i	$y_i{*}A_i$	I_i	$d_i=y_i\text{-ybar}$	$d_i^2A_i$
1	18	1	18	6	-2	72
2	18	5	90	54	2	72
	36		108	60		144
ybar	3	in.				
I	204	in^4				

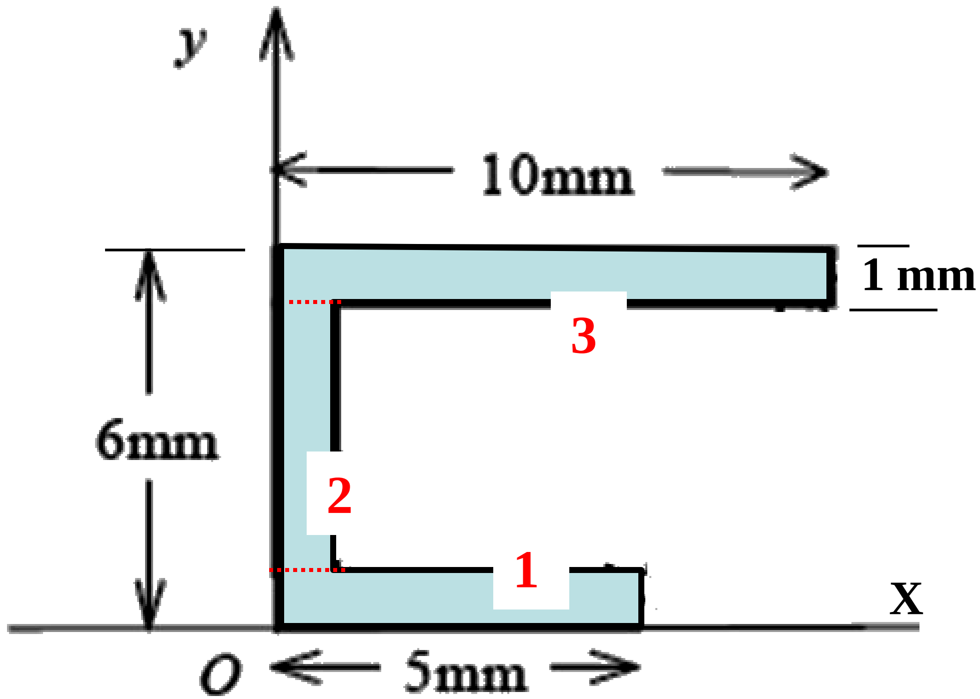
$$I_x=\sum \overline{I_{xi}}+\sum\left(y_i-\bar{y}\right)^2 A_i$$

$$=60\text{ in}^4+144\text{ in}^4=204\text{ in}^4$$

$$r_x=\sqrt{\frac{I_x}{A}}=\sqrt{\frac{204\text{ in}^4}{36\text{ in}^2}}$$

$$=2.38\text{ in.}$$

Find the centroid of the area shown $\overline{X}$, and $\overline{Y}$



Element	Area	x	y	A.x	A.y
1	5	2.5	0.5	12.5	2.5
2	4	0.5	3	2	12
3	10	5	5.5	50	55
Σ	19			64.5	69.5

Hence centroid is $\left(\bar{x} = \frac{64.5}{19} = 3.39mm\ ,\ \bar{y} = \frac{69.5}{19} = 3.66mm\right)$ from origin

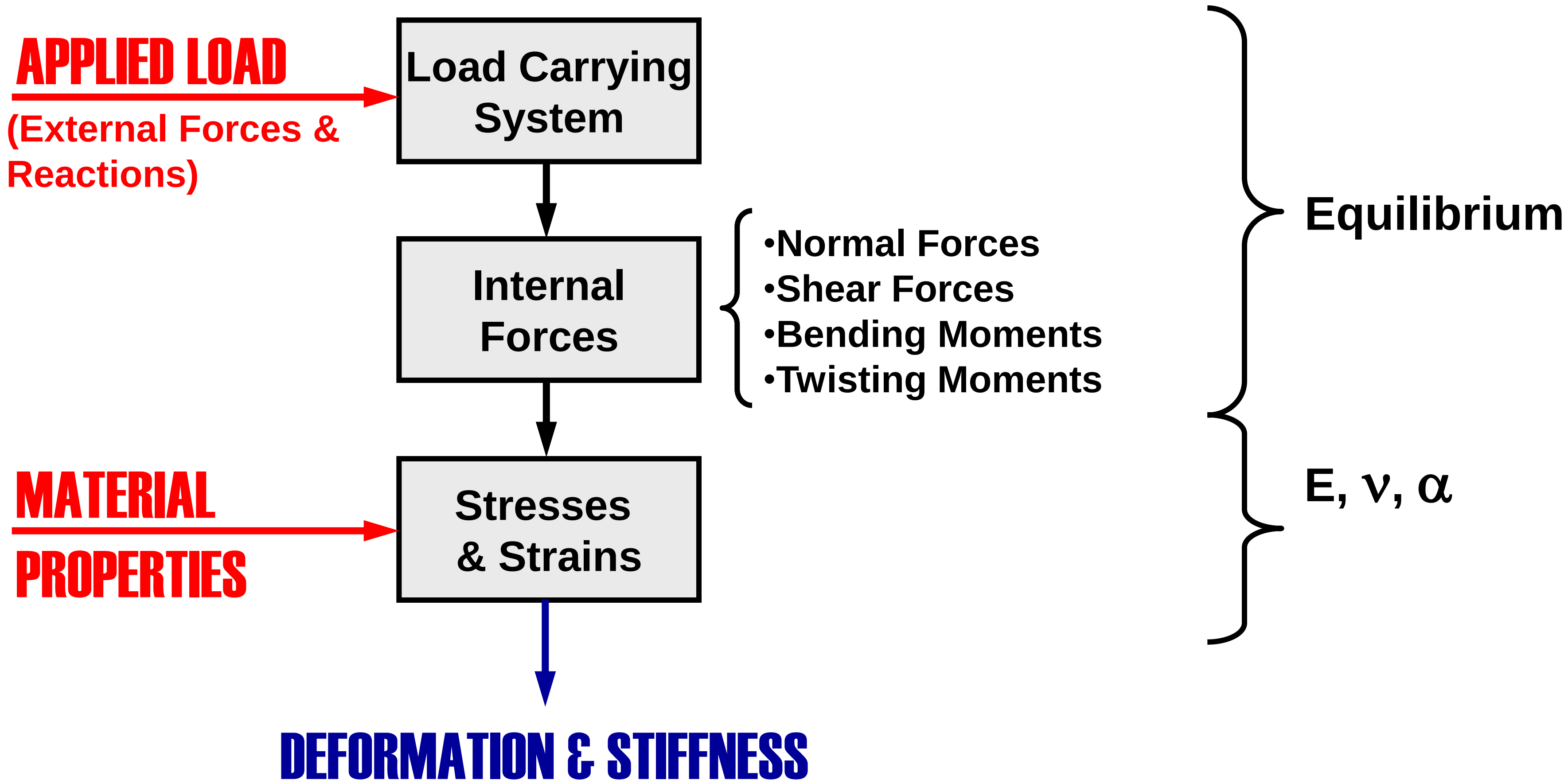
Engineering Mechanics

Statics & Strength of Materials

Tension, Compression and Shear

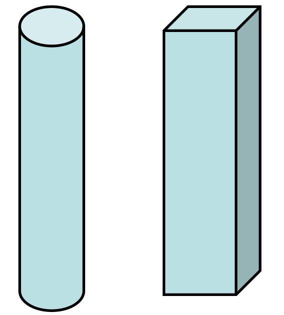
Eng. Iqbal Marie

iqbal@hu.edu.jo

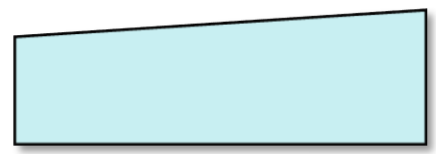

APPLIED LOAD
(External Forces & Reactions)
Load Carrying System
Internal Forces
•Normal Forces
•Shear Forces
•Bending Moments
•Twisting Moments
Equilibrium
MATERIAL PROPERTIES
Stresses & Strains
E, ν, α
DEFORMATION & STIFFNESS

Normal Stress and Strain

Prismatic bar: is a straight structural member having the same cross section throughout its length

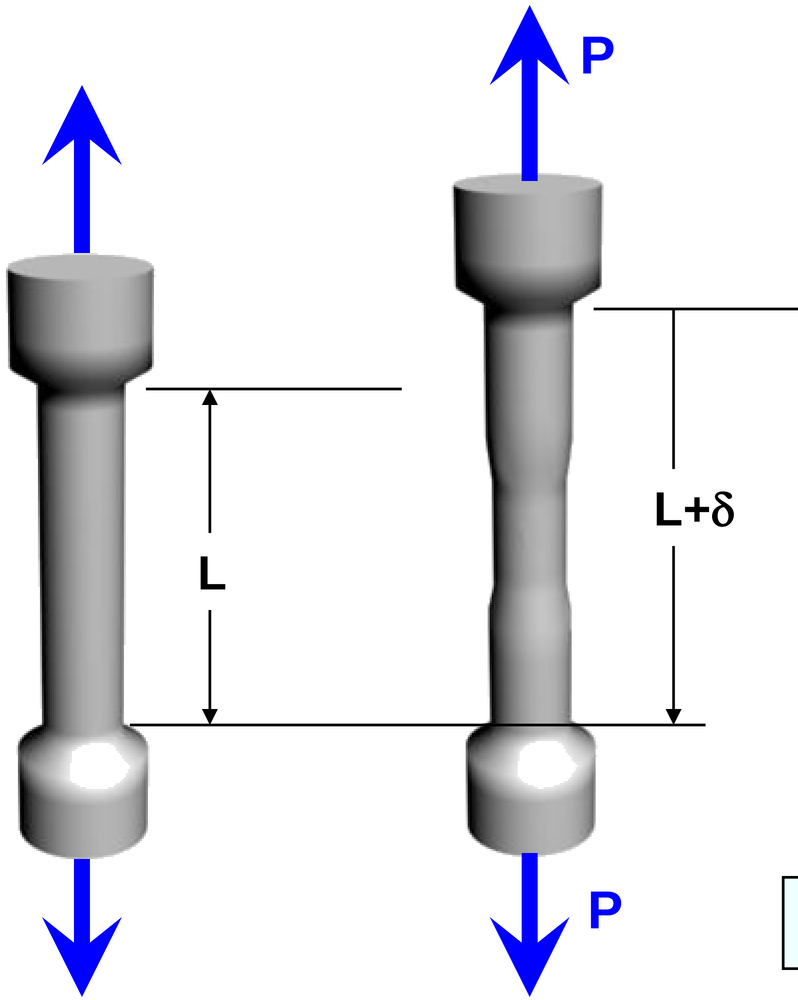


Non prismatic member with non uniform stresses

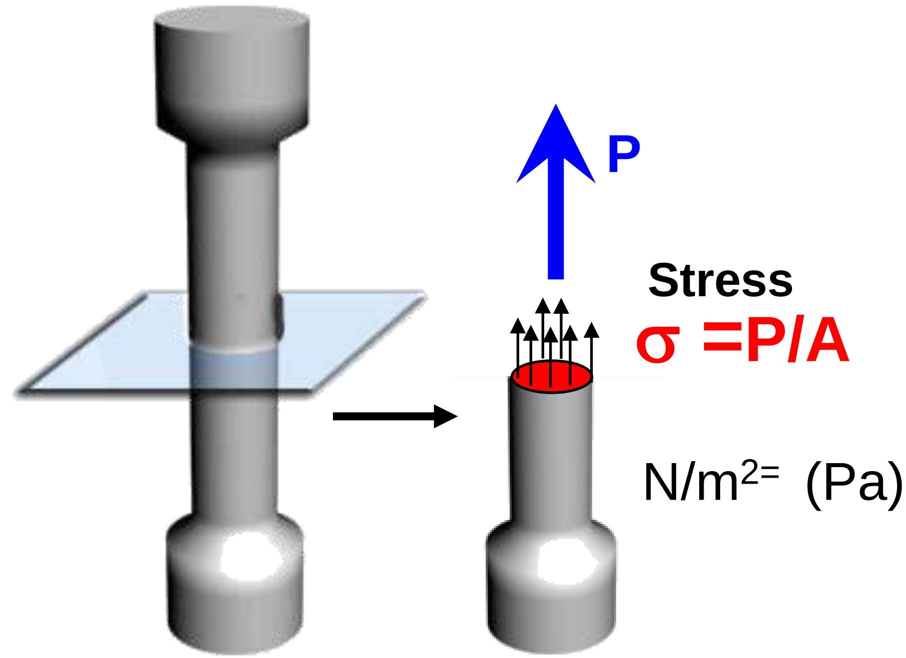


Axial force: A load directed along the axis of the member resulting either tension of compression in the bar.

Normal Stress



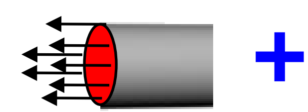
Prismatic Bar in tension



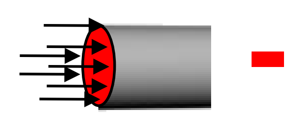
Normal Stress in the Bar

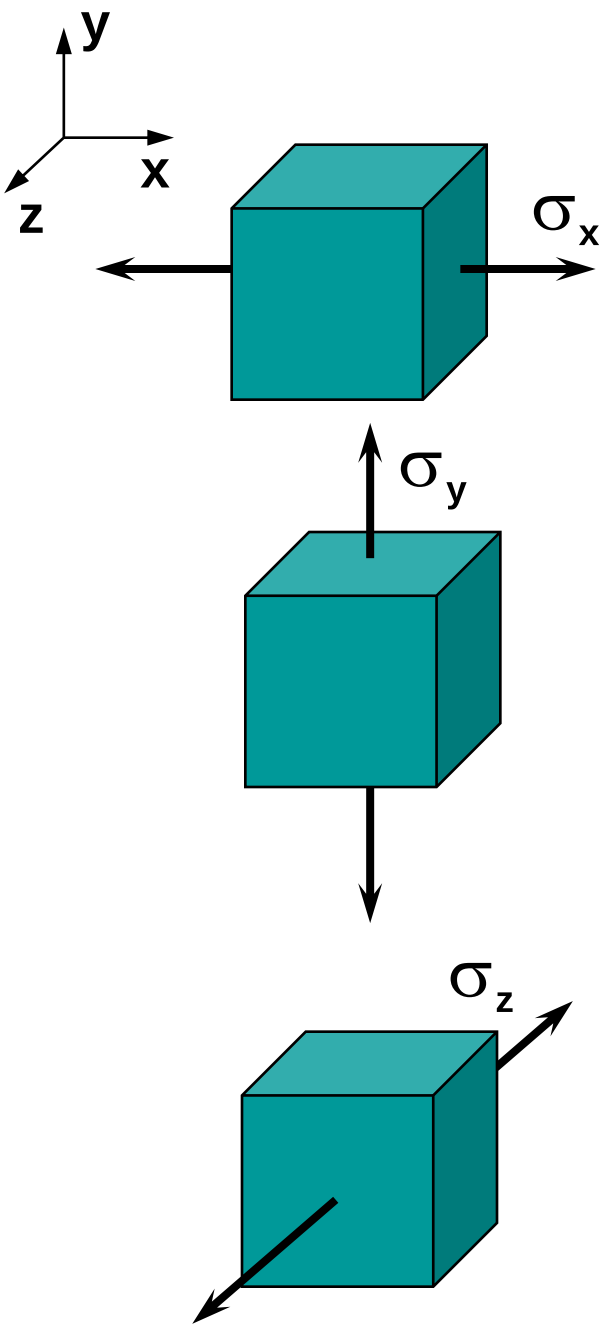
Sign convention:

Tensile Stress: Positive



Compressive stress: Negative

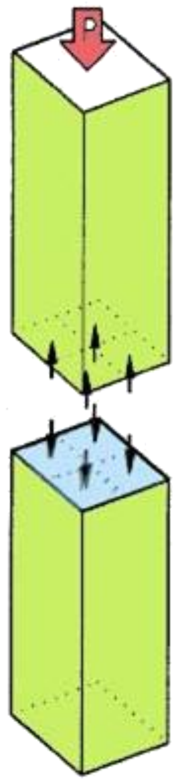




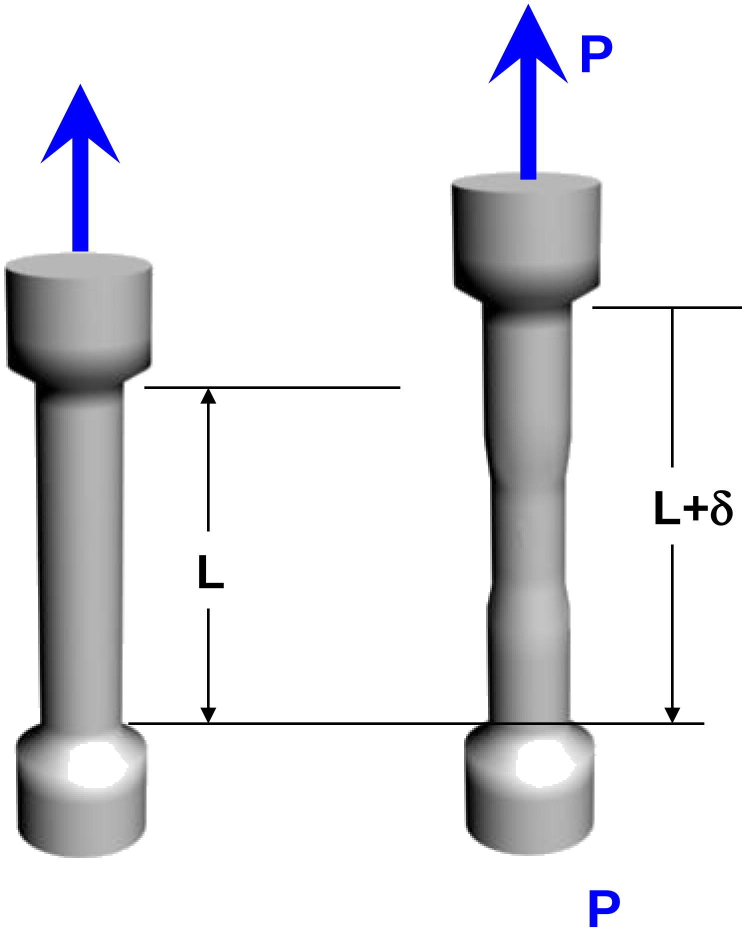
Limitations

The equation $\sigma = P/A$ is valid only if the stress is uniformly distributed over the cross section of the bar. (the load acts at the centroid of the cross sectional area)

For homogeneous materials and uniform stress in prismatic bar



Normal Strain (Axial Strain)



$\varepsilon = \delta / L$

Eg. A steel bar having L = 2.0 m and diameter D= 50mm, when loaded in tension with tensile load P= 30kN the bar elongated by 1.4 mm. What is the axial stress and strain?

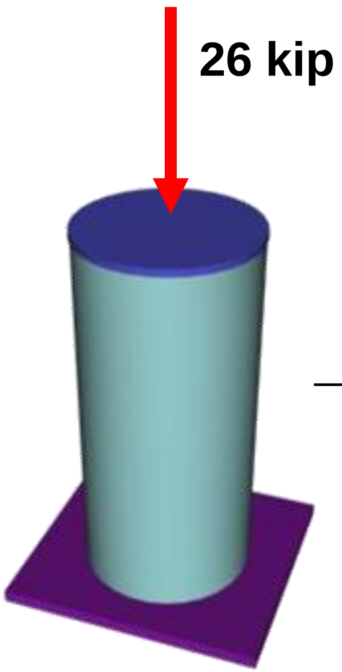
$$\varepsilon = \delta / L = \frac{1.4\ mm}{2*1000mm} = 7 \times 10^{-4}$$

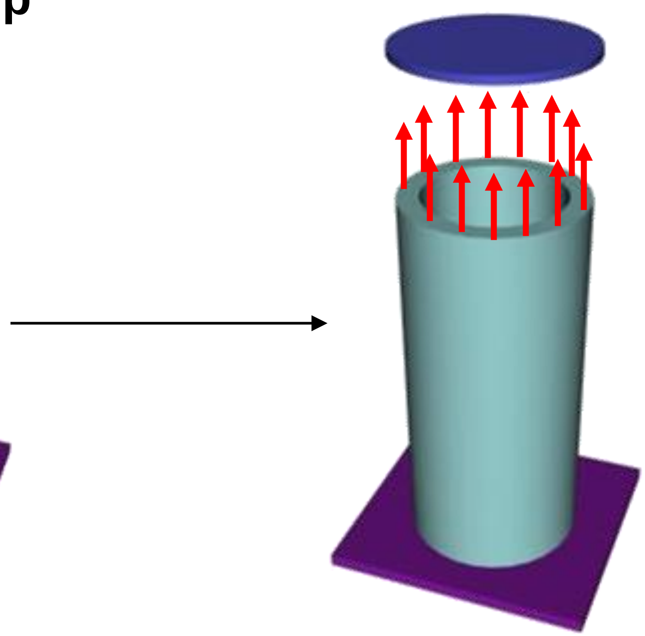
σ =P/A = 30 / (π $(0.05)^2/4$)= 0.05887 kN/m^2 (kPa)

Example 1-1

A short post constructed from a hollow circular tube of Aluminum supports a compressive load of 26 kips. The inner and outer diameters of the tube are d1 = 4.0 in and d2 =4.5 in respectively, and its length is 16 in. the shortening of the post due to the load is 0.012 in.

Determine the compressive stress and strain the post.





Cross section Area (A) = $\pi(4.5^2 - 4.0^2)/4 =$

$3.338\ in^2$

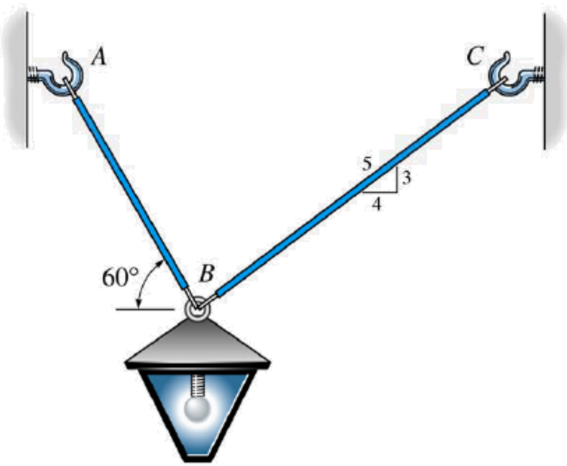
$\sigma = P/A = 26 / 3.338$

$= 7.790\ ksi$

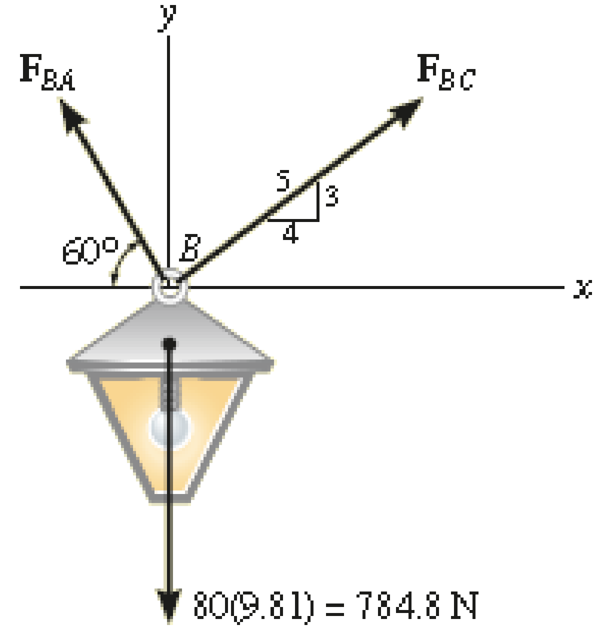
$\varepsilon = \delta / L = 0.012 / 16$

$= 750 \times 10^{-6}\ in/in$

The 80-kg lamp is supported by two rods AB and BC as shown in Fig. 1–17a. If AB has a diameter of 10 mm and BC has a diameter of 8 mm, determine the average normal stress in each rod.



(a)



Internal Loading. We must first determine the axial force in each rod.
the

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad F_{BC}\left(\frac{4}{3}\right) - F_{BA}\cos 60^\circ = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad F_{BC}\left(\frac{3}{5}\right) + F_{BA}\sin 60^\circ - 784.8\text{ N} = 0$$

$$F_{BC} = 395.2\text{ N}, \qquad F_{BA} = 632.4\text{ N}$$

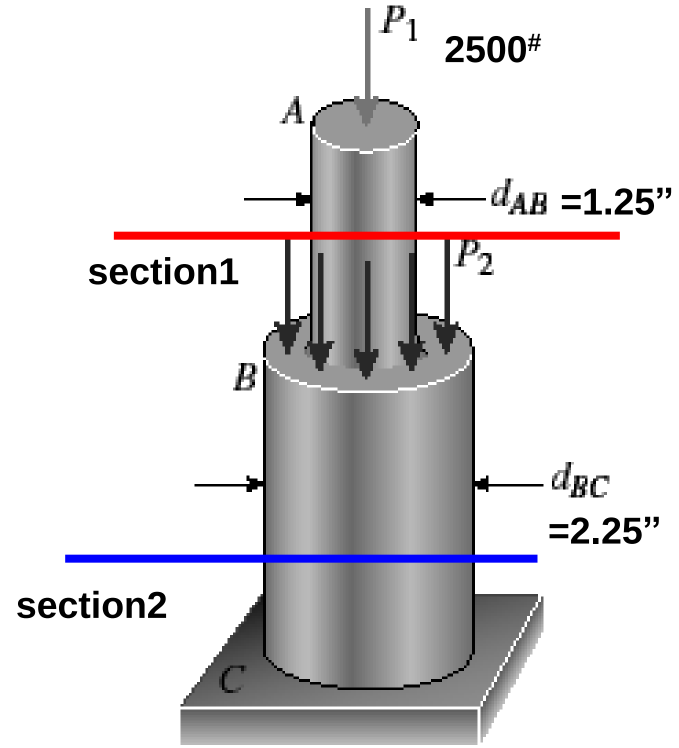
$$\sigma_{BC} = \frac{F_{BC}}{A_{BC}} = \frac{395.2\text{ N}}{\pi(0.004\text{ m})^2} = 7.86\text{ MPa}$$

$$\sigma_{BA} = \frac{F_{BA}}{A_{BA}} = \frac{632.4\text{ N}}{\pi(0.005\text{ m})^2} = 8.05\text{ MPa}$$

Problem 1.2-1 A solid circular post ABC (see figure) supports a load $P_1 = 2500$ lb acting at the top. A second load P_2 is uniformly distributed around the shelf at B. The diameters of the upper and lower parts of the post are $d_{AB} = 1.25$ in. and $d_{BC} = 2.25$ in., respectively.

(a) Calculate the normal stress σ_{AB} in the upper part of the post.

(b) If it is desired that the lower part of the post have the same compressive stress as the upper part, what should be the magnitude of the load P_2?



section1

(a) Normal stress in part AB

$$\sigma_{AB} = \frac{P_1}{A_{AB}} = \frac{2500 \text{ lb}}{\frac{\pi}{4}(1.25 \text{ in.})^2} = 2040 \text{ psi} \quad \longleftarrow$$

section2

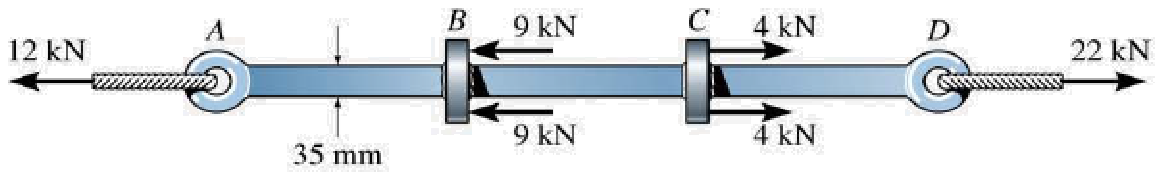
(b) Load P_2 for equal stresses

$$\sigma_{BC} = \frac{P_1 + P_2}{A_{BC}} = \frac{2500 \text{ lb} + P_2}{\frac{\pi}{4}(2.25 \text{ in.})^2}$$

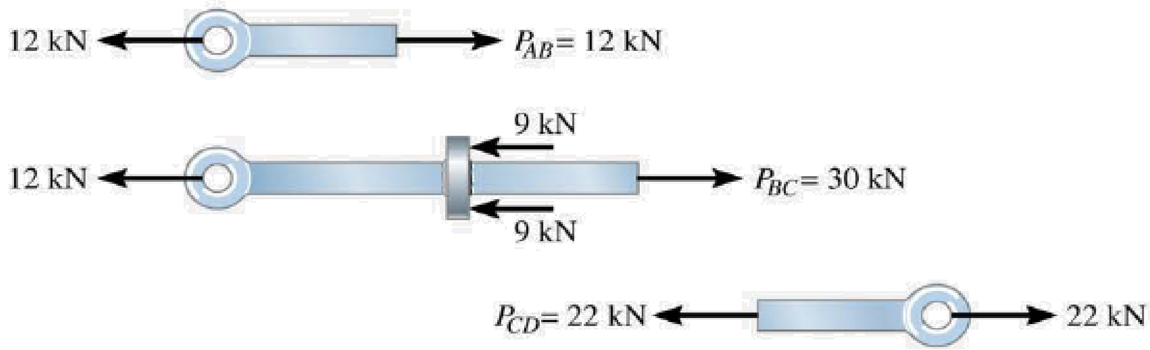
$$= \sigma_{AB} = 2040 \text{ psi}$$

Solve for P_2: $\quad P_2 = 5600 \text{ lb} \quad \longleftarrow$

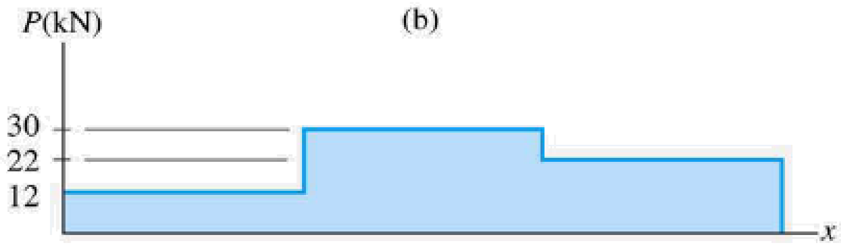
The bar in Fig. has a constant width of 35 mm and a thickness of 10 mm. Determine the maximum average normal stress in the bar when it is subjected to the loading shown.



(a)



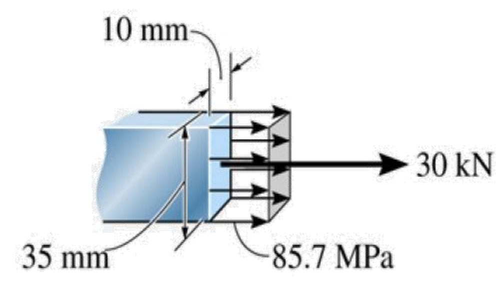
(b)



normal force diagram

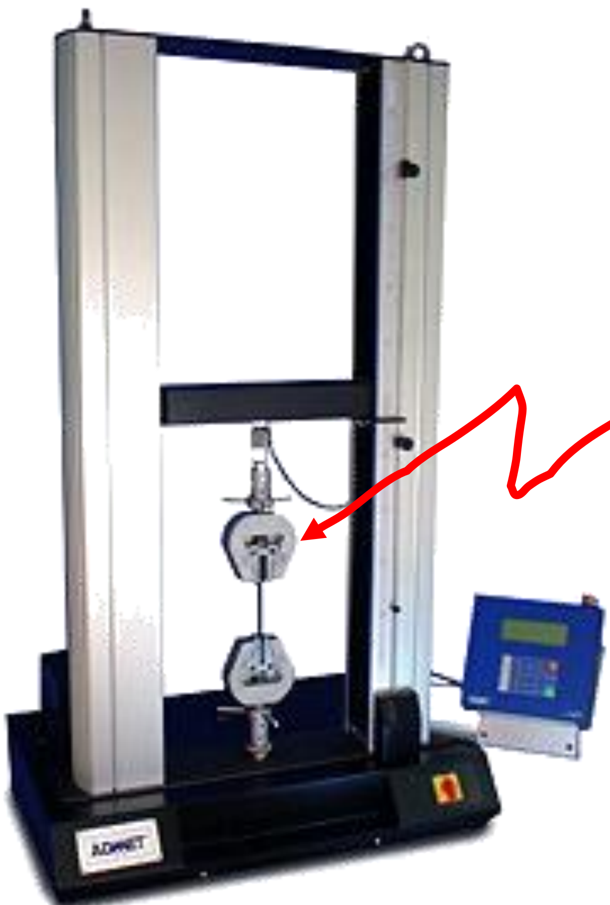
(c)

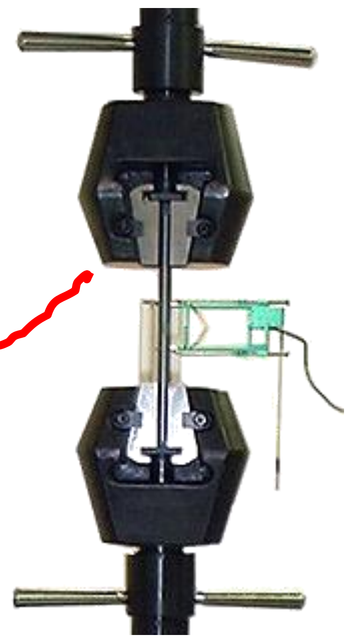
$$\sigma_{BC} = \frac{P_{BC}}{A} = \frac{30(10^3)\ \text{N}}{(0.035\ \text{m})(0.010\ \text{m})} = 85.7\ \text{MPa}$$



Mechanical Properties of Materials

The design of machines and structures so that they will function properly requires that we understand the *mechanical behavior of the materials* being used.





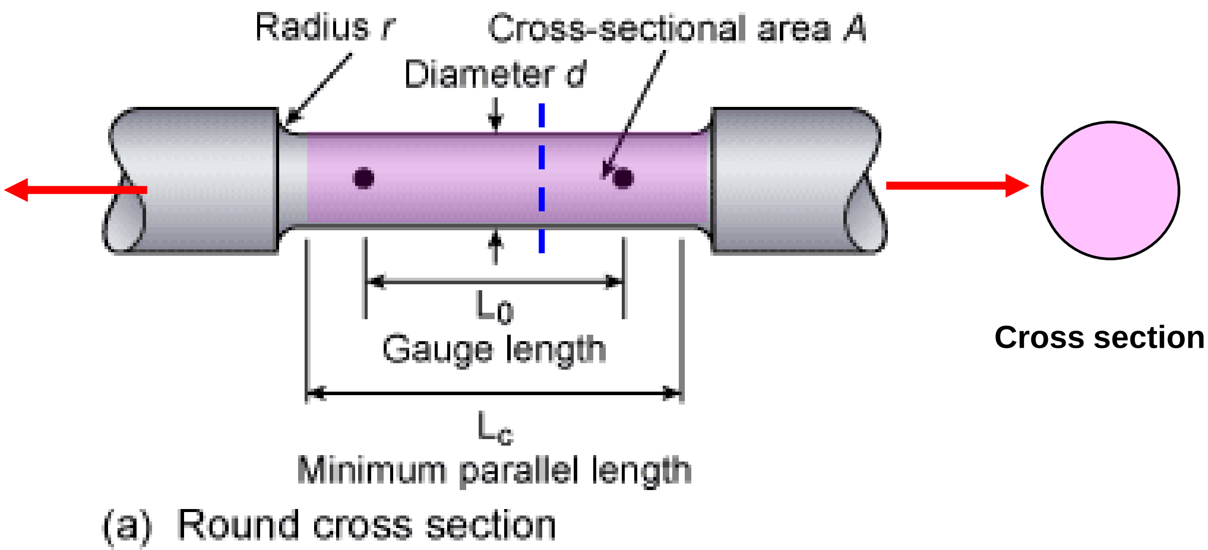
Gripping Devices

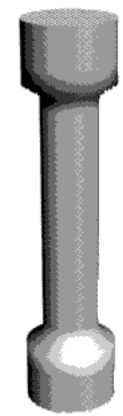
Tensile Testing Machine

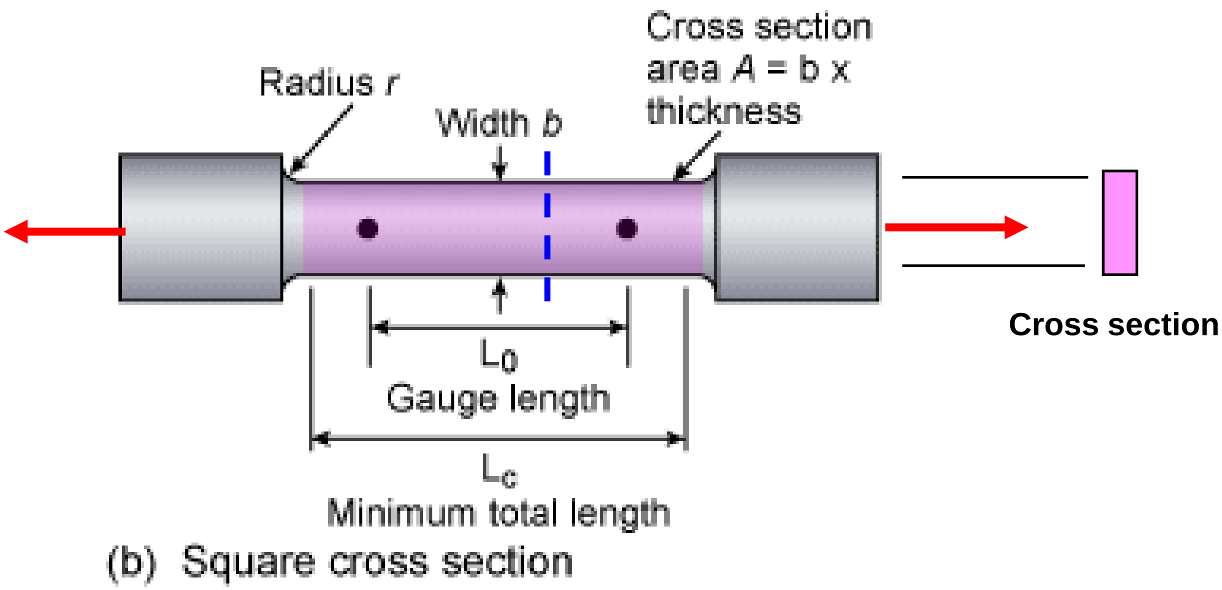
Tensile Testing

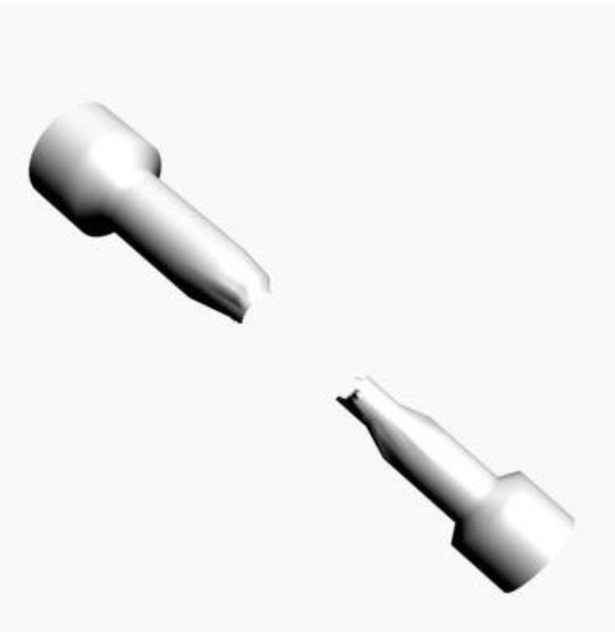
Tensile tests are carried out by gripping the ends of a suitably prepared standardised test piece in a tensile testing machine and then applying a continually increasing uni-axial load until such time as failure occurs

Tensile Test Specimen









Stress- Strain Diagram

σ = normal stress on a plane perpendicular to the longitudinal axis of the specimen

P = applied load

A = original cross sectional area

ε = normal strain in the longitudinal direction

δ = change in the specimen's gage length

L = original gage length

$$\sigma = \frac{P}{A} \qquad \varepsilon = \frac{\delta}{L}$$

- **Engineering stress**

  $\sigma = P/A_0$

- **True stress**

  $\sigma = P/A$

- **Engineering strain**

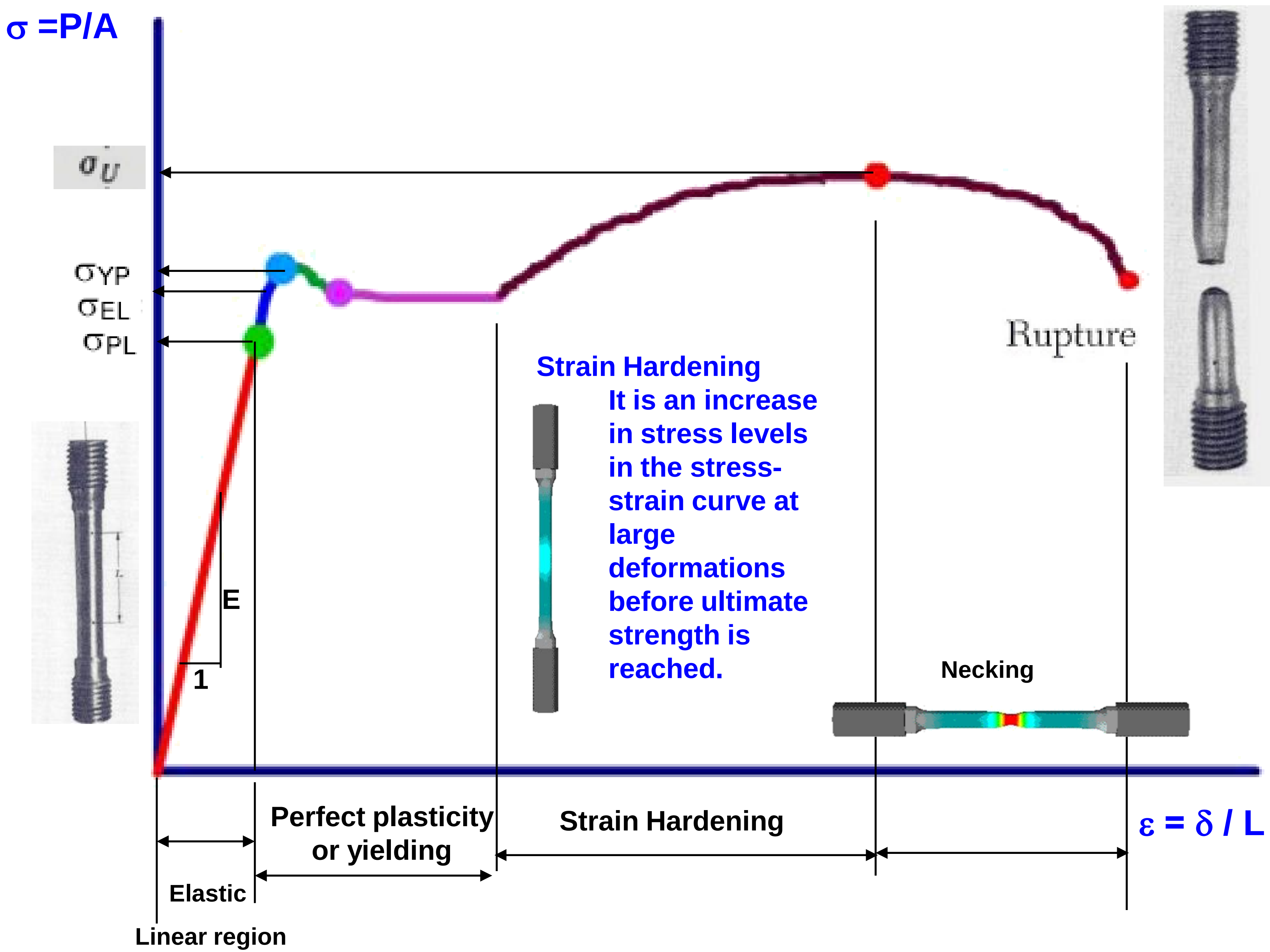
  $\varepsilon = (l - l_0)/l_0$

- **True strain (Logarithmic strain)**

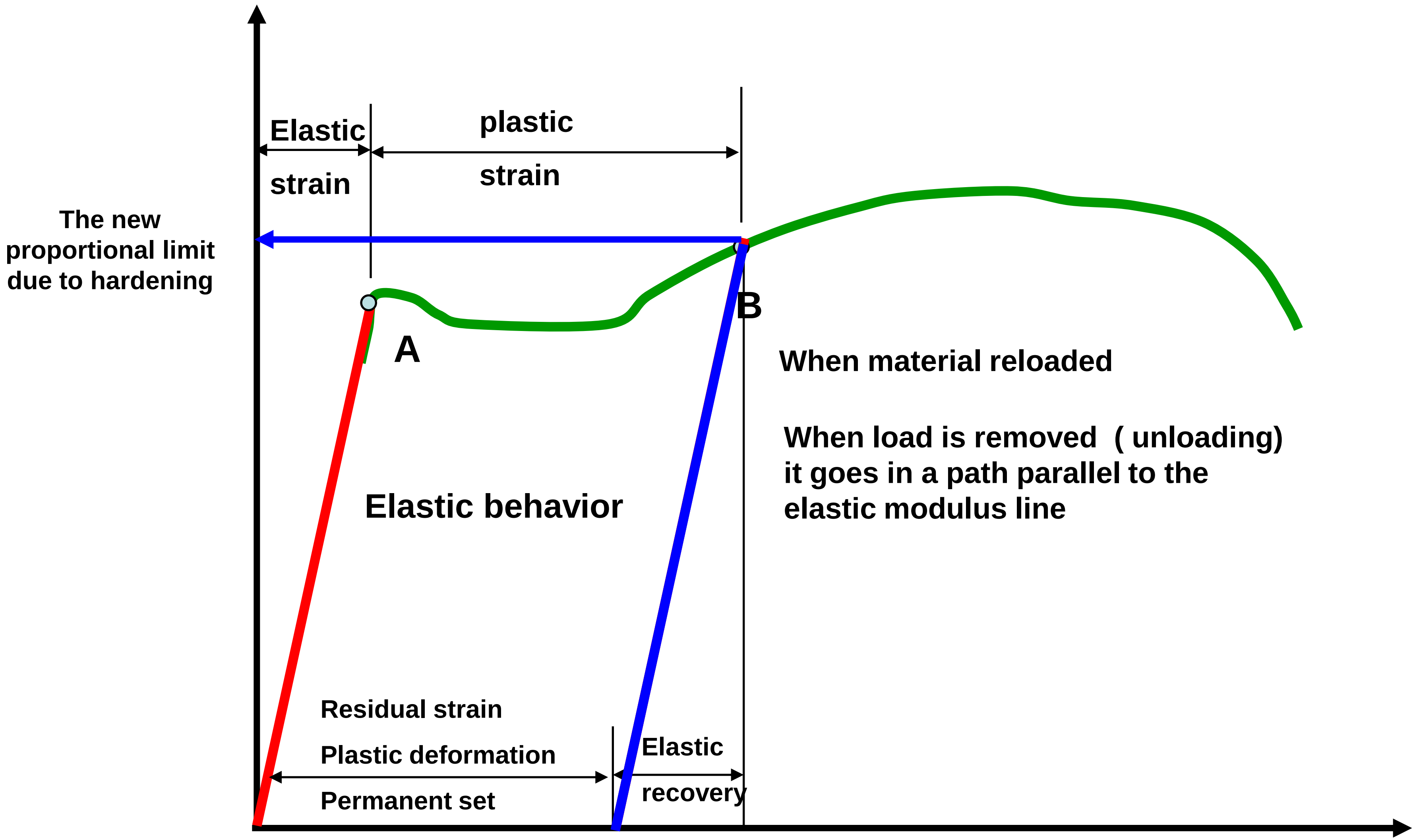
  $\varepsilon = \ln(l/l_0) = \ln(A/A_0)$

- **Volume must be the same $Al = A_0 l_0$**

stress-strain curve or diagram gives a direct indication of the material properties.



σ =P/A
σ_U
σ_{YP}
σ_{EL}
σ_{PL}
E
1
Strain Hardening
It is an increase in stress levels in the stress-strain curve at large deformations before ultimate strength is reached.
Rupture
Necking
Perfect plasticity or yielding
Strain Hardening
ε = δ / L
Elastic
Linear region



Linear relationship between stress and strain

Strain is temporary, meaning that all strain is fully recovered upon removal of the stress

The slope of this is called the elastic modulus

E . Modulus of Elasticity (Young's Modulus) - **Slope of the initial linear portion of the stress-strain diagram. The modulus of elasticity may also be characterized as the "stiffness" or ability of a material to resist deformation within the linear range.**

Proportional limit : is the maximum value of the stress from the stress-strain diagram, where the stress and strain are proportional

Elastic Limit : is the maximum stress for a material to behave elastically, - the specimen will return to its original undeformed shape if the load is removed so long as the stress is below the elastic limit.

Yield Point: This defined as the maximum stress on stress-strain curve, where there is an appreciable increase in strain with no increase in stress. It is generally easier to determine than the proportional limit or elastic

Some materials do not exhibit a distinct yield point

Yield Strength :It is the stress which induces a specified permanent set. This is useful for materials which have no well defined yield point. The offset method is generally used to determine yield stress

Tensile strength: the maximum stress applied to the specimen.

Failure stress: the stress applied to the specimen at failure (usually less than the maximum tensile strength because necking reduces the cross-sectional area).

Ductility

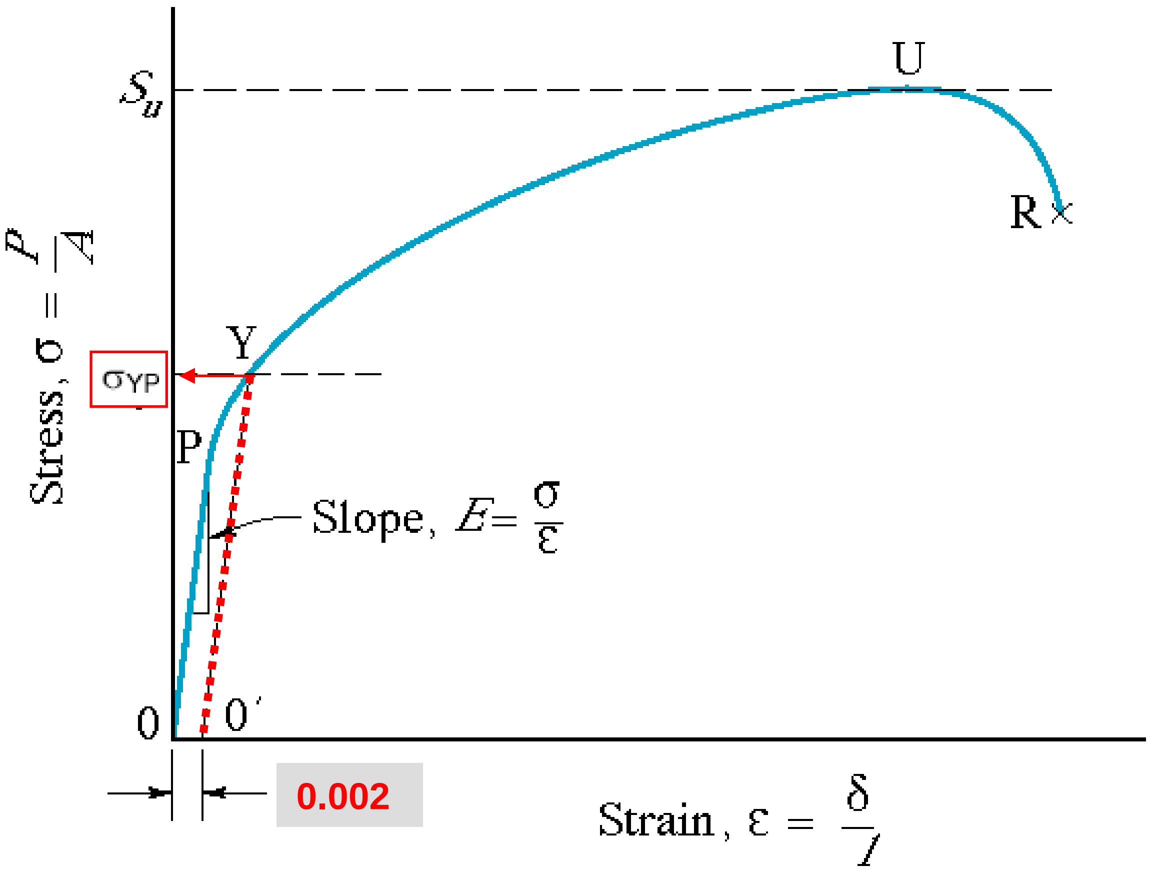
It is the ability of a material to deform plastically.

Two measurements of ductility:

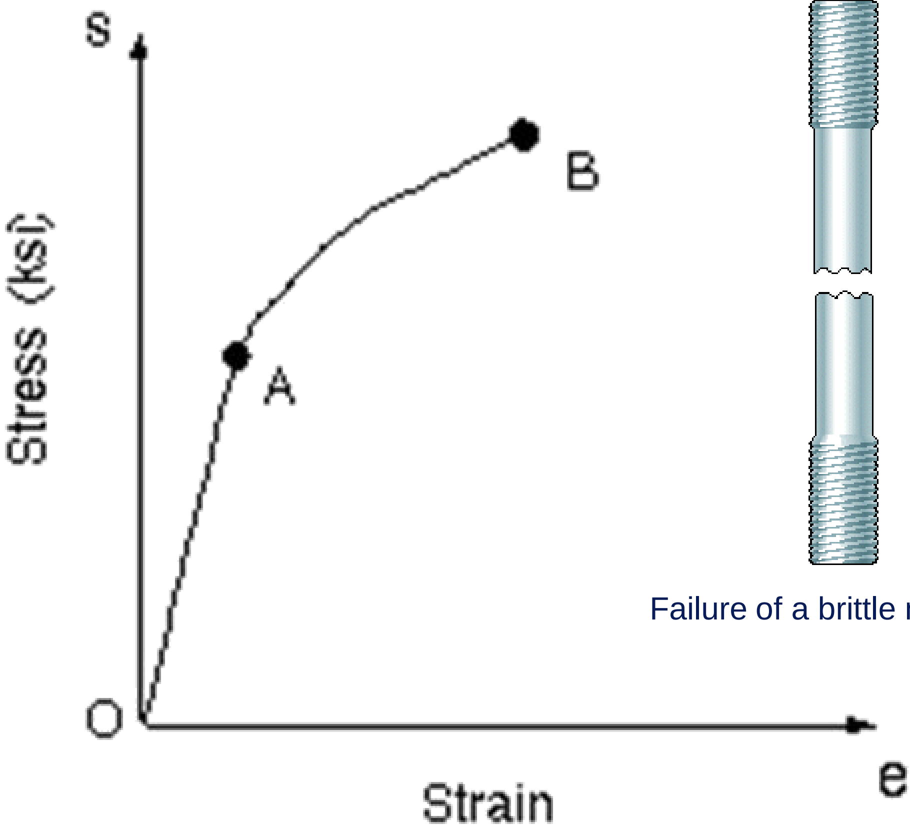
Percent (%) elongation of the member = $(L_f - L_0) / (L_0) * 100.0$

Percent (%) reduction in area at the location of fracture

% Area = $(A_0 - A_f) / (A_0) * 100.0$



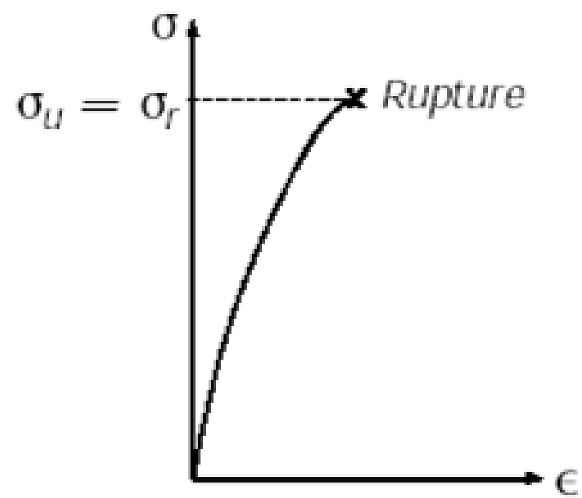
The yield point may be determined by the offset method . A line is drawn on the stress-strain diagram parallel to the initial linear part of the curve but is offset by some standard amount of strain, such as 0.002 or 0.2%). The intersection of the offset line and the stress-strain curve (point A in the figure) defines the yield stress.



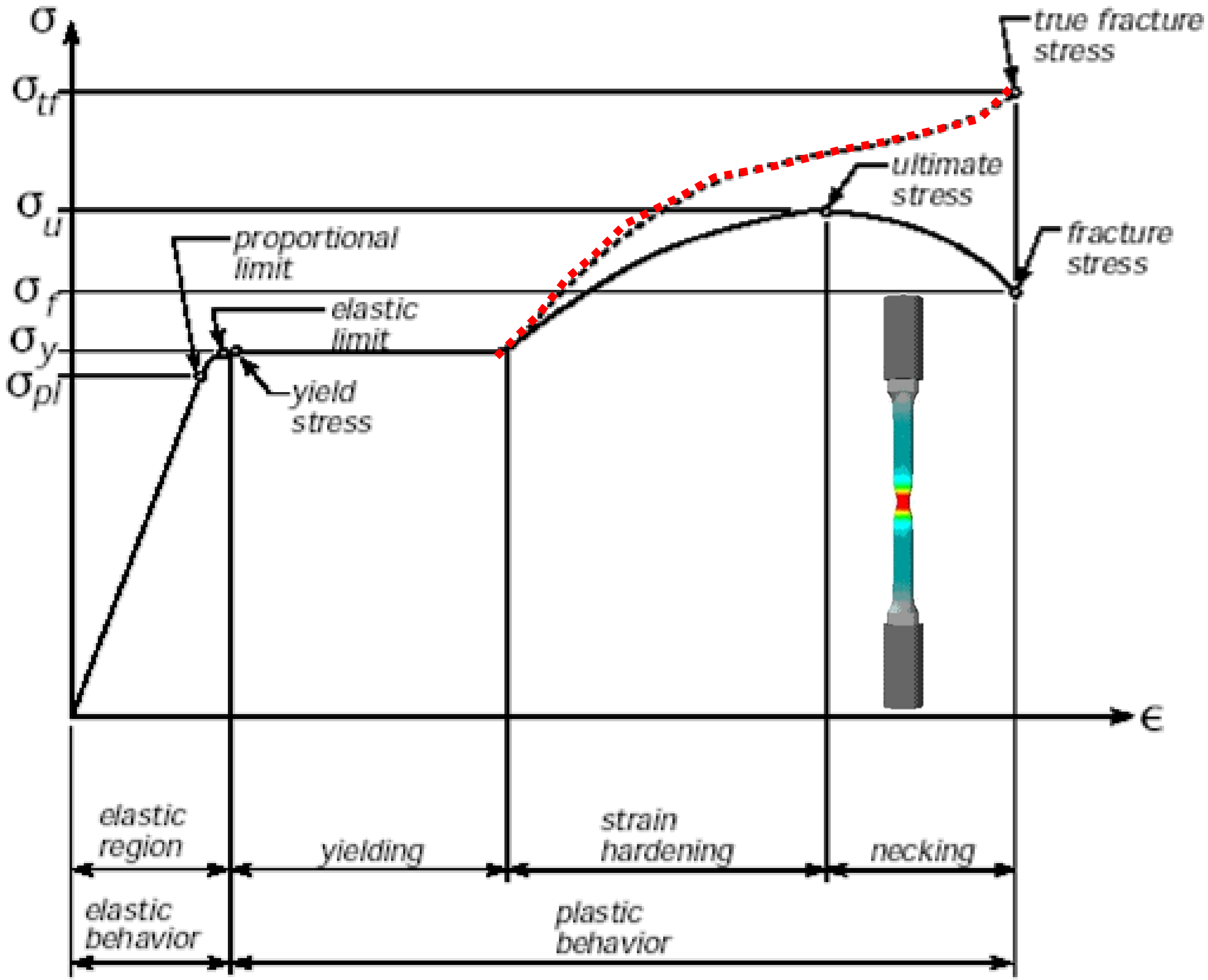
Stress-strain diagram for a brittle material.

Failure of a brittle material

Ordinary glass is a nearly ideal brittle material, because it exhibits almost no ductility whatsoever

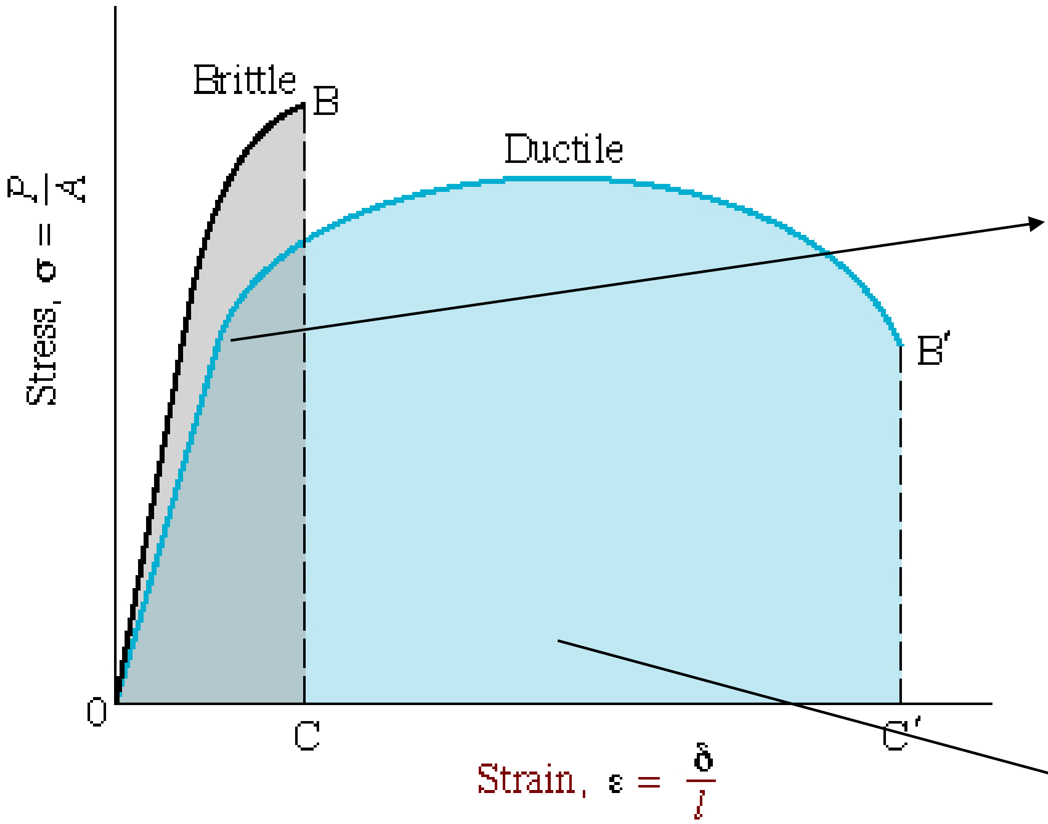


Materials that fail in tension at relatively low values of strain are classified as <u>brittle materials</u>. Examples are concrete, stone, cast iron, glass, ceramic materials, and many common metallic alloys. These materials fail with only little elongation after the proportional limit (point A) is exceeded, and the fracture stress (point B) is the same as the ultimate stress



- The <u>true-stress vs. true-strain curve</u> is a plot of the stress in the sample at its minimum diameter, after necking has begun, vs the local elongation.
- This more accurately reflects the physical processes happening in the material, but is much more difficult to measure than the engineering stress and strain, which divide the applied load by the original cross-sectional area, and the total elongation by the original length.

Brittle and Ductile Metal Comparison

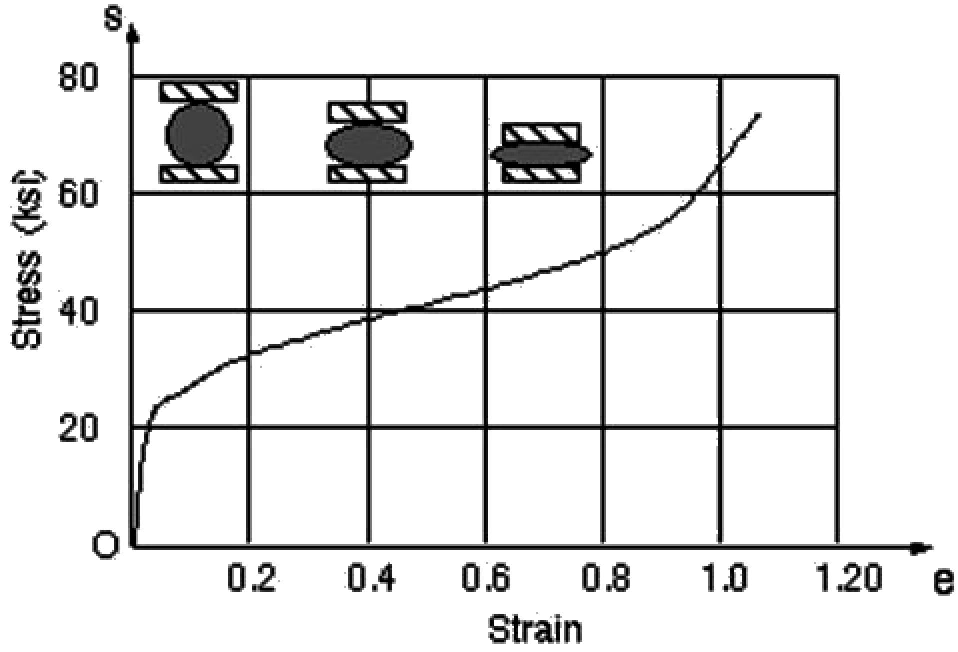


Modulus of resilience: the area under the linear part of the curve, measuring the stored elastic energy

Toughness: the total area under the curve, which measures the energy absorbed by the specimen in the process of breaking

tensile stress-strain diagrams for brittle and ductile metals loaded to fracture.

Compression Stress Strain Diagram



Compression stress-strain diagram for copper.

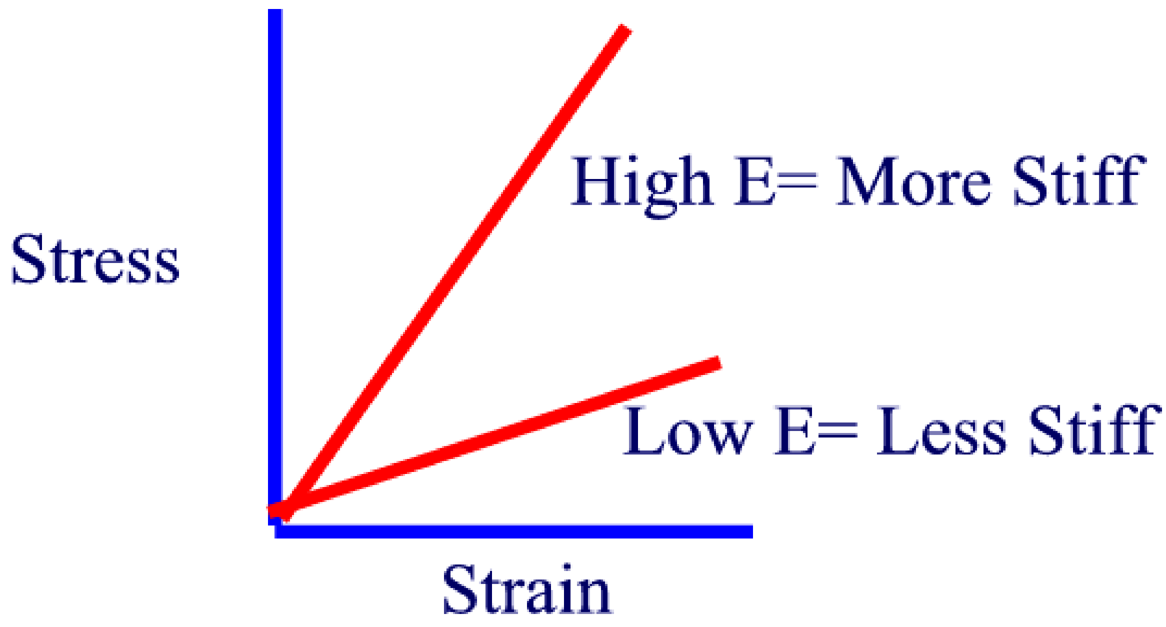
Stress-strain diagrams for compression have different shapes from those for tension. Ductile metals such as steel, aluminum, and copper have proportional limits in compression very close to those in tension, hence the initial regions of their compression stress-strain diagrams are very similar to the tension diagrams. When yielding begins, the behavior is quite different. In a tension test, the specimen is being stretched, necking may occur, and ultimately fracture takes place. When a small specimen of ductile material is compressed, it begins to bulge outward on the sides and become barrel shaped. With increasing load, the specimen is flattened out, thus offering increased resistance to further shortening (which means the stress-strain curve goes upward

Ductile Material – Materials that are capable of undergoing large strains (*at normal temperature*) before failure. An advantage of ductile materials is that visible distortions may occur if the loads before too large. Ductile materials are also capable of absorbing large amounts of energy prior to failure. Ductile materials include mild steel, aluminum and some of its alloys, copper, magnesium, nickel, brass, bronze and many others.

Brittle Material – Materials that exhibit very little inelastic deformation. In other words, materials that fail in tension at relatively low values of strain are considered brittle. Brittle materials include concrete, stone, cast iron, glass and plaster.

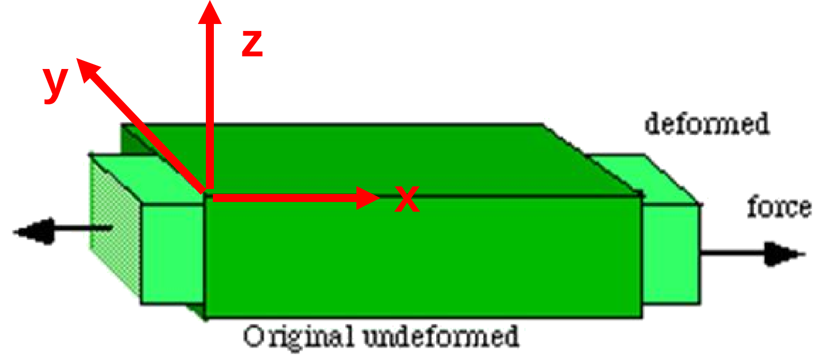
Linear Elasticity, Hooke's Law and Poisson's Ratio

Hooke's Law: $$\sigma_x = E\varepsilon_x \, ,$$

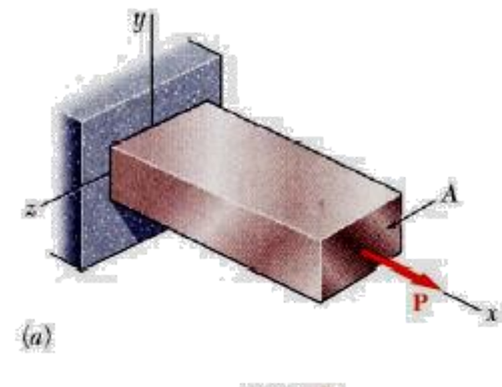


Poisson's ratio is defined as

$$\nu = \left|\frac{\text{lateral strain}}{\text{axial strain}}\right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

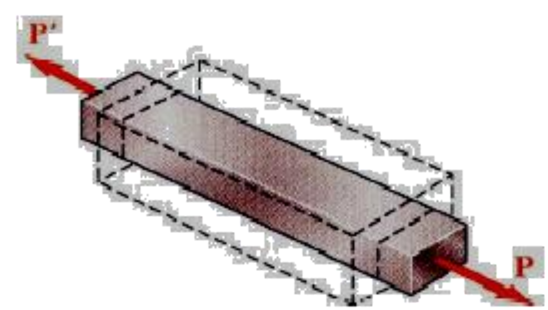


ε' (lateral strain) = - $\nu\ \varepsilon$



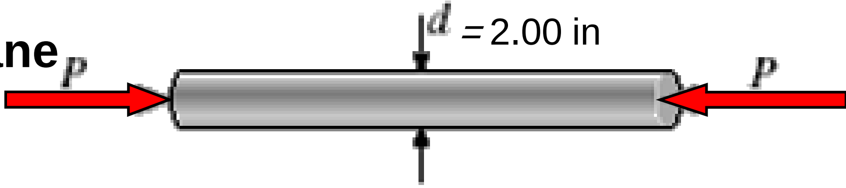
Isotropic – Isotropic materials have elastic properties that are independent of direction. Most common structural materials are isotropic.

Anisotropic – Materials whose properties depend upon direction. An important class of anisotropic materials is fiber-reinforced composites.



Homogeneous – A material is homogeneous if it has the same composition at every point in the body. A homogeneous material may or may not be isotropic.

A high-strength steel bar used in a large crane has diameter $d = 2.00$ in.
Because of clearance requirements, the diameter of the bar is limited to 2.001 in. when it is compressed by axial forces.
What is the largest compressive load P_{max} that is permitted?



E 29 x 10^6 psi

$\nu = 0.29$.

Solution

$$E = 29 \times 10^6 \text{ psi} \qquad \nu = 0.29$$

2.001-2= 0.001” ⟹ $\text{Max. } \Delta d = 0.001 \text{ in.}$

LATERAL STRAIN

$$\varepsilon' = \frac{\Delta d}{d} = \frac{0.001 \text{ in.}}{2.00 \text{ in.}} = 0.0005$$

AXIAL STRAIN

$$\varepsilon = -\frac{\varepsilon'}{\nu} = -\frac{0.0005}{0.29} = -0.001724$$

(shortening)

AXIAL STRESS

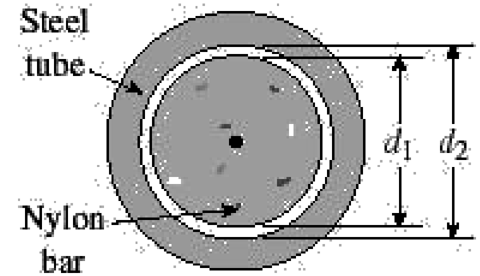
$$\sigma = E\varepsilon = (29 \times 10^6 \text{ psi})(-0.001724)$$
$$= -50.00 \text{ ksi (compression)}$$

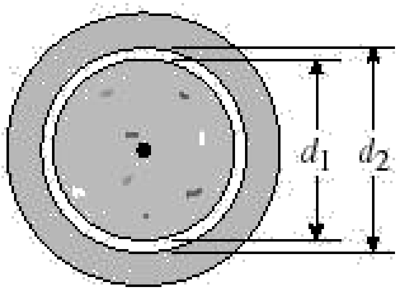
MAXIMUM COMPRESSIVE LOAD

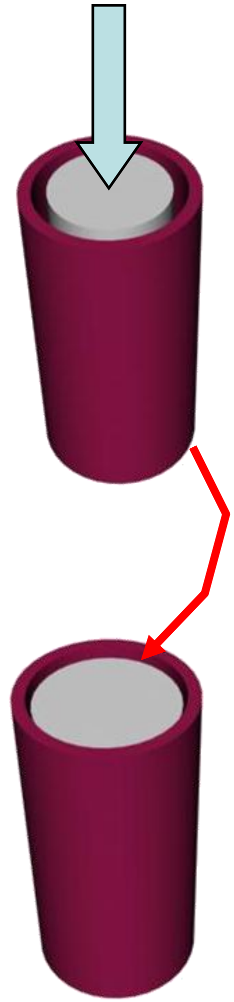
$$P_{max} = \sigma A = (50.00 \text{ ksi})\left(\frac{\pi}{4}\right)(2.00 \text{ in.})^2$$
$$= 157 \text{ k} \leftarrow$$

Problem 1.5-3 A nylon bar having diameter $d_1 = 3.50$ in. is placed inside a steel tube having inner diameter $d_2 = 3.51$ in. (see figure). The nylon bar is then compressed by an axial force P.

At what value of the force P will the space between the nylon bar and the steel tube be closed? (For nylon, assume $E = 400$ ksi and $\nu = 0.4$.)







COMPRESSION

$d_1 = 3.50$ in. $\quad \Delta d_1 = 0.01$ in.

$d_2 = 3.51$ in.

Nylon: $E = 400$ ksi $\quad \nu = 0.4$

LATERAL STRAIN

$$\varepsilon' = \frac{\Delta d_1}{d_1} \text{ (Increase in diameter)}$$

$$\varepsilon' = \frac{0.01 \text{ in.}}{3.50 \text{ in.}} = 0.002857$$

AXIAL STRAIN

$$\varepsilon = -\frac{\varepsilon'}{\nu} = -\frac{0.002857}{0.4} = -0.007143$$

(Shortening)

AXIAL STRESS

$$\sigma = E\varepsilon = (400 \text{ ksi})(-0.007143)$$

$$= -2.857 \text{ ksi}$$

(Compressive stress)

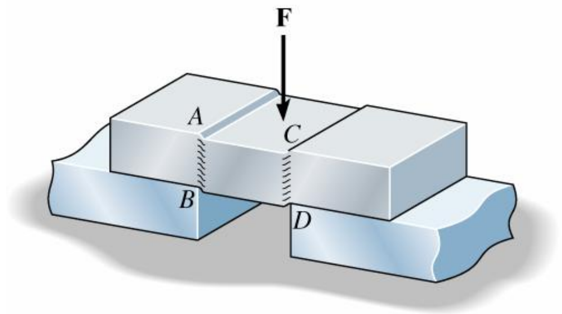
Assume that the yield stress is greater than σ and Hooke's law is valid.

FORCE P (COMPRESSION)

$$P = \sigma A = (2.857 \text{ ksi})\left(\frac{\pi}{4}\right)(3.50 \text{ in.})^2$$

$$= 27.5 \text{ k} \quad \longleftarrow$$

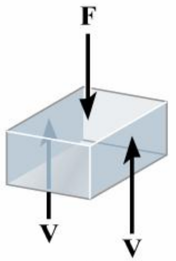
Average Shear Stress



(a)

$$\tau = V/A$$

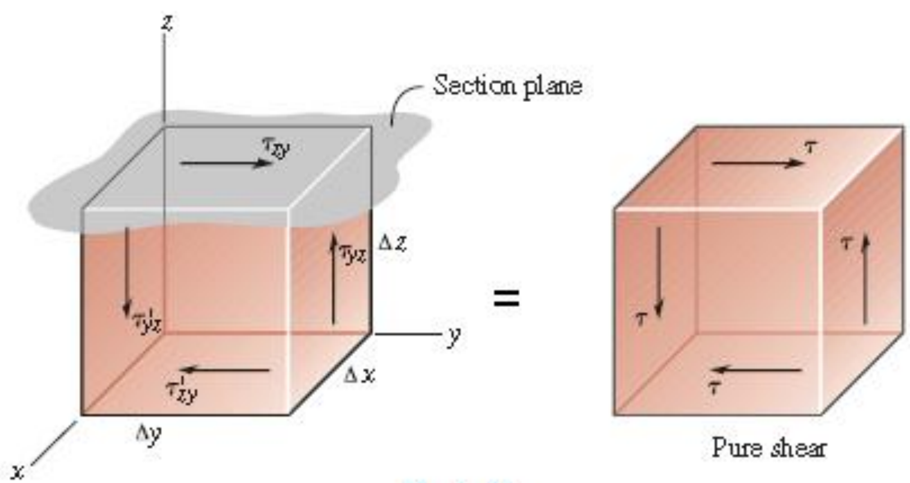
average shear stress at the section, which is assumed to be the *same at each point located on the section*

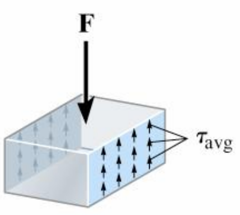


(b)

V = internal resultant shear force at the section determined from the equations of equilibrium

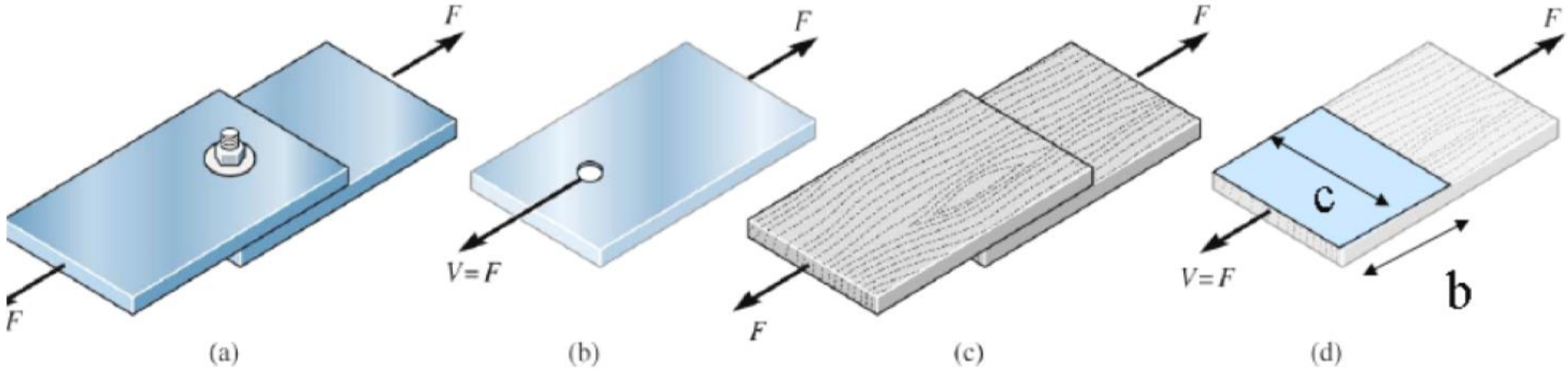
A= area at the section (parallel to the shear force





(c)

Single Shear

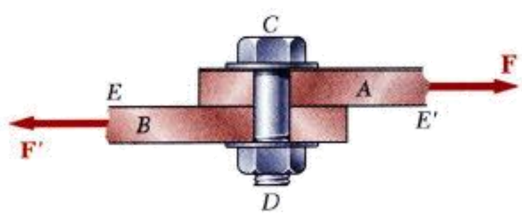


Shear stress on bolt

Shear stress on bonded area

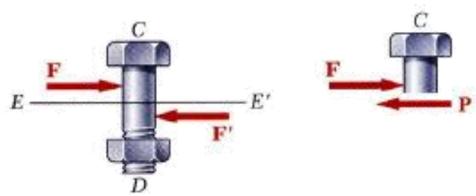
$\tau = F/A = F/\pi r^2$

Where r is the radius of the bolt

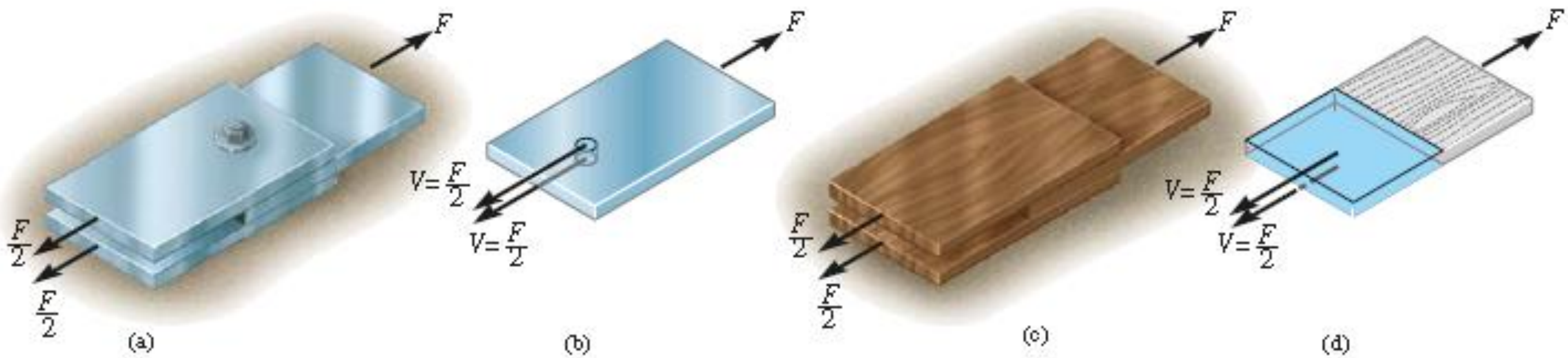


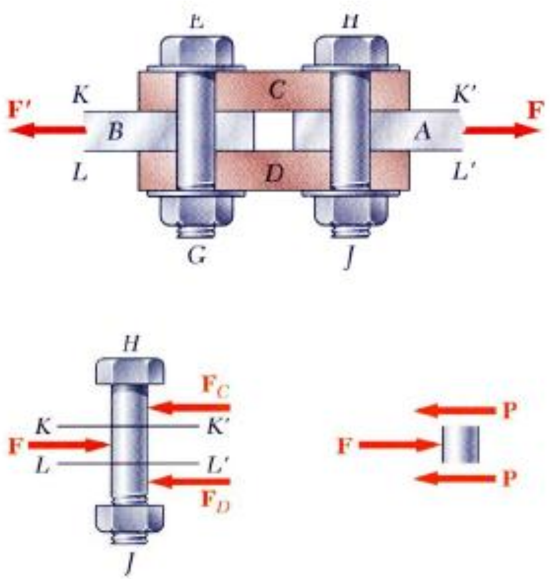
$\tau = F/bc$

Where (bc) is the area of contact subjected to the shear force



Double Shear





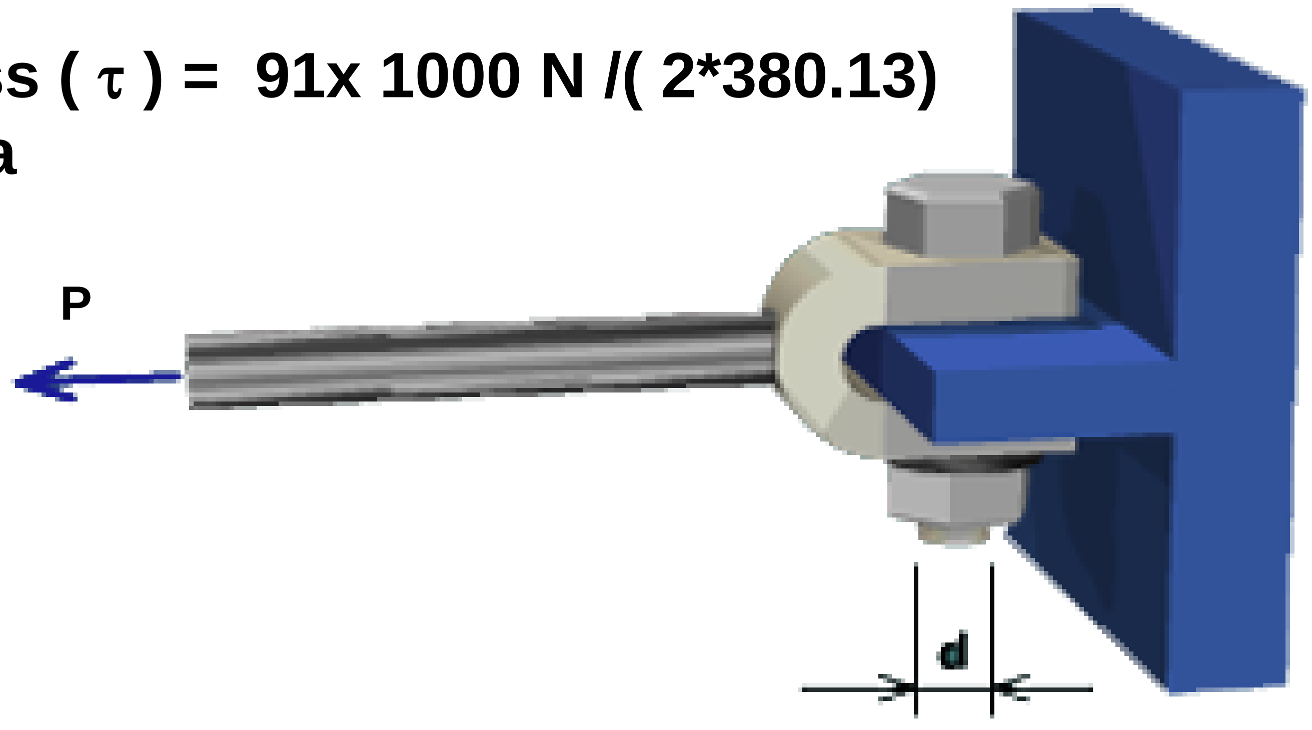
$$\tau=(F/2)/A=F/2A$$

Where A is the parallel area of the bolt subjected to shear force

The bolted connection is subjected to a tensile force of P= 91kN. The diameter of the bolt d= 22mm. Determine the average shear stress in the bolt in (MPa)

Cross section area of bolt = 380.13 mm^2

Shear stress (τ) = 91x 1000 N /(2*380.13) = 119.7 MPa



Allowable Stress and allowable Load

Factors to be considered in design includes :

- **functionality,**
- **strength,**
- **appearance,**
- **economics and**
- **environmental protection.**

Factor of safety considerations:

- uncertainty in material properties
- uncertainty of loadings
- uncertainty of analyses
- number of loading cycles
- types of failure
- maintenance requirements and deterioration effects
- importance of member to integrity of whole structure
- risk to life and property
- influence on machine function

Factor of Safety = $$n = \frac{Actual\ strength}{Required\ strength}$$

The factor of safety must be greater than one to avoid failure

The allowable load = (Permissible load or safe load) = (Allowable stress) (Area)

$$P_{allow} = \sigma_{allow} A$$

Allowable Stress and allowable Load

Factors to be considered in design includes functionality, strength, appearance, economics and environmental protection.

Factor of Safety =

$$n = \frac{Actual\ strength}{Required\ strength}$$

The factor of safety must be greater than one to avoid failure

The allowable load = (Permissible load or safe load) = (Allowable stress) (Area)

$$P_{allow} = \sigma_{allow} A$$

allowable stress

The factor of safety is a number greater than unity (n>1).

The allowable stress for a given material is: the maximum stress the material can take (normally the ultimate or yield stress) divided by the factor of safety.

$$\sigma_{allow} = \frac{\sigma_y \ \underline{\underline{or}} \ \sigma_u}{n}$$

$$\tau_{allow} = \frac{\tau_y \ \underline{\underline{or}} \ \tau_u}{n}$$

Design for Axial Loads and Direct Shear

Analysis: Given the structure and loads, determine stresses and strains.
Design: Given the loads and allowable stresses, determine the properties of the structure.
Design for axial loads and direct shear entails finding the required area to carry the loads

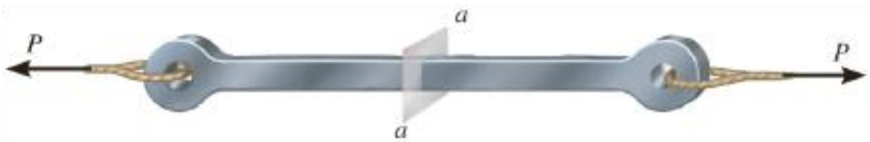
$$\text{Required area} = \frac{\text{Load to be transmitted}}{\text{Allowable stress}} \qquad \text{(i.e., Strength Consideration)}$$

Other design considerations include

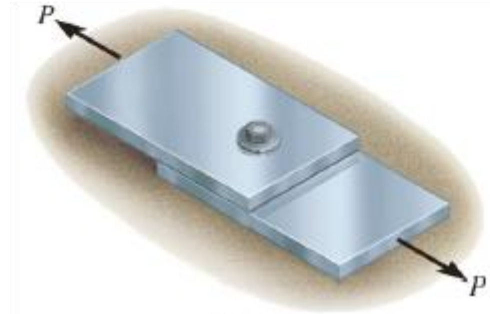
- ***Stiffness***: Designing the structure to resist changes in shape.
- ***Stability***: Designing the structure to resist buckling under compressive loads.
- ***Optimization***: Designing the best structure to meet a particular goal.

Design of Simple Connections

$$A = \frac{P}{\sigma_{allowable}} \quad or \quad A = \frac{V}{\tau_{allowable}}$$

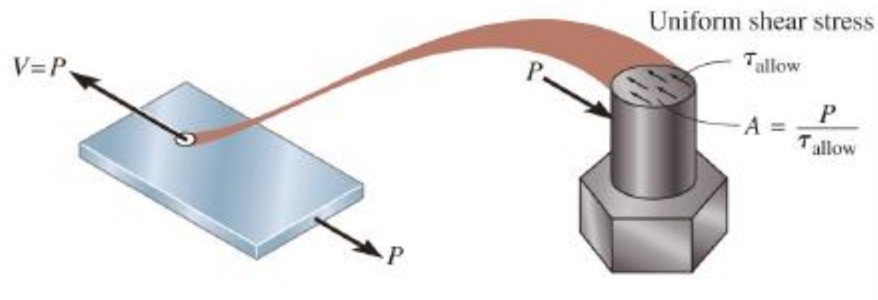


(a)



(a)

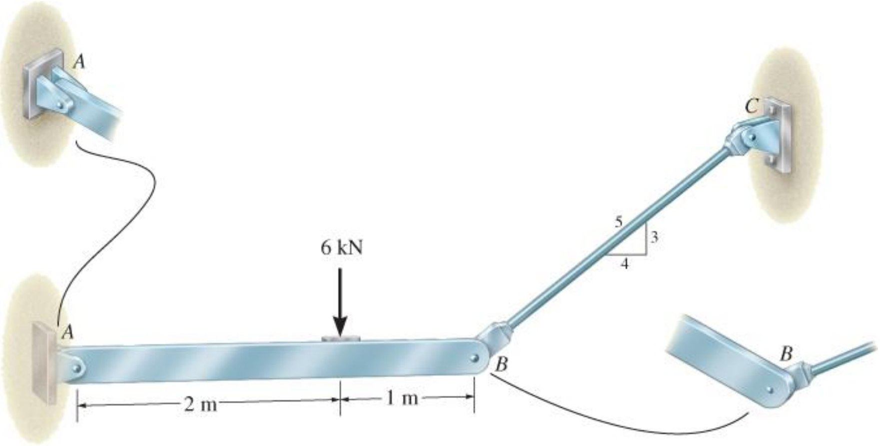
(b)



(b) (c)

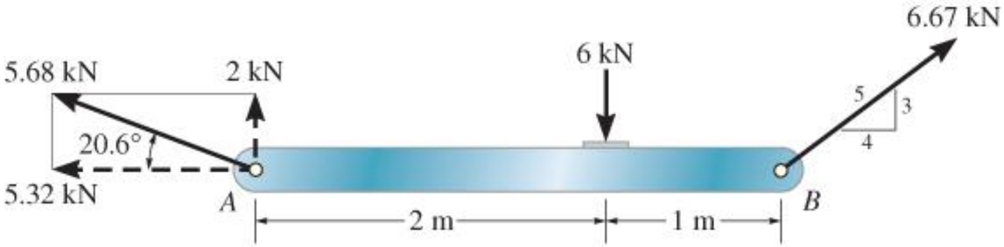
The two members pinned together at *B*. If the pins have an allowable shear stress of $\tau_{allow} = 90$ MPa, and allowable tensile stress of rod *CB* is $(\sigma_t)_{allow} = 115$ MPa

Determine to nearest mm the smallest diameter of pins *A* and *B* and the diameter of rod *CB* necessary to support the load.



(a)

F.B.D



(b)

$$\sigma = \frac{P}{A} = \frac{800\ \text{N}}{(0.04\ \text{m})(0.04\ \text{m})} = 500\ \text{kPa}$$

Diameter of pins:

(c)

$$A_A = \frac{V_A}{T_{\text{allow}}} = \frac{2.84 \text{ kN}}{90 \times 10^3 \text{ kPa}} = 31.56 \times 10^{-6} \text{ m}^2 = \pi(d_A{}^2/4)$$

$d_A = 6.3$ mm

$$A_B = \frac{V_B}{T_{\text{allow}}} = \frac{6.67 \text{ kN}}{90 \times 10^3 \text{ kPa}} = 74.11 \times 10^{-6} \text{ m}^2 = \pi(d_B{}^2/4)$$

$d_B = 9.7$ mm

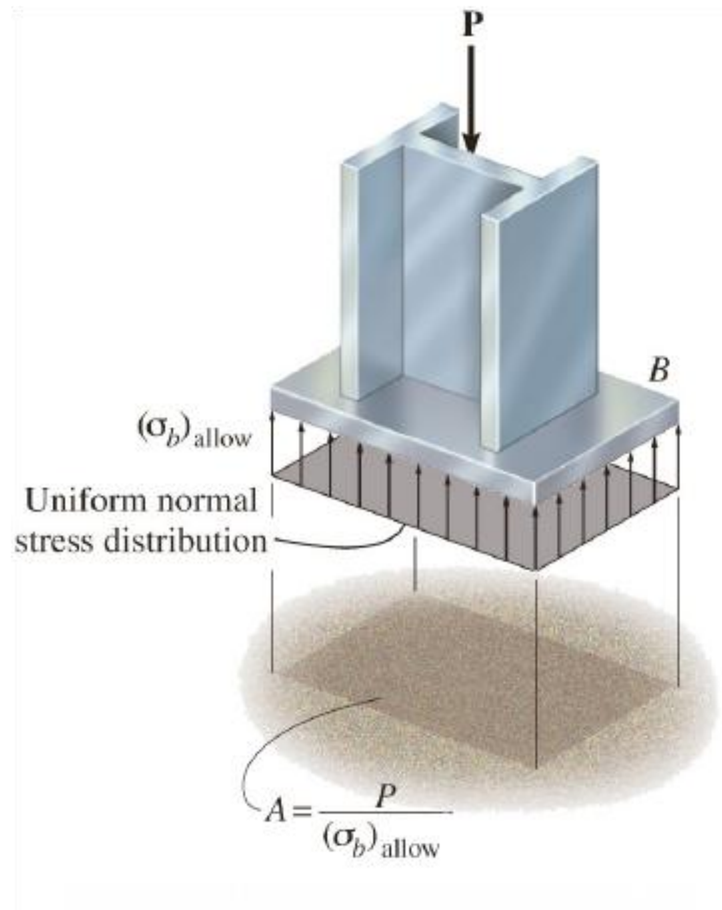
Choose a size larger to nearest millimeter.

$d_A = 7$ mm $\qquad$ $d_B = 10$ mm

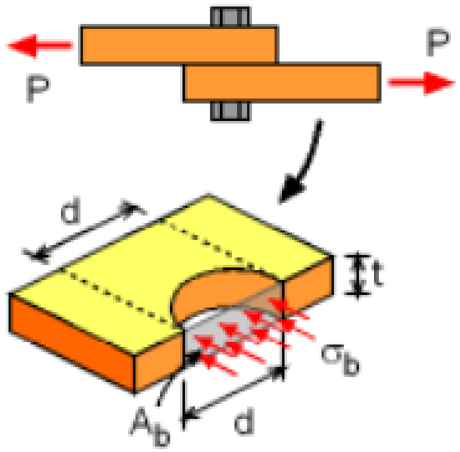
Bearing stress

$$A = \frac{P}{(\sigma_b)_{allowable}}$$

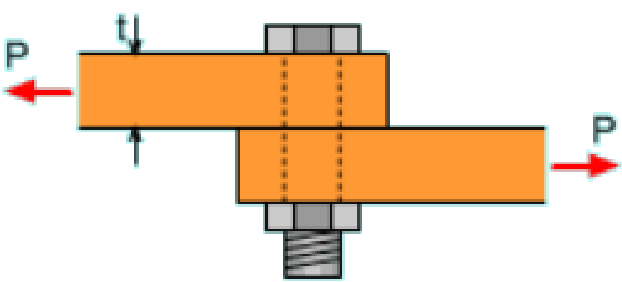
where $(\sigma_b)_{allowable}$ is the allowable bearing stress of the weaker material



Bearing Stress



Bearing Stress Due to a Bolt

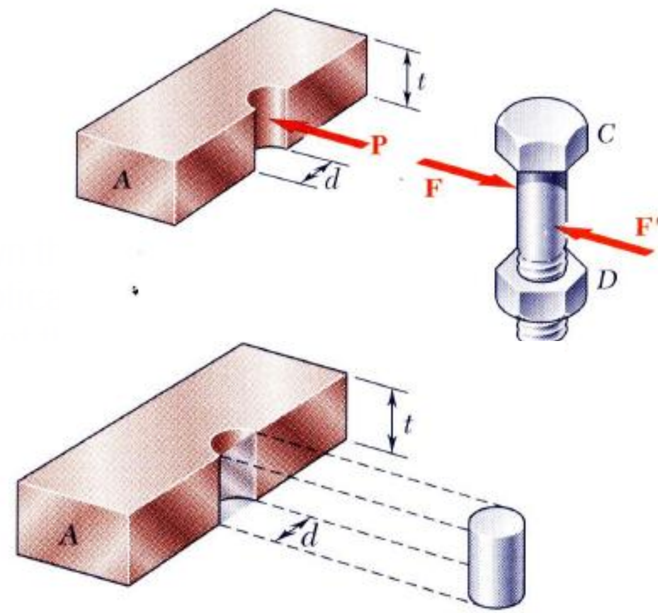


The average bearing stress is the force pushing against a structure divided by the area. Exact bearing stress is more complicated but for most applications, the following equation works well for the average,

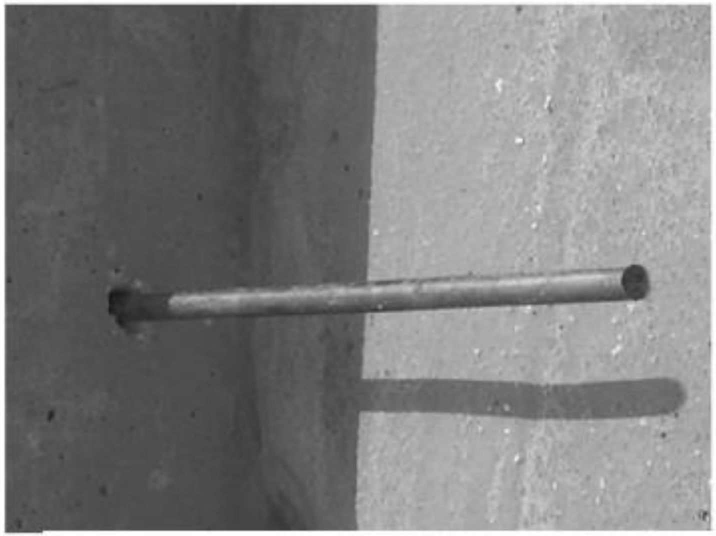
$$\sigma_b = P/A_b$$

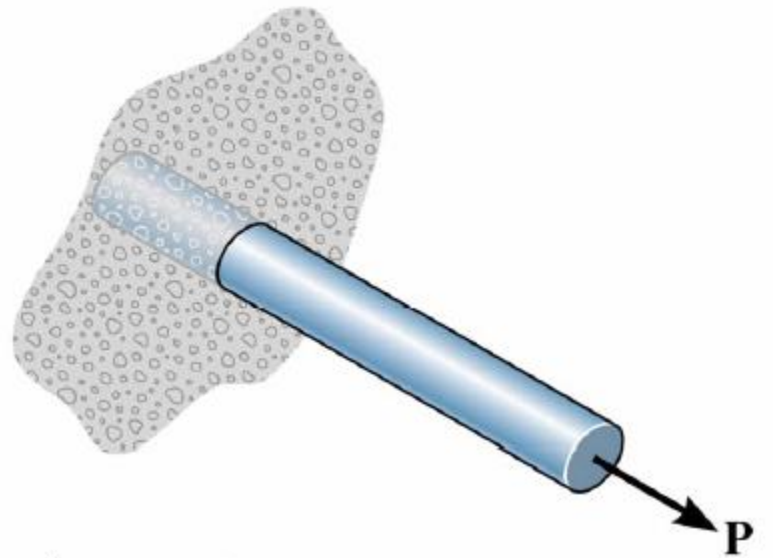
This relationship can be further refined by using the width and height of the bearing area as

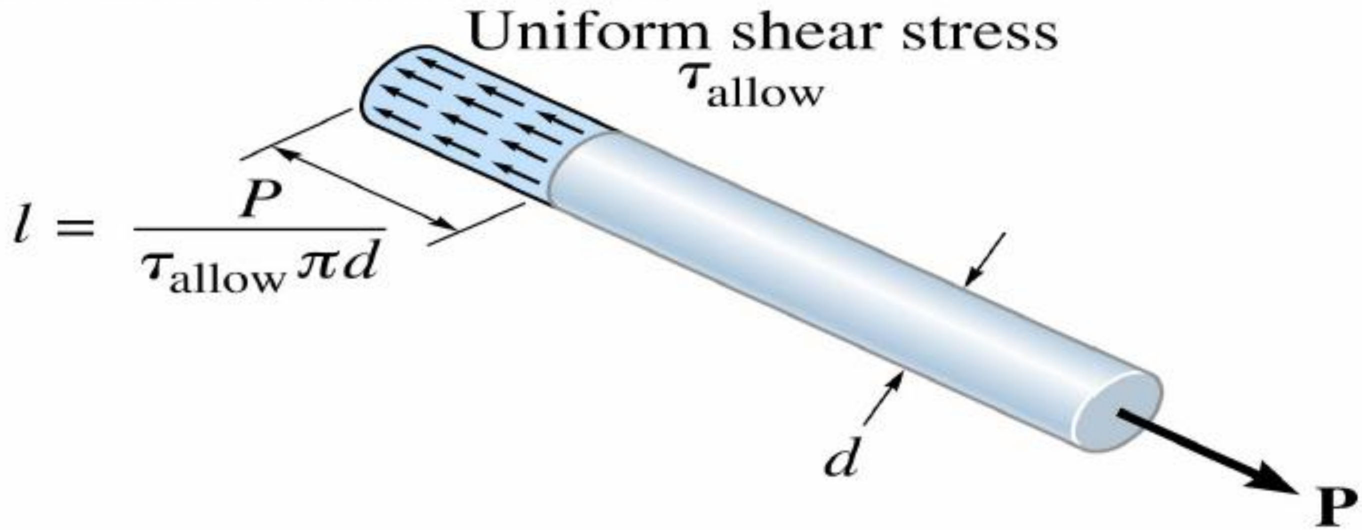
$$\sigma_b = \frac{P}{dt}$$



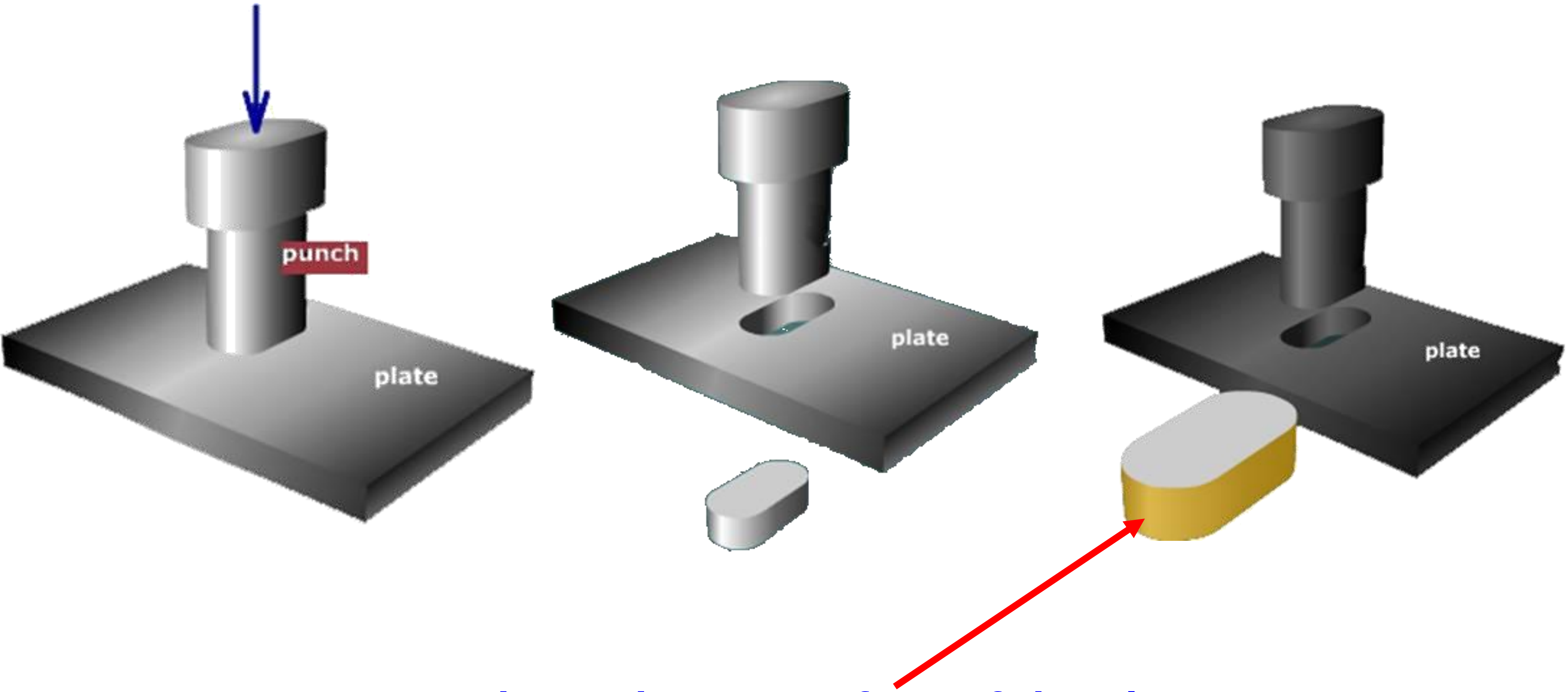
Required Area to resist shear caused by axial load





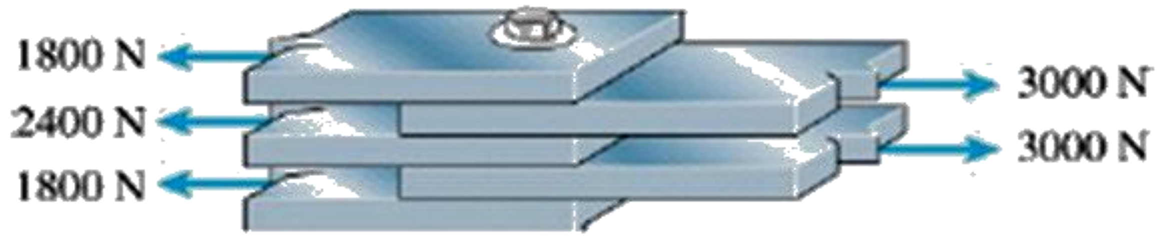


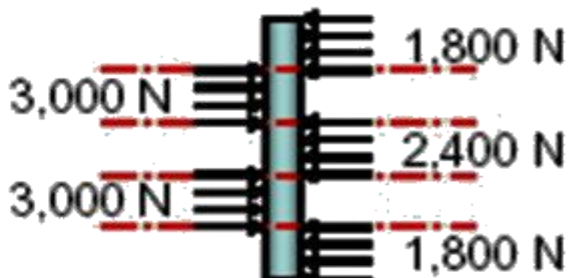
Punch Shear

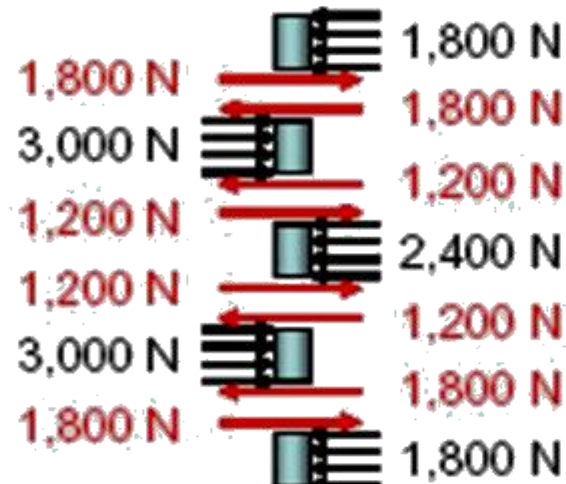


Stress acts on the perimeter surface of the slug. To compute the shear stress at failure, divide the applied load by the area of the slug perimeter

EXAMPLE: The connection shown in the figure consists of five steel plates, each 2.5 mm thick, to be joined by a single bolt. Determine the required diameter of the bolt if the allowable bearing stress, σ_b, is 180.0 MPa and the allowable shear stress, τ_{allow}, is 45.0 MPa?







Maximum Bearing Stress:

$$\sigma_b = \frac{P_b}{t_{plate} \times d_{bolt}} = \frac{3{,}000\text{ N}}{(2.5\times10^{-3}\text{ m})d_{bolt}} = 180\text{ MPa}$$

$$d_{bolt} = \frac{3{,}000\text{ N}}{(2.5\times10^{-3}\text{ m})\times180\times10^{6}\text{ N/m}^2}$$

$$= 0.00667\text{ m} = 6.67\text{ mm}$$

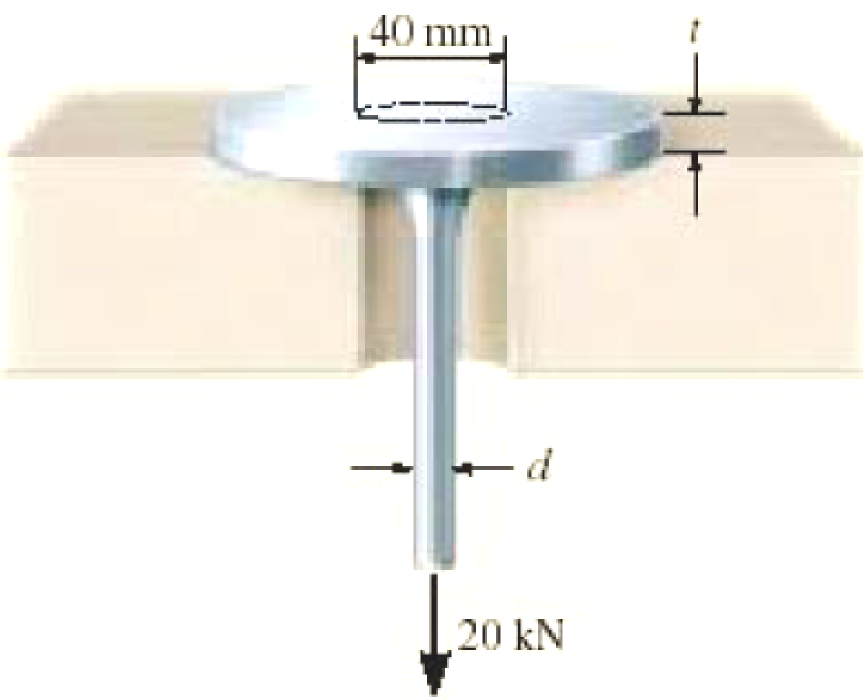
Maximum Shear Stress:

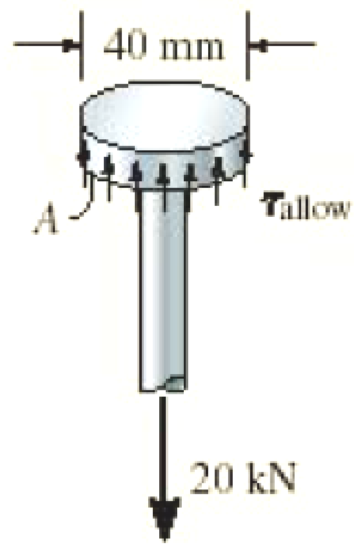
$$\tau = \frac{V}{A_{bolt}} = \frac{1{,}800\text{ N}}{\pi d_{bolt}^2/4} = 45\text{ MPa}$$

$$d_{bolt} = \sqrt{\frac{4\times1{,}800\text{ N}}{\pi\tau_{allow}}} = \sqrt{\frac{4\times1{,}800\text{ N}}{\pi\times45\times10^{6}}}$$

$$= 0.00714\text{ m} = 7.14\text{ mm}$$

The suspender rod is supported at its end by a fixed-connected circular disk as shown If the rod passes through a 40-mm-diameter hole, determine the minimum required diameter of the rod and the minimum thickness of the disk needed to support the 20-kN load. The allowable normal stress for the rod is $\sigma_{allow} = 60$ MPa, and the allowable shear stress for the disk is $\tau_{allow} = 35$ MPa.





Diameter of Rod. By inspection, the axial force in the rod is 20 kN. Thus, the required cross-sectional area of the rod is

$$A = \frac{P}{\sigma_{allow}} = \frac{20(10^3)\text{ N}}{60(10^6)\text{ N/m}^2} = 0.3333(10^{-3})\text{ m}^2$$

So that

$$A = \pi\left(\frac{d^2}{4}\right) = 0.3333(10^{-2})\text{ m}^2$$

$$d = 0.0206\text{ m} = 20.6\text{ mm} \qquad \textit{Ans.}$$

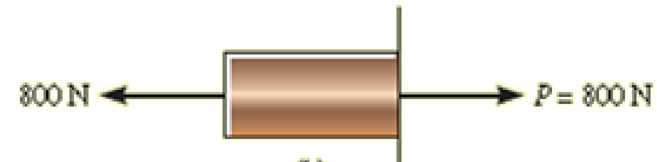
Thickness of Disk. As shown on the free-body diagram of the core section of the disk, Fig. 1–33*b*, the material at the sectioned area must resist *shear stress* to prevent movement of the disk through the hole. If this shear stress is *assumed* to be distributed uniformly over the sectioned area, then, since $V = 20$ kN, we have

$$A = \frac{V}{\tau_{allow}} = \frac{20(10^3)\text{ N}}{35(10^6)\text{ N/m}^2} = 0.571(10^{-3})\text{ m}^2$$

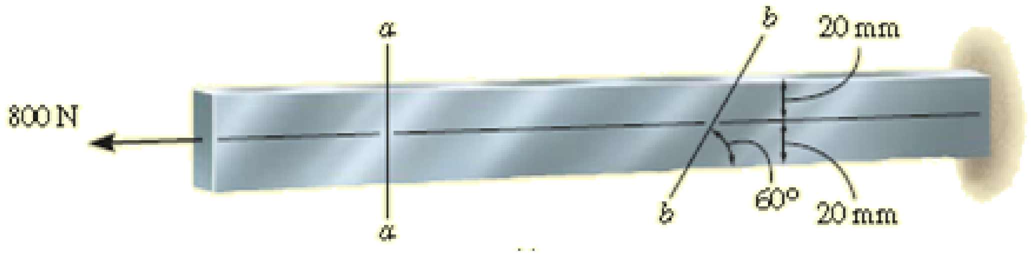
Since the sectioned area $A = 2\pi(0.02\text{ m})(t)$, the required thickness of the disk is

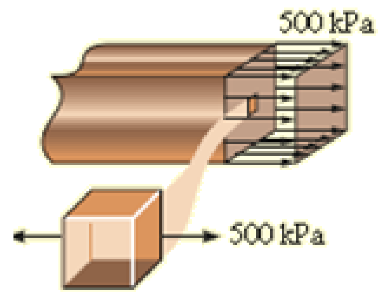
$$t = \frac{0.5714(10^{-3})\text{ m}^2}{2\pi(0.02\text{ m})} = 4.55(10^{-3})\text{ m} = 4.55\text{ mm} \qquad \textit{Ans.}$$

The bar shown has a square cross section for which the depth and thickness are 40 mm. If an axial force of 800 N is applied along the centroidal axis of the bar's cross-sectional area, determine the average normal stress and average shear stress acting on the material along (a) section plane *a–a* and (b) section plane *b–b*.



(b)





(c)

Part (a)

Internal Loading. The bar is sectioned, Fig. 1–24*b*, and the internal resultant loading consists only of an axial force for which $P = 800$ N.

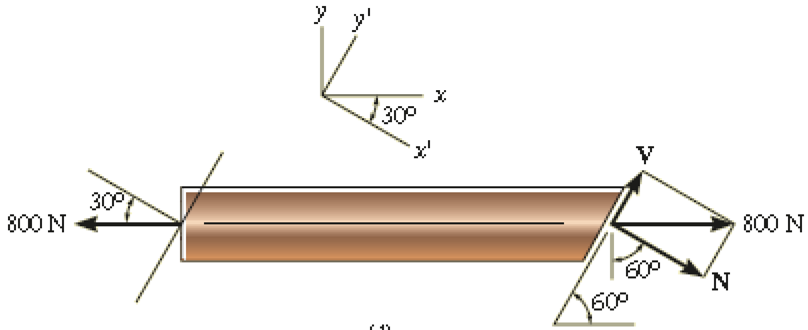
Average Stress. The average normal stress is determined from Eq.1–6.

$$\sigma = \frac{P}{A} = \frac{800\text{ N}}{(0.04\text{ m})(0.04\text{ m})} = 500\text{ kPa} \qquad \textit{Ans.}$$

No shear stress exists on the section, since the shear force at the section is zero.

$$\tau_{avg} = 0 \qquad \textit{Ans.}$$

The distribution of average normal stress over the cross section is shown in Fig. 1–24*c*.



(d)

$$+\searrow\Sigma F_{x'} = 0; \qquad N - 800\text{ N}\cos 30^\circ = 0$$

$$+\nearrow\Sigma F_{y'} = 0; \qquad V - 800\text{ N}\sin 30^\circ = 0$$

Solving either set of equations,

$$N = 692.8\text{ N}$$
$$V = 400\text{ N}$$

Inclined Plane

$$\sigma = \frac{P}{A}\cos^2\theta$$

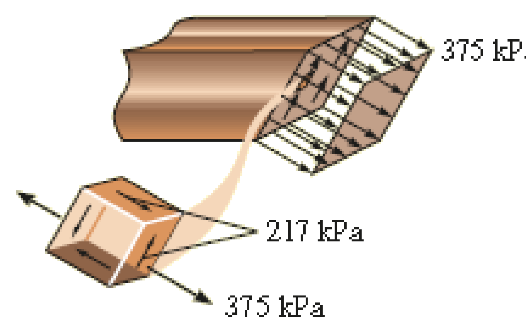
$$\tau = \frac{P}{A}\cos\theta\sin\theta$$

Average Stresses. In this case the sectioned area has a thickness and depth of 40 mm and 40 mm/sin 60° = 46.19 mm, respectively, Fig. 1–24*a*. Thus the average normal stress is

$$\sigma = \frac{N}{A} = \frac{692.8\text{ N}}{(0.04\text{ m})(0.04619\text{ m})} = 375\text{ kPa} \qquad \textit{Ans.}$$

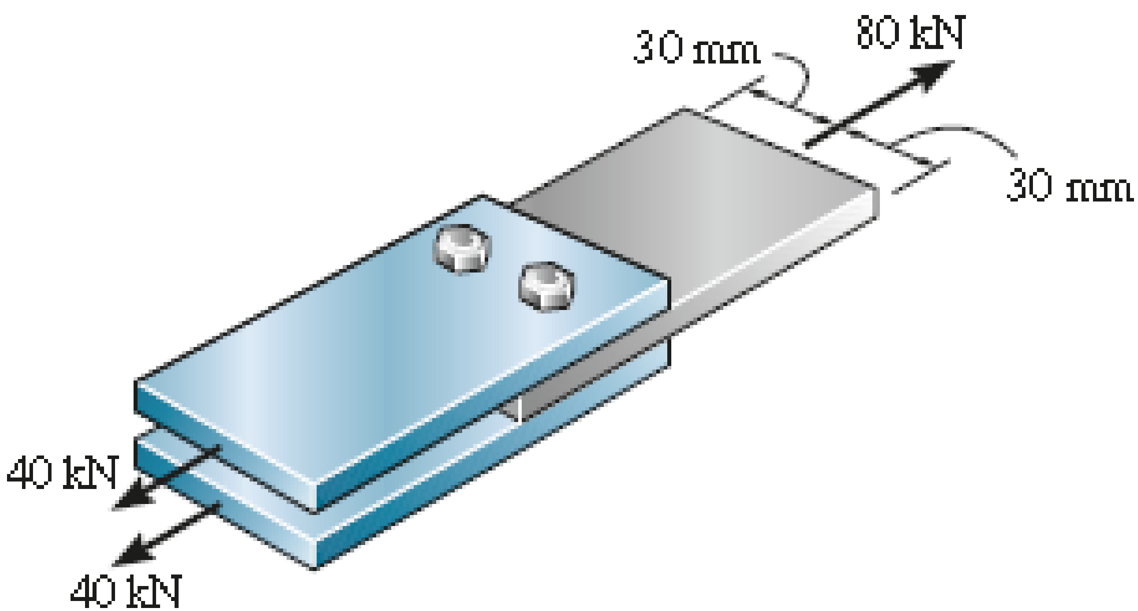
and the average shear stress is

$$\tau_{\text{avg}} = \frac{V}{A} = \frac{400\text{ N}}{(0.04\text{ m})(0.04619\text{ m})} = 217\text{ kPa} \qquad \textit{Ans.}$$

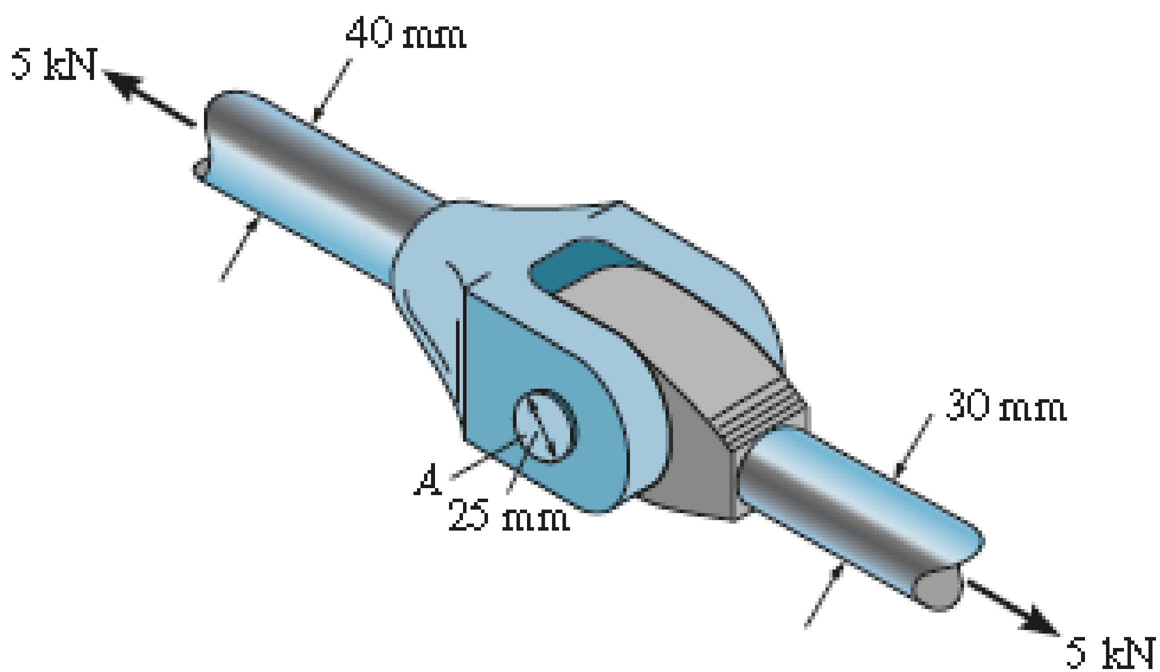


(e)

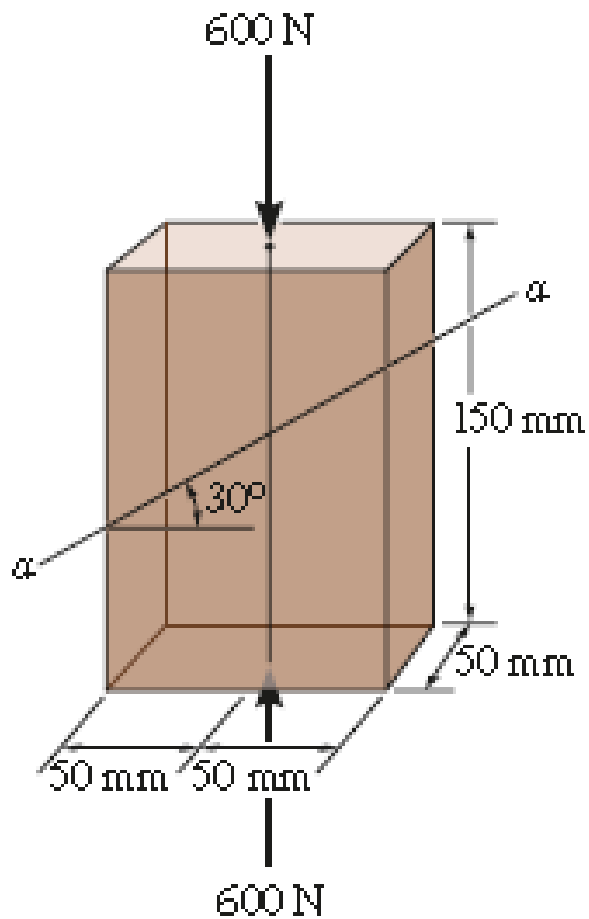
The joint is fastened together using two bolts. Determine the required diameter of the bolts if the allowable shear stress for the bolts is $\tau_{allow} = 110$ MPa. Assume each bolt supports an equal portion of the load.



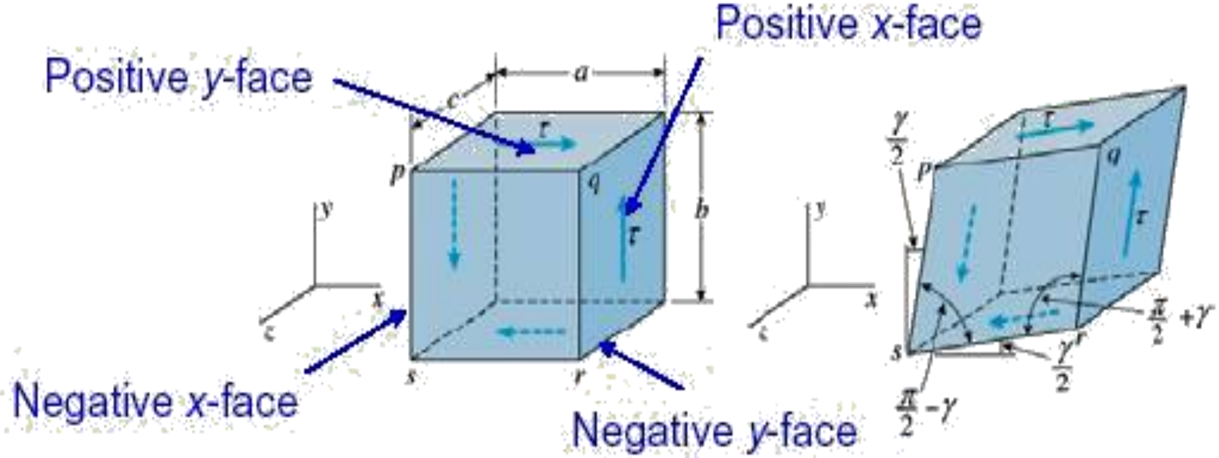
The yoke-and-rod connection is subjected to a tensile force of 5 kN. Determine the average normal stress in each rod and the average shear stress in the pin *A* between the members.



The plastic block is subjected to an axial compressive force of 600 N. Assuming that the caps at the top and bottom distribute the load uniformly throughout the block, determine the average normal and average shear stress acting along section a–a.



Sign Conventions for Shear Stresses and Strains

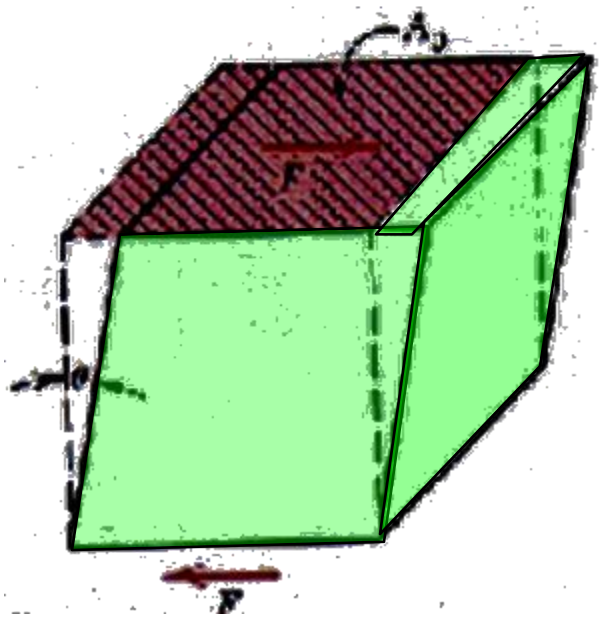


A shear stress is positive if it is acting on a positive face and in the positive direction of one of the coordinate axes, or on a negative face and in the negative direction of one of the coordinate axes. A shear stress is negative if it is acting on a negative face and in the positive direction of one of the coordinate axes, or on a positive face and in the negative direction of one of the coordinate axes.

A shear strain in an element is positive when the angle between two positive faces (or two negative faces) is reduced, and is negative if the angle is increased.

Hooke's Law in Shear

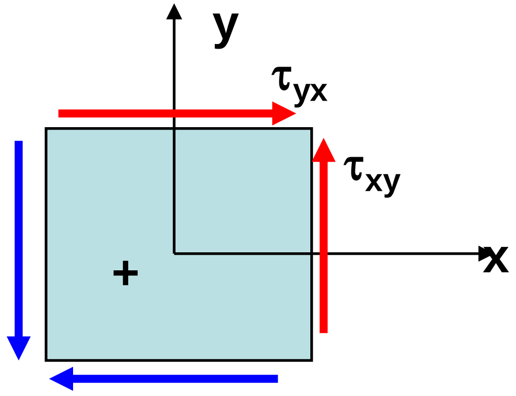
$$\tau = G\gamma$$



The constant ***G*** is called the shear modulus and relates the shear stress and strain in the elastic region .

It is also used to relate shear and elastic moduli•

$$E = 2G(1 + \nu)$$



Engineering Mechanics

Statics & Strength of Materials

Axially deformation and thermal stresses

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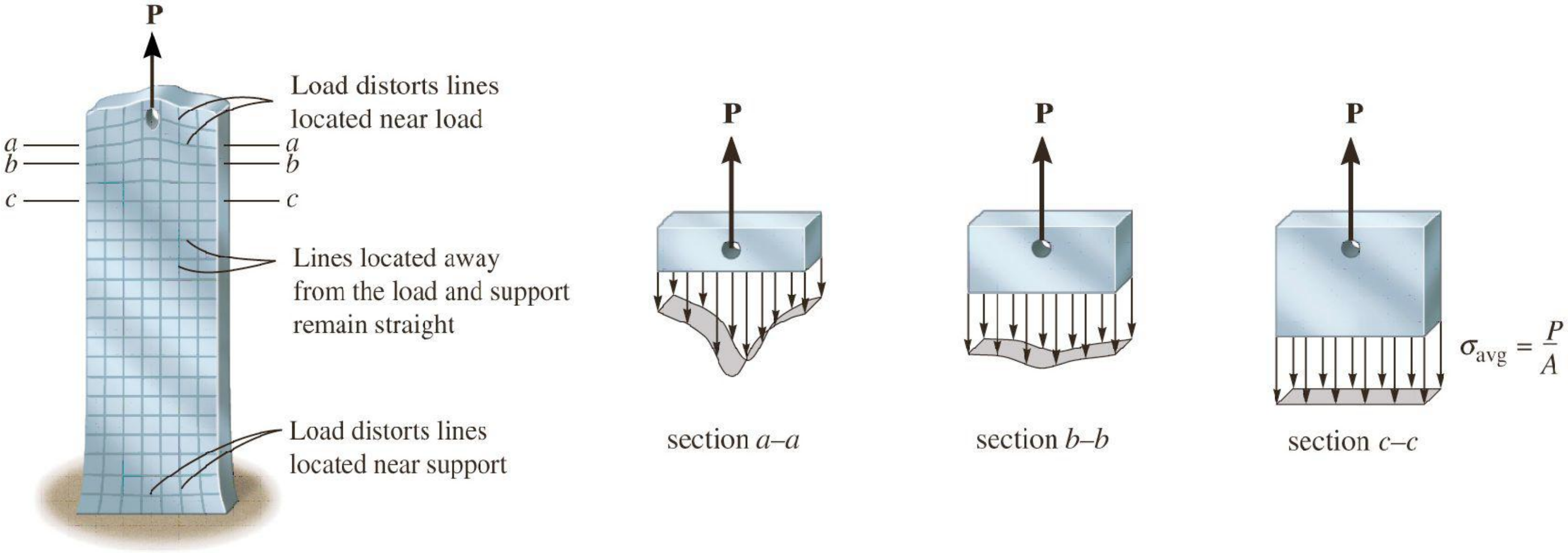
Saint-Venant's Principle (discovered by Barré de Saint-Venant in 1855)

A sufficient distance away from the point of application of the load, the stresses will be identical for any load with the same resultant force.

Localized deformation occurs at each end, and the deformations decrease as measurements are taken further away from the ends



Introduction

Axially loaded Members : **Structural components subjected only to tension or compression**

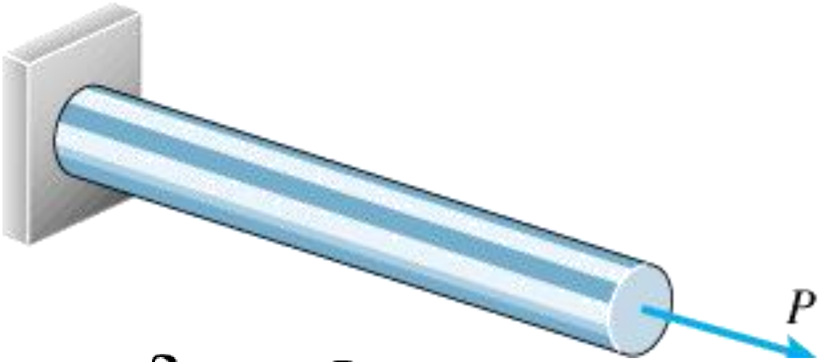
(solid bars, cables, coil springs)

eg.

Truss members

Connecting rods in engines

Columns in buildings



$$\delta = \varepsilon L$$

$$\varepsilon = \frac{\sigma}{E}$$

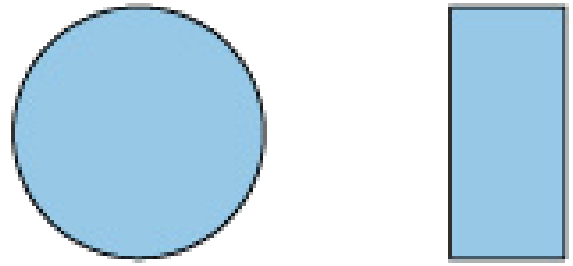
$$\sigma = \frac{P}{A}$$

Typical cross sections of structural members.

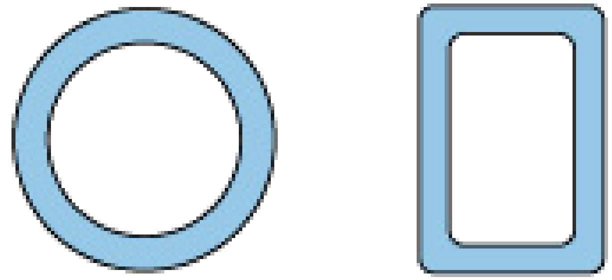
$$\left(\frac{P}{A}\right) = E\left(\frac{\delta}{L}\right) \Rightarrow \delta = \frac{PL}{EA}$$

$$\text{Stiffness}: k = \frac{EA}{L}$$

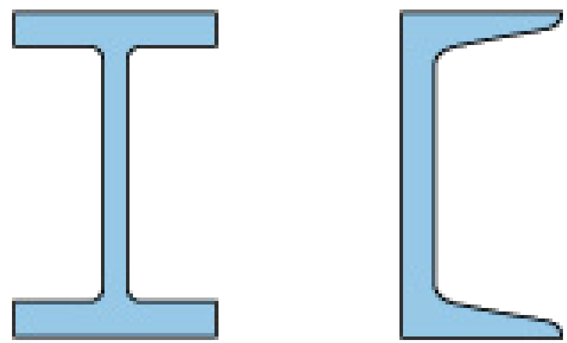
$$\text{Flexibility}: f = \frac{L}{EA}$$



Solid cross sections



Hollow or tubular cross sections

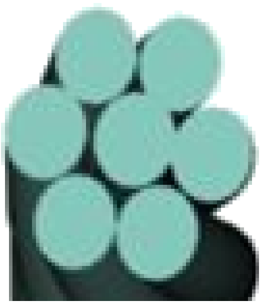


Thin-walled open cross sections

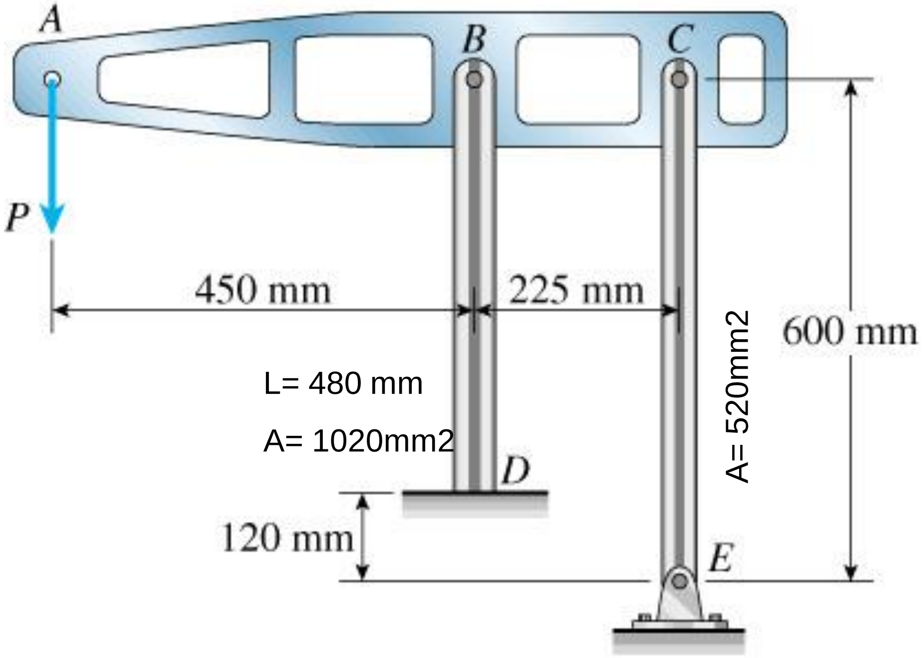
Cables

Cables are used to transmit large tensile force.

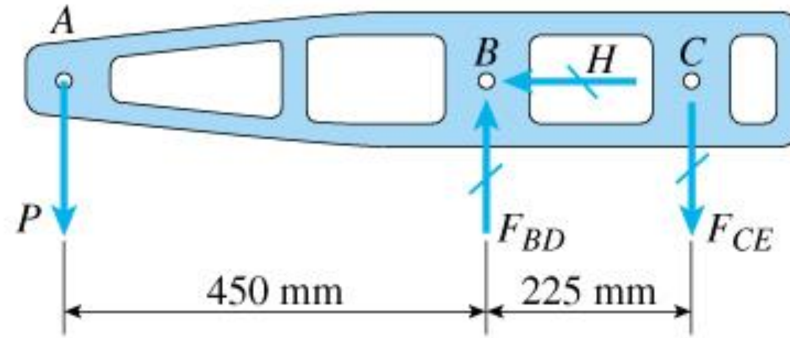
Cables are constructed from large number of wires wound in some particular manner.



The cross sectional area of a cable is equal to the total cross sectional area of the individual wires, called the effective area or metallic area. This area is less than the area of a circle having the same diameter as the cable because there are spaces between the individual wires

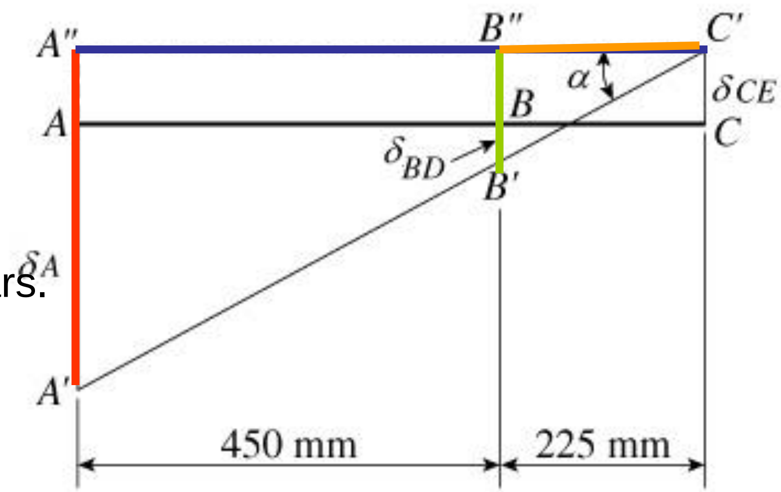


(a)



(b)

The bars are of steel E= 205 GPa



(c)

Horizontal beam ***ABC (rigid)*** supported by two vertical bars.

Find the maximum allowable load P max. if the displacement of point A is limited to 0.1mm

$$\delta = \frac{PL}{AE}$$

F_{CE}= 2P F_{BD}= 3P

δ_{BD} (3P) 480/ [(205e9N/m²) (1020)] = 6.887 P E-6 mm

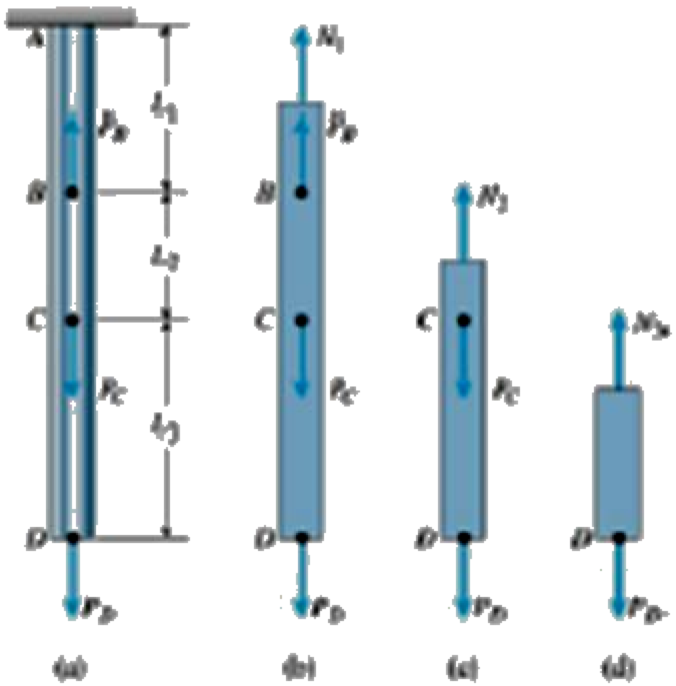
δ_{CE} (2P) 600/ [(205e9N/m2) (520)] = 11.26 P E-6 mm

(δ_A + 11.26e-6P) / (450+225) = (6.887Pe-6 + 11.26Pe-6)/ 225

P = 23200 N

Changes in Lengths of Non-Uniform Bars

1 Bars with Intermediate Axial Loads



(a) Bar with external loads acting at intermediate points; (b) (c), and (d) free-body diagrams (F.B.D.) showing the internal axial forces N_1, N_2, and N_3.

1. Denote the internal forces in segments *AB*, *BC*, and *CD* as N_1, N_2, and N_3, respectively. Draw F.B.D. as shown to expose those internal forces.
2. Determine the internal force in each segment from the FBDs. The internal force remains constant in each segment

$$N_1 = -P_B + P_C + P_D, \quad N_2 = P_C + P_D, \quad N_3 = P_D$$

3. Determine the change in the length of each segment (of length L_1, L_2, and L_3, respectively)

$$\delta_1 = \frac{N_1 L_1}{EA}, \quad \delta_2 = \frac{N_2 L_2}{EA}, \quad \delta_3 = \frac{N_3 L_3}{EA}$$

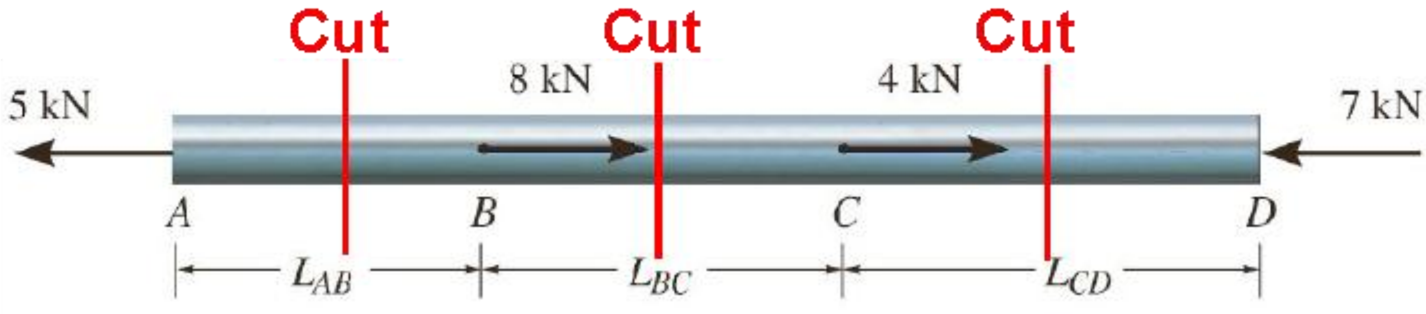
4. The change in length for the entire bar is the sum of the changes in length of all segments

$$\delta = \delta_1 + \delta_2 + \delta_3 = \sum_{i=1}^{3} \frac{N_i L_i}{EA}$$

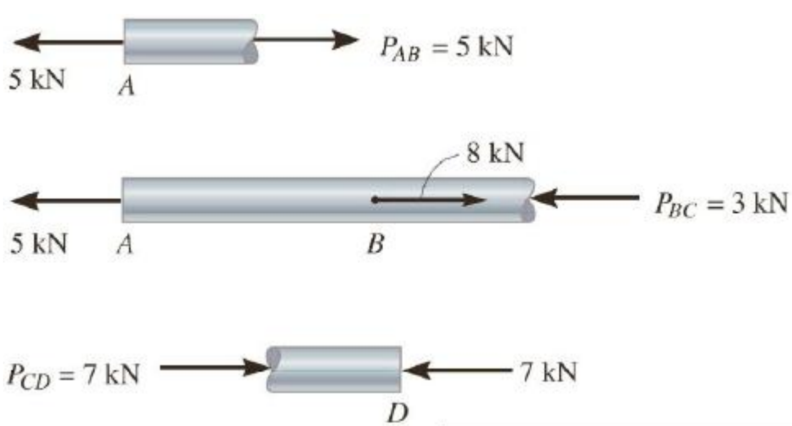
Several constant loads are applied:

Cross-sectional areas are constant
Material is homogeneous and isotropic

$$\delta = \sum \frac{PL}{AE}$$



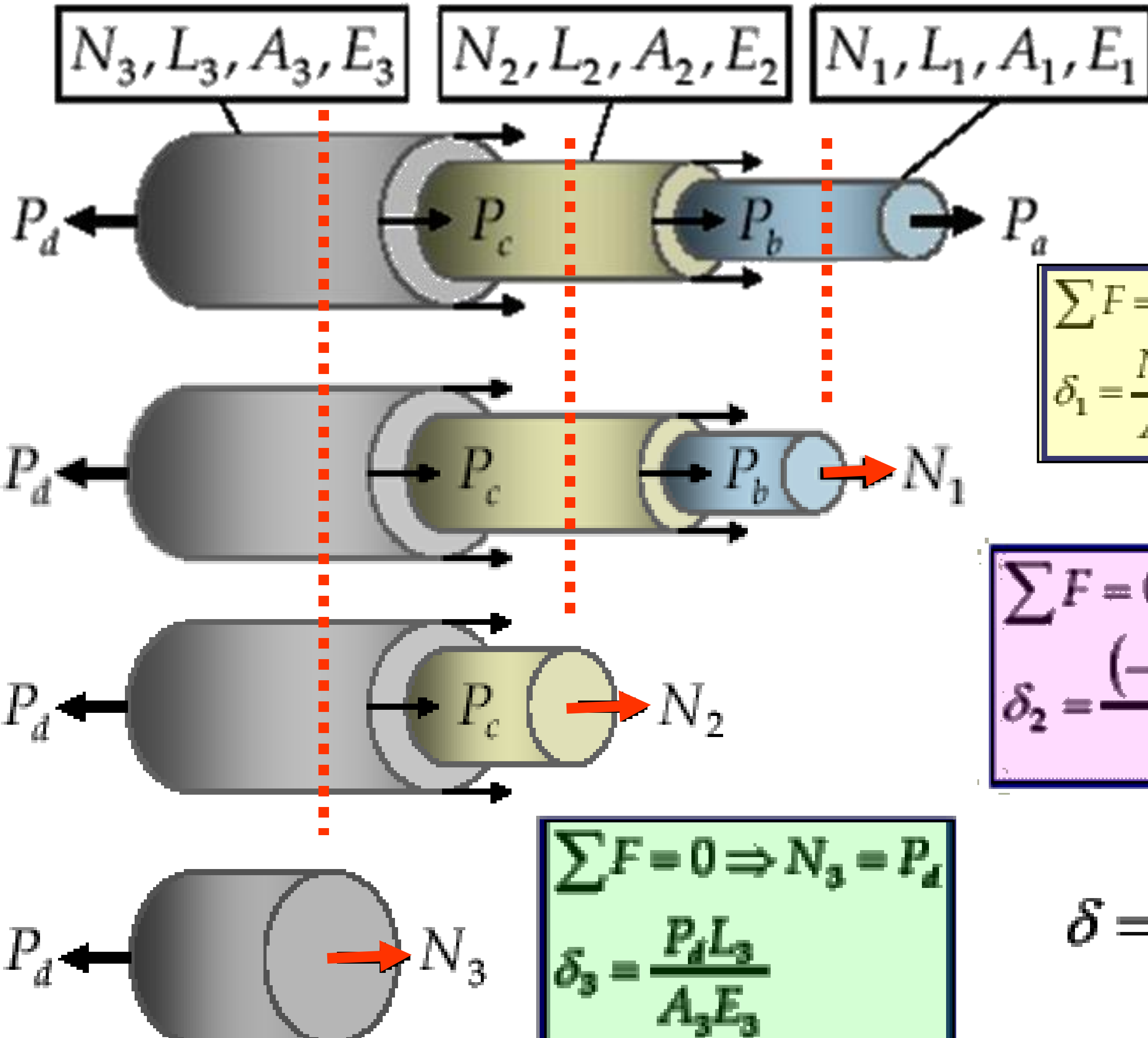
$$\delta_{A/D} = ?$$



$$\delta_{A/D} = \sum \frac{P_i L_i}{A_i E} = \frac{P_{AB} L_{AB}}{AE} + \frac{P_{BC} L_{BC}}{AE} + \frac{P_{CD} L_{CD}}{AE}$$

$$= \frac{(5\text{kN}) L_{AB}}{AE} + \frac{(-3\text{kN}) L_{BC}}{AE} + \frac{(-7\text{kN}) L_{CD}}{AE}$$

2 Bars Consisting of Prismatic Segments Each Having Different Axial Forces, Dimensions, and Materials



$$\sum F = 0 \Rightarrow N_1 = -P_b - P_c + P_d$$

$$\delta_1 = \frac{N_1 L_1}{A_1 E_1} = \frac{(-P_b - P_c + P_d) L_1}{A_1 E_1}$$

$$\sum F = 0 \Rightarrow N_2 = -P_c + P_d$$

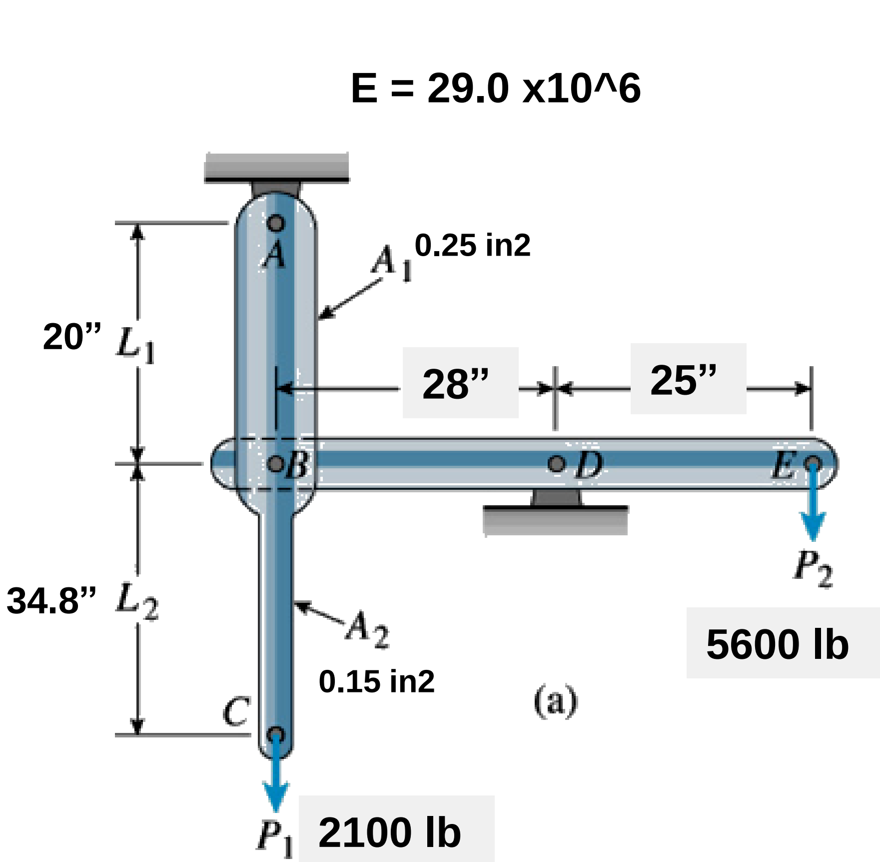
$$\delta_2 = \frac{(-P_c + P_d) L_2}{A_2 E_2}$$

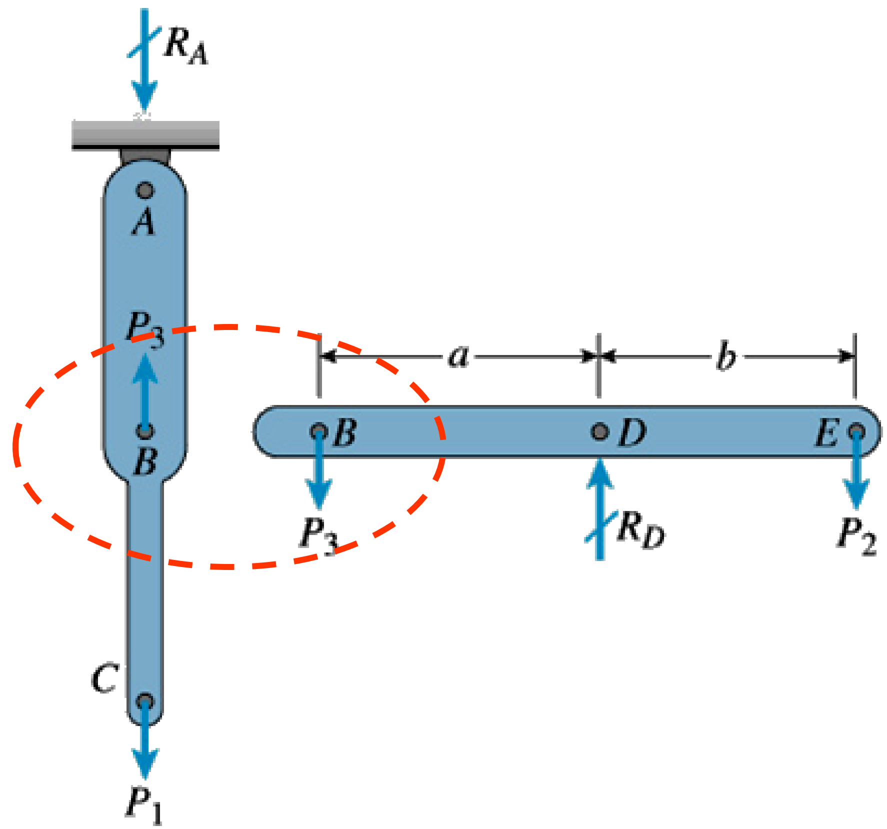
$$\sum F = 0 \Rightarrow N_3 = P_d$$

$$\delta_3 = \frac{P_d L_3}{A_3 E_3}$$

$$\delta = \sum_{i=1}^{3} \delta_i = \sum_{i=1}^{3} \frac{N_i L_i}{E_i A_i}$$

Example : Calculate the vertical displacement δ_C at point C





Use equilibrium equations for member BDE

Then $P_3 = 5000$ lb

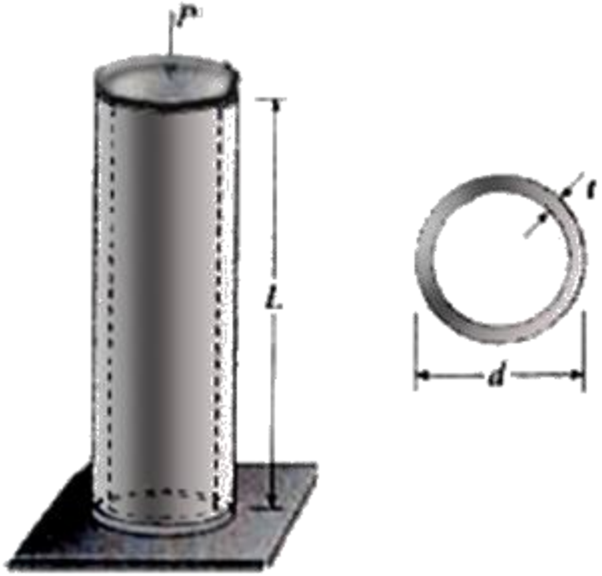
Use equilibrium equations for member ABC

Then $R_A = 2900$ lb

$$\delta = \sum_{i=1}^{3} \delta_i = \sum_{i=1}^{3} \frac{N_i L_i}{E_i A_i}$$

$\delta_C = 0.0088$ in

Required the minimum thickness



$P = 85\text{ k}$

$E = 30{,}000\text{ ksi}$

$L = 8.0\text{ ft}$

$d = 7.5\text{ in.}$

$\sigma_{allow} = 7{,}000\text{ psi}$

$\delta_{allow} = 0.02\text{ in.}$

REQUIRED AREA BASED UPON ALLOWABLE STRESS

$$\sigma = \frac{P}{A} \qquad A = \frac{P}{\sigma_{allow}} = \frac{85\text{ k}}{7{,}000\text{ psi}} = 12.14\text{ in.}^2$$

REQUIRED AREA BASED UPON ALLOWABLE SHORTENING

$$\delta = \frac{PL}{EA} \qquad A = \frac{PL}{E\delta_{allow}} = \frac{(85\text{ k})(96\text{ in.})}{(30{,}000\text{ ksi})(0.02\text{ in.})}$$

$$= 13.60\text{ in.}^2$$

MINIMUM THICKNESS t_{min}

$$A = \frac{\pi}{4}[d^2 - (d-2t)^2] \quad \text{or}$$

$$\frac{4A}{\pi} - d^2 = -(d-2t)^2$$

$$(d-2t)^2 = d^2 - \frac{4A}{\pi} \quad \text{or} \quad d - 2t = \sqrt{d^2 - \frac{4A}{\pi}}$$

$$t = \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \frac{A}{\pi}} \quad \text{or}$$

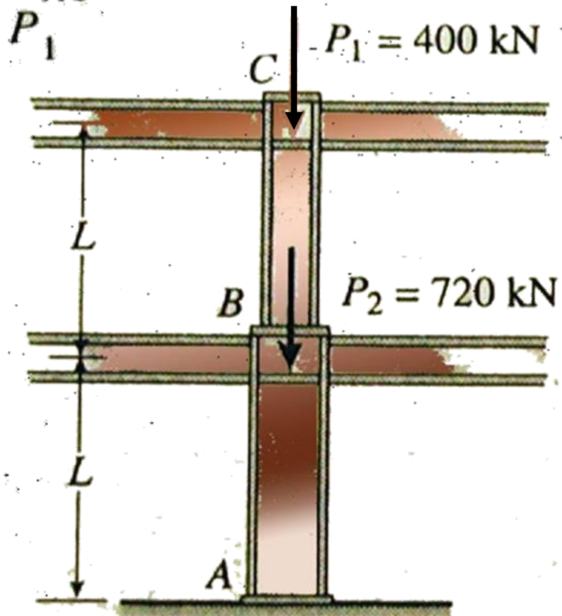
$$t_{min} = \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \frac{A_{min}}{\pi}}$$

SUBSTITUTE NUMERICAL VALUES

$$t_{min} = \frac{7.5\text{ in.}}{2} - \sqrt{\left(\frac{7.5\text{ in.}}{2}\right)^2 - \frac{13.60\text{ in.}^2}{\pi}}$$

$$t_{min} = 0.63\text{ in.} \quad \leftarrow$$

Assuming that $E = 206$ GPa, determine the total shortening δ_{AC} of the two columns due to the combined action of the loads P_1 and P_2.



(a) SHORTENING δ_{AC} OF THE TWO COLUMNS

$$\delta_{AC} = \sum \frac{N_i L_i}{E_i A_i} = \frac{N_{AB} L}{EA_{AB}} + \frac{N_{BC} L}{EA_{BC}}$$

$$= \frac{(1120 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(11{,}000 \text{ mm}^2)} + \frac{(400 \text{ kN})(3.75 \text{ m})}{(206 \text{ GPa})(3{,}900 \text{ mm}^2)}$$

$$= 1.8535 \text{ mm} + 1.8671 \text{ mm} = 3.7206 \text{ mm}$$

$\delta_{AC} = 3.72$ mm ⟵

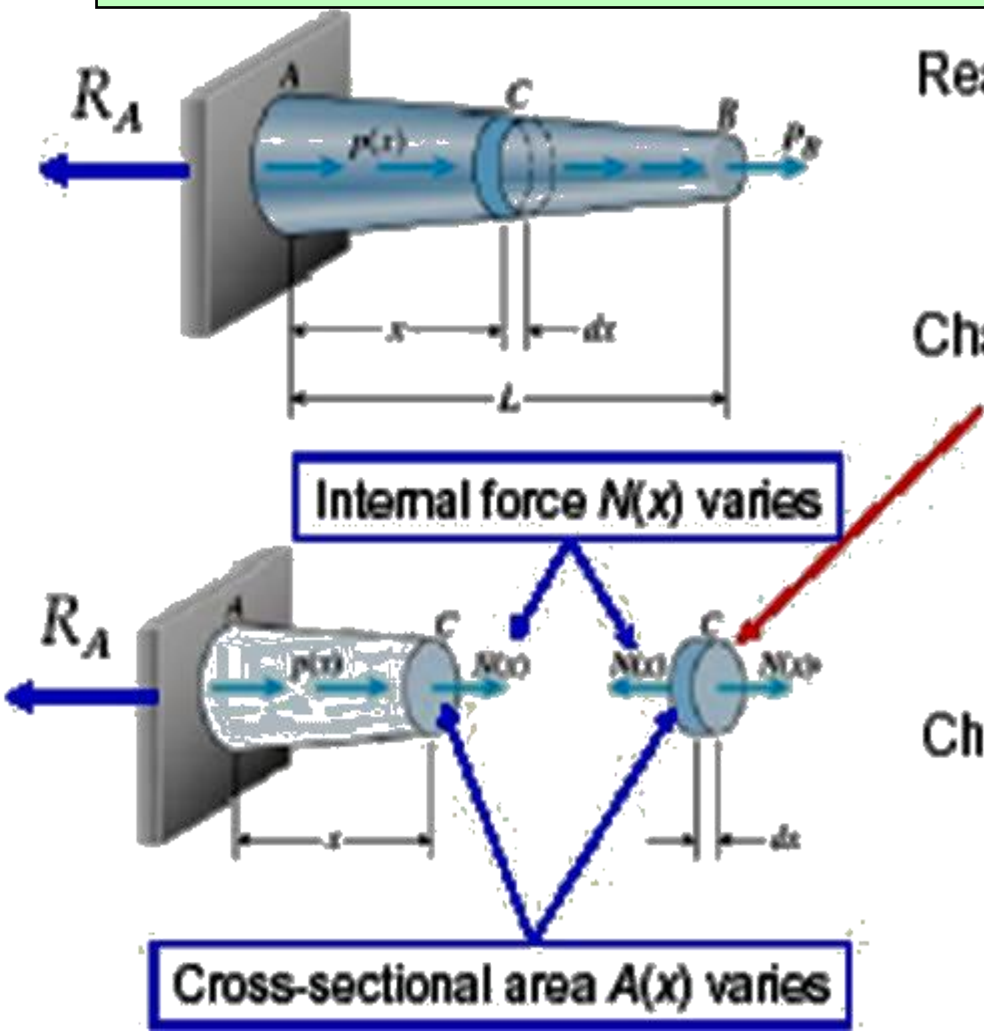
L = length of each column = 3.75 m

$E = 206$ GPa

$A_{AB} = 11{,}000 \text{ mm}^2$

$A_{BC} = 3{,}900 \text{ mm}^2$

3 Bars with Continuously Varying Loads and/or Dimensions



Reaction at *A*:

$$R_A = P_B + \int_0^L p(x)dx$$

Change in length of segment *dx*:

$$N(x) = R_A - \int_0^x p(x)dx = P_B + \int_x^L p(x)dx$$

$$d\delta = \frac{N(x)dx}{EA(x)}$$

Change in length of the entire member:

$$\delta = \sum d\delta = \int_0^L d\delta = \int_0^L \frac{N(x)dx}{EA(x)}$$

Thermal Stress

$$\delta_{Thermal} = \alpha(\Delta T)L$$

$\delta_{Thermal}$ = *deformation in length due to temperature change*

α = *linear coefficient of thermal expansion (1/°C,1/°F,1/°K)*

ΔT = *Temperature change*

L = *original length of the structure member*

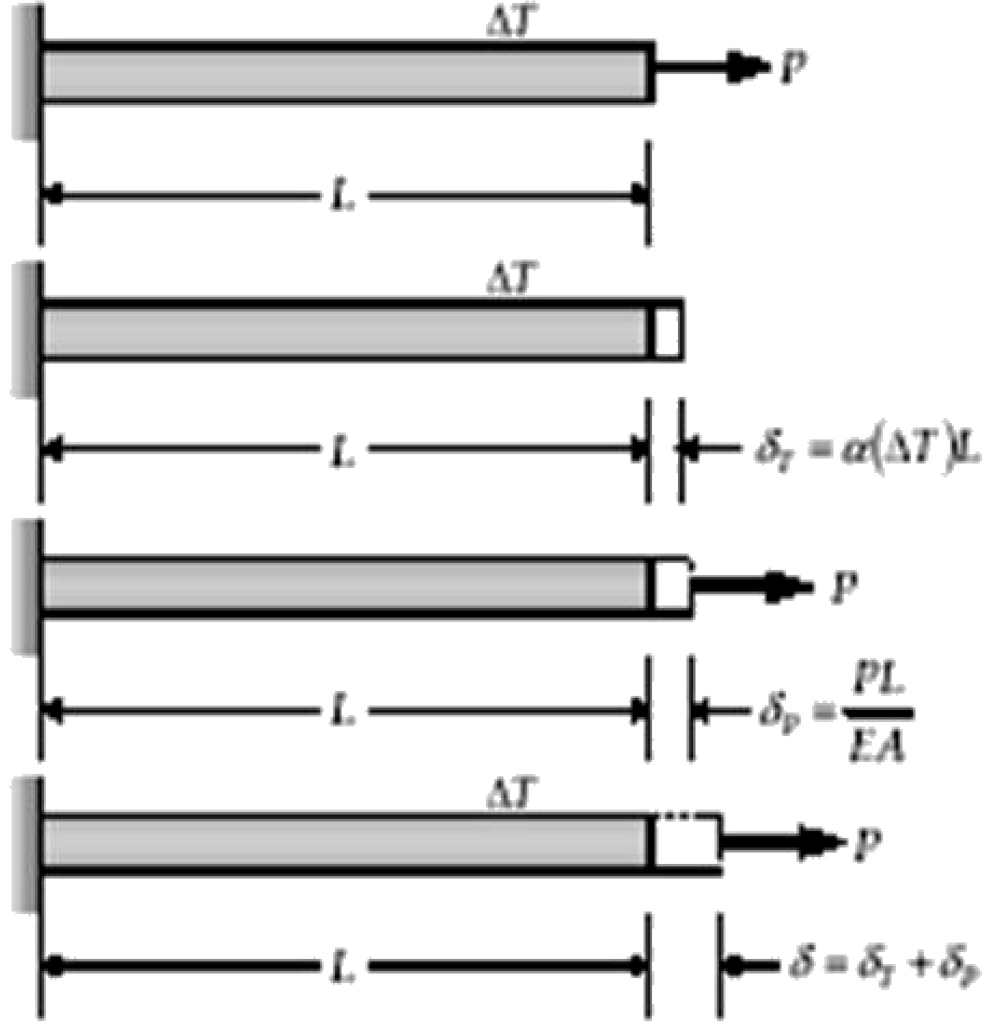
If free expansion is not allowed such as in a statically indeterminate member

$$\sigma = \sigma_{mechanical} + \sigma_{thermal}$$

$$= E\varepsilon_{mechanical} + E\varepsilon_{thermal} = E(\varepsilon_{mechanical} + \alpha\Delta T)$$

Thermal Stresses and Strains

A Bar Subjected to ΔT and P



Thermal expansion

$$\delta_T = \alpha(\Delta T)L$$

Elongation due to load P

$$\delta_P = \frac{PL}{EA}$$

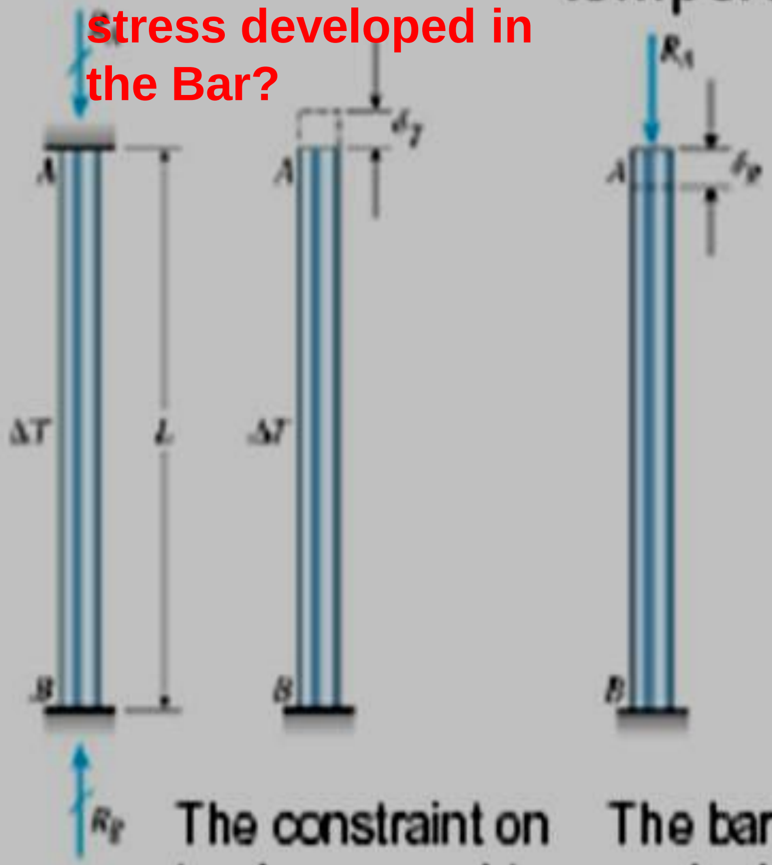
Total Elongation

$$\delta = \delta_T + \delta_P = \alpha(\Delta T)L + \frac{PL}{EA}$$

Stress $$\sigma = E(\varepsilon - \varepsilon_T) = E\varepsilon_P = E\frac{\delta_P}{L} = \frac{P}{A}$$

Example 2.7 Statically indeterminate bar with a uniform temperature increase ΔT.

What is the thermal stress developed in the Bar?



The constraint on top is removed to allow the bar to expand freely as the temperature rises

The bar is then pushed back to its original length by applying the reaction force.

Compatibility: $\delta_{AB} = \delta_T - \delta_R = 0$

Equilibrium: $\sum F_{verti} = 0 \Rightarrow R_A = R_B$

Force-displacement relations $\delta_R = \dfrac{R_A L}{EA}$

Temperature-displacement relation

$$\boxed{\delta_T = \alpha(\Delta T)L}$$

$$\delta_T - \delta_R = \alpha(\Delta T)L - \frac{R_A L}{EA} = 0$$

$$R_A = EA\alpha(\Delta T)$$

$$\sigma_T = \frac{R_A}{A} = \frac{R_B}{A} = E\alpha(\Delta T)$$

Example:

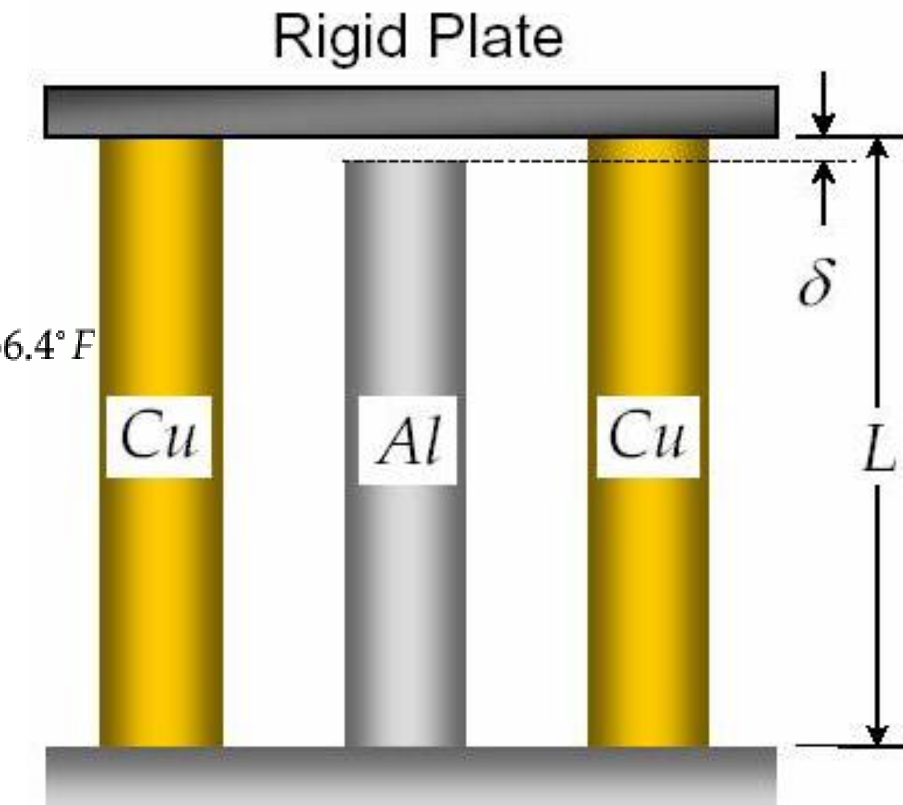
Two copper bars and aluminum bar are fixed at the bottom as shown in the figure. The top ends of all three bars are supposed to be welded to a rigid steel plate. The aluminum bar, however, is a little shorter (δ= 0.1 in.) than the copper bars and it had to be heated to make it extend to the same length as the copper bars to complete the welding

What is the temperature increase, Δ*T* (°*F*), that is needed to bring the aluminum bar to the same length as that of copper bars?

.

$$E_{AL} = 10.6\times10^6 \text{ psi} \quad E_{Cu} = 18.0\times10^6 \text{ psi}$$

$$\alpha_{Al} = 13\times10^{-6} /^\circ F \quad A_{Al} = A_{Cu} = 1.0\,\text{in}^2 \quad L = 30\text{ in}$$



$$\delta = \alpha_{Al}\Delta T L \quad \Rightarrow \quad \Delta T = \frac{\delta}{\alpha_{Al}L} = \frac{0.1}{(13\times10^{-6})(30)} = 256.4^\circ F$$

After the welding is done the temperature returns to normal, what will the stresses be in the aluminum bar and the copper bars, respectively?

The free-body diagram shown below indicates that force in the aluminum bar must balance the forces in the two copper bars. The copper bars will be shortened and the aluminum bar be stretched.

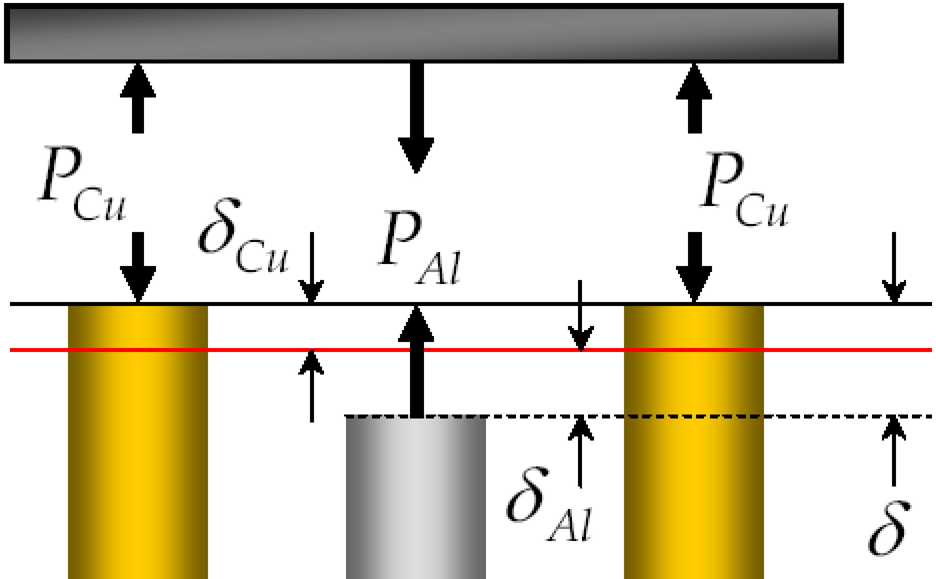
Equilibrium $P_{Al} = 2P_{Cu}$

Compatibility $\delta_{Al} + \delta_{Cu} = \delta$

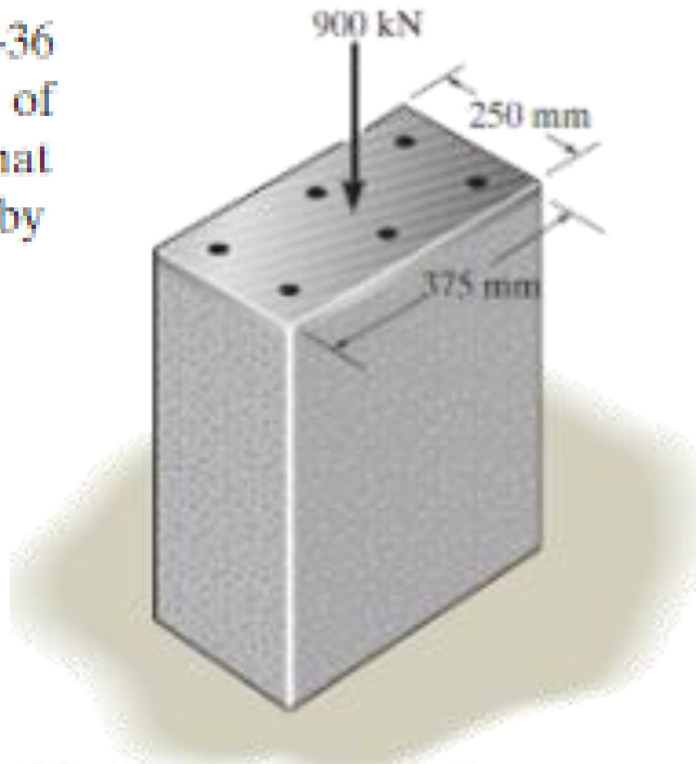
Force/Displacement $\delta_{Al} = \dfrac{P_{Al}L}{E_{Al}A_{Al}} \quad \delta_{Cu} = \dfrac{P_{Cu}L}{E_{Cu}A_{Cu}}$

$P_{Al} = 27{,}300 \text{ lb} \quad P_{Cu} = 13{,}650 \text{ lb}$

$\sigma_{Al} = 27{,}300 \text{ psi} \quad \sigma_{Cu} = 13{,}650 \text{ psi}$



4–42. The post is constructed from concrete and six A-36 steel reinforcing rods. If it is subjected to an axial force of 900 kN, determine the required diameter of each rod so that one-fifth of the load is carried by the steel and four-fifths by the concrete. $E_{st} = 200$ GPa, $E_c = 25$ GPa.



The normal force in each steel rod is

$$P_{st} = \frac{\frac{1}{5}(900)}{6} = 30 \text{ kN}$$

The normal force in concrete is

$$P_{con} = \frac{4}{5}(900) = 720 \text{ kN}$$

Since the steel rods and the concrete are firmly bonded, their deformation must be the same. Thus

$$\delta_{con} = \delta_{st}$$

$$\frac{P_{con}\,L}{A_{con}\,E_{con}} = \frac{P_{st}\,L}{A_{st}\,E_{st}}$$

$$\frac{720(10^3)\,L}{\left[0.25(0.375) - 6(\frac{\pi}{4}\,d^2)\right]\left[25(10^9)\right]} = \frac{30(10^3)\,L}{\frac{\pi}{4}\,d^2\left[200(10^9)\right]}$$

$$49.5\pi\,d^2 = 0.09375$$

$$d = 0.02455 \text{ m} = 24.6 \text{ mm}$$ **Ans**

Engineering Mechanics

Statics & Strength of Materials

Stresses in Beams

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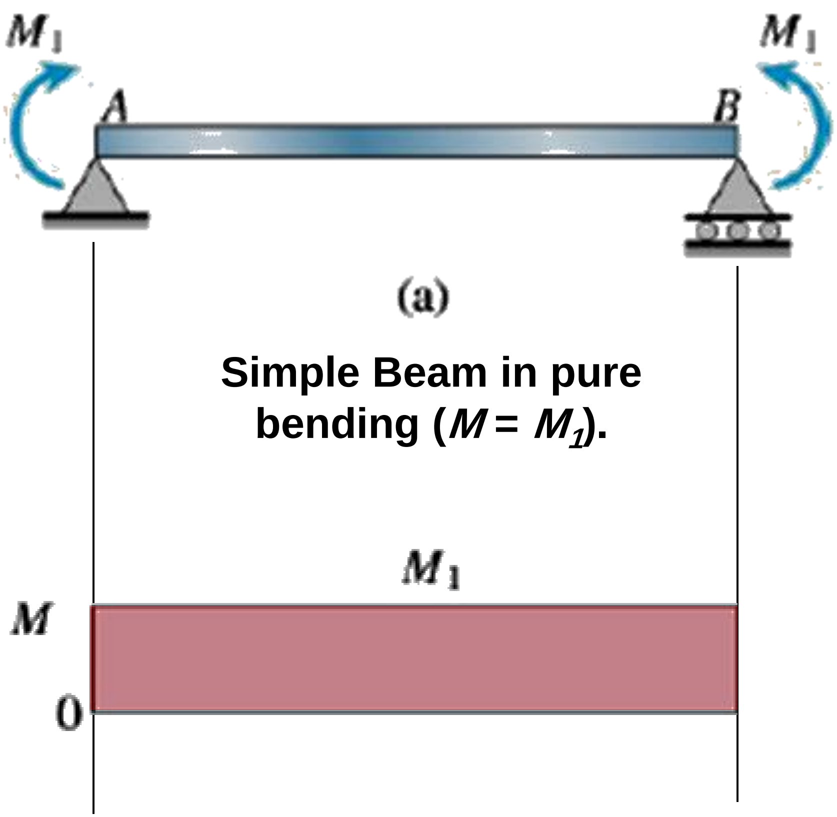
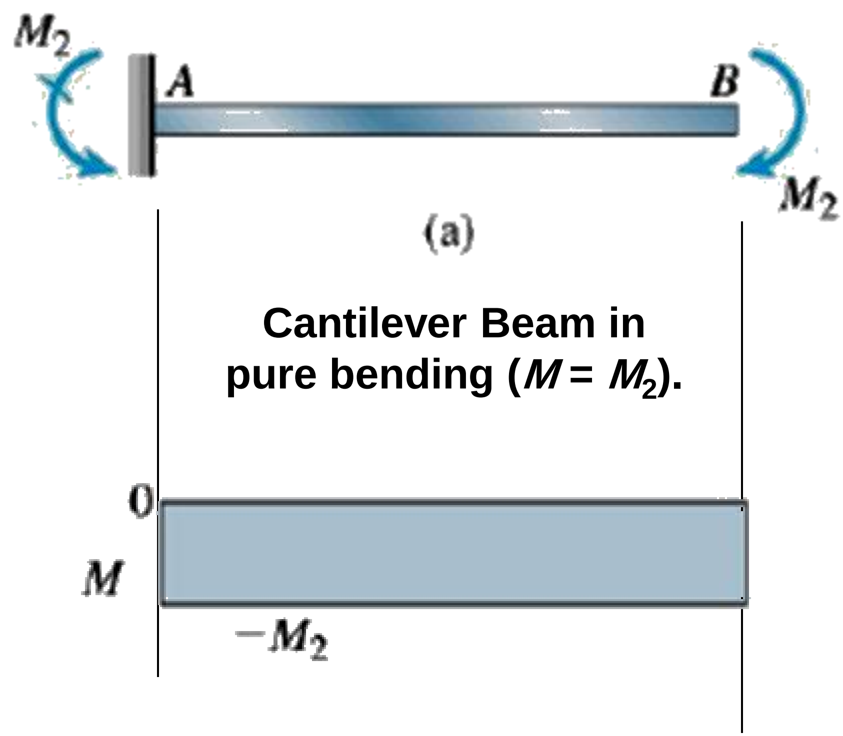
Pure Bending and Nonuniform Bending

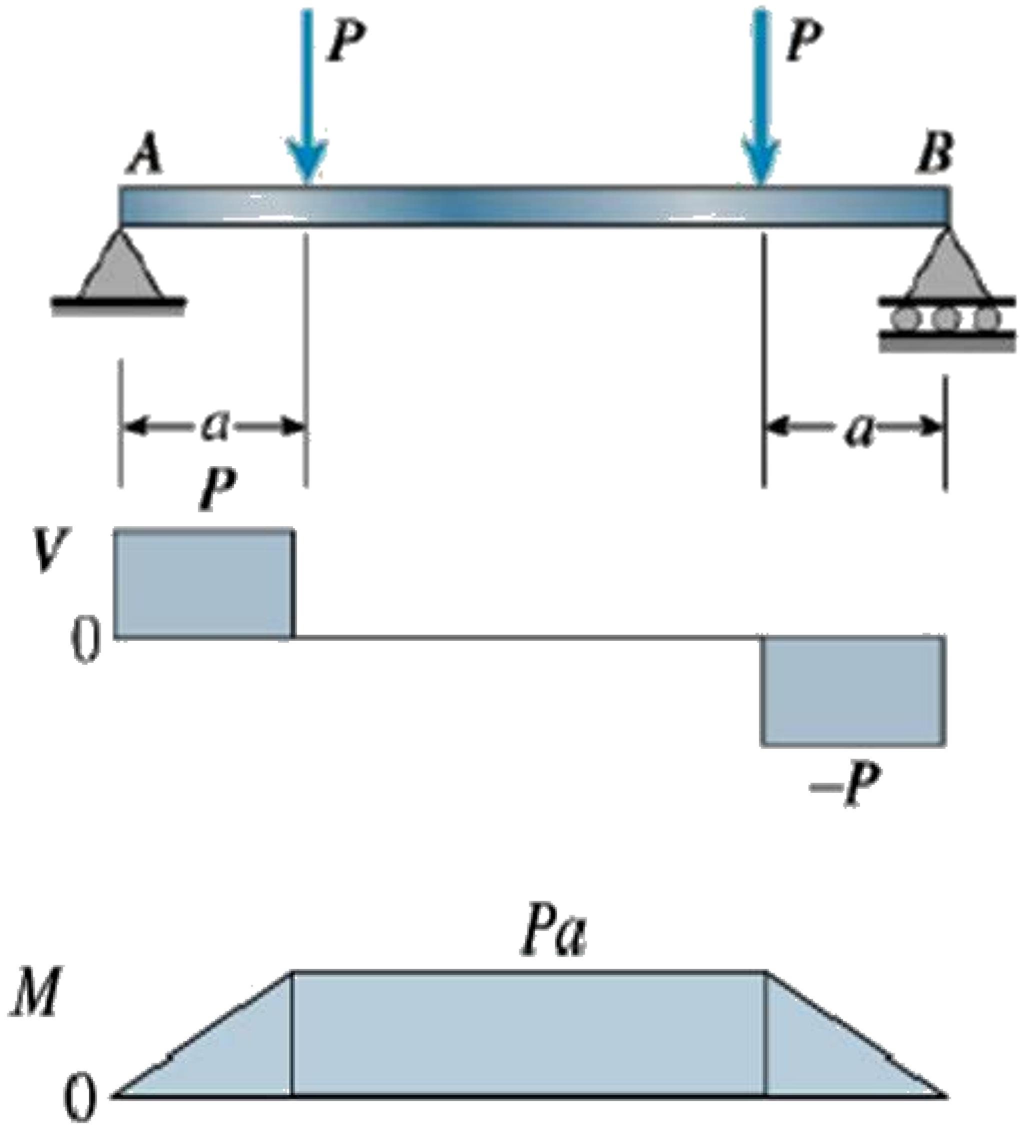
Pure bending = No shear,



Simple Beam in pure bending ($M = M_1$).



Cantilever Beam in pure bending ($M = M_2$).



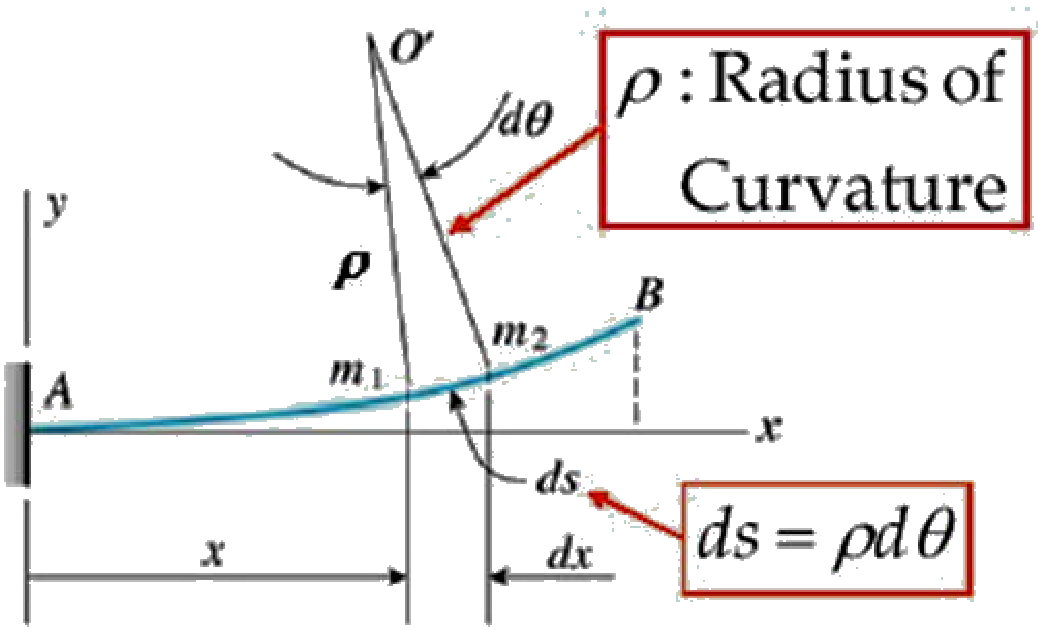
Simple beam with central region in pure bending

Curvature of a Beam

Radius of Curvature : ρ

Curvature : κ

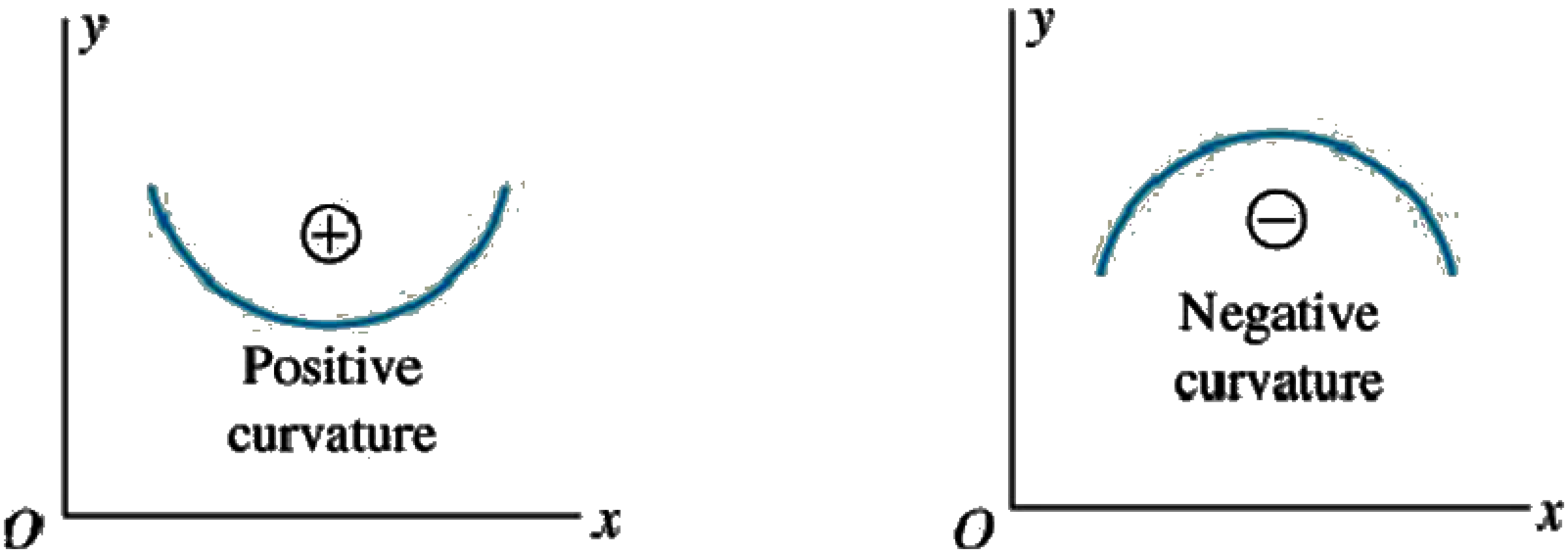
$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}$$

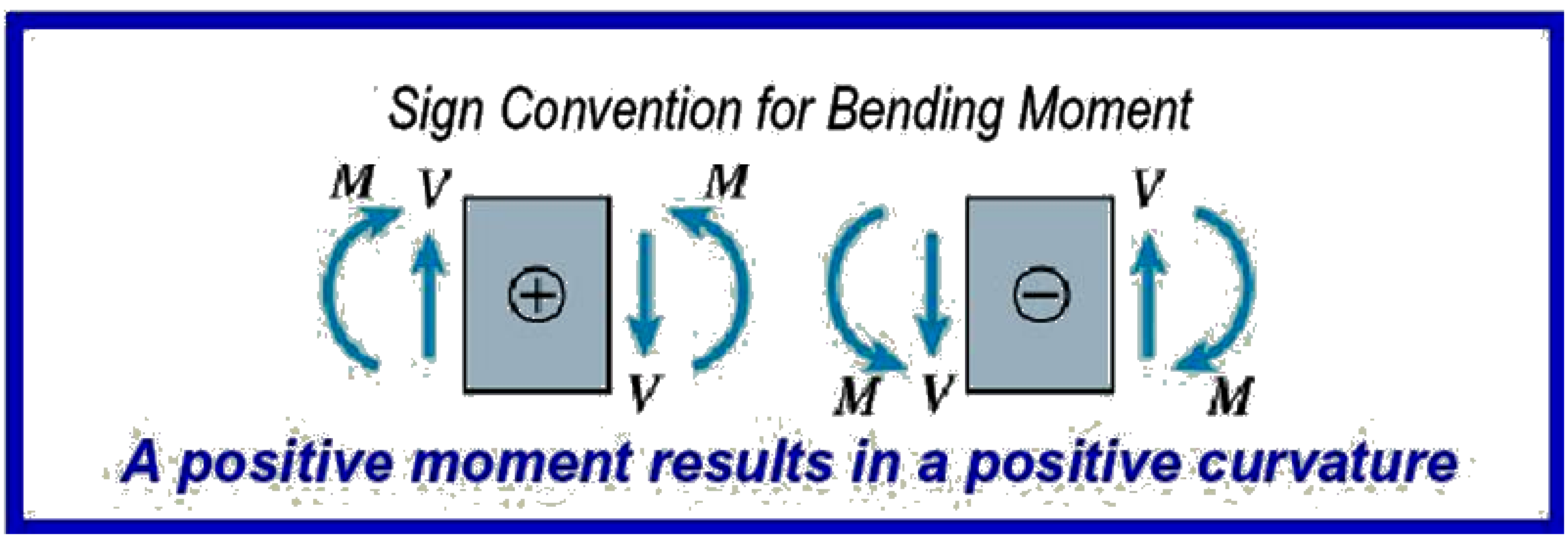


For Infinitesimal Deformation

$$ds \approx dx \quad \Rightarrow \quad \kappa = \frac{1}{\rho} \approx \frac{d\theta}{dx}$$

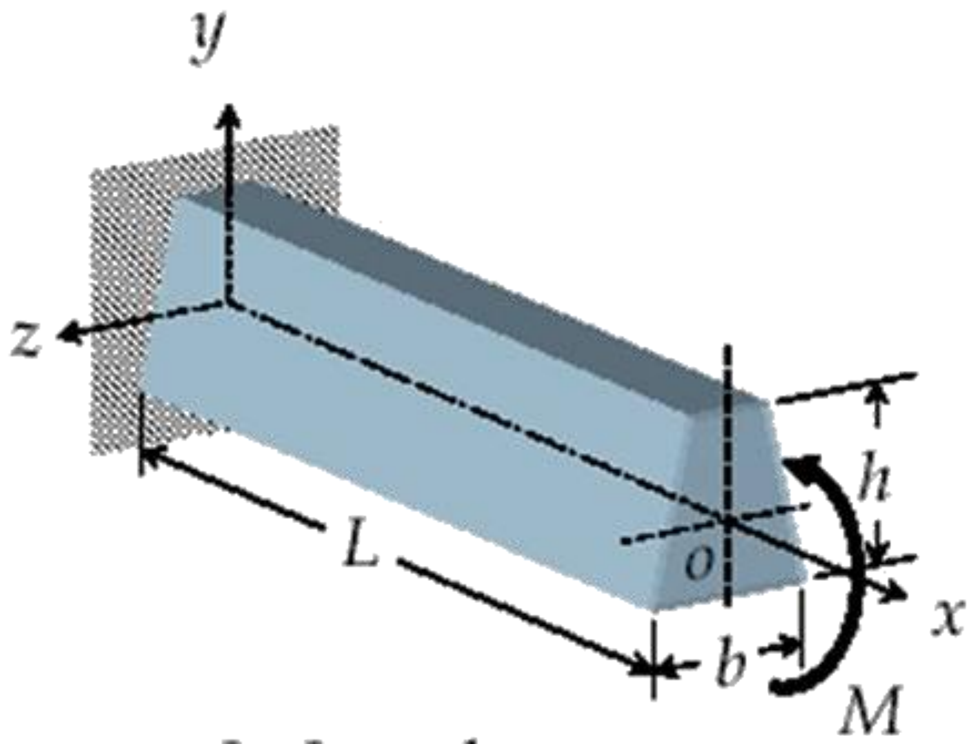
Sign Convention for Curvature





Longitudinal Strains in beams

- ***xy*** **plane is a plane of symmetry**
- **Loading is applied in *xy* plane**
- **Beam deflects in *xy* plane**
- **Thickness of the beam, *h, remains* unchanged**
- **Axis of the beam coincides with the centroidal line of the cross section (**

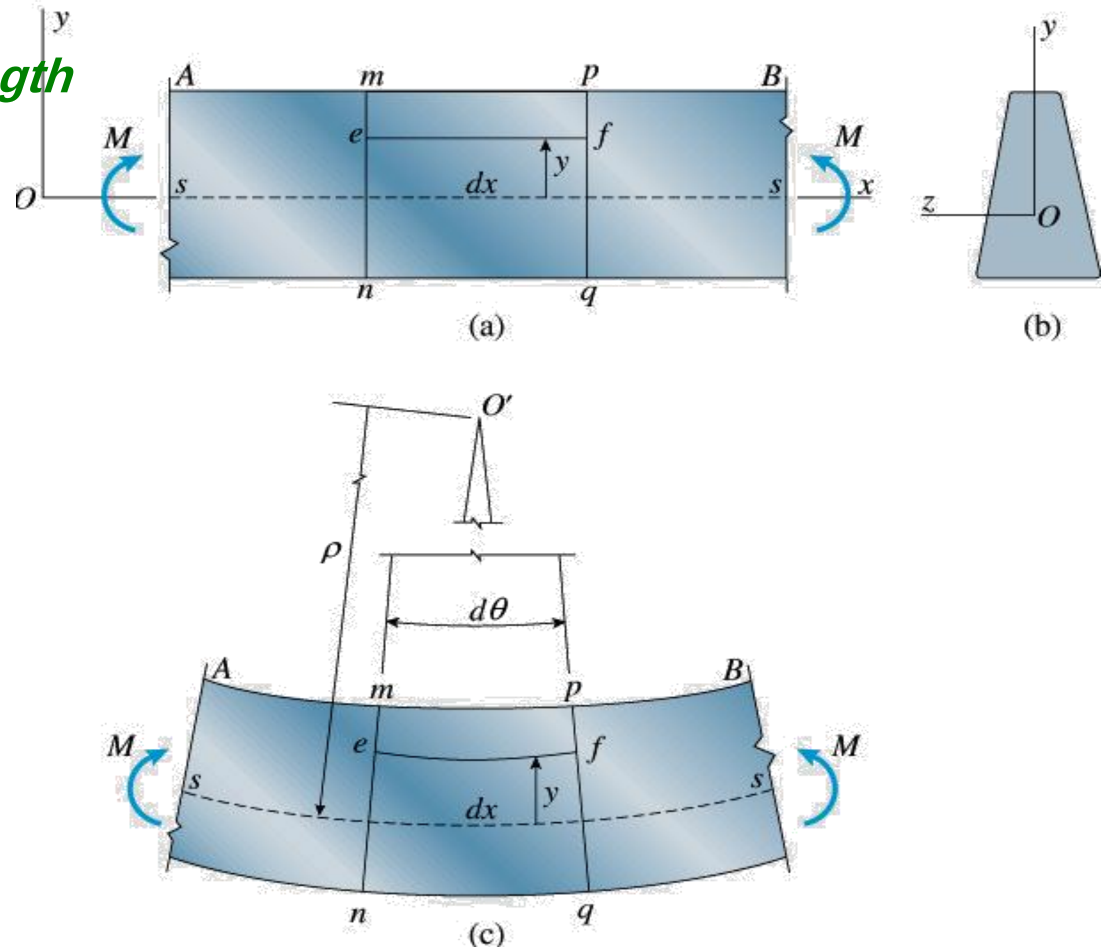


1. Cross sections (mn and pq) remain plane

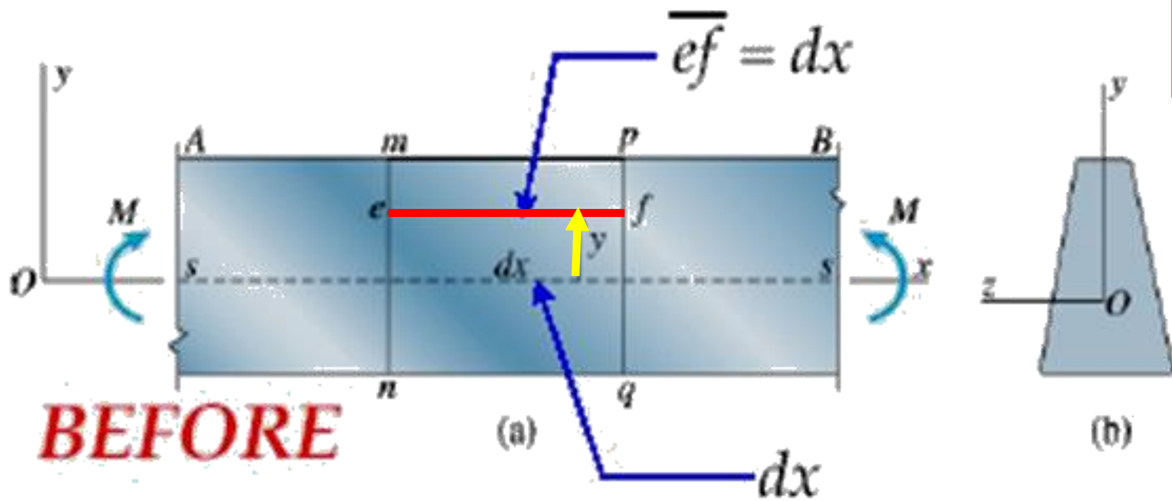
2. Cross sections remain perpendicular to the axis of the beam

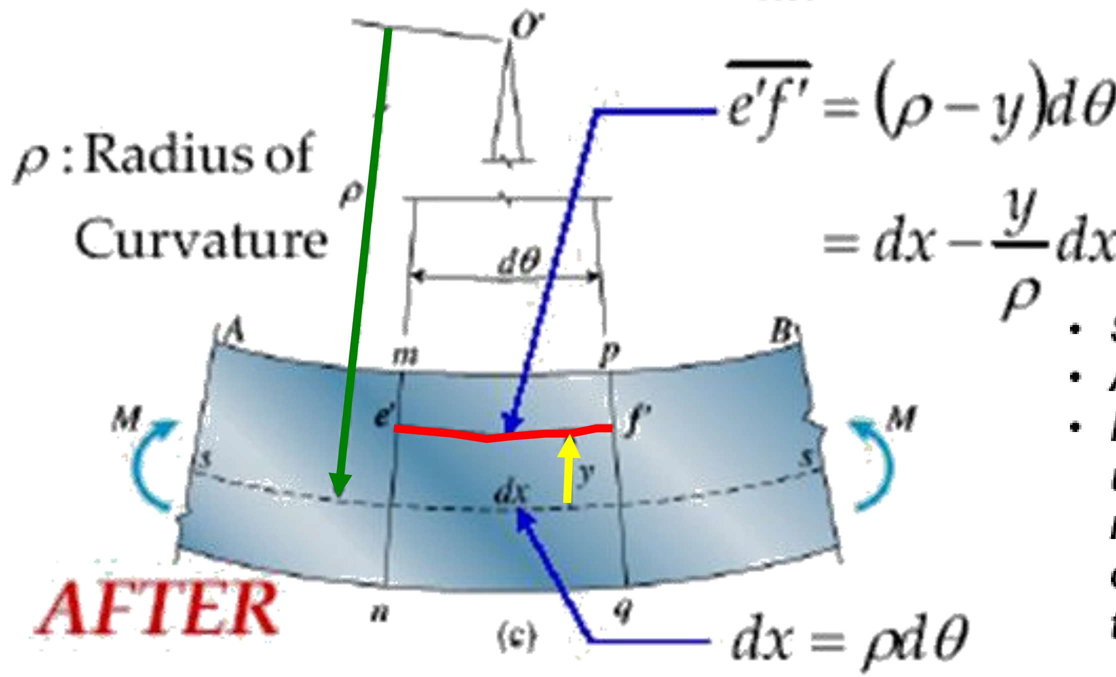
3. For positive moments (hence positive curvature), lines on the lower part of the beam (nq) are elongated; those on the upper part (mp) are shortened

4. Somewhere between top and bottom there is a line whose length does not change, and is called "Neutral Axis"



Normal Strain Due to Bending





$$\varepsilon_x = \frac{\overline{e'f'} - \overline{ef}}{\overline{ef}} = \frac{(\rho - y)d\theta - dx}{dx}$$

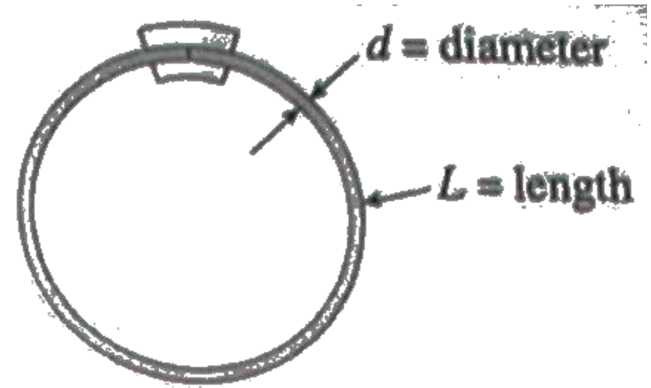
$$= \frac{\left[dx - \frac{y}{\rho}dx\right] - dx}{dx} = -\frac{y}{\rho}$$

$$\varepsilon_x = -\frac{y}{\rho} = -\kappa y$$

- ***Strains vary linearly with y***
- ***Along x-axis (y = 0) strain is zero***
- ***For a positive curvature, strains on upper part of the beam (y > 0) are negative (in compression) and those on lower part (y < 0) are positive (in tension)***

Problem 5.4-2 A copper wire having diameter $d = 3$ mm is bent into a circle and held with the ends just touching (see figure). If the maximum permissible strain in the copper is $\epsilon_{max} = 0.0024$, what is the shortest length L of wire that can be used?

$$d = 3 \text{ mm} \qquad \varepsilon_{max} = 0.0024$$

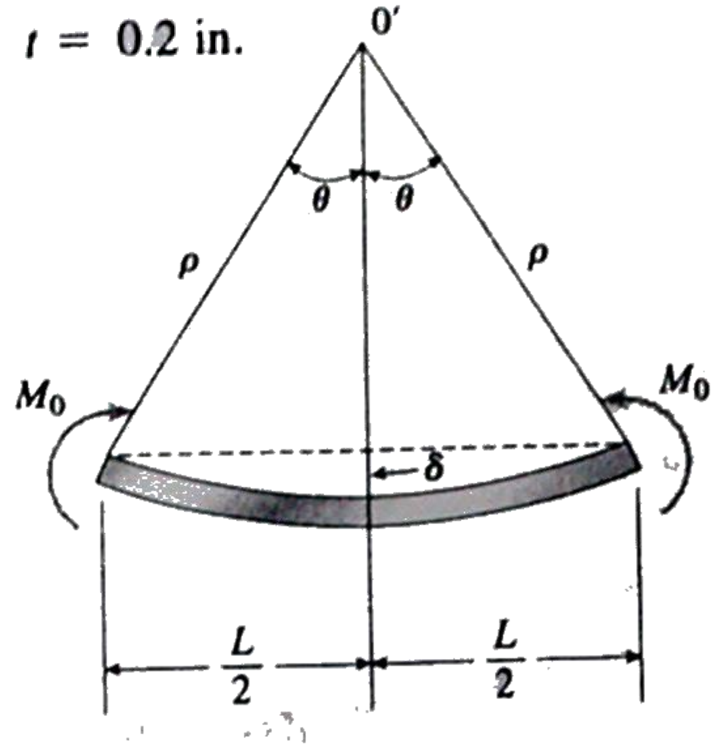


$$L = 2\pi\rho \qquad \rho = \frac{L}{2\pi}$$

$$\varepsilon_{max} = \frac{y}{\rho} = \frac{d/2}{L/2\pi} = \frac{\pi d}{L}$$

$$L_{min} = \frac{\pi d}{\varepsilon_{max}} = \frac{\pi\,(3 \text{ mm})}{0.0024} = 3.93 \text{ m} \quad \longleftarrow$$

$L = 20$ in. $\quad t = 0.2$ in.
$\delta = 0.25$ in.



What is the normal strain?

The deflection curve is very flat (note that $L/\delta = 80$) and therefore θ is a very small angle.

$$\sin\theta = \frac{L/2}{\rho}$$

For small angles, $\theta = \sin\theta = \frac{L/2}{\rho}$ (θ is in radians)

$$\delta = \rho - \rho\cos\theta = \rho(1 - \cos\theta)$$

$$= \rho\left(1 - \cos\frac{L}{2\rho}\right)$$

Substitute numerical values ($\rho =$ inches):

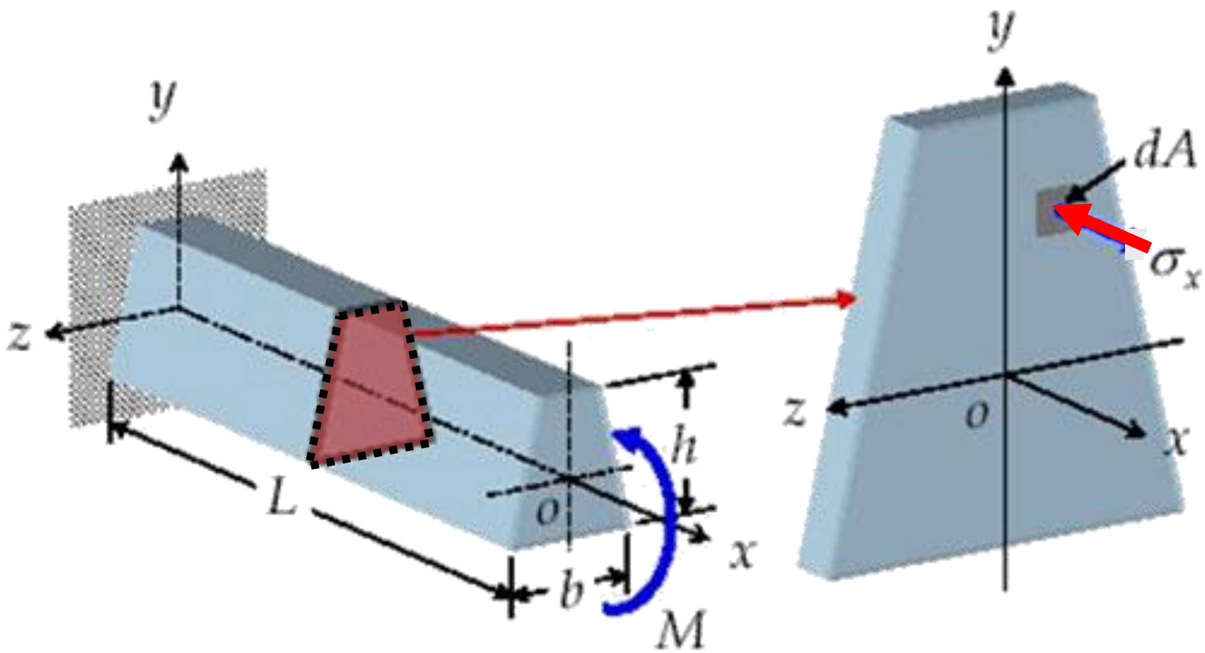
$$0.25 = \rho\left(1 - \cos\frac{10}{\rho}\right)$$

Solve numerically: $\rho = 200.0$ in.

NORMAL STRAIN

$$\varepsilon = \frac{y}{\rho} = \frac{t/2}{\rho} = \frac{0.1 \text{ in.}}{200 \text{ in.}} = 500 \times 10^{-6} \longleftarrow$$

Normal Stress Due to Bending



$$\varepsilon_x = -\frac{y}{\rho} = -\kappa y$$

Linear Elastic →

$$\sigma_x = E\varepsilon_x = -\frac{Ey}{\rho} = -E\kappa y$$

$$\sum F_x = \int_A \sigma_x dA = 0$$

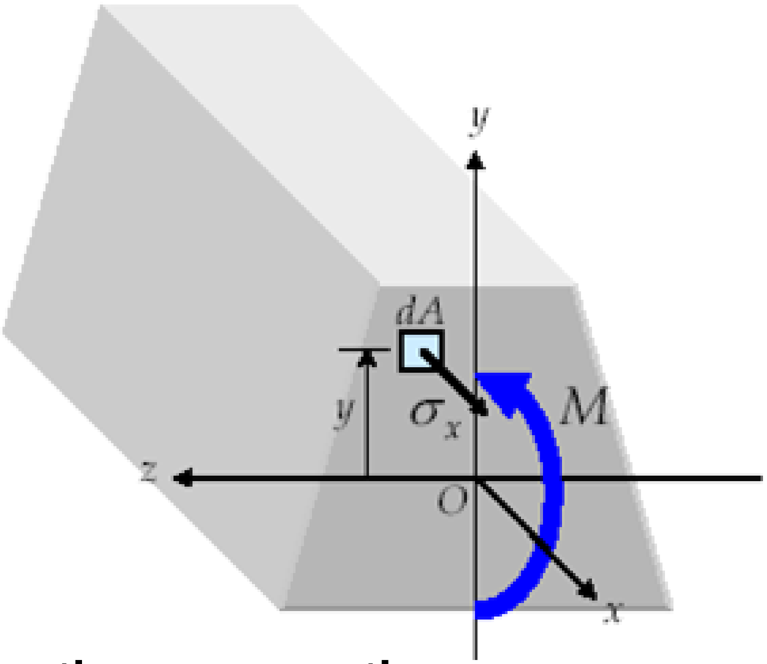
$$\Rightarrow \quad \int_A y dA = 0$$

$$\overline{y} = \int_A y dA \big/ A = 0$$

Centroid

x-axis must pass through the centroid of the cross section

Flexural Formula



$$\sigma_x = E\varepsilon_x = -\frac{Ey}{\rho} = -E\kappa y$$

Moment due to $\sigma_x dA$:

$$dM = (\sigma_x dA)y = -E\kappa y^2 dA$$

The resultant moment of the normal stress over the cross section must equal to the applied moment *M*

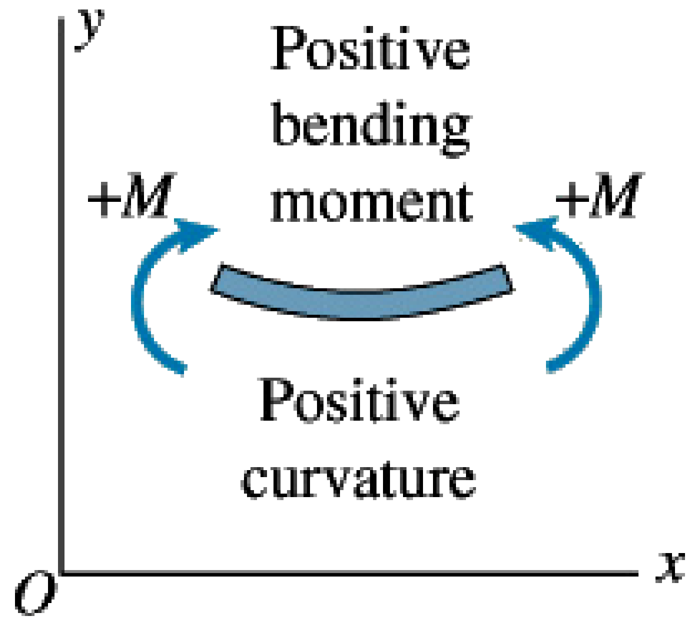
$$M = -\int_A \sigma_x y dA = \kappa E \int_A y^2 dA = \kappa E I_z \quad \Rightarrow \quad \kappa = \frac{1}{\rho} = \frac{M}{EI_z}$$

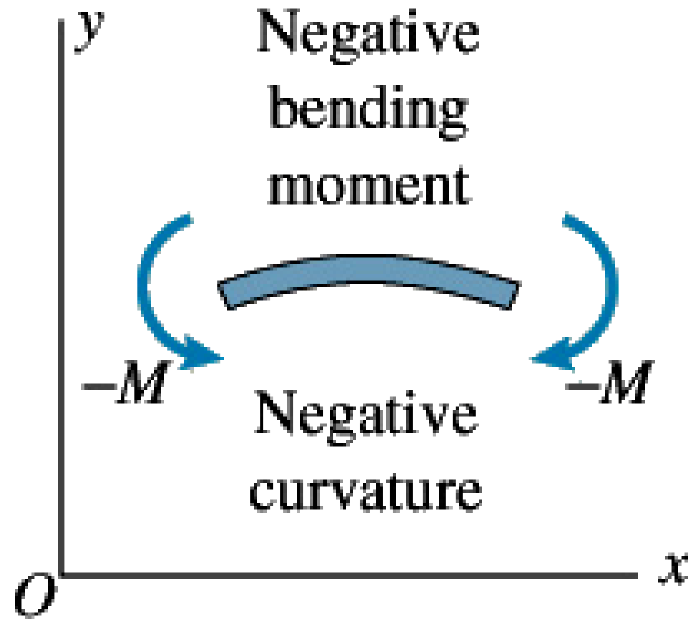
$$\sigma_x = -\kappa E y = -\left(\frac{M}{EI_z}\right)(Ey) = -\frac{My}{I_z}$$

$$\sigma_x = -\frac{My}{I_z}$$

$I_z = \int_A y^2 dA =$ Moment of Inertia (Chapter 12)

$$\sigma_x = -\frac{My}{I_z}$$

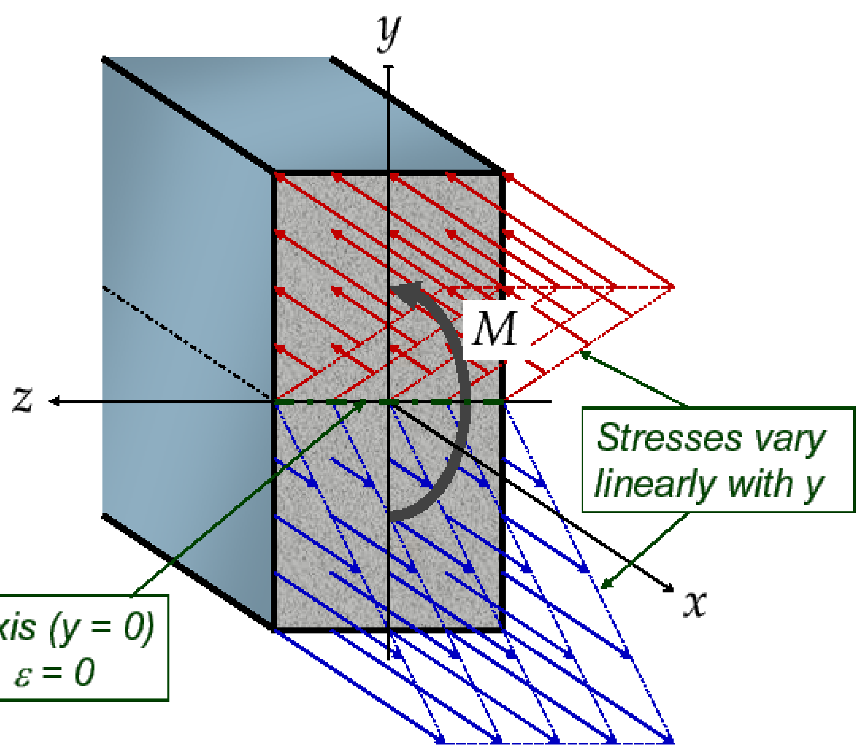


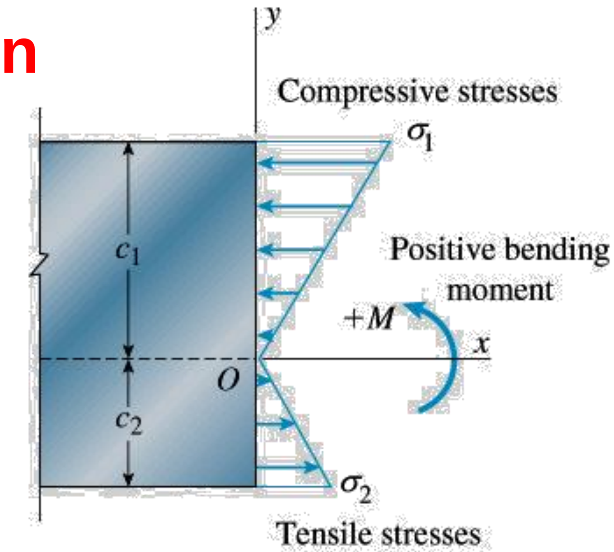


Maximum Stresses at a Cross Section

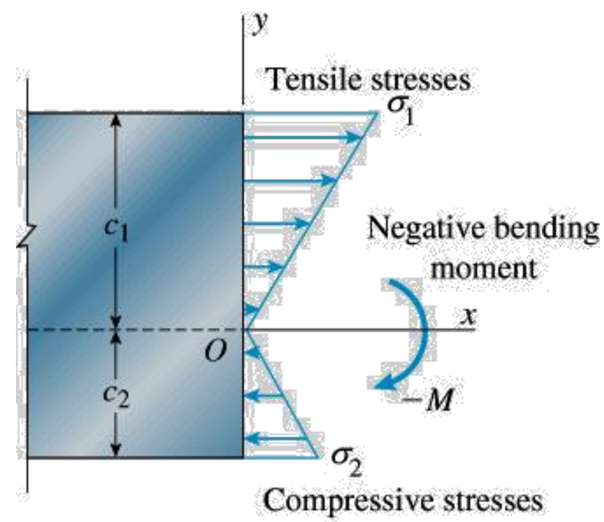
Normal Stresses Due to Bending

$$\sigma_x = -\frac{My}{I_z}$$



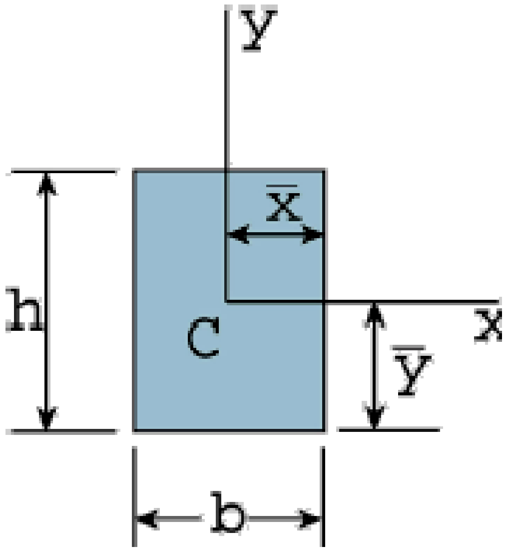


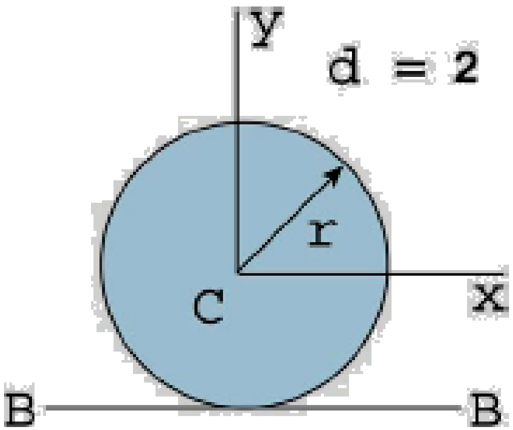
(a)



(b)

$$I_x = \int_A y^2 dA, \quad I_y = \int_A x^2 dA$$



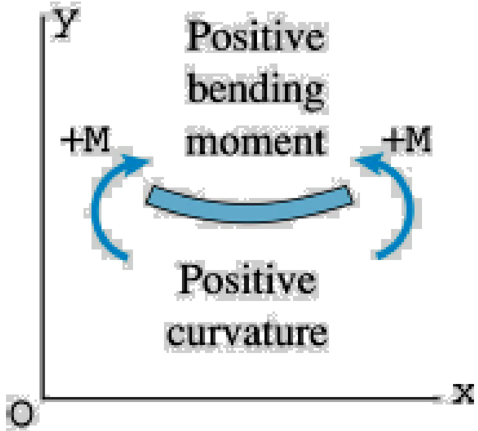


$$I_x = \frac{bh^3}{12}, \quad I_y = \frac{hb^3}{12}$$

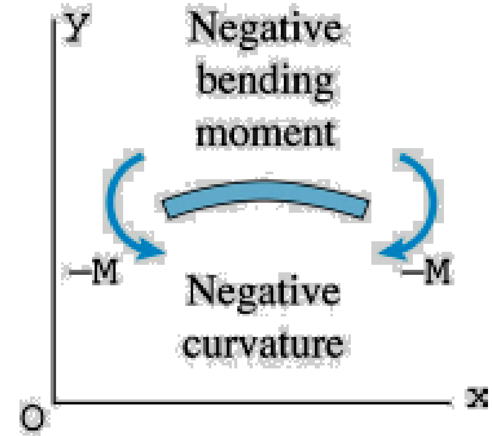
$$I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

Memorize the moments of inertia of these two cross sections!

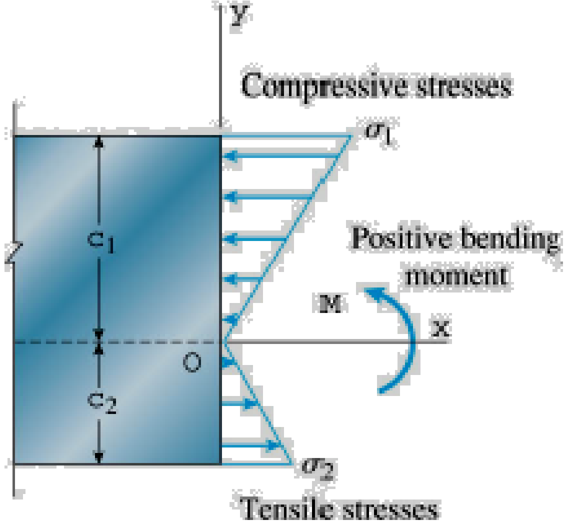
Relationships Between Bending Moments and Curvatures.



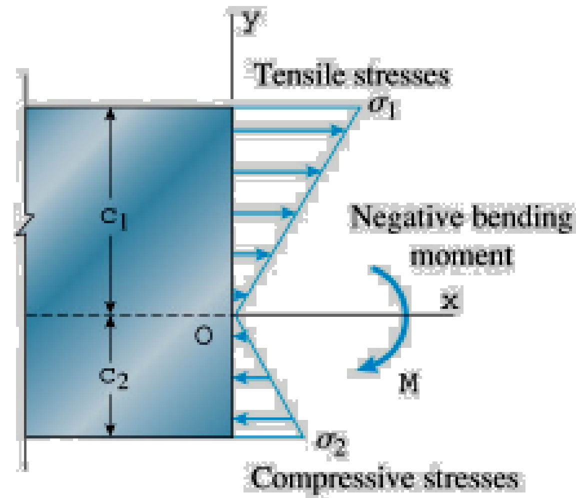
$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$



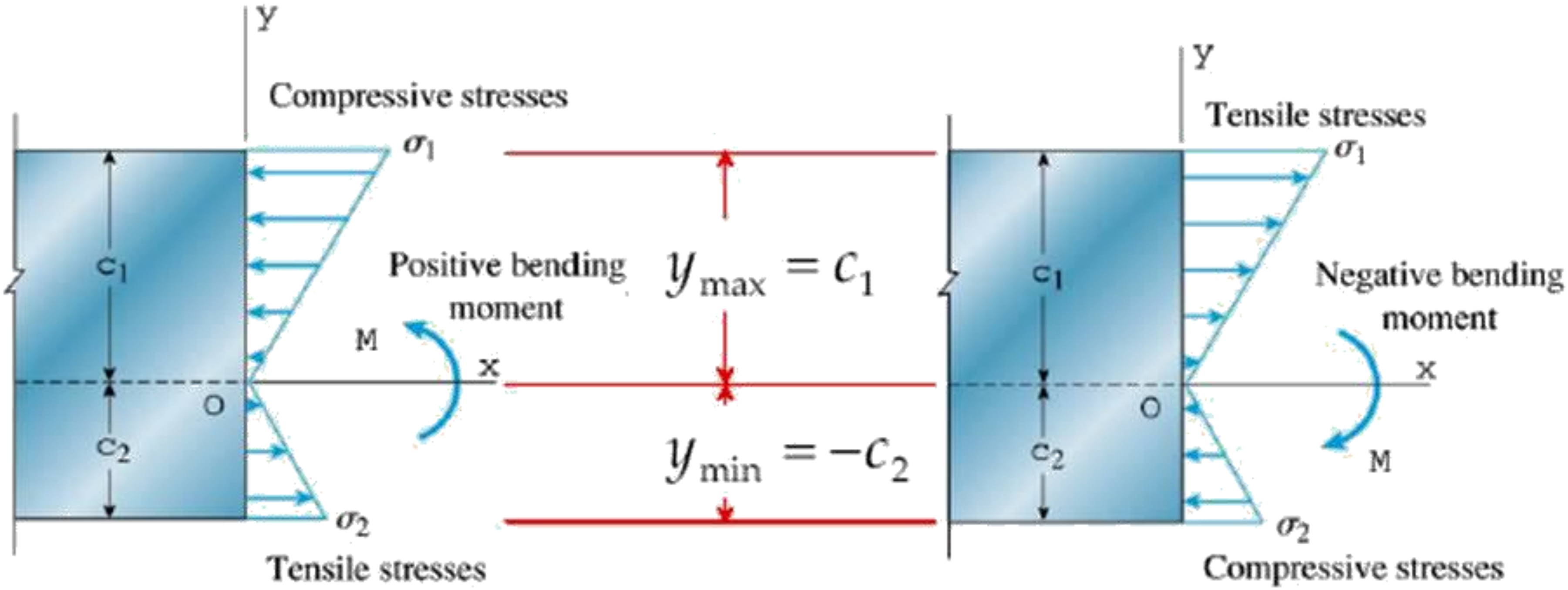
Relationships Between Bending Moments and Normal Stresses



$$\sigma_x = -\frac{My}{I}$$



Maximum Stresses at a Cross Section



$$\sigma_x = -\frac{My}{I}$$

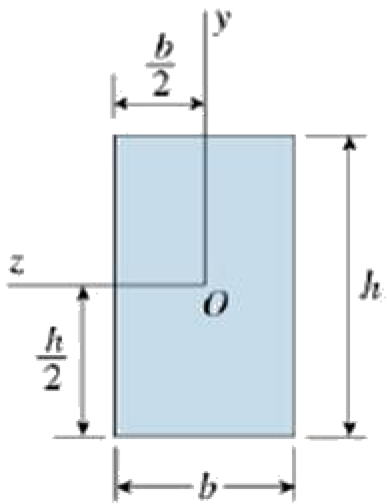
$$\sigma_1 = -\frac{Mc_1}{I} = -\frac{M}{S_1}, \quad S_1 = \frac{I}{c_1}$$

$$\sigma_2 = -\frac{M(-c_2)}{I} = \frac{M}{S_2}, \quad S_2 = \frac{I}{c_2}$$

S_1 and S_2 are known as the "Section Muduli" of the cross-sectional area. (See Appendix E)

Section Moduli for Doubly Symmetric Shapes

Rectangular:

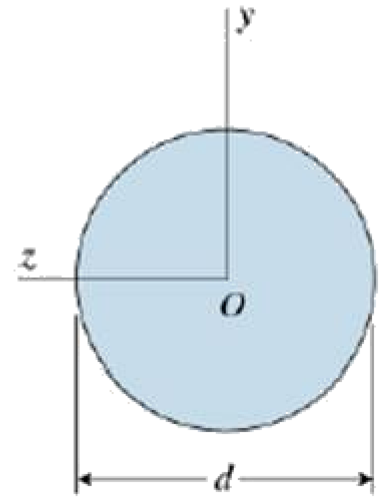


$$I = \frac{bh^3}{12}$$

$$c_1 = c_2 = \frac{h}{2}$$

$$S = \frac{bh^2}{6} = \frac{Ah}{6}$$

Circular:



$$I = \frac{\pi d^4}{64}$$

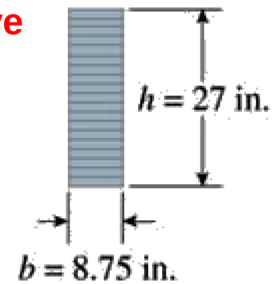
$$c_1 = c_2 = \frac{d}{2}$$

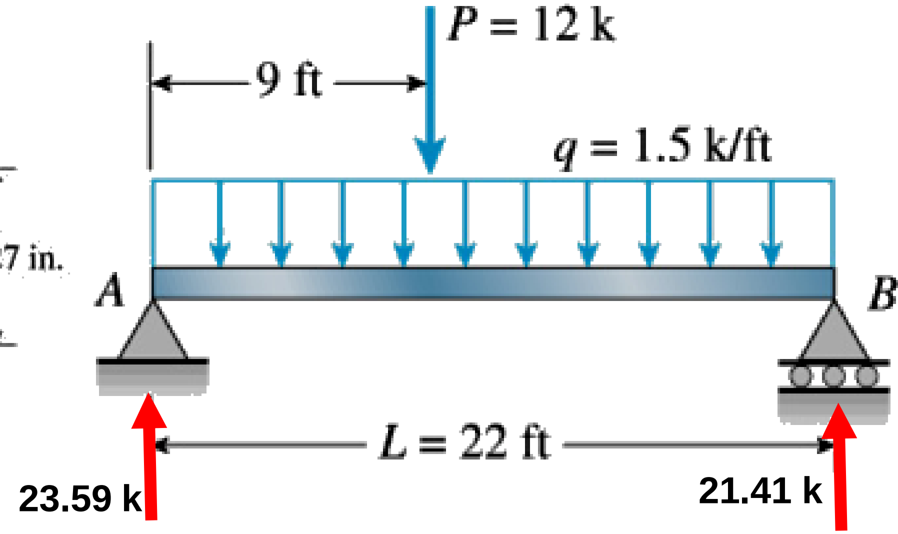
$$S = \frac{\pi d^3}{32} = \frac{Ad}{8}$$

$$c_1 = c_2 \quad \Rightarrow \quad \sigma_1 = -\sigma_2 = -\frac{Mc}{I} = -\frac{M}{S}$$

$$\sigma_{\max} = \frac{M}{S}, \quad S = \frac{I}{c}$$

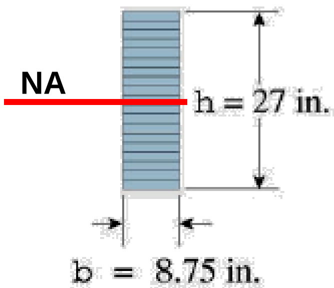
The beam is constructed of a glued laminated wood . **Determine the maximum compressive and tensile stresses in the beam due to bending?**





1. Find reactions
2. Draw shear and moment diagrams
3. Find maximum moment
4. Find the centroid location and moment of inertia
5. Calculate the stresses

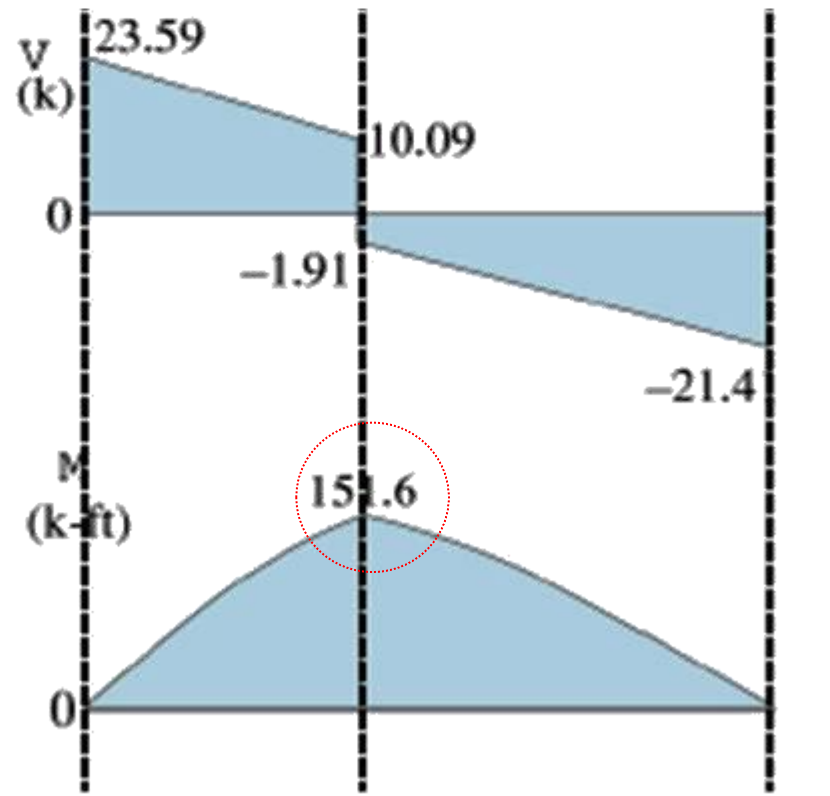
Cross Section:



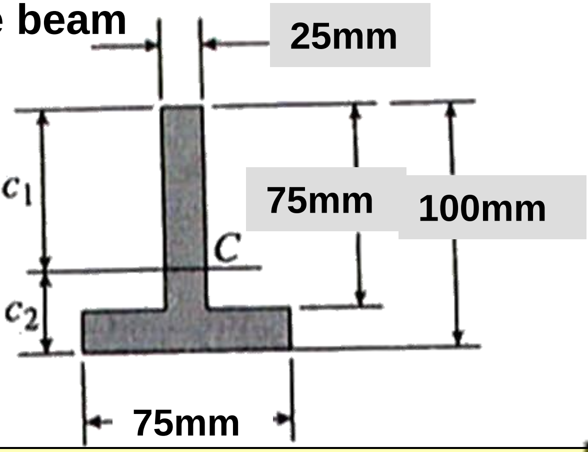
$$I = \frac{(8.75)(27)^3}{12} = 14{,}352 \text{ in}^4$$

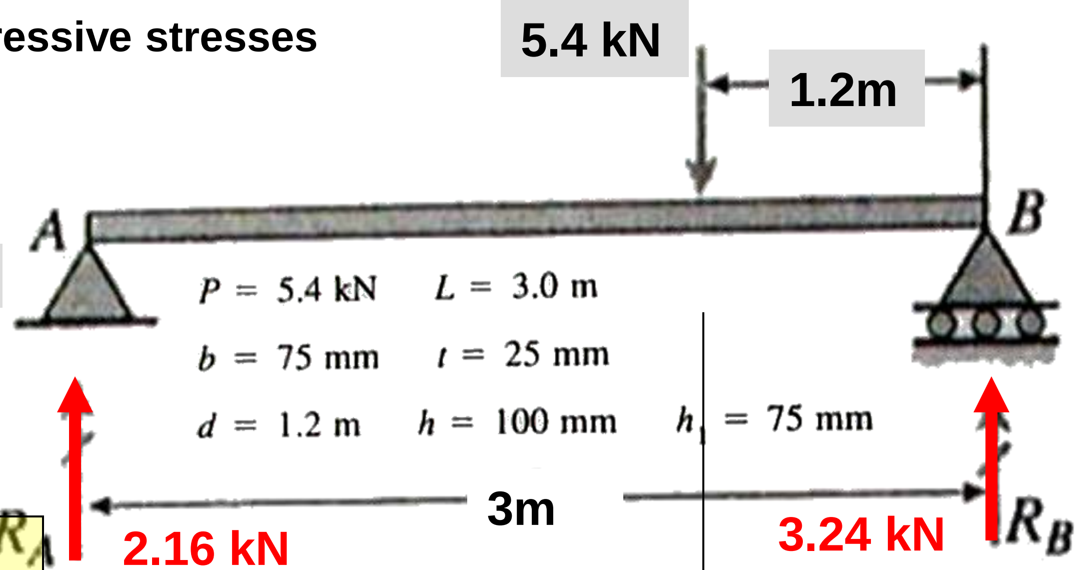
$$S = \frac{I}{c} = \frac{14{,}352}{27/2} = 1{,}063 \text{ in}^3$$

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{151.6 \times 1{,}000 \times 12}{1{,}063} = 1{,}710 \text{ psi}$$



Prob. 5.16 Max. Tensile and compressive stresses in the beam





PROPERTIES OF THE CROSS SECTION

$A = 3750 \text{ mm}^2$

$c_1 = 62.5 \text{ mm} \qquad c_2 = 37.5 \text{ mm}$

$I_C = 3.3203 \times 10^6 \text{ mm}^4$

MAXIMUM TENSILE STRESS

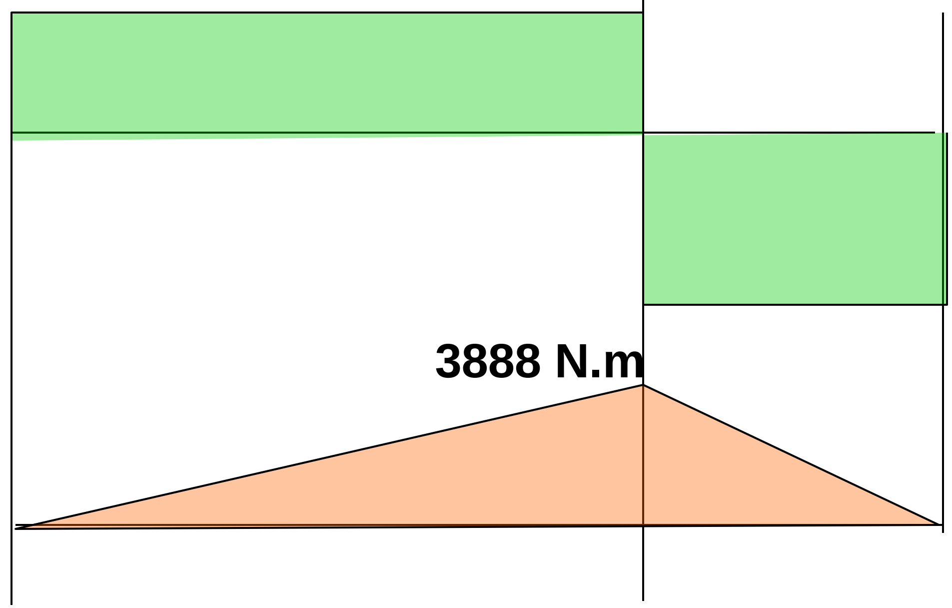
$$\sigma_t = \frac{M_{max} c_2}{I_C} = \frac{(3888 \text{ N} \cdot \text{m})(0.0375 \text{ m})}{3.3203 \times 10^6 \text{ mm}^4}$$

$$= 43.9 \text{ MPa} \longleftarrow$$

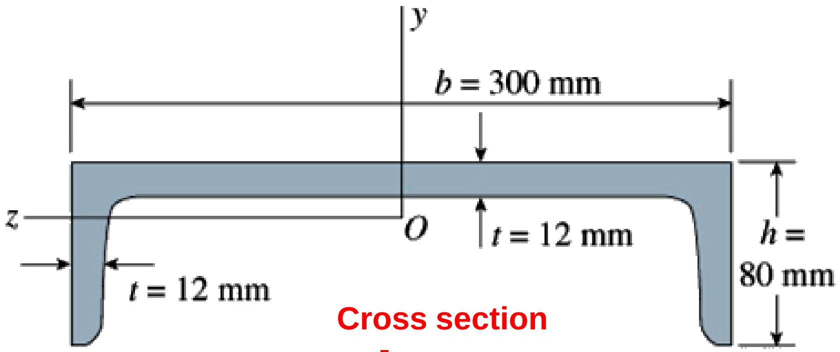
MAXIMUM COMPRESSIVE STRESS

$$\sigma_c = \frac{M_{max} c_1}{I_C} = \frac{(3888 \text{ N} \cdot \text{m})(0.0625 \text{ m})}{3.3203 \times 10^6 \text{ mm}^4}$$

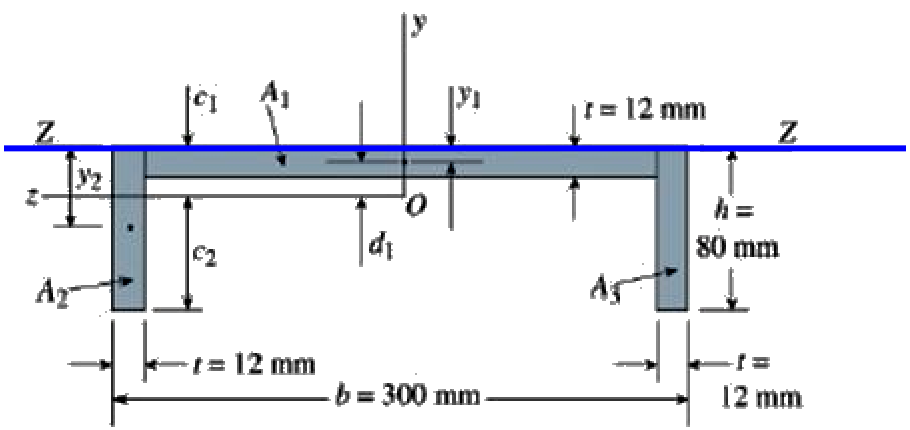
$$= 73.2 \text{ MPa} \longleftarrow$$

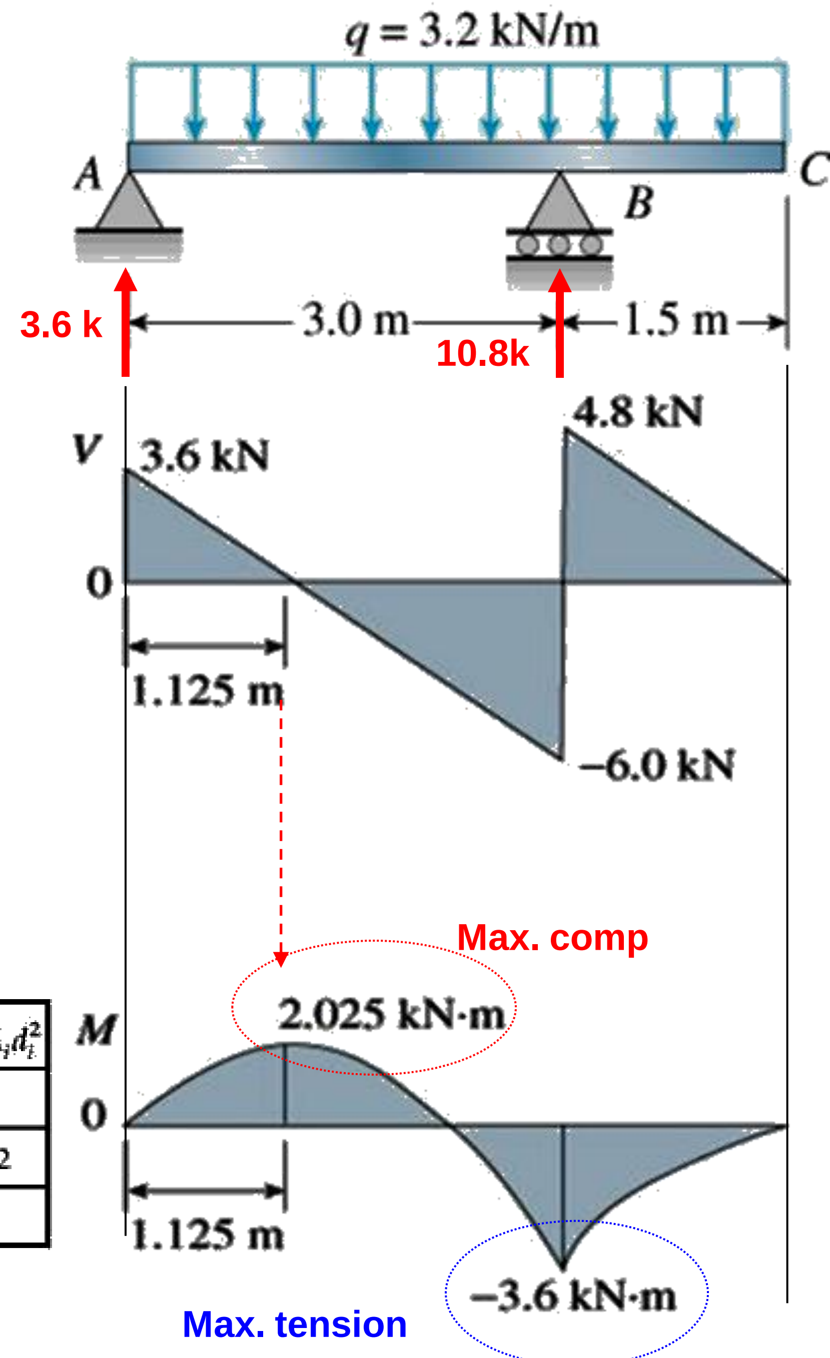


Example 5-4: Determine the maximum tensile and compressive stress in the beam



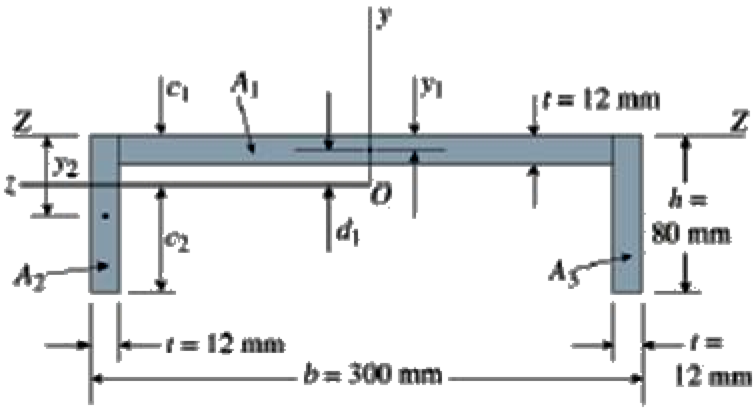
Cross section



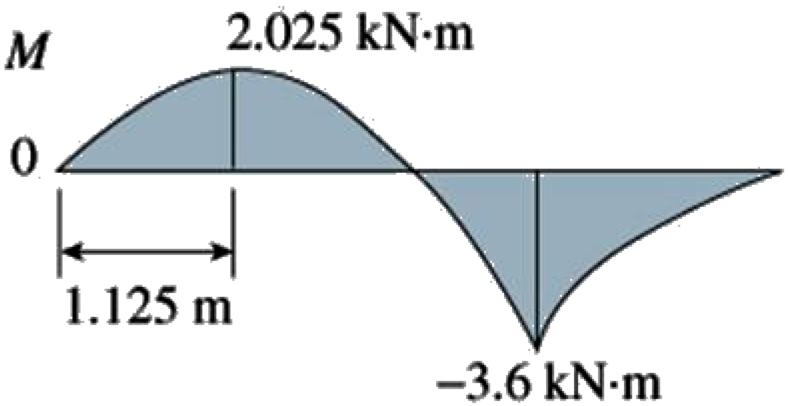


	A_i	$\bar{y}_i$	Q_i	$I_{zi}^{(i)} = bh^3/12$	d_i	$A_i d_i^2$	$I_z^{(i)} = I_{zi}^{(i)} + A_i d_i^2$
①	3,312	6	19,872	39,744	12.48	515,845	555,589
②, ③	960 × 2	40	38,400 × 2	512,000 × 2	- 21.52	444,586 × 2	956,586 × 2
Σ	5,232		97,672				2,468,761

$$\bar{y} = c_1 = \frac{\sum Q_i}{\sum A_i} = 18.48 \text{ mm}, \quad c_2 = h - c_1 = 61.52 \text{ mm}$$



(b)



$$\left(\sigma_{tensile}\right)_{\max} = 50.5\ \text{MPa}$$
$$\left(\sigma_{compressive}\right)_{\max} = -89.8\ \text{MPa}$$

$$I_z = 2.469 \times 10^6\ \text{mm}^4$$

$$c_1 = 18.48\ \text{mm} \quad \Rightarrow \quad S_1 = \frac{I_z}{c_1} = 133{,}600\ \text{mm}^3$$

$$c_2 = 61.52\ \text{mm} \quad \Rightarrow \quad S_2 = \frac{I_z}{c_2} = 40{,}100\ \text{mm}^3$$

at $x = 1.125\ \text{m}, \quad M = 2.025\ \text{kN}\cdot\text{m}$

$$\sigma_1 = -\frac{M}{S_1} = -\frac{2.025\ \text{kN}\cdot\text{m}}{133{,}600\ \text{mm}^3} = -15.2\ \text{MPa}$$

$$\sigma_2 = \frac{M}{S_2} = \frac{2.025\ \text{kN}\cdot\text{m}}{40{,}100\ \text{mm}^3} = 50.5\ \text{MPa}$$

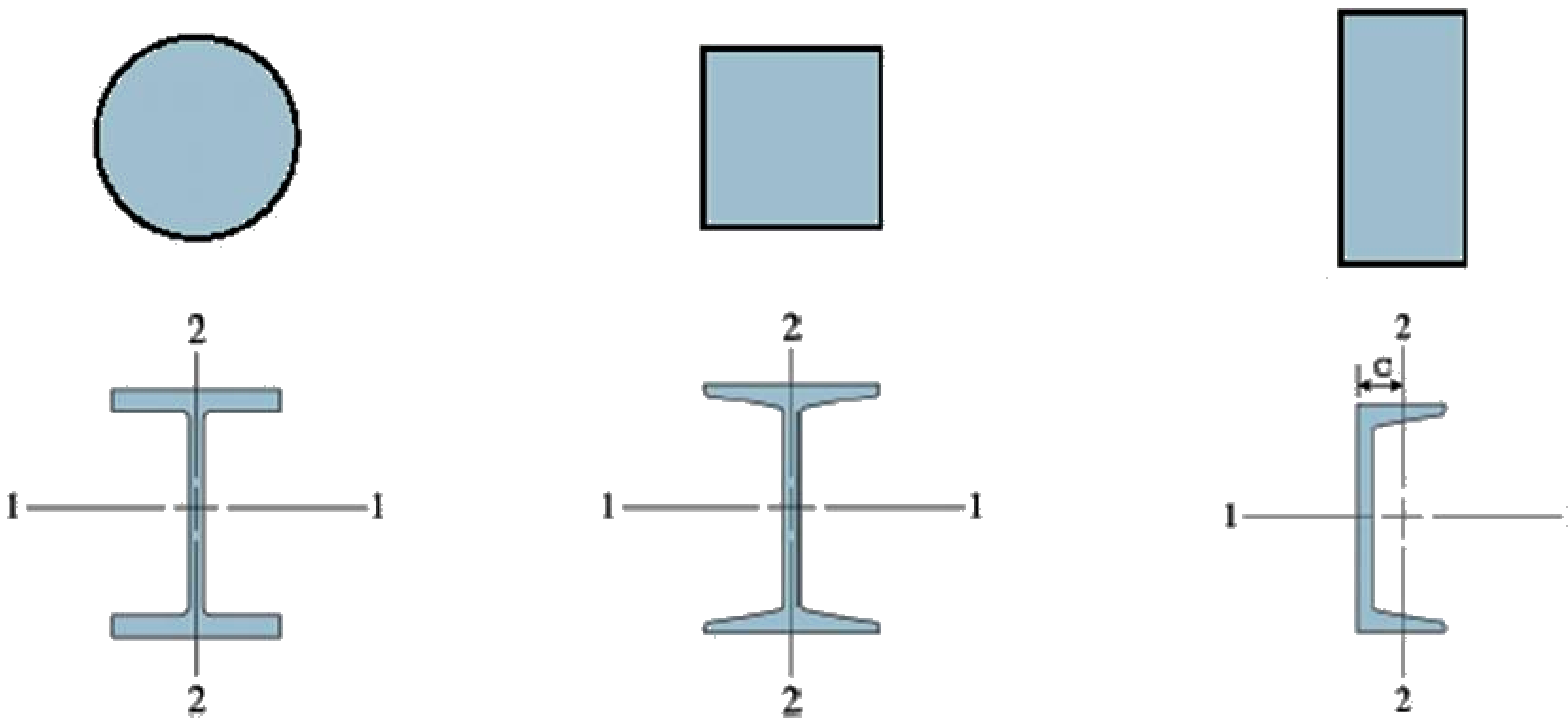
at $x = 3.0\ \text{m}, \quad M = -3.6\ \text{kN}\cdot\text{m}$

$$\sigma_1 = -\frac{M}{S_1} = -\frac{-3.6\ \text{kN}\cdot\text{m}}{133{,}600\ \text{mm}^3} = 26.9\ \text{MPa}$$

$$\sigma_2 = \frac{M}{S_2} = \frac{-3.6\ \text{kN}\cdot\text{m}}{40{,}100\ \text{mm}^3} = -89.8\ \text{MPa}$$

Design of Beams for Bending Stresses

$$S = \frac{M_{\max}}{\sigma_{\text{allow}}}$$

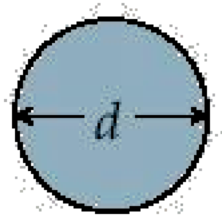


Which cross section is the most efficient one?

Design of Beams for Bending Stresses

$$S = \frac{I}{c} = \frac{M_{\max}}{\sigma_{\text{allow}}}$$

I. Circular Cross Sections

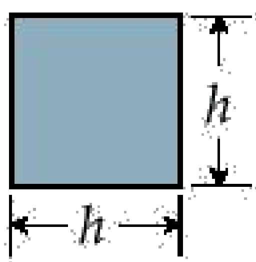


$$Area = \frac{\pi d^2}{4}$$

$$I = \frac{\pi d^3}{64}, \quad c = \frac{d}{2}$$

$$S_{\text{circle}} = \frac{\pi d^3}{32} = 0.0982 d^3$$

II. Square Cross Sections



$$Area = h^2$$

$$I = \frac{h^4}{12}, \quad c = \frac{h}{2}$$

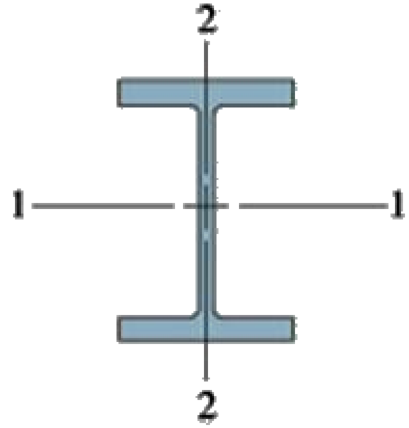
Compare to a circular cross section of identical area

$$Area = h^2 = \frac{\pi d^2}{4} \quad \Rightarrow \quad h = \frac{\sqrt{\pi} d}{2} = 0.886d$$

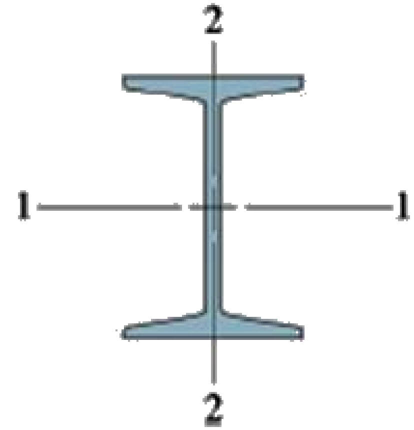
$$S_{\text{square}} = \frac{I}{c} = \frac{h^3}{6} = \frac{1}{6}\left(\frac{\sqrt{\pi} d}{2}\right)^3 = 0.116 d^3 = 1.181 S_{\text{circle}}$$

Properties of Structural-Steel Shapes

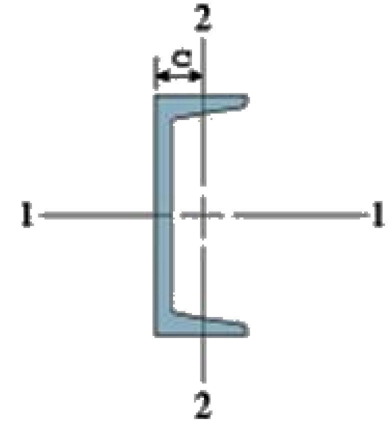
Appendix E, pp. 897 - 902



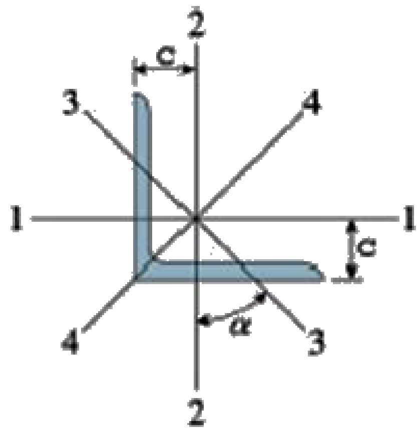
Wide-Flange Sections
(W Shapes)



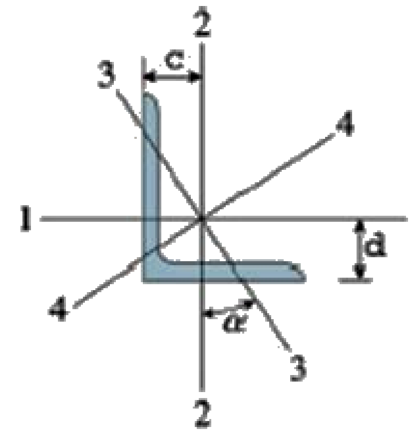
I-Beam Sections
(S Shapes)



Channel Sections
(C Shapes)

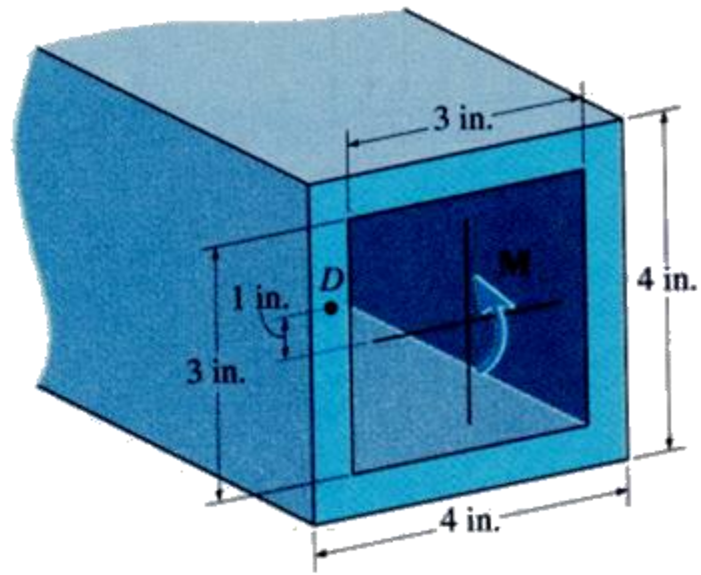


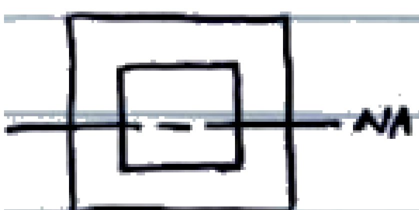
Angle Sections with Equal Legs
(L Shapes)



Angle Sections with Unequal Legs
(L Shapes)

1. Determine the moment M that should be applied to the beam in order to create a compressive stress at point D of 10 ksi.





$$I = \frac{4(4)^3}{12} - \frac{3(3)^3}{12} = 14.58 \text{ in}^4$$

$$\sigma = \frac{-M(1)}{14.58} = -10{,}000$$

$$M = 145{,}800 \text{ in-lb} = \boxed{12{,}150 \text{ ft-lb}}$$

Beams With Axial Loads

$$\sigma = \frac{P}{A}$$

σ – Normal Stress

P – Axial Force

A – Cross Sectional Area

$$\sigma_b = \frac{My}{I}$$

σ_b – Bending stress

M – Calculated bending moment

y – Vertical distance away from the neutral axis

I – Moment of inertia around the neutral axis

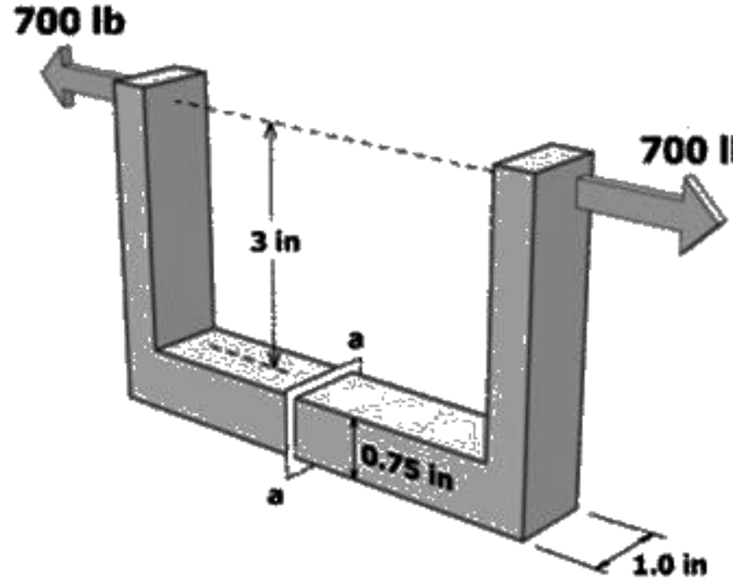
These equations assume linearity of structures

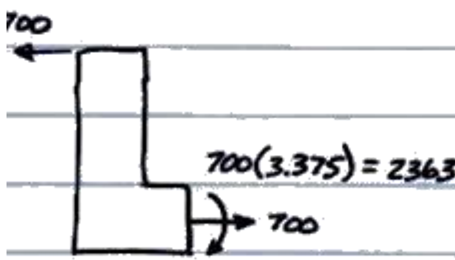
Means supperposition is applied

Therefore if different load types are simultaneously applied

Resultant stress distribution found by superposition

3. Determine the maximum normal stress in the horizontal portion of the bracket. The bracket has a thickness of 1 in. and a height of 0.75 in.

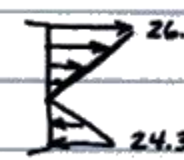




$$\sigma = \frac{700}{.75(1)} - \frac{(-2363)(\pm .375)}{1(.75)^3/12}$$

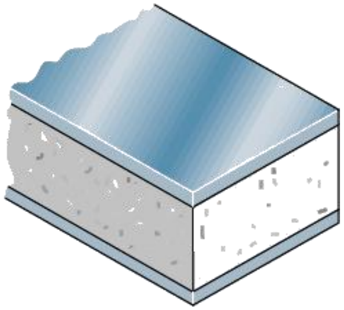
$\sigma_{top} = \boxed{26.1 \text{ Ksi}}$

$\sigma_{bottom} = -24.3 \text{ Ksi}$

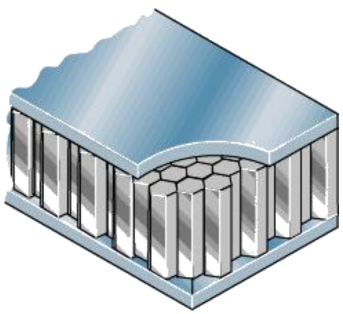


Composite Beams

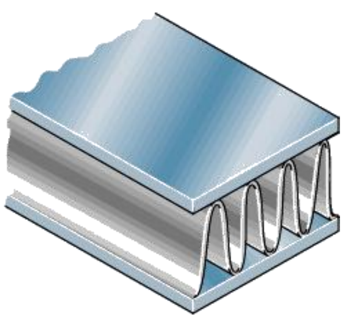
Sandwich beams with: (a) plastic core, (b) honeycomb core, and (c) corrugated core.



(a)

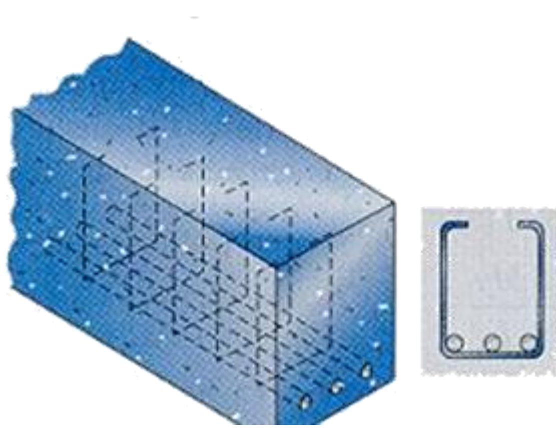


(b)

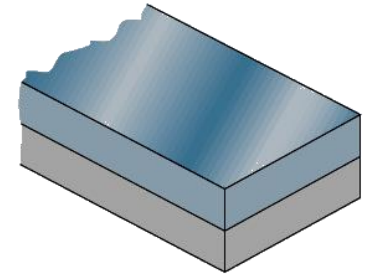


(c)

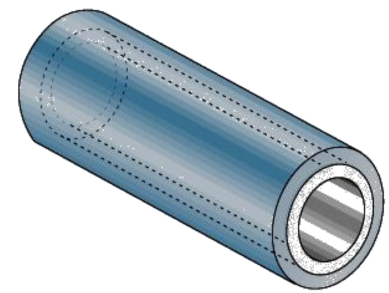
A *composite* beam is composed of two or more elemental structural forms, or different materials, bonded, knitted, or otherwise joined together. *Composite materials or forms* include such heavy handed stuff as concrete (one material) reinforced with steel bars (another material); high-tech developments such as tubes built up of graphite fibers embedded in an epoxy matrix; sports structures like *laminated* skis, the poles for vaulting, even a golf ball can be viewed as a *filament wound* structure encased within another material.



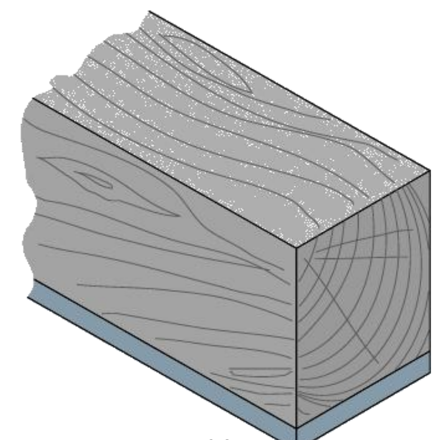
Examples of composite beams: (a) bimetallic beam, (b) plastic-coated steel pipe, and (c) wood beam reinforced with a steel plate.



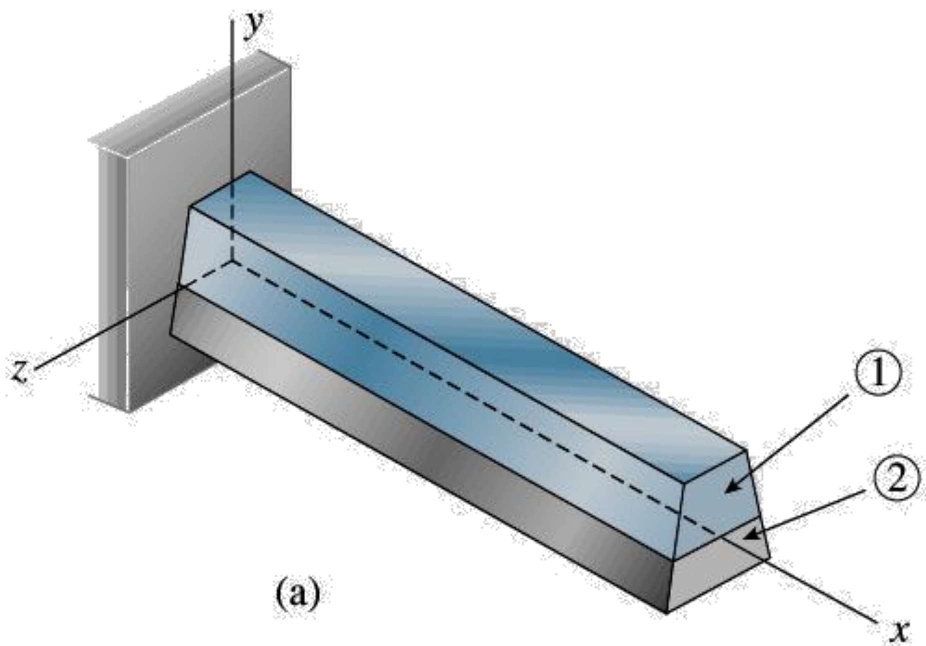
(a)



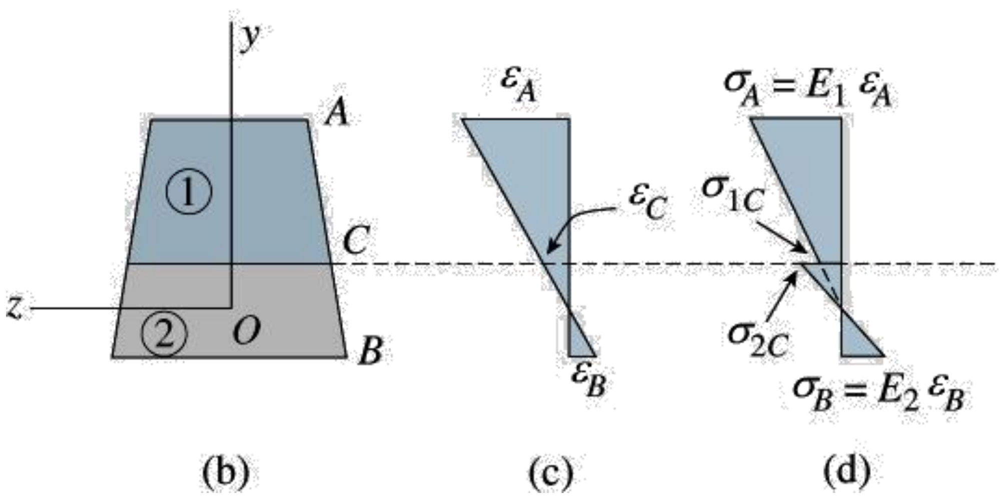
(b)



(c)



(a) Composite beam of two materials, (b) cross section of beam, (c) distributions of strains of ε_x throughout the height of the beam, and (d) distributions of stresses σ_x in the beam for the case where $\mathbf{E_2 > E_1}$.



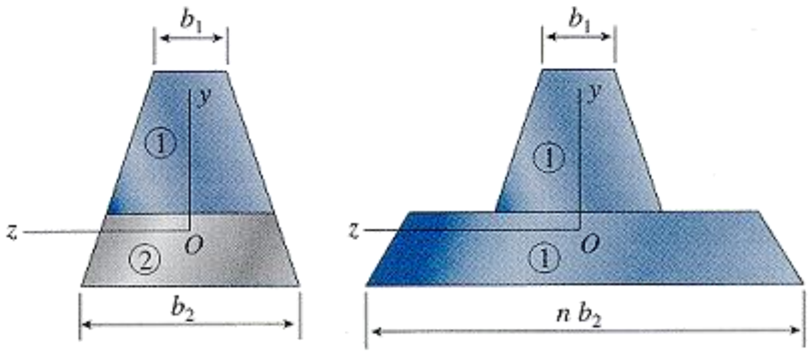
$$\varepsilon_x(y) = -y \cdot \left(\frac{d\phi}{ds}\right) = -(y/\rho)$$

For material #1 we have $\sigma_x = -E_1 \cdot (y/\rho)$ while for #2 $\sigma_x = -E_2 \cdot (y/\rho)$

Transformed Section Method

1. Transform the cross section of a composite beam into an equivalent cross section (**of an imaginary beam composed of only one material**) is called the transformed section
2. Analyze the transformed section as customary for a beam of one material .
3. Convert the stresses back to the original beam .
4. Modular ratio $n = \dfrac{E_2}{E_1}$

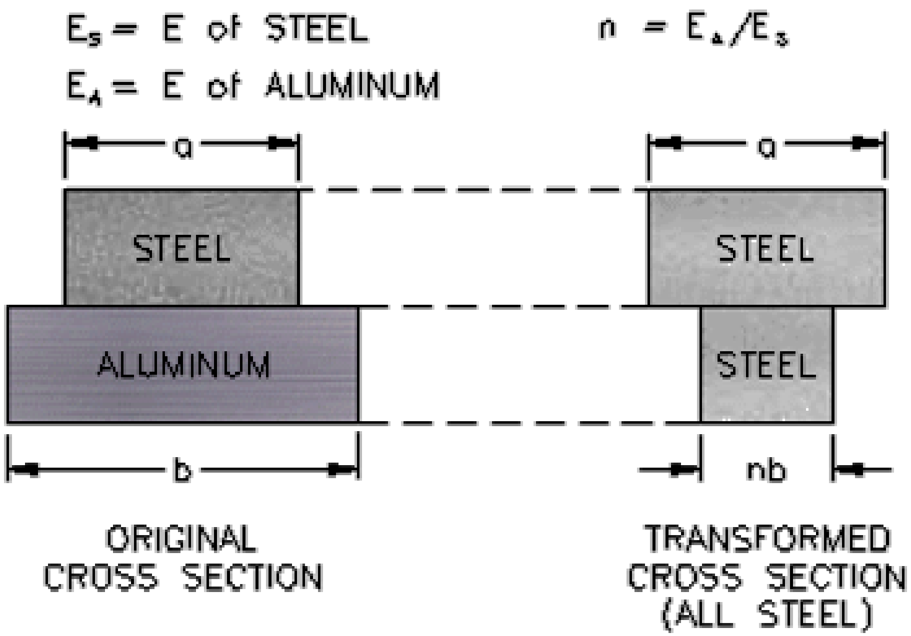
5. The dimensions of area 1 remain unchanged, and the width of area 2 is multiplied by *n*) . all dimensions perpendicular to the neutral axis remain the same(

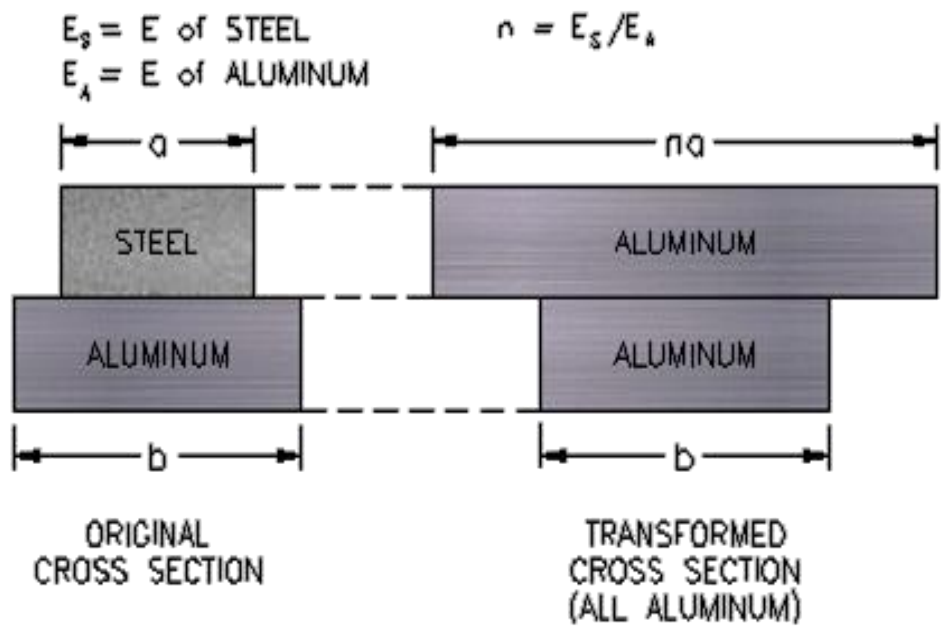


A similar procedure can be used to transform the beam into material 2 or a completely different material. One can also extend this technique to cover beams of more than two materials.

Flexure Formula

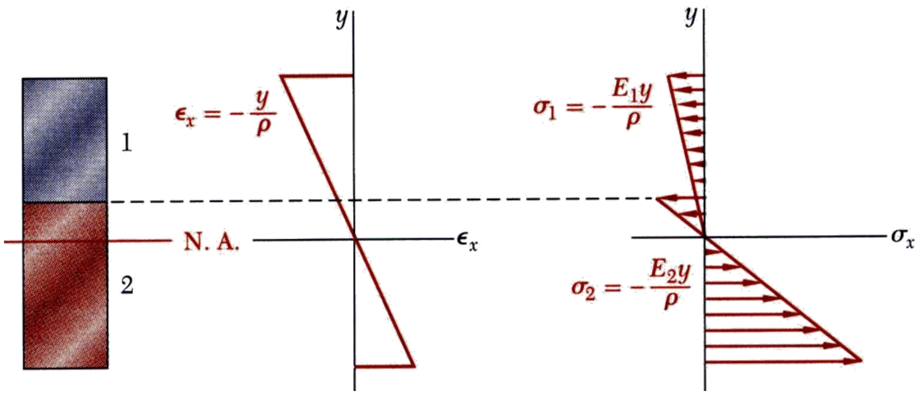
$$\sigma_{x1} = -\frac{My}{I_T} \qquad \sigma_{x2} = -\frac{My}{I_T}n$$

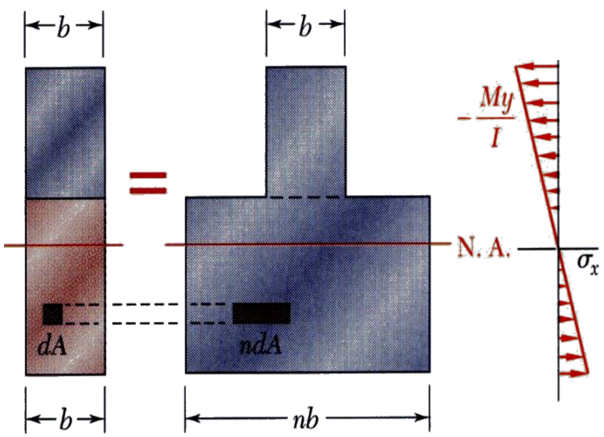




After the section is transformed all calculations are made using the transformed cross section, just as they would be on a beam of one material. The neutral axis of bending is at the centroid of the transformed section and flexure stresses are calculated with the flexure stress formula .

One final step is required to return to the original cross section. If in going from the stress state in the transformed material we find a reduction in area then we must increase the stresses accordingly to carry the same load. Conversely if we increase area then we reduce stress. Those portions of the cross section which were unaltered in the transformation process carry the same stresses on both the original and transformed sections .





$$\sigma_x = -\frac{My}{I}$$

$$\sigma_1 = \sigma_x \qquad \sigma_2 = n\sigma_x$$

- Consider a composite beam formed from two materials with E_1 and E_2.
- Normal strain varies linearly.

$$\varepsilon_x = -\frac{y}{\rho}$$

- Piecewise linear normal stress variation.

$$\sigma_1 = E_1 \varepsilon_x = -\frac{E_1 y}{\rho} \qquad \sigma_2 = E_2 \varepsilon_x = -\frac{E_2 y}{\rho}$$

Neutral axis does not pass through section centroid of composite section.

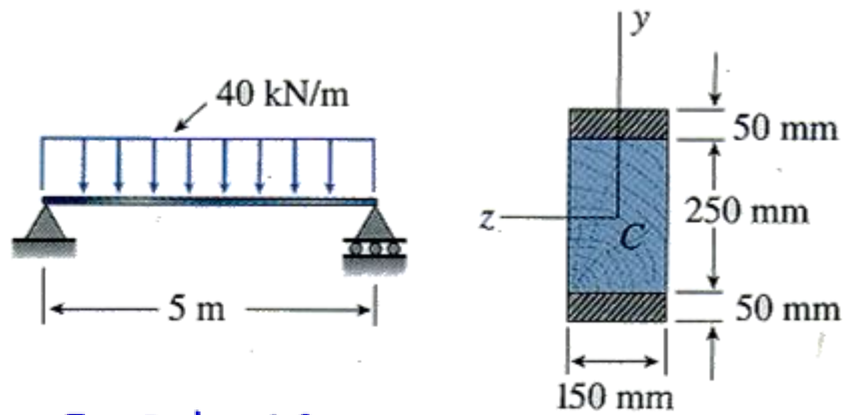
- Elemental forces on the section are

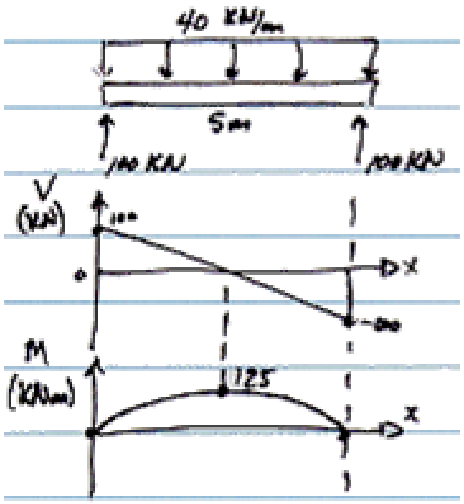
$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \quad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

- Define a transformed section such that

$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho}(n\,dA) \qquad n = \frac{E_2}{E_1}$$

6.3-4 The composite beam shown in the figure is simply supported and carries a total uniform load of 40 kN/m on a span length of 5 m. The beam is built of a wood member having cross-sectional dimensions 150 mm × 250 mm and two steel plates of cross-sectional dimensions 50 mm × 150 mm. Determine the maximum stresses σ_s and σ_w in the steel and wood, respectively, if the moduli of elasticity are E_s = 209 GPa and E_w = 11 GPa.





M_{max} = 125 kNm

From Section 6.2

Wood

E_w = 11 GPa

y_w = .125 m

$I_w = \frac{bh^3}{12} = 0.0001953125\ m^4$

$\sigma_w = \frac{M y_w E_w}{E_w I_w + I_s E_s} = 2.344$ MPa

Steel

E_s = 209 GPa

y_s = .125 m + .05 m = .175 m

$I_s = \frac{bh^3}{12}\Big|_{outside} - \frac{bh^3}{12}\Big|_{inside} = 0.000340625\ m^4$

$\sigma_s = \frac{M y_s E_s}{E_w I_w + E_s I_s} = 62.34$ MPa

$b_s^t = (.15m)\left(\frac{209}{11}\right) = 2.85m$

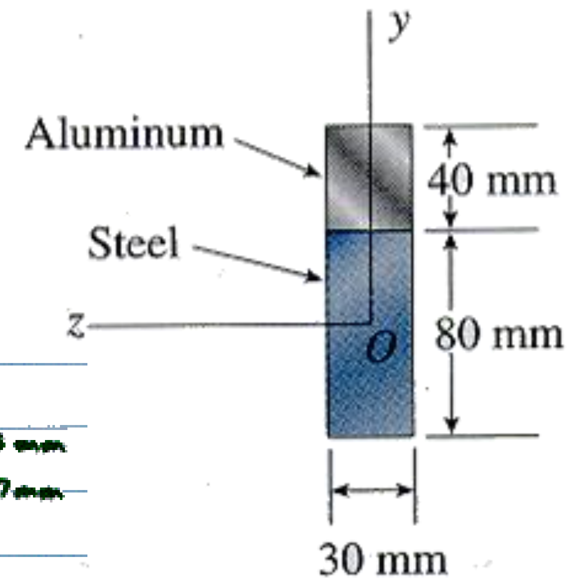
n = 19

$\sigma_w = \frac{M y_w}{I_t} = $ **2.344 MPa**

$\sigma_s = \frac{M y_s}{I_t} n = $ **62.34 MPa**

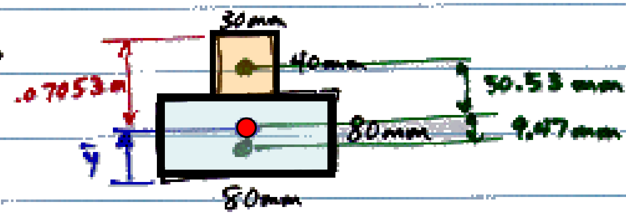
$I_t = \frac{1}{12}(bh^3 - bh_1^3 + th_1^3) = 0.0066672\ m^4$

6.3-8 The cross section of a composite beam made of aluminum and steel is shown in the figure. The moduli of elasticity are $E_a = 75$ GPa and $E_s = 200$ GPa. Under the action of a bending moment that produces a maximum stress of 50 MPa in the aluminum, what is the maximum stress σ_s in the steel?



$E_a = 75\ GPa$ $\quad n = \frac{E_s}{E_a} = 2.667$

$E_s = 200\ GPa$



$$\bar{y} = \frac{y_a A_a + \bar{y}_s A_s}{A_a + A_s} = \frac{(.04)(.08)(.08) + (.08+.02)(.03)(.04)}{(.08)(.08) + (.03)(.04)} = 0.04947\ m$$

$$I_t = I_a + I_s^t$$
$$= \left.\frac{bh^3}{12}\right|_a + (.05053m)^2(.03)(.04) + \left.\frac{bh^3}{12}\right|_s + (.00947)^2(.08)(.08)$$
$$= 7.2112 \times 10^{-6}\ m^4$$

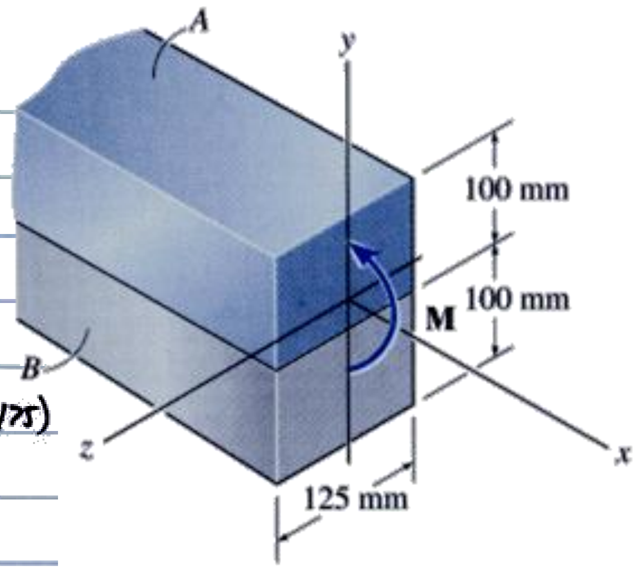
$y_a = .07053\ m$

$y_s = .04947\ m$

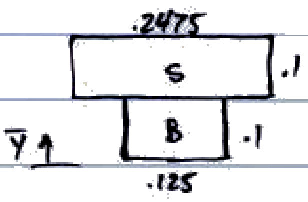
$$\sigma_a = \frac{M y_a}{I_t} = 50\ MPa \Rightarrow M = 5112\ Nm$$

$$\sigma_s = \frac{M y_s n}{I_t} = \boxed{93.52\ MPa}$$

The composite beam is made of steel (A) and brass (B). If the allowable bending stress for the steel is $\sigma_s = 180$ MPa and for the brass $\sigma_b = 60$ MPa, determine the maximum moment M that can be applied to the beam. Assume $E_s = 200$ GPa and $E_b = 101$ MPa.



$$n = \frac{E_S}{E_B} = \frac{200}{101} = 1.980$$



$$\bar{y} = \frac{.05(.1)(.125) + .15(.1)(.2475)}{.1(.125) + .1(.2475)} = .1164 \text{ m}$$

$$I = \frac{.125(.1)^3}{12} + .0664^2(.1)(.125) + \frac{.2475(.1)^3}{12} + .0336^2(.1)(.2475)$$

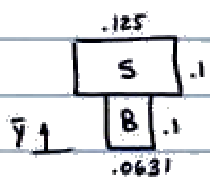
$$= .0001141 \text{ m}^4$$

$$\sigma_s = \frac{nM(.0836)}{.0001141} \le 180\times10^6 \Rightarrow M \le 124{,}100 \text{ Nm}$$

$$\sigma_b = \frac{M(.1164)}{.0001141} \le 60\times10^6 \Rightarrow \boxed{M \le 58{,}800 \text{ Nm}}$$

or

$$n = \frac{101}{200} = .505$$



$$\bar{y} = \frac{.05(.1)(.0631) + .15(.1)(.125)}{.1(.0631) + .1(.125)} = .1164 \text{ m}$$

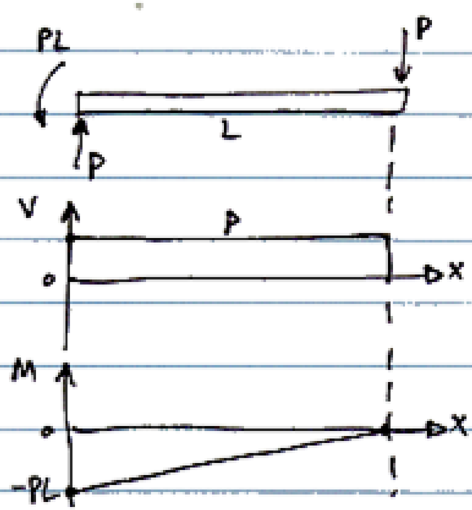
$$I = \frac{.0631(.1)^3}{12} + .0664^2(.1)(.0631) + \frac{.125(.1)^3}{12} + .0336^2(.1)(.125)$$

$$= .00005761 \text{ m}^4$$

$$\sigma_s = \frac{M(.0836)}{.00005761} \le 180\times10^6 \Rightarrow M \le 124{,}000 \text{ Nm}$$

$$\sigma_b = \frac{nM(.1164)}{.00005761} \le 60\times10^6 \Rightarrow \boxed{M \le 58{,}800 \text{ Nm}}$$

A 4-in. wide × 6-in. deep timber cantilever beam 8 ft long is reinforced by bolting two 1/2 × 6-in. steel plates to the sides of the timber beam, as shown. The moduli of elasticity of the timber and steel are 1600 ksi and 29,000 ksi, respectively. Determine the **maximum tensile bending stress in each of the materials** when a static load of 1250 lb is applied to the free end of the beam.



$E_w = 1600\text{ ksi}$ $\quad P = 1250\text{ lb}$

$E_s = 29{,}000\text{ ksi}$ $\quad L = 8\text{ ft}$

$n = E_s/E_w = 18.125$

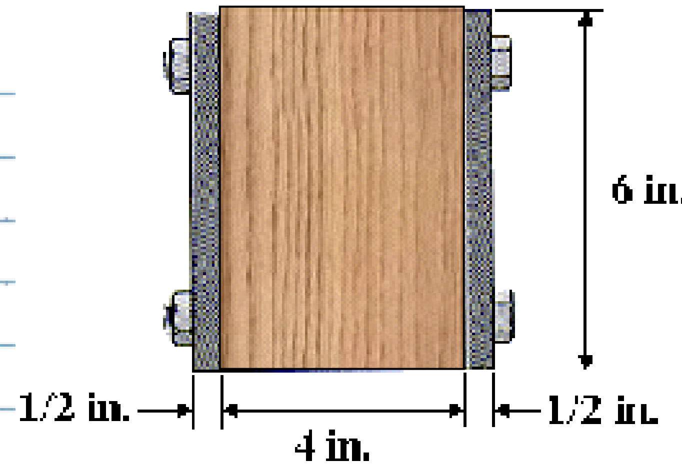
$$I_t = \frac{(22.125'')(6'')^3}{12} = 398.25\text{ in}^4$$

$y_s = y_w = 3''$

$M_{max} = 120{,}000\text{ in-lb}$

$$\sigma_w = \frac{M y_w}{I_t} = \boxed{904\text{ psi}}$$

$$\sigma_s = \frac{M y_s n}{I_t} = \boxed{16{,}380\text{ psi}}$$



What is the thickness t of the steel plates

$M_{max} = \frac{qL^2}{8} = 61.44\ kN \cdot m$

Simple beam : $L = 3.2\ m$ $\quad q = 48\ kN/m$

① wood flanges : $b = 100\ mm$ $\quad h = 300\ mm$

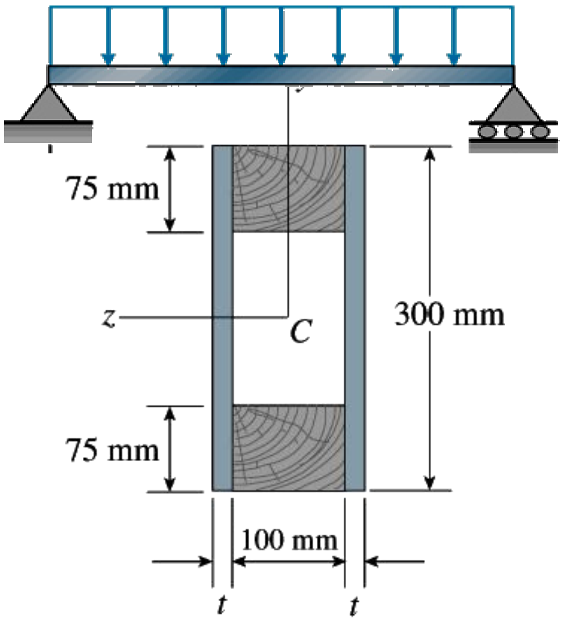
$h_1 = 150\ mm$ $\quad E_w = 10\ GPa$

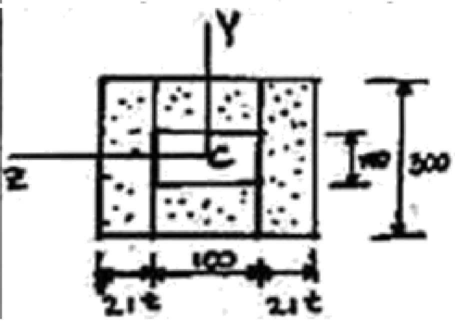
$(\sigma_1)_{allow} = 6.5\ MPa$

② steel plates $\quad t$ = thickness $\quad h = 300\ mm$

$E_s = 210\ GPa$

$(\sigma_2)_{allow} = 120\ MPa$





$n = \frac{E_s}{E_w} = 21$

All dimensions in millimeters

$I_T = \frac{1}{12}(100 + 42t)(300)^3 - \frac{1}{12}(100)(150)^3$

$= 196.9 \times 10^6\ mm^4 + 94.5\,t \times 10^6\ mm^4$

Required thickness based upon the wood ① (Eq. 6-15)

$\sigma_1 = \frac{M(h/2)}{I_T}$ $\quad (I_T)_1 = \frac{M_{max}(h/2)}{(\sigma_1)_{allow}} = 1.418 \times 10^9\ mm^4$

Equate I_T and $(I_T)_1$ and solve for t: $t_1 = 12.92\ mm$

Required thickness based upon the steel ② (Eq. 6-17)

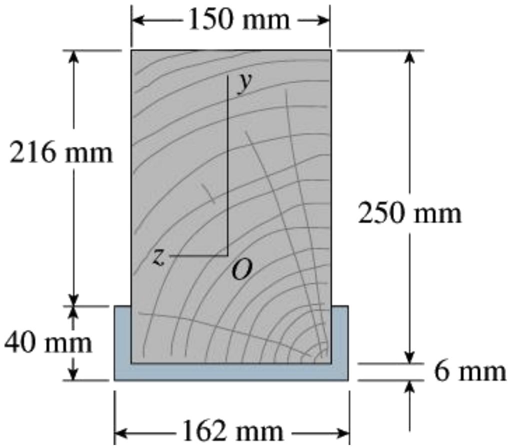
$\sigma_2 = \frac{M(h/2)n}{I_T}$ $\quad (I_T)_2 = \frac{M_{max}(h/2)n}{(\sigma_2)_{allow}} = 1.612 \times 10^9\ mm^4$

Equate I_T and $(I_T)_2$ and solve for t: $t_2 = 14.97\ mm$

Steel governs. $\quad t_{min} = 15.0\ mm$ ←

Problem 6.3-12

What is the maximum allowable bending moment for the beam ?



① Wood beam: $b_w = 150$ mm

$h_w = 250$ mm

$(\sigma_w)_{allow} = 8.5$ MPa

② Aluminum channel: $t = 6$ mm

$b_a = 162$ mm

$h_a = 40$ mm

$(\sigma_a)_{allow} = 40$ MPa

Maximum moment based upon the wood ①

(Eq. 6-15)

$$\sigma_w = \sigma_1 = \frac{M h_1}{I_T} \qquad M_1 = \frac{(\sigma_w)_{allow} I_T}{h_1} = 17.2 \text{ kN·m}$$

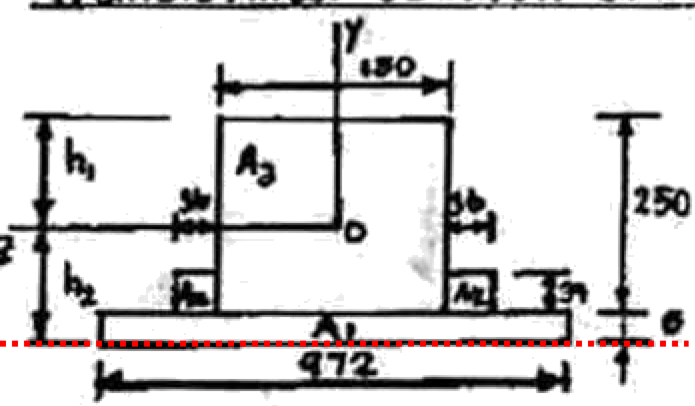
Maximum moment based upon the aluminum ②

(Eq. 6-17)

$$\sigma_a = \sigma_2 = \frac{M h_2 n}{I_T} \qquad M_2 = \frac{(\sigma_a)_{allow} I_T}{h_2 n} = 182 \text{ kN·m}$$

Wood governs $M_{allow} = 17.2$ kN·m

Transformed section (wood)



$$n = \frac{E_a}{E_w} = 6$$

$$n b_a = (6)(162 \text{ mm}) = 972 \text{ mm}$$

$$n t = (6)(6 \text{ mm}) = 36 \text{ mm}$$

All dimensions in millimeters.

Use the base of the cross section as a reference line.

$$h_2 = \frac{\Sigma Y_i A_i}{\Sigma A_i}$$

Area A_1: $Y_1 = 3$ $A_1 = (972)(6) = 5832$

$Y_1 A_1 = 17{,}496 \text{ mm}^3$

Area A_2: $Y_2 = 23$ $A_2 = (36)(34) = 1224$

$Y_2 A_2 = 28{,}152 \text{ mm}^3$

Area A_3: $Y_3 = 131$ $A_3 = (150)(250) = 37{,}500$

$Y_3 A_3 = 4{,}912{,}500 \text{ mm}^3$

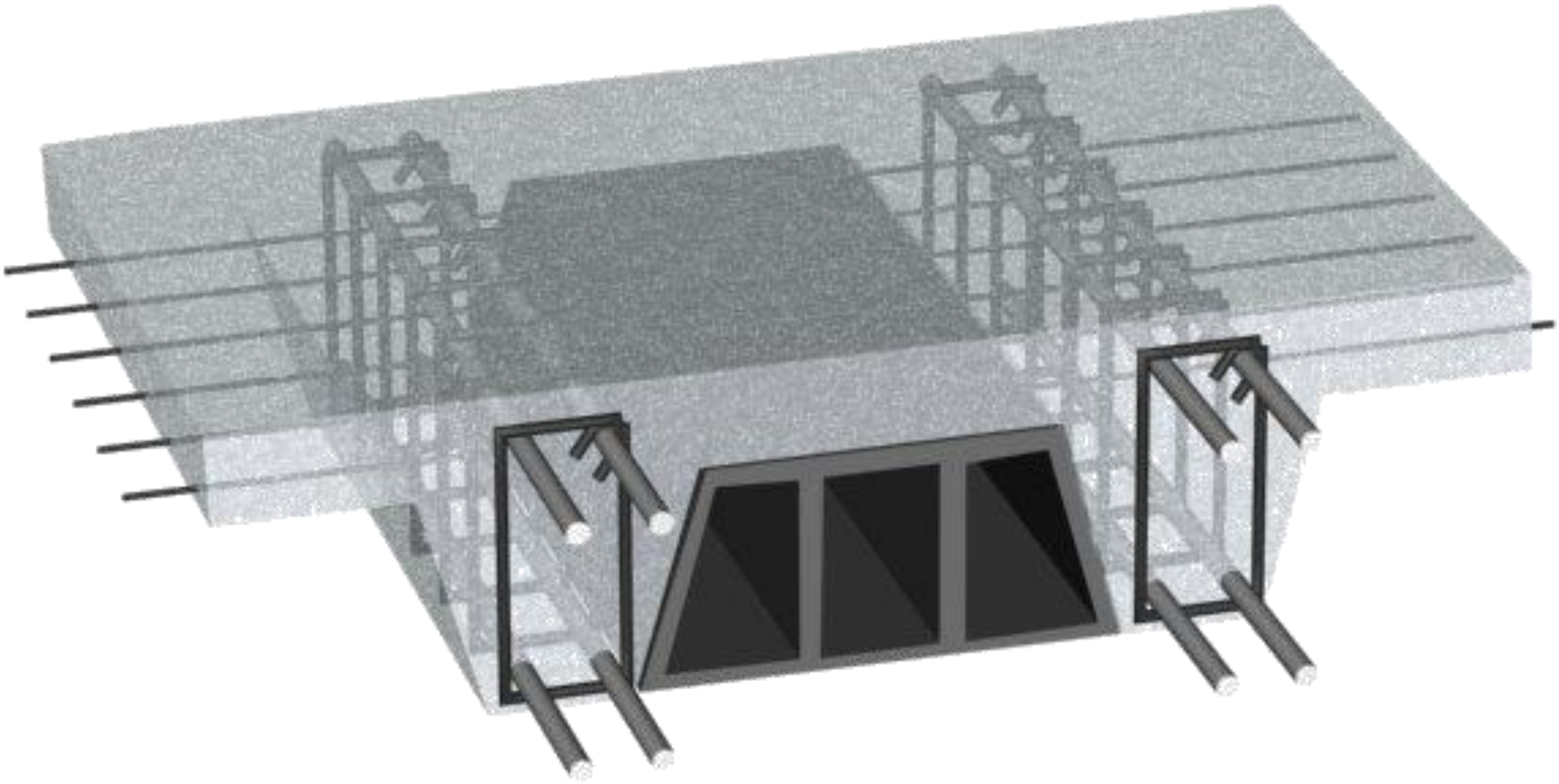
$$h_2 = \frac{Y_1 A_1 + 2 Y_2 A_2 + Y_3 A_3}{A_1 + 2A_2 + A_3} = \frac{4{,}986{,}300 \text{ mm}^3}{45{,}780 \text{ mm}^2}$$

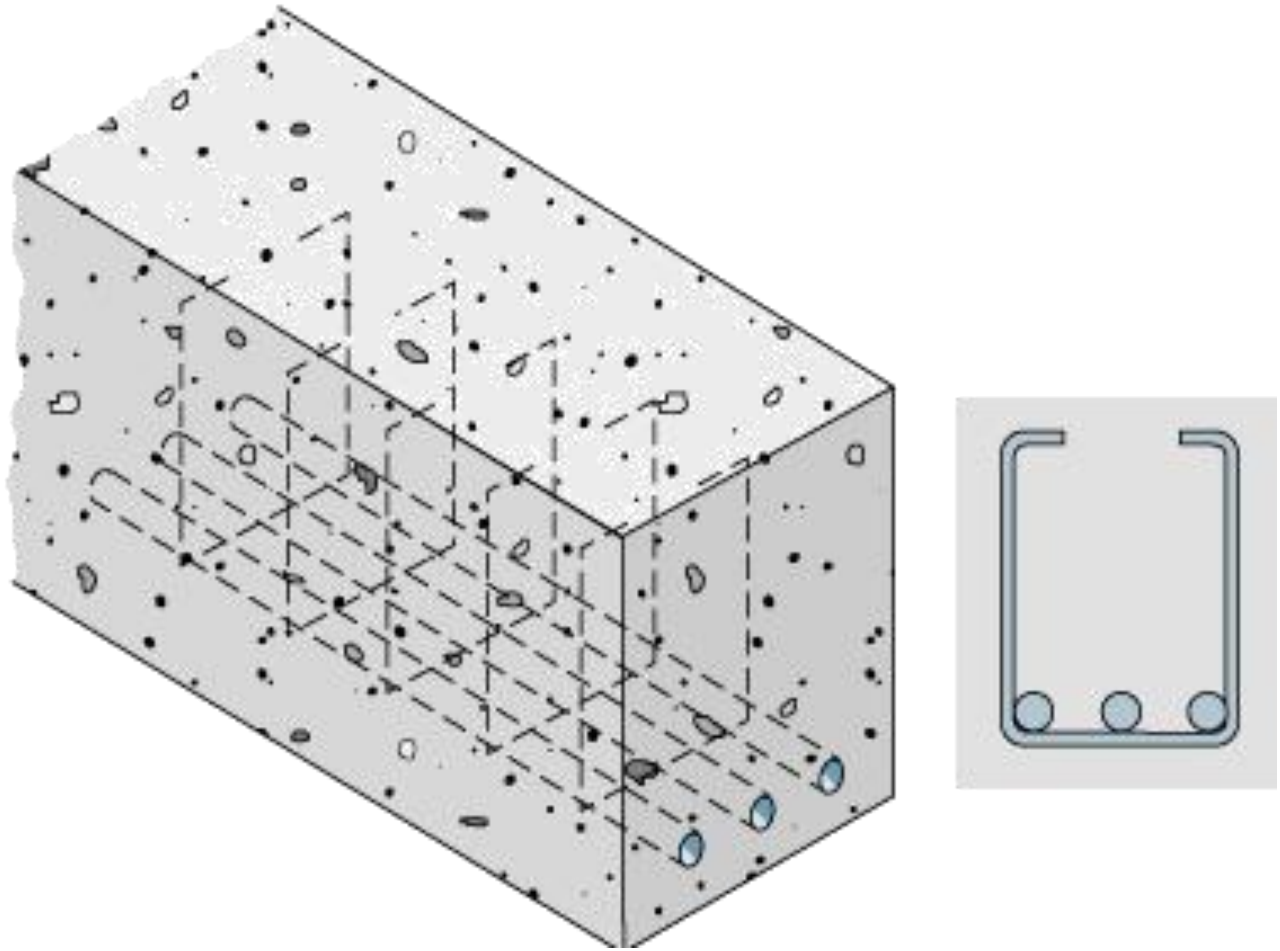
$$= 108.92 \text{ mm}$$

$$h_1 = 256 - h_2 = 147.08 \text{ mm}$$

$$I_T = I_1 + 2I_2 + I_3 = 297.3 \times 10^6 \text{ mm}^4$$

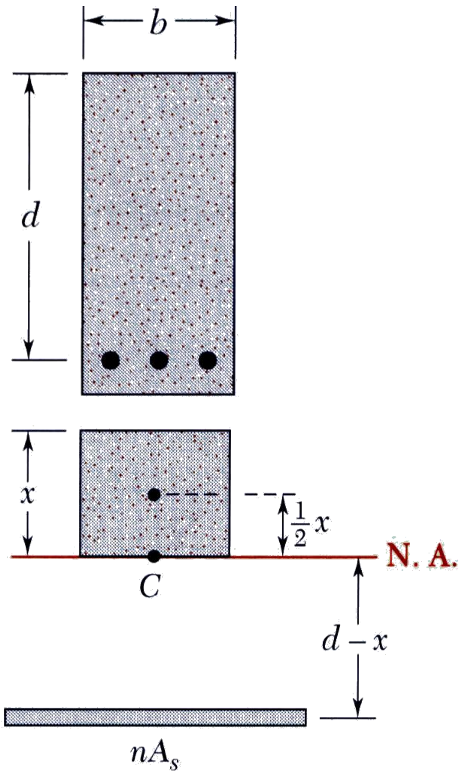
Reinforced concrete sections

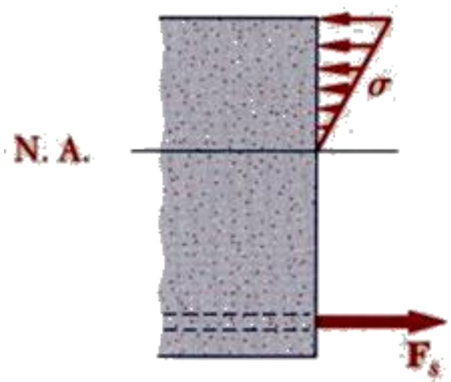




Reinforced concrete beam with longitudinal reinforcing bars and vertical stirrups.

Reinforced Concrete Beams





- Concrete beams subjected to bending moments are reinforced by steel rods.

- The steel rods carry the entire tensile load below the neutral surface. The upper part of the concrete beam carries the compressive load.

- In the transformed section, the cross sectional area of the steel, A_s, is replaced by the equivalent area nA_s where $n = E_s/E_c$.

- To determine the location of the neutral axis,

$$(bx)\frac{x}{2} - n A_s (d - x) = 0$$

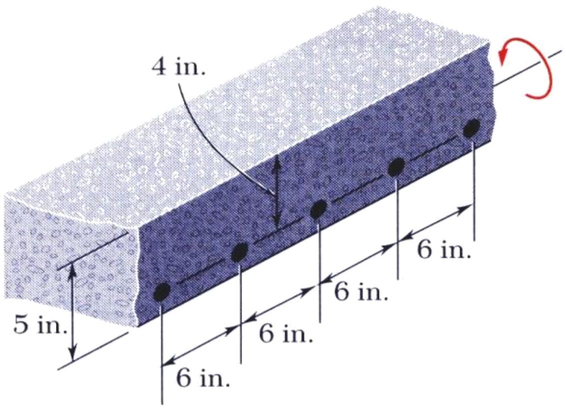
$$\tfrac{1}{2} b x^2 + n A_s x - n A_s d = 0$$

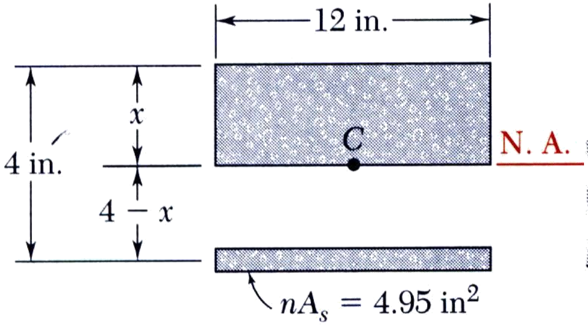
- The normal stress in the concrete and steel

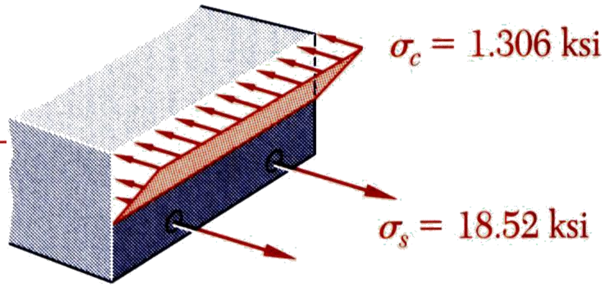
$$\sigma_x = -\frac{My}{I}$$

$$\sigma_c = \sigma_x \qquad \sigma_s = n\sigma_x$$

A concrete floor slab is reinforced with 5/8-in-diameter steel rods. The modulus of elasticity is 29x106psi for steel and 3.6x106psi for concrete. With an applied bending moment of 40 kip*in for 1-ft width of the slab, determine the maximum stress in the concrete and steel.







SOLUTION:

- Transform to a section made entirely of concrete.
- Evaluate geometric properties of transformed section.
- Calculate the maximum stresses in the concrete and steel.

$$n = \frac{E_s}{E_c} = \frac{29 \times 10^6 \text{ psi}}{3.6 \times 10^6 \text{ psi}} = 8.06$$

$$nA_s = 8.06 \times 2\left[\frac{\pi}{4}\left(\frac{5}{8}\text{ in}\right)^2\right] = 4.95\text{in}^2$$

- **Evaluate the geometric properties of the transformed section.**

$$12x\left(\frac{x}{2}\right) - 4.95(4 - x) = 0 \qquad x = 1.450\text{in}$$

$$I = \tfrac{1}{3}(12\text{in})(1.45\text{in})^3 + \left(4.95\text{in}^2\right)(2.55\text{in})^2 = 44.4\text{in}^4$$

- Calculate the maximum stresses.

$$\sigma_c = \frac{Mc_1}{I} = \frac{40\,\text{kip}\cdot\text{in} \times 1.45\text{in}}{44.4\text{in}^4} \qquad \boxed{\sigma_c = 1.306\text{ksi}}$$

$$\sigma_s = n\frac{Mc_2}{I} = 8.06\frac{40\,\text{kip}\cdot\text{in} \times 2.55\text{in}}{44.4\text{in}^4} \qquad \boxed{\sigma_s = 18.52\text{ksi}}$$

Engineering Mechanics

Statics & Strength of Materials

Shear Stresses in Beams

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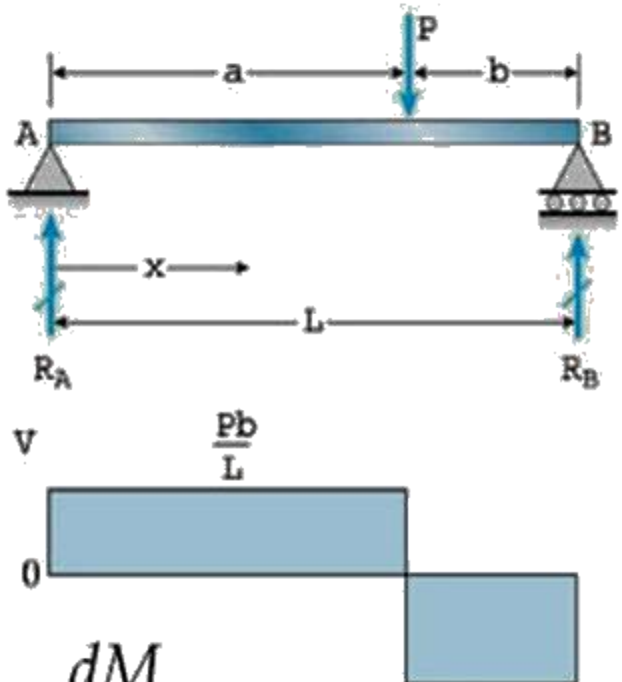
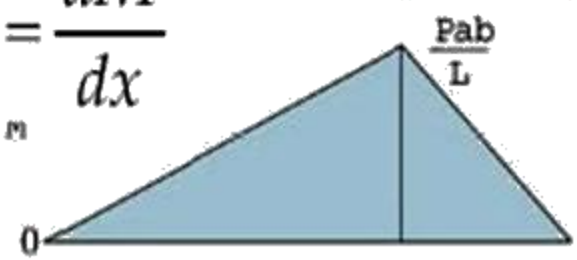
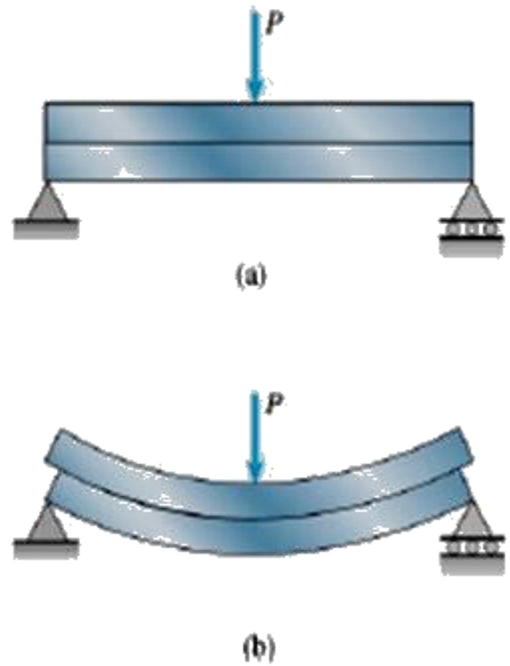
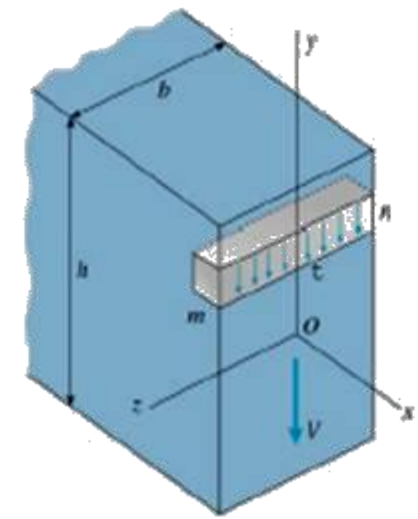
Shear Stresses in Beams

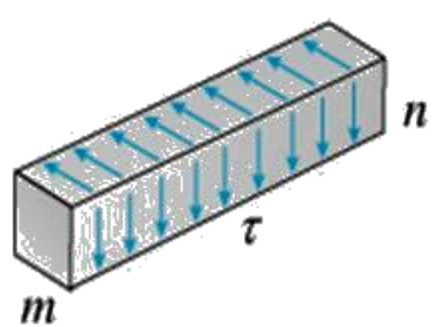


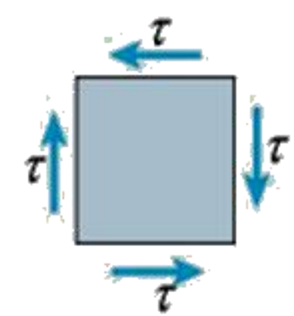
$$V = \frac{dM}{dx}$$



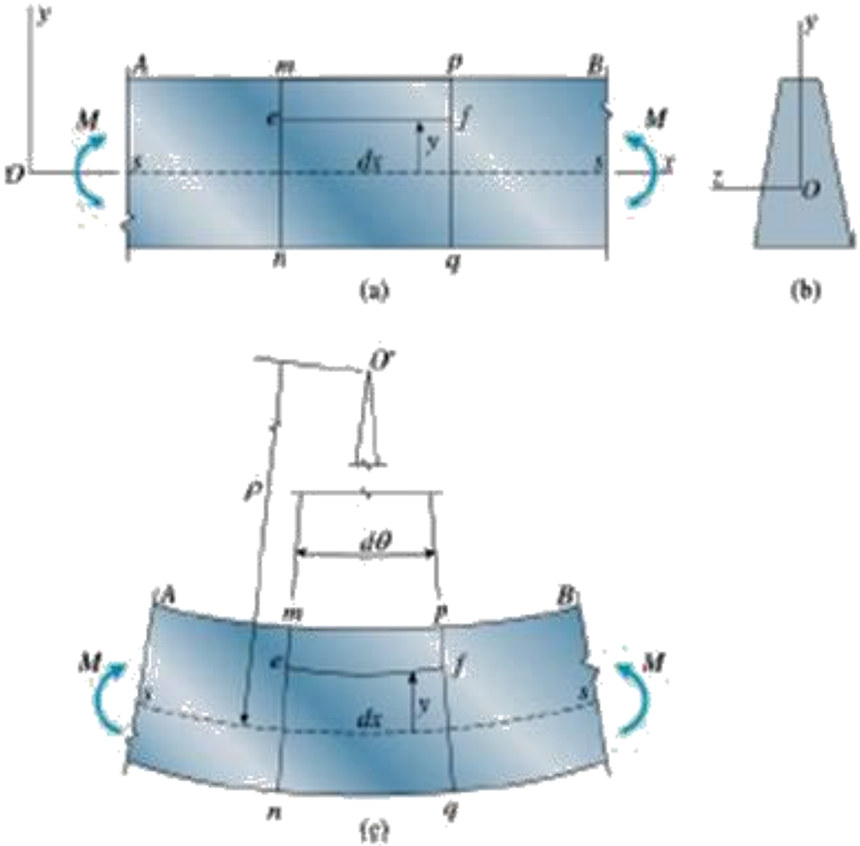




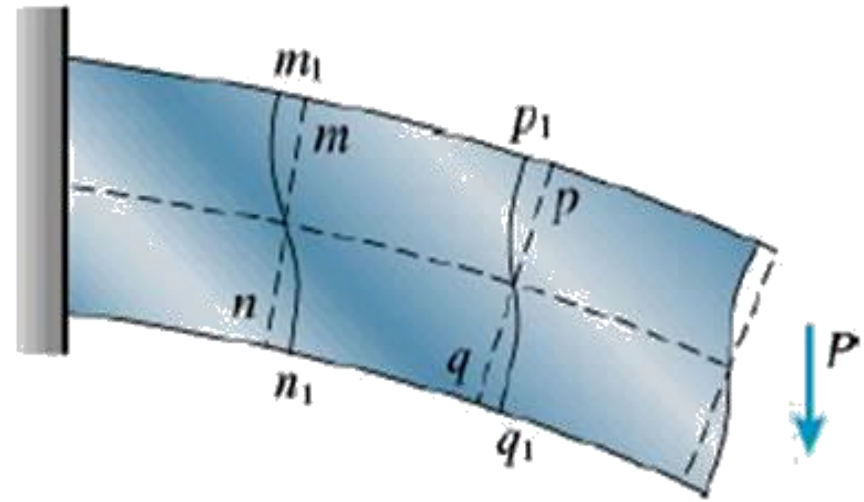




Effects of Shear Strain



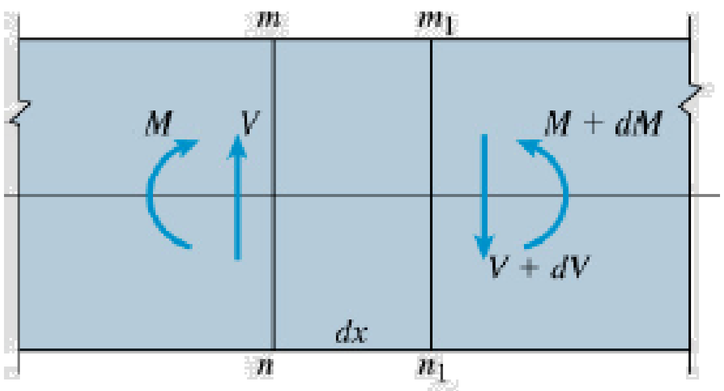
Pure Bending Assumption:
Cross sections remain plane



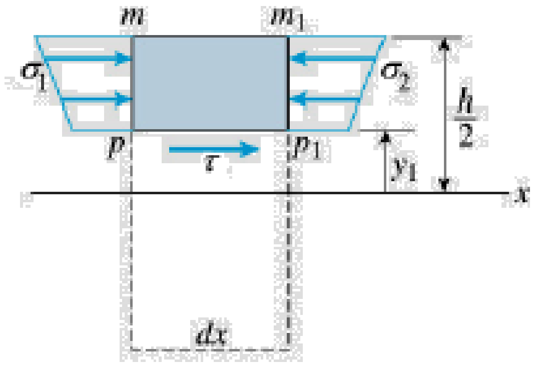
Warping of the cross sections of a beam due to shear strains.

The effect of shear strain becomes negligible when the aspect ratio, L/h, of the beam is greater than 10.

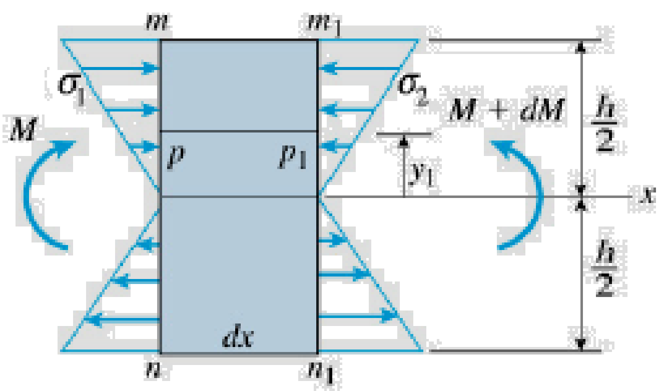
Shear Stresses in Beams



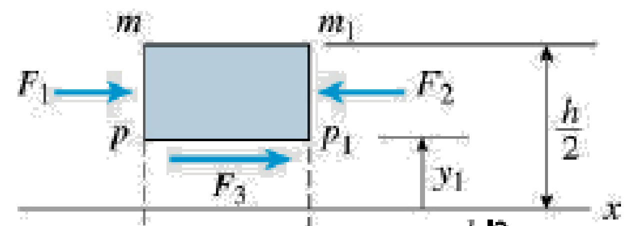
Side view of beam



Side view of subelement



Side view of element



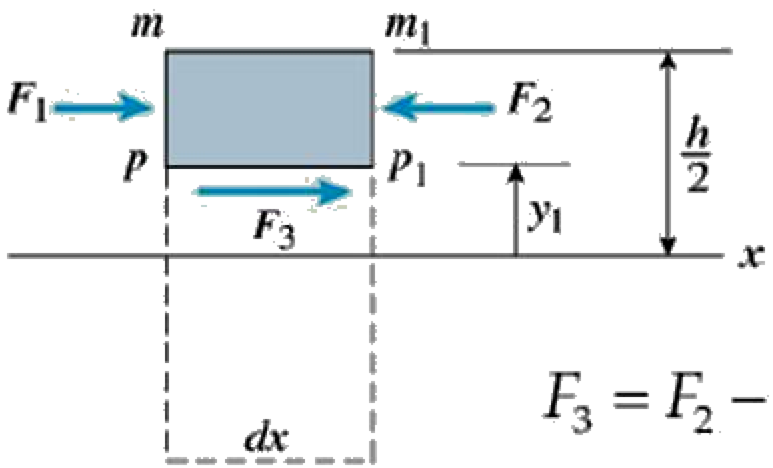
$$F_1 = \int_{y_1}^{h/2} \sigma_1 dA = \int_{y_1}^{h/2} \frac{My}{I} dA$$

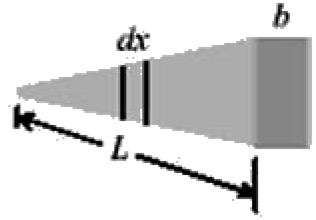
$$F_2 = \int_{y_1}^{h/2} \sigma_2 dA = \int_{y_1}^{h/2} \frac{(M+dM)y}{I} dA$$

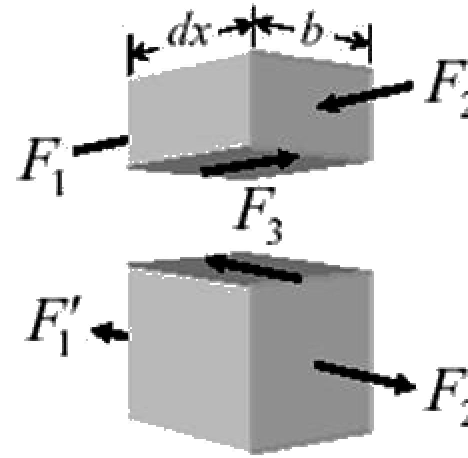
$$\sigma_1 = -\frac{My}{I} \qquad \sigma_2 = -\frac{(M+dM)y}{I}$$

$$F_3 = F_2 - F_1 = \int_{y_1}^{h/2} \frac{(dM)y}{I} dA$$

Shear stresses in beams







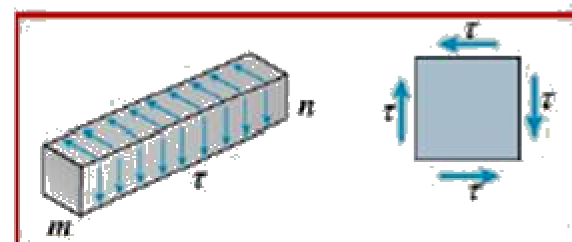
$$F_3 = F_2 - F_1 = \frac{dM}{I}\int_{y_1}^{h/2} y\,dA$$

$$\text{Shear Stress}: \tau = \tau_{ave} = \frac{F_3}{\text{Bottom Area of the sub-element}}$$

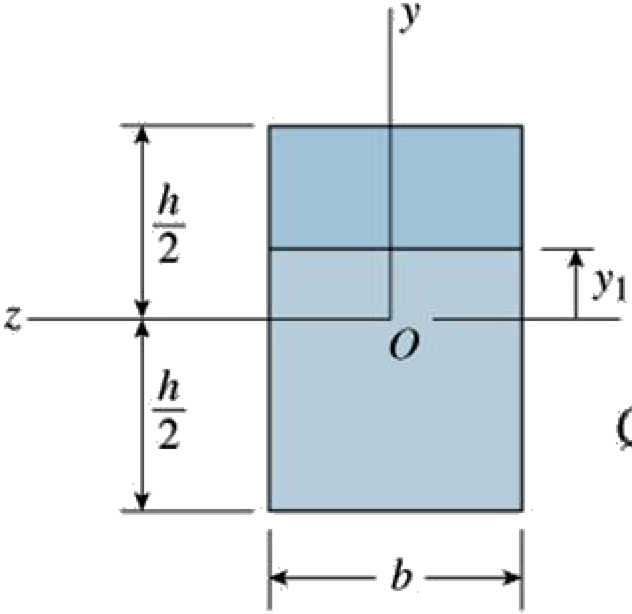
$$\tau_{ave} = \frac{F_3}{bdx} = \left(\frac{dM}{dx}\right)\frac{1}{bI}\int_{y_1}^{h/2} y\,dA = \frac{V}{bI}\int_{y_1}^{h/2} y\,dA$$

$$\text{Let } Q = \int_{y_1}^{h/2} y\,dA$$

$$\boxed{\tau = \frac{VQ}{bI}}$$

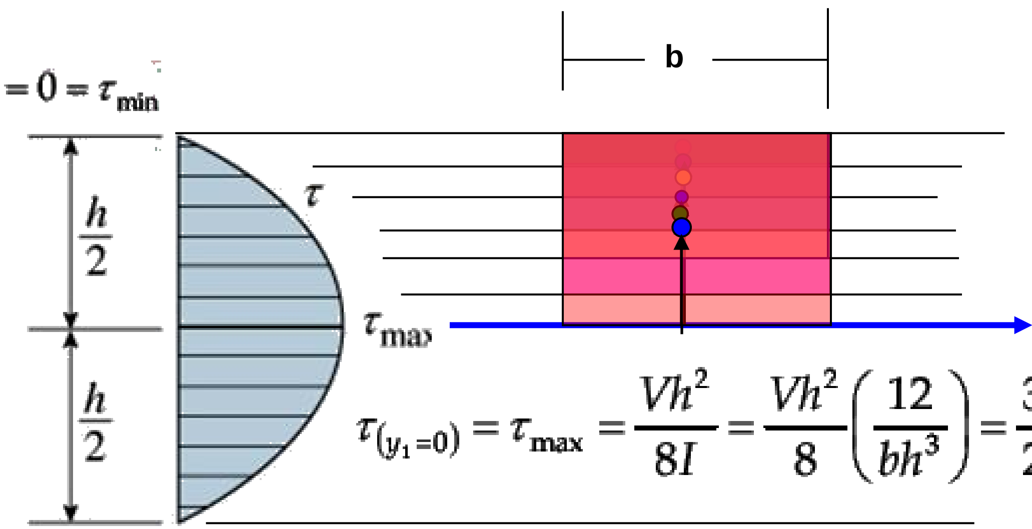


Shear Stresses in Rectangular Beams



$$\tau = \frac{VQ}{bI}$$

$$Q = \int_{y_1}^{h/2} y\,dA = \text{First moment of } A_1 \text{ w.r.t the z-axis}$$

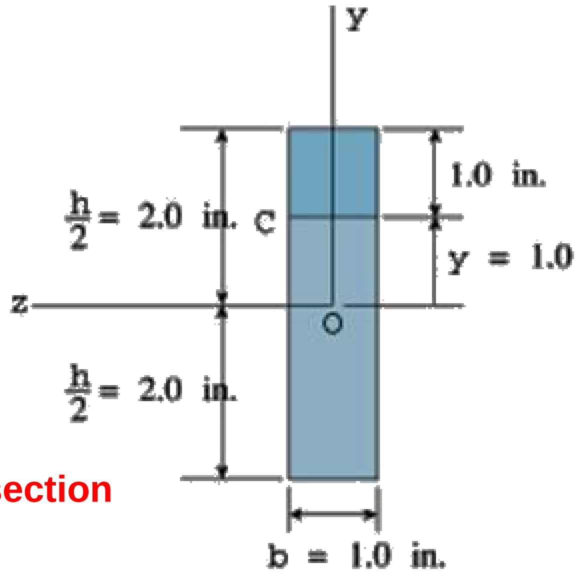


$$\tau_{\left(y_1=\pm\frac{h}{2}\right)} = 0 = \tau_{\min}$$

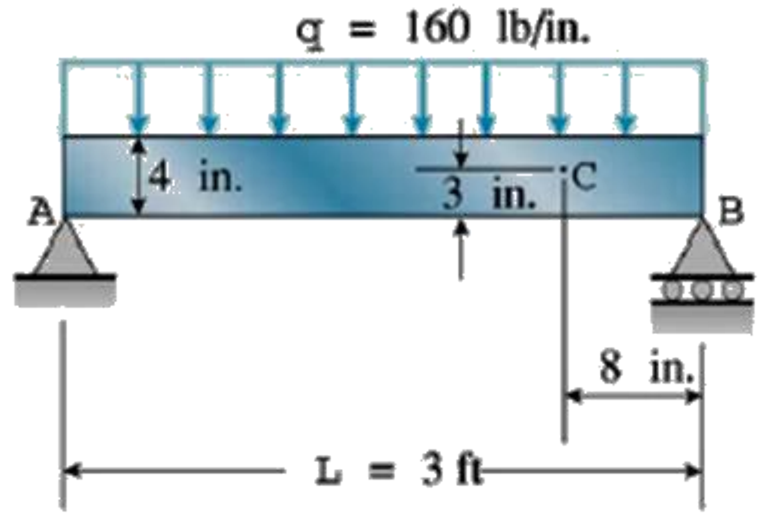
$$\tau_{(y_1=0)} = \tau_{\max} = \frac{Vh^2}{8I} = \frac{Vh^2}{8}\left(\frac{12}{bh^3}\right) = \frac{3}{2}\frac{V}{bh} = 1.5\tau_{ave}$$

Parabolic Distribution

Determine the normal and shear stresses at Point C



Cross section



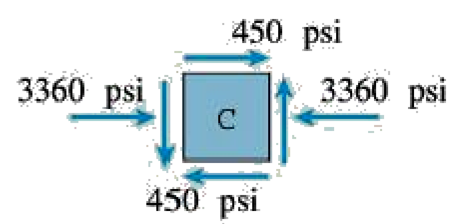
$$I = \frac{bh^3}{12} = 5.333\ \text{in}^4$$

$$Q_C = A_C \bar{y}_C$$
$$= (1.0)(1.5) = 1.5\ \text{in}^3$$

$$M_C = 17{,}920\ \text{lb - in}$$
$$V_C = -1{,}600\ \text{lb}$$

$$\sigma_C = -\frac{M_C y_C}{I}$$
$$= -\frac{(17{,}920)(1.0)}{5.333} = \text{-}3{,}360\ \text{psi}$$

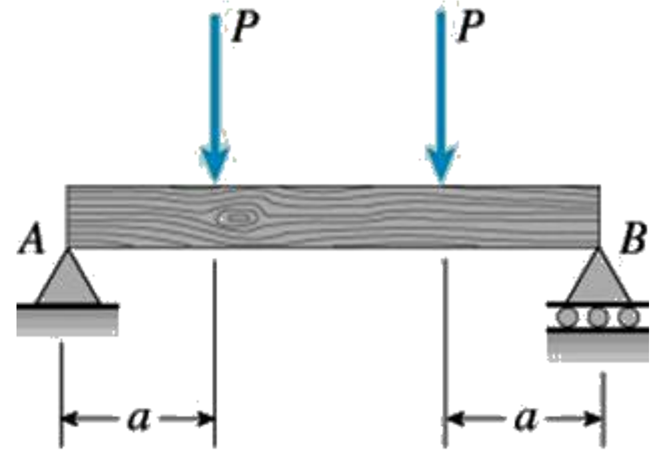
$$\tau_C = \frac{V_C Q_C}{Ib} = \frac{(1{,}600)(1.5)}{(5.333)(1.0)} = \text{-}450\ \text{psi}$$



what is the Maximum Permissible Load

$b = 100\text{ mm} \qquad h = 150\text{ mm}$

$\sigma_{allow} = 11\text{ MPa} \quad \tau_{allow} = 1.2\text{ MPa}$



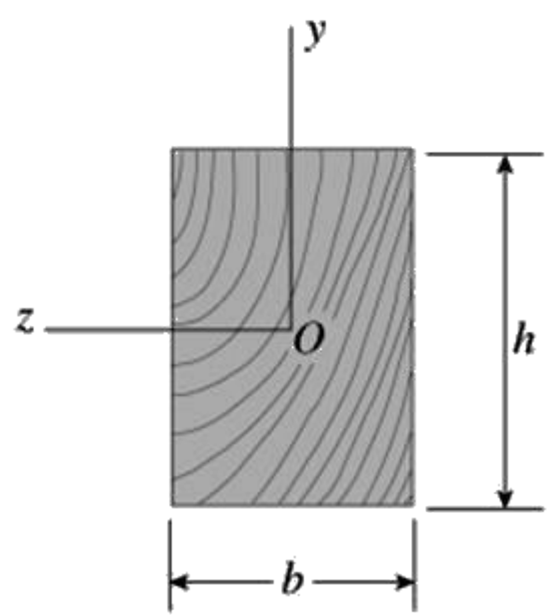
(a)

$$M_{max} = Pa, \quad V_{max} = P$$

$$S = \frac{I}{c} = \frac{bh^3/12}{h/2} = \frac{bh^2}{6}, \qquad A = bh$$

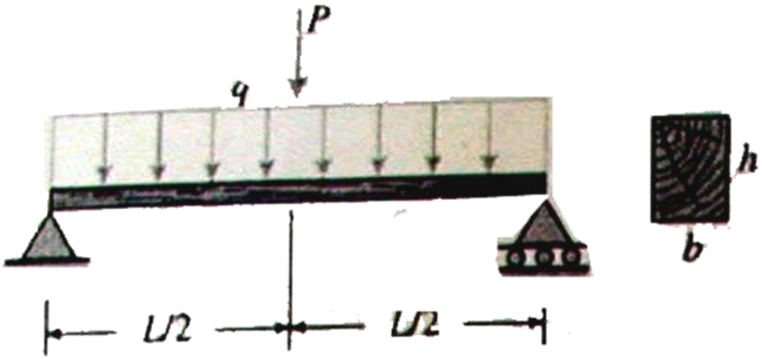
$$\sigma_{max} = \frac{M_{max}}{S} = \frac{6Pa}{bh^2} \quad \Rightarrow \quad P_{max}^{Bending} = \frac{\sigma_{allow} bh^2}{6a}$$

$$\tau_{max} = \frac{3V_{max}}{2A} = \frac{3P}{2bh} \quad \Rightarrow \quad P_{max}^{Shear} = \frac{2\tau_{allow} bh}{3}$$



(b)

$$P_{max}^{Bending} = 8.25\text{ kN}, \qquad P_{max}^{Shear} = 12.0\text{ kN},$$



$b = 140 \text{ mm} \quad h = 240 \text{ mm} \quad A = bh = 33{,}600 \text{ mm}^2$

$S = \frac{bh^2}{6} = 1344 \times 10^3 \text{ mm}^3$

$\gamma = 5.4 \text{ kN/m}^3$

$L = 1.2 \text{ m} \quad q = \gamma bh = 181.44 \text{ N/m}$

Allowable P= ?

(a) Allowable load P based upon bending stress

$\sigma_{allow} = 8.5 \text{ MPa} \quad \sigma = \frac{M_{max}}{S}$

$$M_{max} = \frac{PL}{4} + \frac{qL^2}{8} = \frac{P(1.2 \text{ m})}{4} + \frac{(181.44 \text{ N/m})(1.2 \text{ m})^2}{8}$$

$$= 0.3P + 32.66 \text{ N} \cdot \text{m} \quad (P = \text{newtons}; M = \text{N} \cdot \text{m})$$

$$M_{max} = S\sigma_{allow} = (1344 \times 10^3 \text{ mm}^3)(8.5 \text{ MPa}) = 11{,}424 \text{ N} \cdot \text{m}$$

Equate values of M_{max} and solve for P:

$0.3P + 32.66 = 11{,}424 \quad P = 37{,}970 \text{ N}$

or $P = 38.0 \text{ kN}$ ←

(b) Allowable load P based upon shear stress

$\tau_{allow} = 0.8 \text{ MPa} \quad \tau = \frac{3V}{2A}$

$$V = \frac{P}{2} + \frac{qL}{2} = \frac{P}{2} + \frac{(181.44 \text{ N/m})(1.2 \text{ m})}{2}$$

$$= \frac{P}{2} + 108.86 \text{ (N)}$$

$$V = \frac{2A\tau}{3} = \frac{2}{3}(33{,}600 \text{ mm}^2)(0.8 \text{ MPa}) = 17{,}920 \text{ N}$$

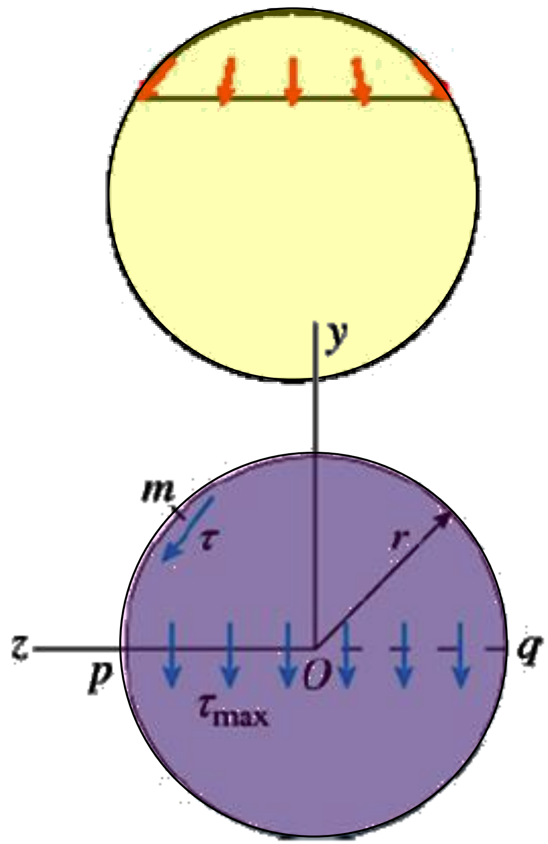
Equate values of V and solve for P:

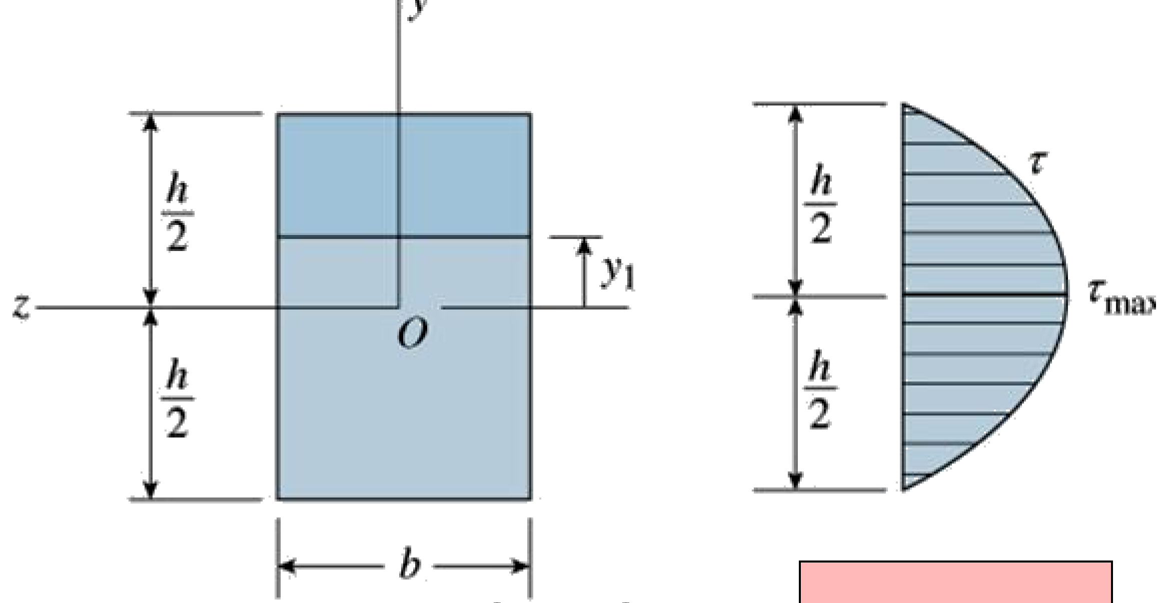
$\frac{P}{2} + 108.86 = 17{,}920 \quad P = 35{,}622 \text{ N}$

or $P = 35.6 \text{ kN}$ ←

Note: The shear stress governs and $P_{allow} = 35.6 \text{ kN}$

Shear Stresses in beams of Circular cross section





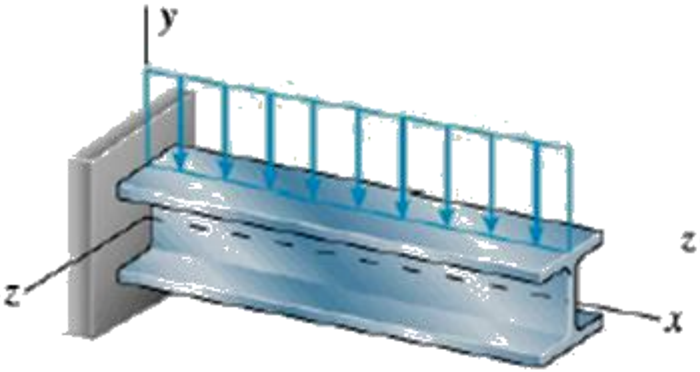
$$\tau_{(y_1=0)} = \tau_{max} = \frac{Vh^2}{8I} = \frac{Vh^2}{8}\left(\frac{12}{bh^3}\right) = \frac{3}{2}\frac{V}{bh} = 1.5\tau_{ave}$$

$$I = \frac{\pi r^4}{4}, \quad Q = A\bar{y} = \left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right) = \frac{2r^3}{3}$$

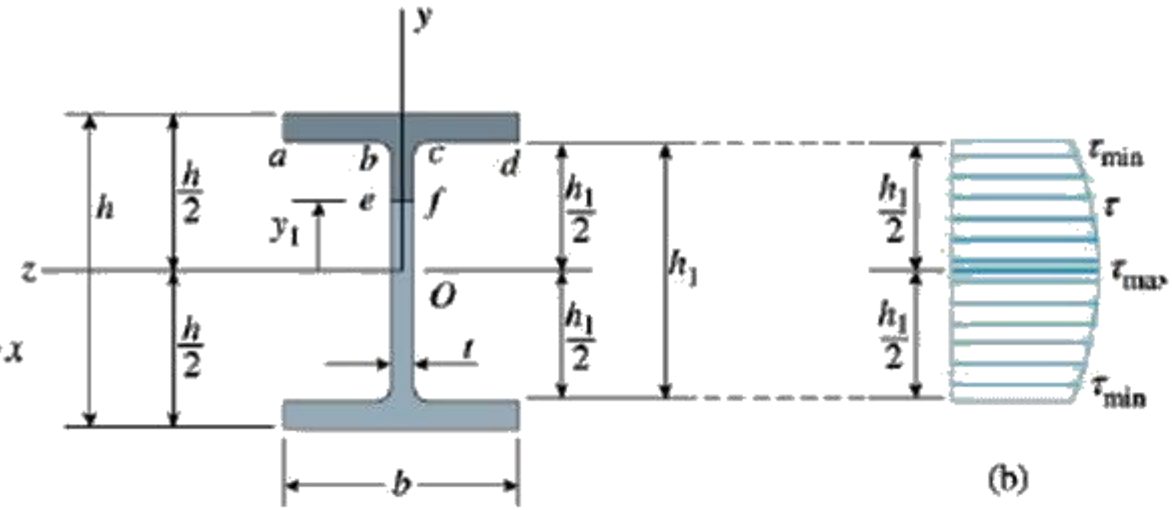
$$\tau_{max} = \frac{VQ}{bI} = \frac{4V}{3\pi r^2} = \frac{4V}{3A} = 1.33\tau_{ave}$$

The exact distribution of shear stress in a beam of circular cross section is very complicated and only that along the neutral axis can be determined relatively easily.

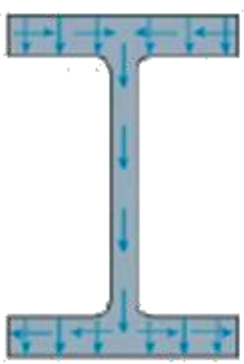
Shear Stresses in the Webs of Beams with Flanges



(a)



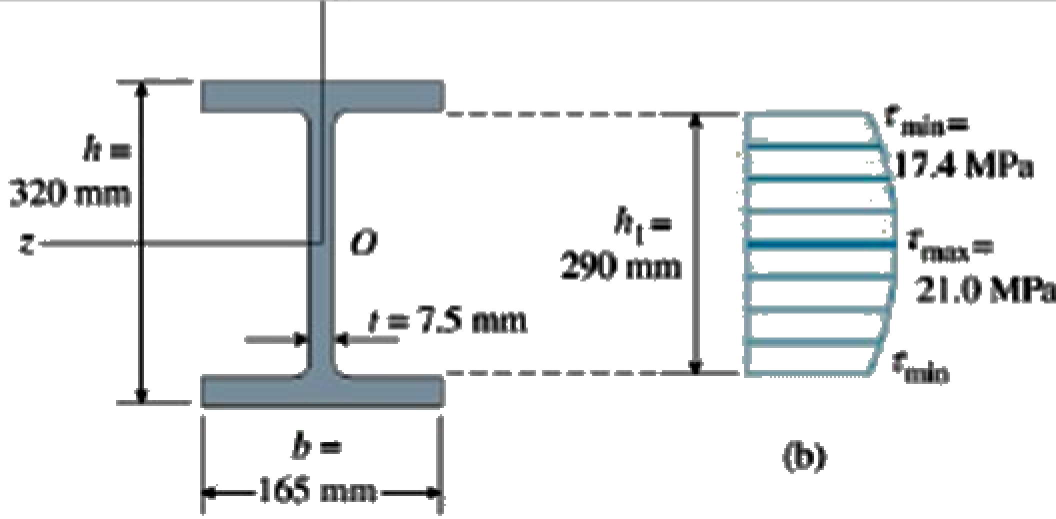
(a)



(b)

$$\tau_{web} = \frac{VQ}{It} = \frac{V}{8It}\left[b\left(h^2 - h_1^2\right) + t\left(h_1^2 - 4y_1^2\right)\right]$$

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = \frac{1}{12}\left(bh^3 - bh_1^3 + th_1^3\right)$$



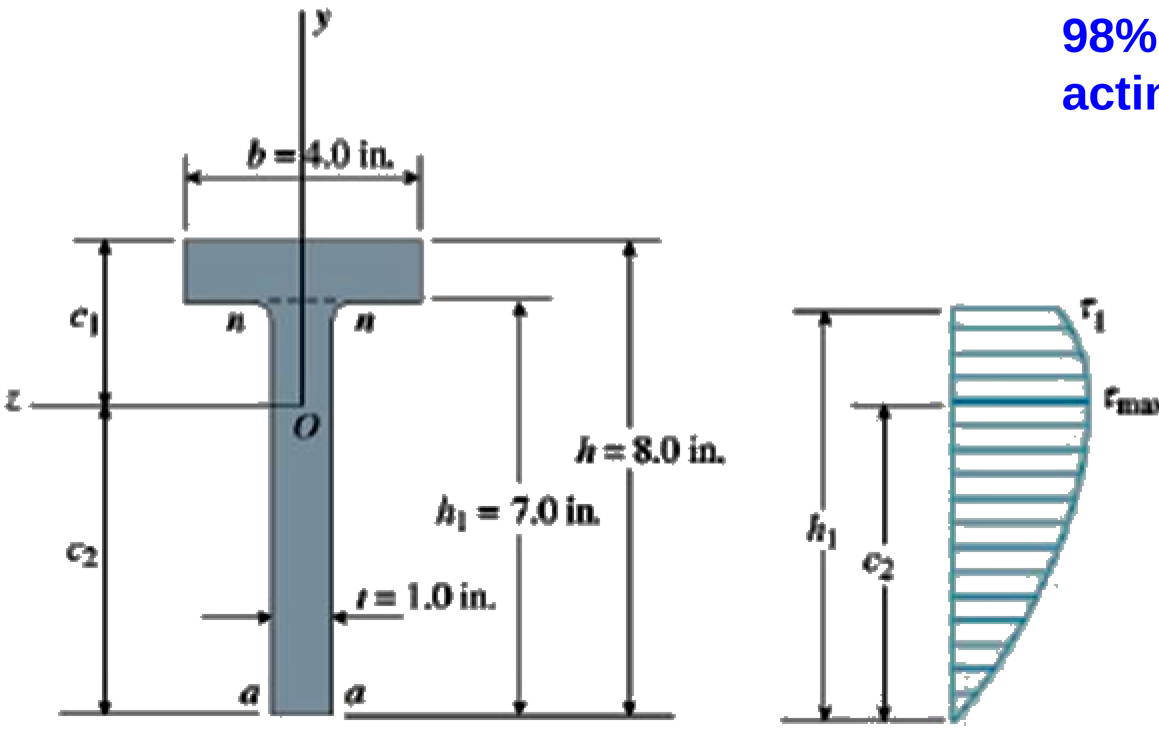
(b)

Shear Force in the Web: = (The are of he shear stress diagram x the thickness of the web)

$$V = [\tau_{min} h_1 + 2/3\, h_1 (\tau_{max} - \tau_{min})]t$$

$$= t\, h_1/3\, (2\tau_{max} + \tau_{min})$$

Shear force in the web is 90% - 98% of the total shear force V acting on the cross section



(a) (b)

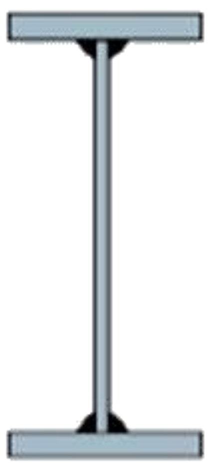
Assuming the web carries all of the shear force…

$$\tau_{avg} = V / th_1$$

Built-Up Beams and Shear Flow

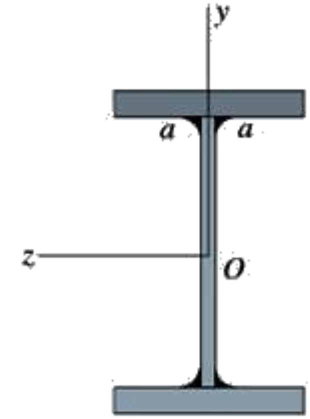
(a)

(b)

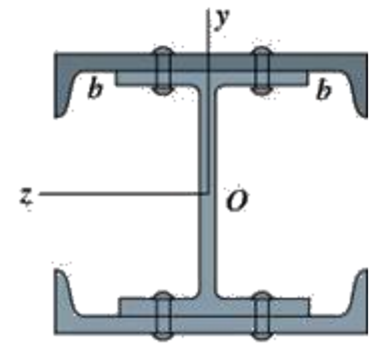


(c)

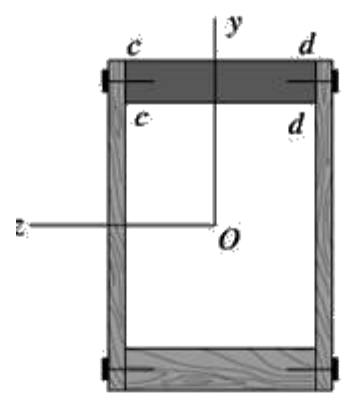
$$f = \frac{VQ}{I}$$



(a)

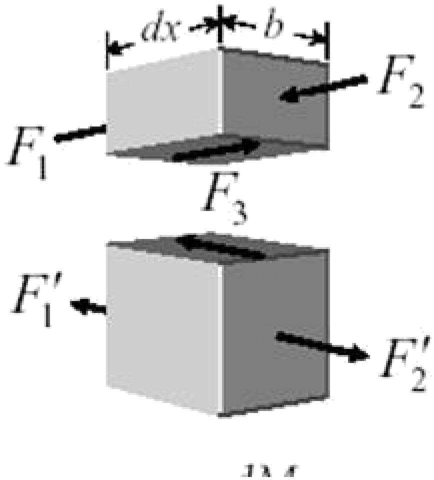


(b)

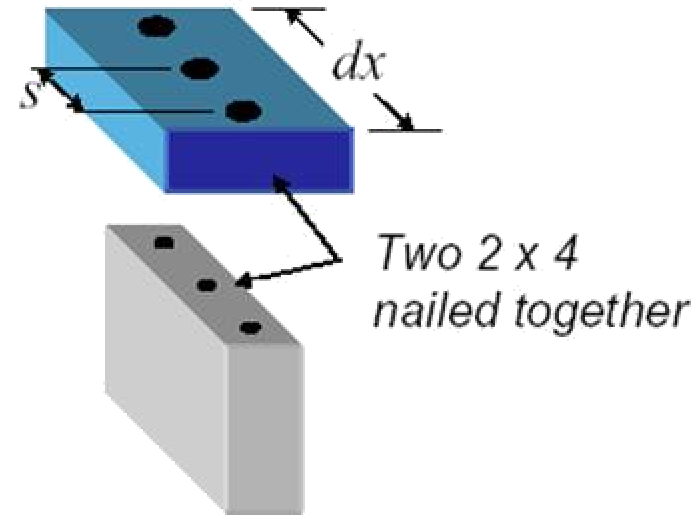


(c)

Shear Flow



Built-Up Beams



$$\text{Shear Force}: \; F_3 = \frac{dM}{I}\int y\,dA$$

$$\text{Shear Stress}: \; \tau = \frac{F_3}{b\,dx} = \frac{dM}{dx\,bI}\int y\,dA = \frac{VQ}{bI}$$

$$\text{Shear Flow}: \; f = \frac{F_3}{dx} = \frac{dM}{dx}\frac{1}{I}\int y\,dA = \frac{VQ}{I}$$

n = Number of rows of nails

F = Strength of each nail

$$f = \frac{nF}{s} = \frac{\text{Shear force provided by nails}}{\text{Nail Spacing}}$$

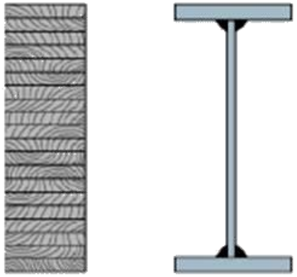
$$\text{Nail Spacing}: \; s = \frac{nF}{f}$$

Shear Force in Fasteners :

In many applications, beam sections consist of several pieces of material that are attached together in a number ways: bolts, rivets, nails, glue, weld, etc. In such so called built-up sections we are interested in knowing the amount of shear stress and the resulting shear force at the cross section of fasteners or over the glued surface .

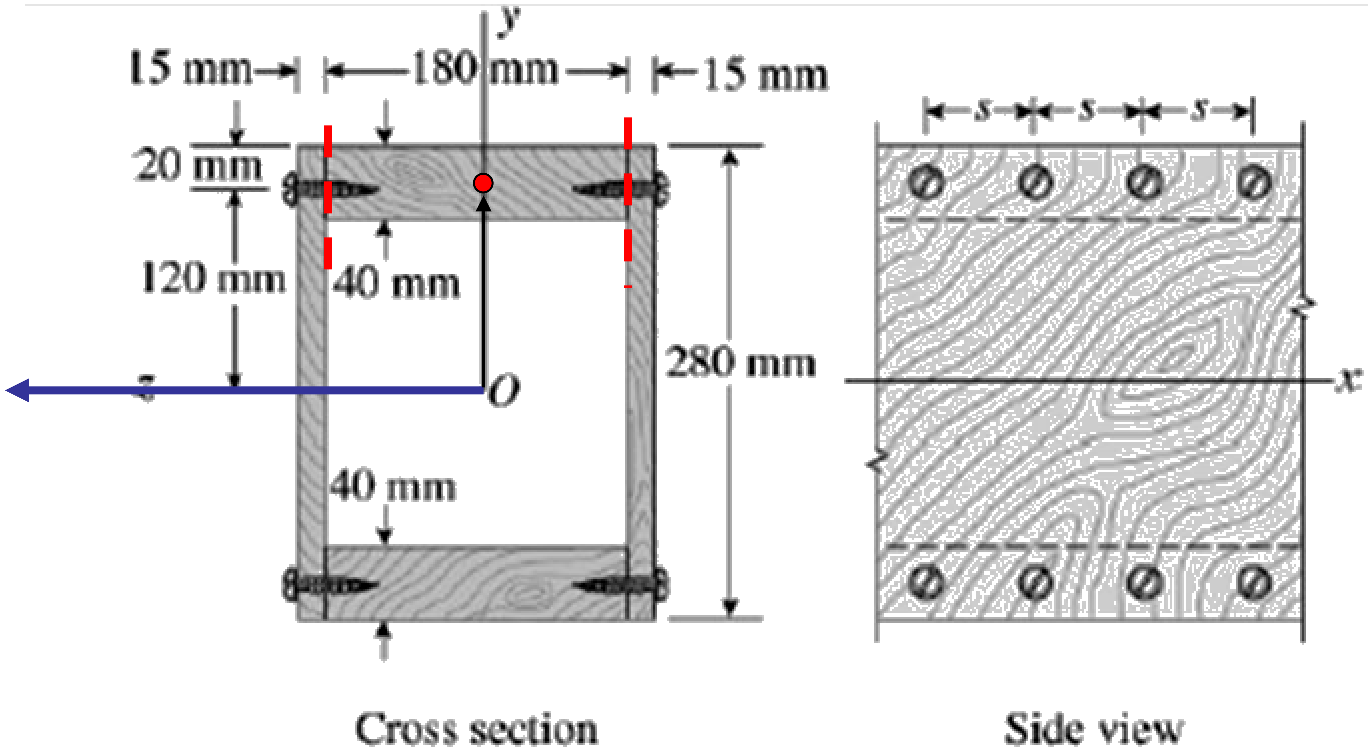
(a)

$$f = \frac{VQ}{I}$$



(b) (c)

Ex. 5-16 The plywood is fastened to the flanges by wood screws having an allowable load in shear of F = 800 N each if the shear force V acting on the cross section = 10.5 kN.
Determine the max. permissible longitudinal spacing s of the screws.



$$I = \frac{(210)(280)^3}{12} - \frac{(180)(200)^3}{12}$$
$$= 264.2 \times 10^6 \text{ mm}^4$$

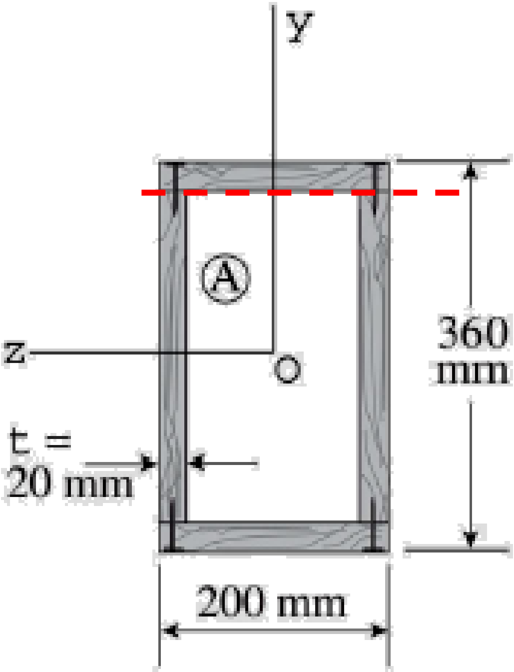
$$Q = (180)(40)(140-20)$$
$$= 864 \times 10^3 \text{ mm}^3$$

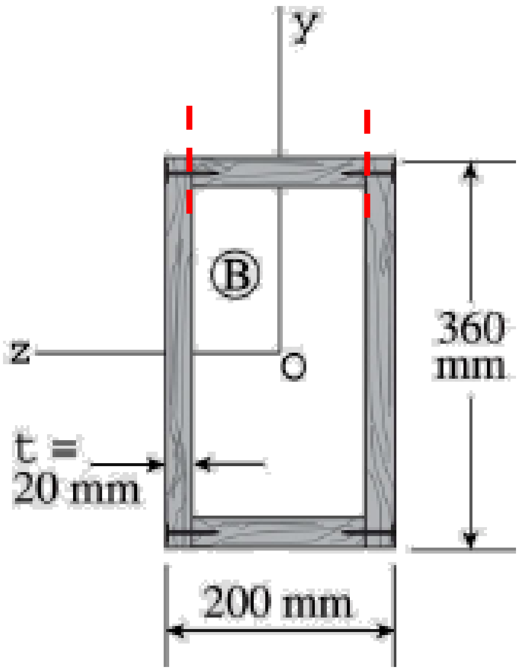
$$f = \frac{VQ}{I} = \frac{(10.5 \times 10^3)(864 \times 10^3)}{264.16 \times 10^6} = 34.3 \frac{\text{N}}{\text{mm}}$$

$$s = \frac{2F}{f} = \frac{2(800)}{34.3} = 46.6 \text{ mm}$$

Use 45 mm

Find the spacing for each case





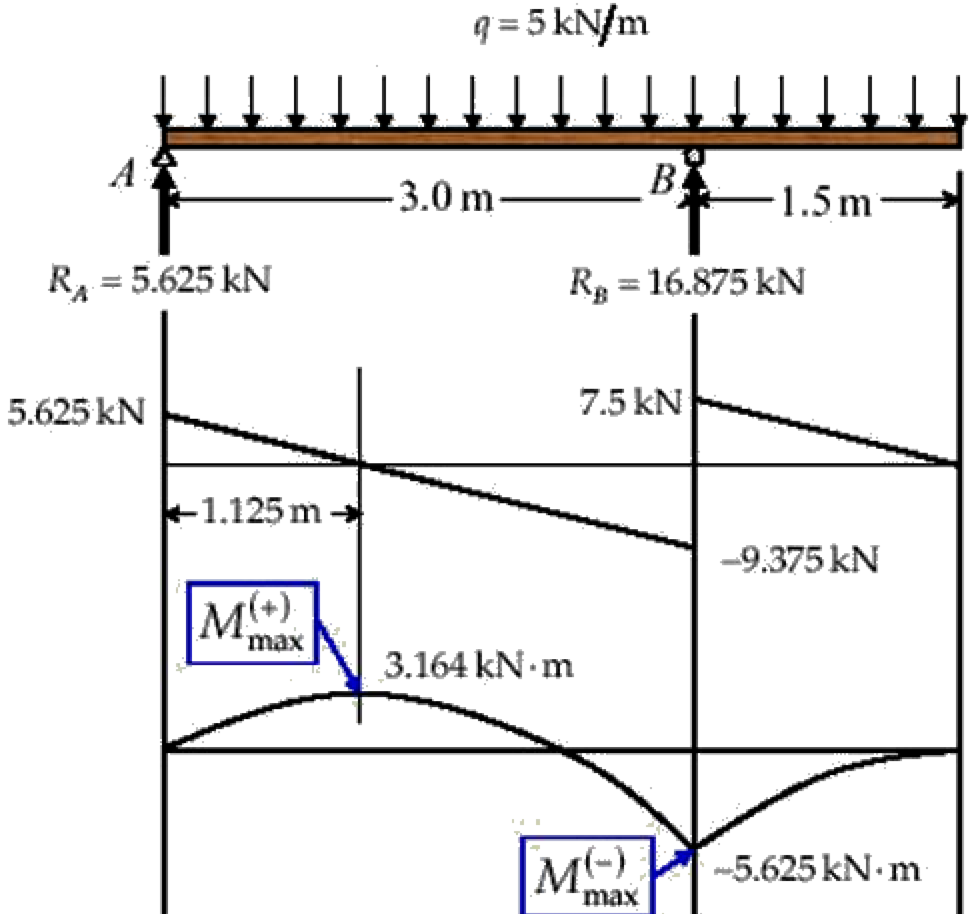
$V = 3.2\text{ kN} \qquad F = 250\text{ N}$

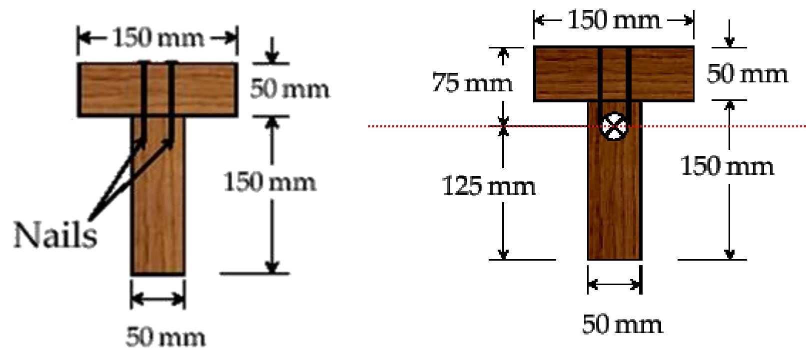
$$I = \frac{(200)(360)^3}{12} - \frac{(160)(320)^3}{12}$$
$$= 340.69\times10^6\text{ mm}^4$$

$$Q = (200)(20)(180-10)$$
$$= 680\times10^3\text{ mm}^3$$
$$s = \frac{2FI}{VQ} = 78.3\text{ mm}$$

$$Q = (160)(20)(180-10)$$
$$= 544\times10^3\text{ mm}^3$$
$$s = \frac{2FI}{VQ} = 97.9\text{ mm}$$

Will the beam be able to support the load if the allowable normal stress of southern pine is 15 MPa? (b) What is the maximum nail spacing if the allowable shear load of each nail is 2,000 N?





$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2}$$

$$= \frac{(150\times 50)(25)+(50\times 150)(50+75)}{(150\times 50)+(50\times 150)} = 75\text{ mm}$$

$$I = 53.125\times 10^6\ \text{mm}^4$$

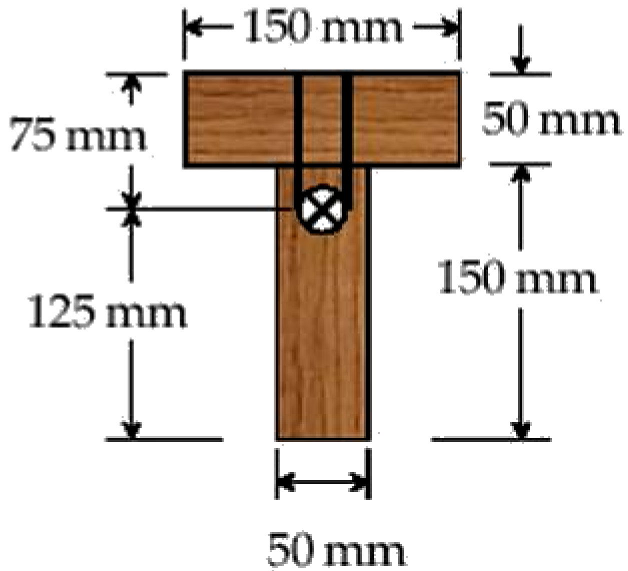
$$\sigma_t = \frac{M^+_{max}}{S_2} = \frac{3{,}164}{425.0\times 10^{-6}} = 7.44\ \text{MPa}$$

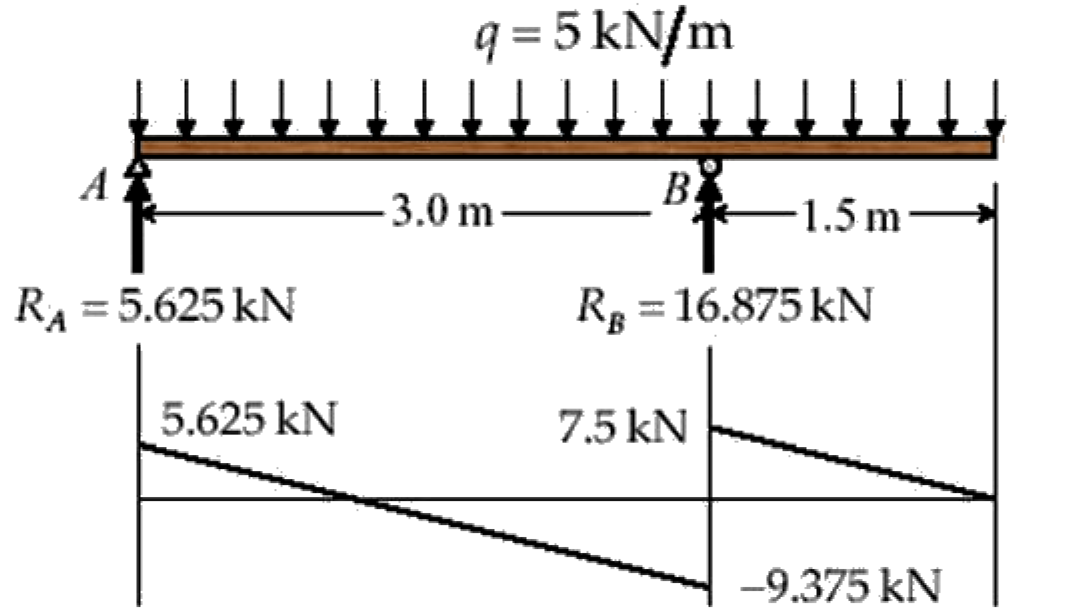
$$\sigma_c = -\frac{M^+_{max}}{S_1} = -\frac{3{,}164}{708.3\times 10^{-6}} = -4.47\ \text{MPa}$$

$$\sigma_t = -\frac{M^-_{max}}{S_1} = -\frac{-5.625}{708.3\times 10^{-6}} = 7.94\ \text{MPa}$$

$$\sigma_c = \frac{M^-_{max}}{S_2} = \frac{-5.625}{425.0\times 10^{-6}} = -13.24\ \text{MPa}$$

Maximum Nail Spacing





Nails are to resist the shear flow between the two 2 x 6

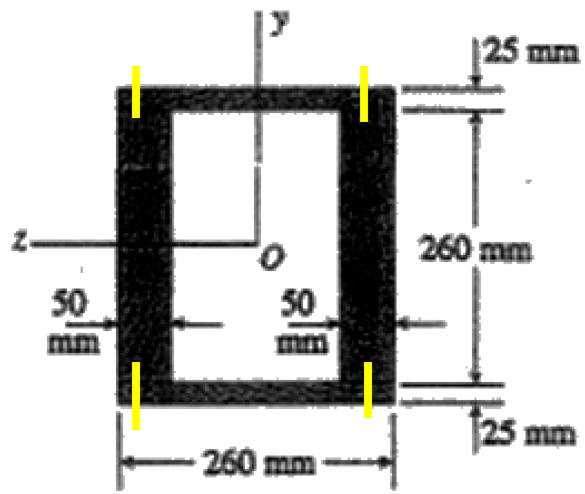
$$Q = (150)(50)(75-25)$$
$$= 375.0\times10^{3}\ \text{mm}^{3}$$

$$f = \frac{VQ}{I} = \frac{(9{,}375)(375.0\times10^{-6})}{53.125\times10^{-6}} = 66.18\times10^{3}\ \frac{\text{N}}{\text{m}}$$

$$s = \frac{2F}{f} = \frac{2\times2{,}000}{66.18\times10^{3}} = 0.0604\ \text{m} = 60.4\ \text{mm}$$

A box beam of wood is constructed of two 260 mm × 50 mm boards and two 260 mm × 25 mm boards (see figure). The boards are nailed at a longitudinal spacing $s = 100$ mm.

If each nail has an allowable shear force $F = 1200$ N, what is the maximum allowable shear force V_{max}?



All dimensions in millimeters.

$b = 260 \quad b_1 = 260 - 2(50) = 160$

$h = 310 \quad h_1 = 260$

s = nail spacing = 100 mm

F = allowable shear force for one nail = 1200 N

f = shear flow between one flange and both webs

$$f_{allow} = \frac{2F}{s} = \frac{2(1200\text{ N})}{100\text{ mm}} = 24\text{ kN/m}$$

$$f = \frac{VQ}{I} \qquad V_{max} = \frac{f_{allow} I}{Q}$$

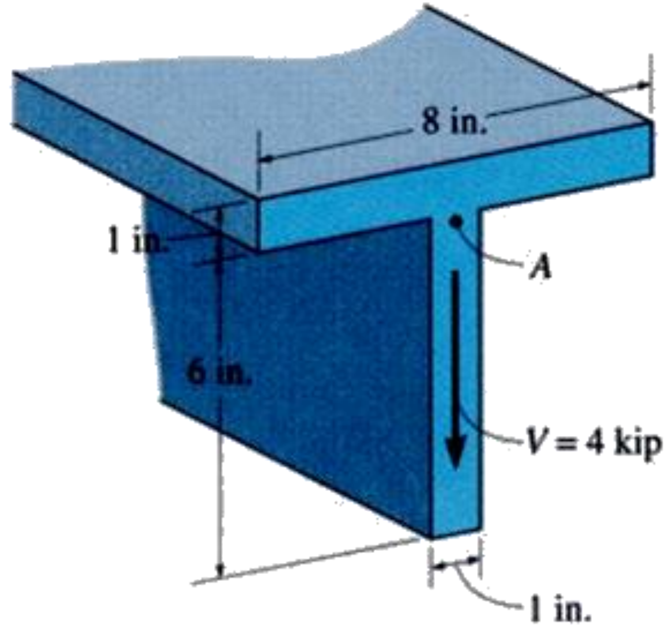
$$I = \frac{1}{12}(bh^3 - b_1 h_1^3) = 411.125 \times 10^6\text{ mm}^4$$

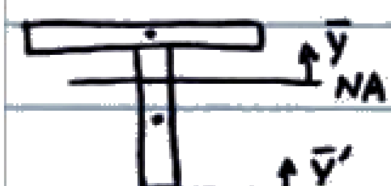
$$Q = Q_{flange} = A_f d_f = (260)(25)(142.5) = 926.25 \times 10^3\text{ mm}^3$$

$$V_{max} = \frac{f_{allow} I}{Q} = \frac{(24\text{ kN/m})(411.125 \times 10^6\text{ mm}^4)}{926.25 \times 10^3\text{ mm}^3}$$

$$= 10.7\text{ kN} \quad \leftarrow$$

4. Determine the shear stress in the beam at point *A*, which is located at the top of the web.



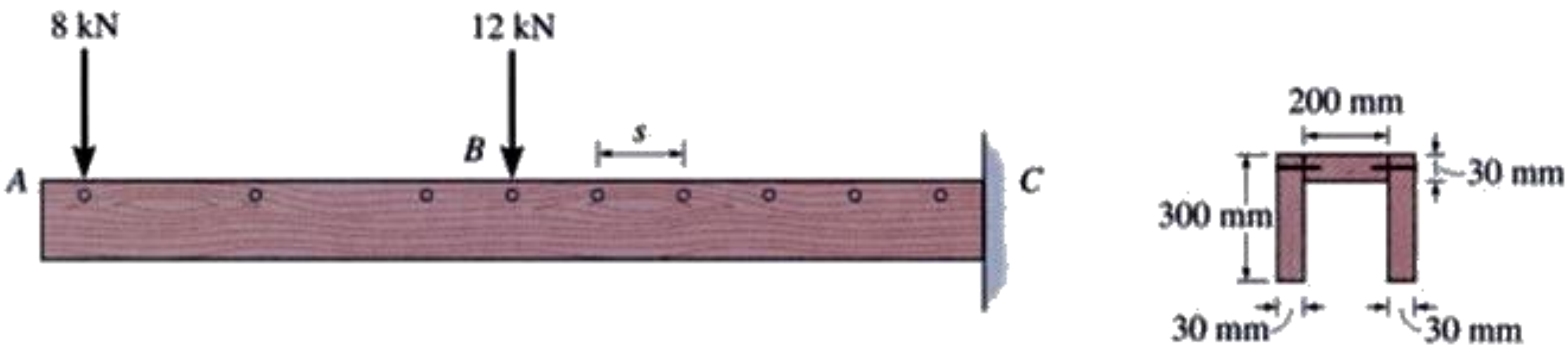


$$\bar{y}' = \frac{3(1)(6) + 6.5(1)(8)}{(1)(6) + (1)(8)} = 5''$$

$$I = \frac{(1)(6)^3}{12} + 2^2(1)(6) + \frac{8(1)^3}{12} + 1.5^2(8)(1) = 60.67\ in^4$$

$$\tau = \frac{4000\,(1.5)(8)(1)}{60.67\,(1)} = \boxed{791.2\ \text{psi}}$$

3. The beam is constructed from three boards as shown. If each nail can support a shear force of 300 N, determine the maximum spacing *s* of the nails within region *BC*.



$$\bar{y} = \frac{150(30)(300)(2) + 285(200)(30)}{30(300)(2) + 200(30)} = 183.75 \text{ mm from bottom}$$

$$I = \left[\frac{.03(.3)^3}{12} + (.18375 - .15)^2(.30)(.03)\right](2) + \frac{.2(.03)^3}{12} + (.285 - .18375)^2(.2)(.03)$$

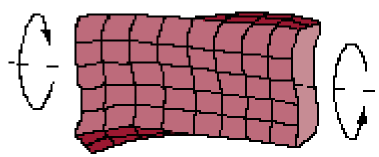
$$q = \frac{20{,}000\,(.285 - .18375)(.2)(.03)}{0.0002175} = \frac{2(300)}{s}$$

$$\boxed{s = 10.74 \text{ mm}}$$

Engineering Mechanics

Statics & Strength of Materials

Torque



Eng. Iqbal Marie

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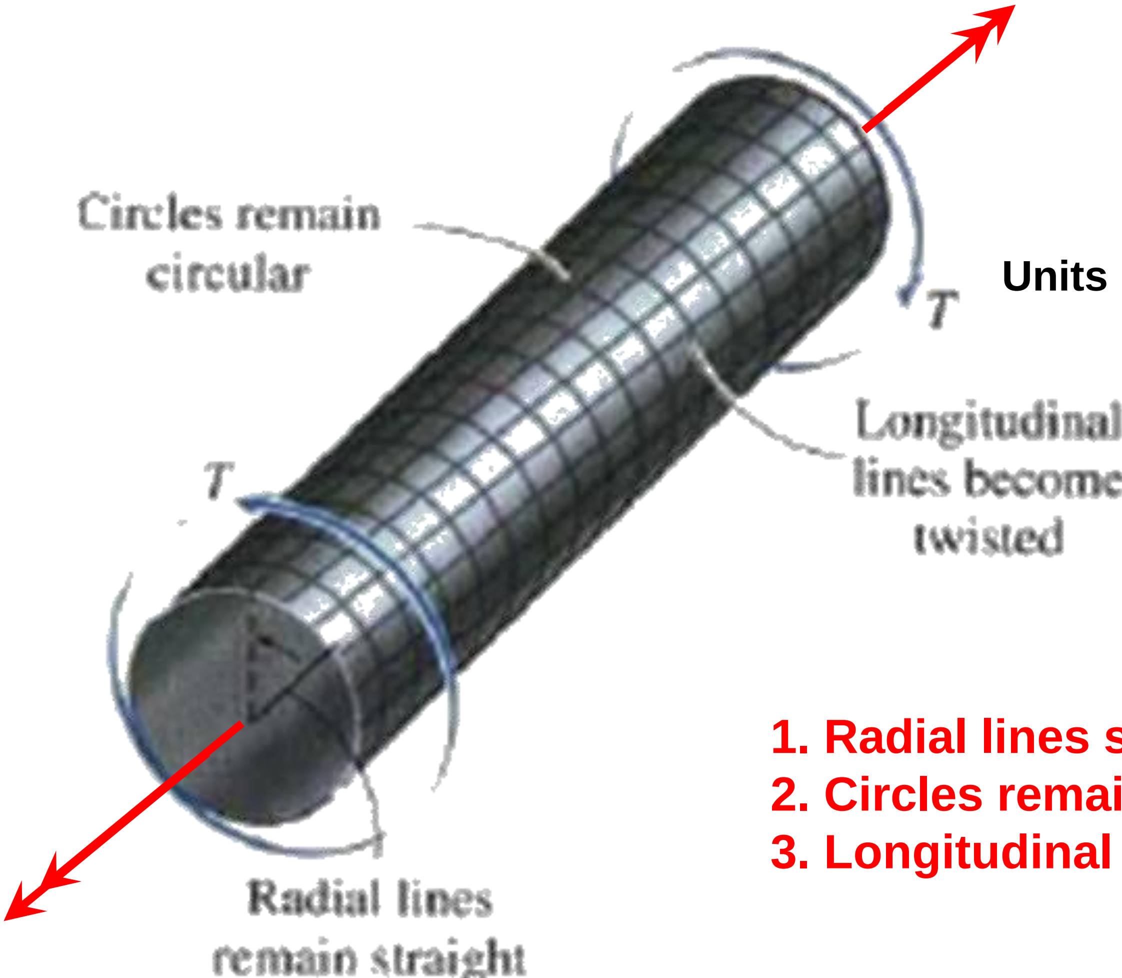
Torsion Deformation of a Circular Shaft

Torque is a moment that twists a member about its longitudinal axis.

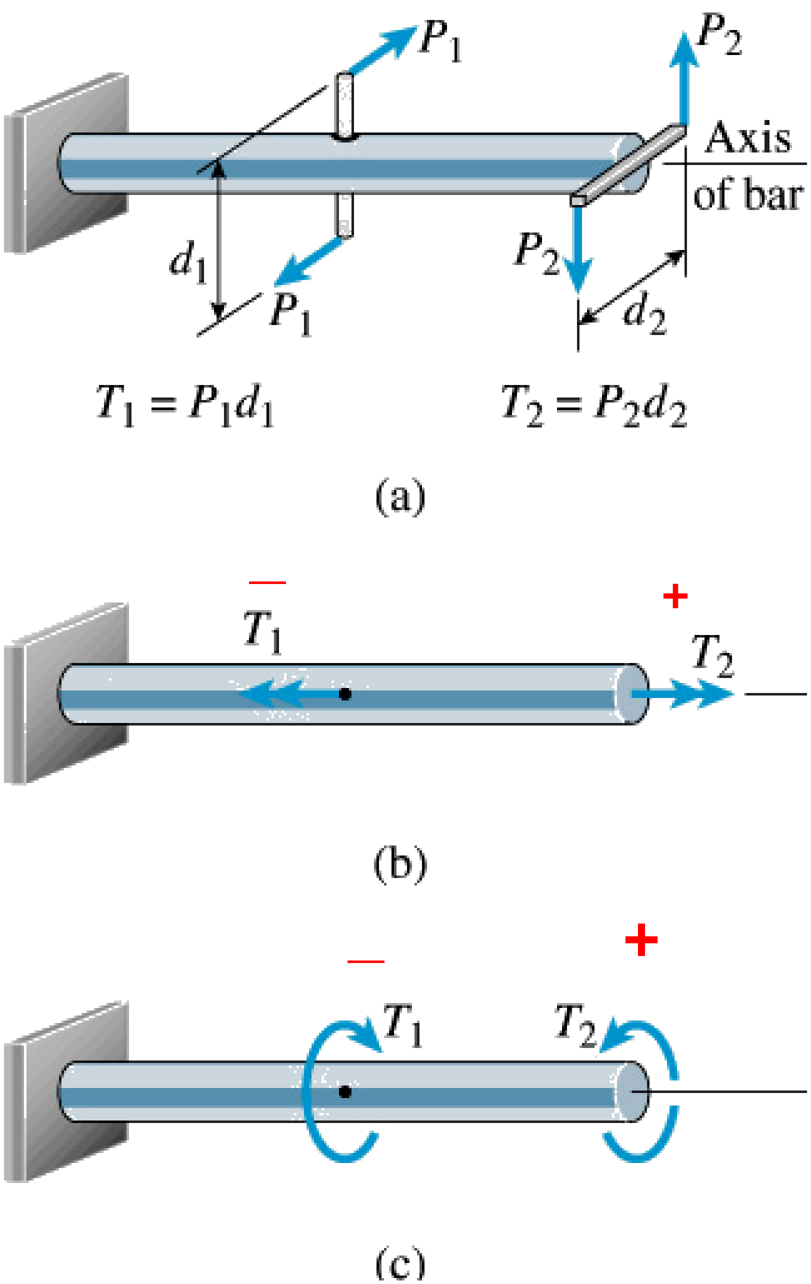
Angle of twist (Φ) is defined as the rotation of a radial line from a fixed end to a cross section some distance x from the end.



Units N.m or lb.ft

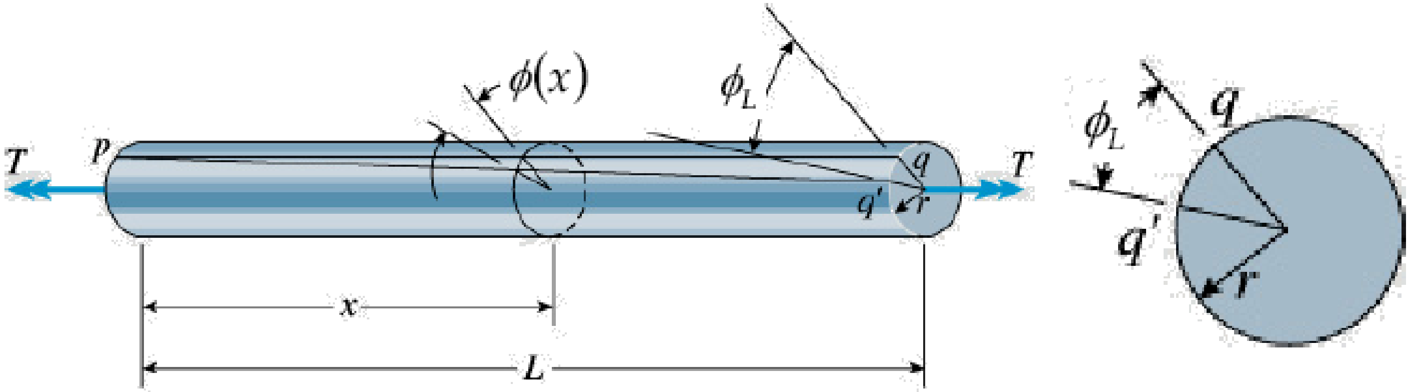
1. **Radial lines stay straight**
2. **Circles remain circular**
3. **Longitudinal lines twist into helix.**

Bar subjected to torsion by torques *T1* and *T2.*

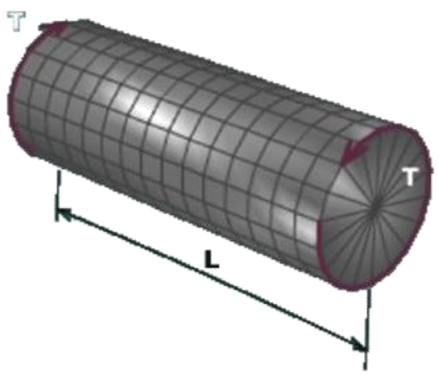


Sign Convention:

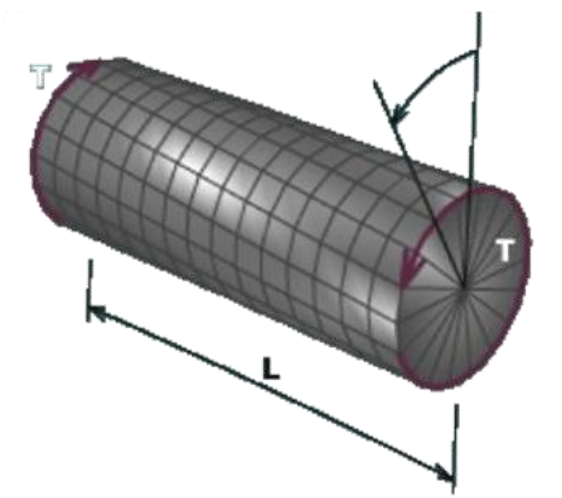
Right-Hand Rule



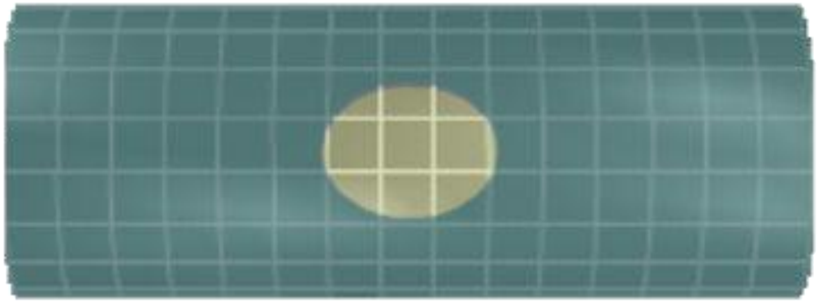
Limitations

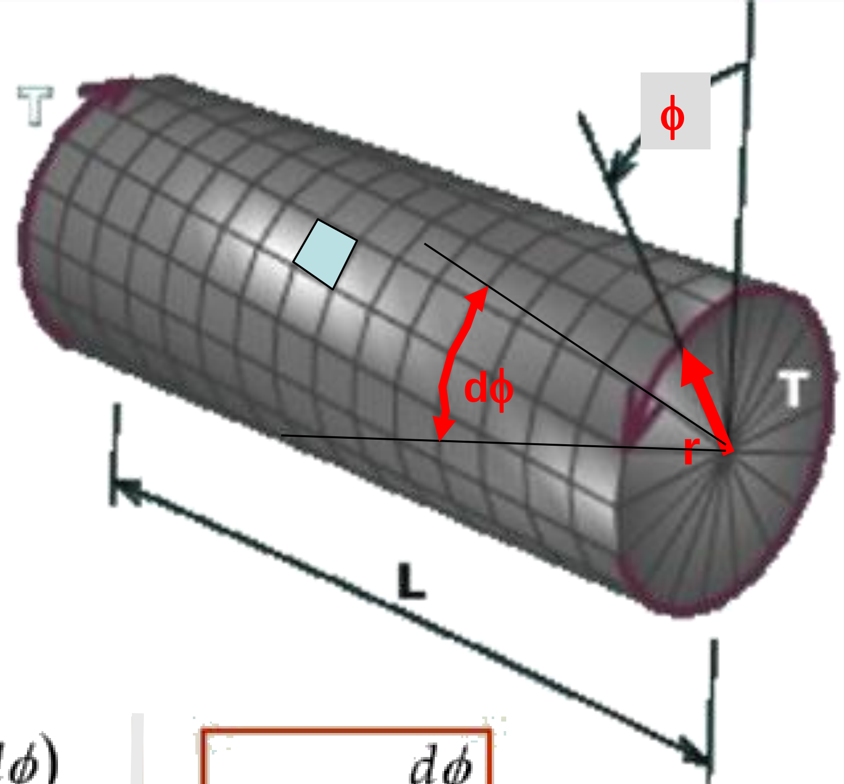


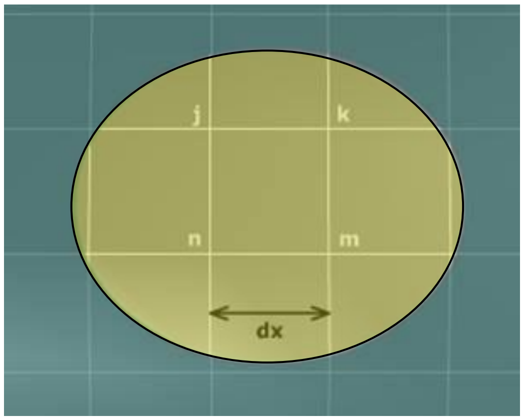
1. **The longitudinal axis of the shaft remains straight**
2. **The shaft does not increase or decrease in length**
3. **Radial lines remain straight and radial as the cross section rotates**
4. **Cross sections rotate about the axis of the member**



The right end will rotate with respect to the left end of the bar. The angle of rotation = Angle of twist ϕ. It changes along the length L of the bar linearly.



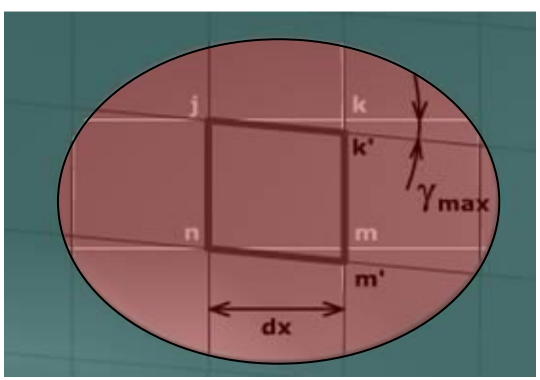




Before torque application

$$kk' = (r)(d\phi) = (dx)(\gamma_{max})$$

$$\gamma_{max} = r\frac{d\phi}{dx}$$



after torque application

θ: Rate of Twist : = *angle of twist per unit length*

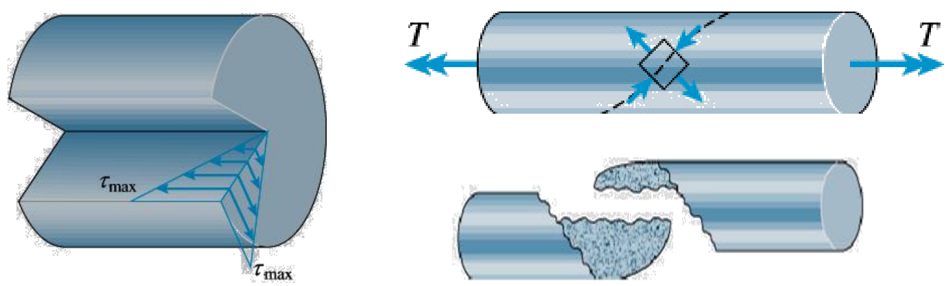
$$\theta = d\phi/dx$$

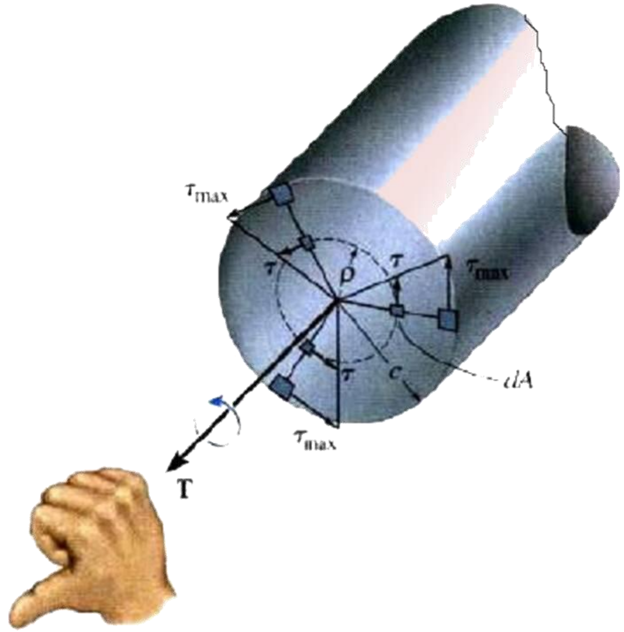
$$\gamma_{max} = r\theta$$

If ϕ is linear, then $\theta = \phi_L/L$

For pure torsion $\gamma_{max} = r\,\phi/L$

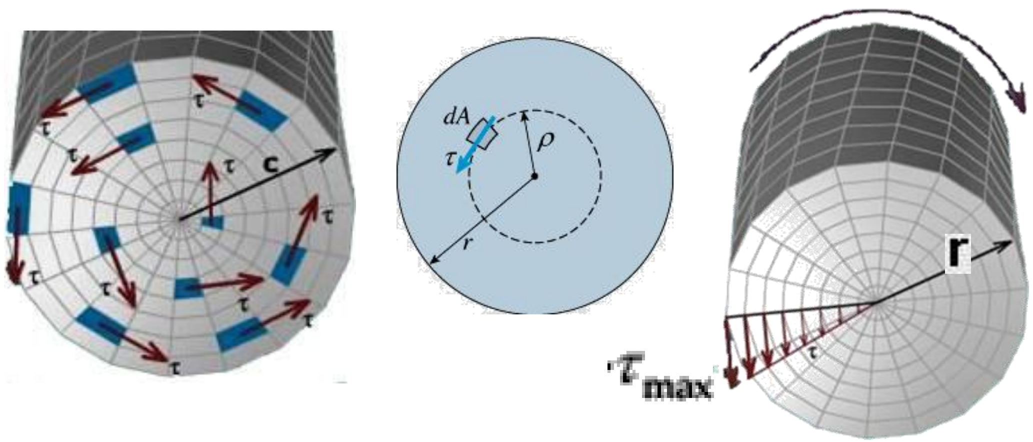
The Torsion Formula





Tensile and compressive stresses acting on a stress element oriented at 45° to the longitudinal axis.

Determination of the resultant of the shear stresses acting on a cross section.



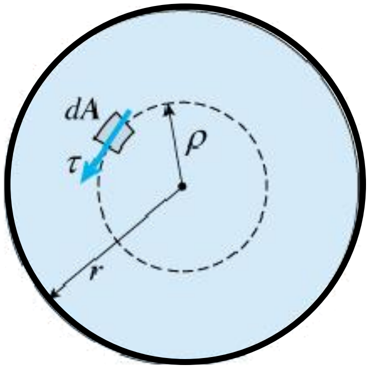
$$dM = (\rho)(\tau)dA = \rho^2 \frac{\tau_{max}}{r} dA$$

$$T = \int_A dM = \frac{\tau_{max}}{r} \int_A \rho^2 dA = \frac{\tau_{max}}{r} I_P$$

$$\tau_{max} = \frac{Tr}{I_P}, \qquad \tau = \frac{T\rho}{I_P}$$

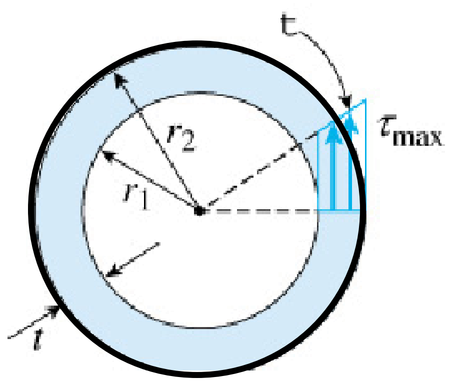
$$I_P = \int_A \rho^2 dA = \text{Polar Moment of Inertia}$$

For Circular Bars



$$I_P = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$$

For Circular Tubes



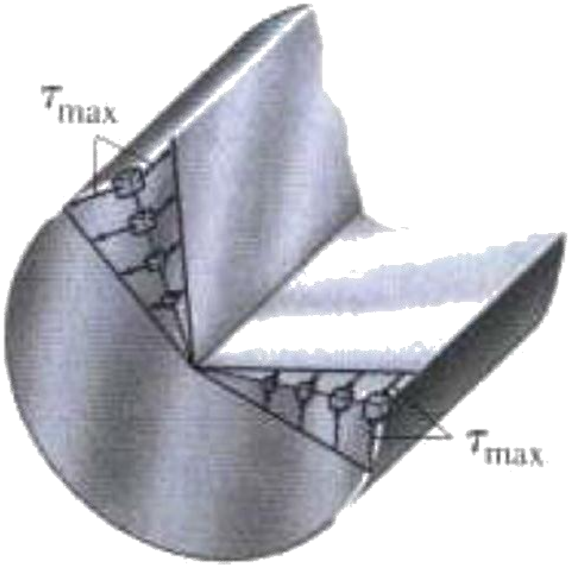
$$I_P = \frac{\pi}{2}\left(r_2^4 - r_1^4\right) = \frac{\pi}{32}\left(d_2^4 - d_1^4\right)$$

$$= \frac{\pi r t}{2}\left(4r^2 + t^2\right) = \frac{\pi d t}{4}\left(d^2 + t^2\right)$$

where $r = \dfrac{r_1 + r_2}{2}$ and $d = \dfrac{d_1 + d_2}{2}$

If $d_1 \approx d_2$, i.e., $t << d$, then

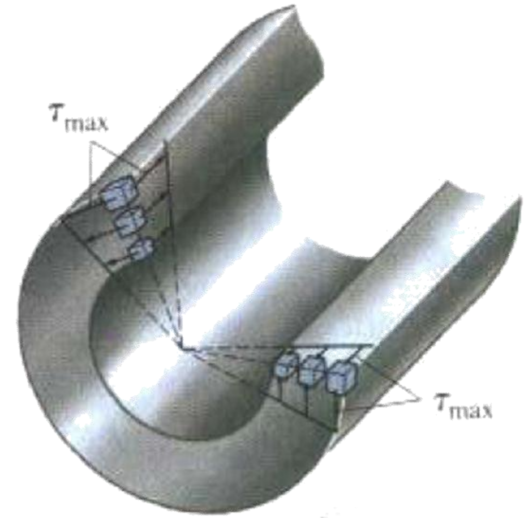
$$I_P \approx 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

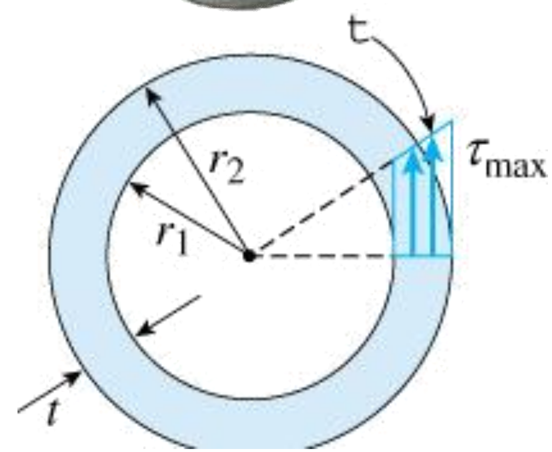


$$\tau = \frac{T\rho}{I_P}, \quad \tau_{max} = \frac{Tr}{I_P}$$

$$\phi = \frac{TL}{GI_P}$$

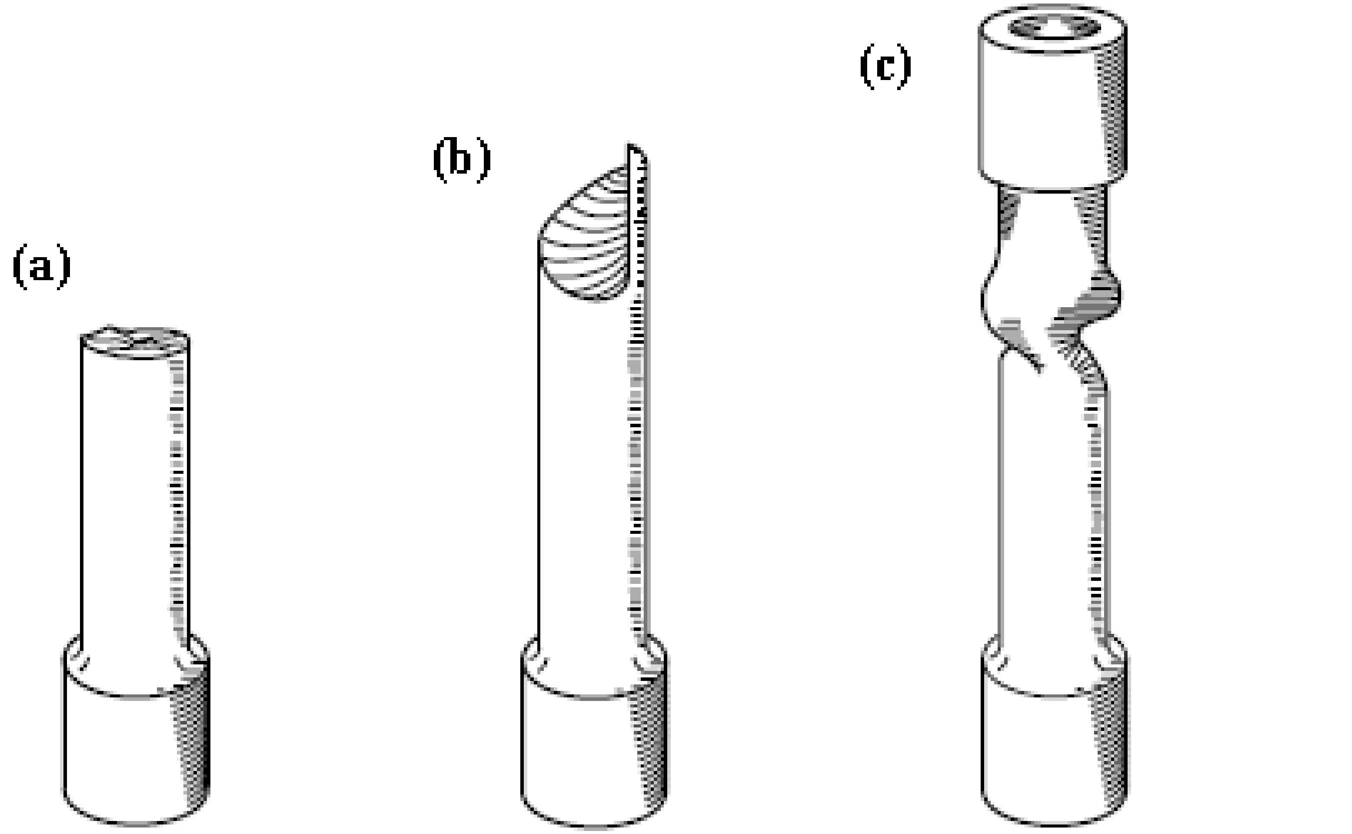
Circular tube in torsion.





$$\tau_{max} = \frac{T\rho}{J}$$

$$J = \frac{\pi}{2}\left(c_o^{\ 4} - c_i^{\ 4}\right)$$



Three failure modes of a torqued specimen: (a,b): material failure in solid shaft, (c): wall buckling in thin walled tubular shaft.

Angle of Twist

$$\tau = G\gamma$$

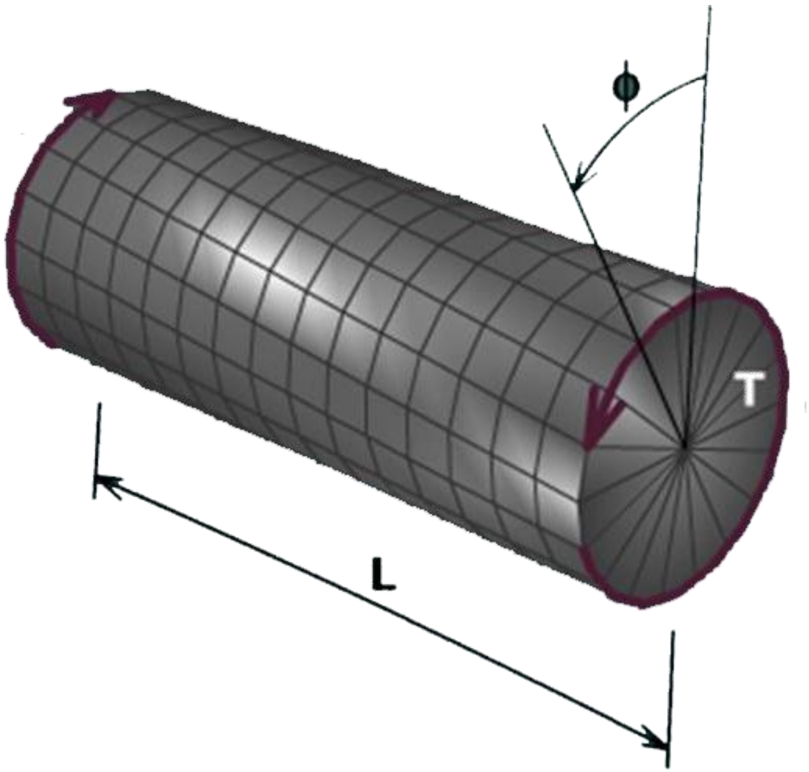
$$\tau = \frac{\rho}{r}\tau_{max} = \frac{T\rho}{I_p}$$

Recall $\tau = G\rho\theta \quad \Rightarrow \quad \theta = \frac{\tau}{G\rho} = \frac{T}{GI_p}$

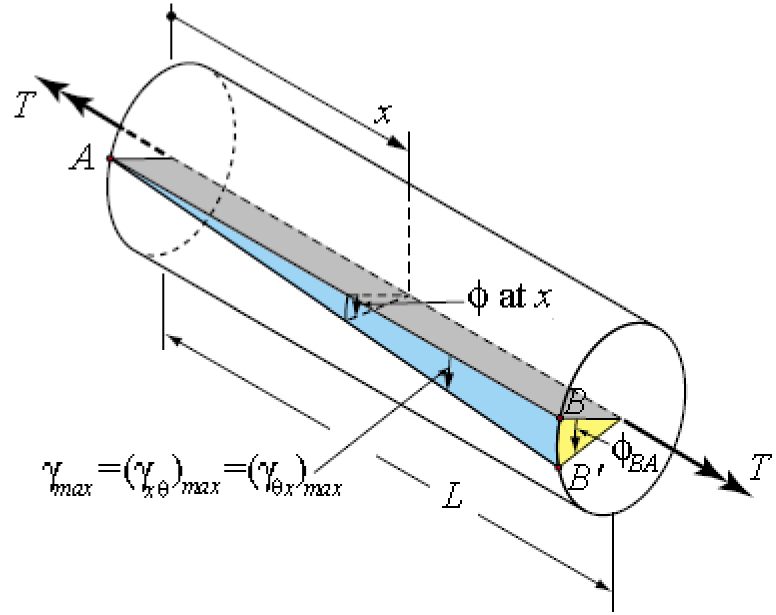
Recall $\phi = \theta L \Rightarrow$

$$\phi = \frac{TL}{GI_p}$$

$k_T = \frac{GI_p}{L}$ = Torsional Rigidity (Stiffness)



This rotation is called the *twist angle φ*. *The twist angle per unit of x-length is* called the *twist rate :* $\theta = d\varphi/dx$.



$$\frac{d\phi}{dx} = \frac{\gamma_{max}}{R} = \frac{\gamma}{\rho} \quad \textbf{so} \quad \gamma = \rho\frac{d\phi}{dx}$$

$$\gamma = \frac{\tau}{G} = \frac{T\rho}{GJ} = \rho\frac{d\phi}{dx}$$

$$\boxed{\frac{d\phi}{dx} = \frac{T}{GJ}}$$

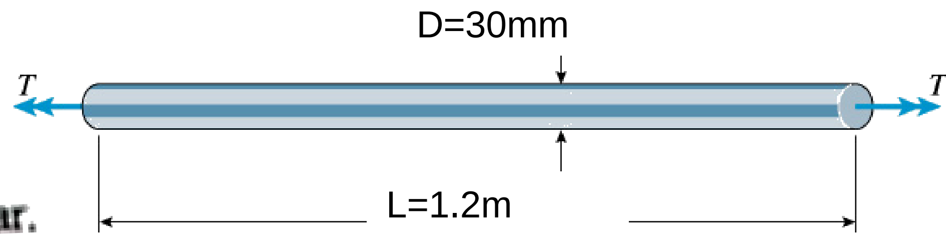
This is the **twist rate formula**. To find the twist angle ϕ_{BA}, where subscripts identify the angle measurement endpoints, integrate along the length of the shaft:

$$\phi_{BA} = \phi_B - \phi_A = \phi_B - 0 = \phi_B = \int_0^L d\phi = \int_0^L \frac{d\phi}{dx}dx = \int_0^L \frac{T}{GJ}dx\,.$$

If T, G and J are constant along the shaft:

$$\phi_{BA} = \frac{T}{GJ}\int_0^L dx = \frac{TL}{GJ}$$

Bar in torsion.



(a) Determine the torsional stiffness of the bar.

(b) If the angle of twist of the bar is 4°, what is the maximum shear stress? What is the maximum shear strain (in radians)?

$L = 1.2$ m $\qquad d = 30$ mm

$G = 28$ GPa $\qquad \phi = 4°$

(a) Torsional stiffness

$$k_T = \frac{GI_P}{L} = \frac{G\pi d^4}{32L} = \frac{(28 \text{ GPa})(\pi)(30 \text{ mm})^4}{32(1.2 \text{ m})}$$

$$k_T = 1860 \text{ N} \cdot \text{m} \quad \longleftarrow$$

(b) Maximum shear stress

$$\phi = 4° = (4°)(\pi/180)rad = \underline{0.069813 \text{ rad}}$$

$$\phi = \frac{TL}{GI_P} \qquad T = \frac{GI_P\phi}{L}$$

$$\tau_{max} = \frac{Tr}{I_P} = \frac{Td}{2I_P} = \left(\frac{GI_P\phi}{L}\right)\left(\frac{d}{2I_P}\right)$$

$$\tau_{max} = \frac{Gd\phi}{2L}$$

$$= \frac{(28 \text{ GPa})(30 \text{ mm})(0.069813 \text{ rad})}{2(1.2 \text{ m})}$$

$$= 24.43 \text{ MPa}$$

Maximum shear strain

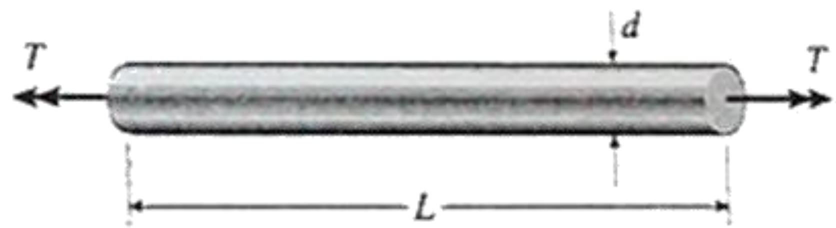
Hooke's Law:

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{24.43 \text{ MPa}}{28 \text{ GPa}}$$

$$\gamma_{max} = 873 \times 10^{-6} \text{ rad} \quad \longleftarrow$$

A plastic bar of diameter $d = 50$ mm is to be twisted by torques T (see figure) until the angle of rotation between the ends of the bar is 5.0°.

If the allowable shear strain in the plastic is 0.012 rad, what is the minimum permissible length of the bar?



$d = 50$ mm

$$\phi = 5.0° = (5.0)\left(\frac{\pi}{180}\right) \text{rad} = 0.08727 \text{ rad}$$

$$\gamma_{allow} = 0.012 \text{ rad}$$

$$L_{min} = \frac{d\phi}{2\gamma_{allow}} = \frac{(50 \text{ mm})(0.08727 \text{ rad})}{(2)(0.012 \text{ rad})}$$

$$L_{min} = 182 \text{ mm} \quad \longleftarrow$$

$$\gamma_{max} = \frac{r\phi}{L} = \frac{d\phi}{2L}$$

(a) SHEAR STRESS AT OUTER SURFACE

(b) SHEAR STRESS AT INNER SURFACE

(c) RATE OF TWIST

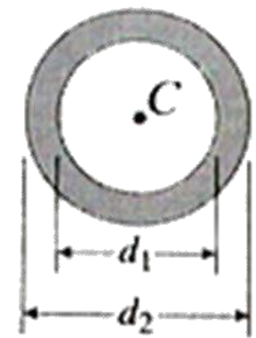
(d) SHEAR STRESS DIAGRAM

$d_2 = 150$ mm $r_2 = 75$ mm

$d_1 = 100$ mm $r_1 = 50$ mm

$G = 75$ GPa

$T = 16$ kN · m



$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = 39.88 \times 10^6 \text{ mm}^4$$

(a) SHEAR STRESS AT OUTER SURFACE

$$\tau_2 = \frac{Tr_2}{I_P} = \frac{(16 \text{ kN} \cdot \text{m})(75 \text{ mm})}{39.88 \times 10^6 \text{ mm}^4}$$

$$= 30.1 \text{ MPa} \quad \longleftarrow$$

(b) SHEAR STRESS AT INNER SURFACE

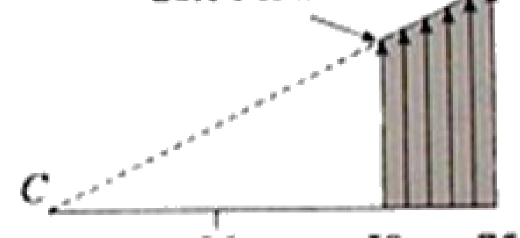
$$\tau_1 = \frac{Tr_1}{I_P} = \frac{r_1}{r_2}\tau_2 = 20.1 \text{ MPa} \quad \longleftarrow$$

(c) RATE OF TWIST

$$\theta = \frac{T}{GI_P} = \frac{16 \text{ kN} \cdot \text{m}}{(75 \text{ GPa})(39.88 \times 10^6 \text{ mm}^4)}$$

$$\theta = 0.005349 \text{ rad/m} = 0.306°\text{/m} \quad \longleftarrow$$

(d) SHEAR STRESS DIAGRAM



Non-Uniform Torsion

Case 1. Bar consisting of Prismatic segments with constant torque throughout each segment

$$\sum T = T_1 + T_2 - T_3 + T_4 = 0$$

Cross sections between CD

$$\sum T = T_1 + T_2 - T_3 + T_{CD} = 0$$
$$\Rightarrow T_{CD} = -T_1 - T_2 + T_3$$

Cross sections between BC

$$\sum T = T_1 + T_2 + T_{BC} = 0$$
$$\Rightarrow T_{BC} = -T_1 - T_2$$

Cross sections between AB

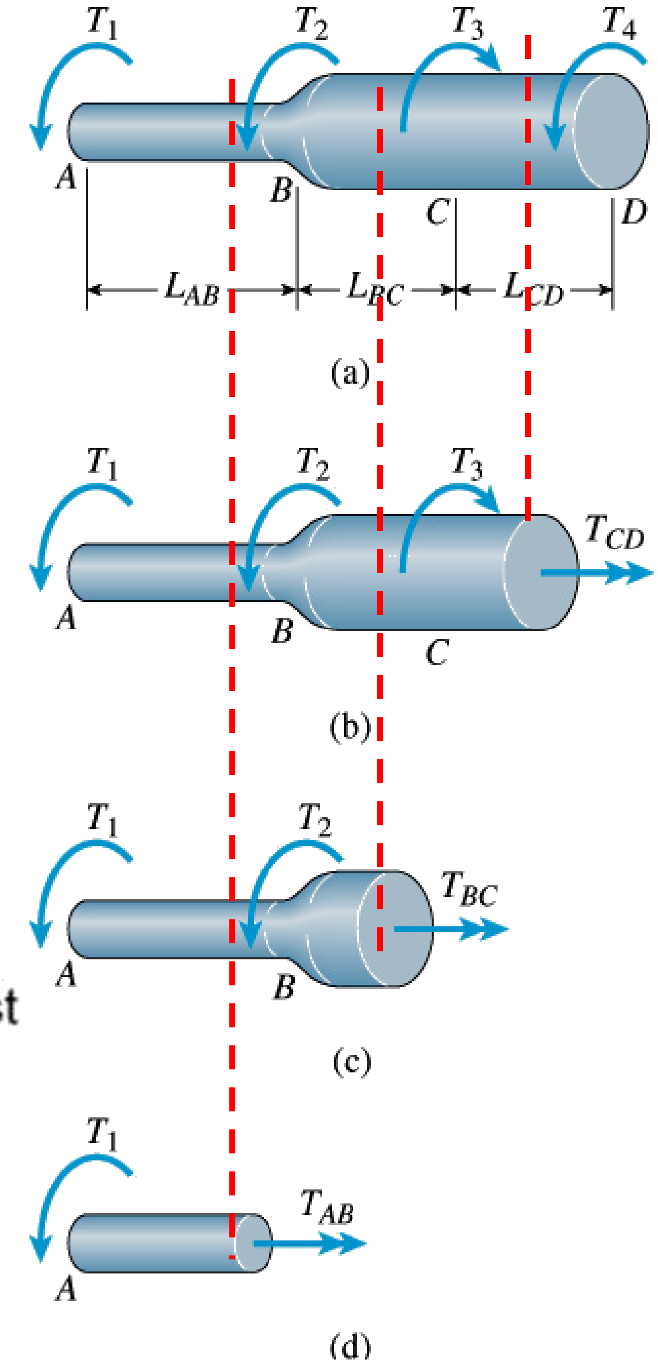
$$\sum T = T_1 + T_{AB} = 0 \Rightarrow T_{AB} = -T_1$$

$$\tau_i = \frac{T_i r_i}{(I_P)_i}$$

$$\phi_i = \frac{T_i L_i}{G_i (I_P)_i}$$

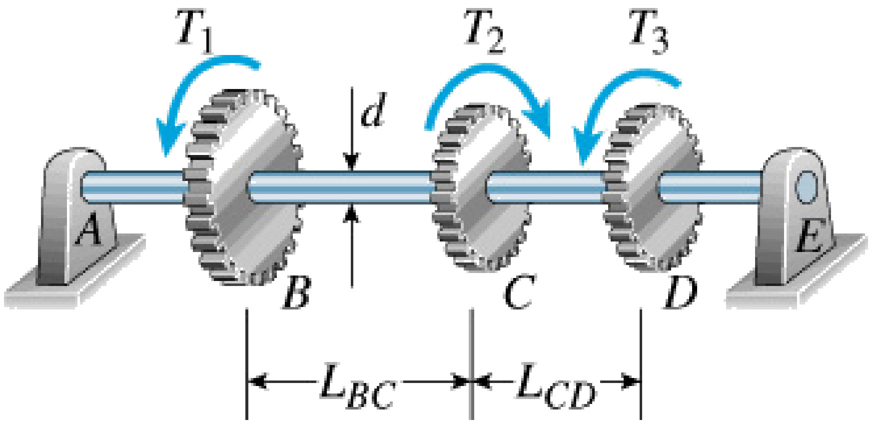
Total angle of twist

$$\phi = \sum_{i=1}^{n} \frac{T_i L_i}{G_i (I_P)_i}$$



Find the shear stress in each part of the shaft

Find the angle of twist of point B relative to D



Shear stress in each segment:

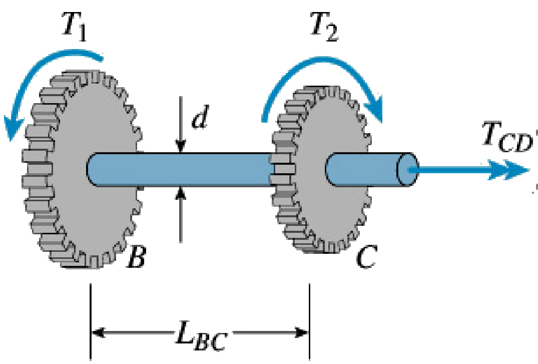
$$\tau_i = \frac{T_i r_i}{(I_P)_i}$$

Angle of twist in each segment

$$\phi_i = \frac{T_i L_i}{G_i (I_P)_i}$$

Total angle of twist

$$\phi = \sum_{i=1}^{n} \frac{T_i L_i}{G_i (I_P)_i}$$

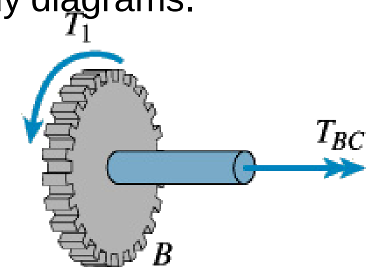


$$\sum T = T_1 - T_2 + T_{CD} = 0$$

$$\Rightarrow T_{CD} = -T_1 + T_2$$

(a)

Free-body diagrams.

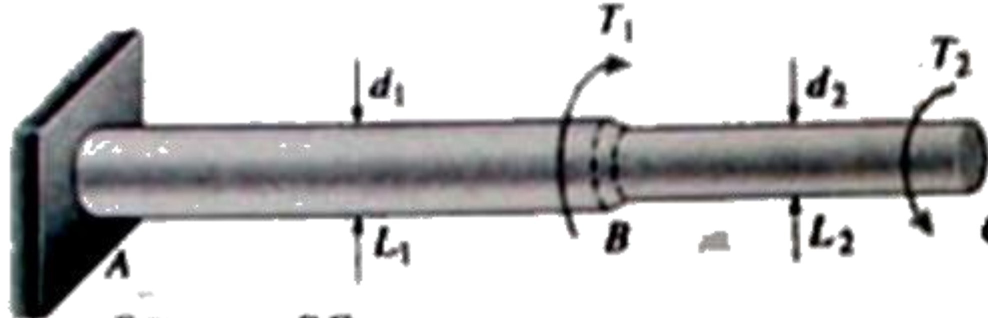


$$\sum T = T_1 + T_{BC} = 0$$

$$\Rightarrow T_{BC} = -T_1$$

(b)

Calculate the following quantities: (a) the maximum shear stress τ_{max} in the shaft, and (b) the angle of twist ϕ_C (in degrees) at end C.



$d_1 = 2.25$ in. $\quad L_1 = 30$ in.

$d_2 = 1.75$ in. $\quad L_2 = 20$ in.

$G = 11 \times 10^6$ psi

$T_1 = 20{,}000$ lb-in.

$T_2 = 8{,}000$ lb-in.

Segment AB

$$T_{AB} = T_2 - T_1 = -12{,}000 \text{ lb-in.}$$

$$\tau_{AB} = \left|\frac{16\,T_{AB}}{\pi d_1^3}\right| = \frac{16(12{,}000 \text{ lb-in.})}{\pi(2.25 \text{ in.})^3} = 5365 \text{ psi}$$

$$\phi_{AB} = \frac{T_{AB}L_1}{G(I_p)_{AB}} = \frac{(-12{,}000 \text{ lb-in.})(30 \text{ in.})}{(11 \times 10^6 \text{ psi})\left(\frac{\pi}{32}\right)(2.25 \text{ in.})^4}$$

$$= -0.013007 \text{ rad}$$

Segment BC

$$T_{BC} = +T_2 = 8{,}000 \text{ lb-in}$$

$$\tau_{BC} = \frac{16\,T_{BC}}{\pi d_2^3} = \frac{16(8{,}000 \text{ lb-in.})}{\pi(1.75 \text{ in.})^3} = 7602 \text{ psi}$$

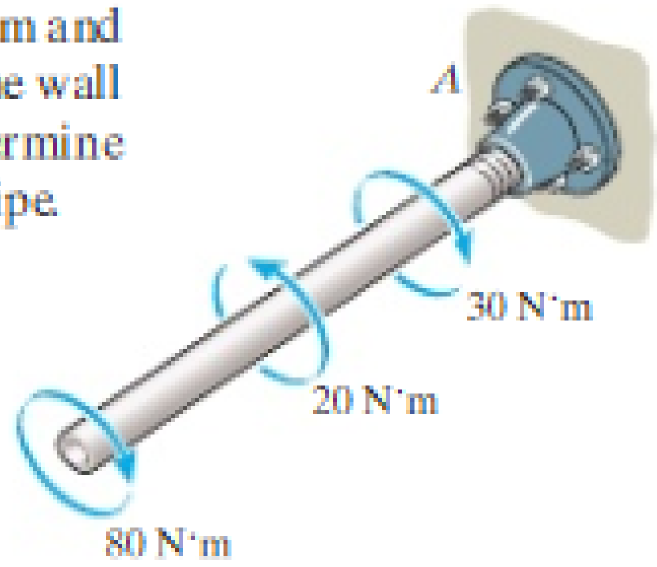
$$\phi_{BC} = \frac{T_{BC}L_2}{G(I_p)_{BC}} = \frac{(8{,}000 \text{ lb-in.})(20 \text{ in.})}{(11 \times 10^6 \text{ psi})\left(\frac{\pi}{32}\right)(1.75 \text{ in.})^4}$$

$$= +0.015797 \text{ rad}$$

$$\phi_C = \phi_{AB} + \phi_{BC} = (-0.013007 + 0.015797) \text{ rad}$$

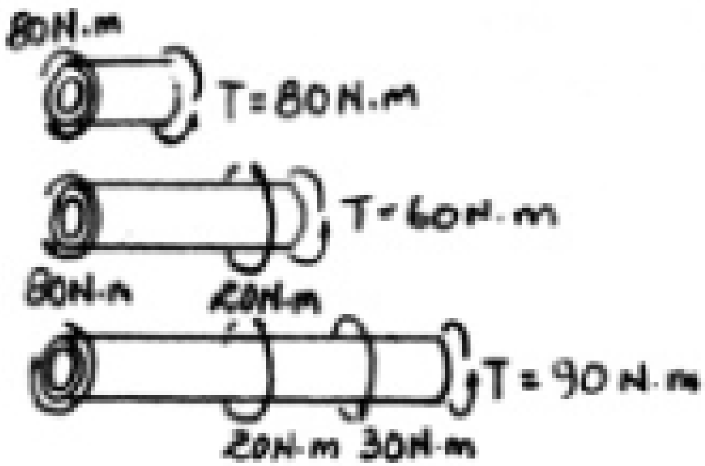
$$= 0.002790 \text{ rad} = 0.16°$$

The copper pipe has an outer diameter of 40 mm and an inner diameter of 37 mm. If it is tightly secured to the wall at A and three torques are applied to it as shown, determine the absolute maximum shear stress developed in the pipe.

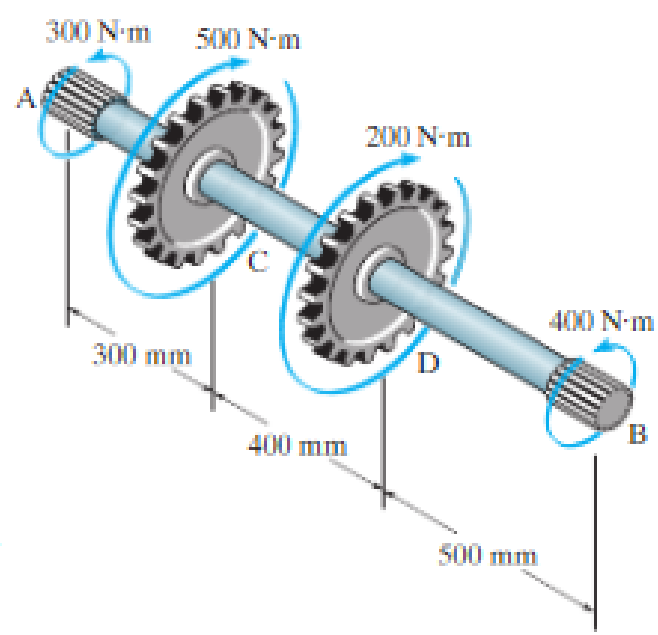


$$\tau_{max} = \frac{T_{max}\, c}{J} = \frac{90(0.02)}{\frac{\pi}{2}(0.02^4 - 0.0185^4)}$$

$$= 26.7 \text{ MPa}$$



The solid 30-mm-diameter shaft is used to transmit the torques applied to the gears. Determine the absolute maximum shear stress on the shaft.

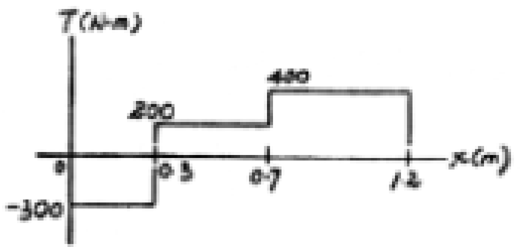


Internal Torque: As shown on torque diagram.

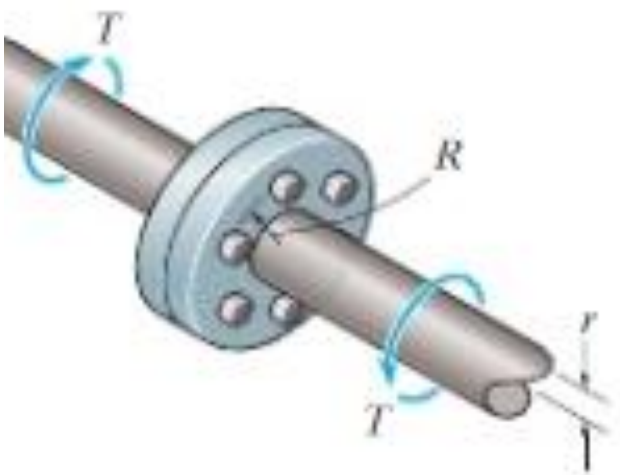
Maximum Shear Stress: From the torque diagram $T_{max} = 400\ \text{N}\cdot\text{m}$. Then, applying torsion Formula.

$$\tau_{max}^{abs} = \frac{T_{max}\, c}{J}$$

$$= \frac{400(0.015)}{\frac{\pi}{2}(0.015^4)} = 75.5\ \text{MPa}$$ **Ans.**



The coupling is used to connect the two shafts together. Assuming that the shear stress in the bolts is *uniform*, determine the number of bolts necessary to make the maximum shear stress in the shaft equal to the shear stress in the bolts. Each bolt has a diameter d.



n is the number of bolts and F is the shear force in each bolt.

$$T - nFR = 0; \qquad F = \frac{T}{nR}$$

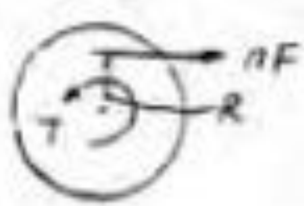
$$\tau_{avg} = \frac{F}{A} = \frac{\frac{T}{nR}}{(\frac{\pi}{4})d^2} = \frac{4T}{nR\pi d^2}$$

Maximum shear stress for the shaft:

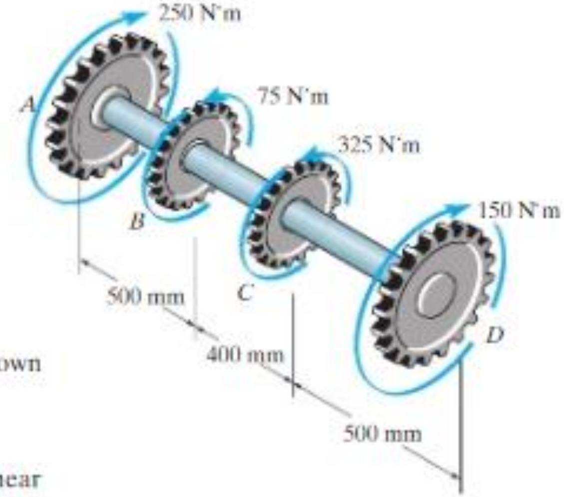
$$\tau_{max} = \frac{Tc}{J} = \frac{Tr}{\frac{\pi}{2}r^4} = \frac{2T}{\pi r^3}$$

$$\tau_{avg} = \tau_{max}; \qquad \frac{4T}{nR\pi d^2} = \frac{2T}{\pi r^3}$$

$$n = \frac{2r^3}{Rd^2}$$



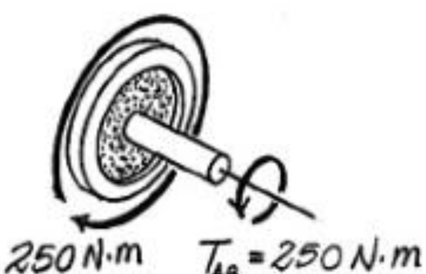
5–14. The solid 50-mm-diameter shaft is used to transmit the torques applied to the gears. Determine the absolute maximum shear stress in the shaft.



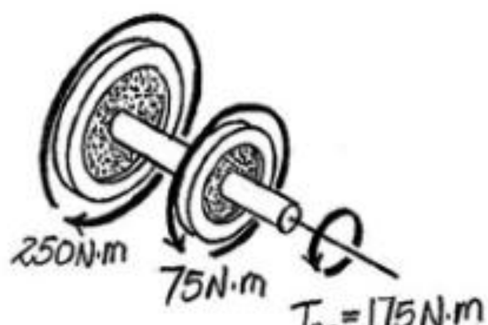
The internal torque developed in segments AB, BC and CD of the shaft are shown in Figs. a, b and c.

The maximum torque occurs in segment AB. Thus, the absolute maximum shear stress occurs in this segment. The polar moment of inertia of the shaft is $J = \frac{\pi}{2}(0.025^4) = 0.1953\pi(10^{-6})\text{m}^4$. Thus,

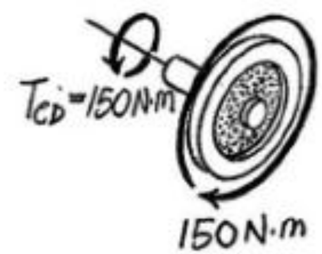
$$\left(\tau_{max}\right)_{abs} = \frac{T_{AB}c}{J} = \frac{250(0.025)}{0.1953\pi(10^{-6})} = 10.19(10^6)\text{Pa} = 10.2\text{ MPa} \qquad \textbf{Ans.}$$



(a)

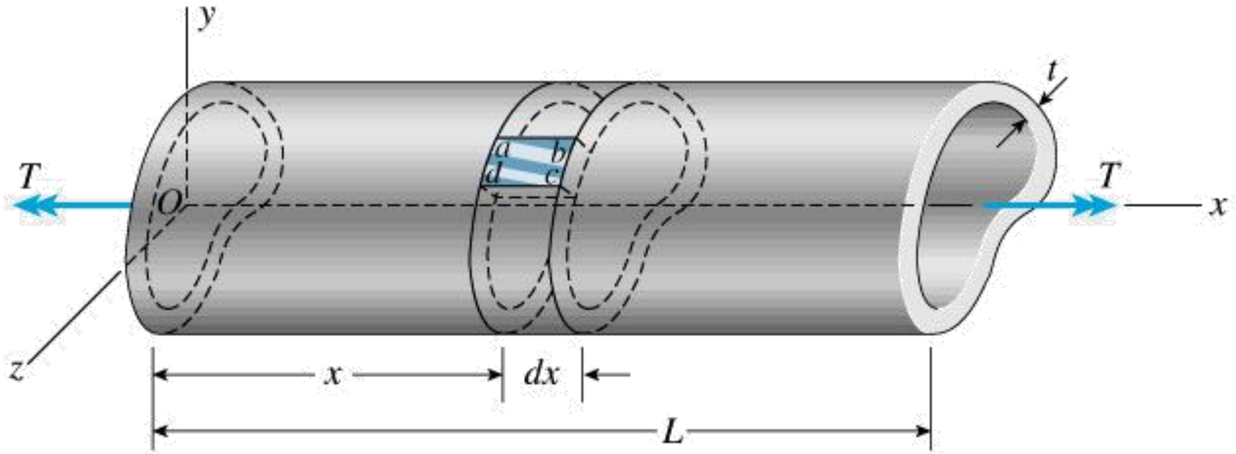


(b)



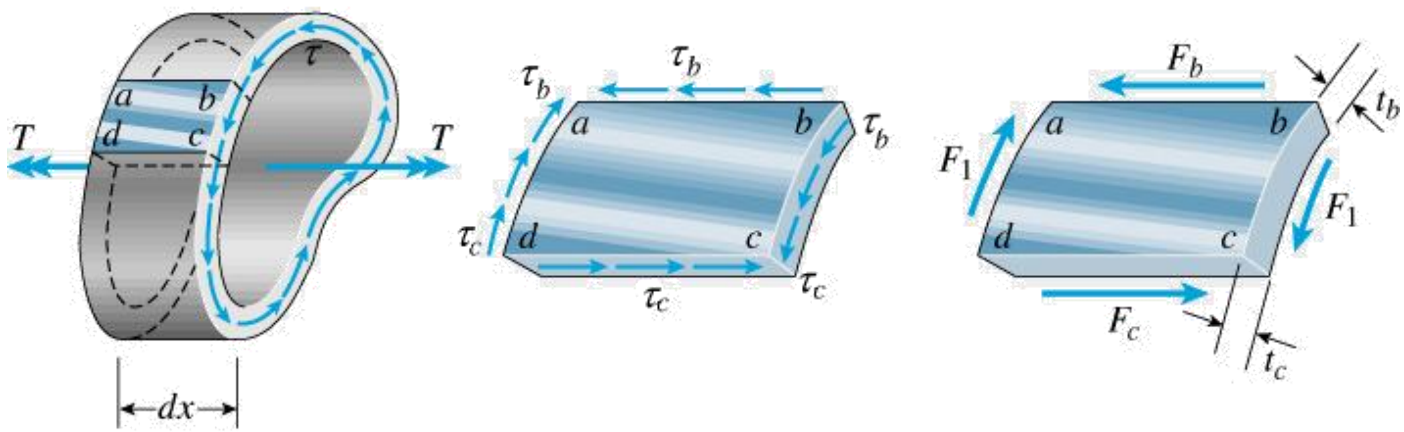
(c)

Thin walled Tubes



(a)

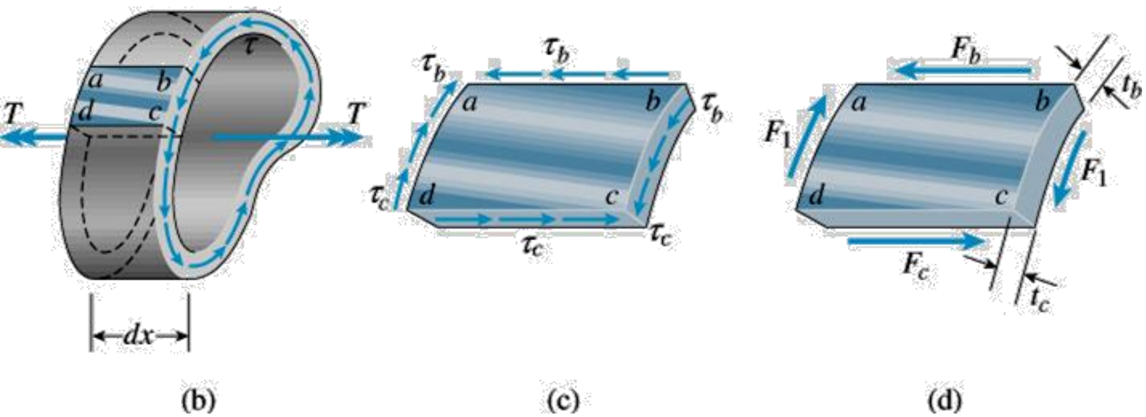
Shear stress and shear flow

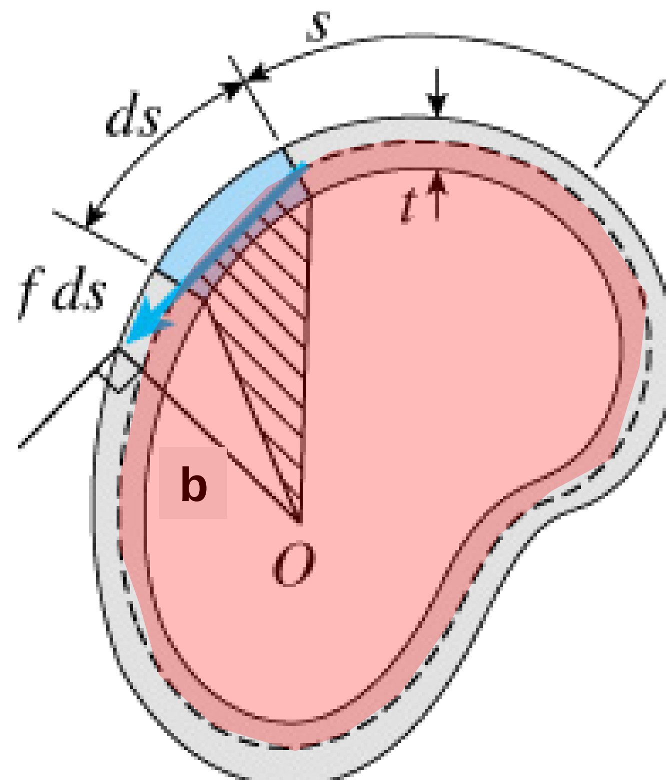


(b) (c) (d)

Shear flow = τ . t

Torsion formula for thin walled tubes





Cross section of thin-walled tube.

$$dF = \tau(t\,ds) = f\,ds$$

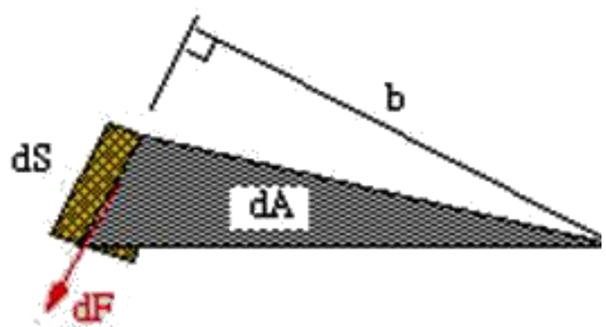
The moment of dF about an arbitrary point O is

$$dM_O = b\,dF = b\,(f\,ds) = f\,(b\,ds)$$

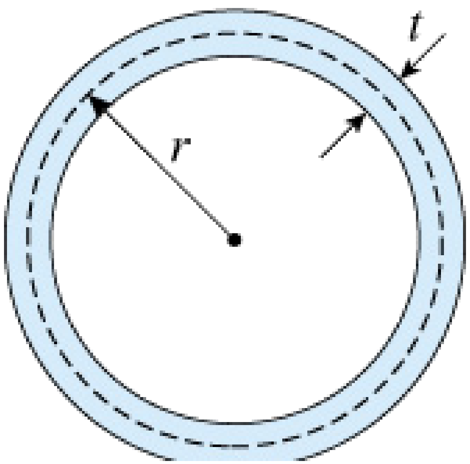
$$dA = \tfrac{1}{2}\,b\,ds \rightarrow b\,ds = 2\,dA$$

now $dM = f\,(2dA)$

$$T = \oint dM_O = \oint f(2dA)$$



$$\tau = \frac{T}{2t\,A_m}$$

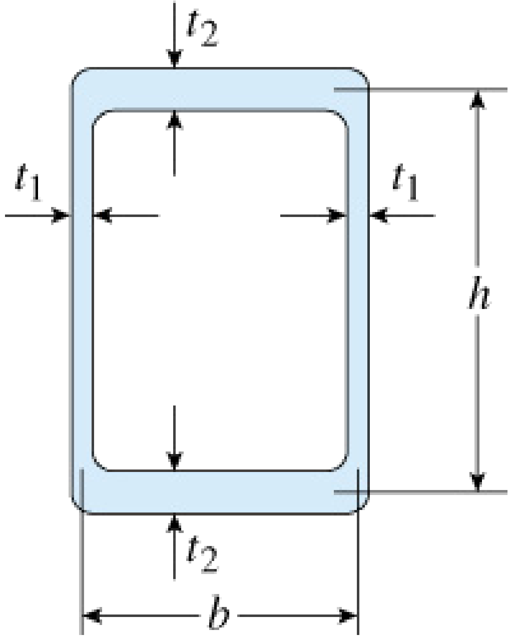


Thin-walled circular tube

$$A_m = \pi r^2$$

Shear stress constant around the cross section

$$\tau = \frac{T}{2t\,A_m} = \frac{T}{2\pi t r^2}$$



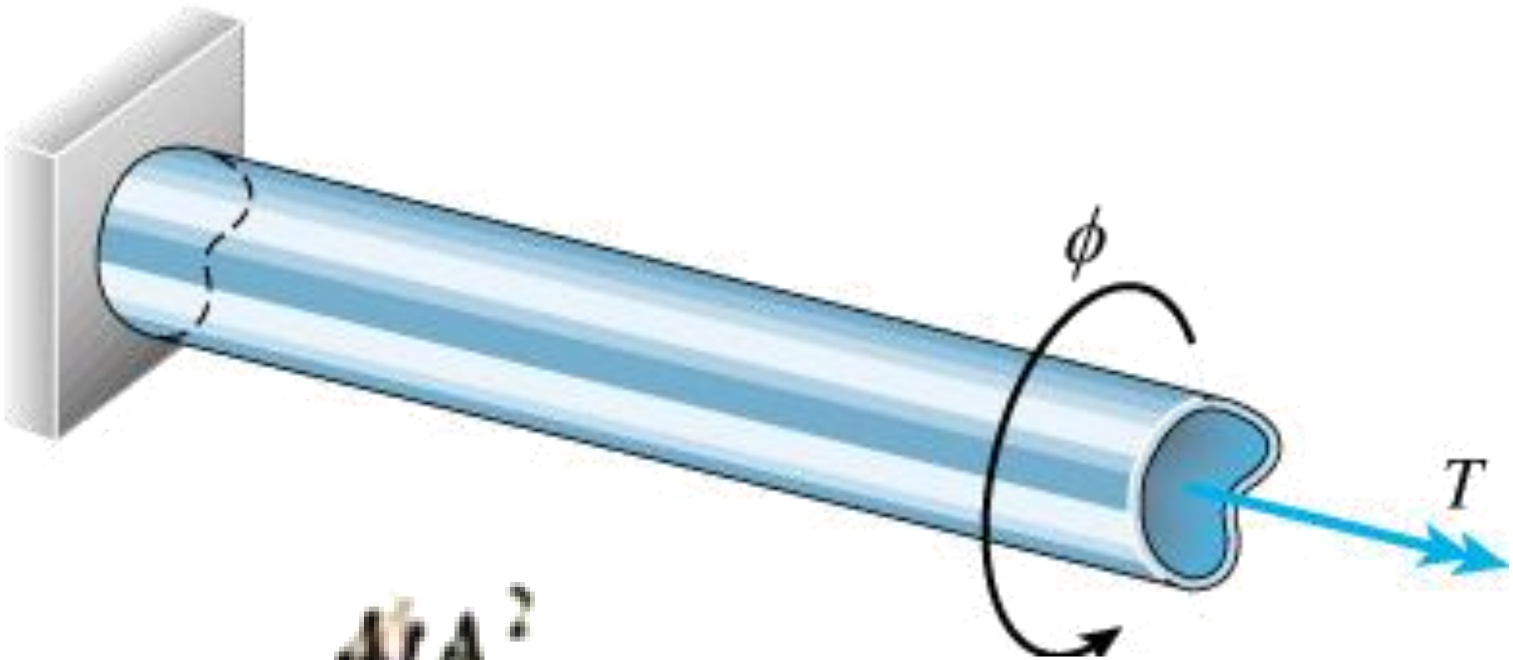
Thin-walled rectangular tube.

$$A_m = b\,h$$

$$\tau_{vert} = \frac{T}{2\,t_1\,b\,h}$$

$$\tau_{Horiz} = \frac{T}{2\,t_2\,b\,h}$$

Angle of twist ϕ for a thin-walled tube



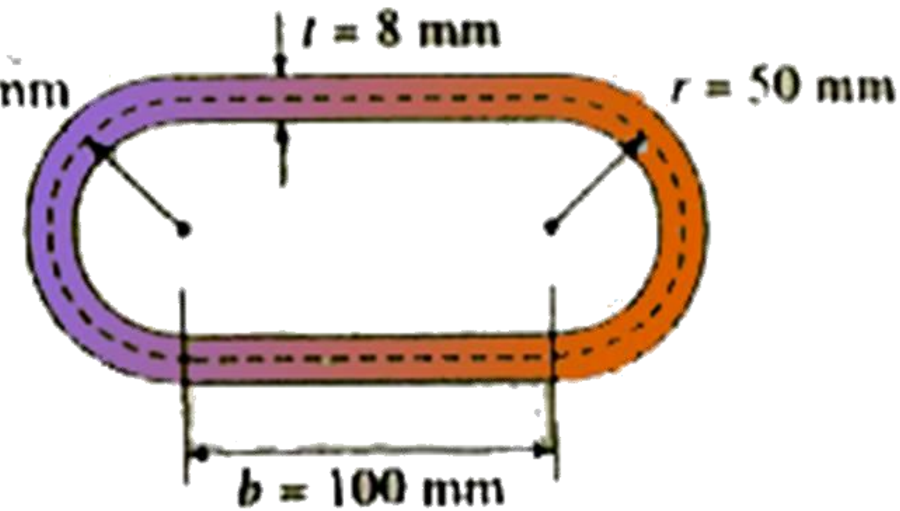
$$J = \frac{4tA_m^2}{L_m}$$

Tensional constant of thin walled tubes of constant thickness

$$\phi = \frac{TL}{GJ}$$

Tensional rigidity

Problem 3.10-6 Calculate the shear stress τ and the angle of twist ϕ (in degrees) for a steel tube ($G = 76$ GPa) having the cross section shown in the figure. The tube has length $L = 1.5$ m and is subjected to a torque $T = 10$ kN · m.



$$A_m = \pi(50\text{ mm})^2 + 2(100\text{ mm})(50\text{ mm})$$
$$= 17{,}850\text{ mm}^2$$

$$L_m = 2b + 2\pi r$$
$$= 2(100\text{ mm}) + 2\pi(50\text{ mm})$$
$$= 514.2\text{ mm}$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4(8\text{ mm})(17{,}850\text{ mm}^2)^2}{514.2\text{ mm}}$$

SHEAR STRESS

$$\tau = \frac{T}{2tA_m} = \frac{10\text{ kN}\cdot\text{m}}{2(8\text{ mm})(17{,}850\text{ mm}^2)}$$

$$= 35.0\text{ MPa} \quad \longleftarrow$$

ANGLE OF TWIST

$$\phi = \frac{TL}{GJ} = \frac{(10\text{ kN}\cdot\text{m})(1.5\text{ m})}{(76\text{ GPa})(19.83 \times 10^6\text{ mm}^4)}$$

$$= 0.00995\text{ rad}$$

$$= 0.570° \quad \longleftarrow$$

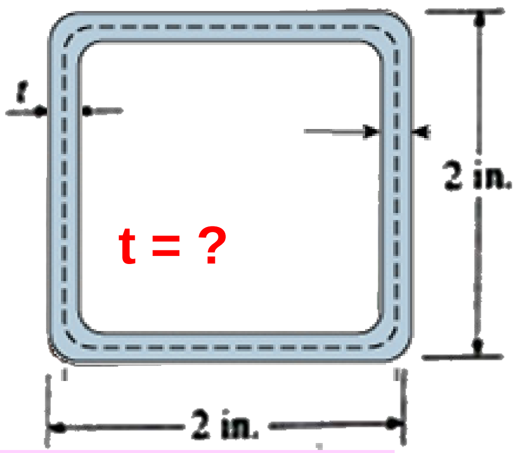
Outer dimensions:

2.0 in. × 2.0 in.

$G = 4 \times 10^6$ psi

$T = 3000$ lb-in.

$\tau_{allow} = 4500$ psi



$$\theta_{allow} = 0.01 \text{ rad/ft} = \frac{0.01}{12} \text{ rad/in.}$$

Centerline dimension $= b - t$

$$A_m = (b-t)^2 \qquad L_m = 4(b-t)$$

$$J = \frac{4tA_m^2}{L_m} = \frac{4t(b-t)^4}{4(b-t)} = t(b-t)^3$$

THICKNESS t BASED UPON SHEAR STRESS

$$\tau = \frac{T}{2tA_m} \qquad tA_m = \frac{T}{2\tau} \qquad t(b-t)^2 = \frac{T}{2\tau}$$

UNITS: t = in. b = in. T = lb-in. τ = psi

$$t(2.0 \text{ in.} - t)^2 = \frac{3000 \text{ lb-in.}}{2(4500 \text{ psi})} = \frac{1}{3} \text{ in.}^3$$

$$3t(2-t)^2 - 1 = 0$$

Solve for t: $t = 0.0915$ in.

THICKNESS t BASED UPON RATE OF TWIST

$$\theta = \frac{T}{GJ} = \frac{T}{Gt(b-t)^3} \qquad t(b-t)^3 = \frac{T}{G\theta}$$

UNITS: t = in. G = psi θ = rad/in.

$$t(2.0 \text{ in.} - t)^3 = \frac{3000 \text{ lb-in}}{(4 \times 10^6 \text{ psi})(0.01/12 \text{ rad/in.})}$$

$$= \frac{9}{10}$$

$$10t(2-t)^3 - 9 = 0$$

Solve for t:

$t = 0.140$ in.

ANGLE OF TWIST GOVERNS

$t_{min} = 0.140$ in. ←