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Strength of Materials Course Contents

Chapter one

Introduction

What is Mechanics?

- **Mechanics** is the science which describes and predicts the conditions of rest or motion of bodies under the action of forces.

- Mechanics is the foundation of most engineering sciences and is an indispensable prerequisite to their study.

Units Of Measurements

- *Kinetic Units*: length, time, mass, and force.
- *International System of Units* (SI):
  The basic units are length, time, and mass which are arbitrarily defined as the meter (m), second (s), and kilogram (kg). Force is the derived unit,

$$F = ma$$

$$1\,\mathrm{N} = (1\,\mathrm{kg})\left(1\frac{\mathrm{m}}{\mathrm{s}^2}\right)$$

$g = 9.81 m/sec^2$

- ***U.S. Customary Units:***
  The basic units are length, time, and force which are arbitrarily defined as the foot (ft), second (s), and pound (lb). Mass is the derived unit,
-

$$m = \frac{F}{a}$$

$$1\text{slug} = \frac{1\text{lb}}{1\text{ft}/\text{s}2}$$

g= 32.2 ft/sec^2

- Conversion units
- 1 ft = 0.308 m
- 1 lb= 4.44 N
- 1slug= 14.6 Kg

Table 1.1: Prefixes and Symbols of Multiplying Factors in SI

Multiplying Factor	*Prefix*	*Symbol*
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{0}	—	—
10^{-3}	milli	m
10^{-6}	micro	µ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

Chapter two

FORCE VECTOR

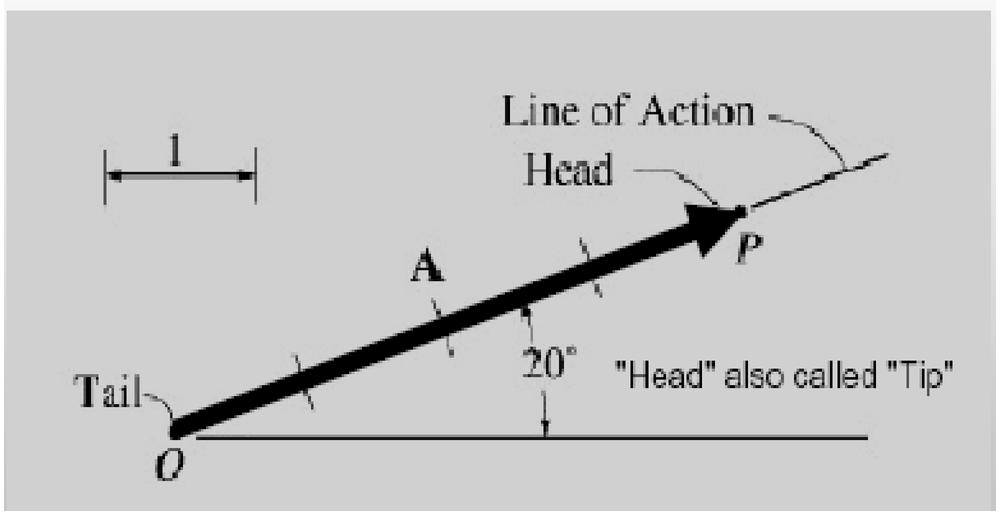
Scalar and Vectors

Scalar: is a quantity which has magnitude only.
Examples of scalars: distance, energy, volume, mass and temperature. .

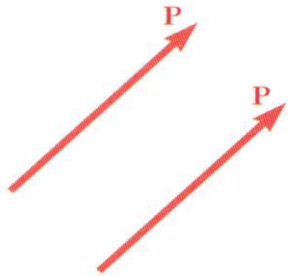
Vectors :are quantities which are fully described by both a magnitude and a direction. Vectors are physical quantities.
Examples of vectors are displacement, velocity, acceleration, force and electric field

Vector notation:

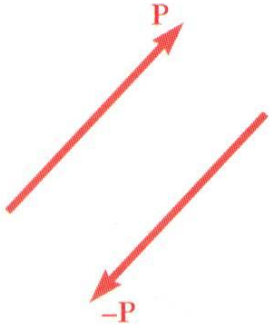
A widely used convention is to denote a vector quantity in bold type, such as **A** ، and that is the convention that will be used. The magnitude of a vector A is written as |**A**|



- ***Equal* vectors** have the same magnitude and direction.

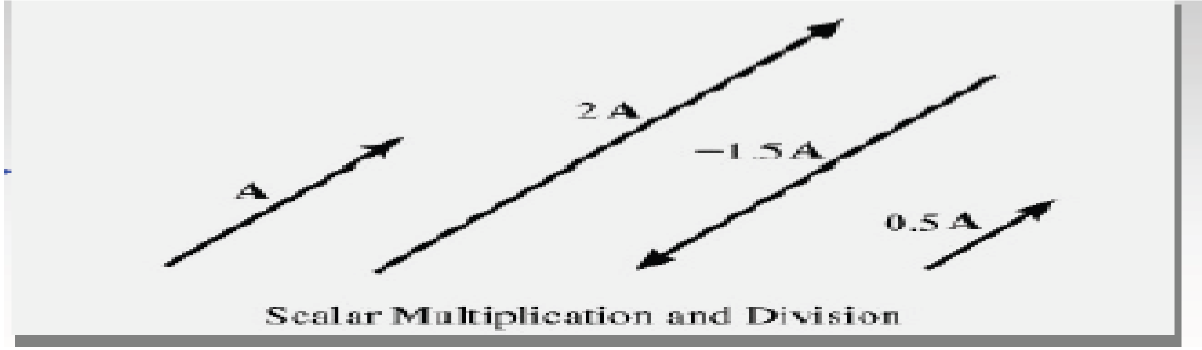


***Negative* vecto**r of a given vector has the same magnitude and the opposite direction.



Multiplication and division of a vector by a scalar

A x m = m|A|



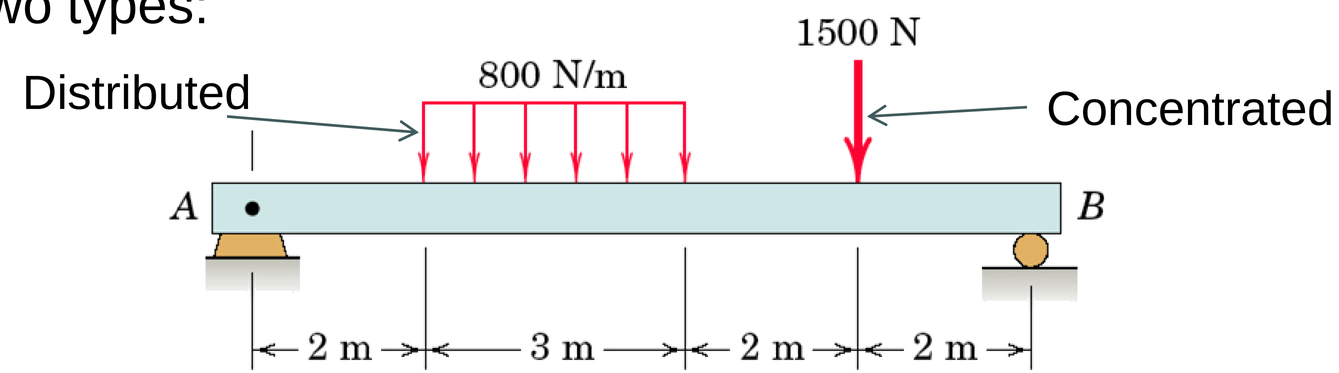
Scalar Multiplication and Division

Introduction: Force Effects

- Force exerted on a body has two effects:
  - The ***external effect***, which is tendency to change the motion of the body or to develop resisting forces in the body
    - Applied forces
    - Reactive forces
  - The ***internal effect***, which is the tendency to deform the body
    - Depends on material strength, elasticity
    - Out of scope of class

Characteristics of Forces

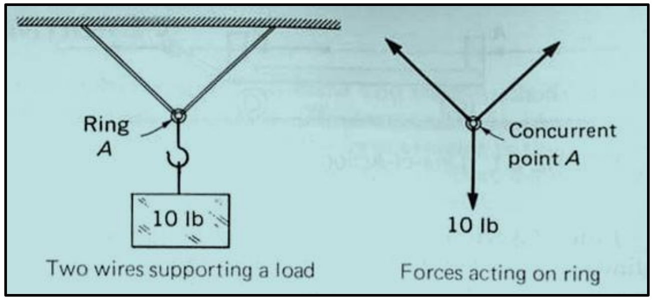
- Force Classification:
  - Contact Force: Produced by direct physical contact
    - Force exerted on a body by a supporting surface
    - Two types:



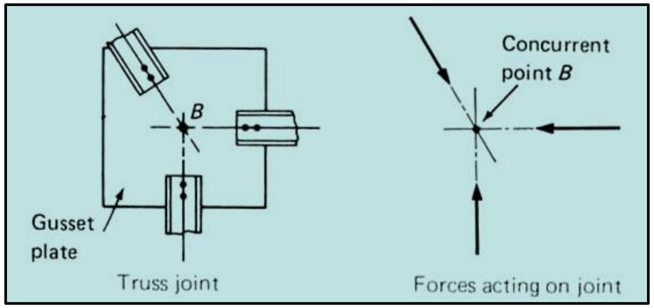
  - Body Force: Generated by virtue of position
    - Weight due to gravitational field
    - Magnetic force due to magnetic field

Characteristics of Forces

- Concurrent Forces:
  - A concurrent force system contains forces whose lines of action *intersect* at a point.
  - Examples:
    - Tension:



    - Compression:



Vector Components

- It is often useful to replace a force by its vector components
- Rectangular Components for 2D Force Systems:
  - Most common resolution of a force vector (using x- and y- Cartesian coordinates)
  - $\mathbf{F}=\mathbf{F}_x+\mathbf{F}_y$
    - $\mathbf{F}_x$, $\mathbf{F}_y$ are vector components of $\mathbf{F}$ in the *x*- and *y*- directions
    - $\mathbf{F}_x = F_x\,\mathbf{i}$, $\mathbf{F}_y = F_y\mathbf{j}$ or $\mathbf{F} = F_x\,\mathbf{i} + F_y\mathbf{j}$
    - $F_x = F\cos\theta$, $F_y = F\sin\theta$,

$$F = \sqrt{F_x^{\,2}+F_y^{\,2}},\;\; \theta = \tan^{-1}\frac{F_y}{F_x}$$

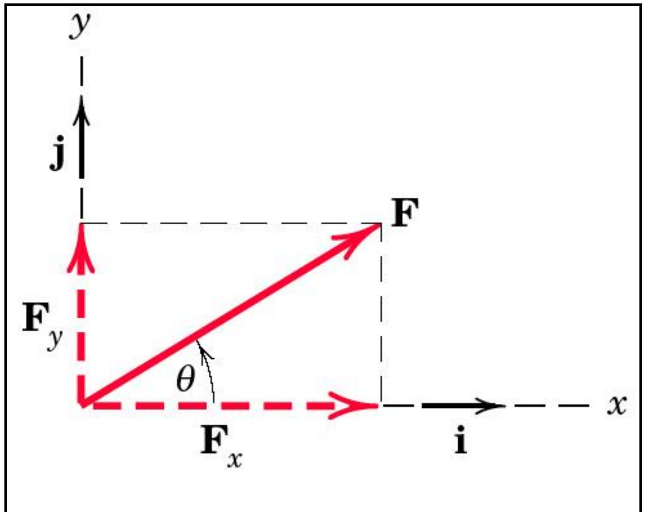


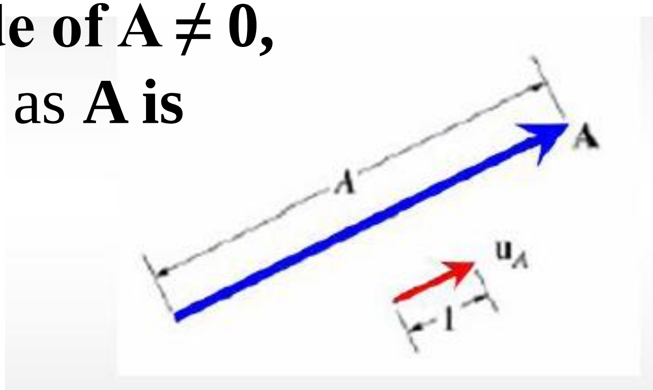
Figure 2/5

Unit vector Representation of a Vector

Direction of **A can be specified using a unit** vector

- Unit vector has a magnitude of 1
- If **A is a vector having a magnitude of** A $\neq$ **0,** unit vector having the same direction as **A is**

$$u_A = \overline{A} / |A|$$



UA is dimensionless. It serves only to indicate direction and sense.

Resultant Vectors

- We can replace multiple forces with a single resultant force
  - This single resultant has the same effect as the original group of forces
  - Multiple ways to compute resultant
    - Parallelogram law
    - Vector addition

Vector Addition

Assume we have three forces, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$

Express each force as a Cartesian Vector

$$\mathbf{F}_1 = F_{1x}\mathbf{i} + F_{1y}\mathbf{j}$$

$$\mathbf{F}_2 = -F_{2x}\mathbf{i} + F_{2y}\mathbf{j}$$

$$\mathbf{F}_3 = F_{3x}\mathbf{i} - F_{3y}\mathbf{j}$$

Vector resultant

$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (F_{Rx}\mathbf{i} + F_{Ry}\mathbf{j})$$

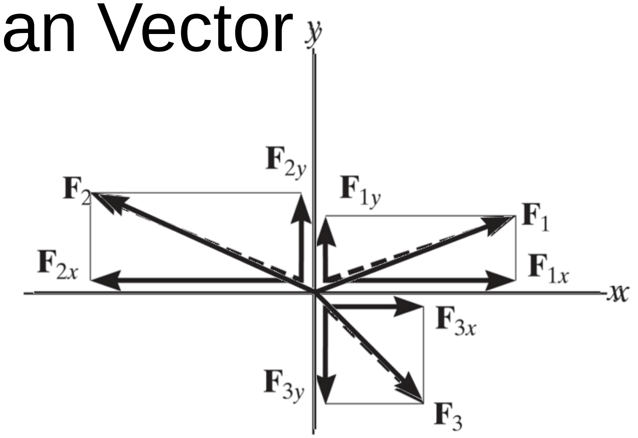
Vector Addition

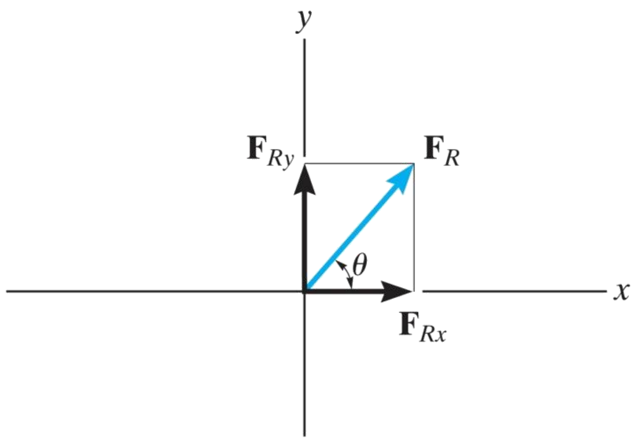
Assume we have three forces, $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$

1. Express each force as a Cartesian Vector
   - $\mathbf{F}_1 = F_{1x}\mathbf{i} + F_{1y}\mathbf{j}$
   - $\mathbf{F}_2 = -F_{2x}\mathbf{i} + F_{2y}\mathbf{j}$
   - $\mathbf{F}_3 = F_{3x}\mathbf{i} - F_{3y}\mathbf{j}$
2. Vector resultant
   - $\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (F_{Rx}\mathbf{i} + F_{Ry}\mathbf{j})$

   Or, using scalar notation:
   - $F_{Rx} = F_{1x} - F_{2x} + F_{3x} \rightarrow F_{Rx} = \Sigma F_x$
   - $F_{Ry} = F_{1y} + F_{2y} - F_{3y} \rightarrow F_{Ry} = \Sigma F_y$
   - Where:

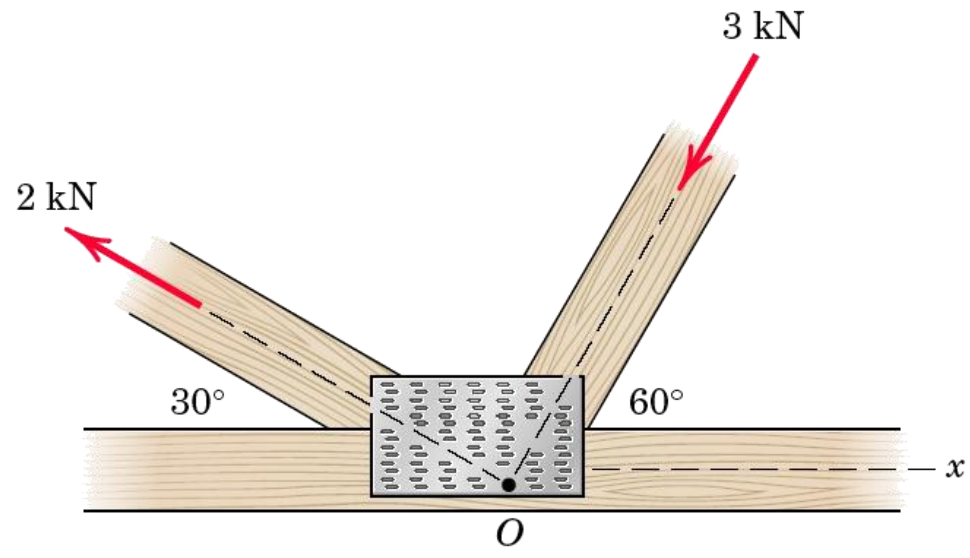
$$F_R = \sqrt{F_{Rx}^2 + F_{Ry}^2} \qquad \theta = \tan^{-1}\left|\frac{F_{Ry}}{F_{Rx}}\right|$$





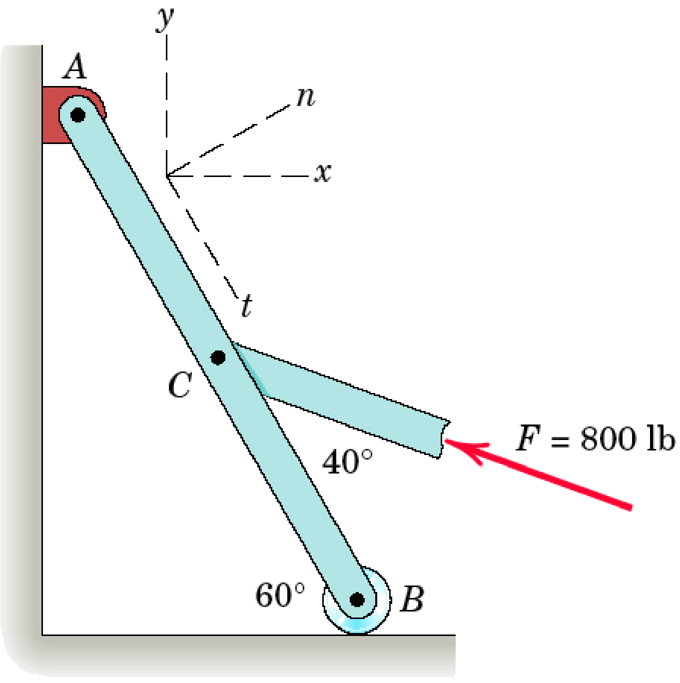
Example 3

Problem 2/7: The two structural members, one of which is in tension and the other in compression, exert the indicated forces on joint O. Determine the magnitude of the resultant R of the two forces and the angle θ which R makes with the positive *x*-axis.



Problem 2/7

<u>Problem 2/12</u>: A force **F** of magnitude 800 lb is applied to point *C* of the bar *AB* as shown. Determine both the *x-y* and *n-t* components of **F**.



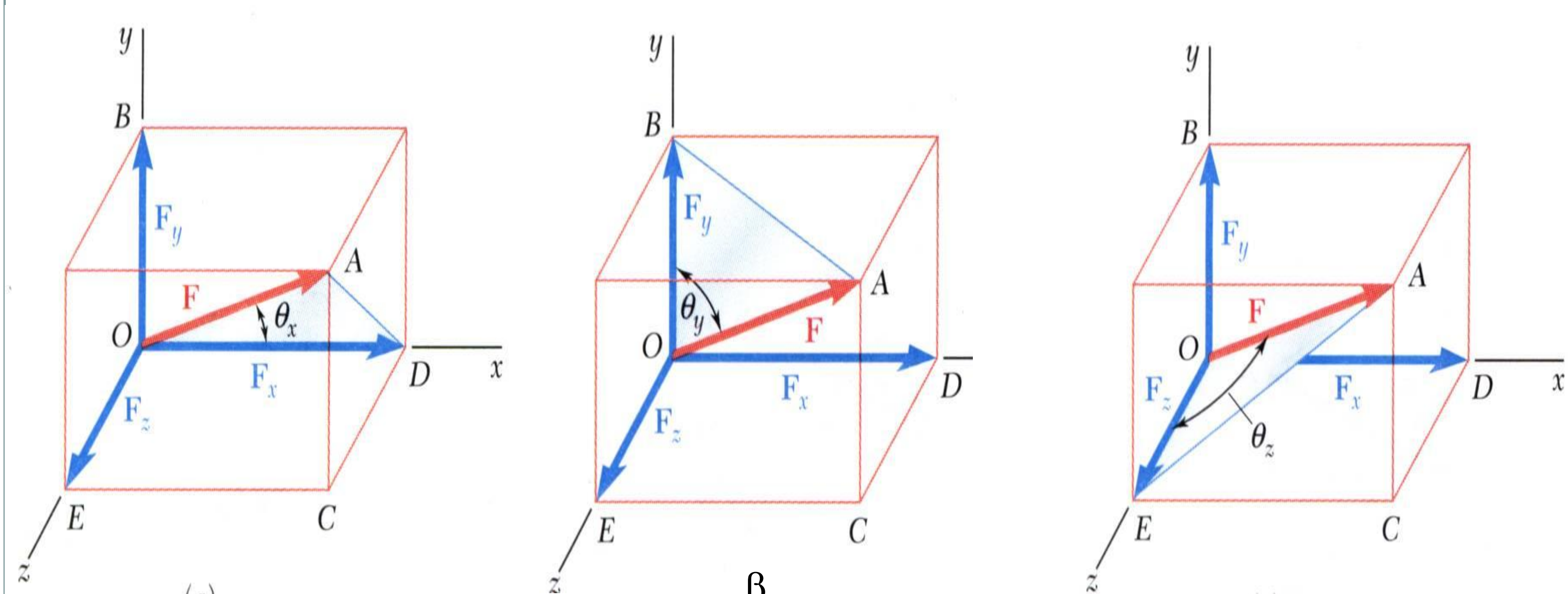
Problem 2/12

Force vector in 3D

Rectangular Components of 3D Forces

- 3D Force Vector – Vector defining a Force in more than one Cartesian Plane defined by its location and rectangular components
- Rectangular Components - Components that fall along the Cartesian coordinate system axes
- Coordinate Angles (α,β,γ)– The angle a vector makes with the individual axes of the Cartesian Coordinate System

Coordinate Angles



α β γ

Note, book uses α, β, γ:

Coordinate Angles

- The values of the three angles are not independent, they are related by the identity:

  $\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$

Resolving a 3D Force Vector into its Rectangular Components

- Given the magnitude of a force vector (F) and its Coordinate angles (α,β,γ):

  $F_x = F\cos(\alpha)$

  $F_y = F\cos(\beta)$

  $F_z = F\cos(\gamma)$

$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}$$

Resultant of a 3D Force Vector from its Rectangular Components

- If given the components of a 3D force (F_x, F_y, F_z), the force can be determined by:
  - Magnitude (F) = $\sqrt{(F_x^2+F_y^2+F_z^2)}$
- The Coordinate Angles of the Force Vector can be found by
  - $\cos(\alpha) = F_x/F$
  - $\cos(\beta) = F_Y/F$
  - $\cos(\gamma) = F_z/F$

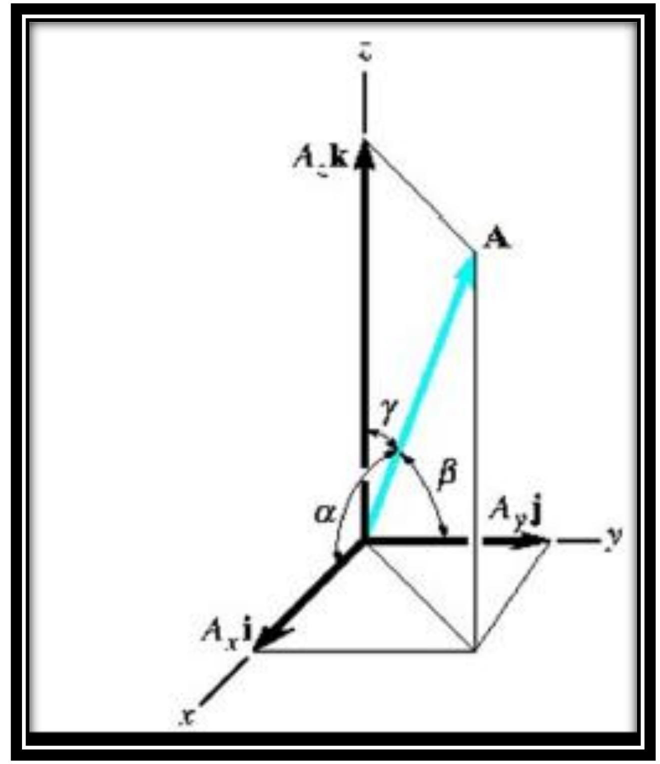
$$\cos\alpha = \frac{A_x}{A} \qquad \cos\beta = \frac{A_y}{A} \qquad \cos\gamma = \frac{A_z}{A}$$

$$\mathbf{u}_A = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}$$

$$U = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

$$1 = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$



Eg. Determine the magnitude and directional cosines of the vector

$$\vec{A} = 700\,\vec{i} - 820\,\vec{j} + 900\,\vec{k}$$

The magnitude of the vector is

$$\vec{A} = 700\,\vec{i} - 820\,\vec{j} + 900\,\vec{k}$$

$$|A| = \sqrt{(700)^2 + (-820)^2 + (900)^2} = 1404.42$$

The directional cosines are

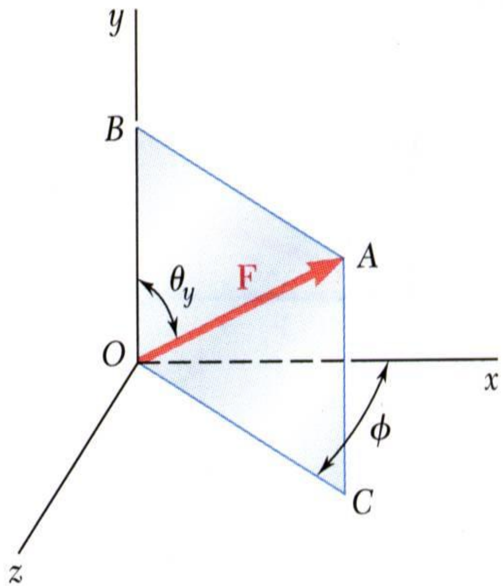
$$\begin{cases} \cos\theta_x = \dfrac{700}{1404.42} = 0.498 \Rightarrow \theta_x = 60.1^\circ \\ \cos\theta_y = \dfrac{-820}{1404.42} = -0.584 \Rightarrow \theta_y = 125.7^\circ \\ \cos\theta_z = \dfrac{900}{1404.42} = 0.641 \Rightarrow \theta_x = 50.1^\circ \end{cases}$$

Check the cosines

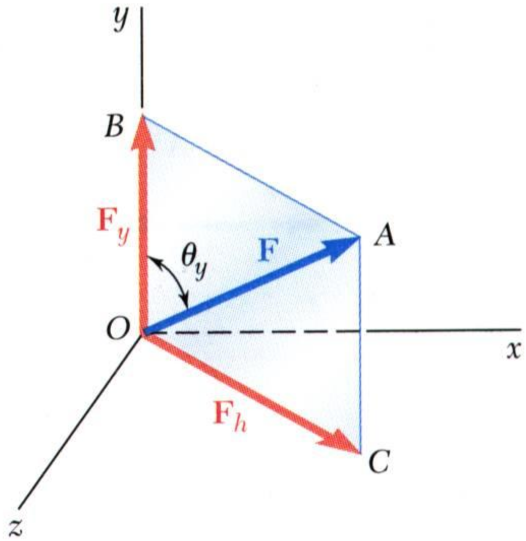
$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

$$(0.498)^2 + (-0.584)^2 + (0.641)^2 = 1$$

Rectangular Components in Space



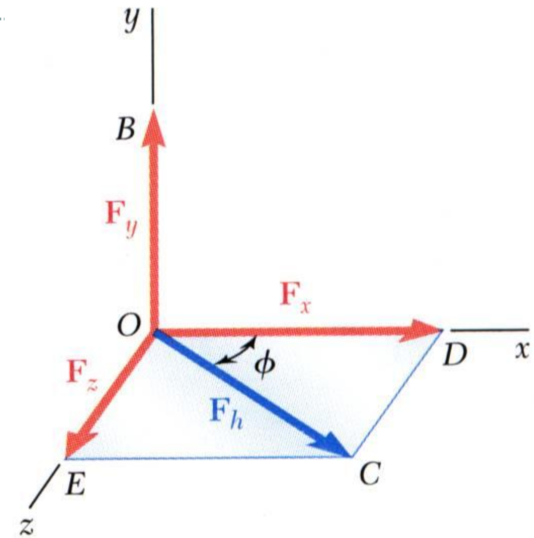
- The vector $\vec{F}$ is contained in the plane *OBAC*.



- Resolve $\vec{F}$ into horizontal and vertical components.

$$F_y = F \cos\theta_y$$

$$F_h = F \sin\theta_y$$



- Resolve F_h into rectangular components

$$F_x = F_h \cos\phi = F \sin\theta_y \cos\phi$$

$$F_y = F_h \sin\phi = F \sin\theta_y \sin\phi$$

ADDITION OF CARTESIAN VECTORS

Once individual vectors are written in Cartesian form, it is easy to add or subtract them. The process is essentially the same as when 2-D vectors are added.

$$\mathbf{F}_R = \Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}$$

For example, if

$\boldsymbol{A} = A_X\, \boldsymbol{i} + A_Y\, \boldsymbol{j} + A_Z\, \boldsymbol{k}$ and

$\boldsymbol{B} = B_X\, \boldsymbol{i} + B_Y\, \boldsymbol{j} + B_Z\, \boldsymbol{k}$, then

$$\boldsymbol{A} + \boldsymbol{B} = (A_X + B_X)\, \boldsymbol{i} + (A_Y + B_Y)\, \boldsymbol{j} + (A_Z + B_Z)\, \boldsymbol{k}$$

or

$$\boldsymbol{A} - \boldsymbol{B} = (A_X - B_X)\, \boldsymbol{i} + (A_Y - B_Y)\, \boldsymbol{j} + (A_Z - B_Z)\, \boldsymbol{k}\,.$$

EXAMPLE 2.8

Express the force **F** shown in Fig. 2–30 as a Cartesian vector.

SOLUTION

Since only two coordinate direction angles are specified, the third angle α must be determined from Eq. 2–8; i.e.,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\cos^2\alpha + \cos^2 60^\circ + \cos^2 45^\circ = 1$$

$$\cos\alpha = \sqrt{1 - (0.5)^2 - (0.707)^2} = \pm 0.5$$

Hence, two possibilities exist, namely,

$$\alpha = \cos^{-1}(0.5) = 60^\circ \qquad \text{or} \qquad \alpha = \cos^{-1}(-0.5) = 120^\circ$$

By inspection it is necessary that $\alpha = 60^\circ$, since $\mathbf{F}_x$ must be in the $+x$ direction.

Using Eq. 2–9, with $F = 200$ N, we have

$$\mathbf{F} = F\cos\alpha\mathbf{i} + F\cos\beta\mathbf{j} + F\cos\gamma\mathbf{k}$$

$$= (200\cos 60^\circ\text{ N})\mathbf{i} + (200\cos 60^\circ\text{ N})\mathbf{j} + (200\cos 45^\circ\text{ N})\mathbf{k}$$

$$= \{100.0\mathbf{i} + 100.0\mathbf{j} + 141.4\mathbf{k}\}\text{ N}$$ *Ans.*

Show that indeed the magnitude of $F = 200$ N.

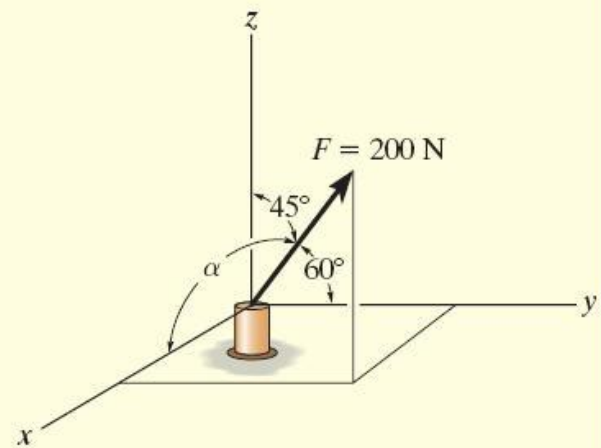
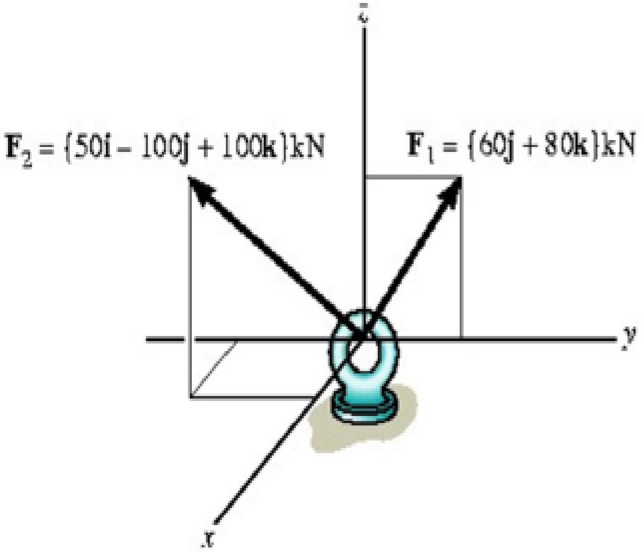


Fig. 2–30

Example 2.9

Determine the magnitude and coordinate direction angles of resultant force acting on the ring



Solution

Resultant force

$\mathbf{F}_R = \sum \mathbf{F}$

$= \mathbf{F}_1 + \mathbf{F}_2$

$= \{60\mathbf{j} + 80\mathbf{k}\}\text{kN}$

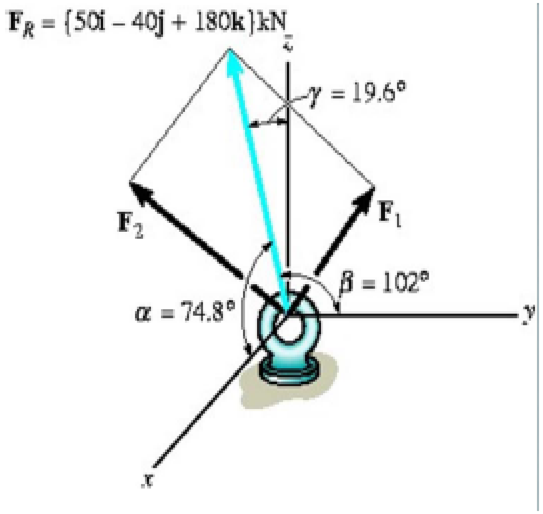
$+ \{50\mathbf{i} - 100\mathbf{j} + 100\mathbf{k}\}\text{kN}$

$= \{50\mathbf{j} -40\mathbf{k} + 180\mathbf{k}\}\text{kN}$

Magnitude of $\mathbf{F}_R$ is found by

$$F_R = \sqrt{(50)^2 + (-40)^2 + (180)^2}$$

$$= 191.0 = 191kN$$



Unit vector acting in the direction of $\mathbf{F}_R$

$$\mathbf{u}_{FR} = \mathbf{F}_R / F_R$$

$= (50/191.0)\mathbf{i} + (40/191.0)\mathbf{j} + (180/191.0)\mathbf{k}$

$= 0.1617\mathbf{i} - 0.2094\mathbf{j} + 0.9422\mathbf{k}$

So that

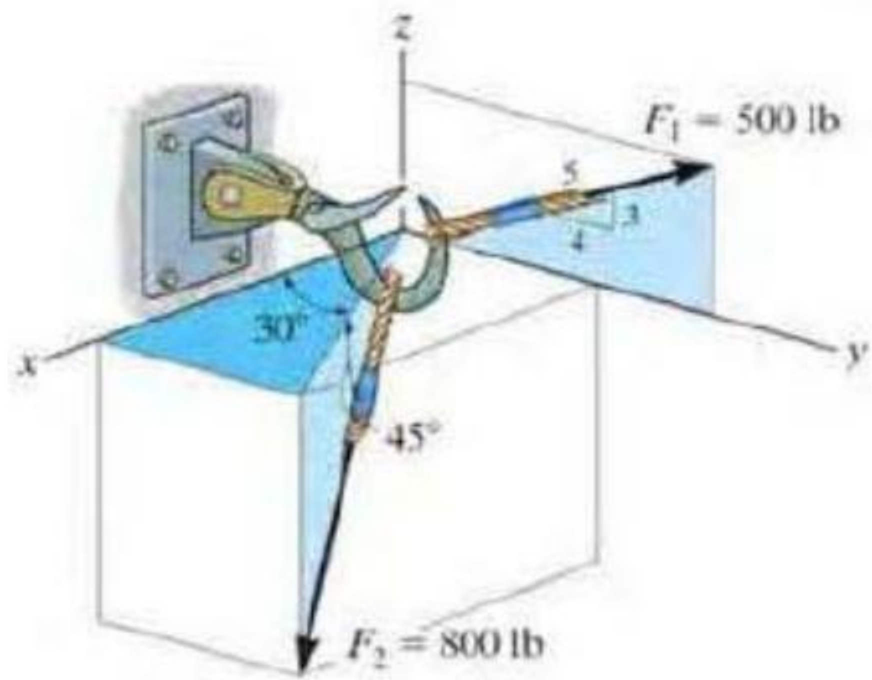
$\cos\alpha = 0.2617$ $\alpha = 74.8^\circ$

$\cos\beta = -0.2094$ $\beta = 102^\circ$

$\cos\gamma = 0.9422$ $\gamma = 19.6^\circ$

*Note $\beta > 90^\circ$ since $\mathbf{j}$ component of $\mathbf{u}_{FR}$ is negative

Determine the resultant force acting on the hook.

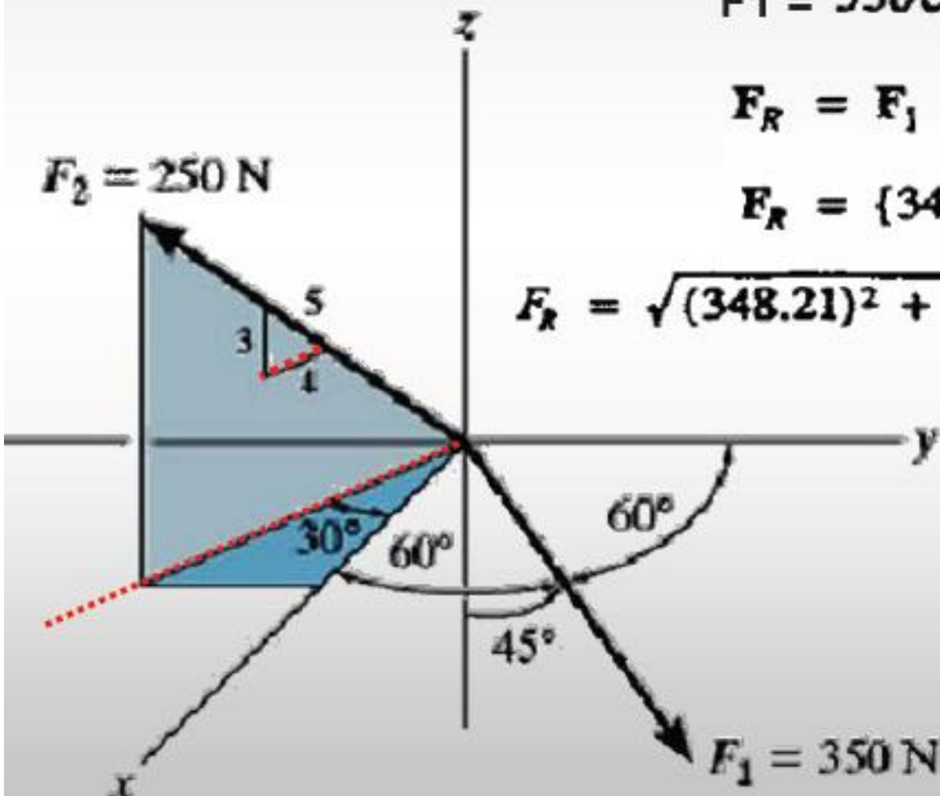


Ans: $\overrightarrow{\mathbf{F_R}} = \overrightarrow{\mathbf{F_1}} + \overrightarrow{\mathbf{F_2}} = \{490\,\vec{\mathbf{i}} + 683\,\vec{\mathbf{j}} - 266\,\vec{\mathbf{k}}\}$ lb

2-71 Determine the magnitude and coordinate direction angles of the resultant force and sketch this vector on the coordinate system.

$$F2 = 250\left(\frac{4}{5}\right)\cos 30°\mathbf{i} - 250\left(\frac{4}{5}\right)\sin 30°\mathbf{j} + 250\left(\frac{3}{5}\right)\mathbf{k}$$

$$F1 = 350\cos 60°\mathbf{i} + 350\cos 60°\mathbf{j} - 350\cos 45°\mathbf{k}$$



$$\mathbf{F}_R = \mathbf{F}_1 + \mathbf{F}_2$$

$$\mathbf{F}_R = \{348.21\mathbf{i} + 75.0\mathbf{j} - 97.487\mathbf{k}\}\ \text{N}$$

$$F_R = \sqrt{(348.21)^2 + (75.0)^2 + (97.487)^2} = 369.29\ \text{N}$$

$$\alpha = \cos^{-1}\left(\frac{348.21}{369.29}\right) = 19.5°$$

$$\beta = \cos^{-1}\left(\frac{75.0}{369.29}\right) = 78.3°$$

$$\gamma = \cos^{-1}\left(\frac{-97.487}{369.29}\right) = 105°$$

Force Magnitude and Two Points on its Line of Action

- Given two points in Space

  (A) with coordinates (X_a, Y_a, and Z_a)

  (B) with coordinates (X_b, Y_b, and Z_b)

- With a Force Vector (**F**) acting at point (A) in the direction of (B)

Force Magnitude and Two Points on its Line of Action

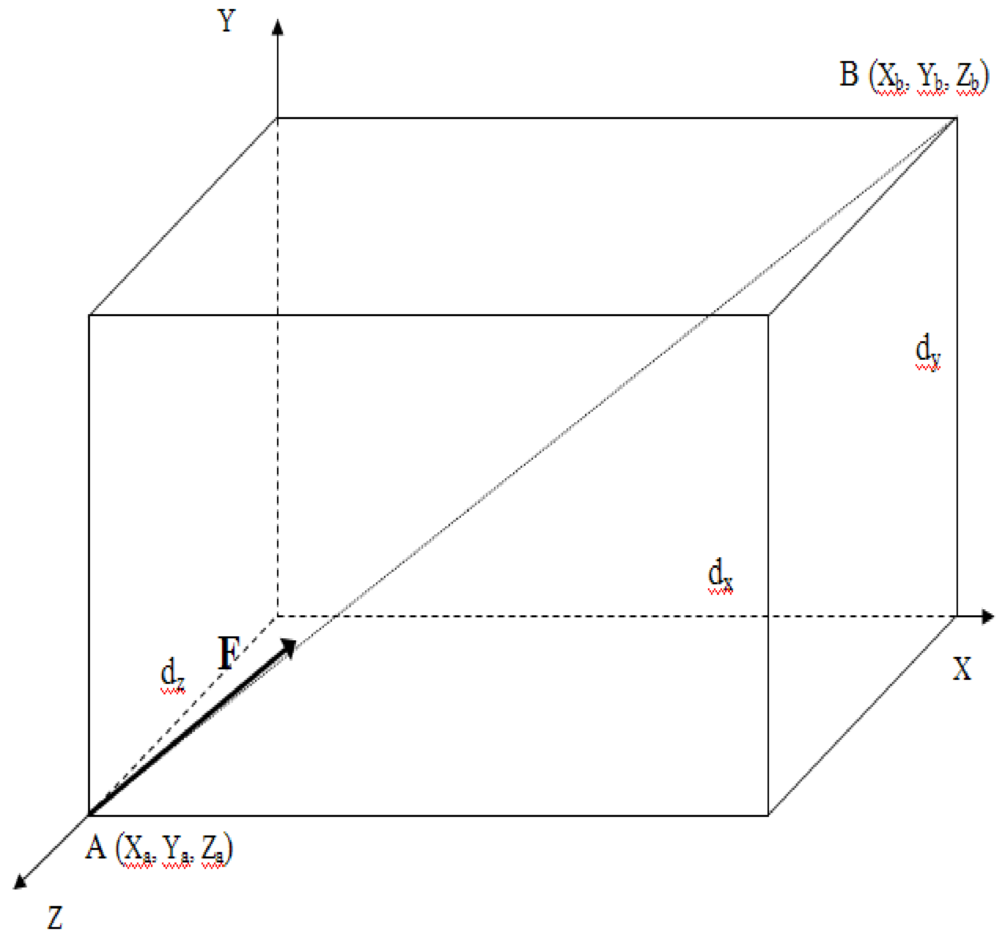
- Calculate the total displacement in rectangular components of Point (B) with respect to Point (A)

  $d_x = X_b - X_a$

  $d_y = Y_b - Y_a$

  $d_z = Z_b - Z_a$

- Total Displacement
- $(d) = \sqrt{(d_x^2 + d_y^2 + d_z^2)}$



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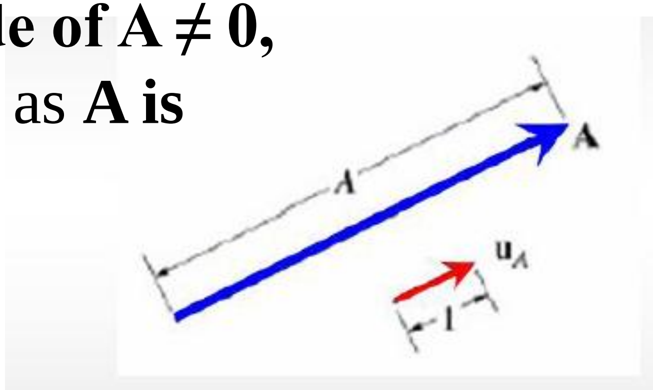


UA is dimensionless. It serves only to indicate direction and sense.

Force Magnitude and Two Points on its Line of Action

- Rectangular Components of the Force vector can then be found by:

  $F_x = F(d_x/d)$

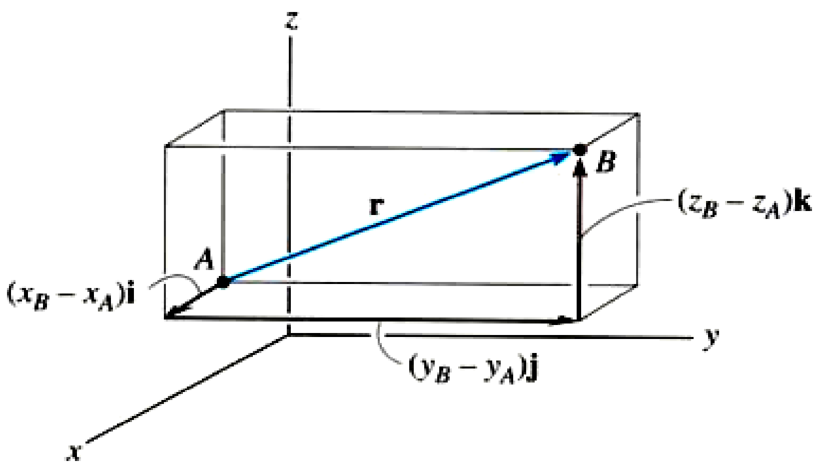
  $F_y = F(d_y/d)$

  $F_z = F(d_z/d)$

- Note: Direction Angles can be found using Rectangular Components of Force or Rectangular Displacements.

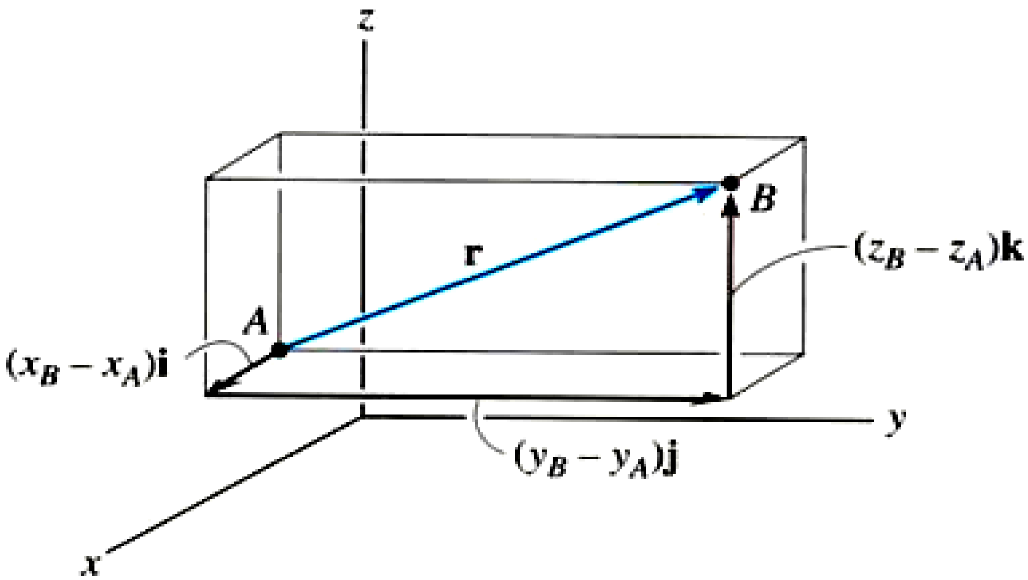
POSITION VECTOR

A position vector is defined as a fixed vector that locates a point in space relative to another point.



Consider two points, A and B, in 3-D space. Let their coordinates be (X_A, Y_A, Z_A) and (X_B, Y_B, Z_B), respectively.

POSITION VECTOR

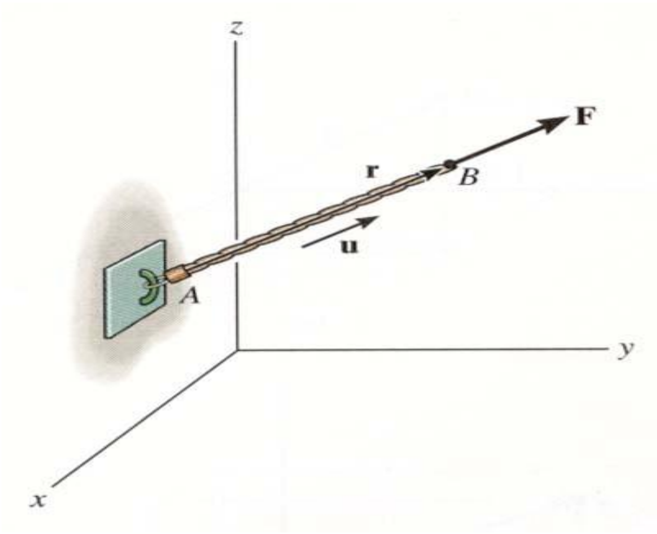


The position vector directed from A to B, $\boldsymbol{r}_{AB}$, is defined as

$$\boldsymbol{r}_{AB} = \{(X_B - X_A)\,\boldsymbol{i} + (Y_B - Y_A)\,\boldsymbol{j} + (Z_B - Z_A)\,\boldsymbol{k}\}\text{m}$$

Please note that B is the ending point and A is the starting point. ALWAYS subtract the "tail" coordinates from the "tip" coordinates!

Optional Way for 3D forces when a FORCE VECTOR DIRECTED ALONG A LINE (Section 2.8) using position and unit vectors:

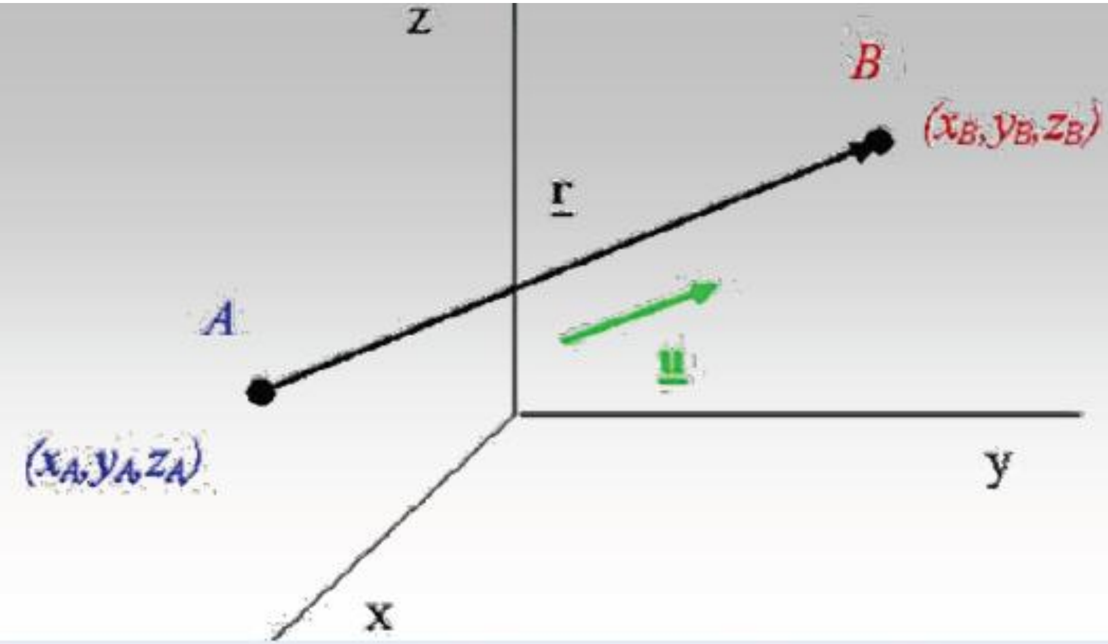


If a force is directed along a line, then we can represent the force vector in Cartesian coordinates by using a unit vector and the force's magnitude. So we need to:

a) Find the position vector, $\boldsymbol{r}_{AB}$, along two points on that line.

b) Find the unit vector describing the line's direction, $\boldsymbol{u}_{AB} = (\boldsymbol{r}_{AB}/|r_{AB}|)$.

c) Multiply the unit vector by the magnitude of the force, $\boldsymbol{F} = F\ \boldsymbol{u}_{AB}$.

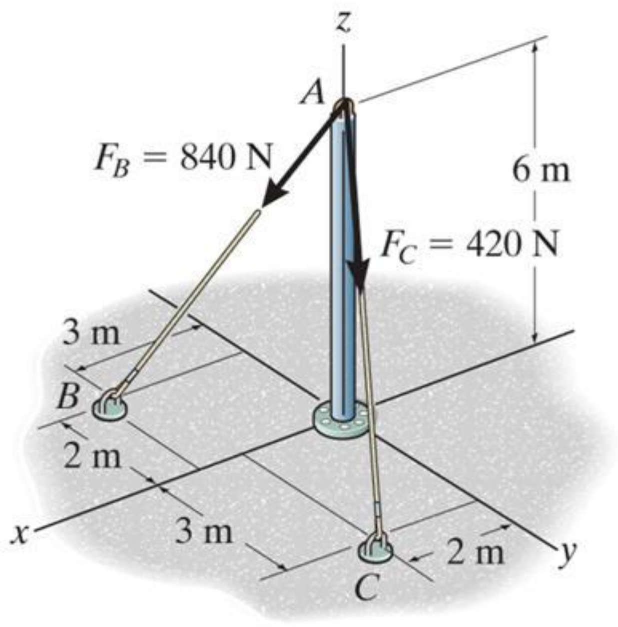


$$\boldsymbol{r} = (x_B - x_A)\boldsymbol{i} + (y_B - y_A)\boldsymbol{j} + (z_B - z_A)\boldsymbol{k}$$

$$\boldsymbol{u} = \frac{\boldsymbol{r}}{r} = \frac{(x_B - x_A)\boldsymbol{i} + (y_B - y_A)\boldsymbol{j} + (z_B - z_A)\boldsymbol{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}$$

$$F = F\frac{\boldsymbol{r}}{r} = F\boldsymbol{u}$$

EXAMPLE



Given: The 420 N force along the cable AC.

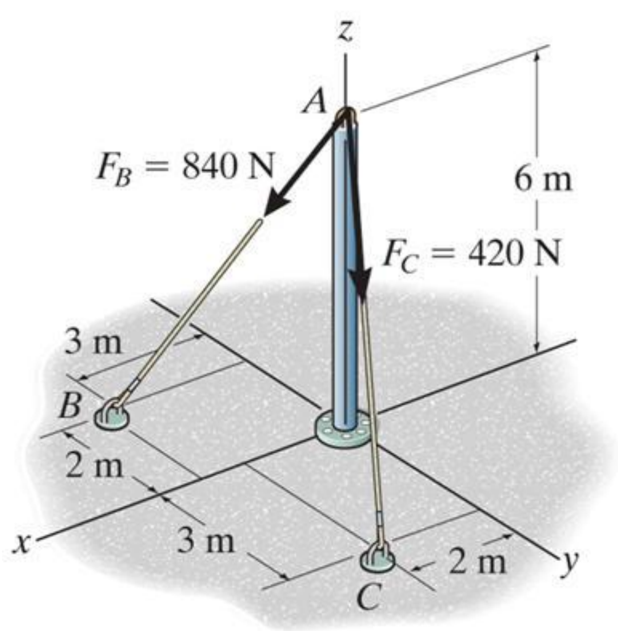
Find: The force $\boldsymbol{F_{AC}}$ in the Cartesian vector form.

Plan:

1. Find the position vector $\boldsymbol{r_{AC}}$ and the unit vector $\boldsymbol{u_{AC}}$.

2. Obtain the force vector as $\boldsymbol{F_{AC}}$ = 420 N $\boldsymbol{u_{AC}}$.

EXAMPLE (continued)



As per the figure, when relating A to C, we will have to go 2 m in the x-direction, 3 m in the y-direction, and -6 m in the z-direction. Hence,

$\boldsymbol{r}_{AC} = \{2\,\boldsymbol{i} + 3\,\boldsymbol{j} - 6\,\boldsymbol{k}\}$ m.

(We can also find $\boldsymbol{r}_{AC}$ by subtracting the coordinates of A from the coordinates of C.)

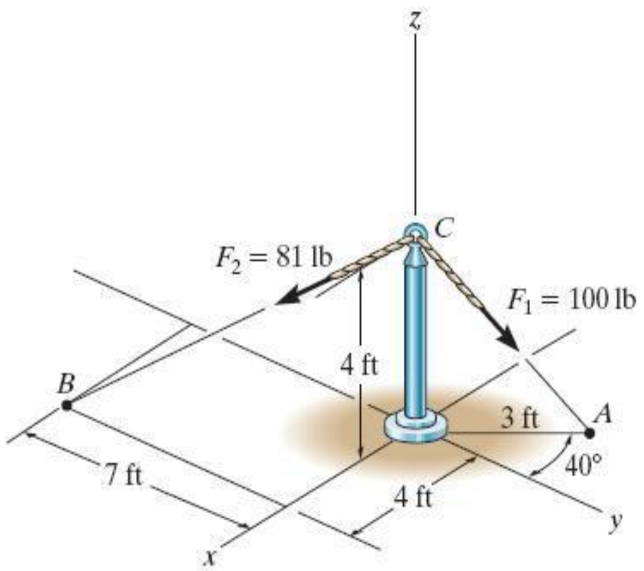
$r_{AC} = (2^2 + 3^2 + 6^2)^{1/2} = 7$ m

Now $\boldsymbol{u}_{AC} = \boldsymbol{r}_{AC}/r_{AC}$ and $\boldsymbol{F}_{AC} = 420\,\boldsymbol{u}_{AC}$ N $= 420\,(\boldsymbol{r}_{AC}/r_{AC})$

So $\boldsymbol{F}_{AC} = 420\{(2\,\boldsymbol{i} + 3\,\boldsymbol{j} - 6\,\boldsymbol{k}) / 7\}$ N

$= \{120\,\boldsymbol{i} + 180\,\boldsymbol{j} - 360\,\boldsymbol{k}\}$ N

Example:



Given: Two forces are acting on a pipe as shown in the figure.

Find: The magnitude and the coordinate direction angles of the resultant force.

<u>Plan</u>:

1) Find the forces along CA and CB in the Cartesian vector form.

2) Add the two forces to get the resultant force, $\boldsymbol{F_R}$.

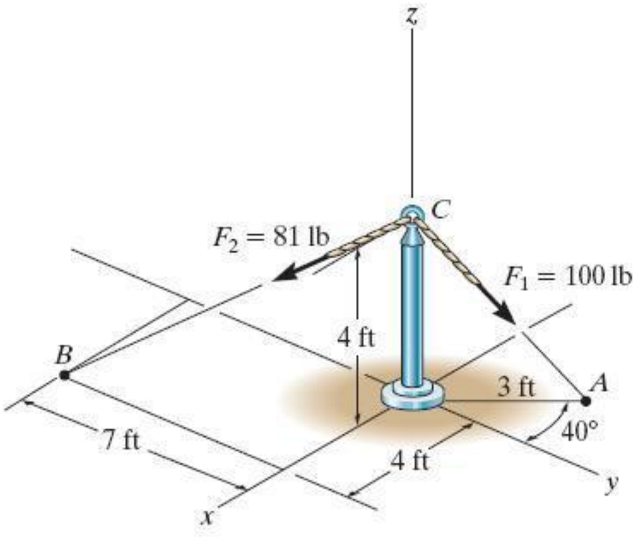
3) Determine the magnitude and the coordinate angles of $\boldsymbol{F_R}$.

(continued)

$\boldsymbol{F_{CA}} = 100 \text{ lb } (\boldsymbol{r_{CA}}/r_{CA})$

$\boldsymbol{F_{CA}} = 100 \text{ lb } (-3 \sin 40° \, \boldsymbol{i} + 3 \cos 40° \, \boldsymbol{j} - 4 \, \boldsymbol{k})/5$

$\boldsymbol{F_{CA}} = (-38.57 \, \boldsymbol{i} + 45.96 \, \boldsymbol{j} - 80 \, \boldsymbol{k}) \text{ lb}$



$\boldsymbol{F_{CB}} = 81 \text{ lb } (\boldsymbol{r_{CB}}/r_{CB})$

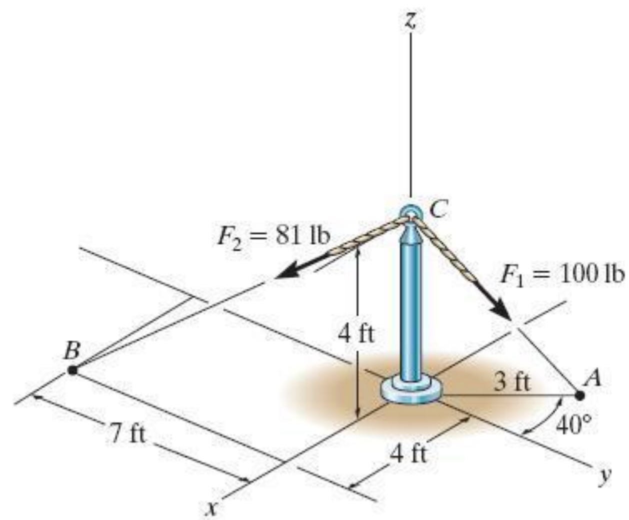
$\boldsymbol{F_{CB}} = 81 \text{ lb } (4 \, \boldsymbol{i} - 7 \, \boldsymbol{j} - 4 \, \boldsymbol{k})/9$

$\boldsymbol{F_{CB}} = \{36 \, \boldsymbol{i} - 63 \, \boldsymbol{j} - 36 \, \boldsymbol{k}\} \text{ lb}$

(continued)

$\boldsymbol{F_R} = \boldsymbol{F_{CA}} + \boldsymbol{F_{CB}}$

$= \{-\,2.57\,\boldsymbol{i} - 17.04\,\boldsymbol{j} - 116\,\boldsymbol{k}\}$ lb



$F_R = (2.57^2 + 17.04^2 + 116^2)$

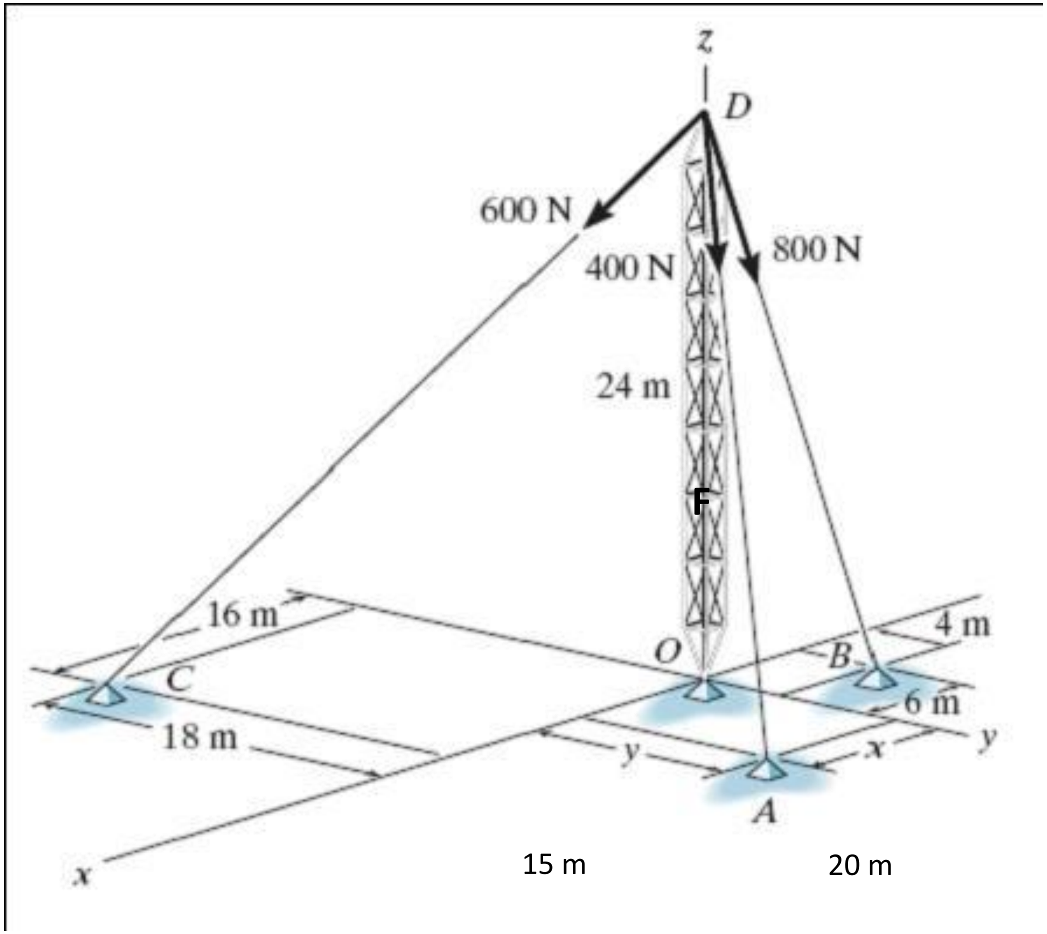
$= 117.3$ lb $= 117$ lb

$\alpha = \cos^{-1}(-2.57/117.3) = 91.3°$

$\beta = \cos^{-1}(-17.04/117.3) = 98.4°$

$\gamma = \cos^{-1}(-116/117.3) = 172°$

Example :the tower is held in place by three cables if the force of each cable acting on the tower as shown .determine the magnitude and coordinate angles of the resultant force

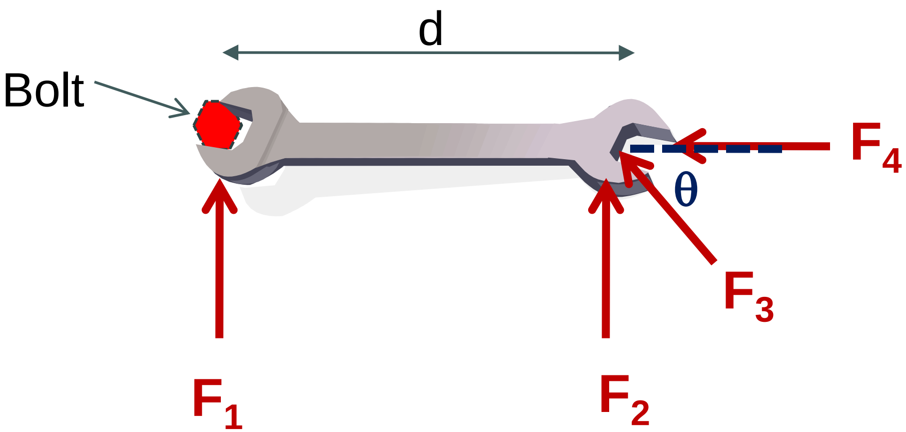


Introduction: Moments

***<u>The moment</u>* of a force about a point or an axis provides a measure of the tendency of the force to cause a body to rotate about the point or axis.**

Introduction: Moments

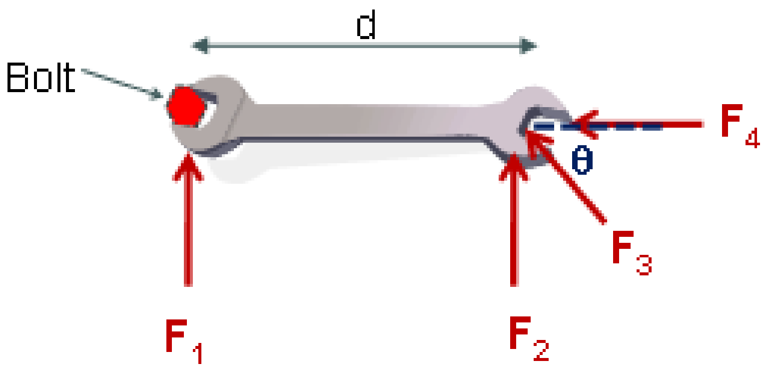
- Moment: *the measure of a force's ability to cause rotation*
  - Depends upon:
    - Magnitude of force
    - Direction of force
    - Rotational point
    - Location of applied force



Introduction: Moments

- In mathematical terms, *the magnitude of the moment is given by* :

$$M=Fd$$



Where

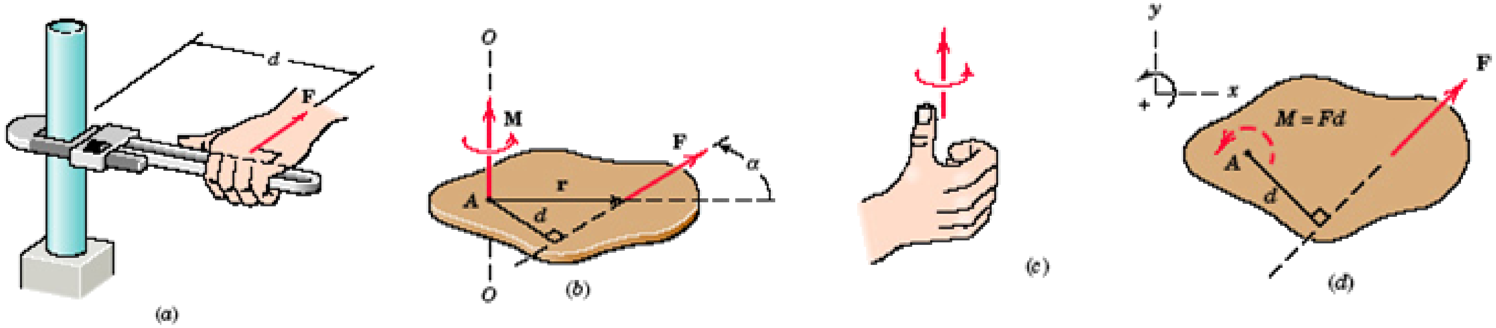
- F: is the force component that cause rotation, (its usually the normal component to the moment arm).
- d: is the distance between force and rotation point (or the moment arm.

- **THUS**, for F_3

$$M=(F_3 \sin\theta)d$$

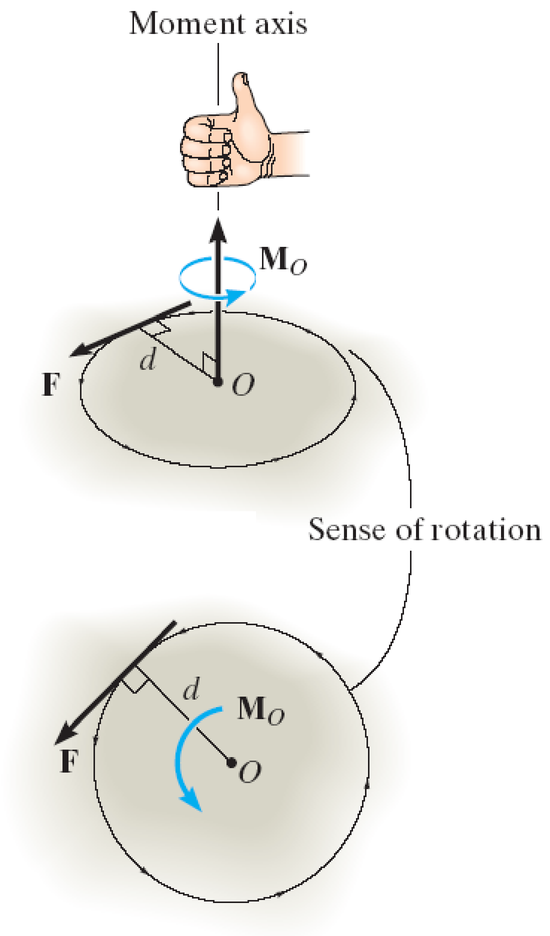
Moment about a point

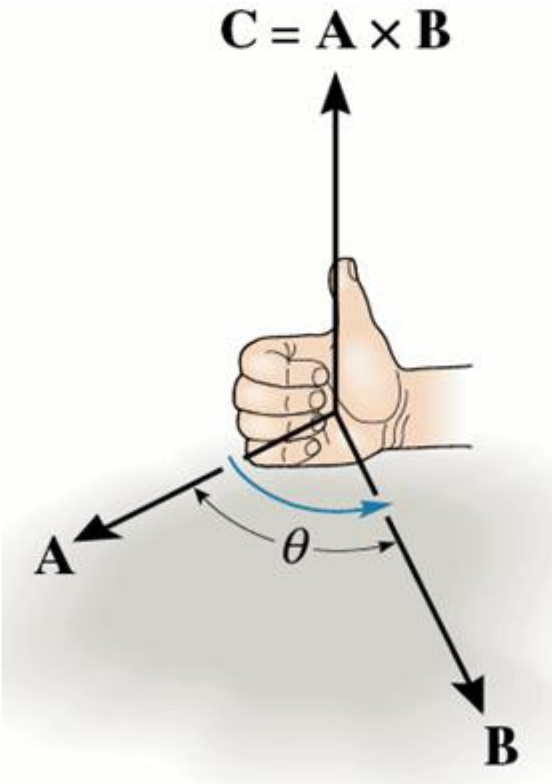
- Moment is a vector
  - **Magnitude** of moment is proportional to force applied (F) and perpendicular distance (moment arm, d)
    M=Fd (measured in N·m or lb·ft)
  - **Direction** is always perpendicular to the plane of the body – describes the direction of rotation
    - In x-y plane, direction will always be either + or – z-direction, typically described as clockwise (CW) or counter-clockwise (CCW) rotation

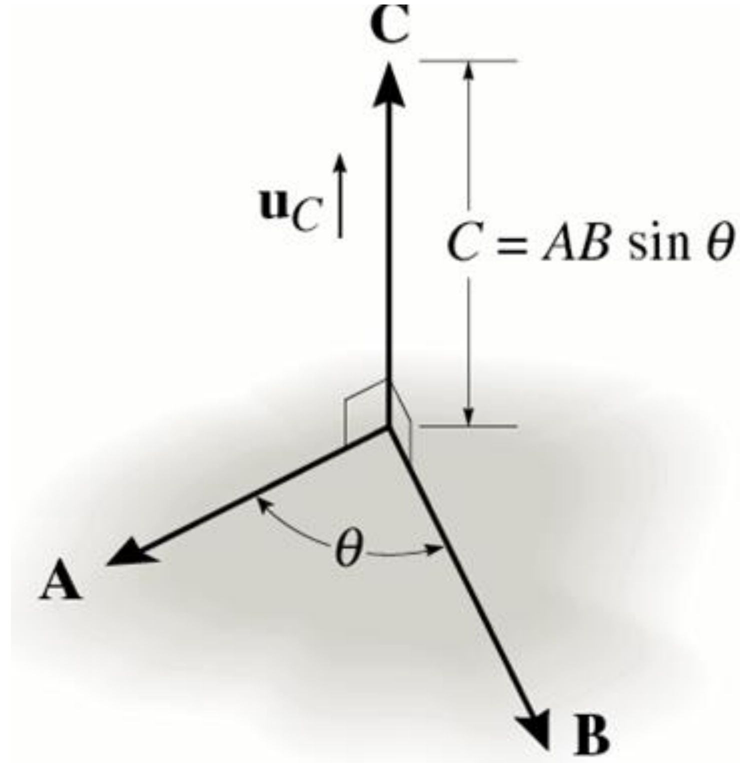


Moment about a point

- Right-hand rule
  - In 2-D, the direction of M_O is either clockwise or counter-clockwise depending on the tendency for rotation.
  - Your thumb points along the moment axis
  - Your fingers curl in the direction of the rotational tendency
  - Typical sign convention: CCW is positive, CW is negative

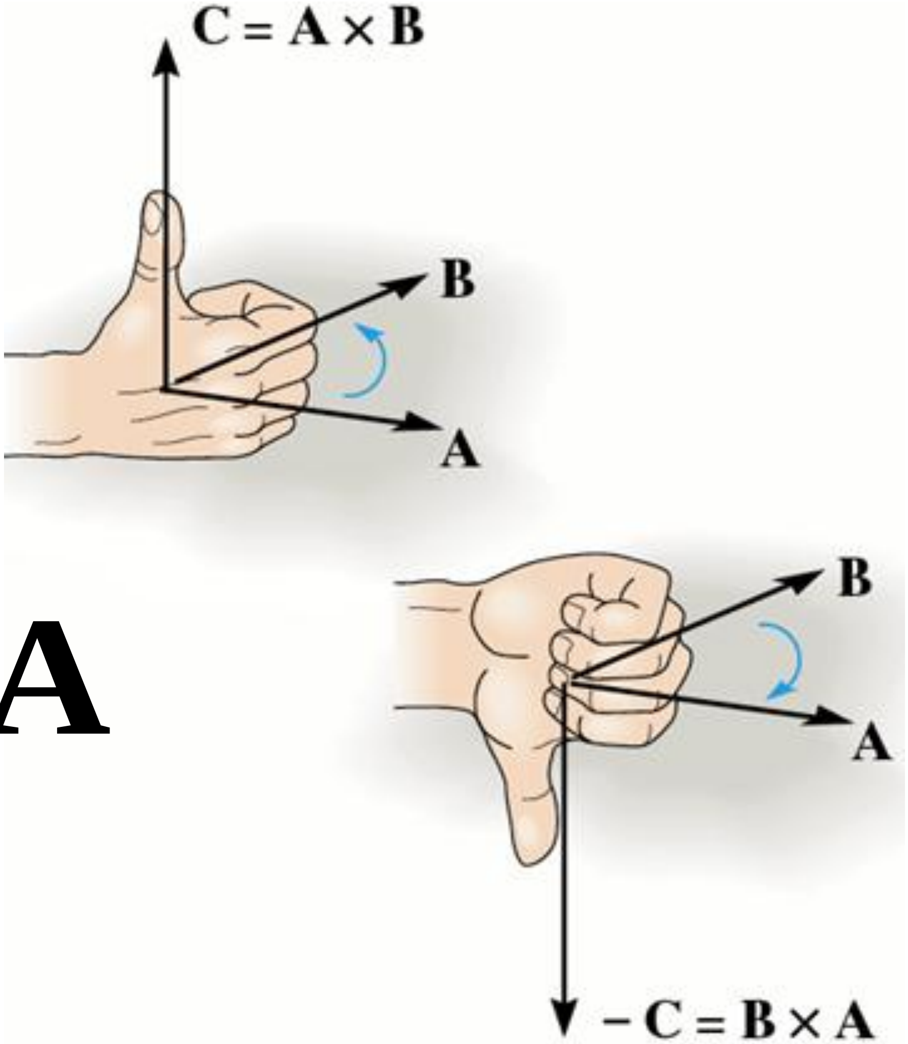



C = A × B
A
θ
B


C
$\mathbf{u}_C$
$C = AB \sin \theta$
A
θ
B

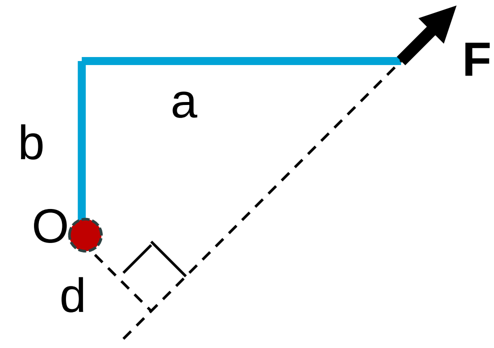
REMEMBER:

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$



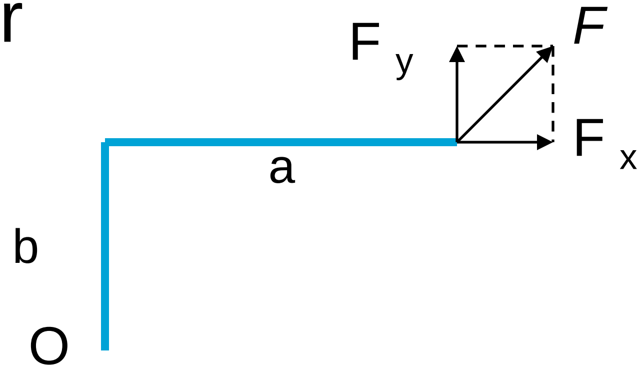
Moment about a point

- Which direction will this force (F) tend to rotate the beam about point *O*?



Counter-clockwise

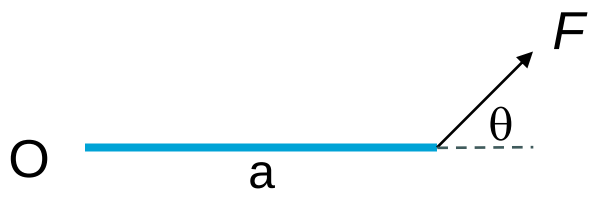
- Often, it is easier to analyze by breaking the force up into its components (d may be difficult or time-consuming to find)

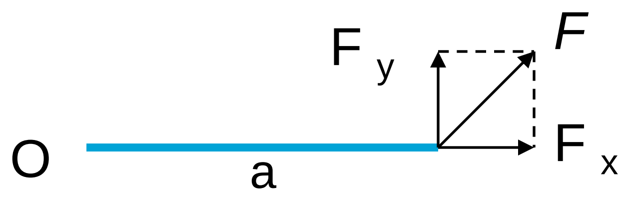


- Now: $M_O = (F_Y\ a) – (F_X\ b)$
  - Note the sign convention

Moment about a point

- This process is also useful when examining the component of force contributing to rotation





- What component of the force, **F** will cause rotation about *O*?
- Solution – split up into x- and y- components and treat separately. Does F_x contributed to rotation? Does F_y?

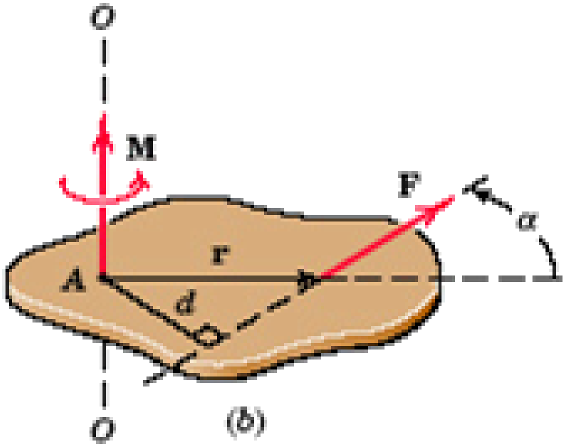
$$M = Fa\sin\theta$$

Computing Moments Using a Cross Product

- Vector approach to computing moment

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

  - **F** – the force contributing to rotation
  - **r** – position vector which the point of rotation (A) with any point on line of action of the Force vector, **F**.



Computing Moments Using a Cross Product

- Recall: Cross product calculation

$$\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k} \text{ and } \mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(r_y \cdot F_z - r_z \cdot F_y\right)\mathbf{i} + \left(r_z \cdot F_x - r_x \cdot F_z\right)\mathbf{j} + \left(r_x \cdot F_y - r_y \cdot F_x\right)\mathbf{k}$$

- Note - $\mathbf{M} = \mathbf{r} \times \mathbf{F} \neq \mathbf{F} \times \mathbf{r}$

Determinant form:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

For Element $\hat{i}$:

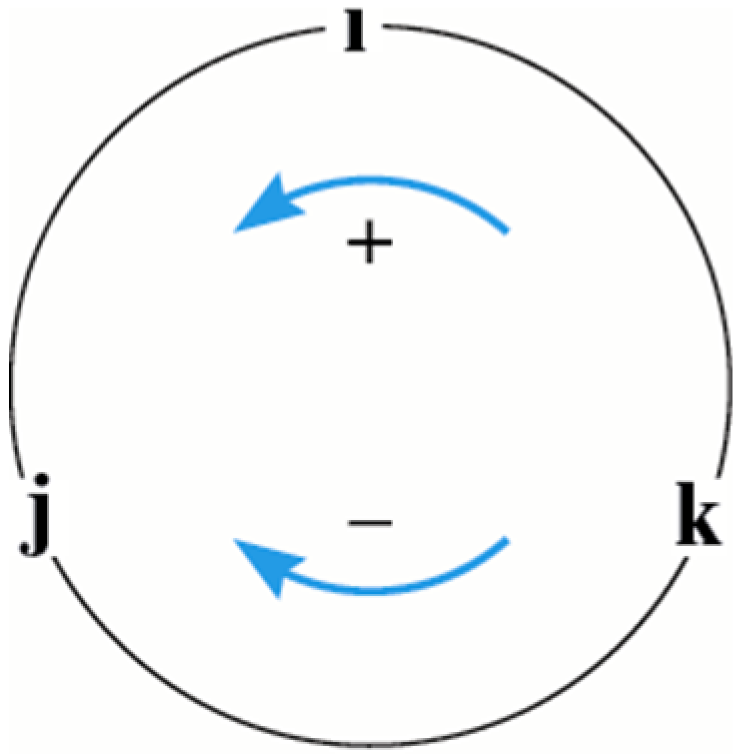
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}\ (A_yB_z - A_zB_y)$$

For Element $\hat{j}$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\hat{j}\ (A_xB_z - A_zB_x)$$

For Element $\hat{k}$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{k}\ (A_xB_y - A_yB_x)$$


l
+
j
−
k

Varignon's Theorem

The moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.

- Similar to a resultant force having the same net effect as the original forces, a similar resultant can be obtained from the addition of moments.

Force: $$\mathbf{R} = \mathbf{F_1} + \mathbf{F_2} + \ldots + \mathbf{F_n}$$

Moment: $$\mathbf{M_o} = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times \mathbf{F_1} + \ldots + \mathbf{r} \times \mathbf{F_n}$$

~OR~ $$\left|\mathbf{M_o}\right| = Rd_R = F_1 \cdot d_1 + \ldots + F_n \cdot d_n$$

Varignon's Theorem

- In this case:

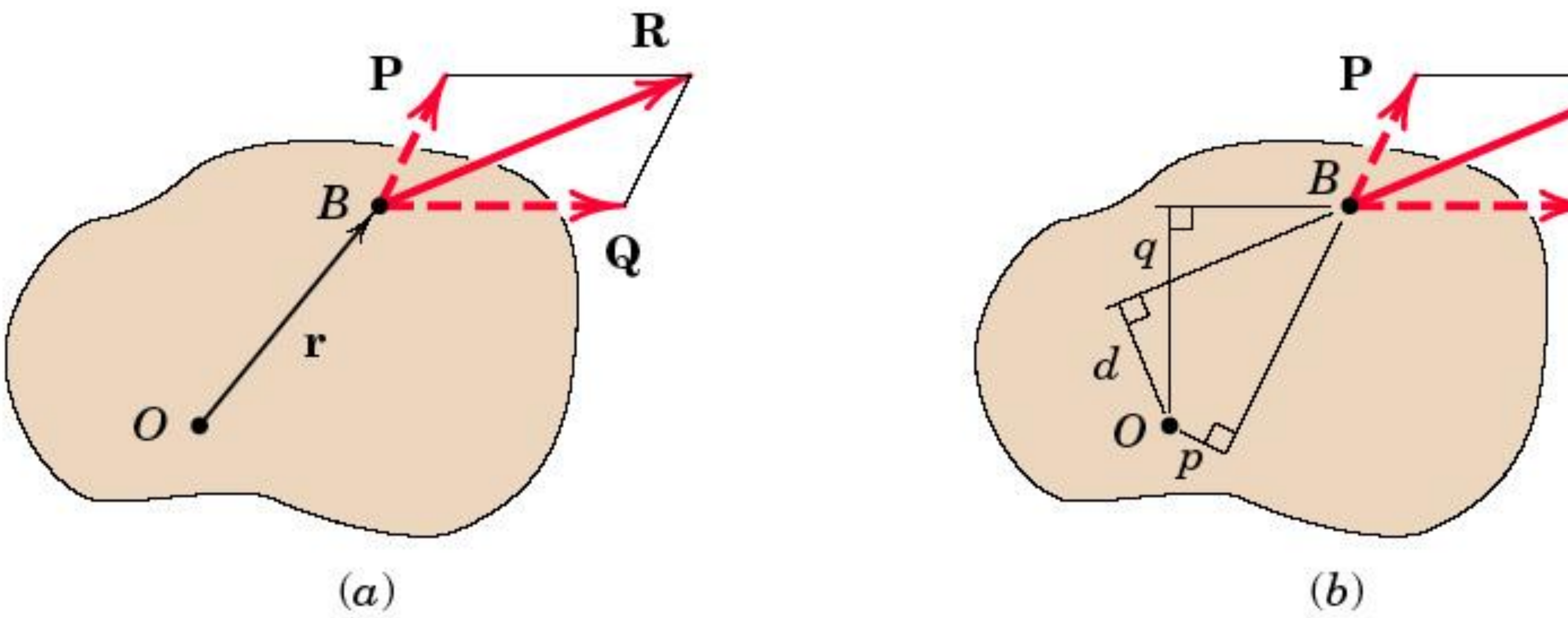
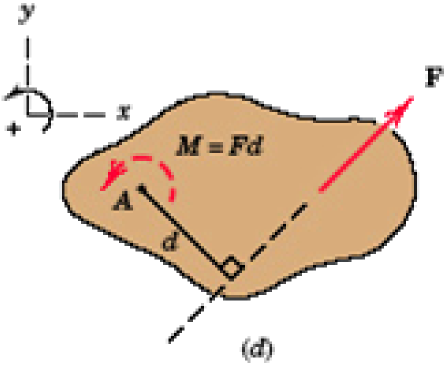


Figure 2/9

$$\mathbf{M_o} = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q}$$

A few notes about moments

- Be consistent! Make sure you always write down your sign convention.
- Don't forget you can always <u>move</u> a force vector <u>through its line of action</u>. This can be useful for simplifying your distance measurements.



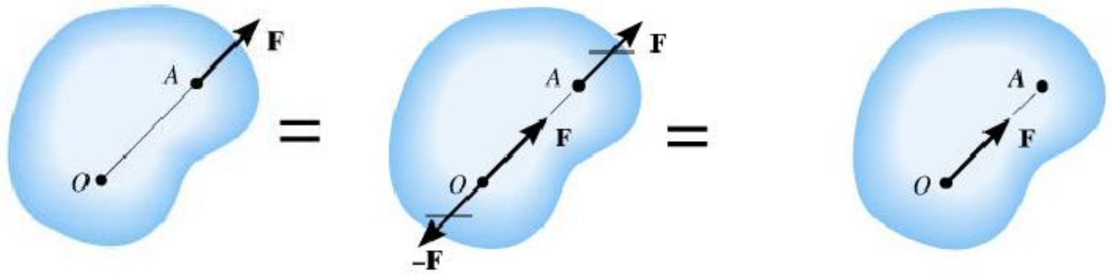
- There are multiple ways to solve for the magnitude and direction of a moment. Be familiar with all approaches, as one may be better for a particular problem.

Force-Couple Systems

- When a number of forces and couple moments are acting on a body, it may be easier to understand their overall effect if they are combined into a single force and couple moment having the same external effect
  - The two force and couple systems are called **equivalent systems** since they have the same **external** effect on the body
- To ensure the two systems are equivalent, must pay attention to forces and their effects
  - If you move a force from one point to another, you must consider any moments the force was inducing

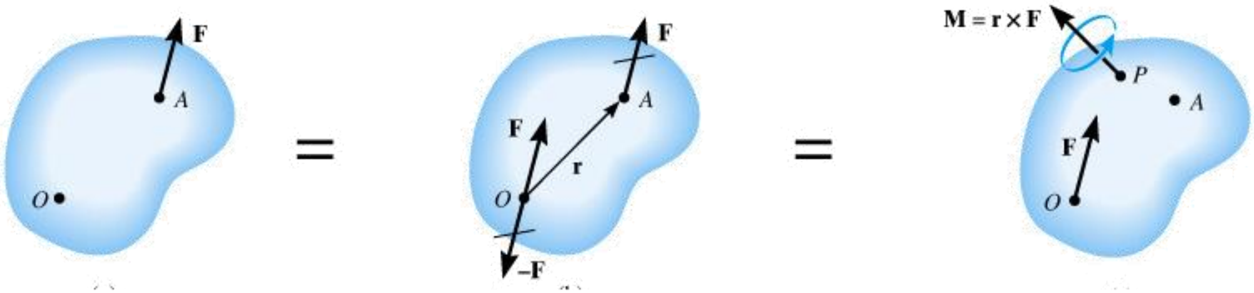
Moving a Force

- Along the same line of action:



(recall the principle of transmissibility)

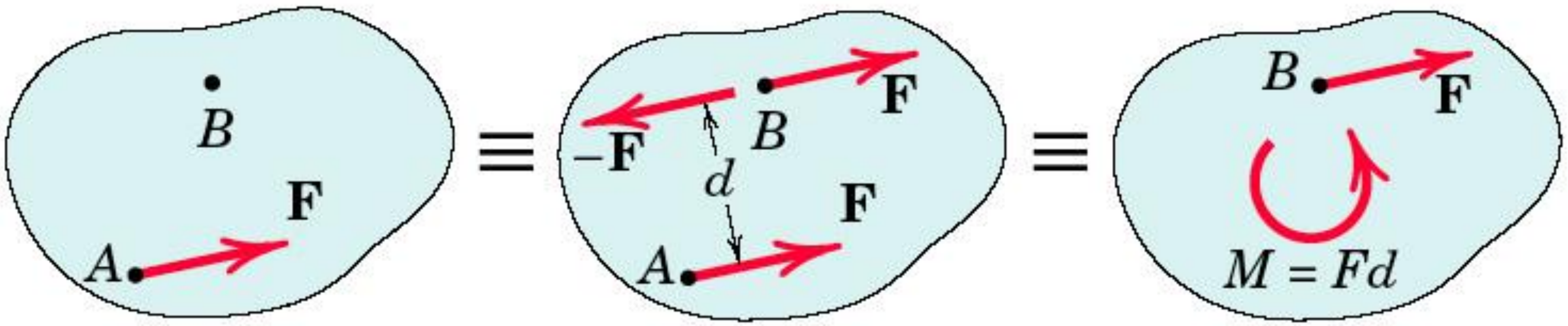
- Off the line of action



This requires creating an additional couple moment to account for original effect. The moment is a free vector, so can be applied at any point P in the body

Equivalent Force-Couple Systems

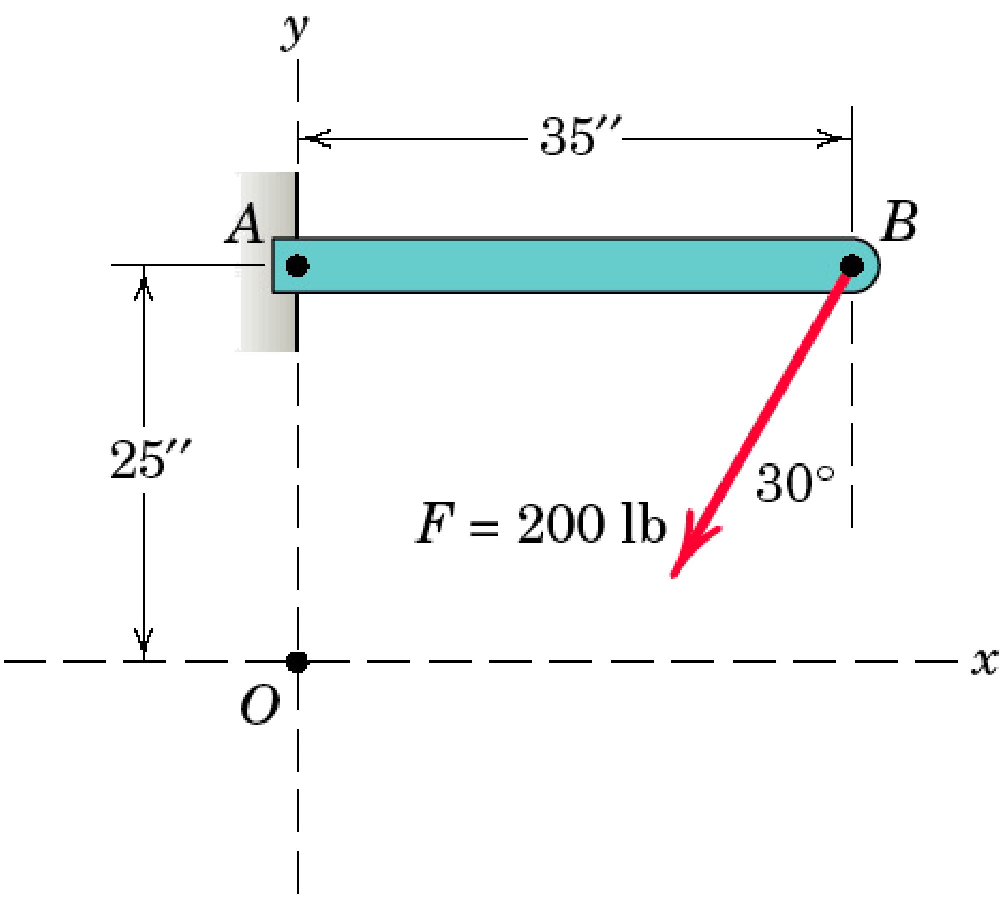
- To find an equivalent force-couple system at Point *B* to that shown in (I):
  1. Compute the moment at *B* resulting from force **F**
  2. Move the force to the point *B*
  3. Include a moment equal to that computed in (1)



(I) (II) (III)

Example 1

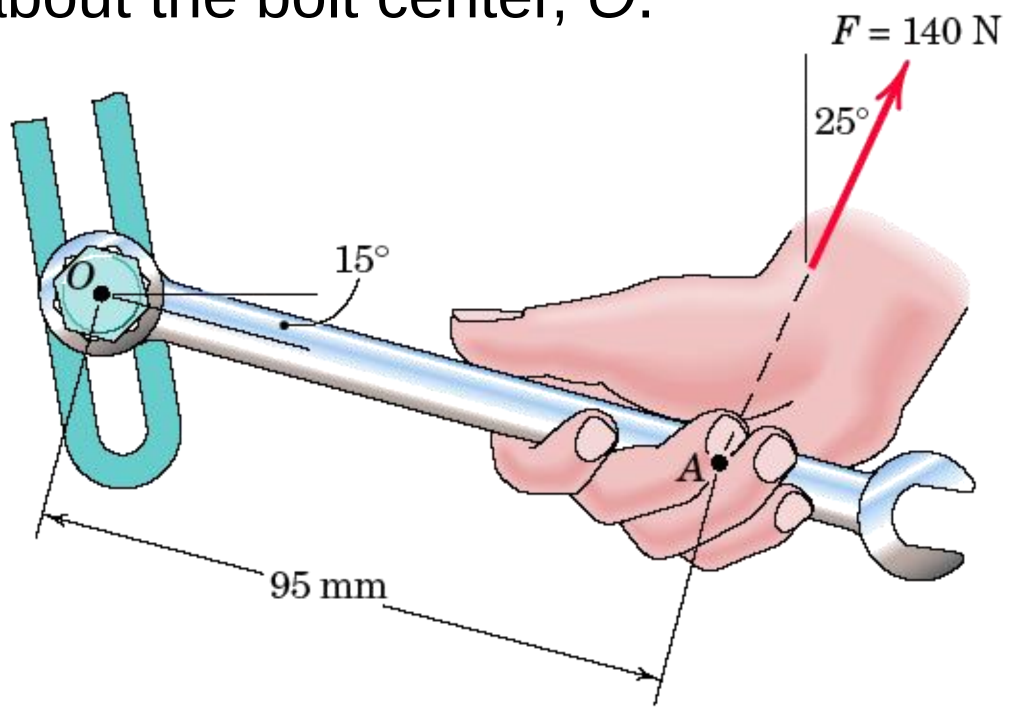
<u>Problem 2/30</u>: Determine the moment of the 200-lb force about point *A* and about point *O.*



Problem 2/30

Example 2

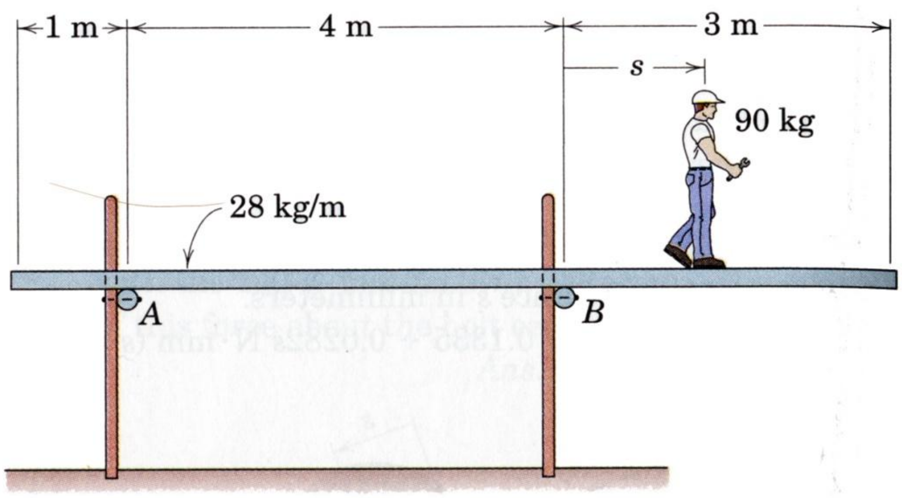
Problem 2/37: A mechanic pulls on the 13-mm combination wrench with the 140-N force shown. Determine the moment of this force about the bolt center, *O*.



Problem 2/37

Example 3

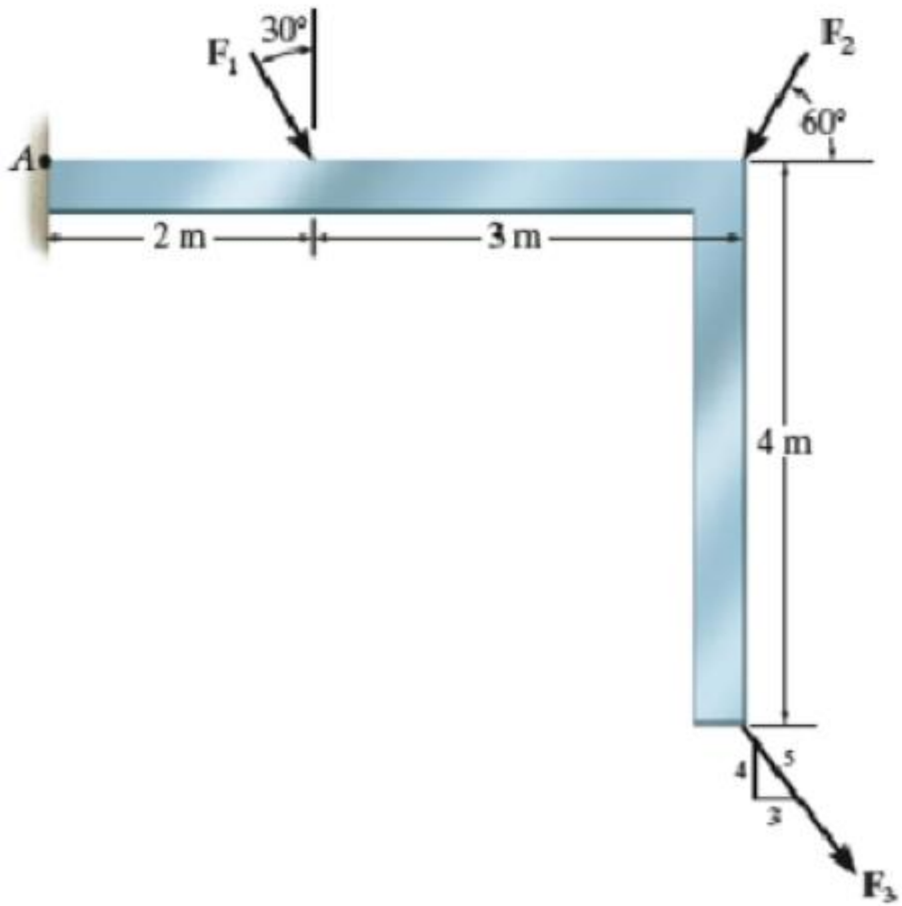
Problem 2/44: The uniform work platform, which has a mass per unit length of 28 kg/m, is simply supported by cross rods *A* and *B*. The 90-kg construction worker starts from point *B* and walks to the right. At what location *s* will the combined moment of the weights of the man and platform about point *B* be zero?



Problem 2/44

Example:

If the resultant moment about point A is 4800 N · m clockwise, determine the magnitude of F3 if F1 = 300 N and F2 = 400 N.

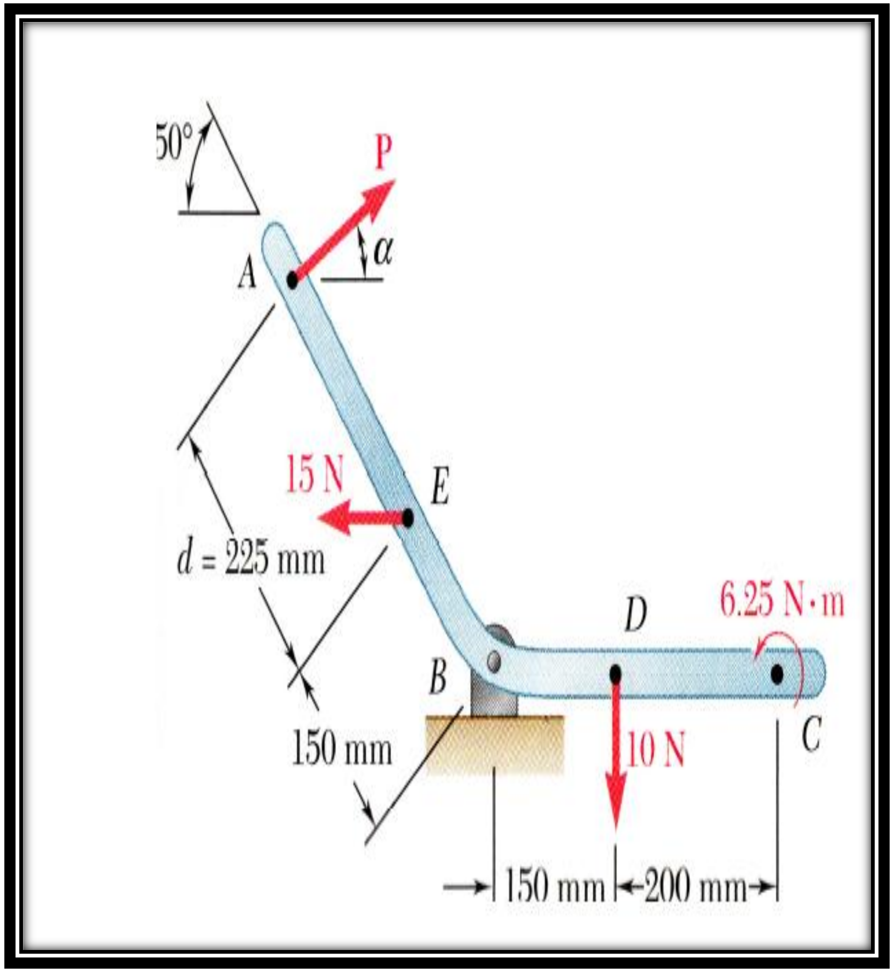


Example :

Three forces and a couple act on crank ABC.

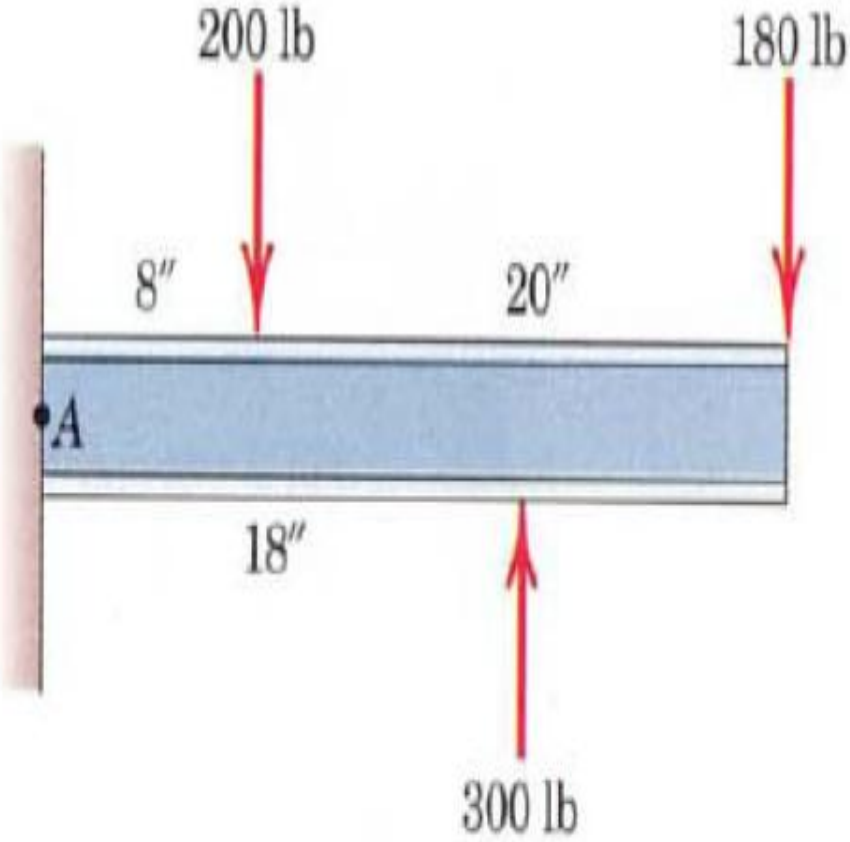
For P = 35 N and α = 40°

Determine the equivalent system consisting of the force resultant, **F**R, and the resultant moment, **M**R, about point B.

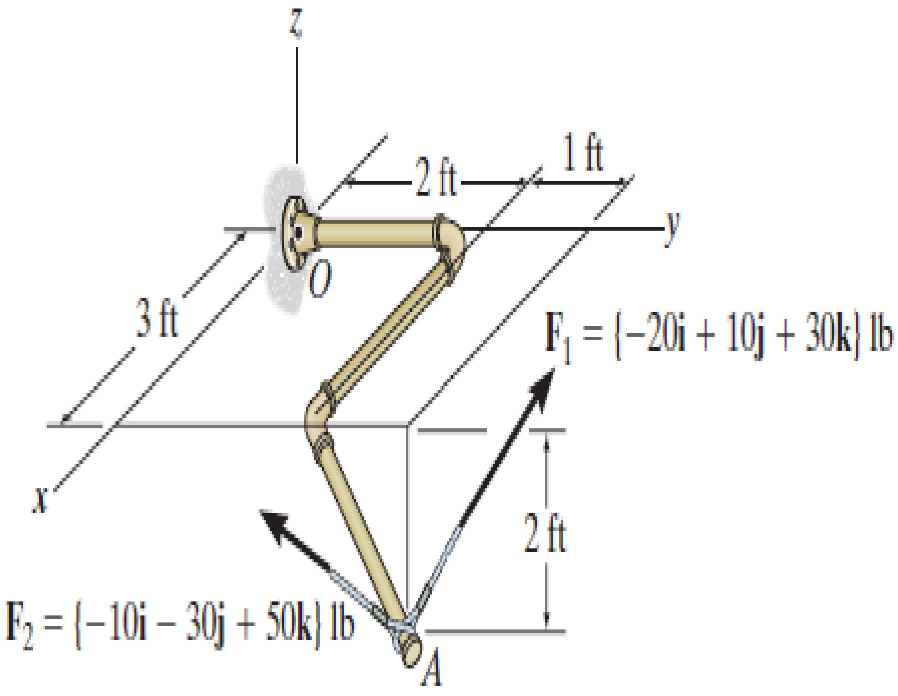


Example 3

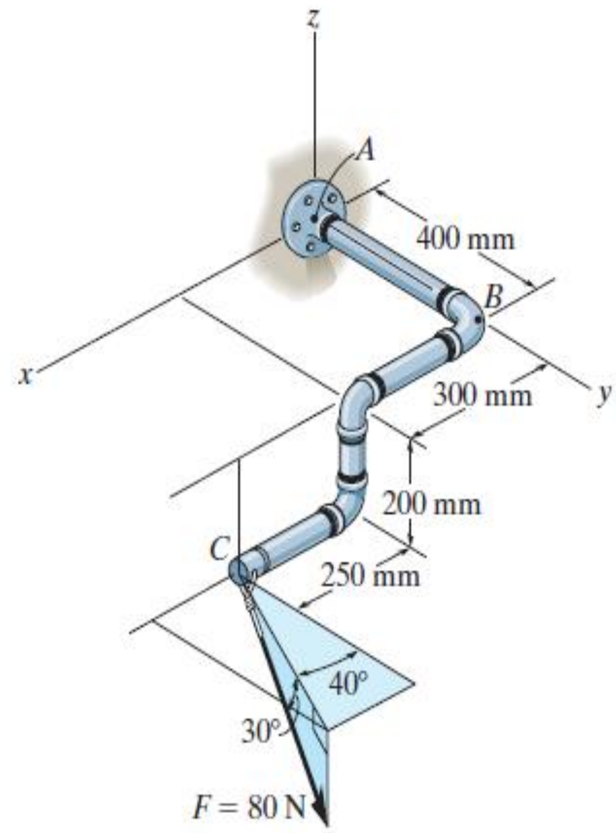
Problem 2/67: Reduce the given loading system to a force Couple system at point A . Then determine the distance X to the right of point A at which the resultant of three forces acts.



Example : Determine the moment produced by F1 about point *O*. Express the result as a Cartesian vector



Example: The pipe assembly is subjected to the 80-N force. Determine the moment of this force about point *A*.



Conditions for Rigid Body Equilibrium

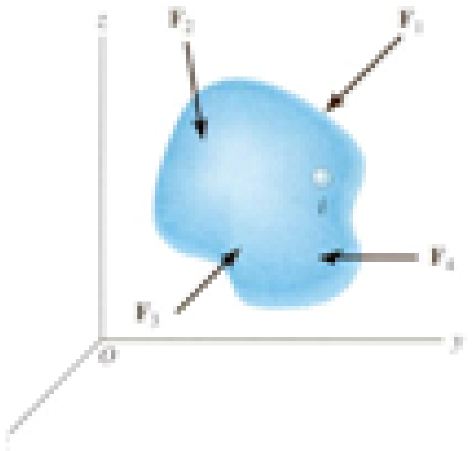
For a rigid body to be in equilibrium:

1- The *net force* must be *zero*.

2- The *net moment* about any *arbitrary point O* must be equal to *zero*.

OR

$$\sum \mathbf{F} = 0 \quad \text{and} \quad \sum \mathbf{M}_O = 0$$



Forces on a rigid body

Applying Equilibrium Conditions

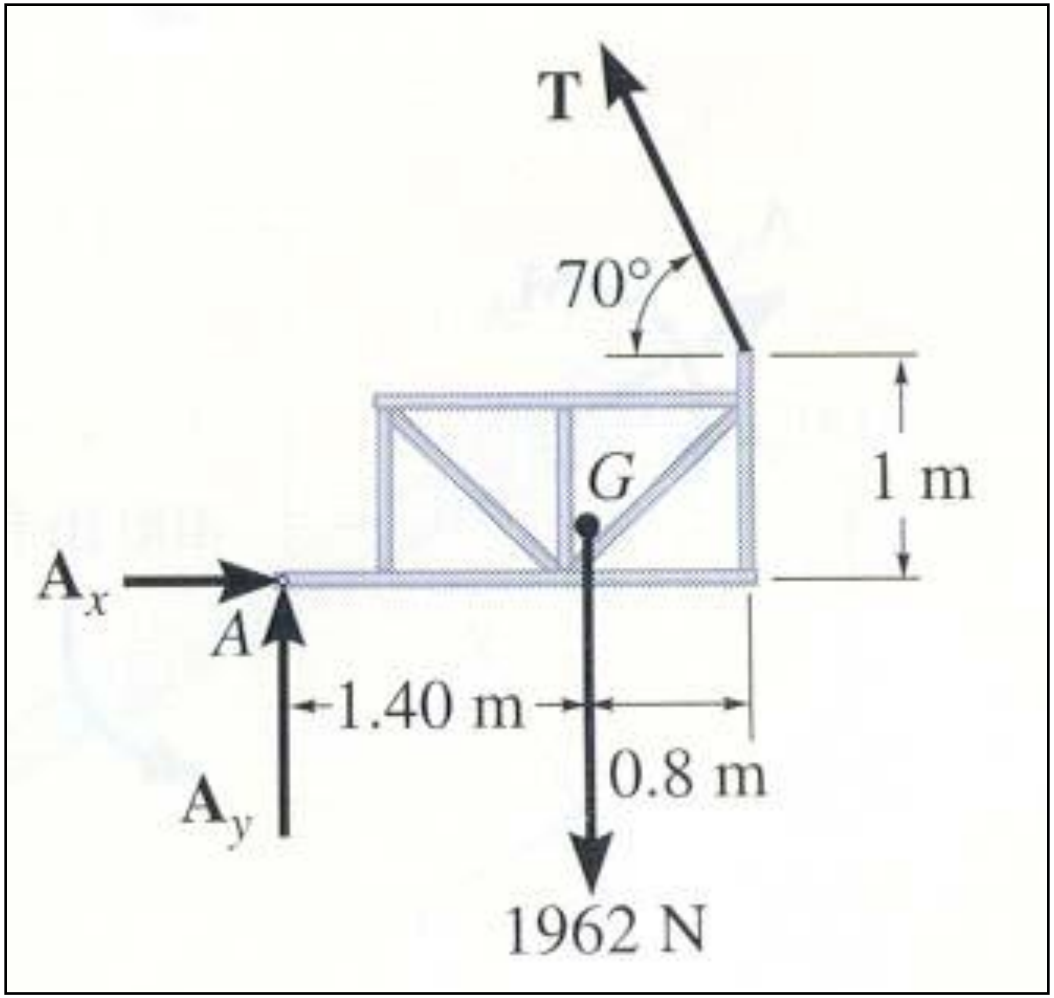
- Necessary and sufficient conditions for equilibrium of a rigid body

$$\sum F = 0; \quad \sum M_O = 0$$

- Equilibrium in Two-Dimensions with (*x*-*y*) coordinates:

$$\sum F_X = 0$$
$$\sum F_Y = 0$$
$$\sum M_O = 0$$

Solving Rigid Body Equilibrium Problems



Solve for unknowns using the equilibrium conditions

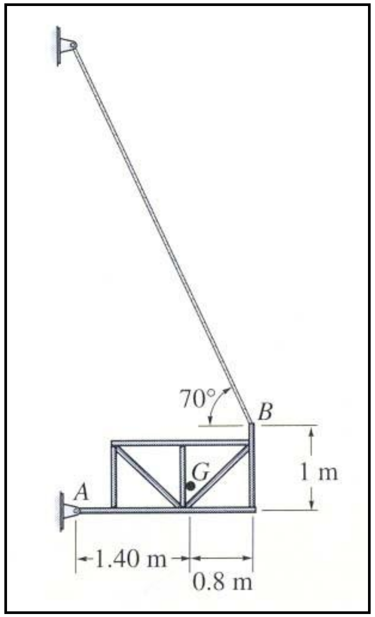
$$\sum \mathbf{F} = 0$$

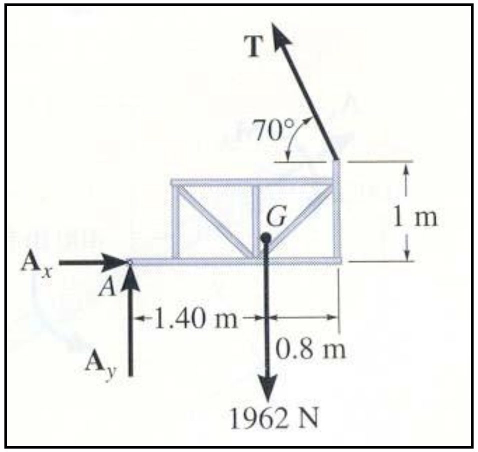
$$\sum \mathbf{M} = 0$$

Free body diagram

- Free body diagram: a schematic that shows all external *forces and moments* acting on this body

Recall: Solving Rigid Body Equilibrium Problems





- To analyze a physical system, first we need to create an <u>idealized model</u>.
- Next, we need to draw a <u>free-body diagram showing all the external (applied and reactive) forces</u>.
- Finally, we need to <u>apply the equations of equilibrium</u> to solve for any unknowns.

Free Body Diagrams

- Weight and the Center of Gravity
  - If the weight of the body is important to the analysis, it will be reported in the problem statement
  - The weight of a body is an external force, and its effect is shown as a single resultant force acting through the body's center of gravity
- Supports/Reaction Forces
  - Any item you "remove" when drawing FBDs (i.e. the wall or support joint) must be replaced by appropriate representative forces and moments (if necessary) which describe the effect of those objects
  - As a general rule, if a support prevents translation of a body in a given direction, then a force is developed on the body in the opposite direction. Similarly, if rotation is prevented, a couple moment is exerted on the body.
  - Common support reactions should be fully understood. These will be covered next lecture.

If the FBD is drawn correctly then solving the problem is trivial.

Free Body Diagrams

- Helpful tips:
  - Draw the forces exerted **on** the body to by isolated *by* the bodies to be removed
  - When smooth surfaces of two bodies are in contact, the force exerted by one body on the other is **normal** to the tangent to the surfaces and is compressive
  - When rough surfaces are in contact, in addition to a **normal** force, a **frictional** force, acting tangent to the surface an opposing motion, is also present

Support reactions

general rule:

if a <u>support prevents translation</u> of a body in a given direction, then <u>a force is developed</u> on the body in the opposite direction. Similarly, if <u>rotation is prevented, a couple moment</u> is exerted on the body.

Common Support Reactions

Figure 3/1,
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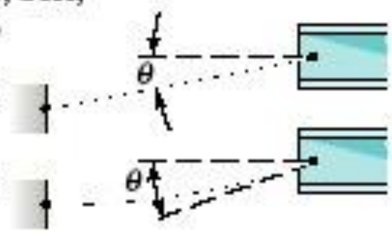
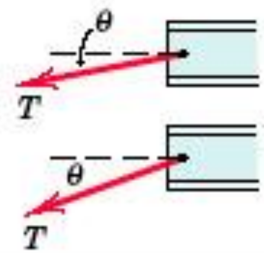
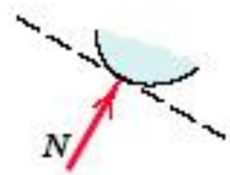
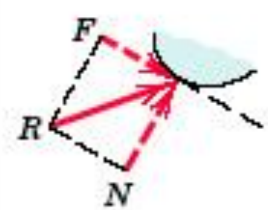
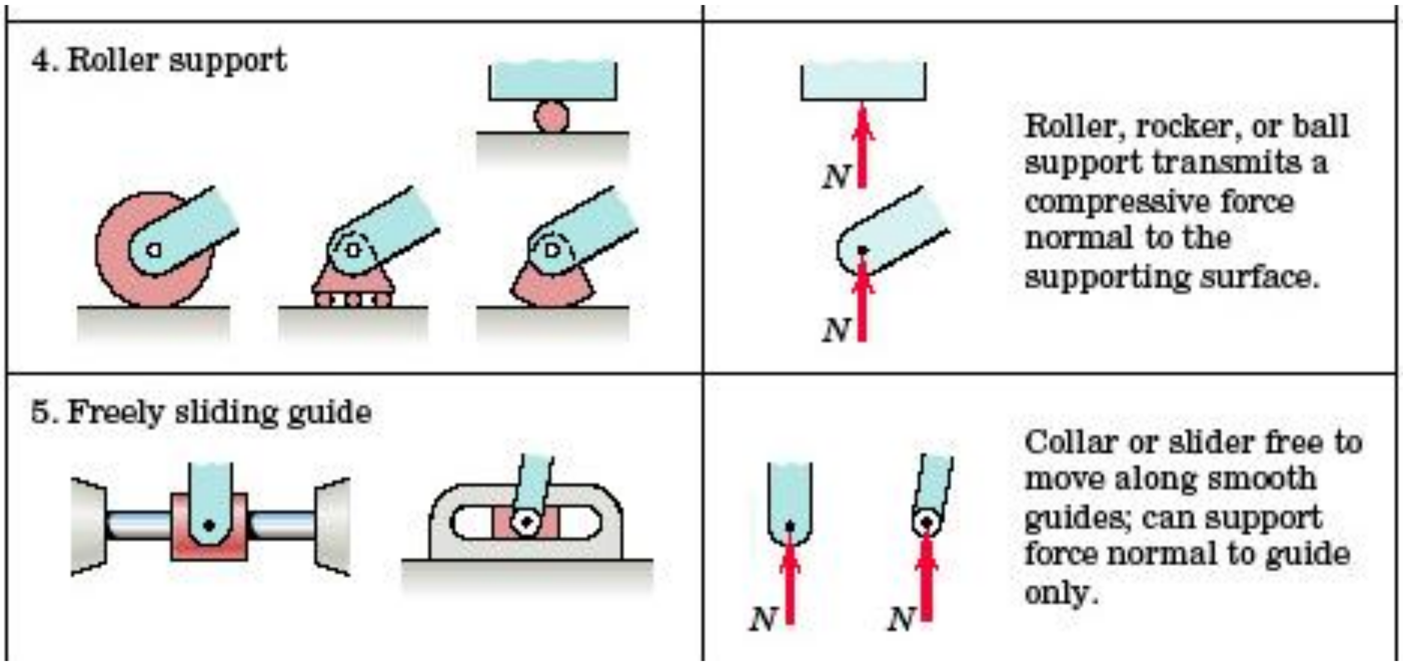
MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component *F* (frictional force) as well as a normal component *N* of the resultant contact force *R*.

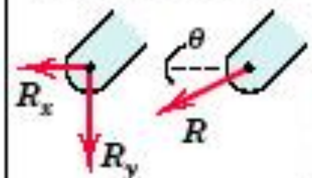
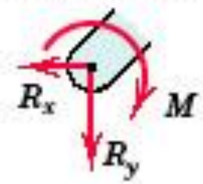
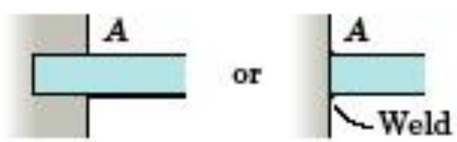
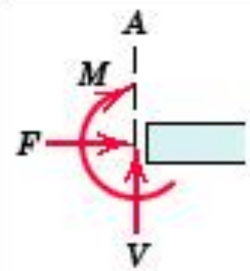
Common Support Reactions

Figure 3/1,
Page 111-112



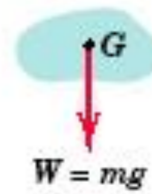
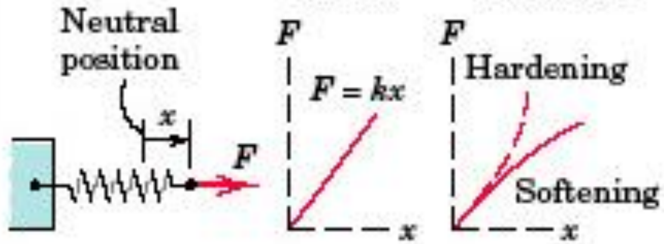
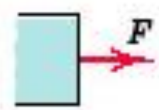
Common Support Reactions

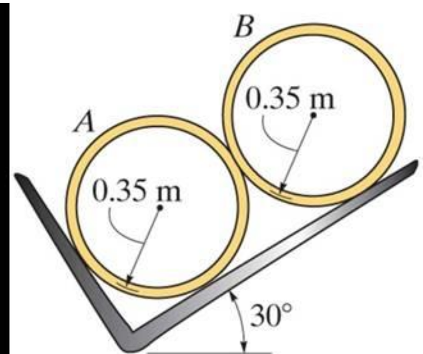
Figure 3/1,
Page 111-112

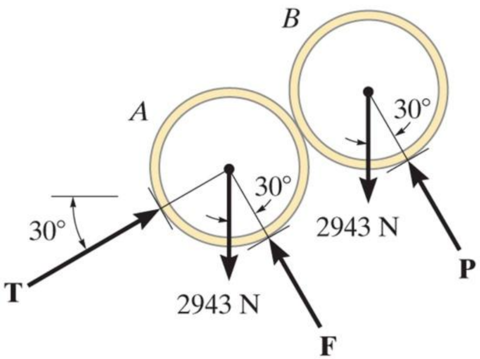
MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
6. Pin connection 	Pin free to turn  Pin not free to turn  A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support 	 A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.

Common Support Reactions

Figure 3/1, Page 111-112

8. Gravitational attraction 	 The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action 	 Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.



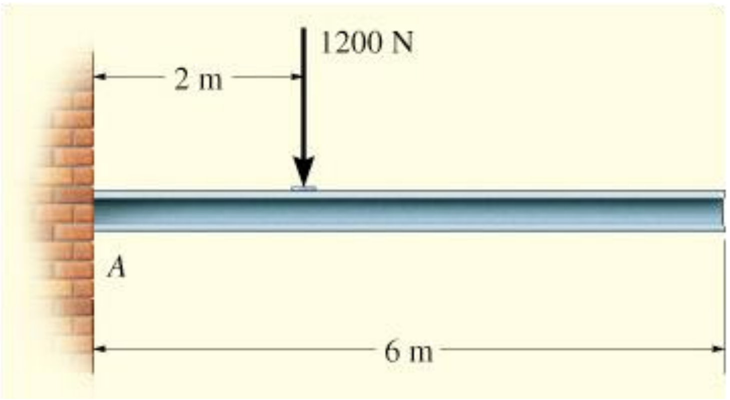


Two smooth pipes, each having a mass of 300 kg, are supported by the tines of the tractor fork attachment.

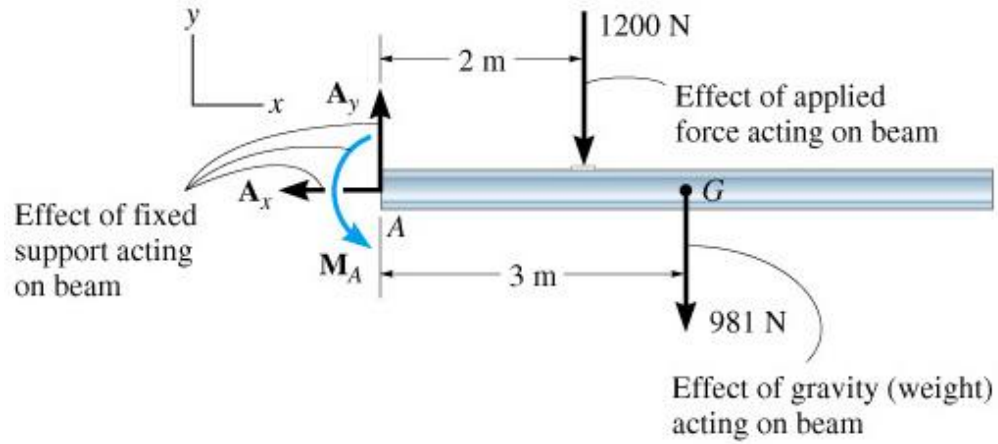
How can we determine all the reactive forces ?

Again, how can we make use of an idealized model and a free body diagram to answer this question?

Free Body Diagrams



Idealized model



Free-body diagram

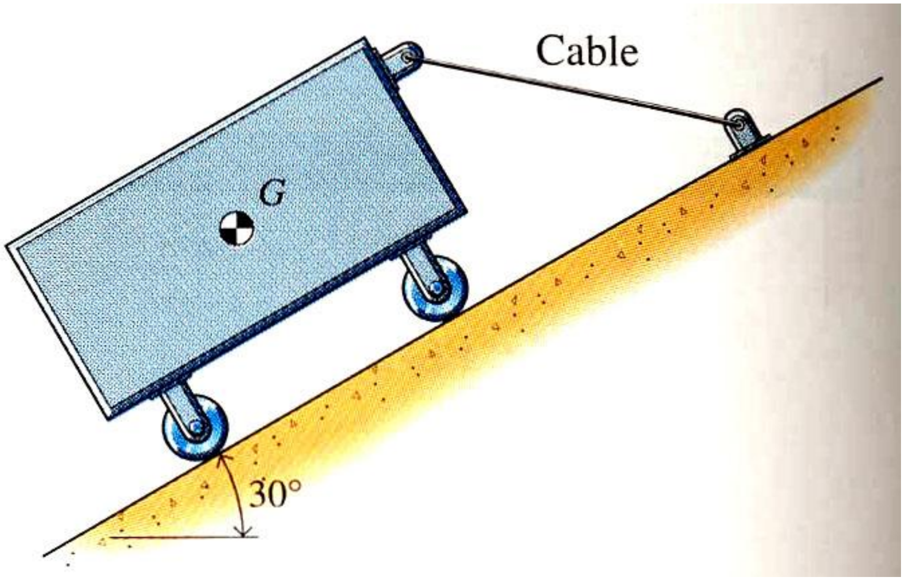
All known forces and couple moments should be labeled with their magnitudes and directions. For the unknown forces and couple moments, use letters like $\mathbf{A_x}$, $\mathbf{A_y}$, $\mathbf{M_A}$, etc.

Summary

- No equilibrium problem should be solved without **first** drawing a FBD, so as to account for all the forces and couple moments that act on the body
- Internal forces are never shown on the FBD (i.e., they act in equal but opposite collinear pairs and therefore cancel one another out.)
- If a support **prevents translation** in a particular direction, then the support exerts a **force** on the body in that direction
- If **rotation is prevented**, then the support exerts a **couple moment** on the body.
- The weight of a body is an external force, and its effect is shown as a single resultant force acting through the body's center of gravity
- **Couple moments** can be placed anywhere on the FBD since they are **free vectors**. *Forces* can act at any point along their lines of action since they are **sliding vectors**.

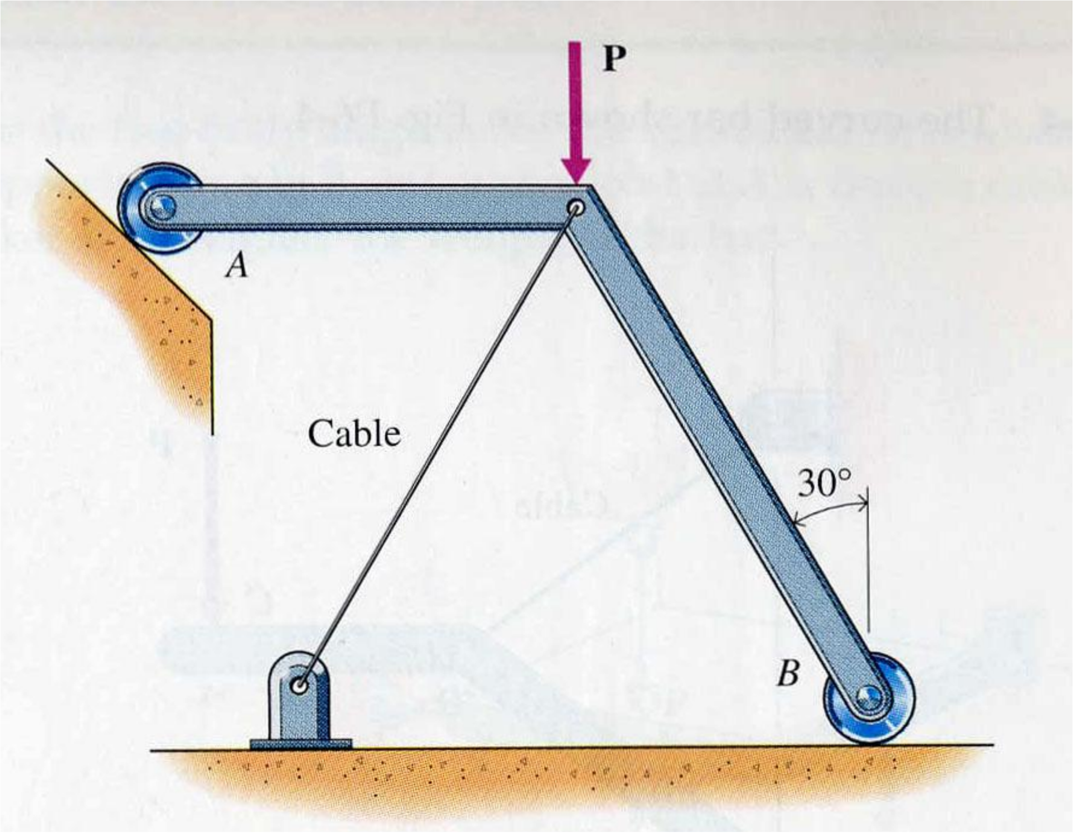
Example 1

Draw the free-body diagram of the cart.



In Class Activity #2

Draw the free-body diagram of the bar AB.



Statical Determinancy

- In 2D there are 3 equations of equilibrium
- A system with 3 or fewer unknowns can be solved using these equations. Such systems are called **statically determinate** systems.
- A system with more than 3 unknowns cannot be solved using equations of equilibrium alone. Such systems are called **statically indeterminate** systems.

In Class Activity #1

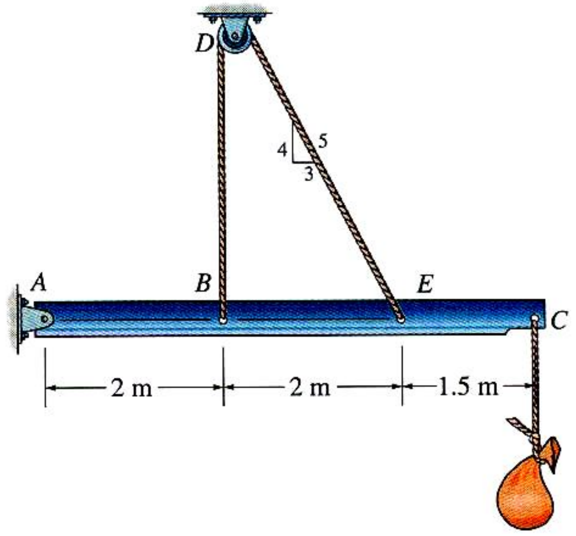
The beam and the cable (with a frictionless pulley at D) support an 80 kg load at C. In a FBD of only the beam, there are how many unknowns?

A) 2 forces and 1 couple moment

B) 3 forces and 1 couple moment

C) 3 forces

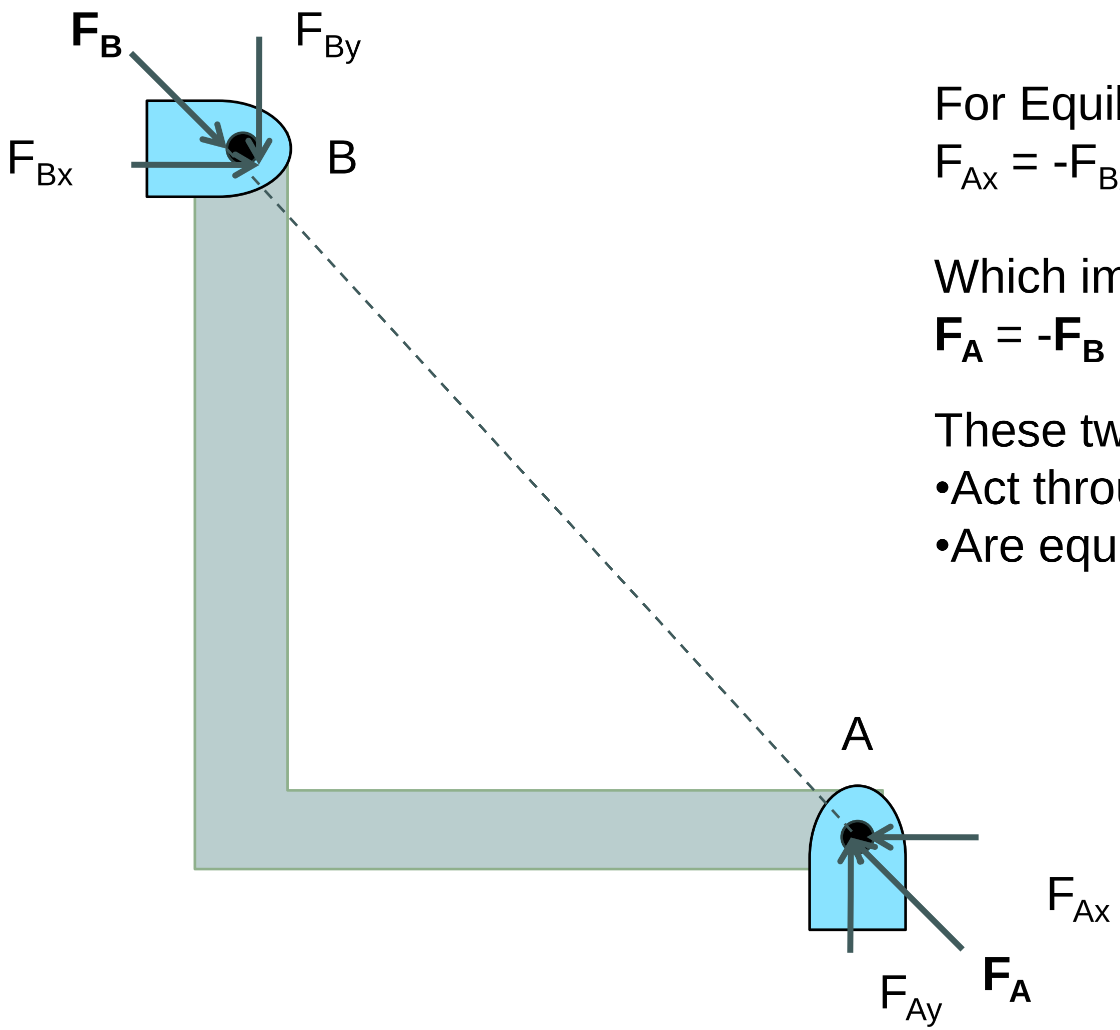
D) 4 forces



Understanding Equilibrium Problems

1. If we have more unknowns than the number of independent equations, then we have a statically indeterminate situation. We cannot solve these problems using just statics.
2. The order in which we apply equations may affect the simplicity of the solution. For example, if we have two unknown vertical forces and one unknown horizontal force, then solving $\sum F_X = 0$ first allows us to find the horizontal unknown quickly.
3. If the answer for an unknown comes out as a negative number, then the sense (direction) of the unknown force is opposite to that assumed when starting the problem.
4. Apply the moment equation of equilibrium ($\sum M_O = 0$) about a point *O* that lies at the intersection of the lines of action of two unknown forces. In this way, the moments of these unknowns are zero about *O*, and a direct solution for the third unknown can be determined.

Two Force Members



For Equilibrium:
$F_{Ax} = -F_{Bx}$ and $F_{Ay} = -F_{By}$

Which implies:
$\mathbf{F_A} = -\mathbf{F_B}$

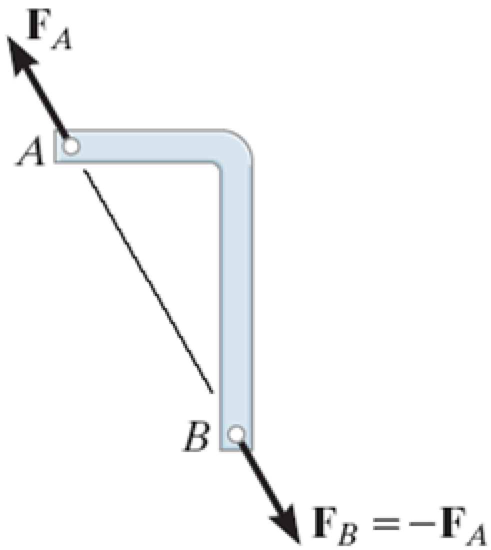
These two forces:
- Act through the same line of action
- Are equal and opposite

Two-Force Members

Two- Force Members

Member subjected to no couple moments with forces applied at only two points.

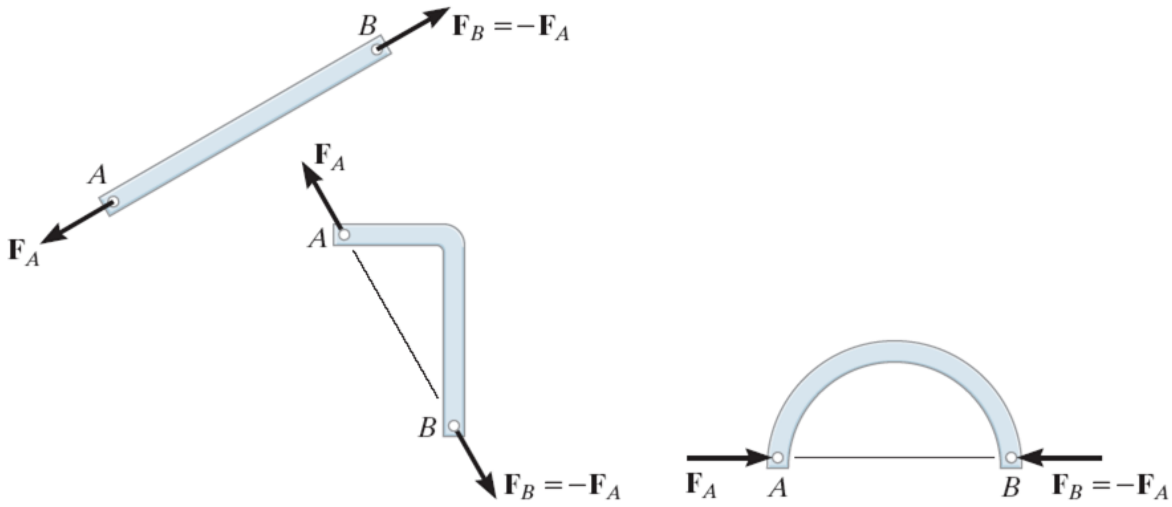
- For static equilibrium, the sum of the moments about *A* must be zero. The moment of F_B must be zero. It follows that the line of action of F_B must pass through *A*.
- Similarly, the line of action of F_A must pass through *B* for the sum of moments about *B* to be zero.



Only the force magnitude must be determined

Two-Force Members

- More Examples:

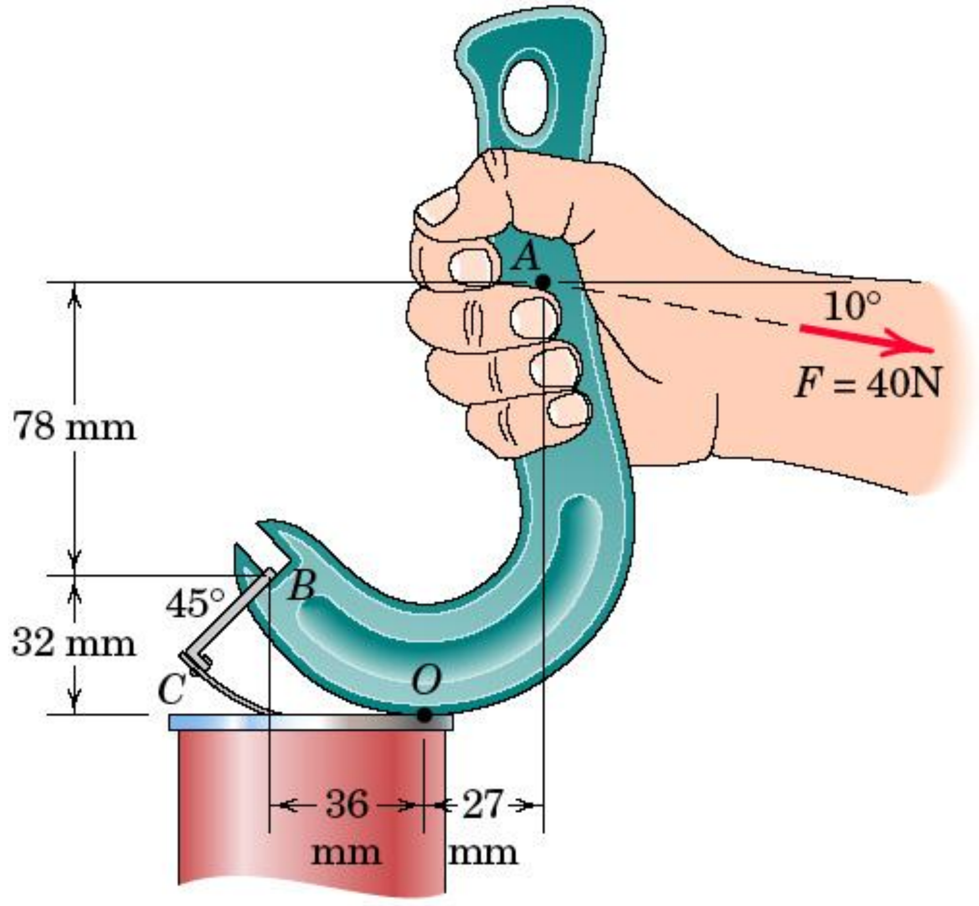


Two-force members

- Note: magnitude of two forces must be equal, direction opposite

Example 1

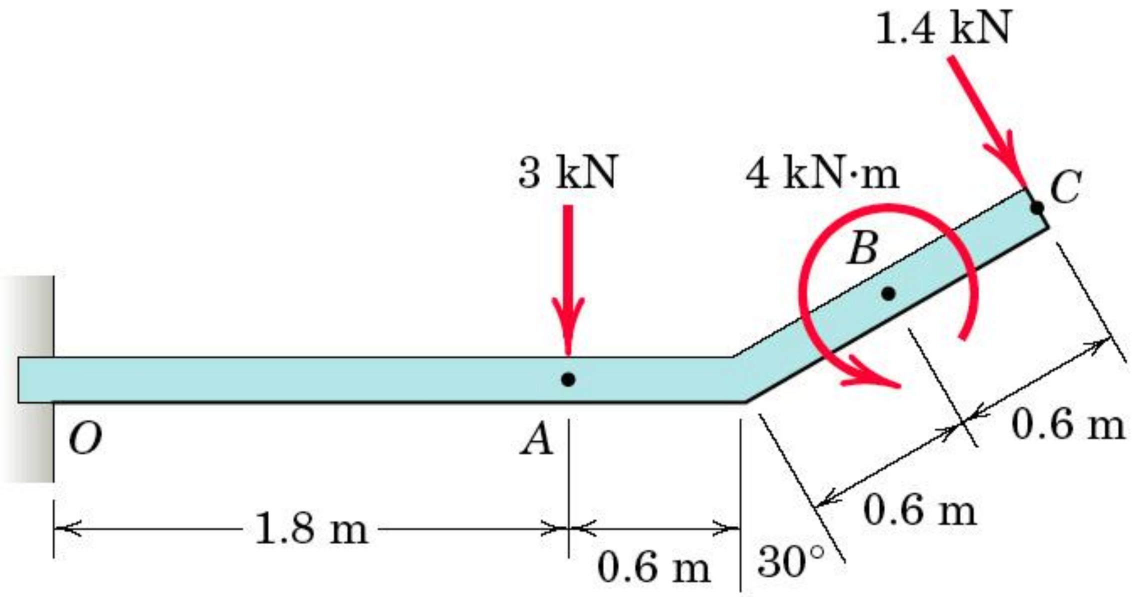
<u>Problem 3/12</u>: The device shown is designed to aid in the removal of pull-tab tops from cans. If the user exerts a 40-N force at *A*, determine the tension *T* in the portion *BC* of the pull tab.



Problem 3/12

Example 2

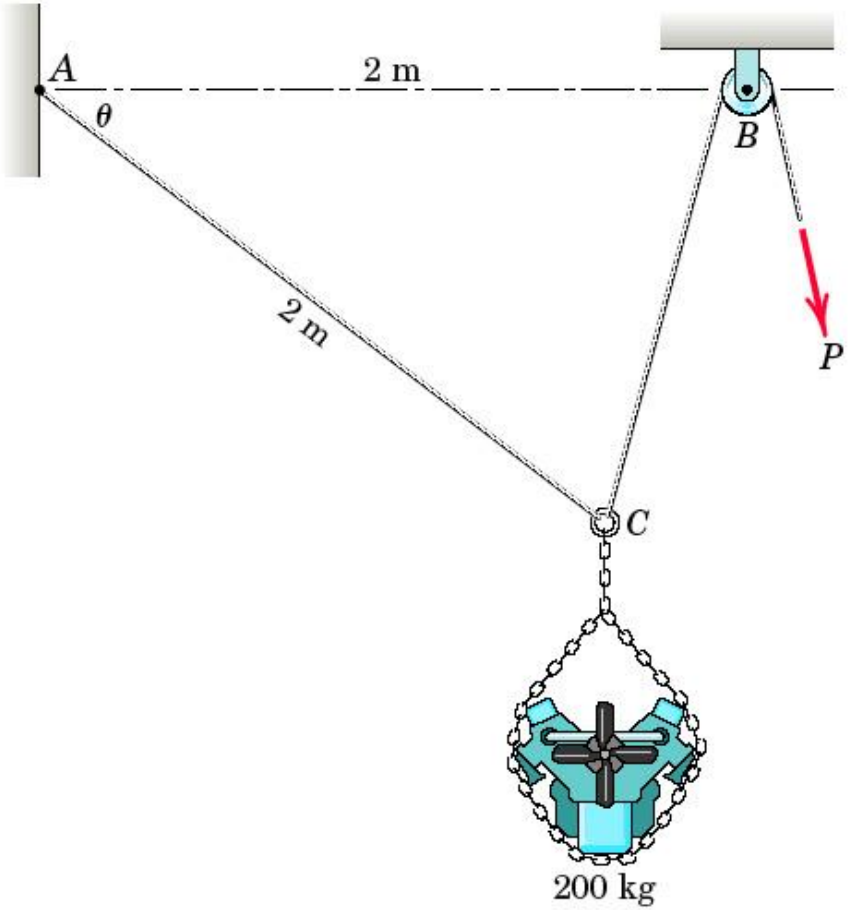
Problem 3/17: The uniform beam has a mass of 50-kg per meter of length. Compute the reactions at the support *O*. The force loads shown lie in a vertical plane.



Problem 3/17

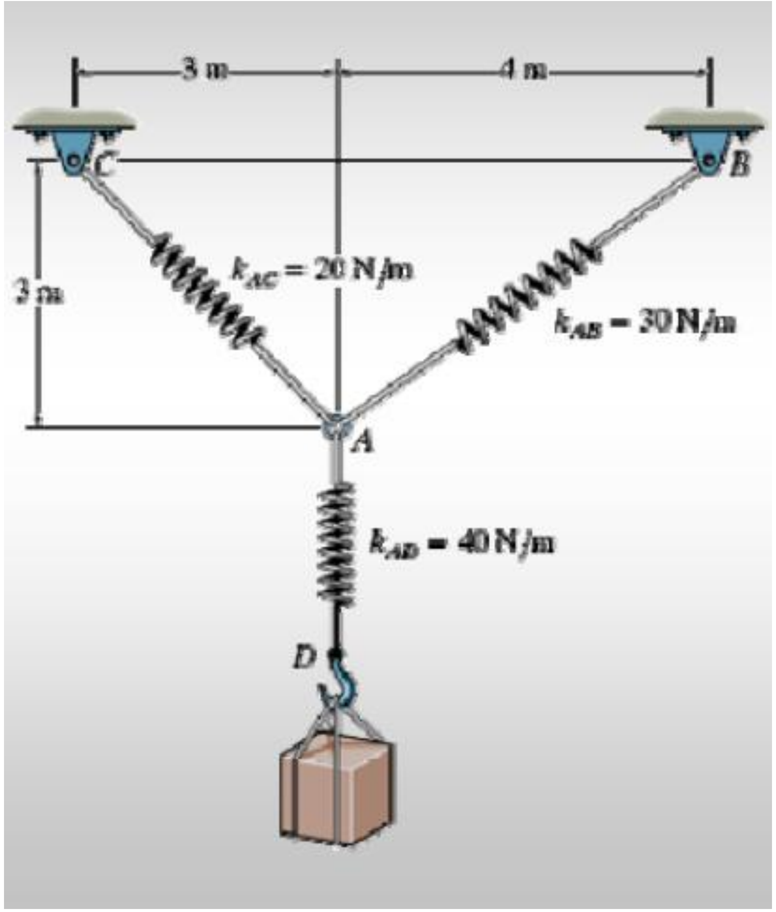
Example 1

Problem 3/1: Determine the force *P* required to maintain the 200-kg engine in the position for which θ = 30°. The diameter of the pulley at *B* is negligible.



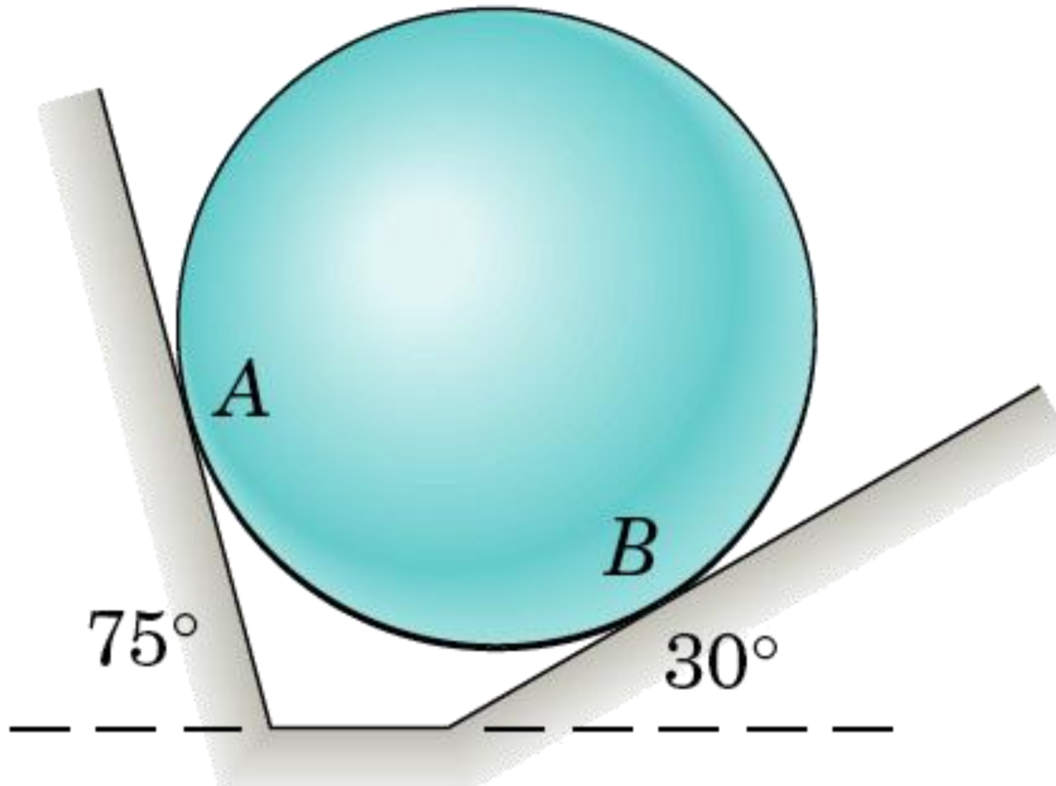
Problem 3/1

Example: Determine the stretch in each spring for equilibrium of the 2-kg block. The springs are shown in the equilibrium position



A note about your HW

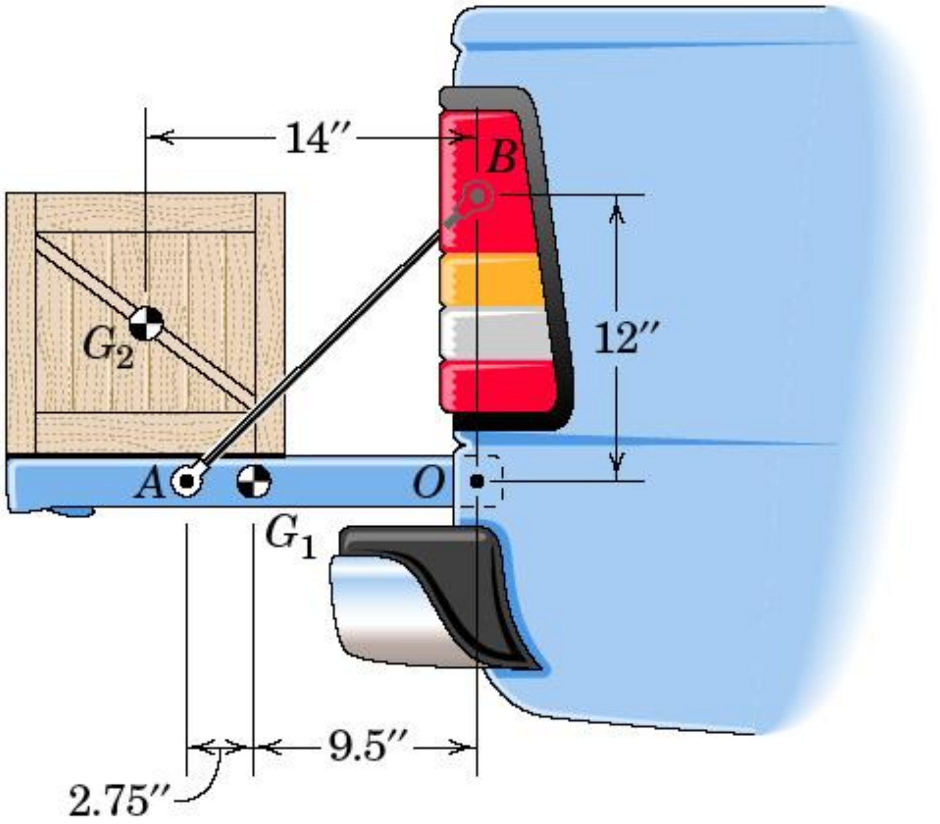
<u>Problem 3/8</u>: The 20-kg homogeneous smooth sphere rests on the two inclines as shown. Determine the contact forces at *A* and *B.*



Problem 3/8

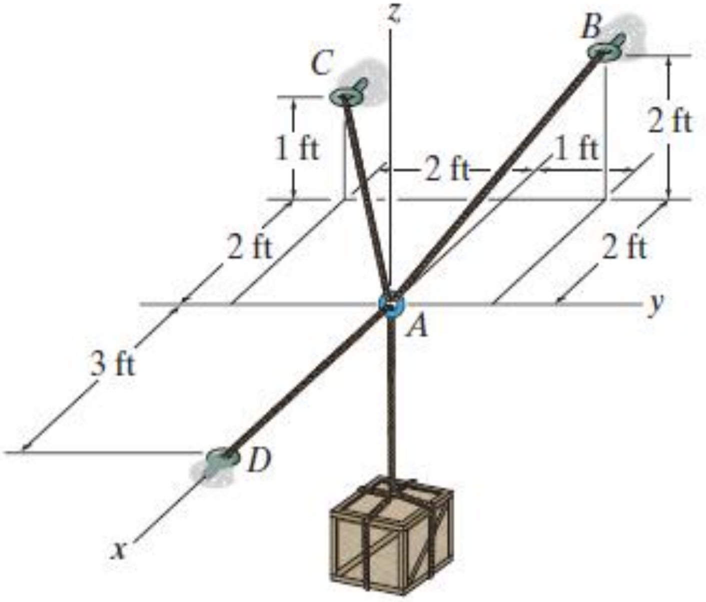
Example 3

<u>Problem 3/9</u>: A 120-lb crate rests on the 60-lb pickup tailgate. Calculate the tension *T* in each of the two restraining cables, one of which is shown. The centers of gravity are at G_1 and G_2. The crate is located midway between the two cables.



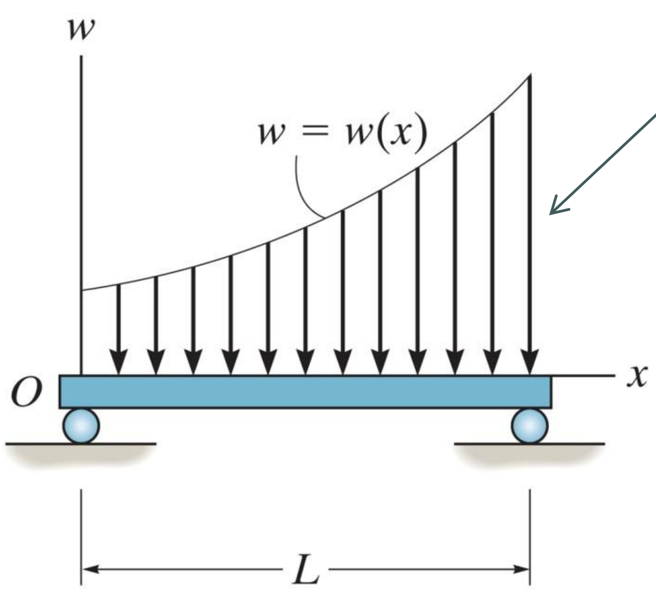
Problem 3/9

Example: Determine the tension developed in cables AB , AC, and AD required of the 300-Ib crate.



Composite Sections: Concept

- In many situations a surface area of a body is subjected to a distributed load. Such forces are caused by winds, fluids, or the weight of items on the body's surface.



This is a load (w) along one axis of a flat rectangular body.

In such cases, w is a function of x and has units of force per length.

Our goal is to determine the equivalent resultant load and its location

Finding the Magnitude of the Resultant Force

Consider an element of length dx

- The force magnitude dR acting on it is given as dR = w(x) dx
- The net force on the beam is then given by:

$$\downarrow + \mathbf{R} = \int_L w(x)dx$$

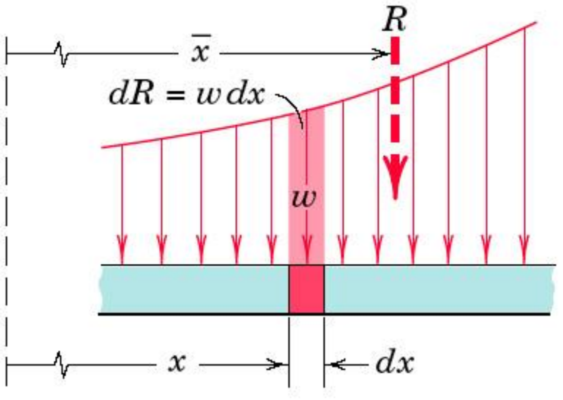


Figure 5/21

Note:

We are computing the area under the loading curve

Finding the location of the resultant

- To determine the location that the equivalent resultant force should be applied, compute the centroid of the curve:

$$\bar{x} = \frac{\int_L xw\,dx}{R}$$

- Note: you only need the x-centroid for this geometry

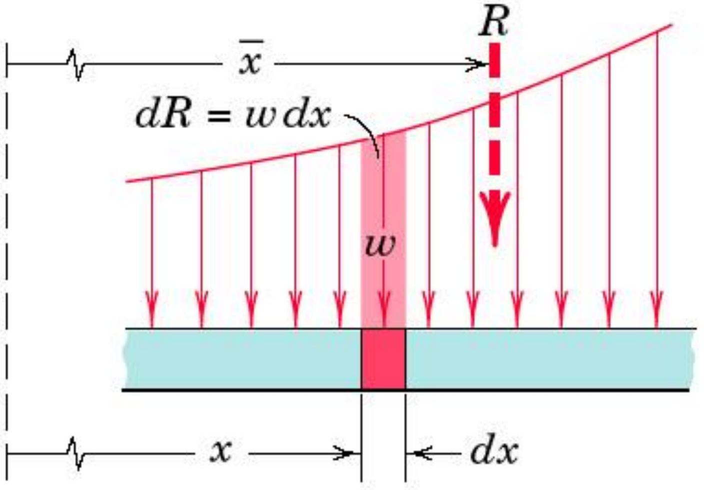


Figure 5/21

Finding the location of the resultant

Why do we use the centroid?

- Once again, examine the differential region
  - The force dR will produce a moment of (x*dR) about a point, O
  - The total moment about O will be

$$M_O = \int_L x\,dR = \int_L xw\,dx$$

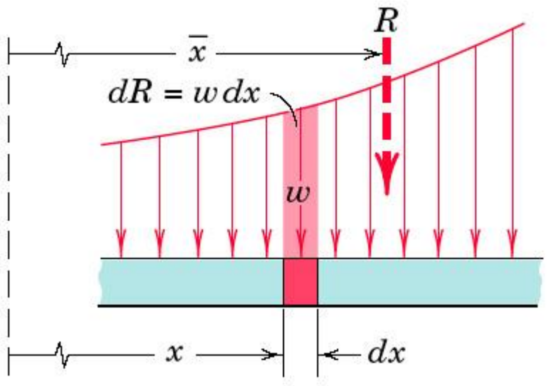


Figure 5/21

  - Similarly, an equivalent resultant force **R** should produce a moment acting from x_R as $M_O = x_R R$
  - Equating these two, we get: $x_R R = \int_L xw\,dx \quad or \quad x_R = \bar{x} = \dfrac{\int_L xw\,dx}{R}$

Do we always need integration?

- Summary:
  - When we have a distributed load, we need to find the area under the load curve to get the equivalent resultant force, R.
  - Similarly, we must compute the centroid to determine the location of the force
- And when we have distributed loads with known geometries?
  - Compute area, centroid location using known equations

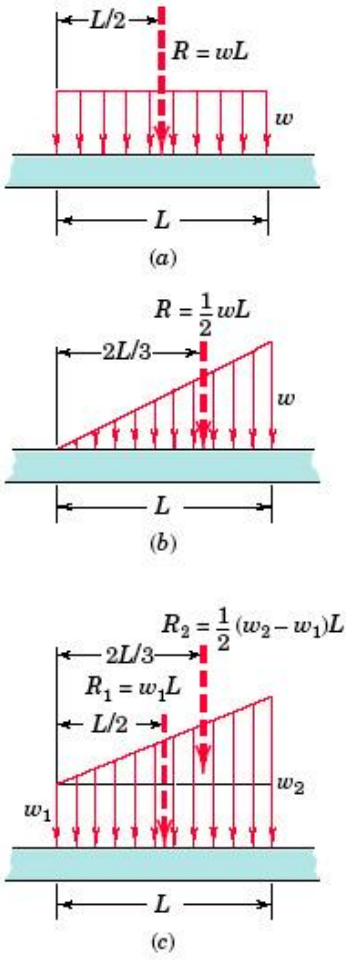
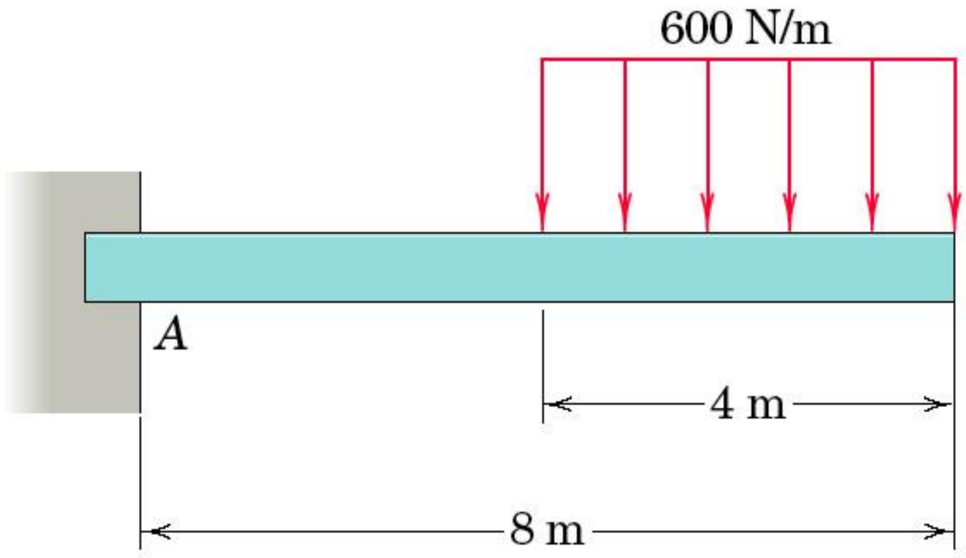


Figure 5/20

Example 1

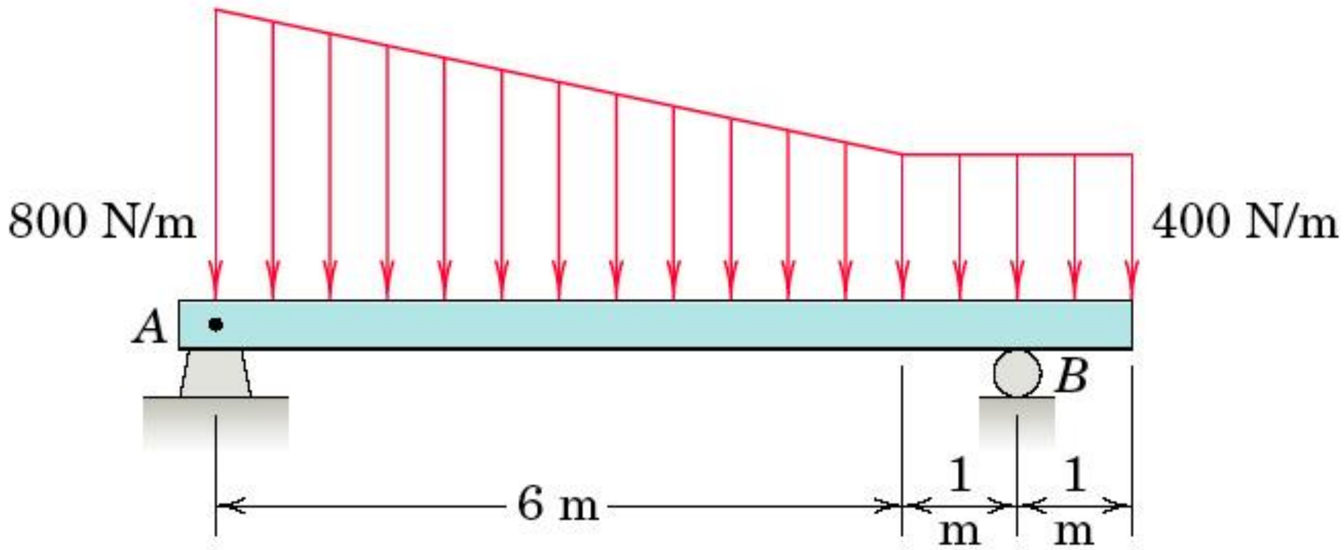
Problem 5/95: Calculate the supporting force R_A and moment M_A at *A* for the loaded cantilever beam.



Problem 5/95

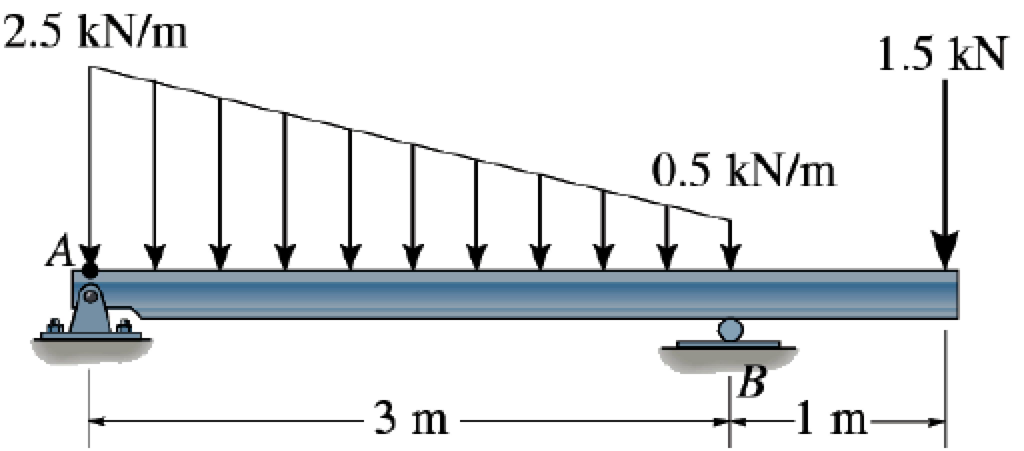
Example 3

Problem 5/103: Determine the reactions at the supports of the beam which is loaded as shown.



Problem 5/103

Replace the force system acting on the beam by an equivalent force and specify its location from point A.



Chapter 4

STRUCTURES

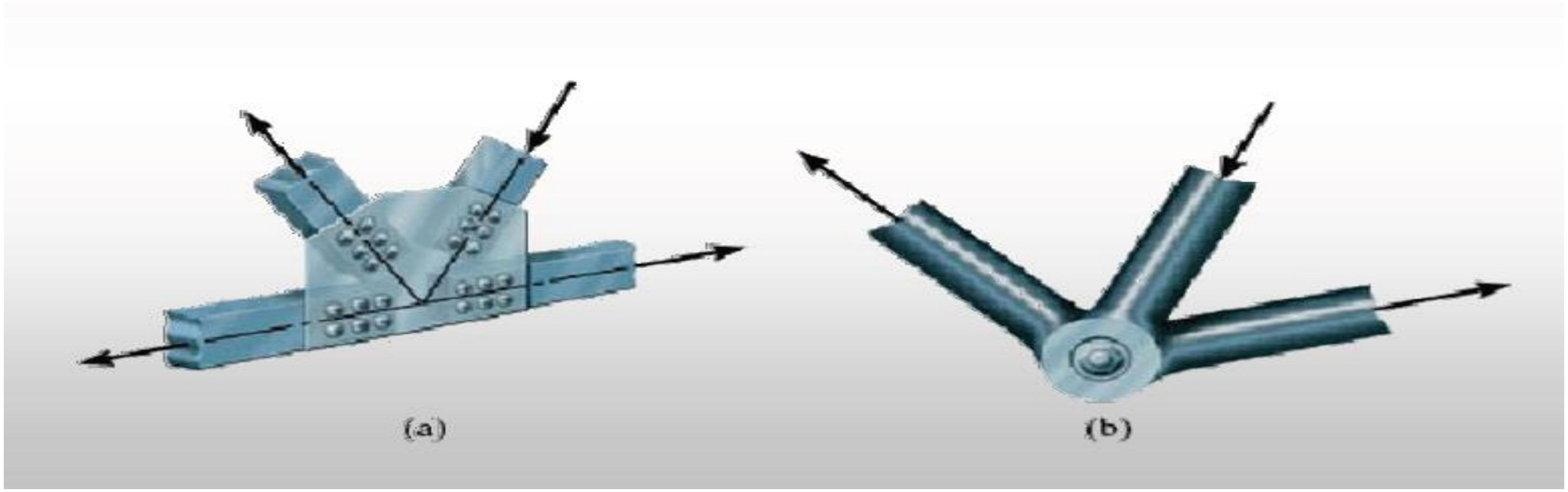
TRUSS

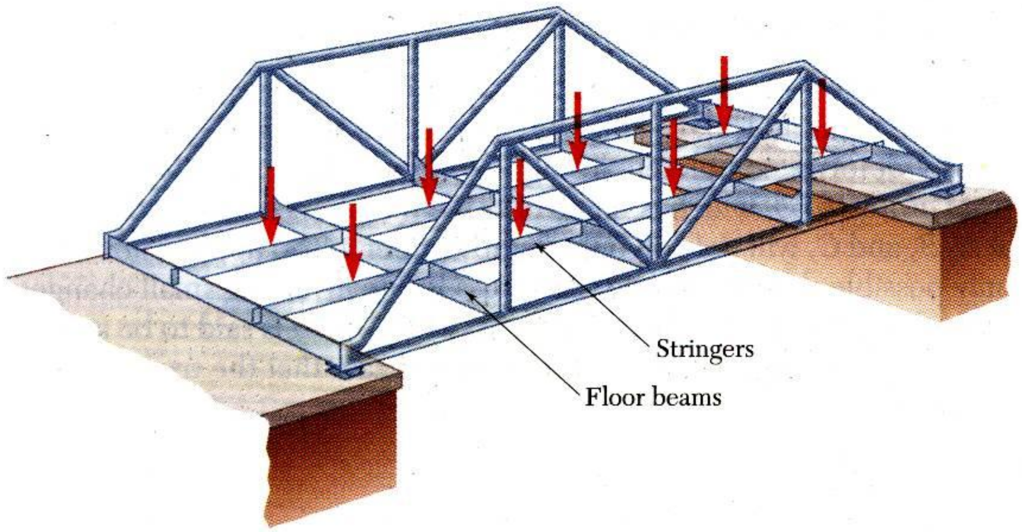
Simple Trusses

structure composed of straight, slender members joined at their endpoints

•joint connection can consist of pin through the ends of the members

•ends of members can bolted or welded to a gusset plate





Members of a truss are slender and not capable of supporting large lateral loads. Loads must be applied at the joints.

Trusses: Introduction

- Trusses are structures composed entirely of two force members . They consists generally of triangular sub-element and are constructed and supported so as to prevent any motion.
- Trusses are used in a variety of structures, including cranes, frames of aircraft or space stations, bridges, roofs

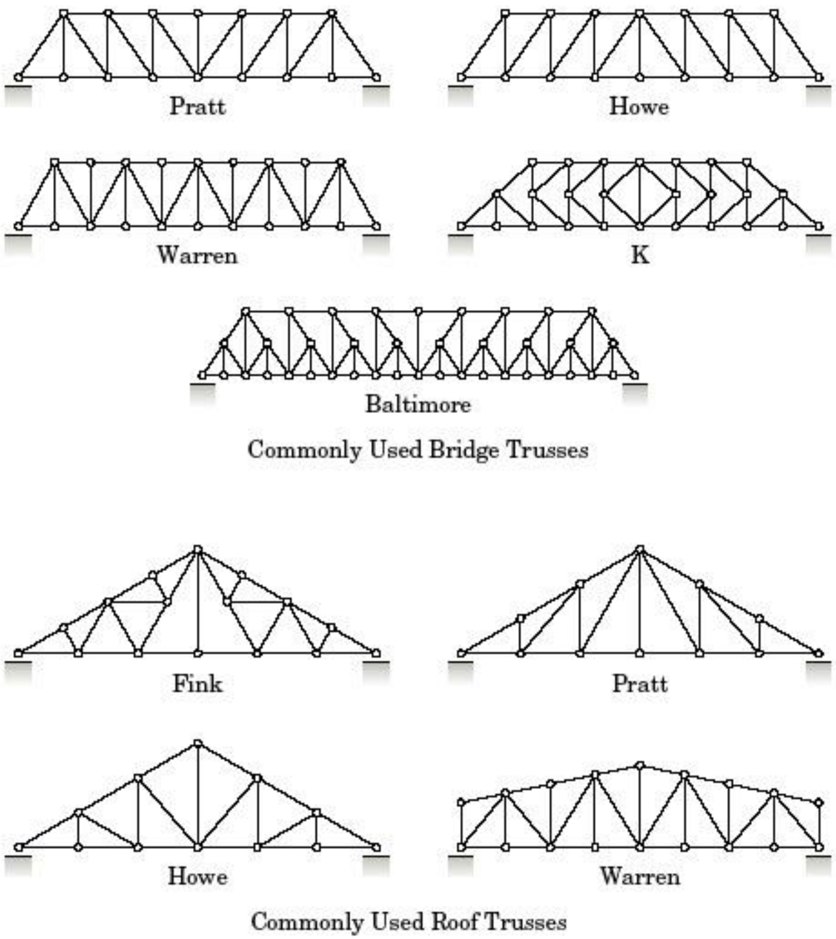
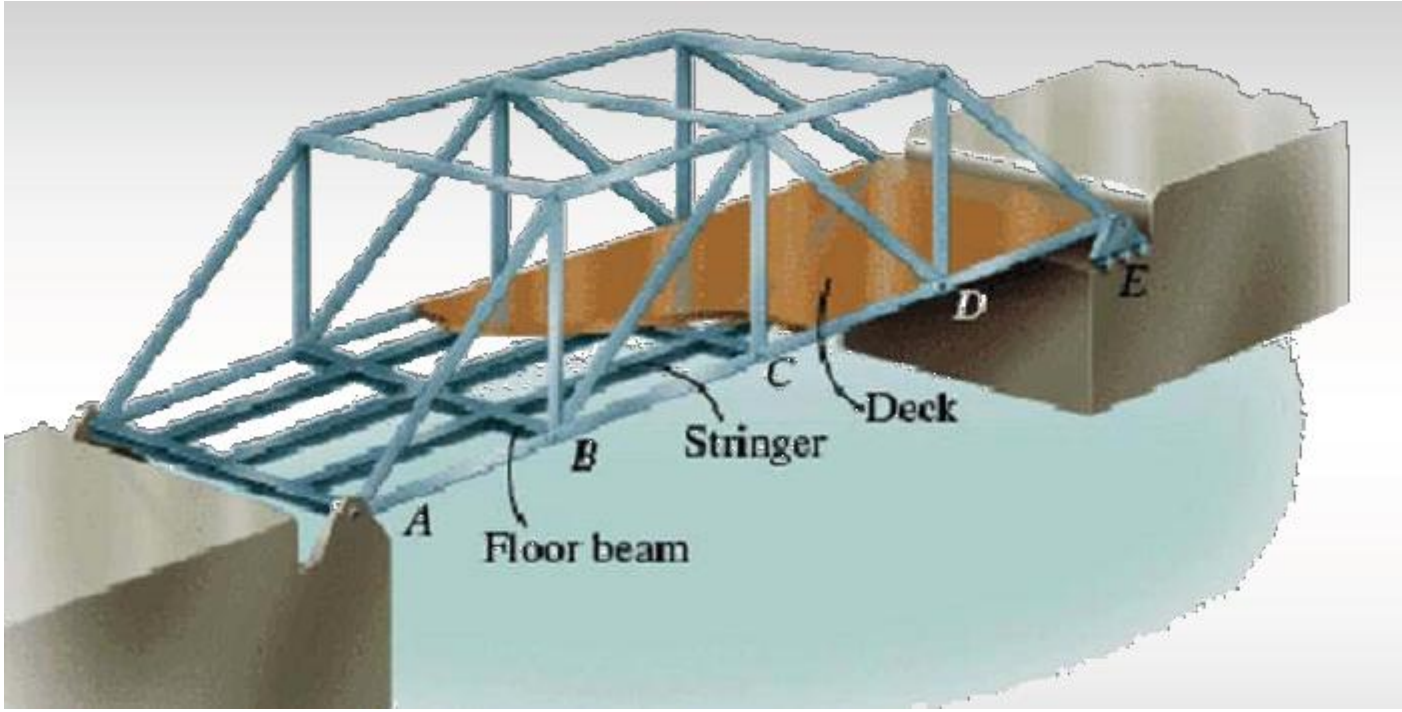


Figure 4/2

Bridge Trusses



Trusses: Introduction

- ***Objective:*** Examine internal forces in each member
  - Determining load on each internal member of a truss helps in design and optimization of truss

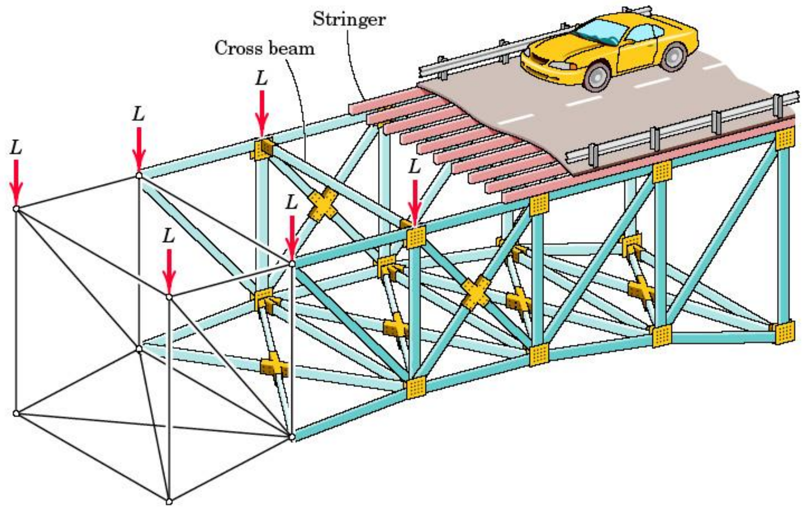


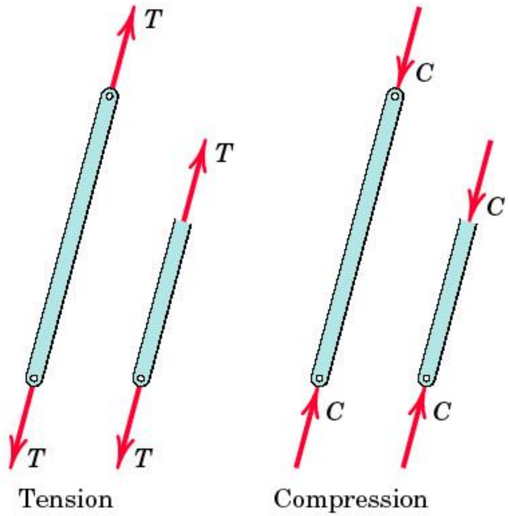
Figure 4/1

Modeling trusses

- Assumptions when using trusses

1. Truss members are connected together only at their ends
2. Truss are connected together by frictionless pins
3. The truss structure is loaded only at the joints
4. The weights of the members may be neglected (weight of members significantly lower than applied weight and support forces)

- Each member acts as a *two-force member*, and thus the forces at the ends of the member must be directed along the axis of the member:
  - Tension
  - Compression



Two-Force Members

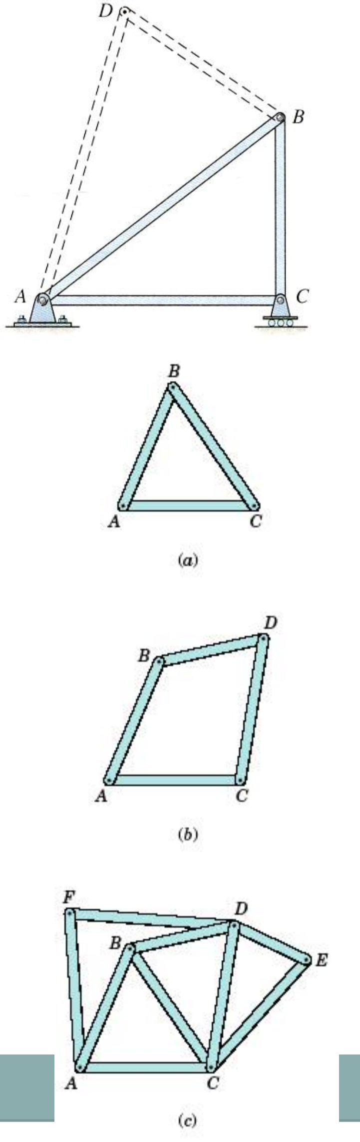
Figure 4/4

Elongation:
Tensile force (T)

Shortening:
Compressive force (C)

Simple Truss

- A simple truss is a planar truss which begins with a triangular element and can be expanded by adding two members and a joint. For these trusses, the number of members (M) and the number of joints (J) are related by the equation
  $M = 2J - 3.$
- Triangular-based frames are considered *rigid* while 4+ bars pinned to construct a frame is considered *collapsible*.
- If more members are present than are necessary, additional members are considered *redundant*.



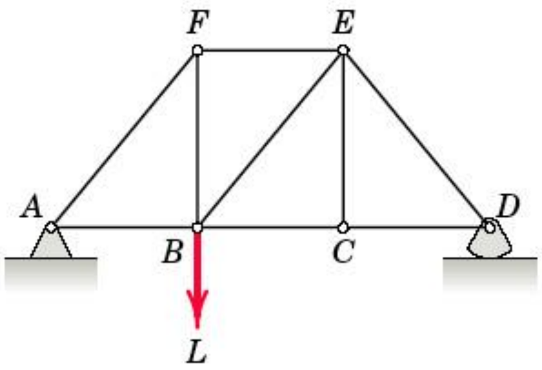
Statical Determinancy

- *External Redundancy*: occurs if a truss has more **external supports** than are necessary to ensure a stable equilibrium condition
- Internal Redundancy: truss has more **internal members** than are necessary to prevent collapse
  - Determined by the conditions
    - If m< 2j-3, then the truss is unstable and will collapse under load
    - If m> 2j-3, then the truss has more unknowns than equations and is an indeterminate structure.
    - If m= 2j-3, then a simple plane truss is rigid and solvable – this condition is not sufficient for a non-simple truss

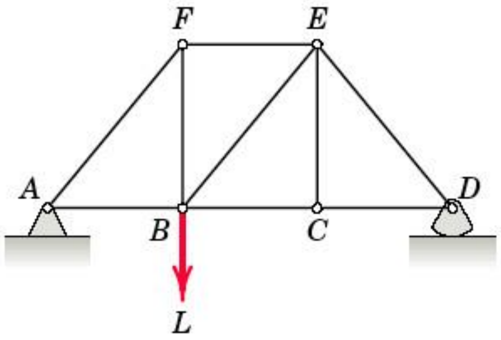
Truss Analysis

Method of Joints

1. Find the external reactions (may or may not be necessary)
2. Begin at a joint that has two or fewer unknowns and at least one known, and step through the truss, joint by joint
3. Draw a FBD at each joint and solve force Eqbm. Eqns. Only (no moment equations) $\Sigma F_x = 0$ and $\Sigma F_y = 0$
4. Record the force and its character *(C: compression or T: tension)* for each member
5. The final joint may not always yield new info but can serve as an equilibrium check



Method of Joints



(a)

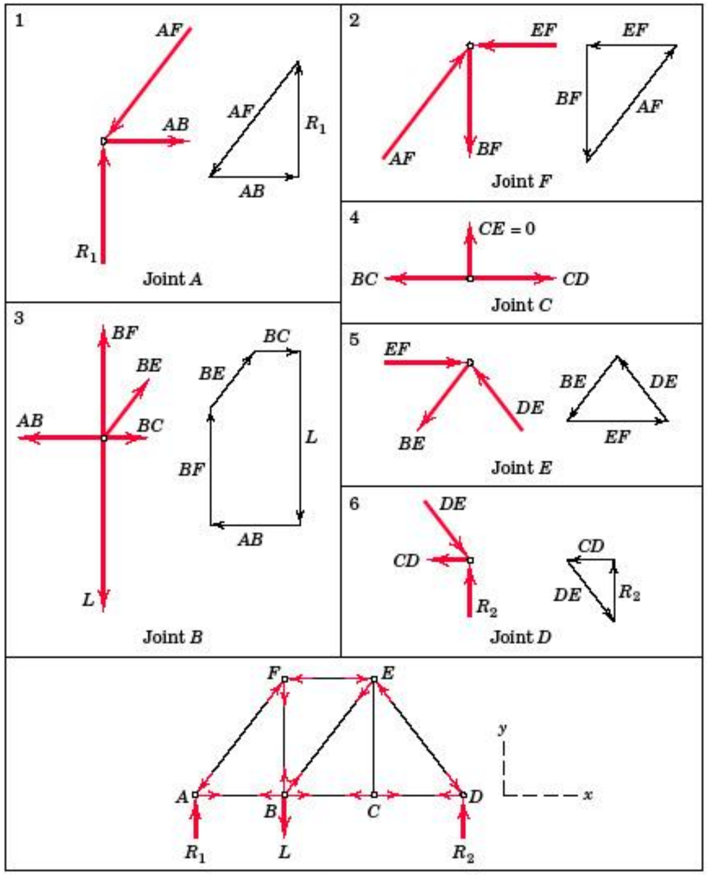


Figure 4/8

Special Case: Zero Force Members

- *Zero-force members* support no loading
  - increase the stability of the truss during construction
  - provide support if the applied loading is changed.
  - can generally be determined by inspection of each of the joints
- Examples:
  - If a joint has only two non-collinear members and there is no external load or support reaction at that joint, then those two members are zero-force members.
  - If three members form a truss joint for which two of the members are collinear and there is no external load or reaction at that joint, then the third non-collinear member is a zero force member.

Special Case: Zero Force Members

Examples:

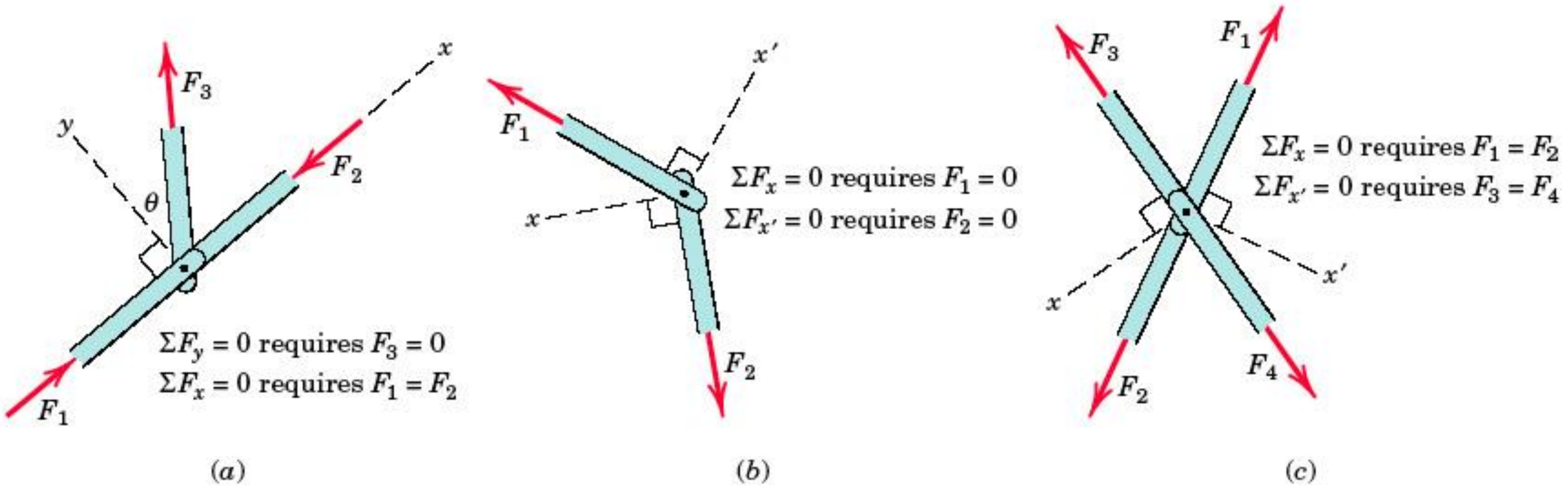
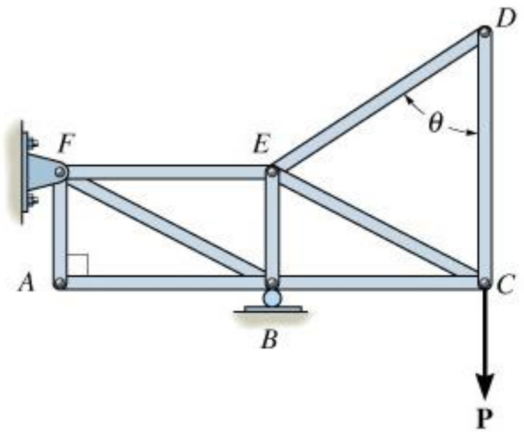
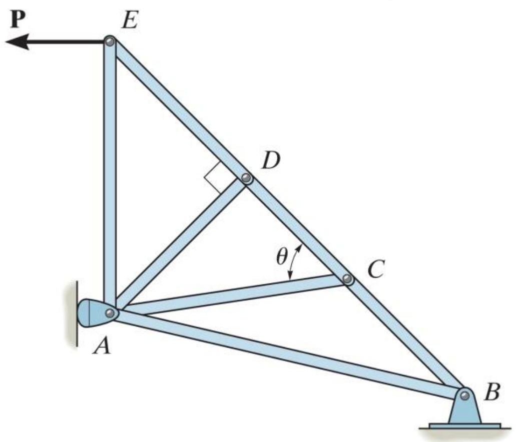


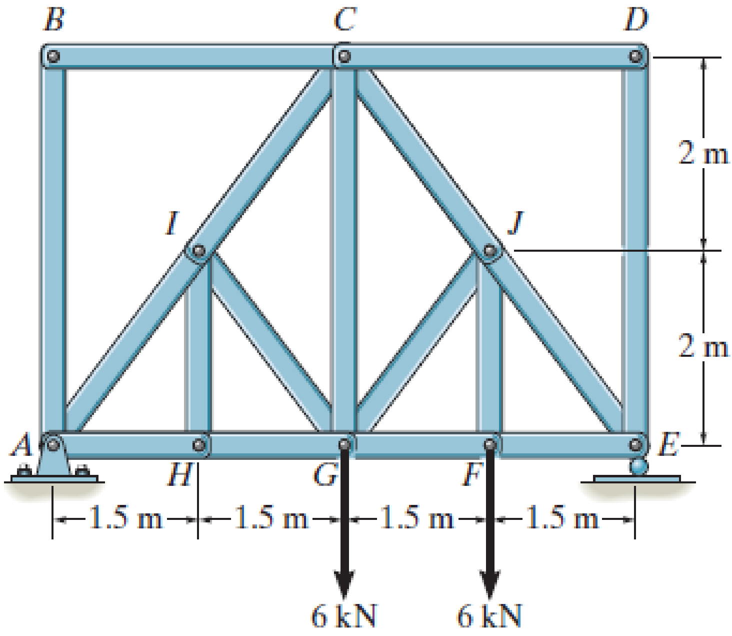
Figure 4/9

Examples

- Determine all zero members in the following structures







Correct Characterization?

Characterization – sense of force (i.e. tension vs. compression)

<u>Two approaches – choose your own adventure:</u>

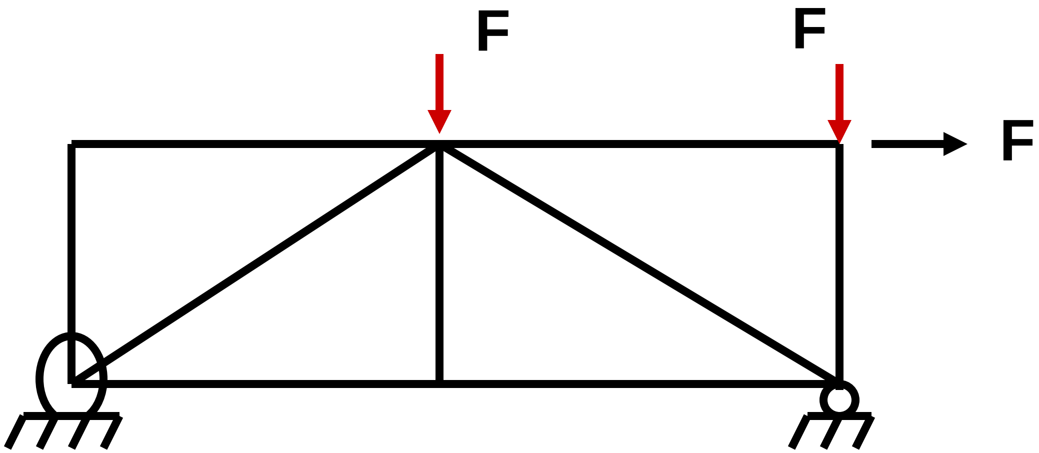
1. *Always assume the unknown member forces* acting on the joint's FBD to be in *tension* (ie "pulling" on the pin).

   Numerical Solutions yields:
   - *positive scalars* for member in *tension* (T)
   - *negative scalars* for members in *compression* (C)

2. *Determined by "inspection"*

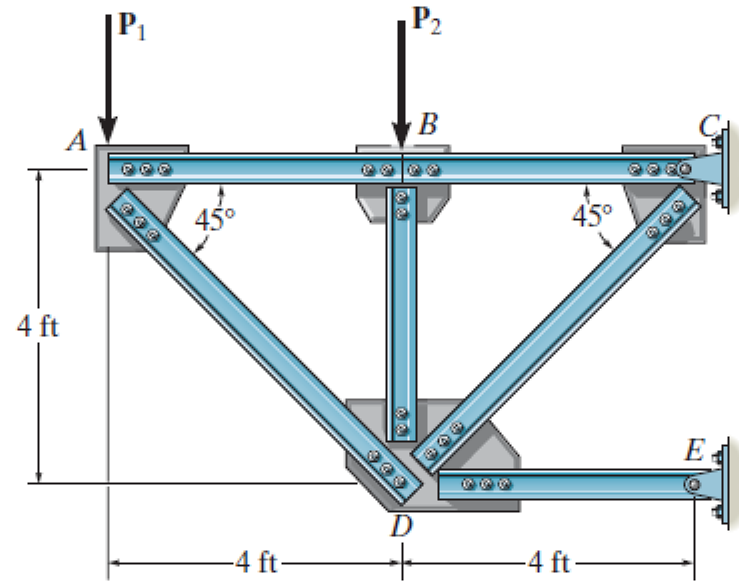
   Numerical Solutions yields:
   - *positive scalar* indicates that the sense is *correct*
   - *negative scalar* indicates that the sense shown on the FBD must be reversed.

In Class Activity



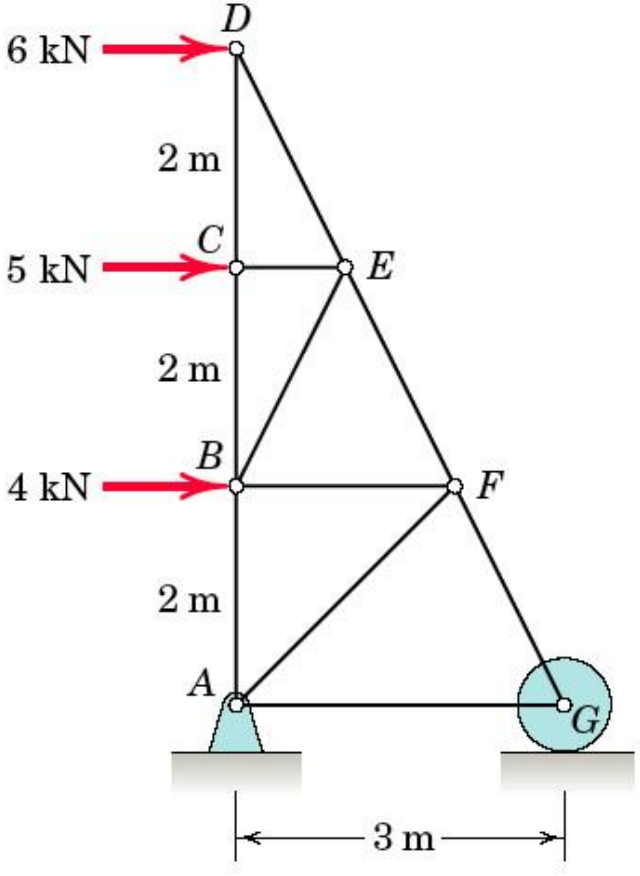
Determine the number of zero force members in the truss

6–3. The truss, used to support a balcony, is subjected to the loading shown. Approximate each joint as a pin and determine the force in each member. State whether the members are in tension or compression. Set $P_1 = 800$ lb, $P_2 = 0$.



Example 2

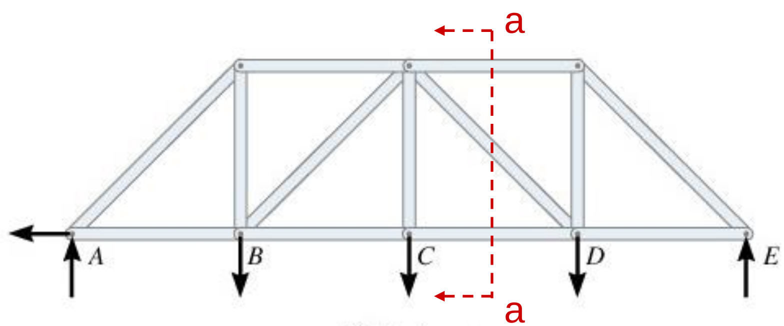
<u>Problem 4/17</u>: Determine the forces in members *AF, BE, BF*, and *CE* of the loaded truss.



Problem 4/17

Application

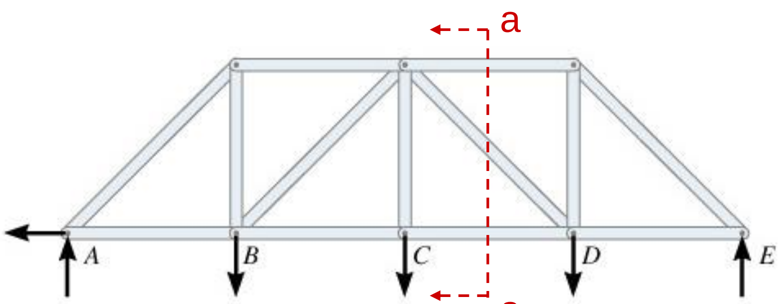
- Long trusses are often used to construct bridges
- The method of joints requires many joints to be analyzed before we can determine the forces in the middle part of the truss
- ***Method of sections*** is used instead of that.



Bridge truss

Method of Sections

- The method of joints is most effective when the forces in all the members of a truss are to be determined.
  - However, if the force in only a few members are needed, then the method of sections is more efficient.
- In the Method of Sections, **a truss is divided into two parts** by making an imaginary "cut / section" (shown here as a-a) through the truss.



Bridge truss

Method of Sections

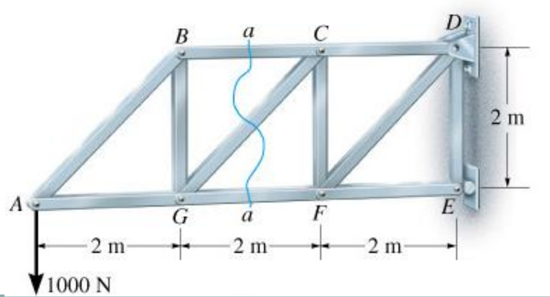
Procedure for Analysis

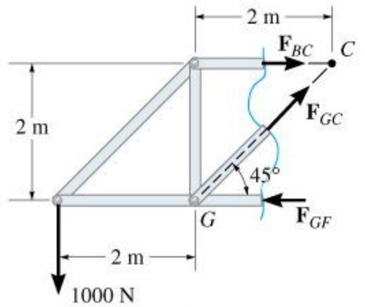
1. In general, find the external reactions first (not always necessary)
2. If possible, pass a section through the desired member and up to two other members, isolating a portion of the truss (***maximum cut through three unknown members***)
3. Apply 2-D rigid body equilibrium conditions on isolated region (you can employ $\Sigma F_x=0$, $\Sigma F_y=0$, $\Sigma M=0$)
4. Apply the Moment equation (about any point)

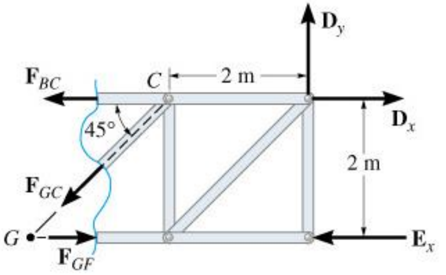
$$\Sigma M_o=0$$

5. Solve for unknowns

Note: Method of Joints and Sections may be used in combination







Correct Characterization?

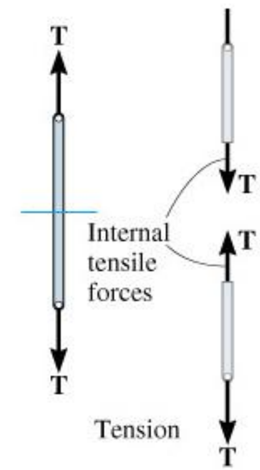
As with Method of Joints, two options

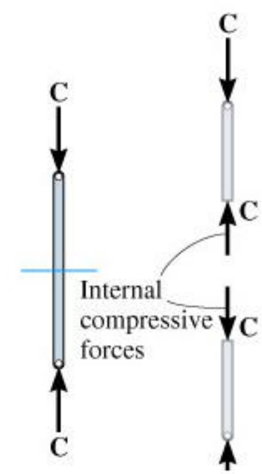
- Always assume the unknown member forces acting on the joint's FBD to be in *tension* (ie "pulling" on the pin).

  Numerical Solutions yields:
  - *positive scalars* for member in *tension* (T)
  - *negative scalars* for members in *compression* (C)

- Determined by "inspection"

  Numerical Solutions yields:
  - *positive scalar* indicates that the sense is *correct*
  - *negative scalar* indicates that the sense shown on the FBD must be reversed.





In Class Activity

1. In the method of sections, generally a "cut" passes through no more than ______ members in which the forces are unknown.

   A) 1 B) 2

   C) 3 D) 4

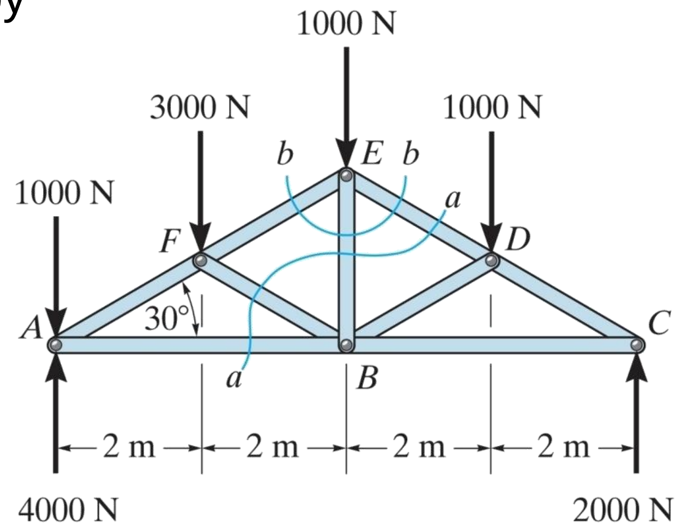
2. Can you determine the force in member ED by making the cut at section a-a? Explain your answer.

   A) No, there are 4 unknowns.

   B) Yes, using $\Sigma M_D = 0$.

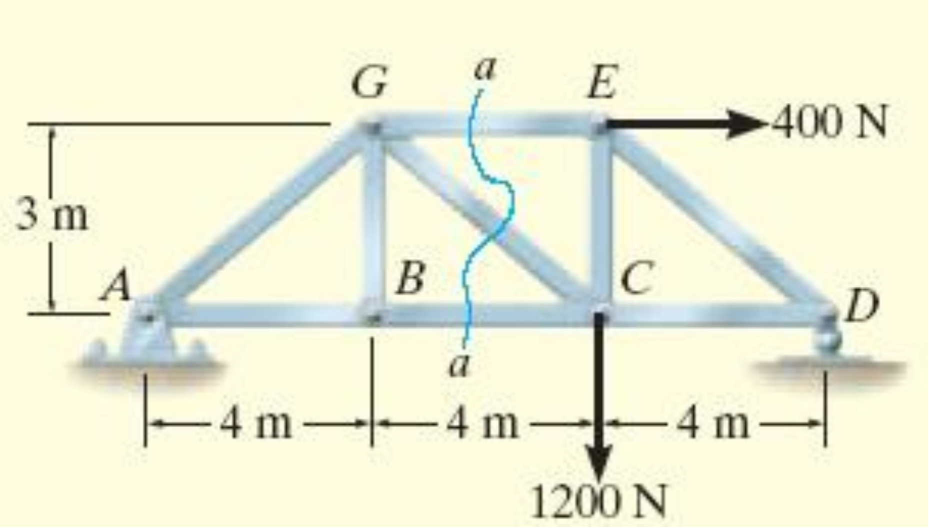
   C) Yes, using $\Sigma M_E = 0$.

   D) Yes, using $\Sigma M_B = 0$.



Example

Determine the force in members GE, GC, and BC of the truss. Indicate whether the members are in tension or compression



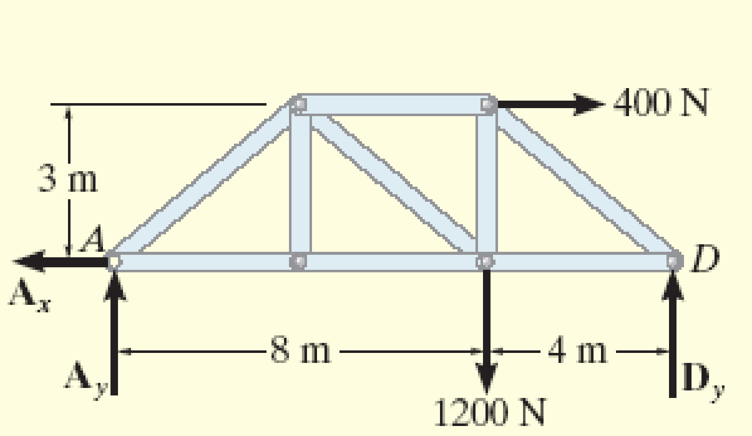
Solution

Choose section a-a since it cuts through the three members
Draw FBD of the entire truss

$$+\rightarrow \sum F_x = 0; \qquad 400N - A_x = 0 \Rightarrow A_x = 400N$$

$$\sum M_A = 0; \qquad -1200N(8m) - 400N(3m) + D_y(12m) = 0 \Rightarrow D_y = 900N$$

$$+\uparrow \sum F_y = 0; \qquad A_y - 1200N + 900N = 0 \Rightarrow A_y = 300N$$

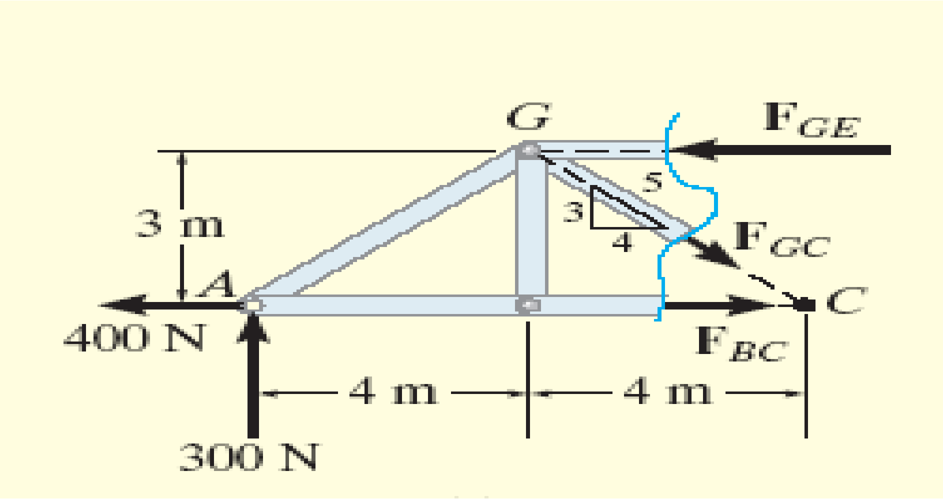


Draw FBD for the section portion

$$\sum M_G = 0; \quad -300N(4m) - 400N(3m) + F_{BC}(3m) = 0 \Rightarrow F_{BC} = 800N(T)$$

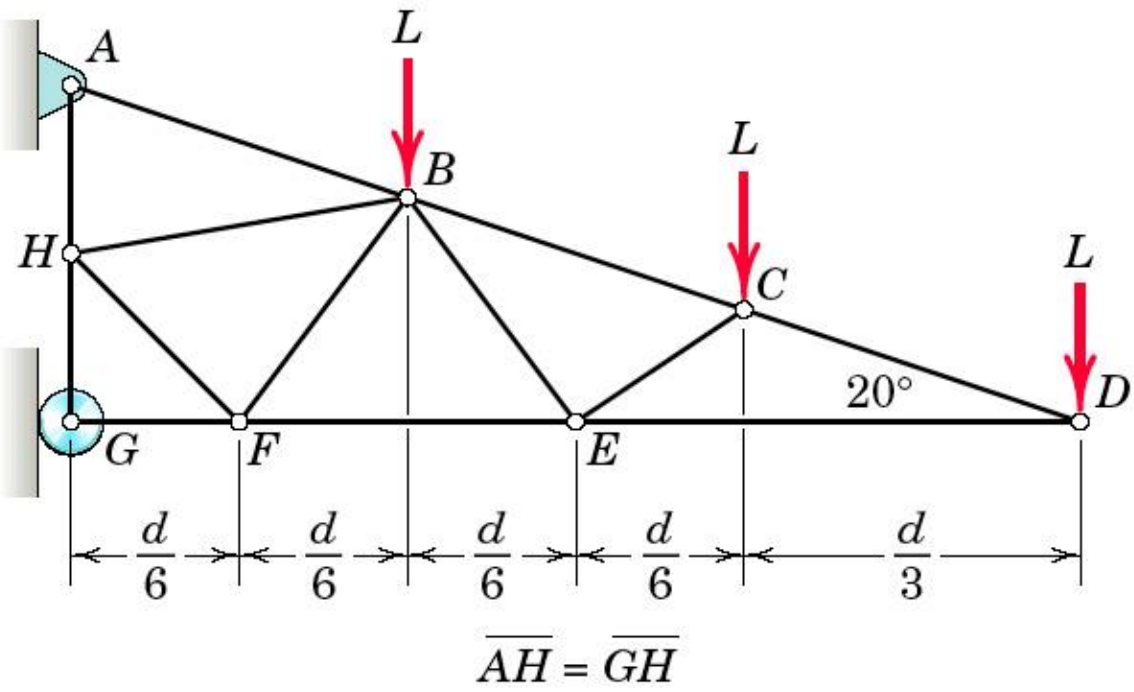
$$\sum M_C = 0; \quad -300N(8m) + F_{GE}(3m) = 0 \Rightarrow F_{GE} = 800N(C)$$

$$+\uparrow \sum F_y = 0; \quad 300N - \frac{3}{5}F_{GC} = 0 \Rightarrow F_{GC} = 500N(T)$$



Example 1

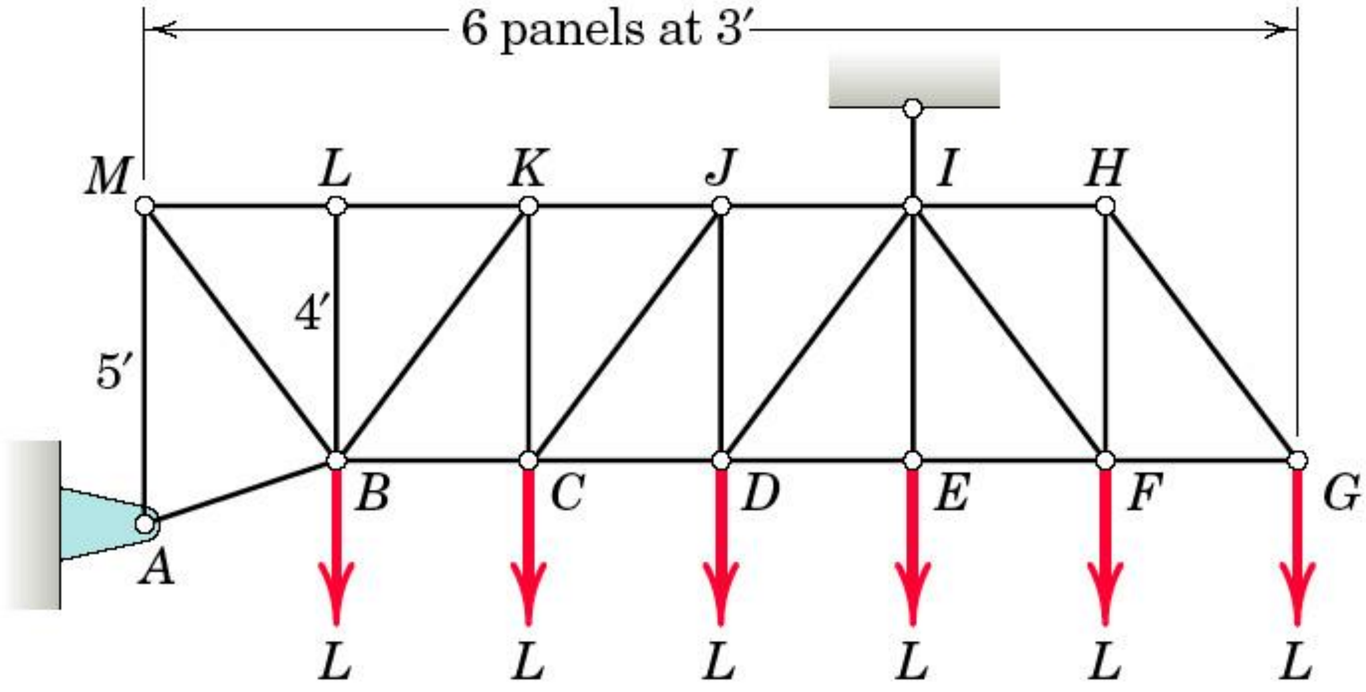
Problem 4/34: Calculate the forces in member *BE* of the loaded truss.



Problem 4/34

Example 2

<u>Problem 4/41</u>: Determine the forces in members *CD, CJ*, and *DJ.*



Problem 4/41

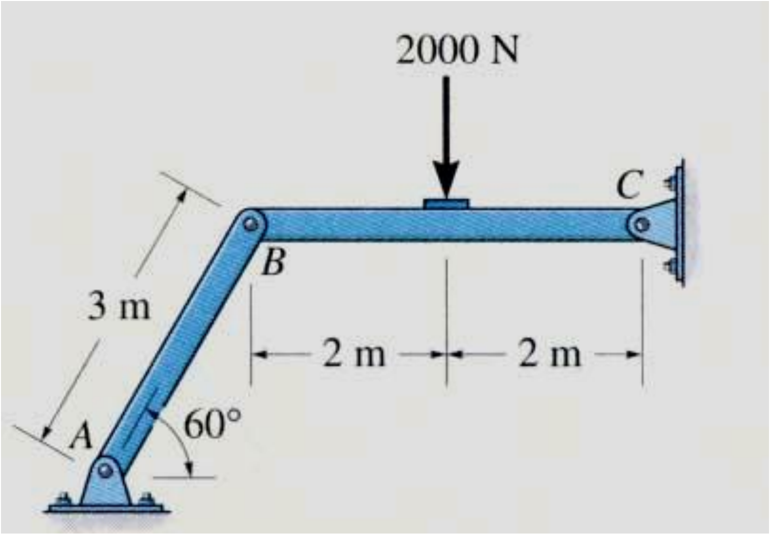
Engineering Mechanics

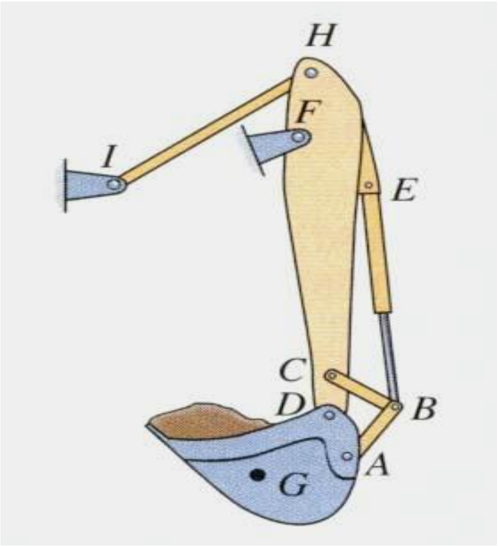
FRAMES AND MACHINES

Overview

- Goals:
  - Draw the free body diagram of a frame or machine and its members
  - Determine the forces acting at the joints and supports of a frame or machine
- Overview
  - Definition of Frames & Machines
  - Force Representation and Free Body Diagrams

FRAMES AND MACHINES: DEFINITIONS





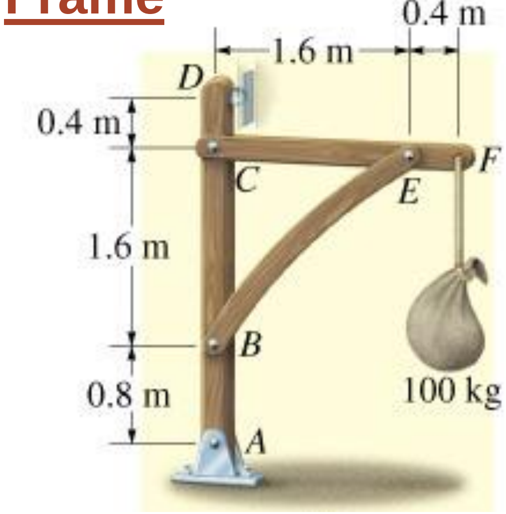
Frames and machines are two common types of structures that have at least one multi-force member.

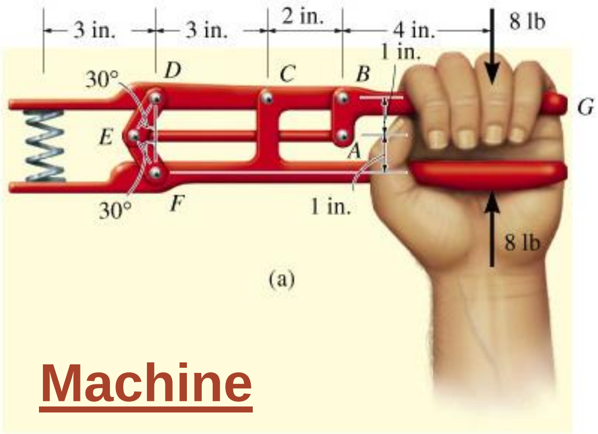
Multi force members: Members on which three or more forces acting on it

Application

- Frames :Structures which are designed to support applied loads and are fixed in position
- Frame are commonly used to support various external loads
- Machines Structure which contain moving parts and are designed to transmit input force
- Machine are used in a variety of applications

Frame





Machine

Frames and Machines

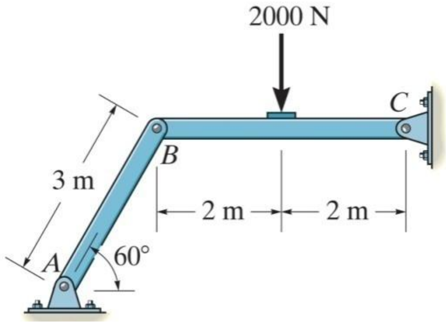
- Frames and machines are two common structures often composed of pin-connected ***multi-force members*** (ie, members that are subjected to $\geq$ 2 forces)
  - ***Frames:*** stationary -- support loads
  - ***Machines:*** contain moving parts -- designed to transmit and alter the effect of forces.

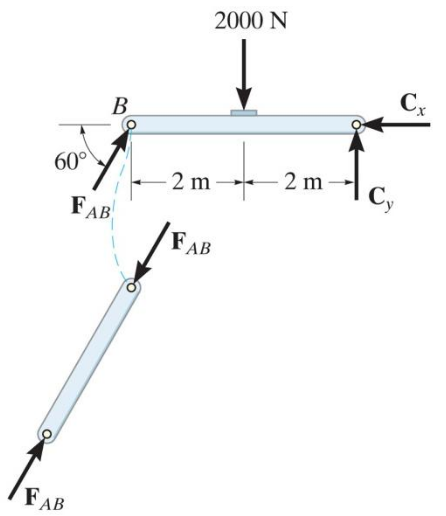
Approach to Analyze a Frame or Machine

1. In general, find the external reactions first
2. Dismember the frame/machine into individual members. Draw the FBD of each member, as necessary.
3. Apply the equations of equilibrium to solve for the unknowns. Number of unknowns must equal number of equations.

These problems can be challenging at first – many unknowns make it difficult to know where to begin!

Hint: Always <u>start with what you know</u> – this often means starting where an external load is applied.



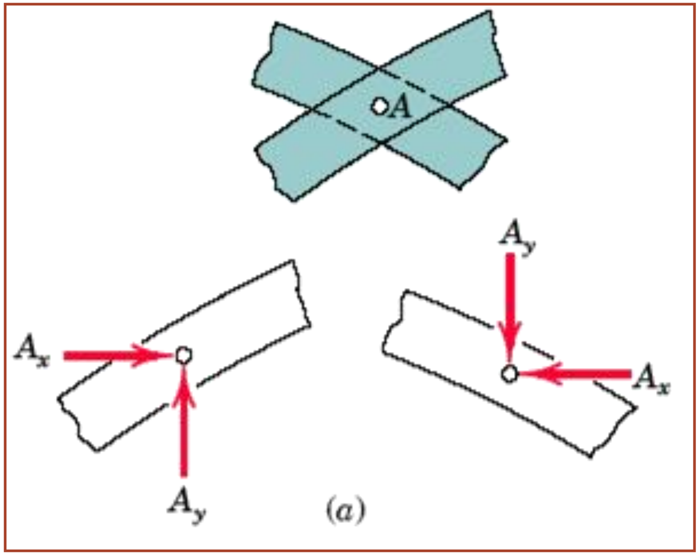


Approach to Analyze a Frame or Machine

Hints:

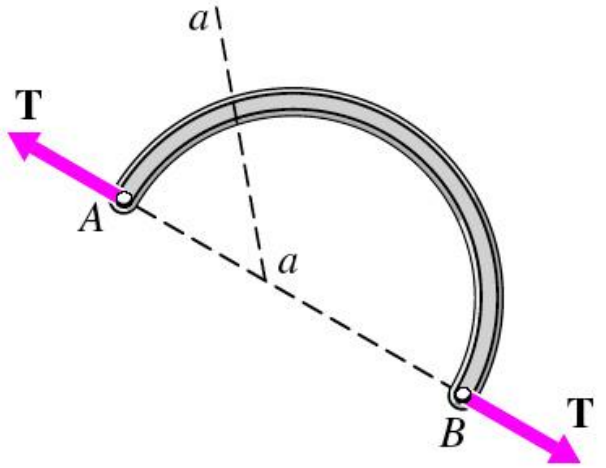
a) Identify any two-force members

b) Forces on contacting surfaces (*usually between a pin and a member*) are equal and opposite



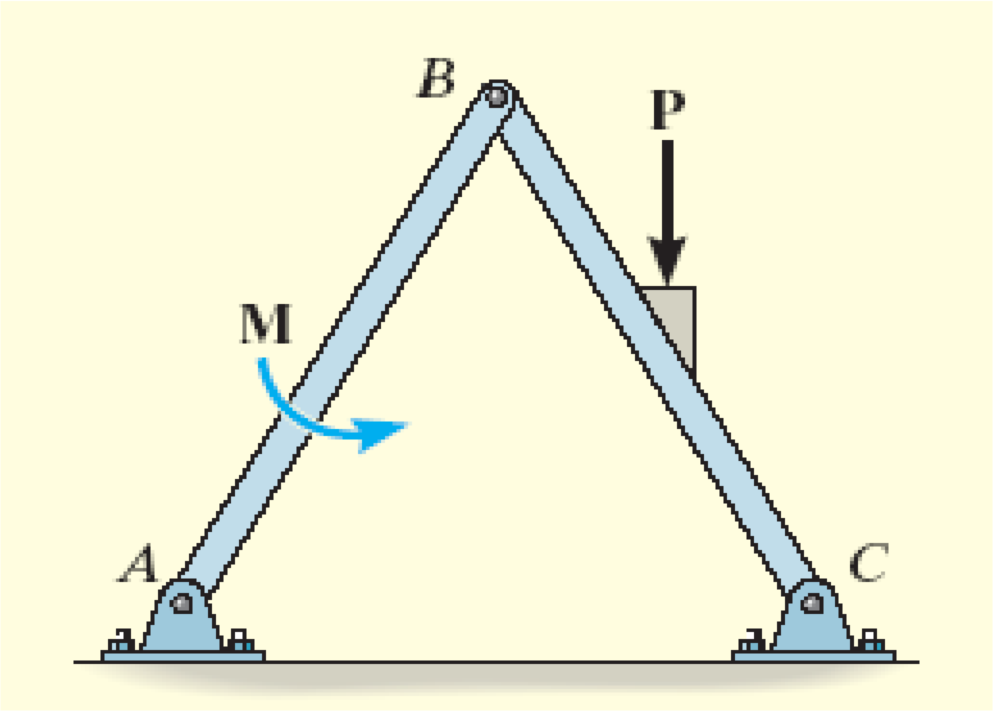
c) For a joint with more than two members or an external force, it is advisable to draw a FBD of the joint

d) Take advantage of symmetry where applicable



Example:

For the frame, draw the free-body diagram of (a) each member, (b) the pin at B and (c) the two members connected together.

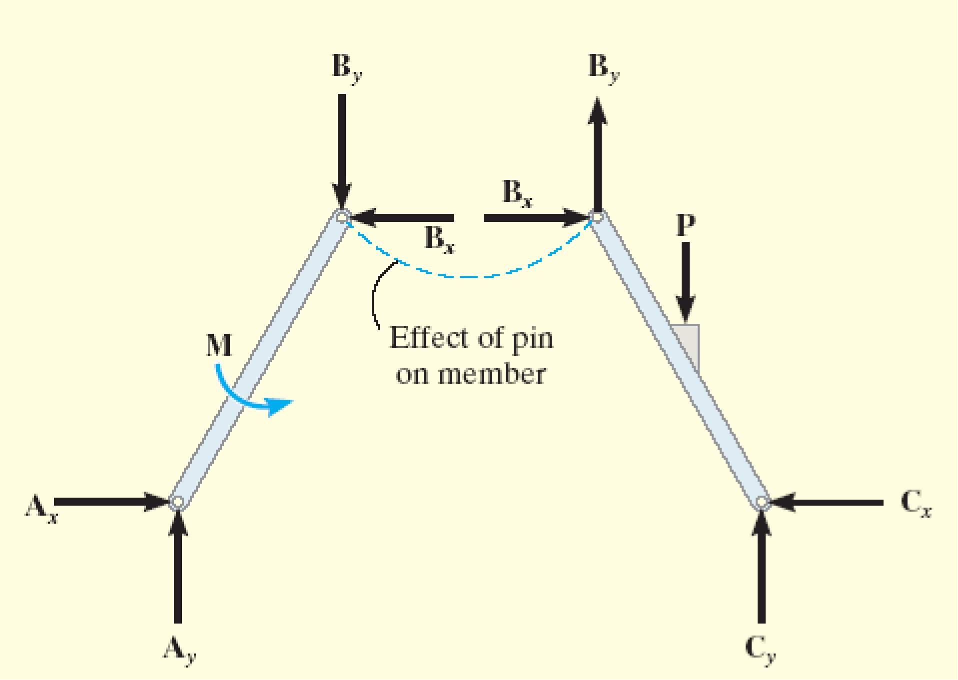


Solution

<u>Part (a)</u>

BA and BC are not two-force

AB is subjected to the resultant forces from the pins

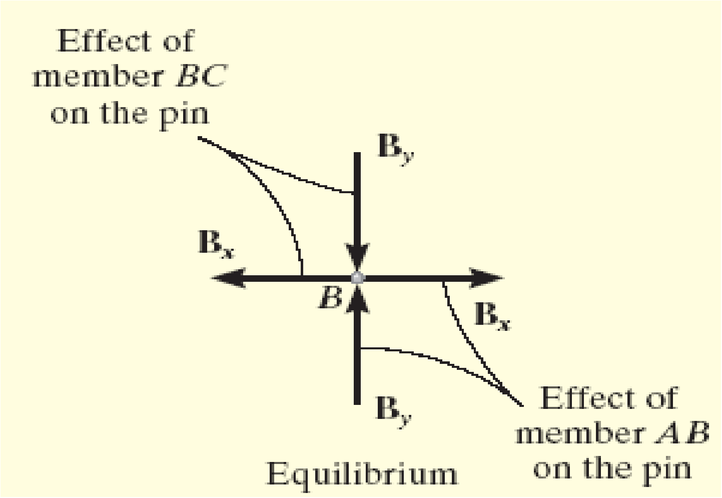


Part (b)

Pin at B is subjected to two forces, force of the member BC and AB on the pin

For equilibrium, forces and respective components must be equal but opposite

$\mathbf{B}_x$ and $\mathbf{B}_y$ shown equal and opposite on members AB

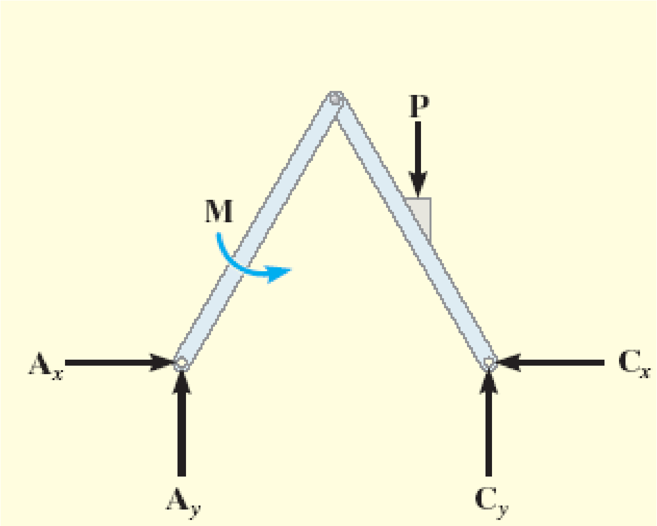


Part (c)

B$_x$ and **B**$_y$ are not shown as they form equal but opposite internal forces

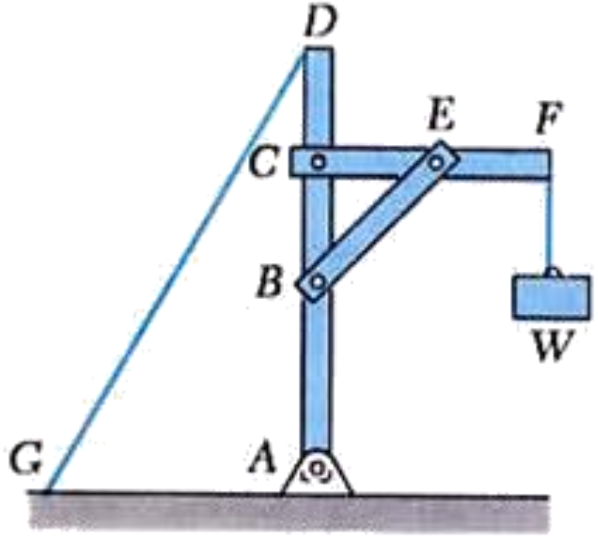
Unknown force at A and C must act in the same sense

Couple moment **M** is used to find reactions at A and C

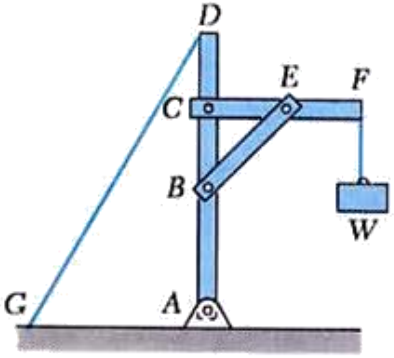


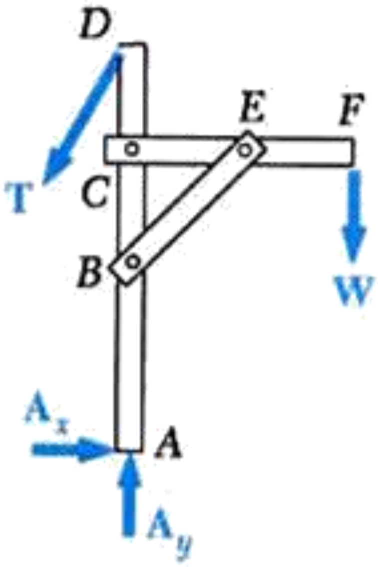
In Class Activity #2

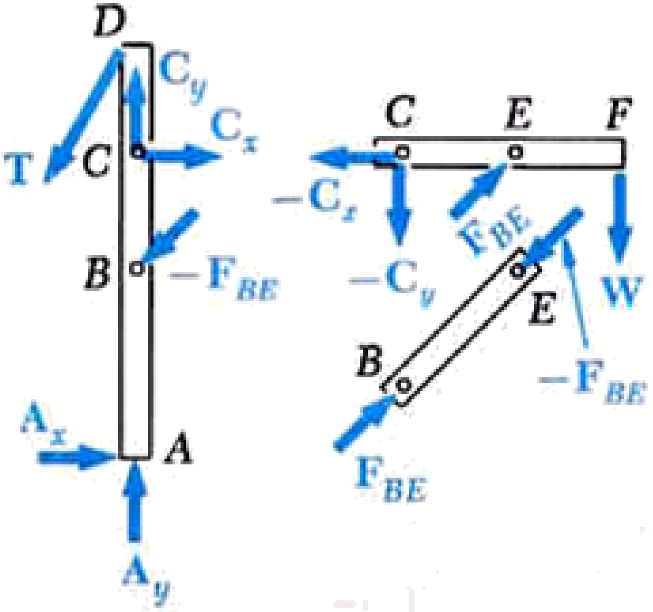
- Draw the necessary FBDs to solve for the forces at each joint in this problem. (Note – you need not solve the problem)



In Class Activity #2

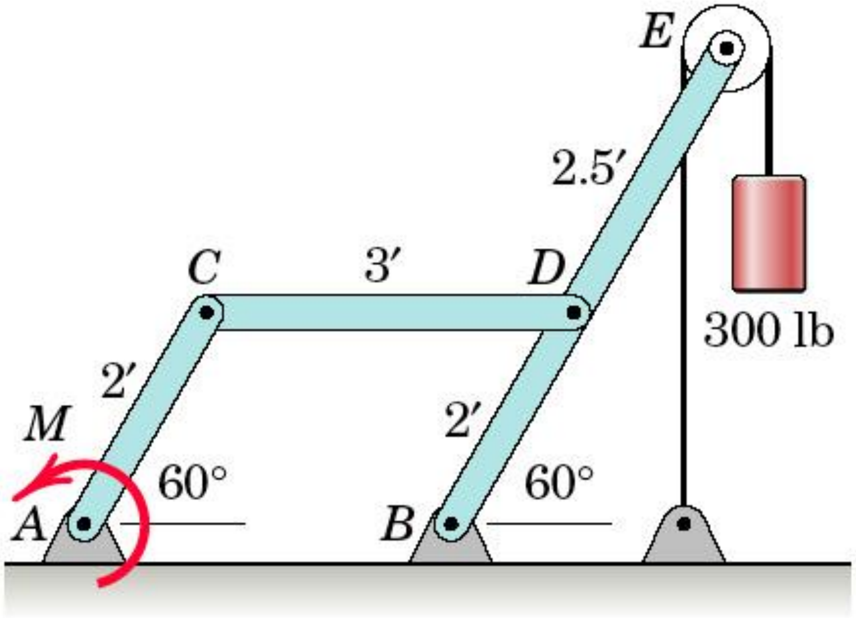






Example 2

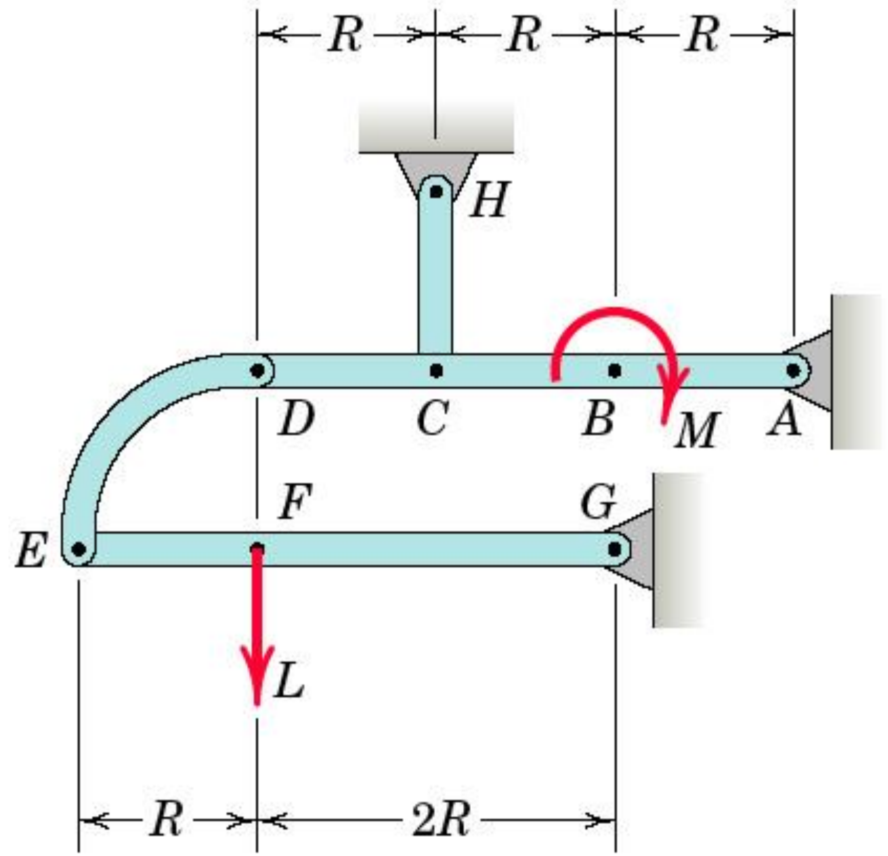
Problem 4/78: Determine the moment *M* which must be applied at *A* to keep the frame in static equilibrium in the position shown. Also calculate the magnitude of the pin reaction at *A*.



Problem 4/78

Example 3

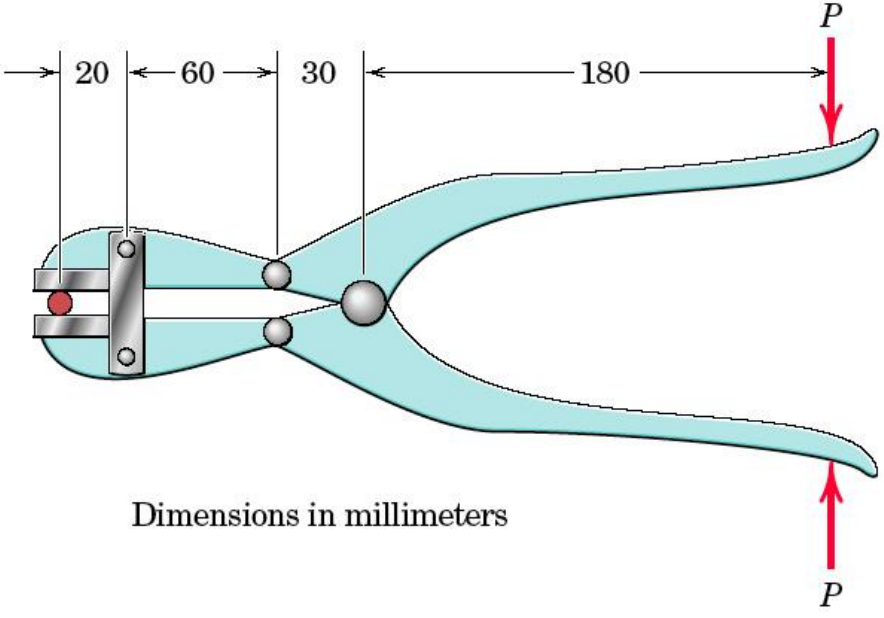
<u>Problem 4/74</u>: Given the values of the load *L* and dimension *R*, for what value of the couple *M* will the force in the link *CH* be zero?



Problem 4/74

Example 1

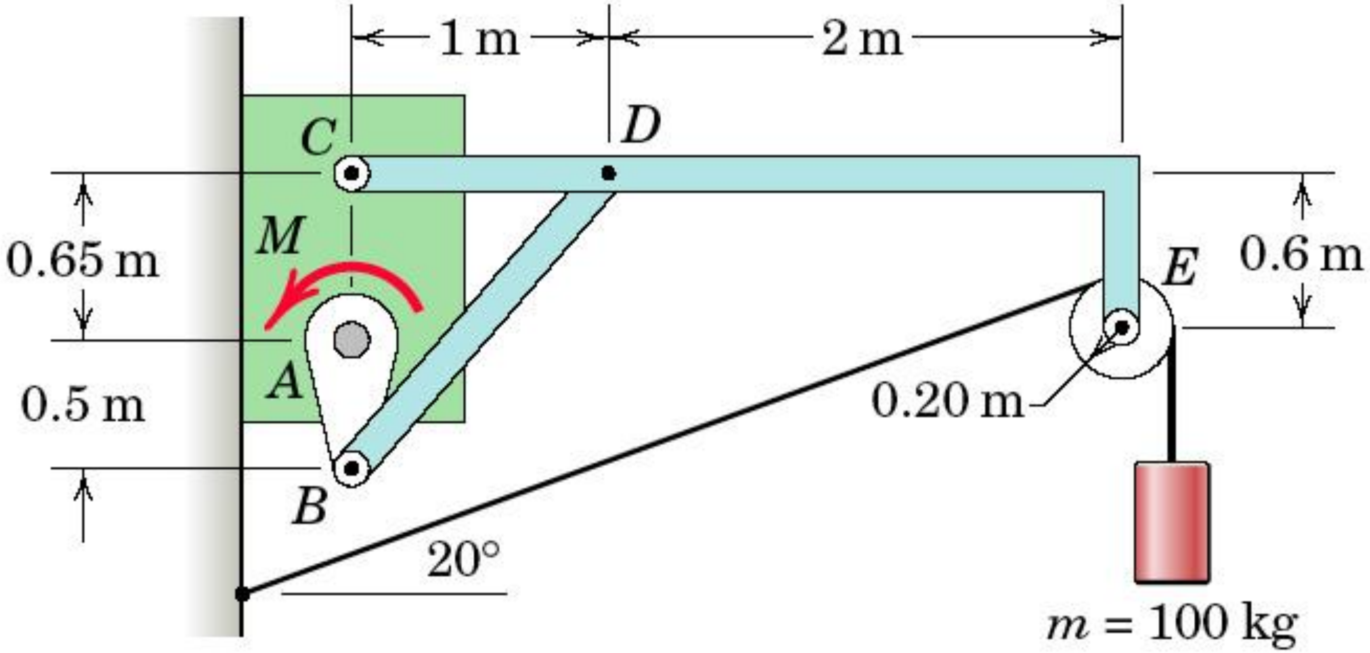
<u>Problem 4/87:</u> A small bolt cutter operated by hand for cutting small bolts and rods is shown in the sketch. For a hand grip P = 150 N, determine the force Q developed by each jaw on the rod to be cut.



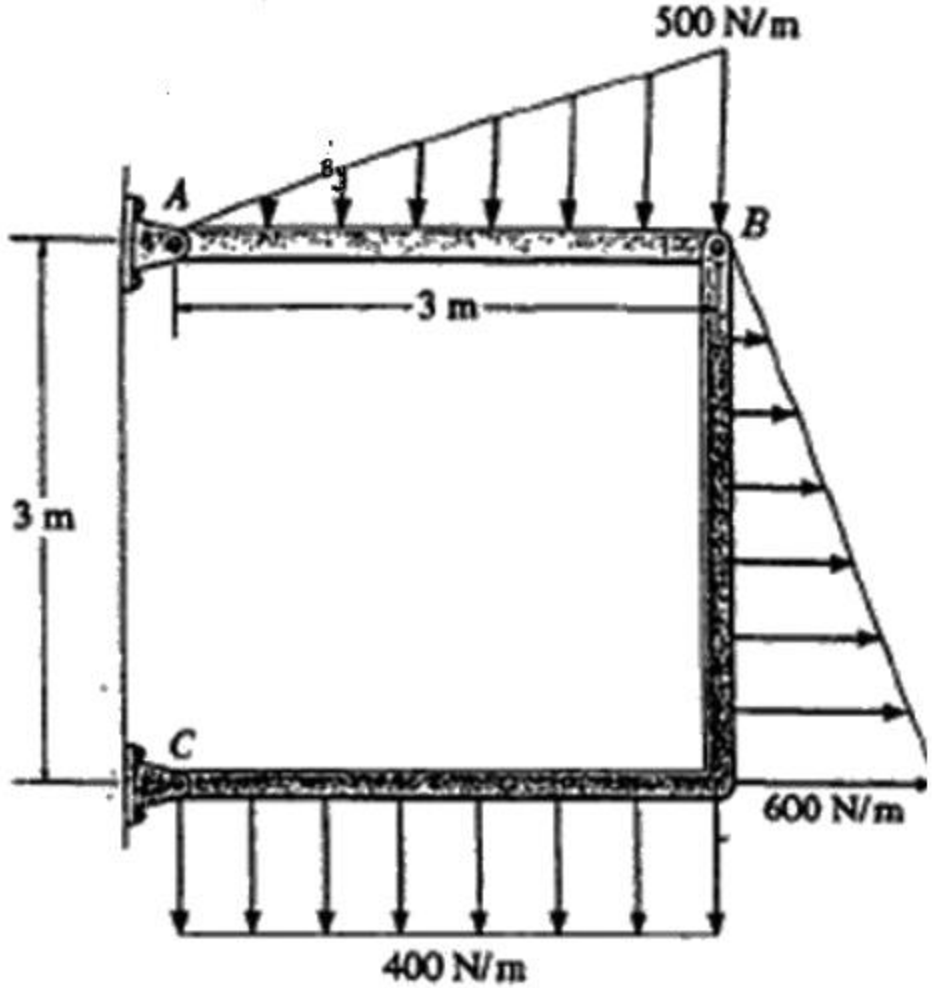
Problem 4/87

Example 2

<u>Problem 4/90</u>: When the crank *AB* is vertical, the beam *CD* is horizontal and the cable makes a 20 angle with the horizontal. Compute the moment *M* required for equilibrium of the frame.



Example: Find the reaction at pin A and B If (A and C is pin And B is internal hinge

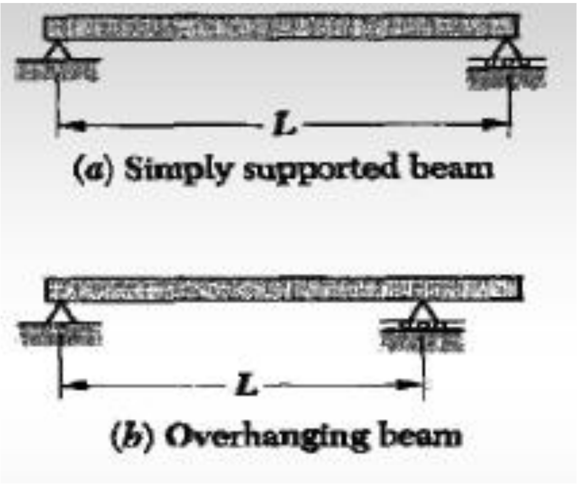


Internal force

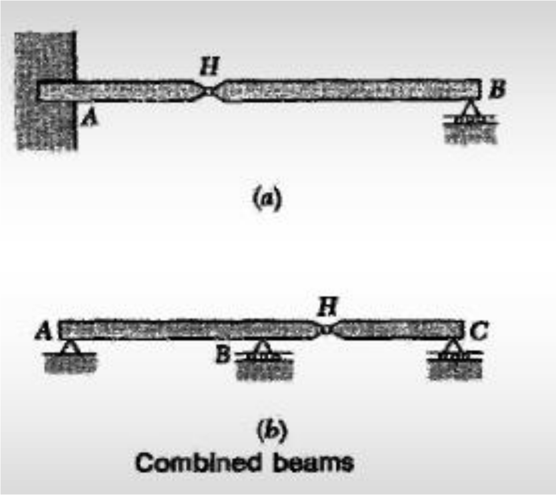
SHEAR AND MOMENT DIAGRAM

Beams – Types

❖A beam can be classified as statically indeterminate beam, which can not be solved with equilibrium equations. It requires a compatibility condition. A



❖A combination beam can be either statically
determinate or indeterminate. These two beams are statically determinate, because the hinge provides another location, where the moment is equal to zero.



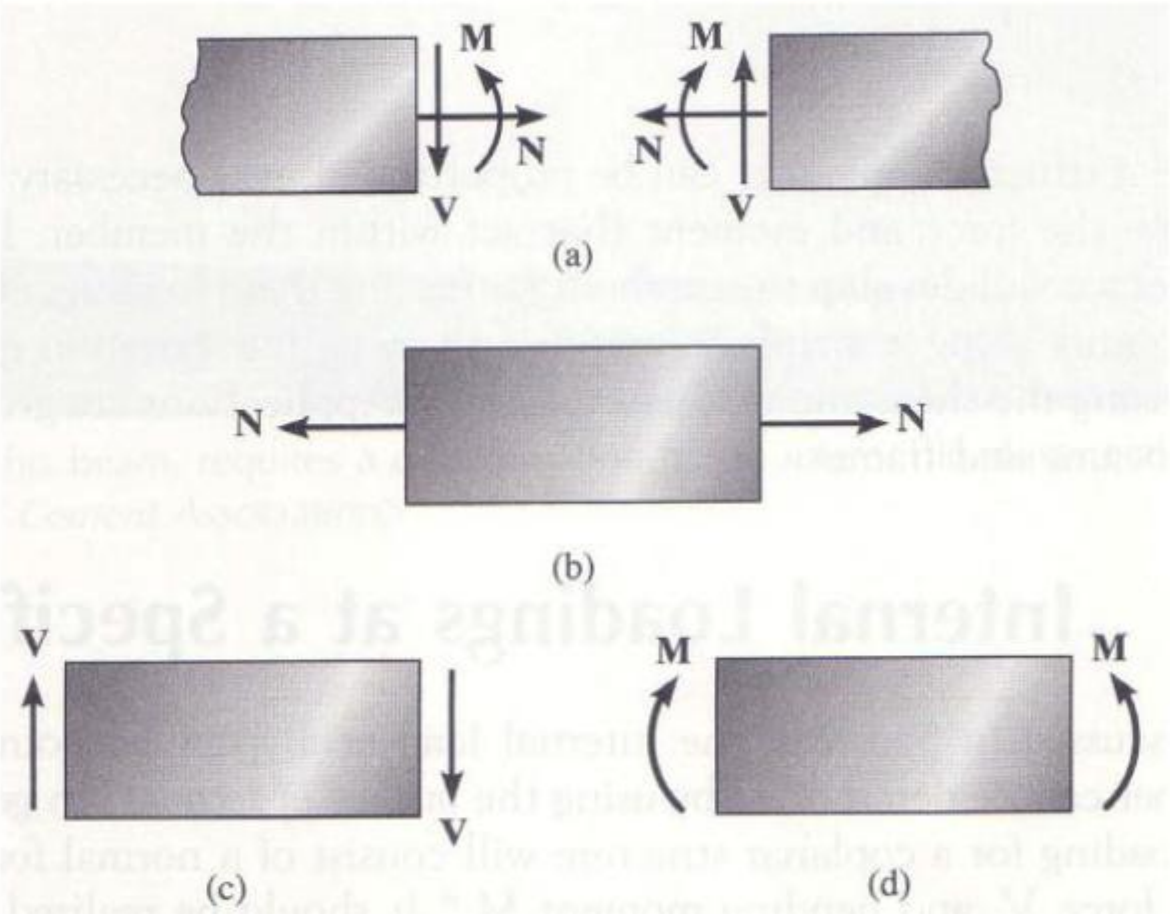
Internal Loadings in Structural Members

In this chapter, we will determine the normal force, shear, and moment at a point in a structural component.

A Shear Force : indicates how a force applied perpendicular to the axis (i.e. parallel to cross section)

A Bending Moment: will show how the applied loads to a beam create a moment variation along the length of the beam.

Sign Convention



(1) Positive shear, V, tends to rotate the component clockwise. Note that the shear is in opposite directions on either side of a cut through the component; nevertheless, each of the two shear components tends to rotate its respective section clockwise. Therefore, each is positive.

(2) Positive normal force, N, tends to elongate the components. Again, note that the normal forces act in opposite directions on either side of the cut; nevertheless, each of the two normal components tends to elongate its respective section. Therefore, each is positive.

(3) Positive moment, M, tends to deform the component into a dish-shaped configuration such that it would hold water. Again, note that the moment acts in opposite directions on either side of the cut; nevertheless, each of the two moments tends to form a dish of its respective section. Therefore, each is positive.

General Solution Scheme

The general scheme for finding the internal set of forces is (2-D)

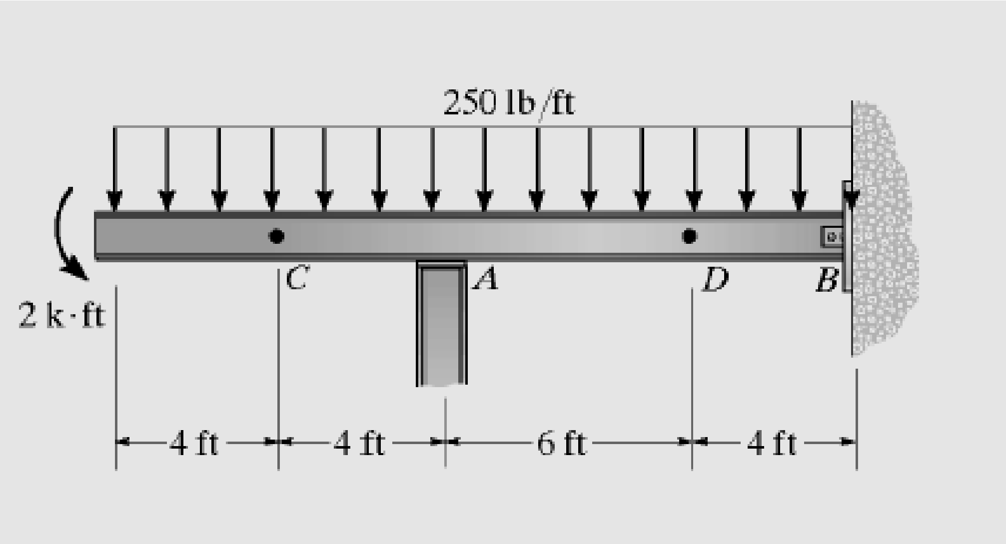
a) Draw the free-body diagram

b) Determine the support reactions

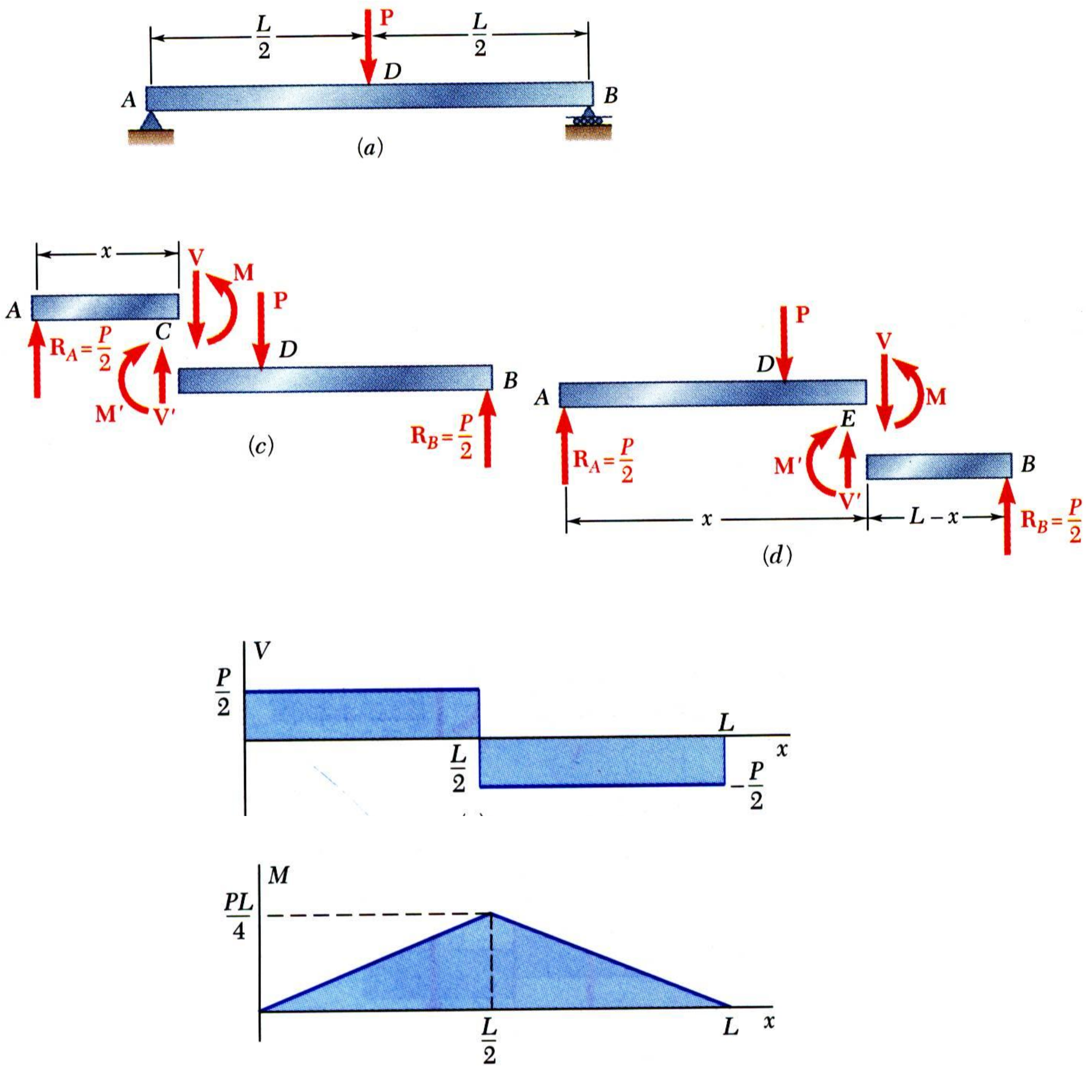
c) Apply the equations of equilibrium

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0$$

Example: Determine the internal shear, axial force , and bending moment in the beam at point C and D. Assume the support at A is a roller and B is a pin



Shear and Bending Moment Diagrams



- Cut beam at *C* and consider member *AC*,

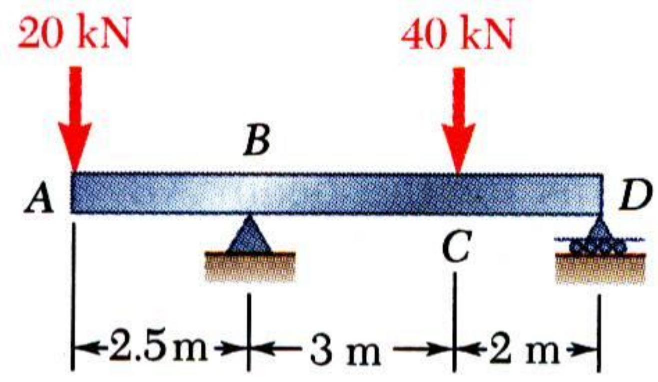
$$V = +P/2 \quad M = +Px/2$$

- Cut beam at *E* and consider member *EB*,

$$V = -P/2 \quad M = +P(L-x)/2$$

- For a beam subjected to <u>concentrated loads</u>, shear is constant between loading points and moment varies linearly.

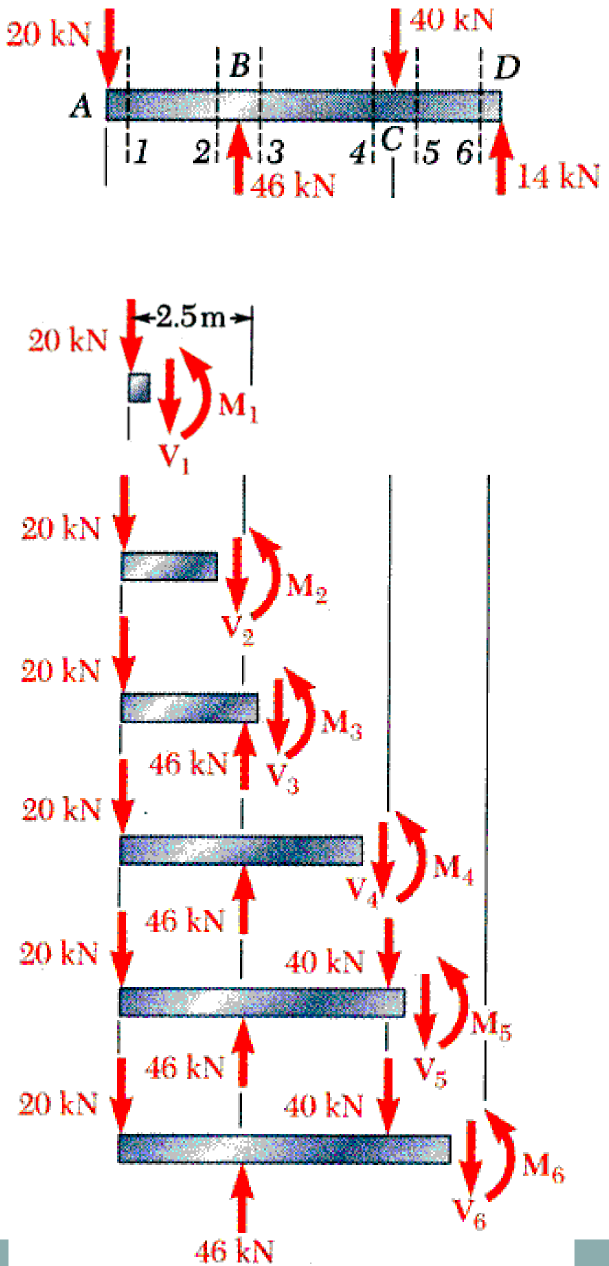
Example: Draw the shear and bending moment diagrams for the beam and loading shown.



SOLUTION:

- Taking entire beam as a free-body, calculate reactions at *B* and *D*.

- Find equivalent internal force-couple systems for free-bodies formed by cutting beam on either side of load application points
- Plot results



SOLUTION:

- Taking entire beam as a free-body, calculate reactions at *B* and *D*.

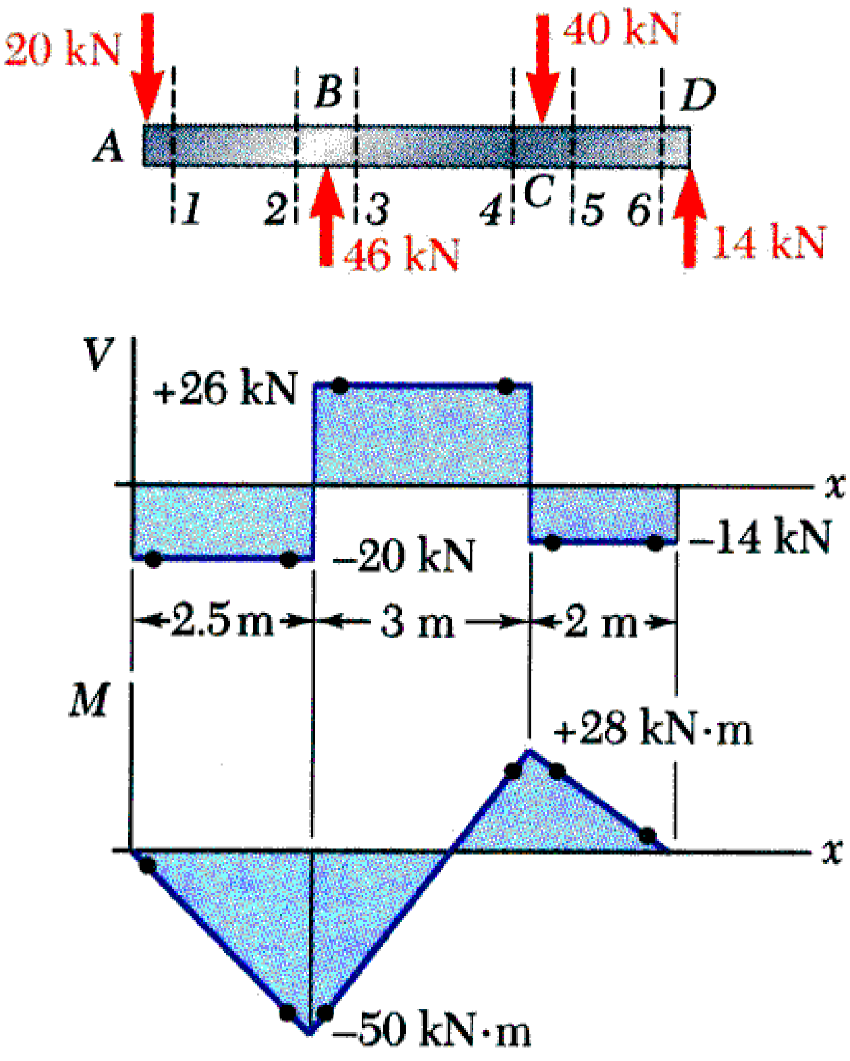
- Find equivalent internal force-couple systems at sections on either side of load application points.

$$\sum F_y = 0: \quad -20\,\text{kN} - V_1 = 0 \qquad \boxed{V_1 = -20\,\text{kN}}$$

$$\sum M_2 = 0: \quad (20\,\text{kN})(0\,\text{m}) + M_1 = 0 \qquad \boxed{M_1 = 0}$$

Similarly,

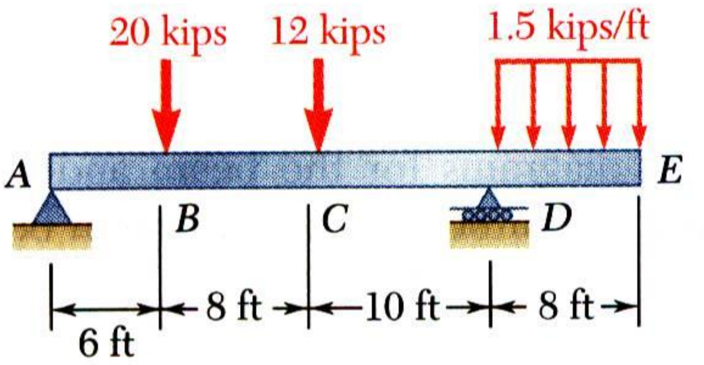
$$\boxed{\begin{array}{ll} V_3 = 26\,\text{kN} & M_3 = -50\,\text{kN}\cdot\text{m} \\ V_4 = 26\,\text{kN} & M_4 = -50\,\text{kN}\cdot\text{m} \\ V_5 = 26\,\text{kN} & M_5 = -50\,\text{kN}\cdot\text{m} \\ V_6 = 26\,\text{kN} & M_6 = -50\,\text{kN}\cdot\text{m} \end{array}}$$



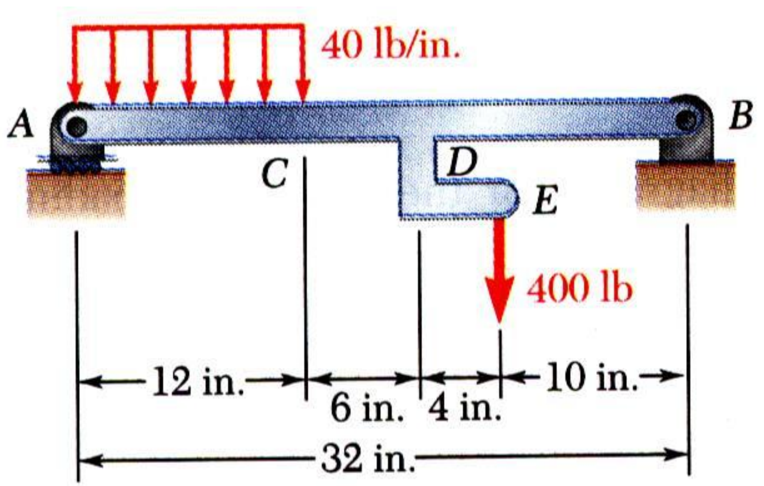
- Plot results.

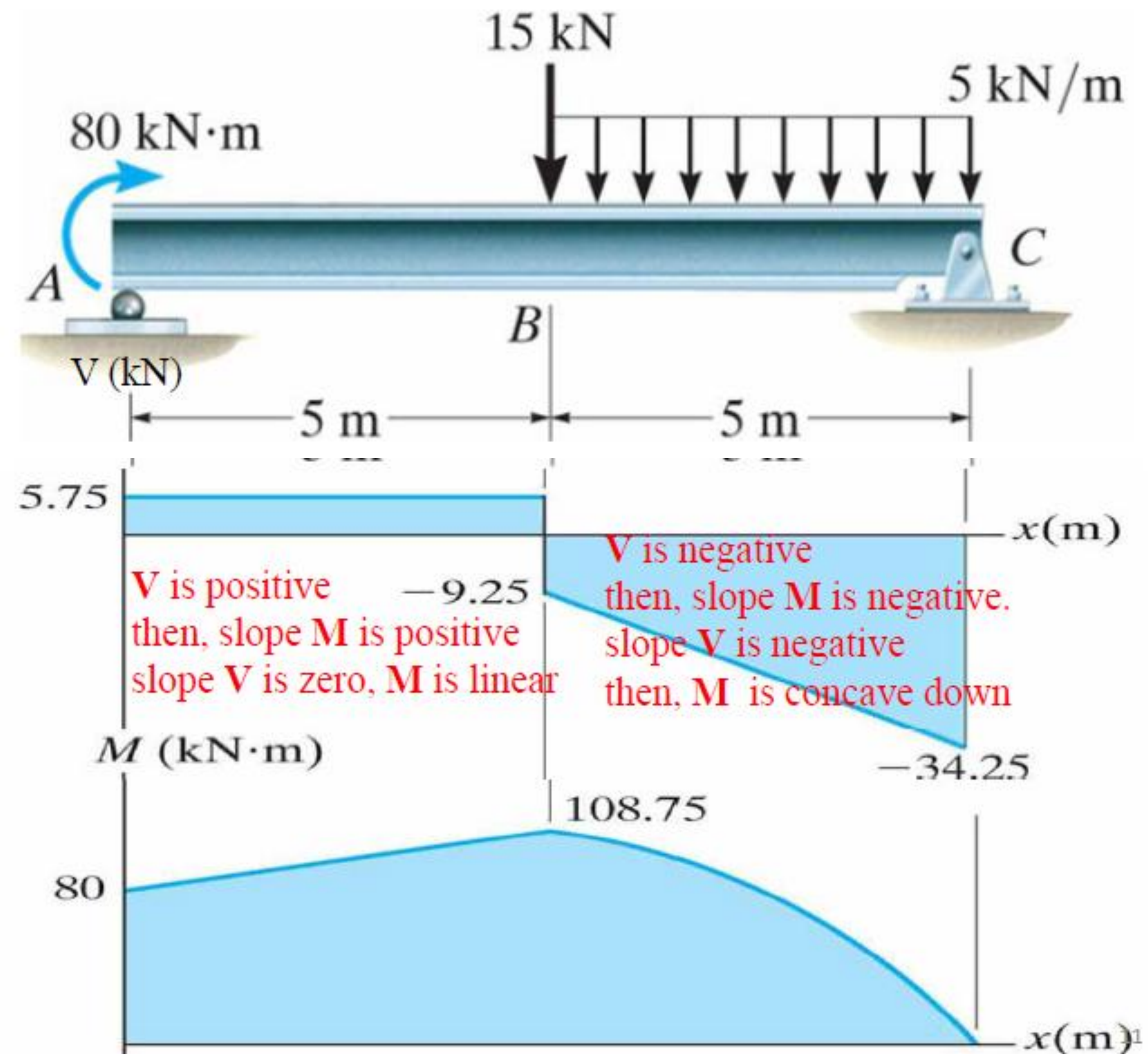
  Note that shear is of constant value between concentrated loads and bending moment varies linearly.

Example: Draw the shear and bending moment diagrams for the beam and loading shown.

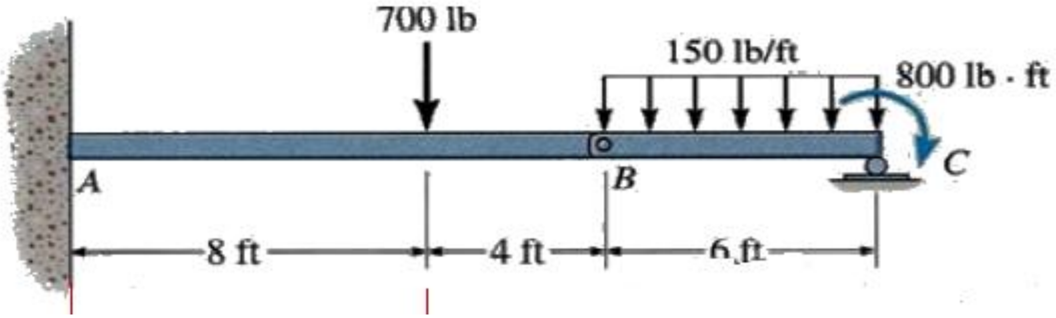


Example: Draw the shear and bending moment diagrams for the beam and loading shown.




15 kN
5 kN/m
80 kN·m
A
B
C
V (kN)
5 m
5 m
5.75
x(m)
V is negative
then, slope M is negative.
slope V is negative
then, M is concave down
V is positive
then, slope M is positive
slope V is zero, M is linear
−9.25
M (kN·m)
−34.25
108.75
80
x(m)

Example: Draw the shear and bending moment diagrams for the beam and loading shown. If support A is fixed and C is roller and B is internal hinge



Engineering mechanics

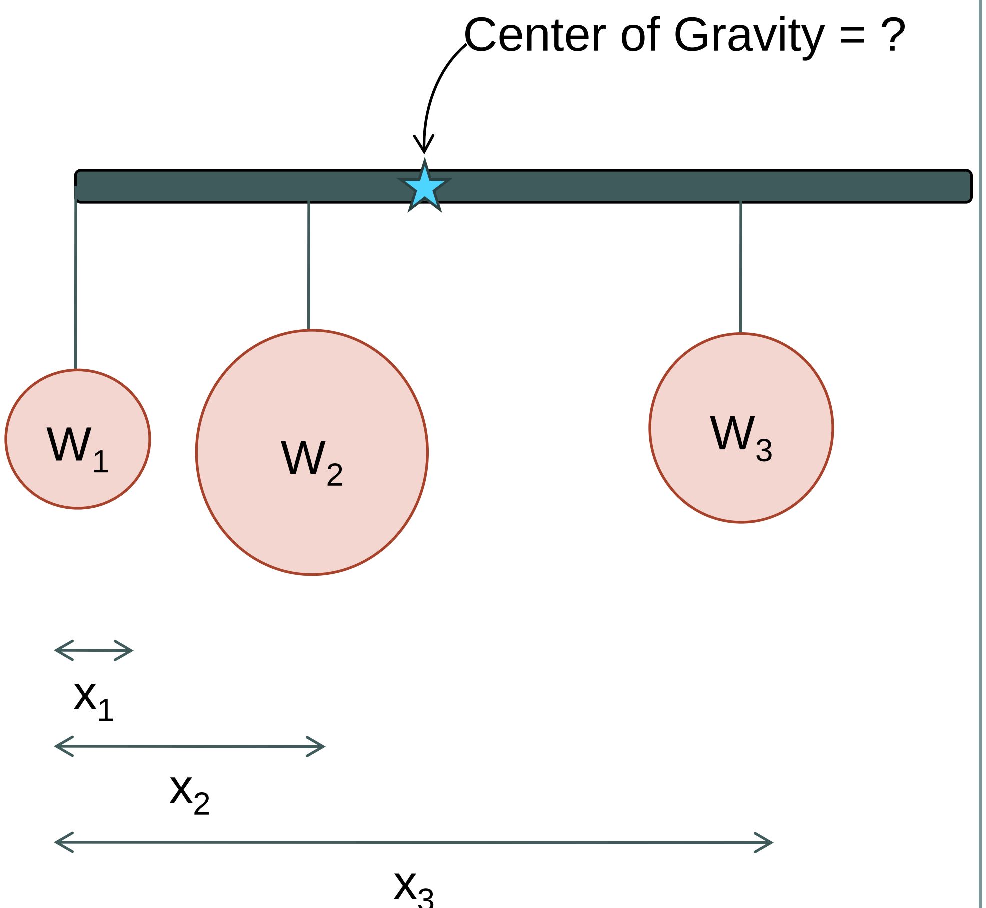
CENTER OF GRAVITY AND CENTROID

Overview

- Goals:
  - Understand the concepts of center of gravity, center of mass, and centroid
  - Be able to determine the location of these points for a system of particles or a body using Method of Integration
- Overview
  - Definition of center of gravity, center of mass, centroid
  - Distinction between these ideas
  - Method of Integration

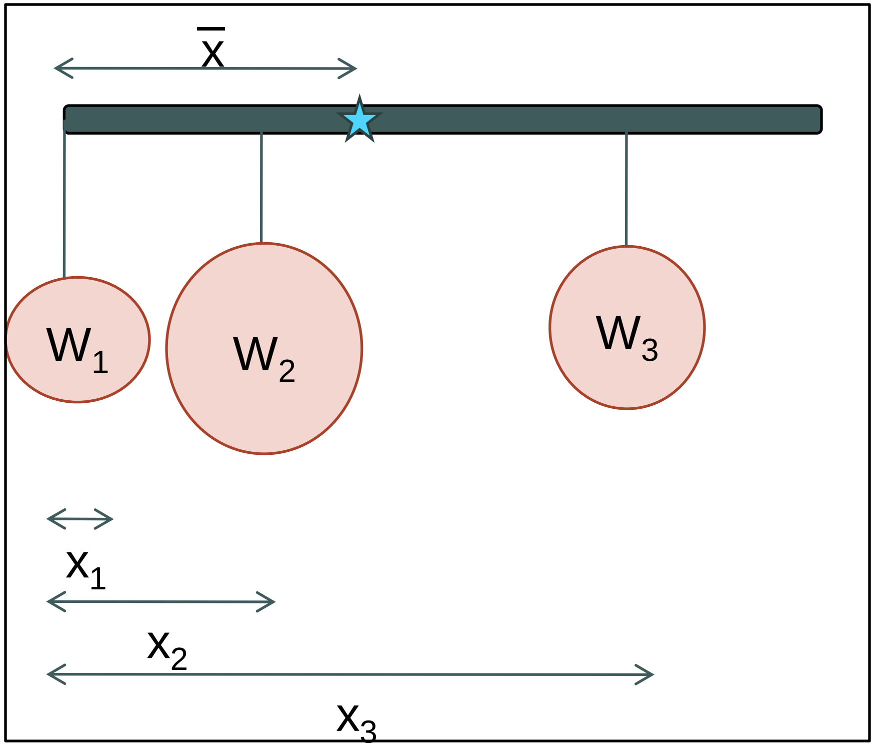
Center of Gravity

- The center of gravity (CG) of a system is the point where the system behaves as a single particle
- The center of gravity is a point which locates the resultant weight of a system of particles or body
  - From the definition of a resultant force, the sum of moments due to individual particle weight about any point is the same as the moment due to the resultant weight located at CG.



Calculating Center of Gravity

- Consider a system of n particles as shown in the figure. The net or the resultant weight is given as $W_R = \sum W$.
- Summing the moments about the y-axis, we get
  - $\bar{x} W_R = x_1 W_1 + x_2 W_2 + \ldots x_n W_n$
  - where x_1 represents x coordinate of W_1, and x-bar is the center of gravity



In general form:

$$\bar{x} = \frac{\sum x_{ci} W_i}{\sum W_i} \qquad \bar{y} = \frac{\sum y_{ci} W_i}{\sum W_i}$$

Calculating Center of Mass

- By dividing top and bottom by gravity, we find an equivalent expression

$$W = mg \rightarrow$$

$$\bar{x} = \frac{\sum x_{c_i} m_i}{\sum m_i} \qquad \bar{y} = \frac{\sum y_{c_i} m_i}{\sum m_i}$$

- We see that center of mass and center of gravity, for our purposes, coincide
  - In non-uniform gravitational fields, center of gravity could be different from center of mass
  - *Center of mass* is more commonly used expression

Calculating Center of Mass

- Rigid Bodies:
  - A rigid body can be considered to be made up of an infinite number of particles
  - Using the same principles we get the coordinates of CG (or CM) by simply replacing the discrete summation sign (∑) by the continuous summation sign (∫) and W by dW

$$\bar{x} = \frac{\int x_c m}{\int m} \qquad \bar{y} = \frac{\int y_c m}{\int m}$$

  - Where x_c, y_c are the locations of the **local** centers of mass of the individual components
  - Similarly, the coordinates of the center of mass and the center of volume, area, or length can be obtained by replacing W by m, V, A, or L, respectively

Centroid

- The centroid C is a point which defines the *geometric* center of an object
- The centroid coincides with the center of mass or the center of gravity only if the material of the body is homogeneous (density or specific weight is constant throughout the body)

Goal: Compute the Centroid/CM, etc

- Examine an arbitrary geometry
  - This geometry must be described by equation or series of eqns.
- Use the appropriate equation:

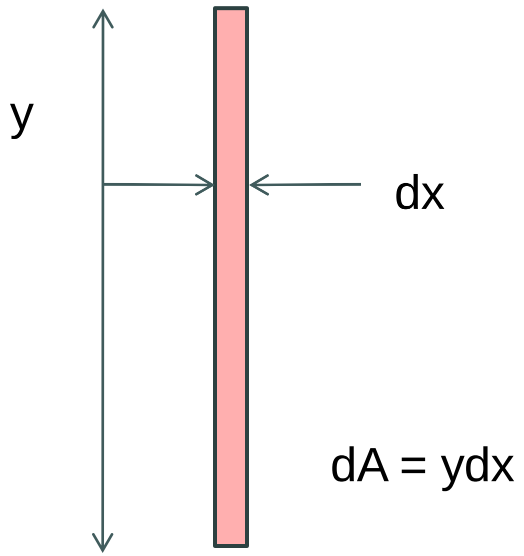
$$\bar{x} = \frac{\int x_c dA}{\int dA} \quad \bar{y} = \frac{\int y_c dA}{\int dA} \quad or \quad \bar{x} = \frac{\int x_c dA}{A} \quad \bar{y} = \frac{\int y_c dA}{A}$$

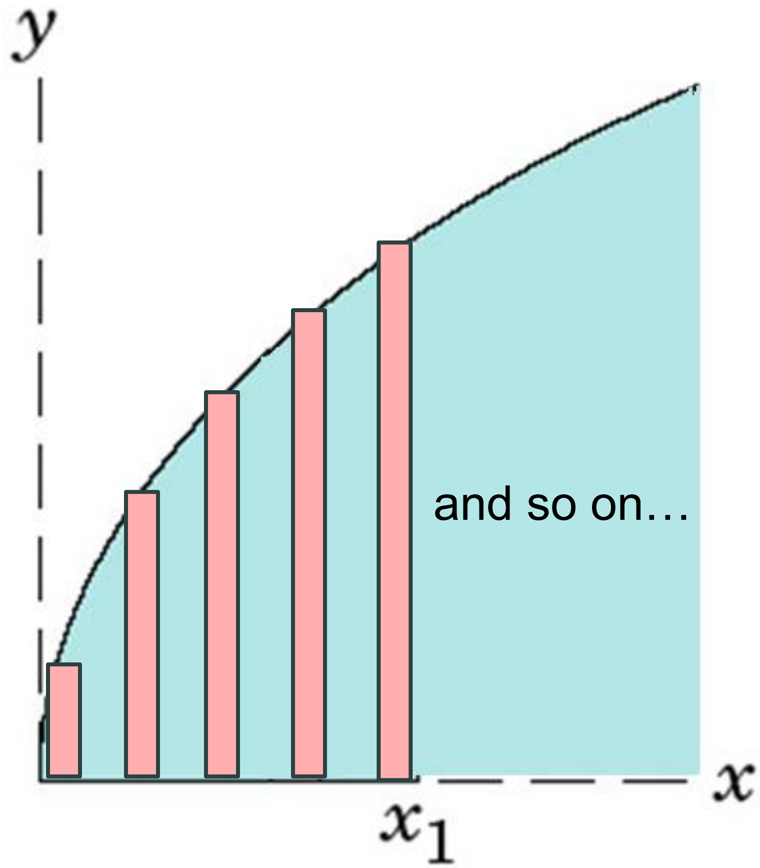
  Where the coordinates of the center of weight or center of gravity, volume, mass, or length can be obtained by replacing A by W, V, m, or L, respectively

- Solve for the centroid/center of mass, etc
  - Approach: consider geometry as sum of differential elements

"Sum of differential elements"???

- Assume geometry can be represented by series of rectangles stacked horizontally or vertically
- Rectangles have infinitesimal height **or** width





Centroid by Integration

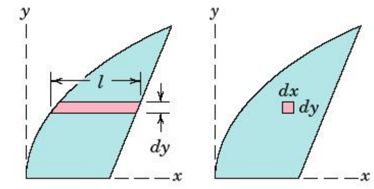
Steps for determining the centroid of the area

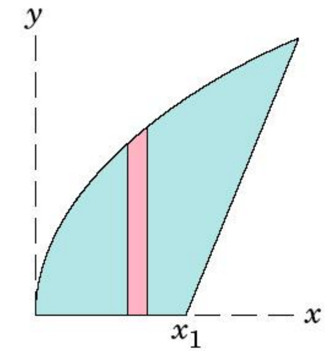
1. Choose an appropriate differential element dA at a general point (x,y).

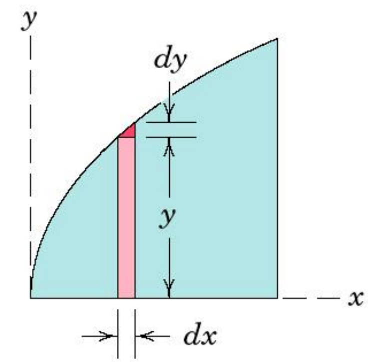
   Hint: Generally, if y is easily expressed in terms of x (e.g., $y = x^2 + 1$), use a vertical rectangular element. If the converse is true, then use a horizontal rectangular element

2. Express dA in terms of the differentiating element dx (or dy).
3. Determine coordinates (x , y) of the centroid of the rectangular element in terms of the general point (x,y).
4. Express all the variables and integral limits in the formula using either x or y depending on whether the differential element is in terms of dx or dy, respectively, and integrate.

These steps will become clearer after doing a few examples.

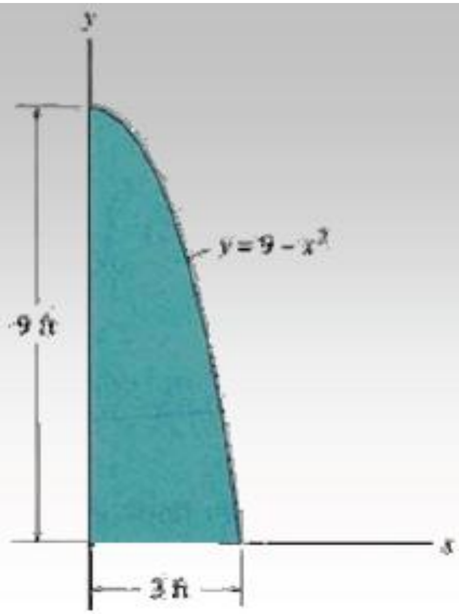


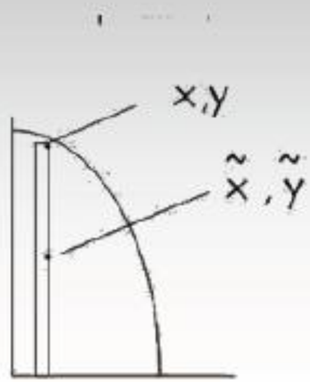




Given: The area as shown.

Find: The centroid location ($\bar{x}$, $\bar{y}$)





1. Since y is given in terms of x, choose dA as a vertical rectangular strip.

2. $dA = y\,dx = (9 - x^2)\,dx$

3. $\tilde{x} = x$ and $\tilde{y} = y / 2$

4. $\bar{x} = (\int_A \tilde{x}\, dA) / (\int_A dA)$

$$= \frac{{}_0\!\int^3 x\,(9 - x^2)\,dx}{{}_0\!\int^3 (9 - x^2)\,dx} = \frac{[\,9\,(x^2)/2 - (x^4)/4\,]_0^3}{[\,9x - (x^3)/3\,]_0^3}$$

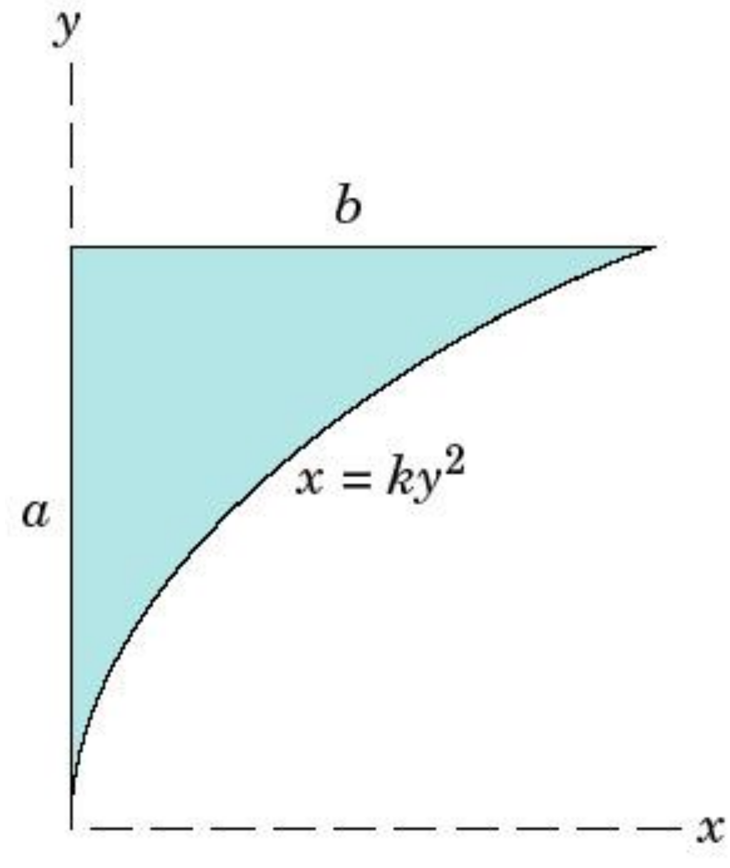
$$= (9 (9) / 2 - 81 / 4) / (9 (3) - (27 / 3))$$

$$= 1.13 \text{ ft}$$

$$\bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} = \frac{½\ {}_0\!\int^3 (9 - x^2)\,(9 - x^2)\,dx}{{}_0\!\int^3 (9 - x^2)\,dx} = 3.60 \text{ ft}$$

Example 1

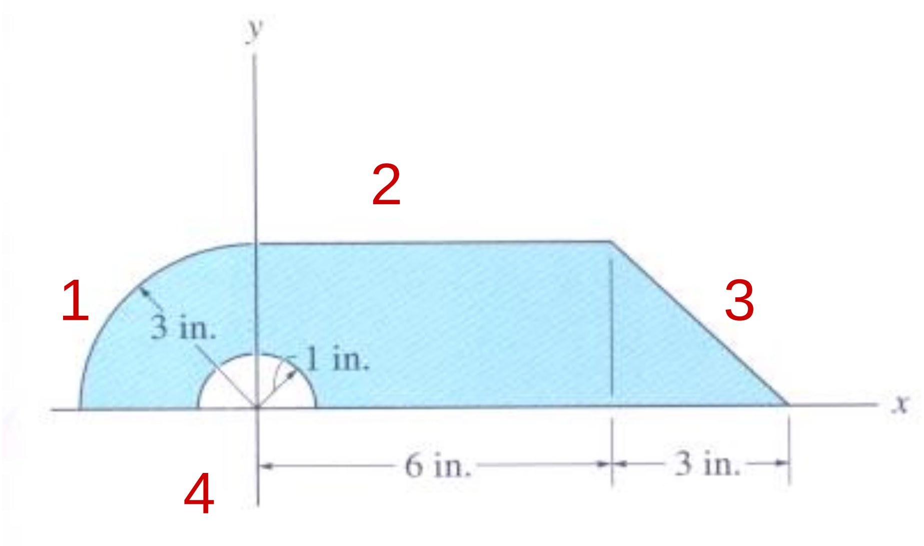
Problem 5/6: Determine the coordinates of the centroid of the shaded area.



Problem 5/6

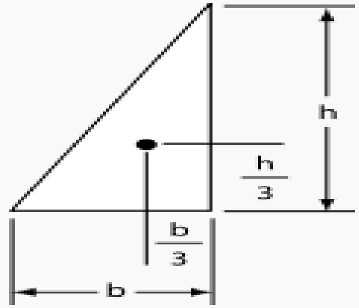
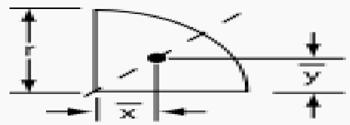
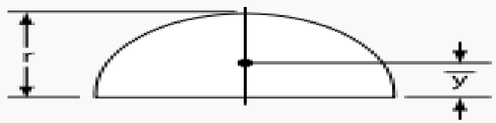
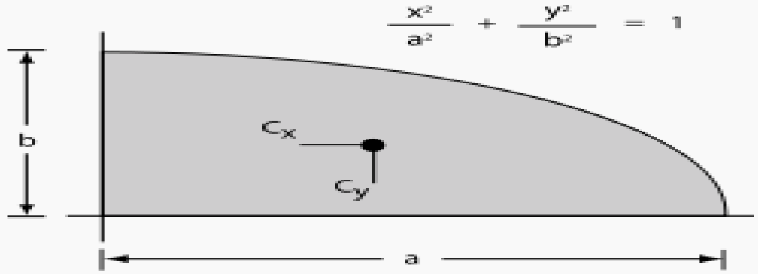
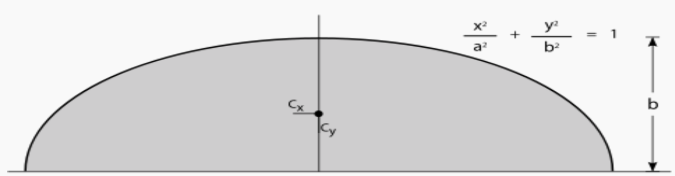
Composite Sections: Concept

- Many objects can be considered as composite bodies made up of a series of connected "simpler" shaped parts or holes, like a rectangle, triangle, and semicircle.
- Knowing the location of the centroid, C, or center of gravity, G, of the simpler shaped parts, we can easily determine the location of the C or G for the more complex composite body.



To do so, we can consider each part to be a "particle" and treat the object similarly to the integration approach

This is a simple, effective, and practical method of determining the location of the centroid or center of gravity.

Shape	Figure	$\bar{x}$	$\bar{y}$	Area
Right-triangular area		$\frac{b}{3}$	$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$

Calculating Center of Gravity

- Recall our initial equation from last week:

$$\bar{x} = \frac{\sum x_c A_n}{\sum A_n} \qquad \bar{y} = \frac{\sum y_c A_n}{\sum A_n}$$

- By examining a composite comprised of several simple geometries with known areas and centroids, we can easily compute the centroid of the composite

Procedure for Composite Analysis

1. Divide the body into known shapes. Holes are considered pieces with negative weight or size.
2. Make a table with the first column for segment number, the second column for weight, mass, or size (depending on the problem), the next set of columns for the moment arms, and, finally, several columns for recording results of simple intermediate calculations.
3. Fix the coordinate axes and determine the coordinates of the center of gravity or centroid of each piece, and fill-in the table.
4. Sum the columns to get x-bar, y-bar, and z-bar.

Use table D-3 (p. 501-2) to find centroid locations for common shapes

Creating a Composite Table

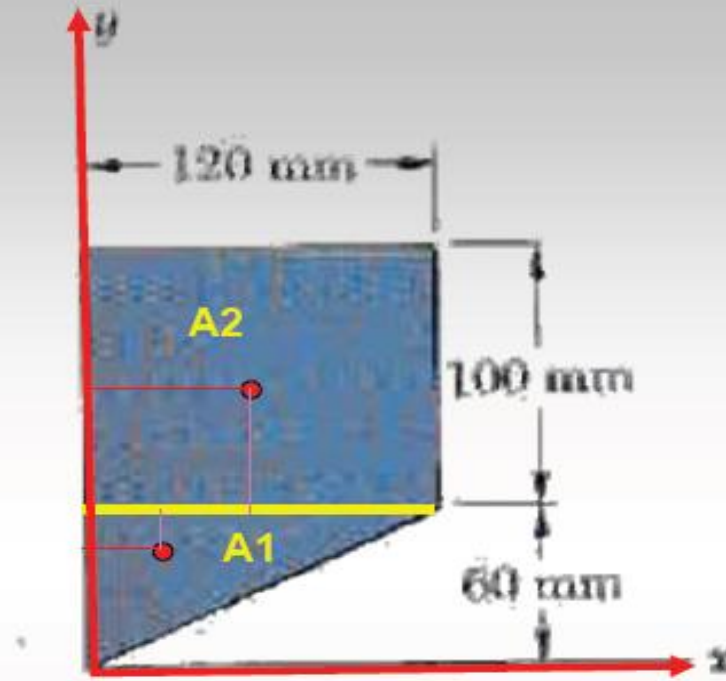
$$\bar{x} = \frac{\sum x_c A}{\sum A} \qquad \bar{y} = \frac{\sum y_c A}{\sum A}$$

Segment	Area A (in^2)	x_c (in)	y_c (in)	A x_c (in^3)	A y_c (in^3)
1					
2 …					
	ΣA			ΣAx_c	ΣAy_c

Find the centroid of the given body

$$\bar{x} = \frac{1}{A_T} \sum \bar{x}_i A_i$$

$$\bar{y} = \frac{1}{A_T} \sum \bar{y}_i A_i$$



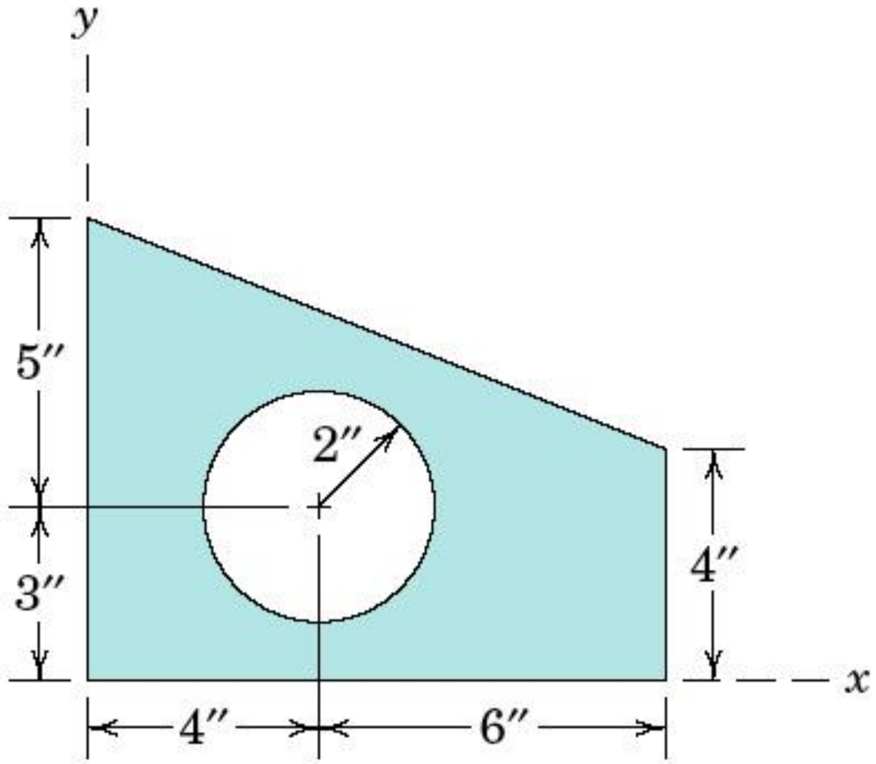
Body	Area(mm^2)	x (mm)	y(mm)	x*Area (mm^3)	y*Area (mm^3)
Triangle	3600	40	40	144000	144000
Square	12000	60	110	720000	1320000
Sum	15600			864000	1464000

centroid (x) 55.38 mm

centroid (y) 93.85 mm

Example 2

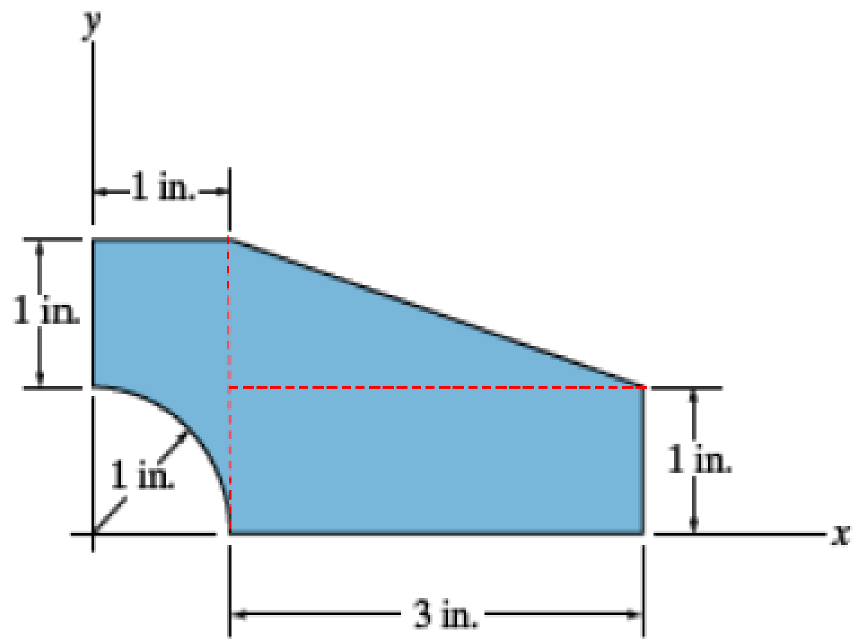
<u>Problem 5/52</u>: Determine the x- and y-coordinates of the centroid of the shaded area.



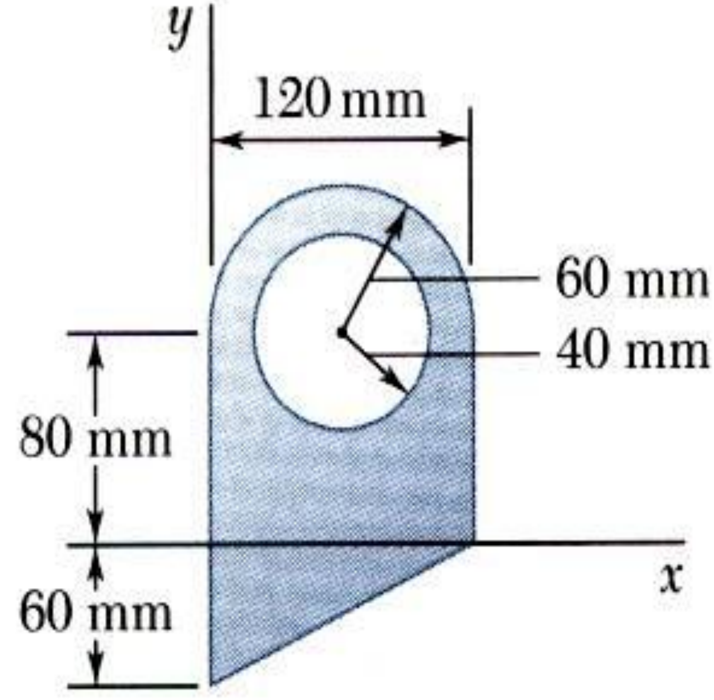
Problem 5/52

Example:

Determine the x- and y-coordinates of the centroid of the shaded area.



Example: Locate the centroid (x,y) of the shaded area shown in fig



Moment Of Inertia

Overview

- Goal
  - To understand the physical and mathematical meanings of the moment of inertia
  - To develop a method for determining the moment of inertia for an area
- Overview
  - Moment of Inertia (MOI) Concept
  - Mathematical Description
  - Examples

Moment of Inertia: Physical Concept

- ***The Moment of Inertia (MOI)*** is a term used to describe the capacity of a cross-section to resist bending.
  - The **larger** the Moment of Inertia the **less** the beam will bend.

- The moment of inertia of an object can change if its shape changes.

Moment of Inertia: Mathematical Description

Moment of Inertia for Area

- MOI is always considered with respect to a reference axis such as *x-x* or *y-y*. It is a mathematical property of a section concerned with a surface area and how that area is *distributed* about the *reference axis*.

Moment of Inertia: Mathematical Description

Moment of Inertia ~ Elastic Beam

- Consider a plate subject to a stress, σ, where

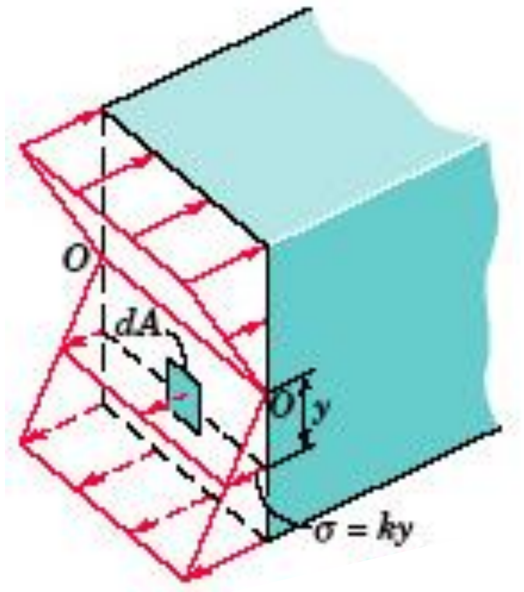
$$\sigma = k \cdot y$$

- The force on a differential area, *dA* is equal to

$$dF = \sigma \cdot dA = k \cdot y \cdot dA$$

- The moment about the x-axis due to this force is

$$dM = y \cdot dF = y^2 \cdot k \cdot dA$$



Area Moment of Inertia

- Recall for beam: $dM = y \cdot dF = y^2 \cdot k \cdot dA$
- MOI for Area: (*by definition*)

$$d\bar{I}_x = y^2 \cdot dA$$

$$\bar{I}_x = \int y^2 dA$$

$$\bar{I}_y = \int x^2 dA$$

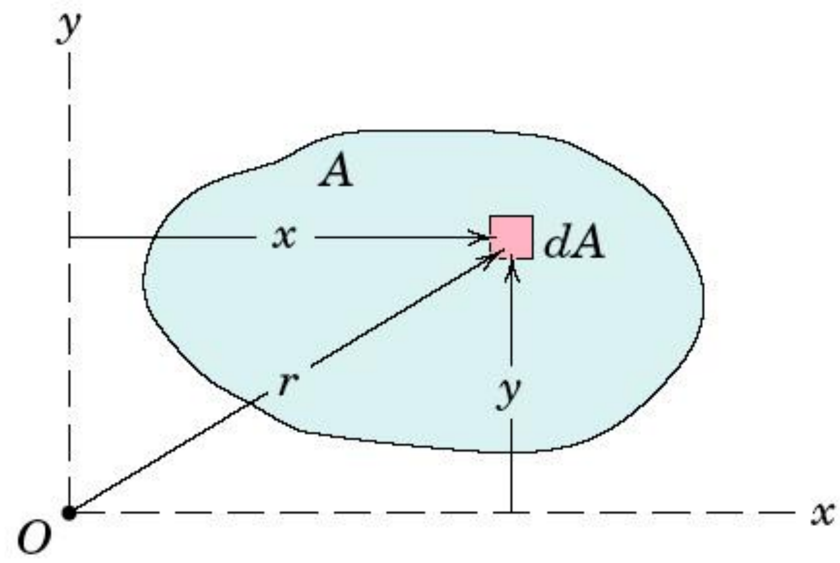


Figure A/2

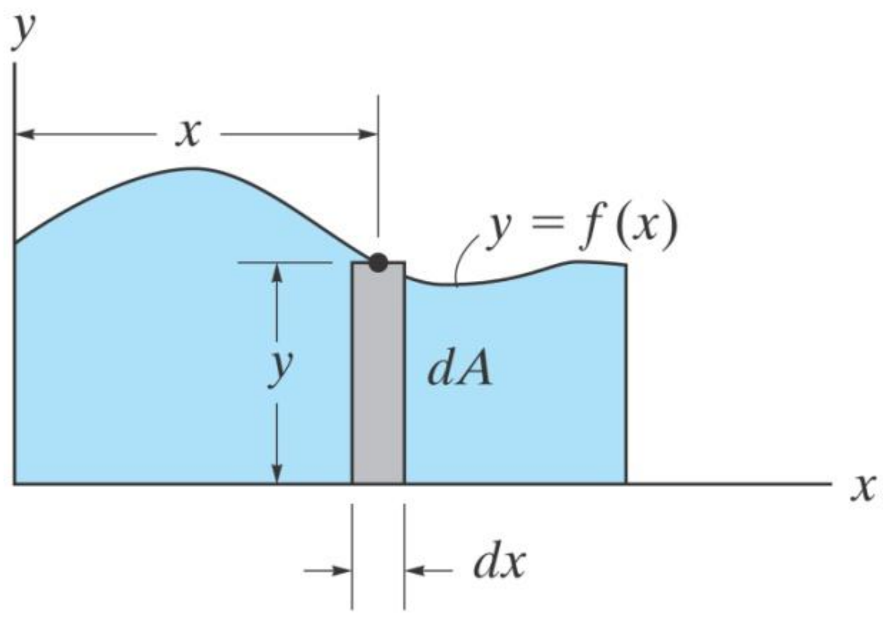
Note –these equations compute the *area* moment of inertia with respect to the *centroid* of the area.

MOI for an Area by Integration

$I_y = \int x^2 \, dA$

and

$I_x = \int y^2 \, dA$



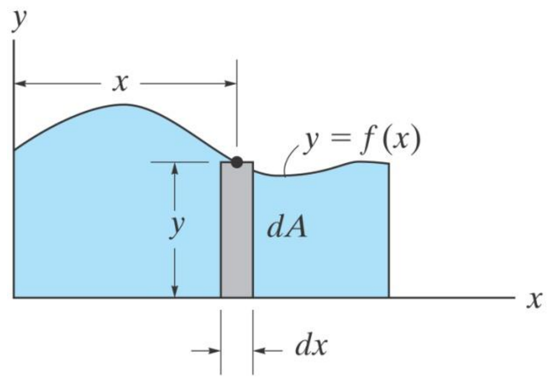
MOI for an Area by Integration

For simplicity, the area element used has a differential size in only one direction (dx or dy). This results in a single integration and is usually simpler than doing a double integration with two differentials, dx·dy.

The step-by-step procedure is:

1. Choose the element dA: (=ydx or =xdy)
   There are two choices: a vertical strip or a horizontal strip.

2. Integrate to find the MoI. For example, given the element shown:



$I_y = \int x^2 \, dA$ and
$I_x = \int y^2 \, dA$

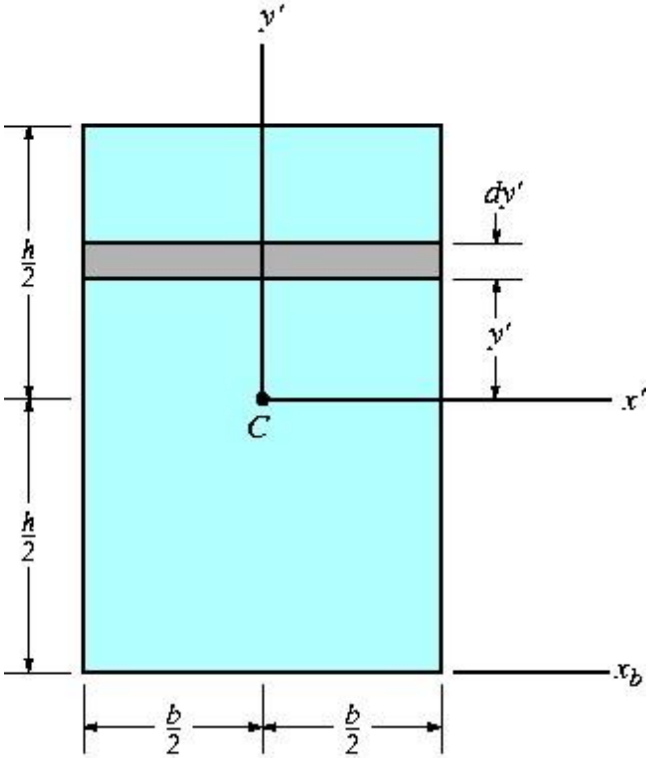
Since in this case the differential element is dx, y needs to be expressed in terms of x and the integral limit must also be in terms of x. As you can see, choosing the element and integrating can be challenging. It may require a trial and error approach plus experience.

Example

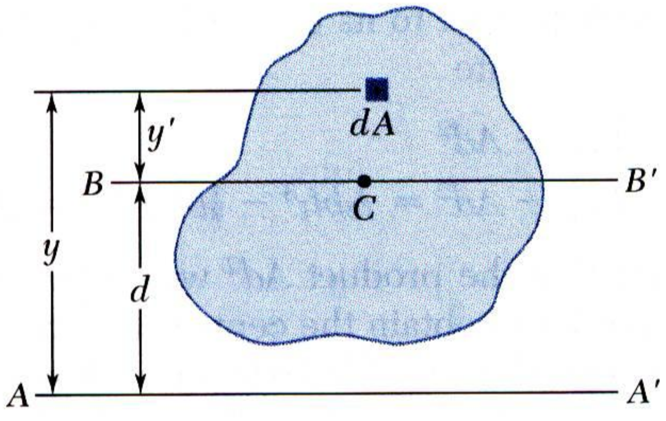
Compute the moment of inertia of the rectangular cross sectional area

A- about the x´ and y´ centroidal axis

B- about the x and y axis



Parallel Axis Theorem



- Consider moment of inertia *I* of an area *A* with respect to the axis *AA'*

$$I = \int y^2 dA$$

The axis *BB'* passes through the area centroid and is called a *centroidal axis*

$$I = \int y^2 dA = \int (y' + d)^2 dA$$
$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

$$I = \bar{I} + Ad^2$$ *parallel axis theorem*

Transfer of Axes: Parallel Axis Theorem

- The MOI equation assumes the axis intersects a specific location on the geometry, either the base of the geometry or the centroid.
- To compute the MOI about an arbitrary (x, y) axis, we must account for the additional distance

$$I_x = \bar{I}_x + A{d_x}^2$$

$$I_y = \bar{I}_y + A{d_y}^2$$

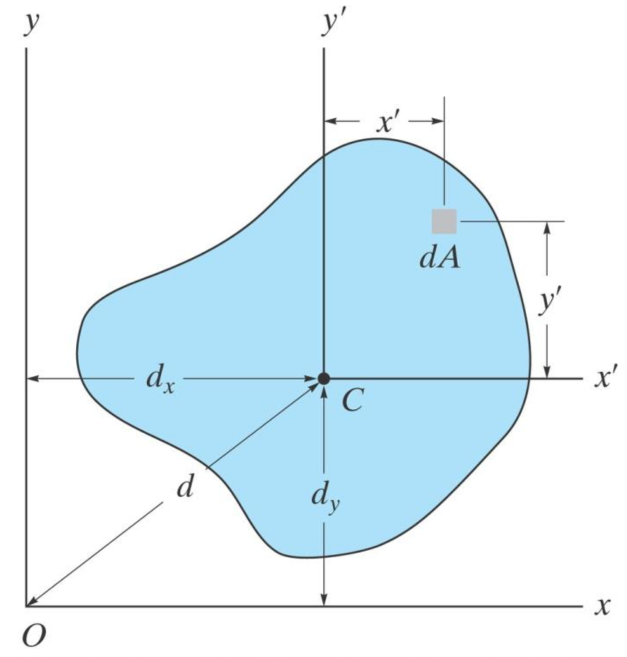
$$I_z = \bar{I}_z + Ad^2$$

- Note: this transfer of axes requires that the two axes be parallel (i.e. x is parallel to x_o)

Transfer of Axes: Parallel Axis Theorem

$$I_x = \bar{I}_x + Ad_y^{\,2}$$

$$I_y = \bar{I}_y + Ad_x^{\,2}$$



In Class Activity

What are the SI units for the Moment of Inertia for an area?

A) m^3

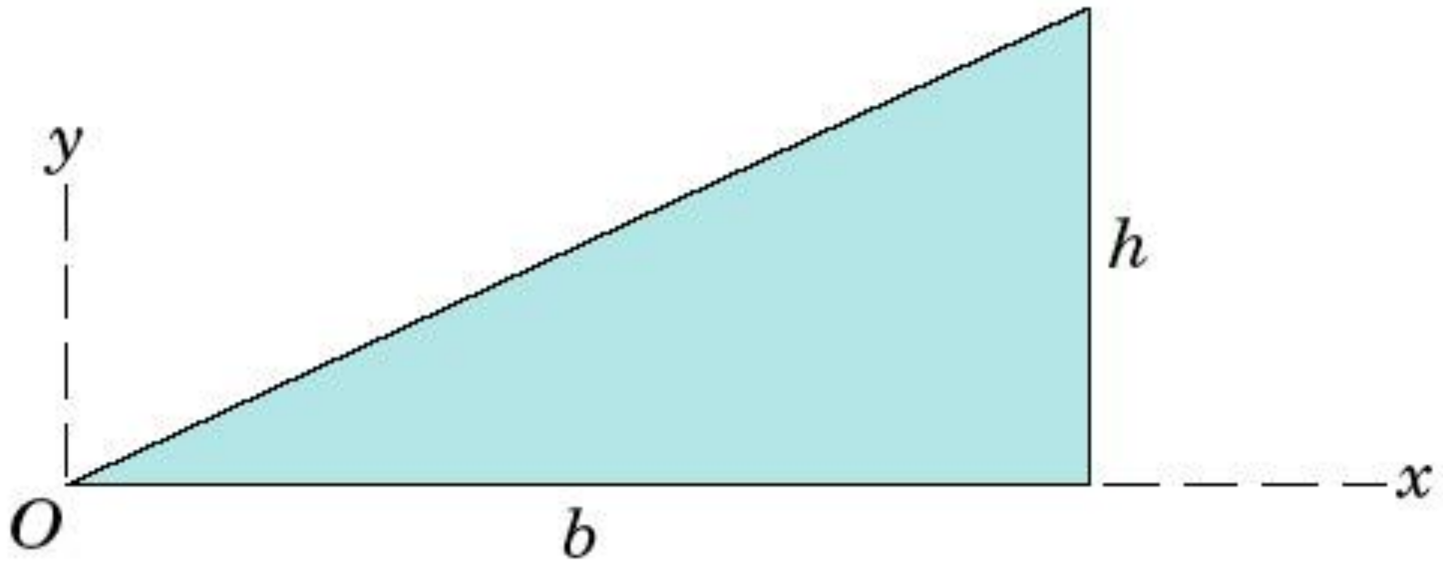
B) m^4

C) $kg \cdot m^2$

D) $kg \cdot m^3$

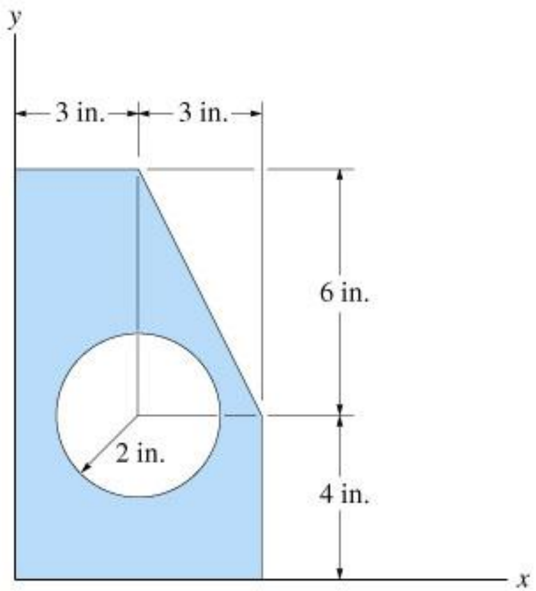
Example 1

Problem A/5: Determine by direct integration the moments of inertia about the *y*-axis.



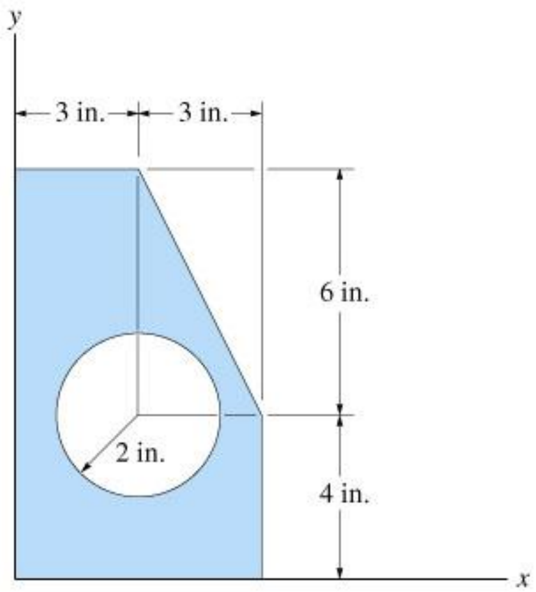
Problem A/5

Moment of Inertia for Composite Sections

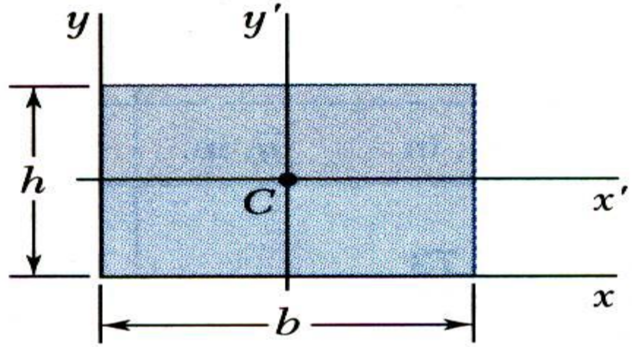
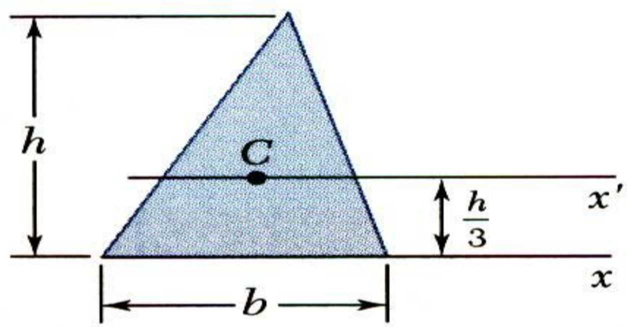
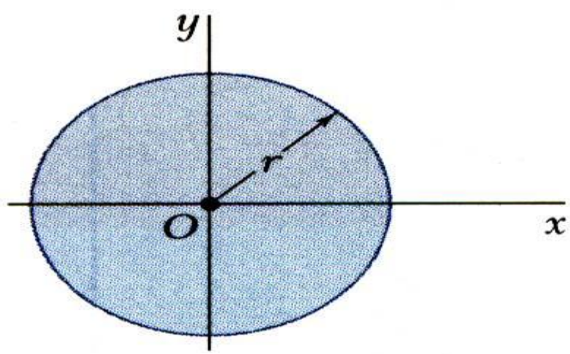


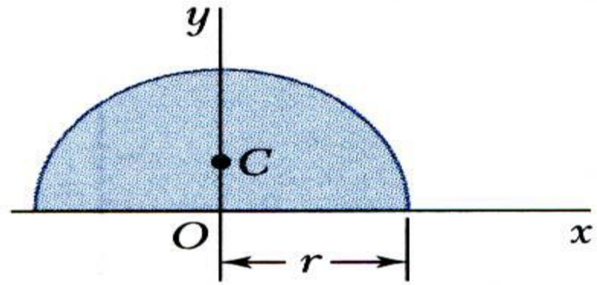
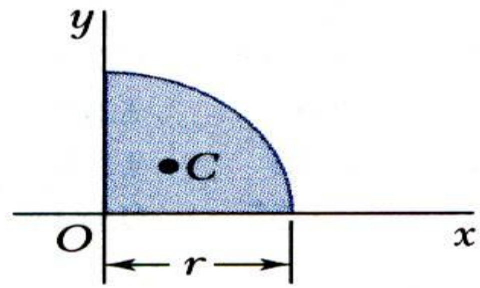
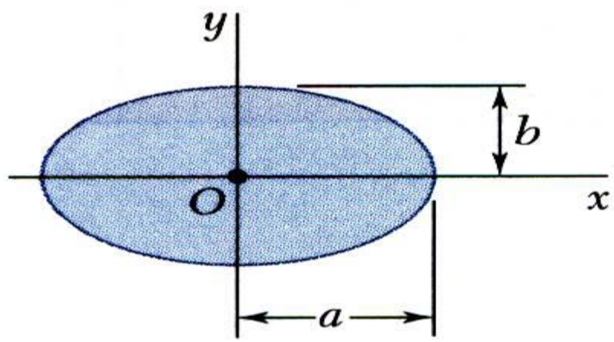
- The MoI of a combination of "simple" shaped areas like rectangles, triangles, and circles can be computed by taking advantage of what we know about the individual pieces
- For example, the MoI on the left can be computed from info about a rectangle minus a triangle and circle.
  - *Very similar to centroid computations!*

Moment of Inertia for Composite Sections

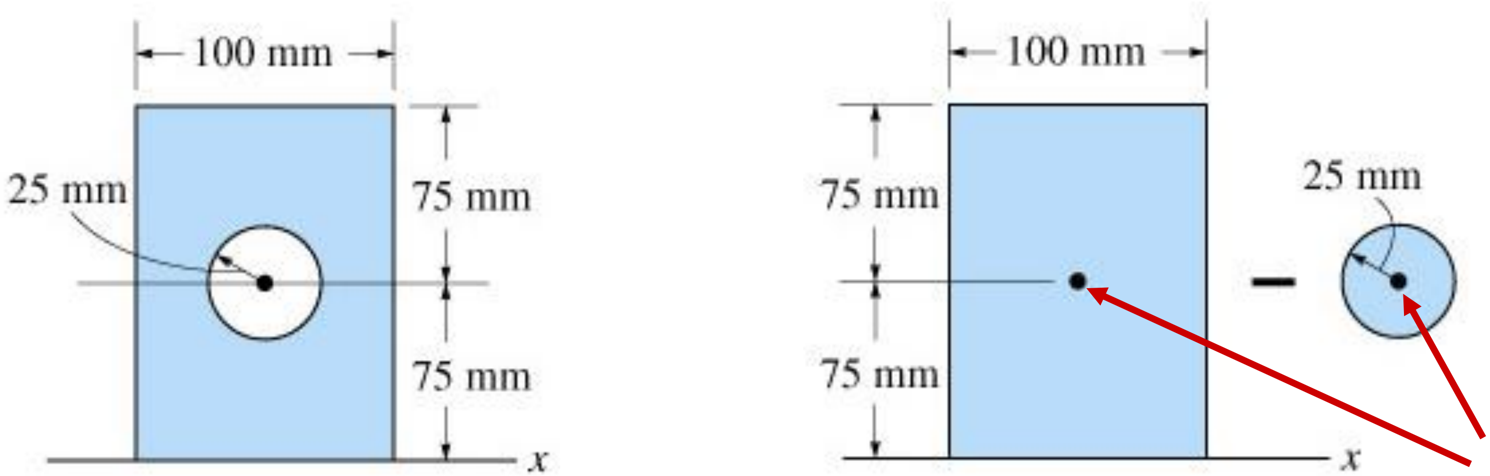


- The MoI of these "simpler" shaped areas about their centroidal axes are found in most engineering handbooks as well as ***Appendix D3***.
  - But note that these MoI's are written in terms of specific axes (*most of available tables are about the centroid axes*).
  - MoI is axis-dependent
- Using this data and the parallel-axis theorem, the MoI for a composite area can easily be calculated.
  - The challenge is correctly computing the distance from the centroid to the desired axes and tabulating your results

Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$
Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$

Semicircle		$I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$
Quarter circle		$I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$
Ellipse		$\overline{I}_x = \frac{1}{4}\pi a b^3$ $\overline{I}_y = \frac{1}{4}\pi a^3 b$ $J_O = \frac{1}{4}\pi a b(a^2 + b^2)$

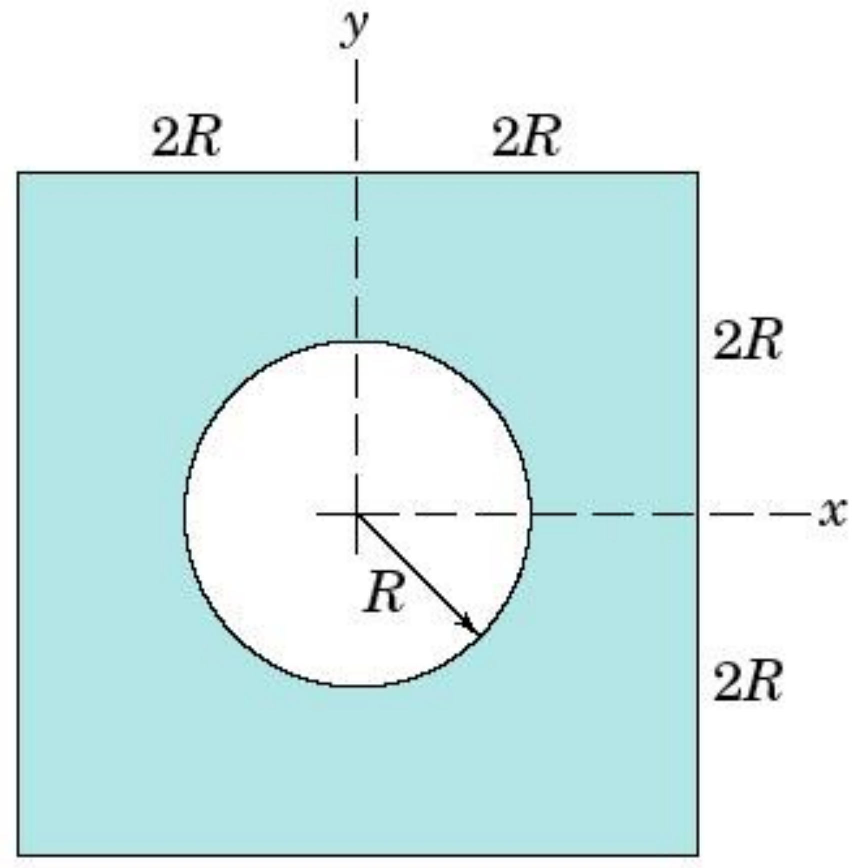
Steps for Analysis



1. Divide the given area into its simpler shaped parts.
2. Locate the centroid of each part and indicate the perpendicular distance from each centroid to the desired reference axis.
3. Determine the MoI of each "simpler" shaped part about the desired reference axis using known MoI's and the parallel-axis theorem ($I_X = I_{X'} + A (d_x)^2$)
4. The MoI of the entire area about the reference axis is determined by performing an algebraic summation of the individual MoIs obtained in Step 3. (Note that MoI of a **hole** is **subtracted**)

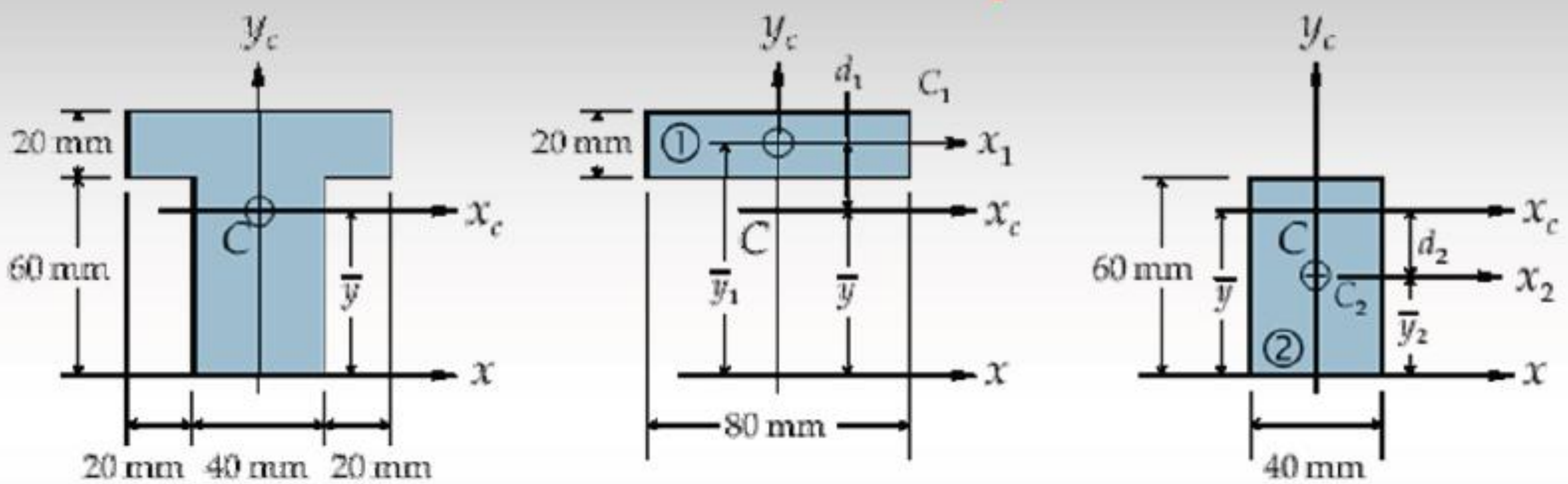
Example

Problem A/36: Determine the moment of inertia about the *x*-axis of the square area without and with the central circular hole.



Problem A/36

Example :Compute the moment of inertia about centroidal X-axis



	A_i	$\bar{y}_i$	$Q_i = A_i\bar{y}_i$	$I_{xi}^{(i)}$	$d_i = \bar{y}_i - \bar{y}$	$A_i d_i^2$	$I_x^{(i)} = I_{xi}^{(i)} + A_i d_i^2$
①	1,600	70	112,000	$I_{x1}^{(1)} = \frac{(80)(20)^3}{12} = 53{,}333$	24	921,600	974,933
②	2,400	30	72,000	$I_{x2}^{(2)} = \frac{(40)(60)^3}{12} = 720{,}000$	-16	614,400	1,334,400
$\sum$	4,000		184000				2,309,333

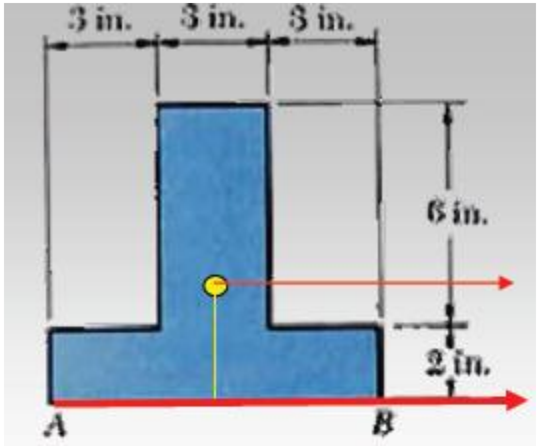
$$\bar{y} = \frac{\sum Q_i}{\sum A_i} = \frac{184{,}000}{4{,}000} = 46 \text{ mm}$$

Example

Find the moment of inertia about centroidal x-axis for cross sectional area shown

Bodies	A_i	y_i	y_i*A_i	I_i	$d_i=y_i$-ybar	$d_i^2A_i$
1	18	1	18			
2	18	5	90			
	36		108			

$$\bar{y} = \frac{\sum y_i A_i}{\sum A_i} = \frac{108 \text{ in}^3}{36 \text{ in}^2} = 3.0 \text{ in.}$$



Bodies	A_i	y_i	y_i*A_i	I_i	$d_i=y_i$-ybar	$d_i^2A_i$
1	18	1	18	6	-2	72
2	18	5	90	54	2	72
	36		108	60		144

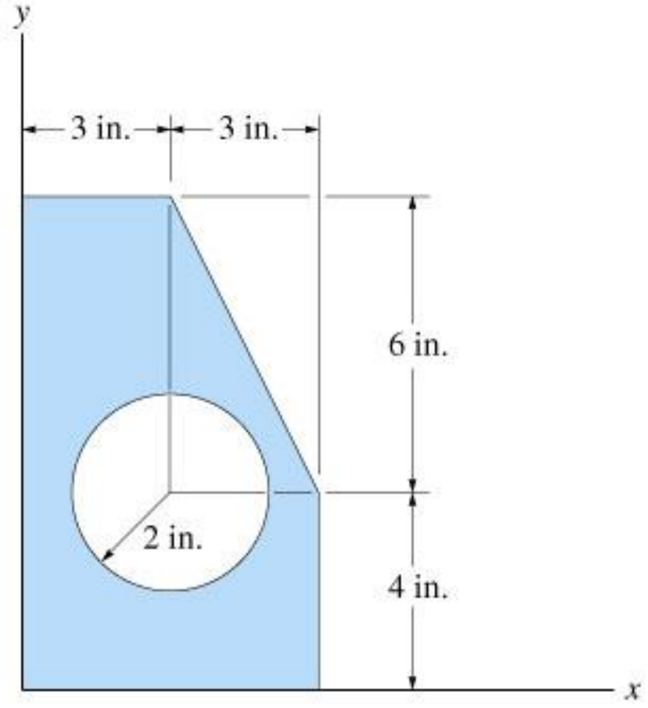
$$I_x = \sum \overline{I_{xi}} + \sum \left(y_i - \bar{y}\right)^2 A_i$$

$$= 60 \text{ in}^4 + 144 \text{ in}^4 = 204 \text{ in}^4$$

In Class Activity

Given: The shaded area as shown in the figure.

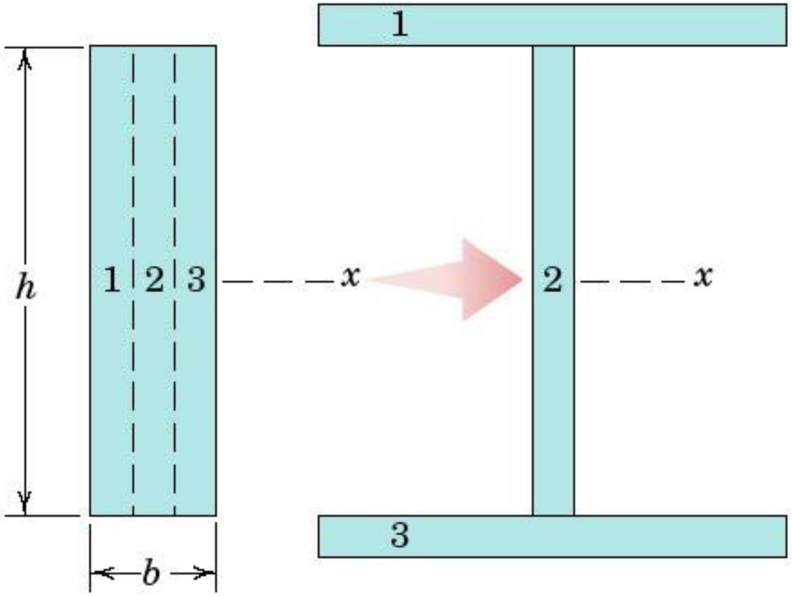
Find: The moment of inertia for the area about the x-axis.



Example 2

<u>Problem A/46:</u>

The rectangular area shown in part *a* of the figure is split into three equal areas which are then arranged as shown in part *b* of the figure. Determine an expression for the moment of inertia of the area in part *b* about the centroidal *x*-axis. What percent increase *n* over the moment of inertia for area *a* does this represent if *h* = 200 mm and *b* = 60mm?



(*a*) (*b*)

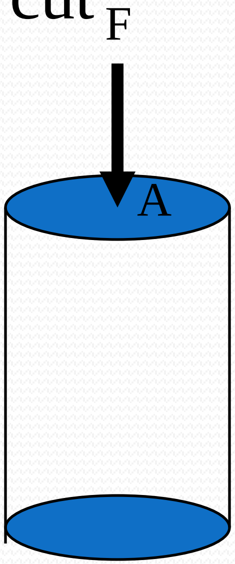
Problem A/46

MECHANICAL PROPERTIES OF MATERIALS

Normal stress

- Stress: is the intensity of the internal force over the cut section

  or the force per unit area

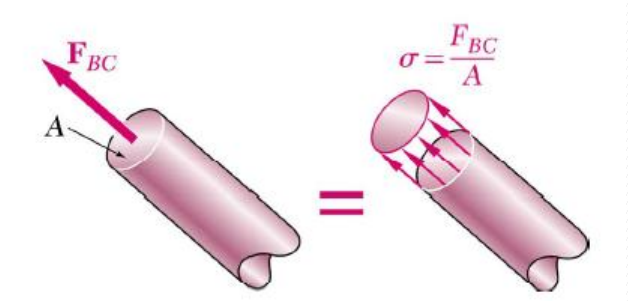


- Stress = F / A

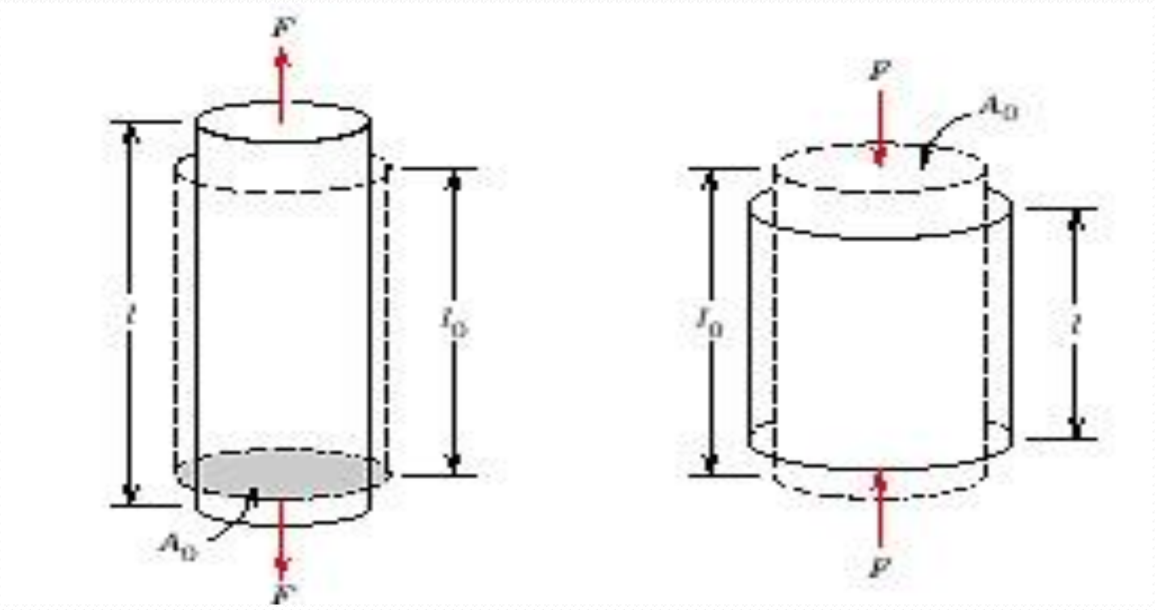
  unit: usually N / m^2 = Pascal (Pa)

  or N/mm^2=Mpa

- Normal stress(Direct stress) : $\sigma = F / A$

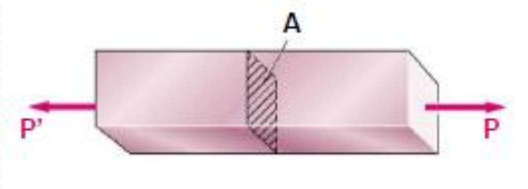


- Direct stress may be tensile, σ_t or compressive, σ_c and result from forces acting perpendicular to the plane of the cross-section

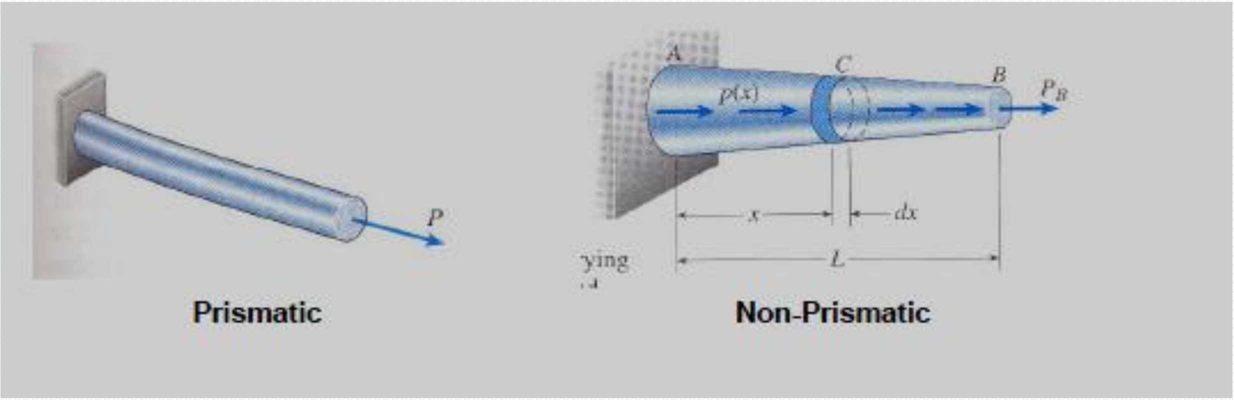


Homogenous: material is the same throughout the bar

Cross-section: section perpendicular to longitudinal axis of bar



Prismatic: cross-section does not change along axis of bar



Normal strain

- When loads are applied to a body, some deformation will occur resulting to a change in dimension.
- Consider a bar, subjected to axial tensile loading force, F. If the bar extension is Δl and its original length (before loading) is L_o, then tensile strain is:

Direct or Normal Strain (Contd.)

- Direct Strain (ε) = $\dfrac{\text{Change in Length}}{\text{Original Length}}$

i.e. $\varepsilon = \Delta L / Lo$

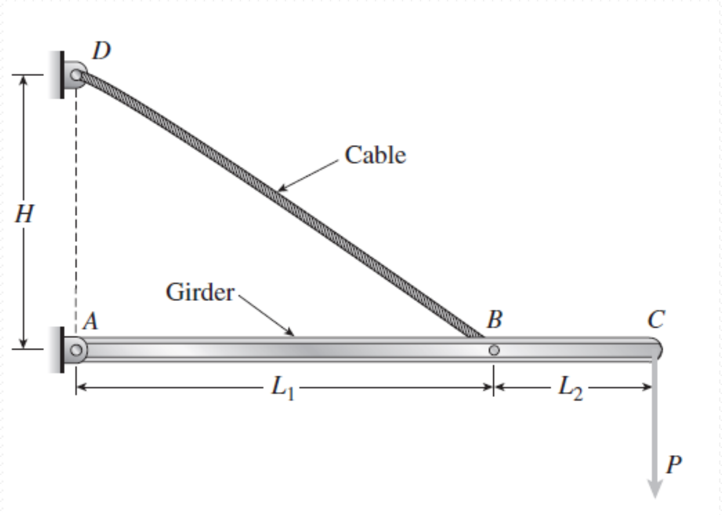
Strain(cont.)

- As strain is a ratio of lengths, it is dimensionless.
- Similarly, for compression by amount, Δl: Compressive strain = $-\Delta l/L$
- **Note:** Strain is positive for an increase in dimension and negative for a reduction in dimension.

Example:

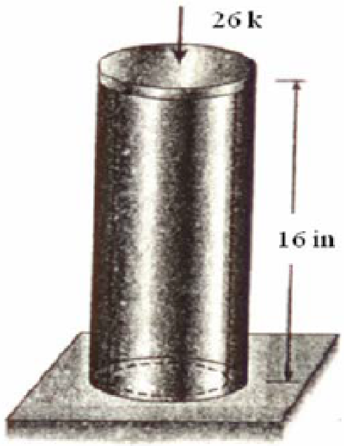
A loading crane consisting of a steel girder ABC supported by a cable BD is subjected to a load P (see figure). The cable has an effective cross-sectional area A = 0.471 in^2 . The dimensions of the crane are H =9 ft, L1= 12 ft, and L2= 4 ft.

If the load P =9000 lb, what is the average tensile stress in the cable?



Example: for a hollow circular tube of aluminum supports a compressive load of 240 kN, with *d1 = 90 mm and d2 = 130 mm, its length is 1 m, the* shortening of the tube is 0.55 mm, determine the stress and strain

$$A = \frac{\pi}{4}(d_2^2 - d_1^2) = \frac{\pi}{4}(130^2 - 90^2) = 6{,}912 \text{ mm}^2$$



$$\sigma = \frac{P}{A} = \frac{240{,}000 \text{ N}}{6{,}912 \text{ mm}^2} = 34.7 \text{ MPa (comp.)}$$

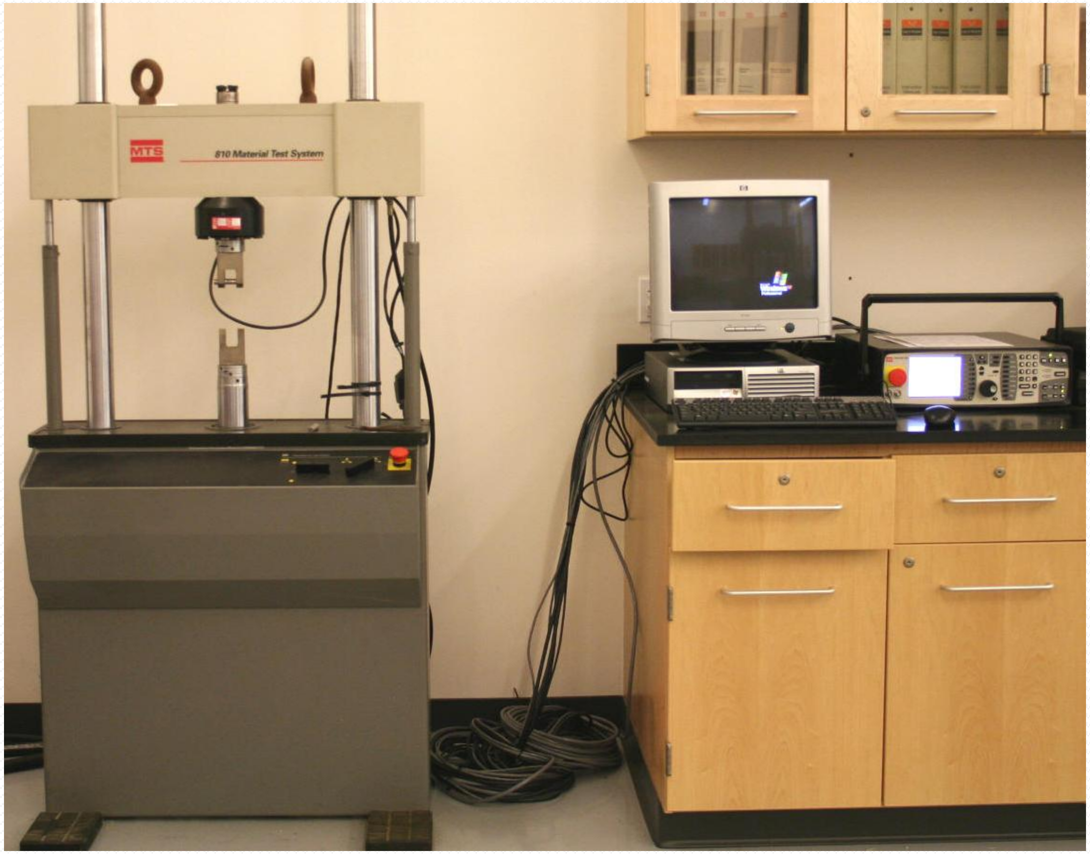
the compressive strain is

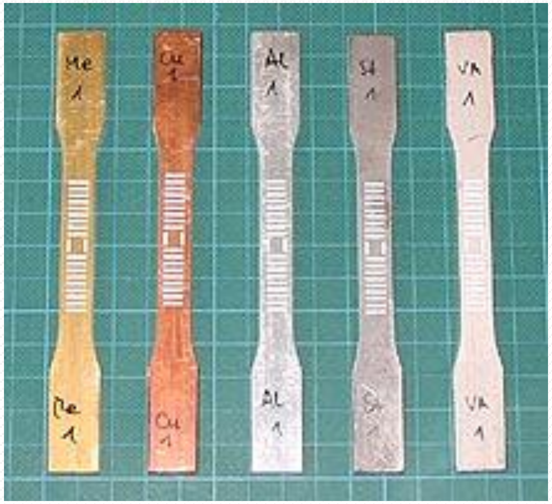
$$\varepsilon = \frac{\delta}{L} = \frac{0.55 \text{ mm}}{1{,}000 \text{ mm}} = 550 \times 10^{-6} = 550\,\mu \text{ m/m}$$

Stress-Strain Test

- In order to understand the mechanical behaviour of materials we need to perform experimental testing in the lab
- A tensile test machine is a typical equipment of a mechanical testing lab

Tensile Test



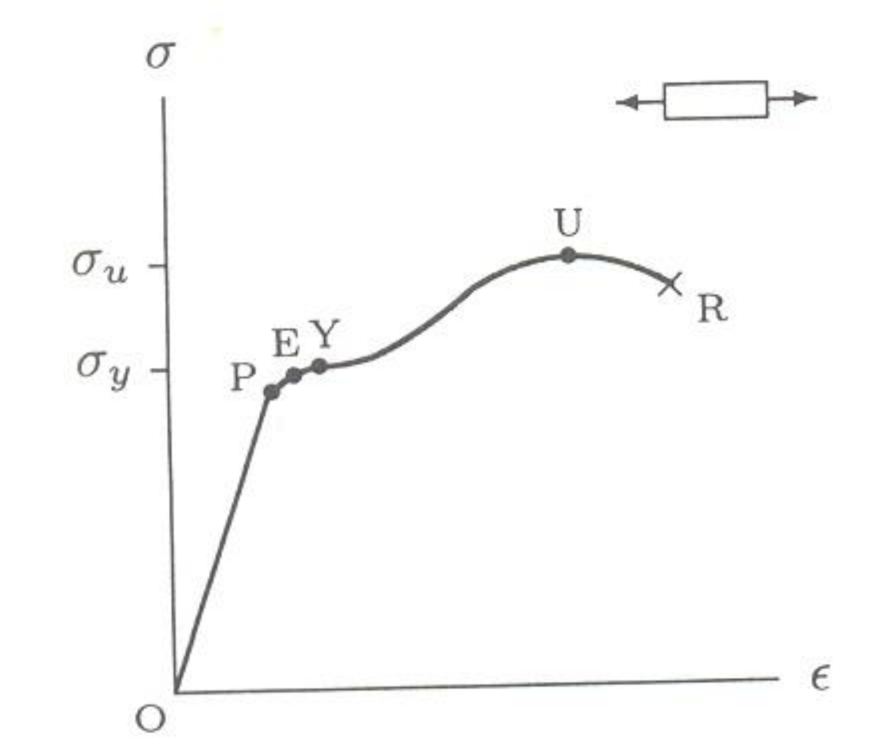


Uniaxial (simple) Tension Test

- The specimen is prepared then fixed in the equipment.
- A tensile force of known magnitude is applied.
- The deformation (elongation) in the specimen is measured.
- By knowing the original length and the cross sectional area, the strain and the tensile stress are calculated.
- Repeating the test for different specimens with different dimensions to calculate the Young’s modulus

Stress-Strain Diagram

- As a result of the uniaxial tensile test, the stress strain diagram can be established.
- Stress-strain diagram of each material can explain different mechanical properties (hardness, stiffness, ductility, brittleness, ….).



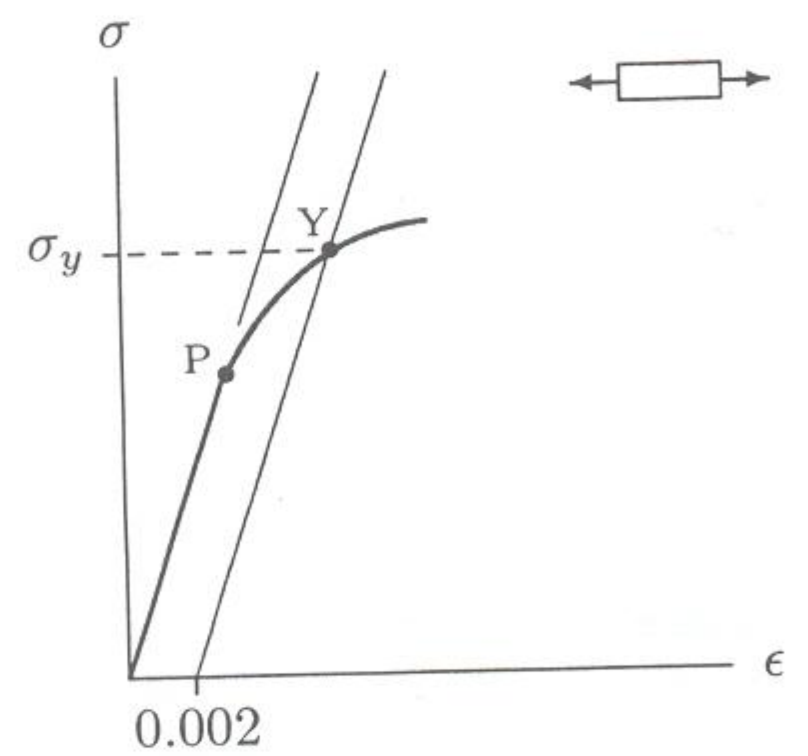
Stress-Strain Diagram

- Point **O** is the origin: corresponds no load, no deformation.
- Point **P** corresponds the proportionality limit: between **O** and **P** the stress and strain are linearly proportional.
- Point **E** corresponds the elastic limit: the stress corresponding this limit is the greatest stress that can be applied without causing permanent deformation.
- Point **Y** is the yield point: the stress at this point is called the **yield strength** of the material. At this level considerable elongation (yielding) can occur without a corresponding increase of load.
- Point **U** is the highest stress point on the s - e curve. The stress at this point is called the **ultimate strength** of the material.
- Point **R** is the rapture or failure point. The stress that correspond this point is called the **rapture strength** of the material.

Determination of the yield strength

"Offset Method"

- ***Offset method*** is used to determine the apparent yield strength of the material.
- Drawing a parallel line to the linear section (the σ – ε curve.
- This line is crossing the strain axis at 0.2% (0.002).
- The intersection of this line with σ – ε curve taken as ***the apparent yield strength***.



Stress-Strain Diagram

- **Elastic deformation**:
  - Elasticity is the ability of the material to resume its original shape and dimensions.
  - If the applied stress is equal or less the **yield strength** then the deformation is called " **Elastic Deformation**"
  - If the point **Y** corresponds the point **P,** and the σ– ε curve is straight line, then the material is called a ***linearly elastic*** material.
  - If the point **Y** does not correspond the point **P,** and the σ – ε curve is straight line followed by a non-uniform or curved line, then the material is called a ***non-linear elastic*** material
    - The slope of the straight line is the Young's modulus (E).

Stress-Strain Diagram (cont)

- **Elastic Region (Point 1 –2)**

  - The material will return to its original shape after the material is unloaded(like a rubber band).

  - The stress is linearly proportional to the strain in this region.

$$\sigma = E\varepsilon$$

or

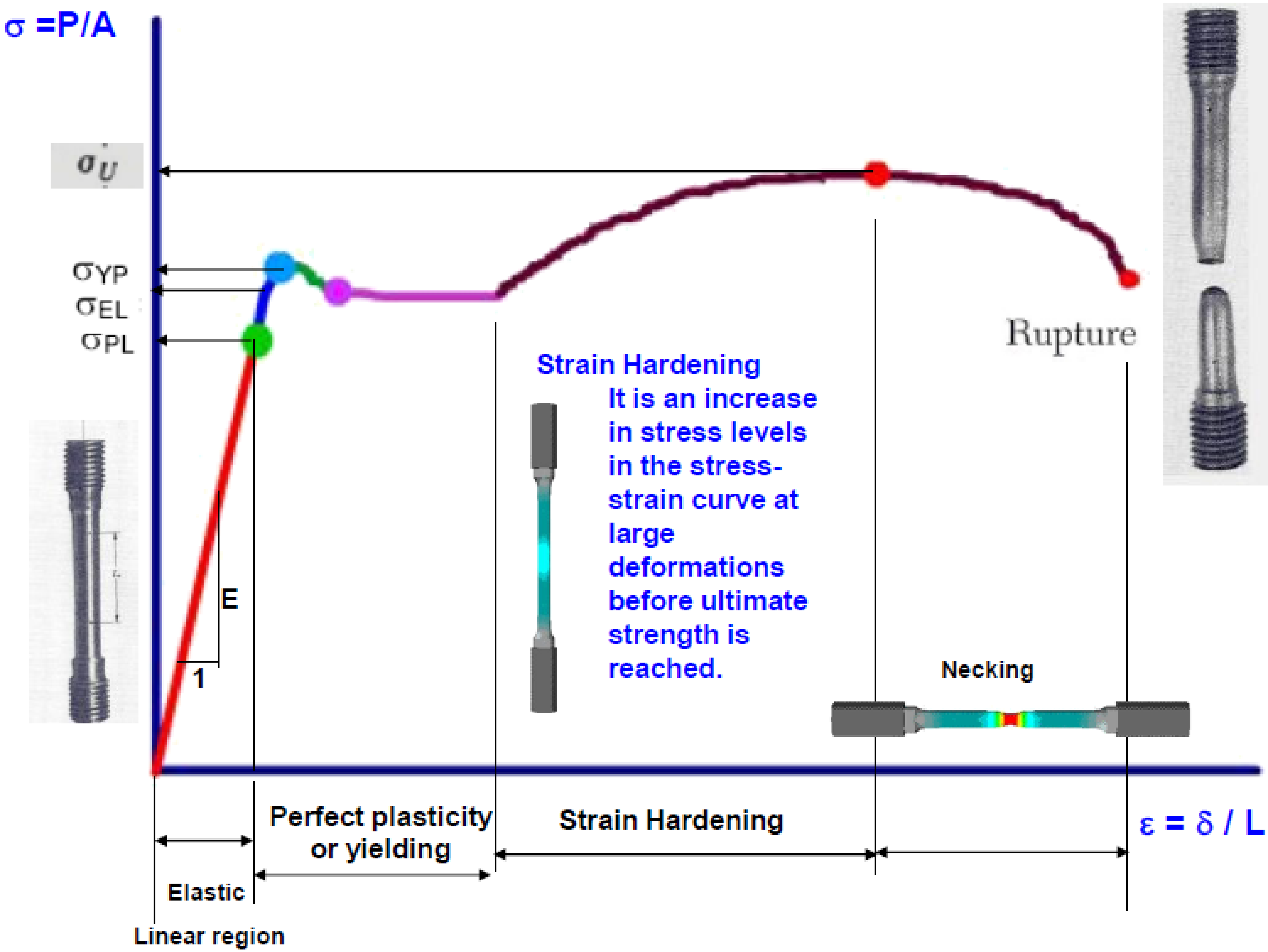
$$E = \frac{\sigma}{\varepsilon}$$

σ : Stress(psi)

E : *Elastic modulus* (*Young's Modulus*) (psi)

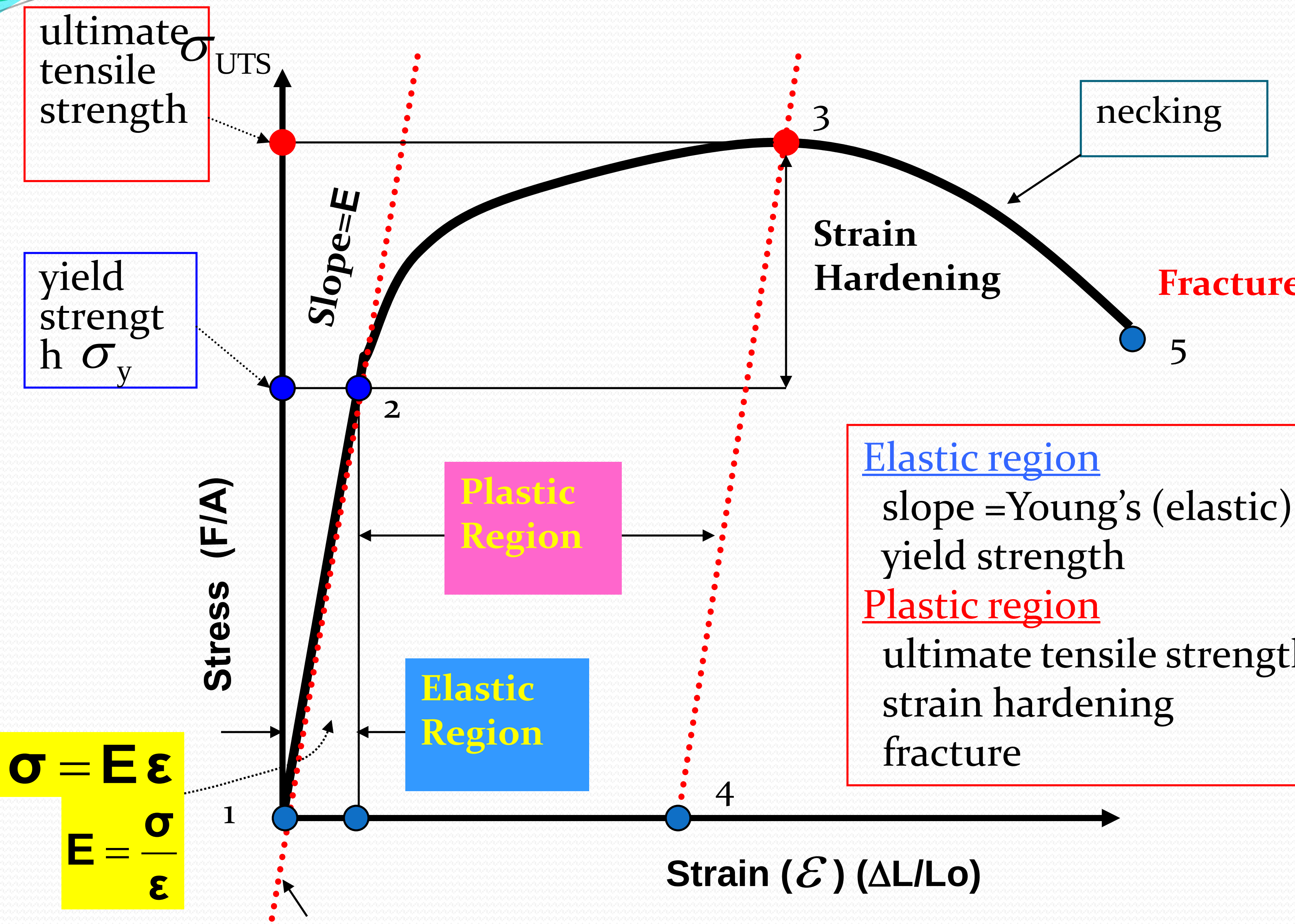
ε : Strain (in/in)

- Point 2 : Yield Strength : a point where permanent deformation occurs. (If it is passed, the material will no longer return to its original length.)



σ =P/A
σU
σYP
σEL
σPL
E
1
Rupture
Strain Hardening
It is an increase in stress levels in the stress-strain curve at large deformations before ultimate strength is reached.
Necking
Perfect plasticity or yielding
Strain Hardening
Elastic
Linear region
ε = δ / L

Stress-Strain Diagram



Elastic region
- slope =Young's (elastic) modulus
- yield strength

Plastic region
- ultimate tensile strength
- strain hardening
- fracture

Stress-Strain Diagram (cont)

- **Strain Hardening**

  - If the material is loaded again from Point 4, the curve will follow back to Point 3 with the same Elastic Modulus (slope).

  - The material now has a higher yield strength of Point 4.

  - Raising the yield strength by permanently straining the material is called <u>Strain Hardening.</u>

Stress-Strain Diagram (cont)

- **Tensile Strength (Point 3)**

  - The largest value of stress on the diagram is called <u>Tensile Strength</u>(TS) or <u>Ultimate Tensile Strength</u> *(UTS)*

  - It is the maximum stress which the material can support without breaking.

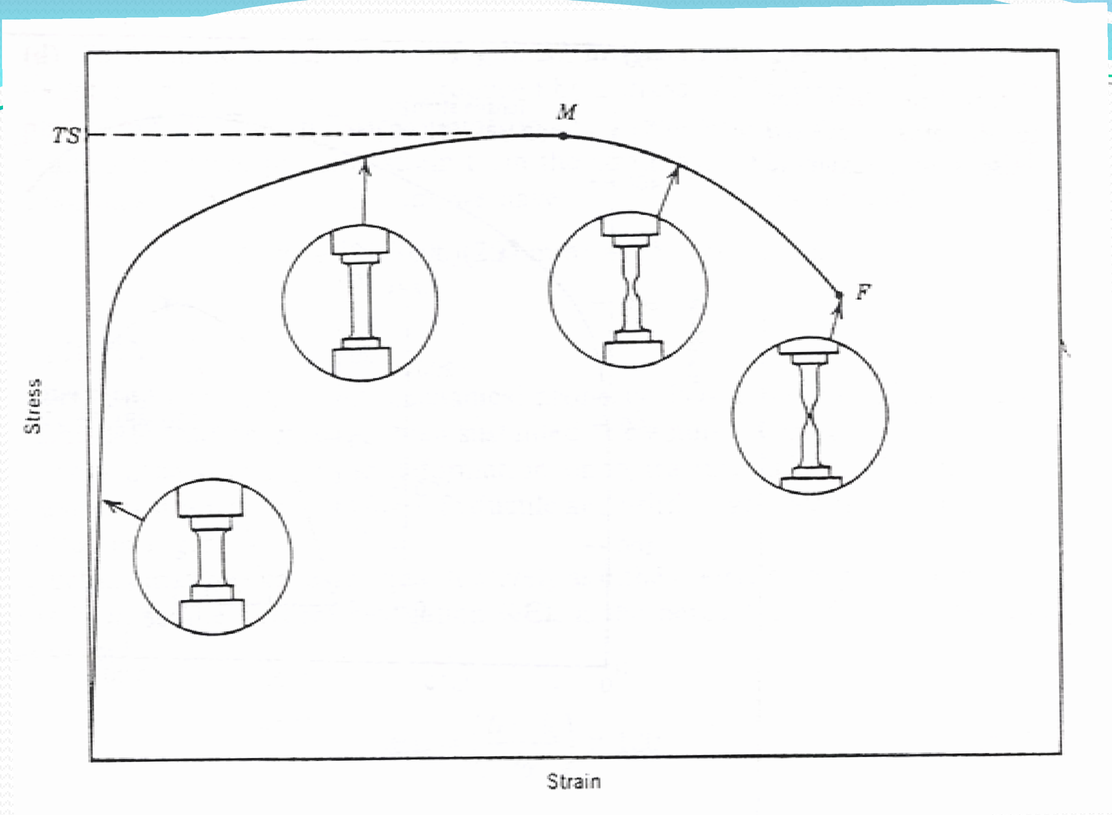
- **Fracture (Point 5)**

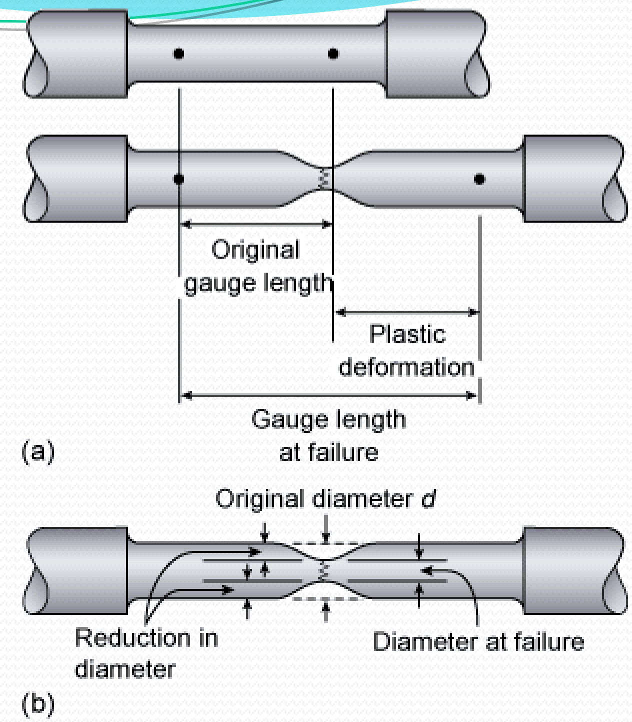
  - If the material is stretched beyond Point 3, the stress decreases as necking and non-uniform deformation occur.

  - Fracture will finally occur at Point 5.

Stress-Strain Diagram

- **Necking**:
  - Once the material is subjected to a stress equal or greater than the ultimate strength of the material, more and more deformation is happening, even by reducing the load.
  - After ultimate strength, the cross sectional area start to decrease. This phenomenon is known by **Necking**.
  - The material failed to hold any load and then rapture.



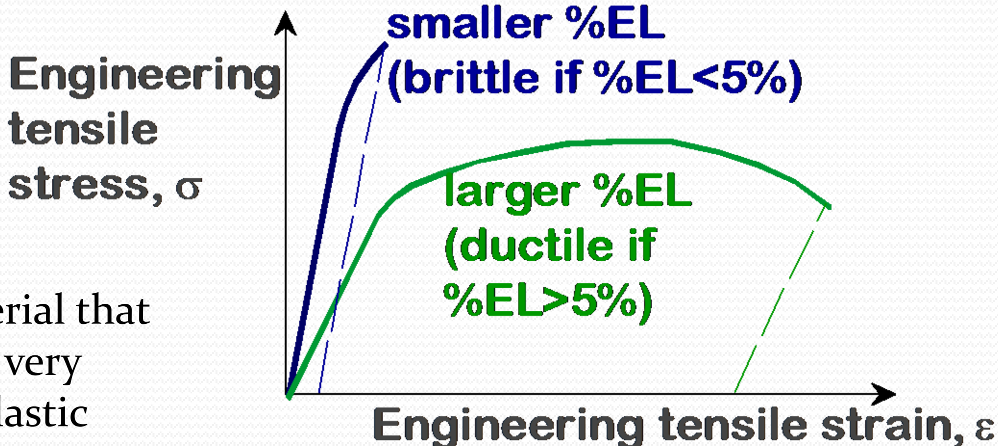


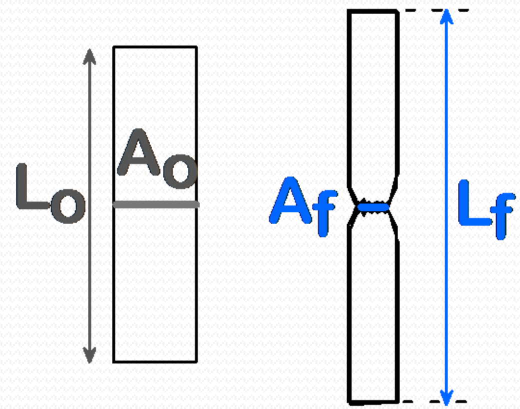
Typical engineering stress-strain behavior to fracture, point F. the tensile strength is indicated at point M.

Ductility

Ductility is a measure of the plastic deformation that has been sustained at fracture:

$$\%EL = \frac{l_f - l_o}{l_o} x100$$





A material that suffers very little plastic deformation is brittle.

- Another ductility measure:

$$\%AR = \frac{A_o - A_f}{A_o} x100$$

- Ductility may be expressed as either percent elongation (% plastic strain at fracture) or percent reduction in area.

Resilience

- *Is the capacity of a material to absorb energy when it is deformed elastically and then, upon loading, to have energy recovered. The associated property is the modulus of resilience, Ur, which is the strain energy per unit volume required to stress a material from an unloaded state up to the yield point of yielding.*

Resilience

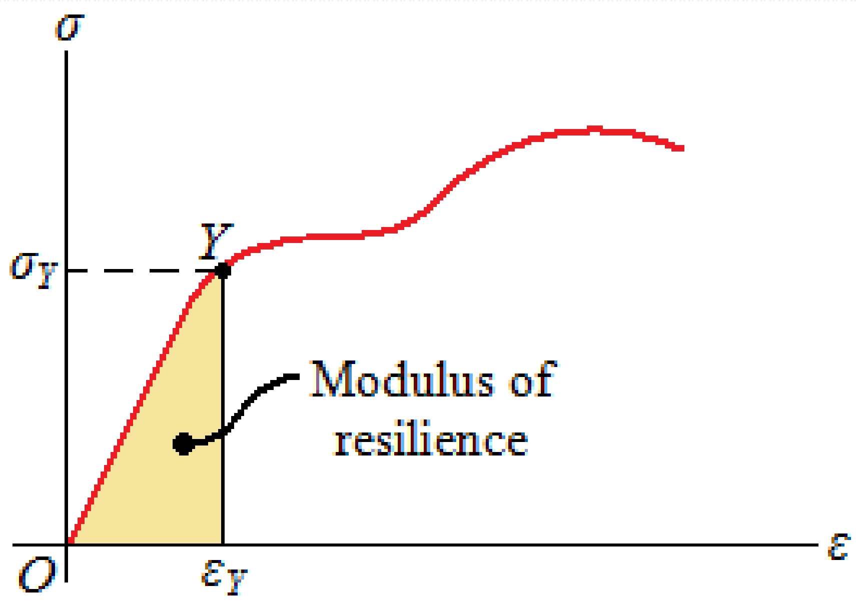
- *Computationally, the modulus of resilience for a specimen subjected to a uni-axial tension is just the area under the engineering stress-strain curve taken to yield,*

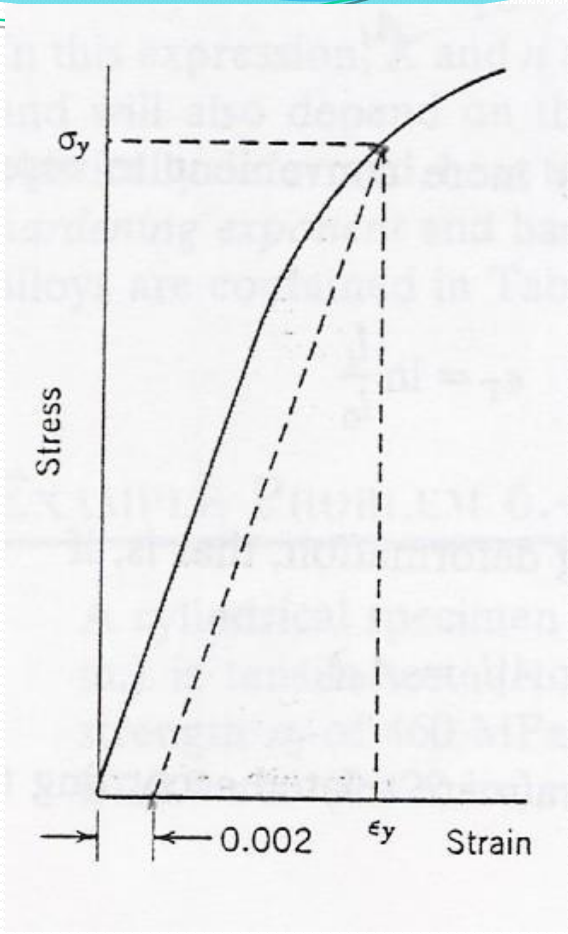
$$U_r = \int_0^{\varepsilon_y} \sigma d\varepsilon$$

Assuming linear elastic region,

$$U_r = \frac{1}{2}\sigma_y \varepsilon_y$$

In which ε_y is the strain at yielding.
The unit of resilience is joules per cubic meter (J/m3).

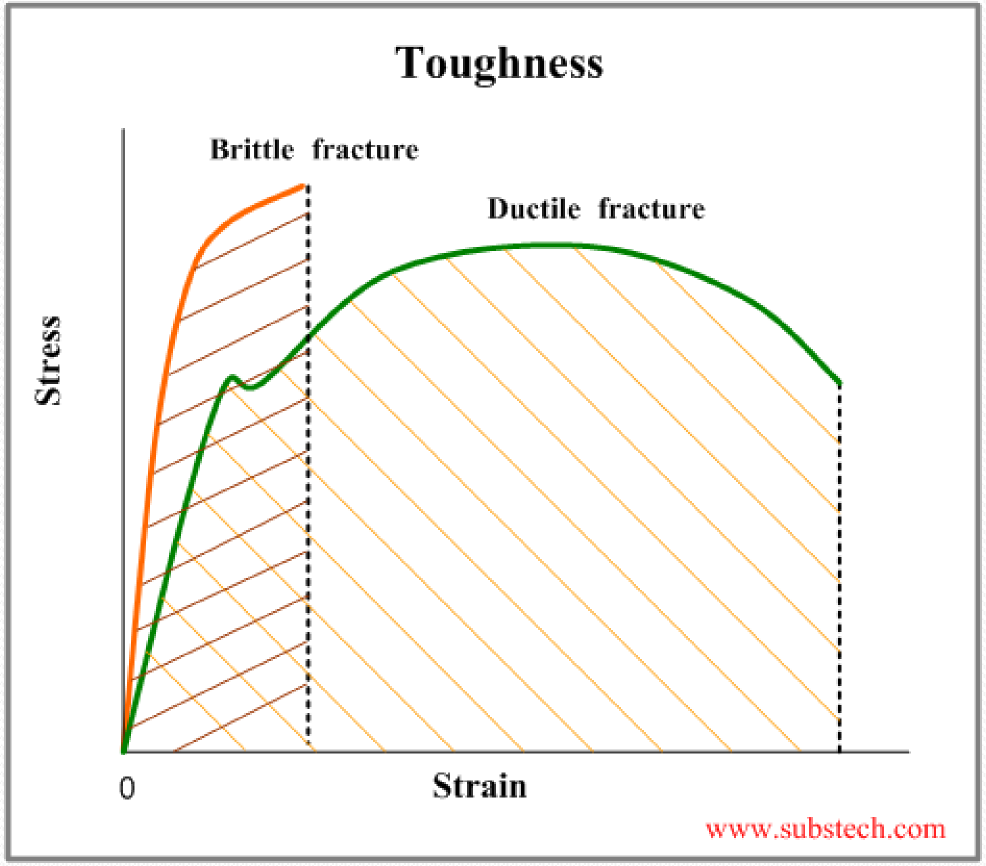




How modulus of resilience is determined from the stress-strain behavior of a material.

Toughness

- *It is a measure of the ability of a material to absorb energy up to fracture. It is the area under the stress-strain curve up to the point of failure. The unit of toughness is the same as for resilience (joules per cubic meter (J/m3)).*



True stress and true strain

- If the stress is calculated based on the original unreformed area, then the stress called conventional stress.
- The true or actual stress is the stress calculated based on the deformed cross-sectional area.

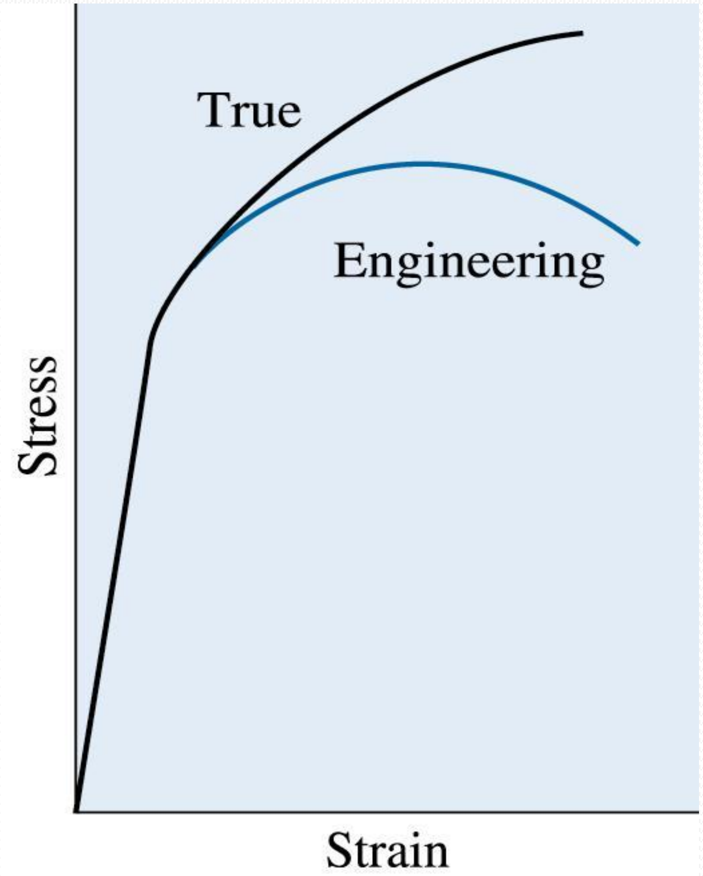
- True stress is defined as the load F divided by the instantaneous cross-sectional area Ai over which deformation is occurring

$$\sigma_T = \frac{F}{A_i}$$

- *And true strain is defined by*

$$\varepsilon_T = ln\frac{l_i}{l_o}$$

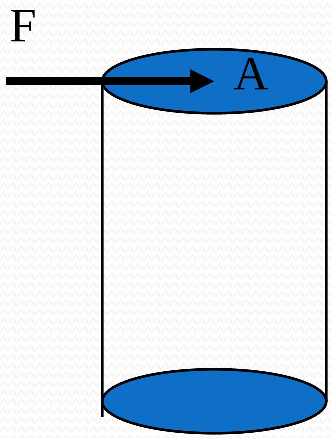
True stress and true strain(continue)



- The relation between the true stress-true strain diagram and engineering stress-engineering strain diagram.
- The curves are identical to the yield point.

Shear stress

- When tangential force is applied, the stress is called "SHEAR STRESS"
- $\tau = F/A$



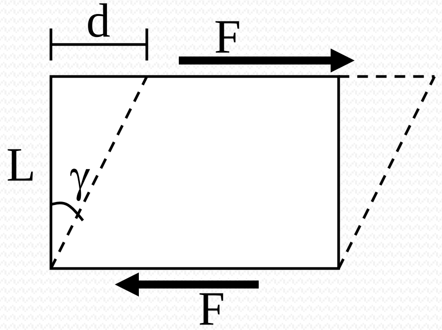
- The forces tend to make

one part of the material slide

over the other part.

The Concept of strain

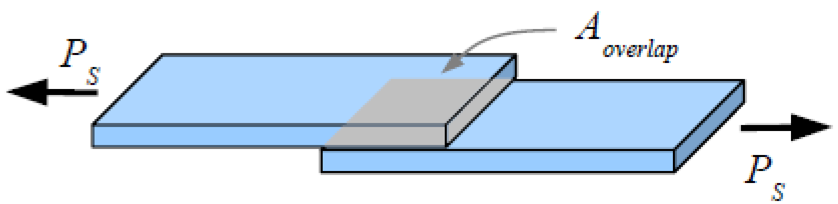
- Shear strain is the distortion produced by shear stress on an element or rectangular block
- *Average shear strain:* is defined as the ratio of ***d*** to ***L***



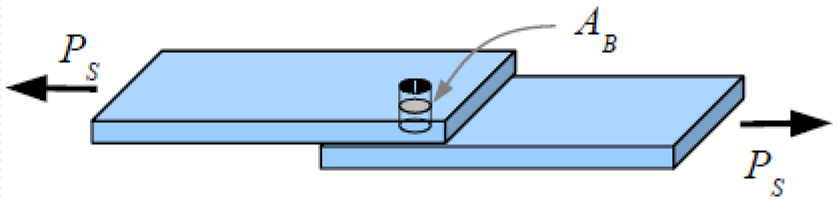
- Note that $\tan(\gamma) = d / L$
- γ is very small
- For small angle

  $\tan(\gamma) = \gamma$
- γ is defined as the average shear strain.

SHEARING STRESS

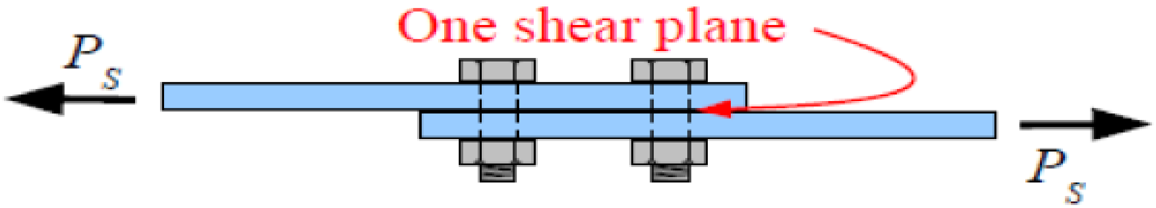


$\tau = P_s/A_{overlap}$

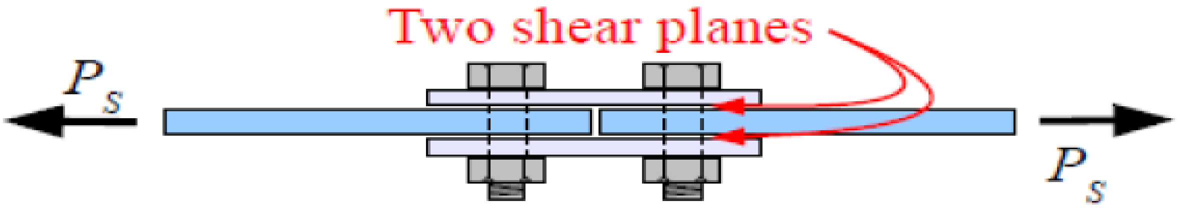


$\tau = P_s/A_B$

Where, $A_B = \pi d^2/4$, where d is the bolt diameter

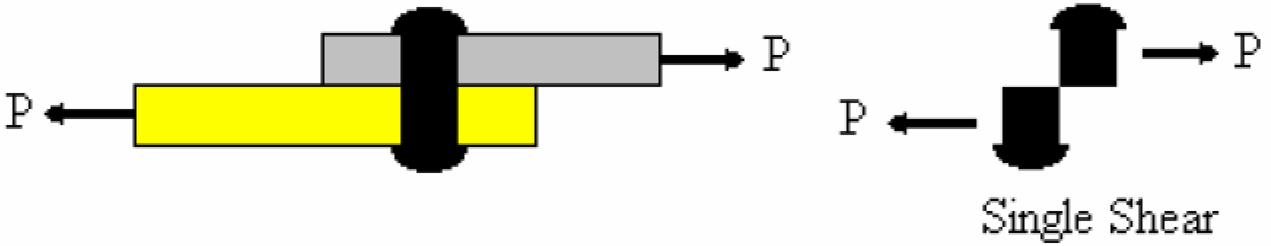


$\tau = P_s/NA_B$ **N: Number of bolt**



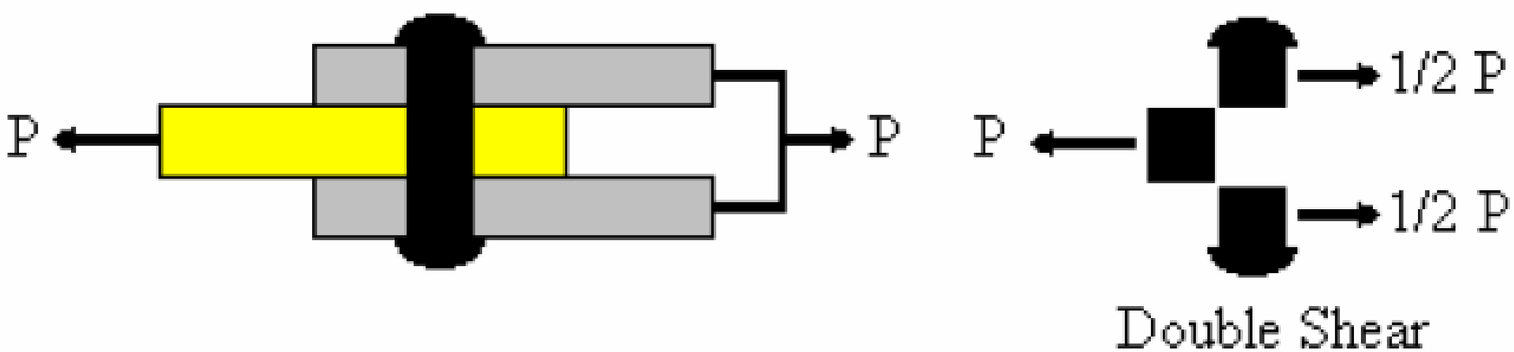
$\tau = P_s/nNA_B$ **n: Number of shear planes**

Single Shear



$$\tau = \frac{P}{A}$$

Double Shear



$$\tau = \frac{P}{2A}$$

Example: The anchor shackle supports a cable force of 600 lb. If the pin has a diameter of 0.25 in., determine the average shear stress in the pin.

$$d = 0.25\,in$$

$$r = 0.125\,in$$

$$A = \pi r^2$$

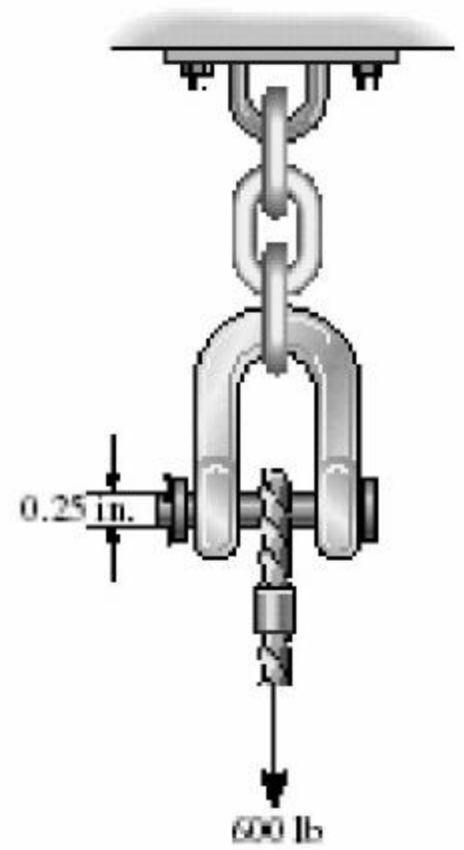
$$= \pi (0.125)^2$$

$$= 0.04909\,in^2$$

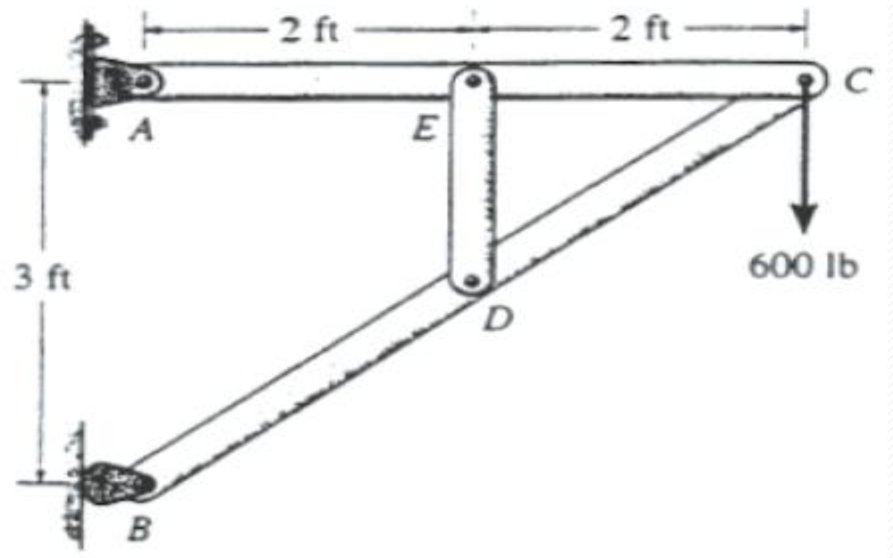
For double shear stress :

$$\tau = V/_{2A} = \frac{0.6\,lb}{2(0.04909\,in^2)}$$

$$= 6.11\ ksi$$



Example: The frame supports the loading shown. The pin at A has a diameter of 0.25 in. if it is subjected to double shear, determine the average shear stress in the pin



Entire frame

$\Sigma F_y = 0;\ A_y = 600\ \text{lb}$

$\Sigma M_B = 0;\ A_x = 800\ \text{lb}$

$$F_A = \sqrt{(600)^2 + (800)^2} = 1000\ \text{lb}$$

$$\tau_A = \frac{F_A/2}{A} = \frac{1000/2}{\frac{\pi}{4}(0.25)^2} = 10.2\ \text{ksi}$$

Stress-Strain Diagram

- Some materials exhibit linearly elastic behavior when they subjected to shear loading.
- For such materials, the shear stress is linearly proportional to shear strain, such that

$$\tau = G\,\gamma$$

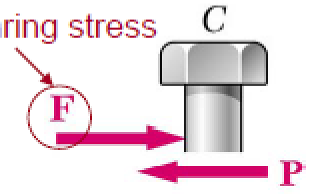
Where, $\boldsymbol{\tau}$ is the shear stress, $\boldsymbol{\gamma}$ is the shear strain, and **G** is the shear modulus or the modulus of rigidity.

- $\boldsymbol{\tau}$ is usually plotted on the y-axis, $\boldsymbol{\gamma}$ is plotted on the x-axis, and the slope of the straight line is G

Bearing stress

- Bearing stress is a normal stress that is produced by the compression of one surface against another. The bearing area is defined as the projected area of the curved bearing surface.
- Bearing stress is a normal stress, not a shearing stress.
- • Bearing stress is in the members that a bolt connects (not in the bolt itself), along a bearing surface.

Thus, $$\sigma_b = \frac{P}{A_b}$$



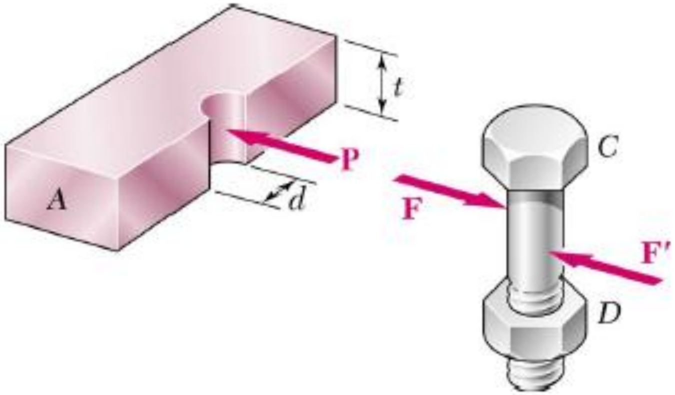
where

A_b = projected area where bearing pressure is applied

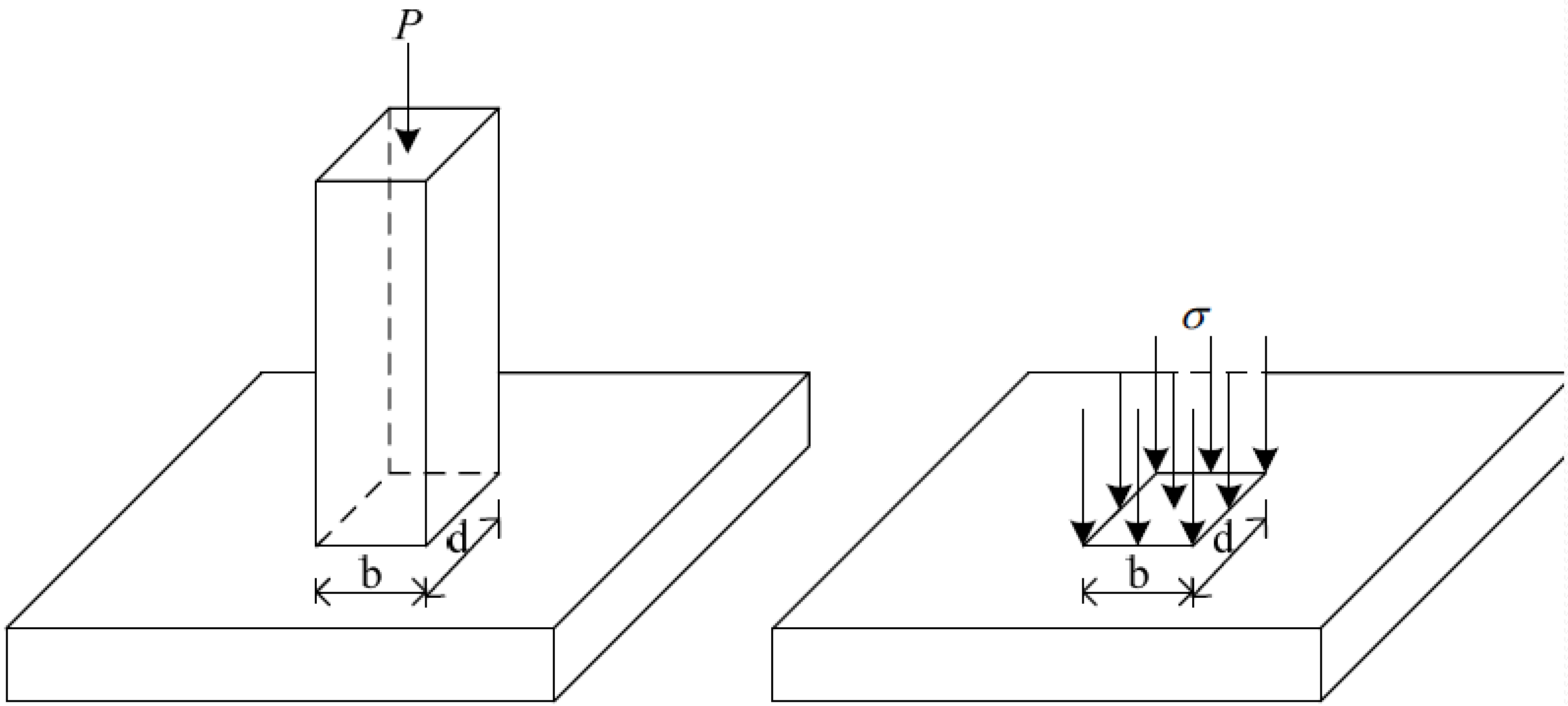
P = bearing force

- For "single shear" case

$$\sigma_b = \frac{P}{A_b} = \frac{P}{td}$$



$$\sigma = \frac{\text{Force}}{\text{Bearing Area}} = \frac{P}{b \times d}$$ Eqn



Allowable stress and factor of safety

- The structure must be designed to withstand a maximum possible level stress known as working stress
- Safety against unpredictable conditions can be achieved by considering a factor of safety.
- The factor of safety (n) is the ratio of the ultimate strength of the material to the allowable stress..
- The allowable stress is usually less than the maximum ultimate stress. So that the factor of safety is greater than one.
- Mathematically,

$$\sigma_{all} = \frac{\sigma_u}{n}$$

1- Determination of the ultimate strength of a material.

$$\sigma_U = \frac{P_U}{A}$$

2- Allowable stress; factor of safety

$$\text{Factor of safety} = \text{F.S} = \frac{\text{Ultimate stress}}{\text{Allowable stress}}$$

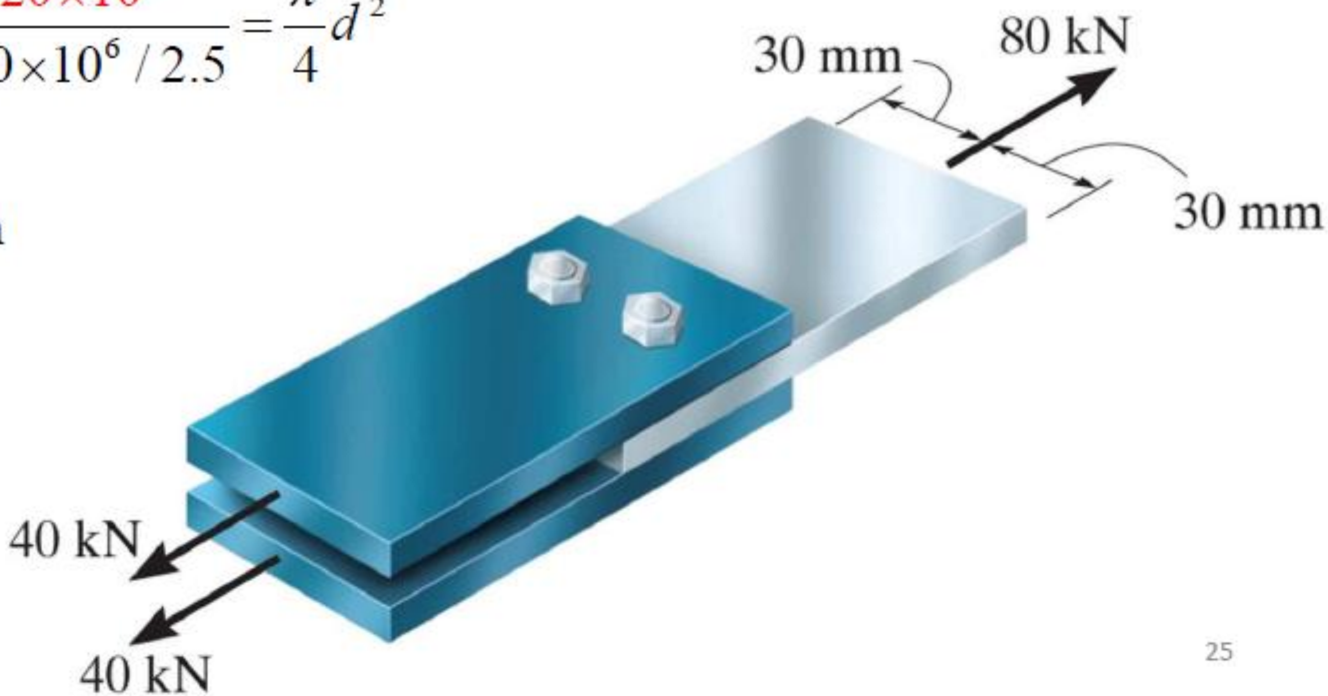
Example : Determine the required diameter of the bolts if the failure shear stress is $\tau_{Fail} = 350$ MPa. use a factor of safety F.S $= 2.5$.

Solution :

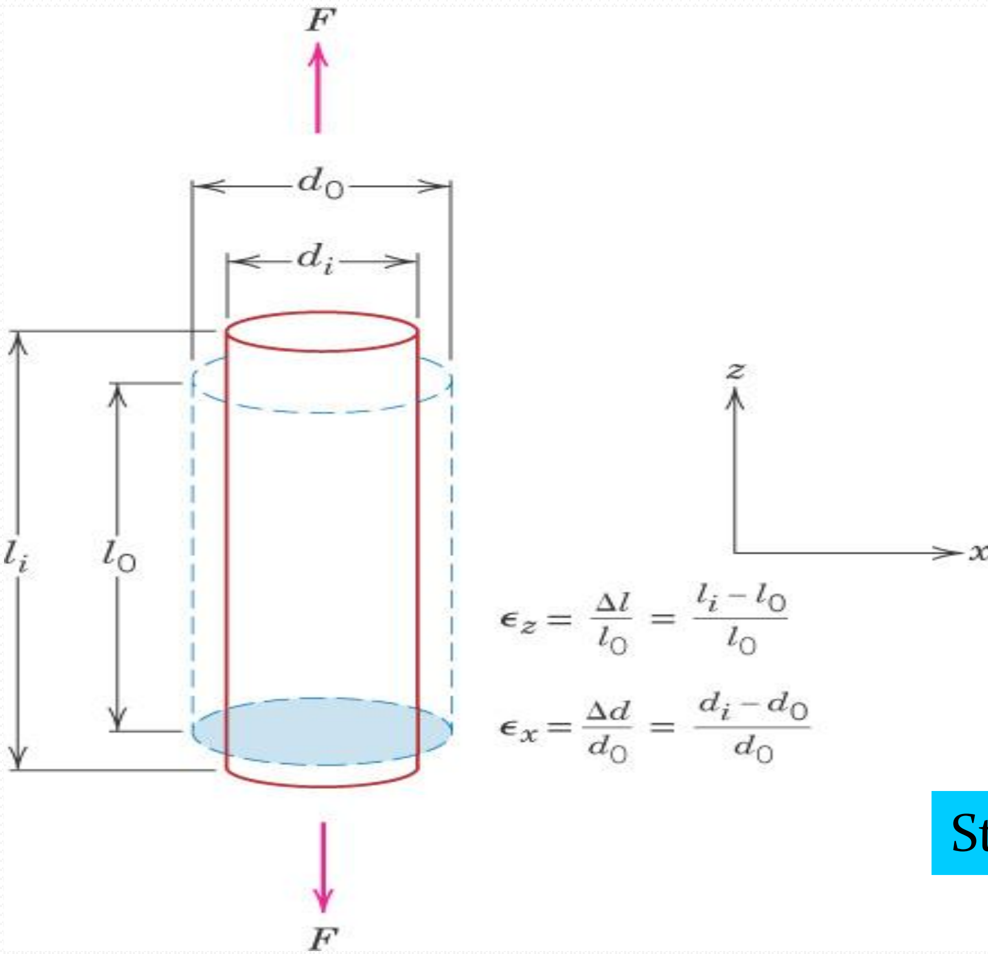
$$A = \frac{V}{\tau_{all}} = \frac{20\times10^3}{350\times10^6/2.5} = \frac{\pi}{4}d^2$$

then, we get

$d = 13.5$ mm



Engineering Strain



Strain is dimensionless.

Poisson's Ratio

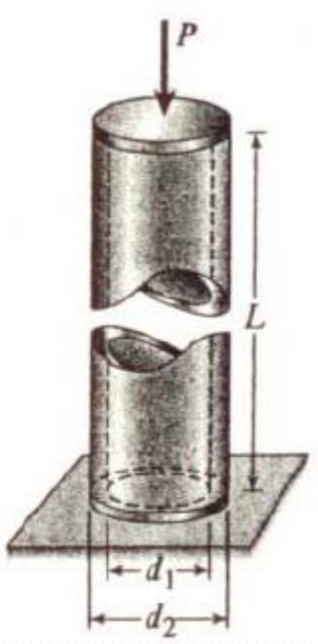
- When a load is applied on ay material, it deformed in two directions:
  - The direction of the load line of action which produces axial strain.
  - The normal direction to the load line of action which produces lateral strain.
- with the elastic range: the ratio of the lateral strain to the axial strain is constant and known as ***Poisson's ratio***.
- Mathematically, ***Poisson's ratio*** expressed as

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x}$$

Example:

- A steel pipe with $L = 1.2\ m$, $d_2 = 150\ mm$, $d_1 = 110\ mm$, $P = 620\ kN$, $E = 200\ GPa$, $\nu = 0.3$

determine (a) δ, (b) ε', (c) Δd_2 and Δd_1



Solution:

$A = \pi\,(d_2^{\,2} - d_1^{\,2}) / 4 = \pi\,(150^2 - 110^2) / 4 = 8{,}168\ \text{mm}^2$

$\sigma \;=\; -P / A \;=\; -\,620\ \text{kN} / 8{,}168\ \text{mm}^2 = -\,75.9\ \text{PMa}$ (comp)

$\varepsilon \;=\; \sigma / E \;=\; -\,75.9\ \text{MPa} / 200{,}000\ \text{MPa} \;=\; -\,379.5 \times 10^{-}$

(a) $\delta \;=\; \varepsilon L \;=\; (-\,379.5 \times 10^{-6})\,(1{,}200\ \text{mm}) \;=\; -\,0.455\ \text{mm}$

(b) $\varepsilon' \;=\; -\,\nu\,\varepsilon \;=\; -\,(0.3)\,(-379.5 \times 10^{-6}) \;=\; 113.9 \times 10^{-6}$

(c) $\Delta d_2 \;=\; \varepsilon'\,d_2 \;=\; (113.9 \times 10^{-6})\,(150\ \text{mm}) \;=\; 0.0171\ \text{mm}$

$\Delta d_1 \;=\; \varepsilon'\,d_1 \;=\; (113.9 \times 10^{-6})\,(110\ \text{mm}) \;=\; 0.0125\ \text{mm}$

- **RELATIONSHIP AMONG v, E AND G**

$$G = \frac{E}{2(1+v)}$$

- G → modulus of rigidity
- E → modulus of elasticity
- v → poisson's ratio

Deformation under axial loading

- From Hooke's Law:

$$\sigma = E\varepsilon \qquad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

- From the definition of strain

$$\varepsilon = \frac{\delta}{L}$$

- Equating and solving for the deformation,

$$\delta = \frac{PL}{AE}$$

- With variations in loading, cross-section or material properties,

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

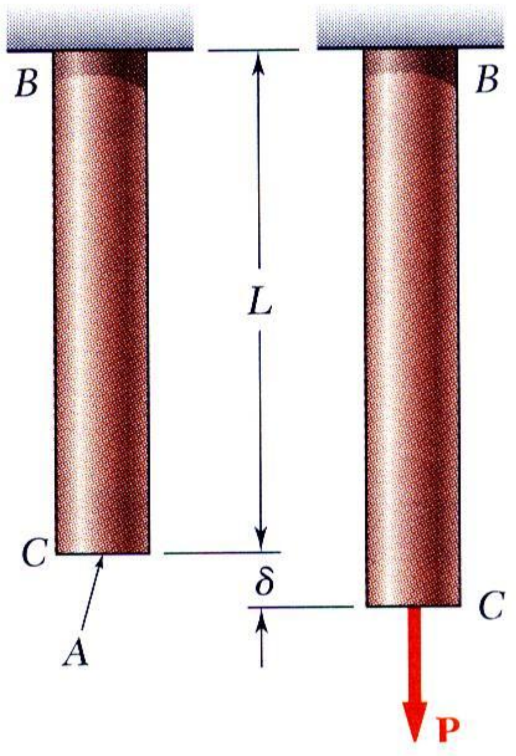
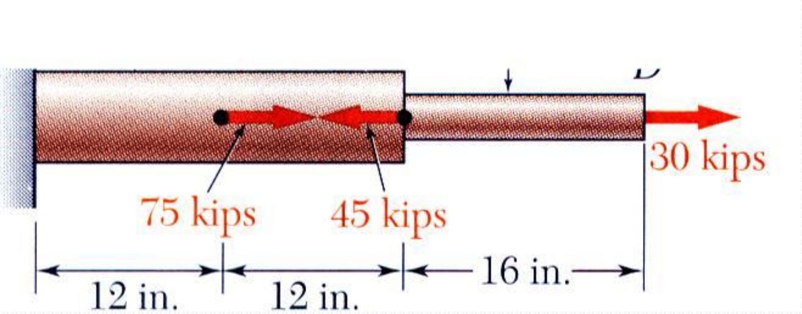


Fig. 2.22

Example1:

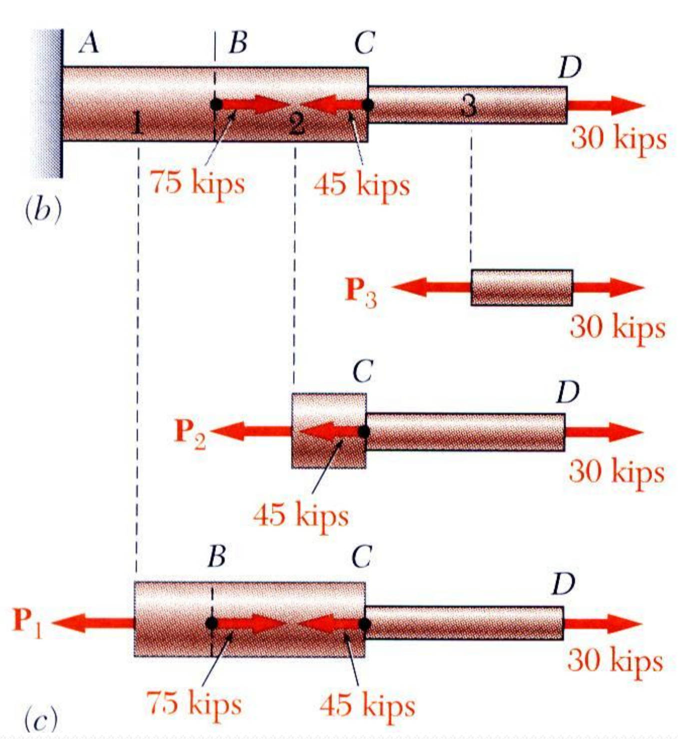
Determine the deformation of the steel rod shown under the given loads.



$E = 29 \times 10^{-6}\,\text{psi}$

$D = 1.07\,\text{in.} \quad d = 0.618\,\text{in.}$

- SOLUTION:
- Divide the rod into components at the load application points.
- Apply a free-body analysis on each component to determine the internal force
- Evaluate the total of the component deflections.



$$L_1 = L_2 = 12\,\text{in.} \qquad L_3 = 16\,\text{in.}$$

$$A_1 = A_2 = 0.9\,\text{in}^2 \qquad A_3 = 0.3\,\text{in}^2$$

Apply free-body analysis to each component to determine internal forces

$$P_1 = 60 \times 10^3\,\text{lb}$$

$$P_2 = -15 \times 10^3\,\text{lb}$$

$$P_3 = 30 \times 10^3\,\text{lb}$$

- Evaluate total deflection,

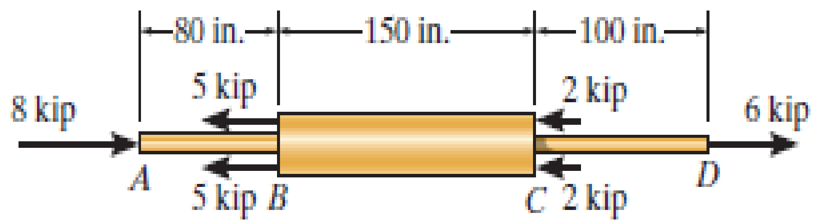
$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3}\right)$$

$$= \frac{1}{29\times10^6}\left[\frac{\left(60\times10^3\right)12}{0.9} + \frac{\left(-15\times10^3\right)12}{0.9} + \frac{\left(30\times10^3\right)16}{0.3}\right]$$

$$= 75.9\times10^{-3}\,\text{in.}$$

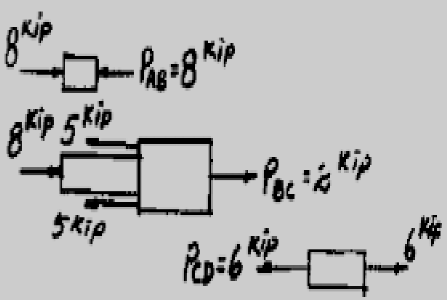
$$\boxed{\delta = 75.9\times10^{-3}\ \text{in.}}$$

Example: The copper shaft is subjected to the axial loads shown. Determine the displacement of end *A* with respect to end *D* if the diameters of each segment are d_{BC} = 1 in., d_{AB} = 0.75 in., and d_{CD} = 0.5 in. Take E_{cu} = 18(103) ksi.



$$\delta_{A/D} = \Sigma\frac{PL}{AE} = \frac{-8(80)}{\frac{\pi}{4}(0.75)^2(18)(10^3)} + \frac{2(150)}{\frac{\pi}{4}(1)^2(18)(10^3)} + \frac{6(100)}{\frac{\pi}{4}(0.5)^2(18)(10^3)}$$

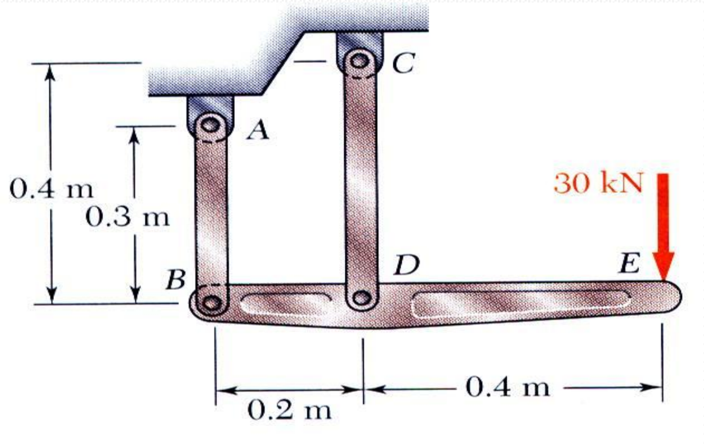
$$= 0.111 \text{ in.}$$ **Ans.**



The positive sign indicates that end *A* moves away from end *D*.

Example II

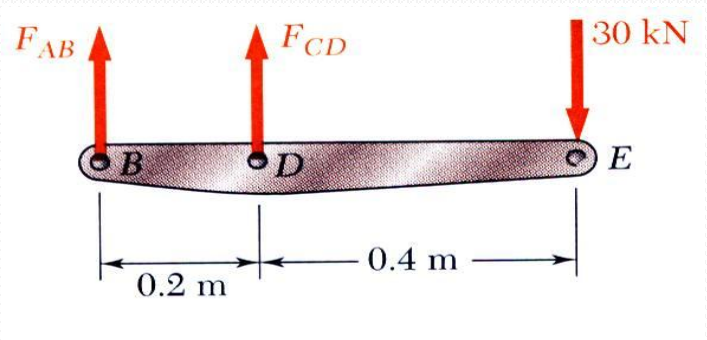
The rigid bar *BDE* is supported by two links *AB* and *CD*. Link *AB* is made of aluminum (E = 70 GPa) and has a cross-sectional area of 500 mm^2. Link *CD* is made of steel (E = 200 GPa) and has a cross-sectional area of (600 mm^2). For the 30-kN force shown, determine the deflection a) of *B*, b) of *D*, and c) of *E*.



- SOLUTION:
- Apply a free-body analysis to the bar *BDE* to find the forces exerted by links *AB* and *DC*.
- Evaluate the deformation of links *AB* and *DC* or the displacements of *B* and *D*.
- Work out the geometry to find the deflection at E given the deflections at B and D.

SOLUTION:

Free body: Bar *BDE*



$$\sum M_B = 0$$

$$0 = -(30\,\text{kN} \times 0.6\,\text{m}) + F_{CD} \times 0.2\,\text{m}$$

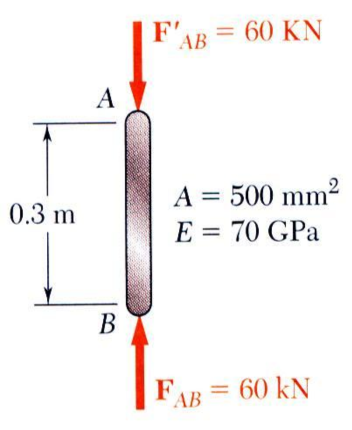
$$F_{CD} = +90\,\text{kN} \quad \textit{tension}$$

$$\sum \mathrm{M_D} = 0$$

$$0 = -(30\,\text{kN} \times 0.4\,\text{m}) - F_{AB} \times 0.2\,\text{m}$$

$$F_{AB} = -60\,\text{kN} \quad \textit{compression}$$

Displacement of *B*:

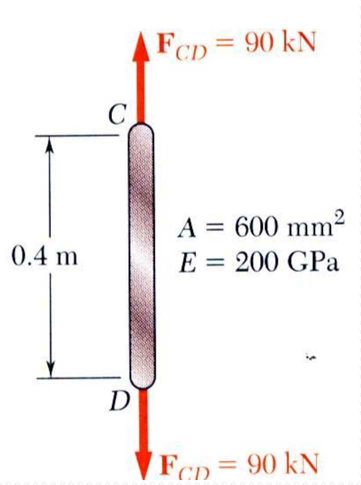


$$\delta_B = \frac{PL}{AE}$$

$$= \frac{(-60\times10^3\,\text{N})(0.3\,\text{m})}{(500\times10^{-6}\,\text{m}^2)(70\times10^9\,\text{Pa})}$$

$$= -514\times10^{-6}\,\text{m}$$

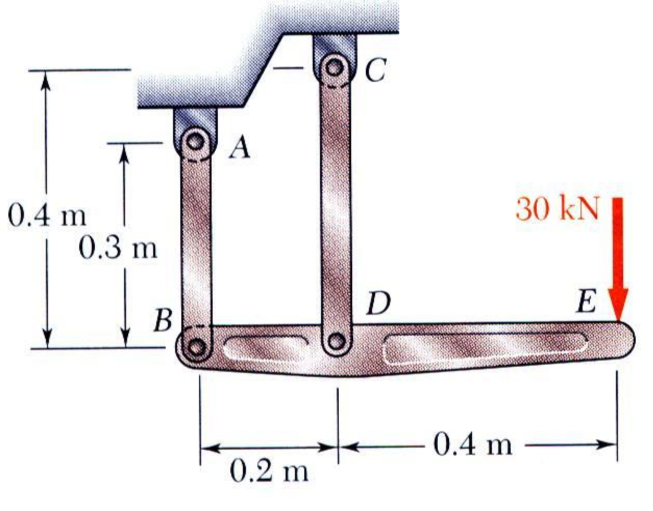
Displacement of *D*:

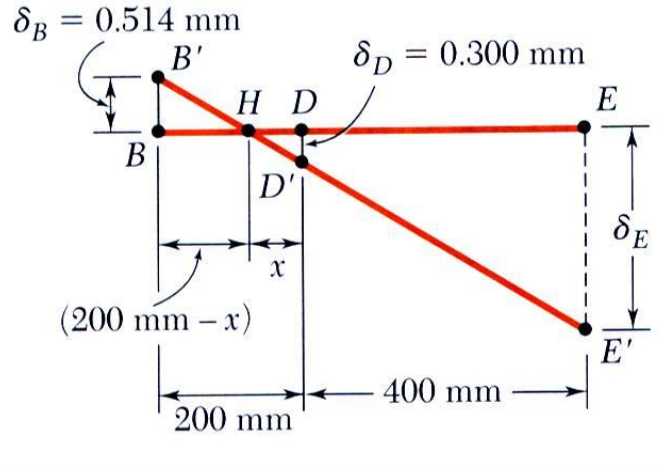


$$\delta_D = \frac{PL}{AE}$$

$$= \frac{(90\times10^3\,\text{N})(0.4\,\text{m})}{(600\times10^{-6}\,\text{m}^2)(200\times10^9\,\text{Pa})}$$

$$= 300\times10^{-6}\,\text{m}$$





Displacement of D:

$$\frac{BB'}{DD'} = \frac{BH}{HD}$$

$$\frac{0.514\,\text{mm}}{0.300\,\text{mm}} = \frac{(200\,\text{mm}) - x}{x}$$

$$x = 73.7\text{ mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD}$$

$$\frac{\delta_E}{0.300\,\text{mm}} = \frac{(400 + 73.7)\text{mm}}{73.7\text{ mm}}$$

$$\delta_E = 1.928\,\text{mm}$$

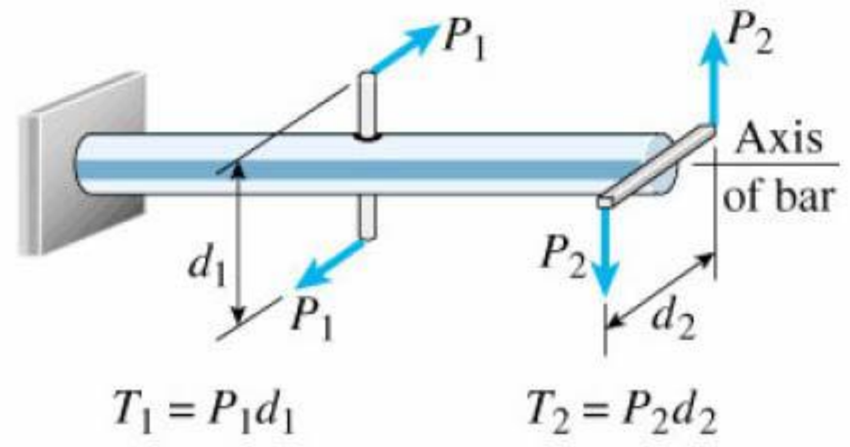
$$\boxed{\delta_E = 1.928\,\text{mm} \downarrow}$$

Torsion Of Circular Sections

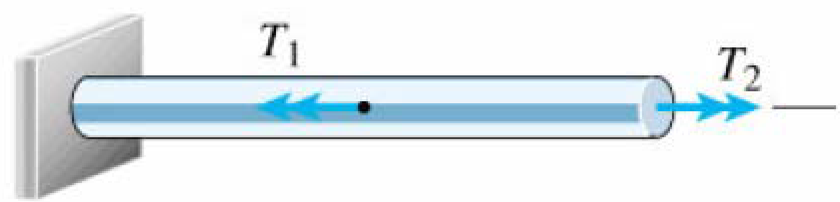
Torsion

- Torsion : twisting of a structural member when it is loaded by couples that produce rotation about its longitudinal axis

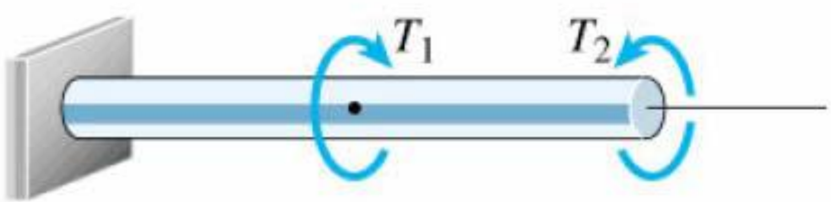
$T1 = P1\ d1$
$T2 = P2\ d2$
the couples *T1*, *T2 are called* torques, twisting couples or twisting moments
unit of *T* : *N-m*, *lb-ft*



(a)



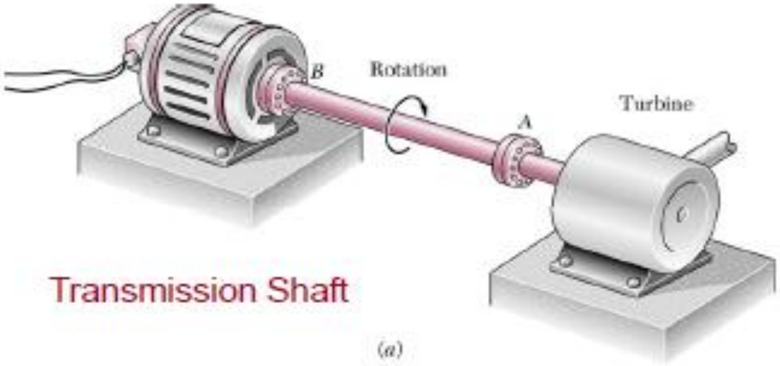
(b)



(c)

in this chapter, we will develop formulas for the stresses and deformations produced in circular bars subjected to torsion, such as drive shafts, thin-walled members

- <u>Turbine</u> exerts torque *T* on the shaft
  - Shaft transmits the torque to the generator



Torsion of circular shaft (cont.)

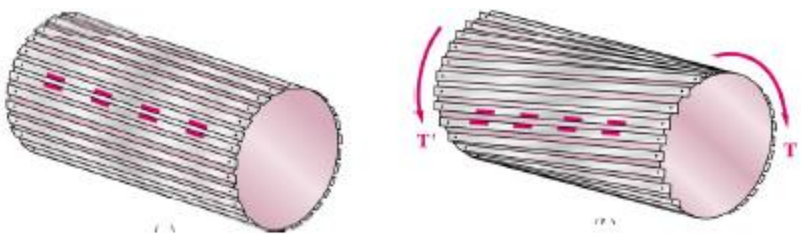
We assume

- Bar in pure torsion
- small rotation (the length and radius will not change

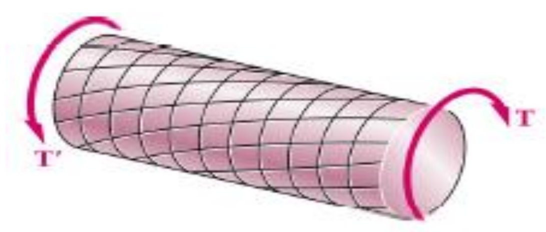
Torsional Deformation of a Circular Bar

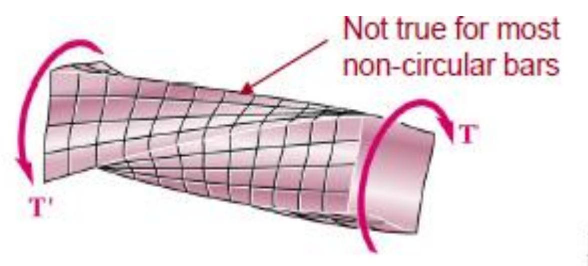
How does the bar deform

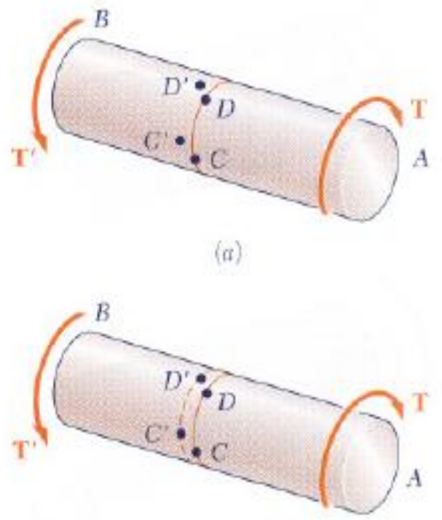
- Cross section of the bar remain the same shape ,bar is simply rotating



Cross-section remains perpendicular to axis of cylinder (cylinder does not wrap)



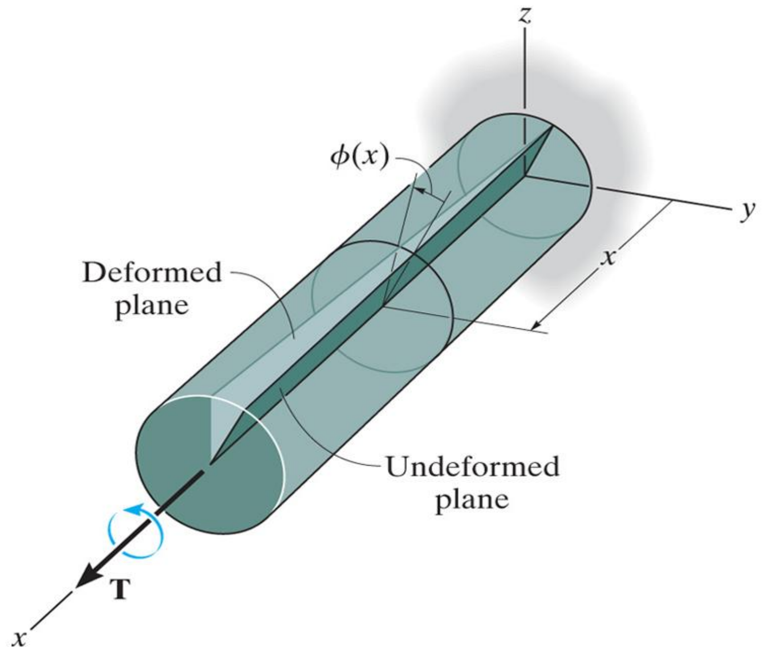




(a)

$$CD = C'D'$$

A circular plane remains circular plane



The angle of twist $\phi(x)$ increases as x increases.

Torsional Deformation of a Circular Bar

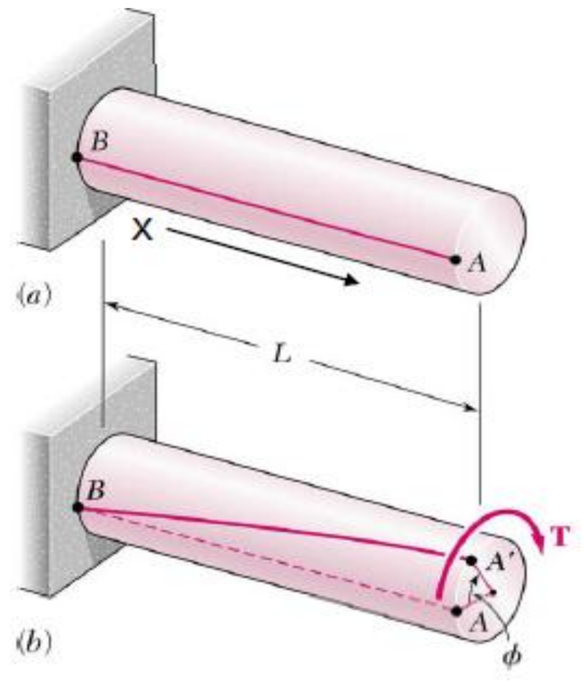
- consider a bar or shaft of circular cross section twisted by a couple *T*, assume the left-hand end is fixed and the right-hand end will rotate a small angle γ, called angle of twist

A moves to A'
-ϕ= angle of twist (in radians)

- From observation, the angle of twist of the shaft is proportional to the applied torque and to the shaft length.

$$\phi \propto T$$

$$\phi \propto L$$



Shearing Strain

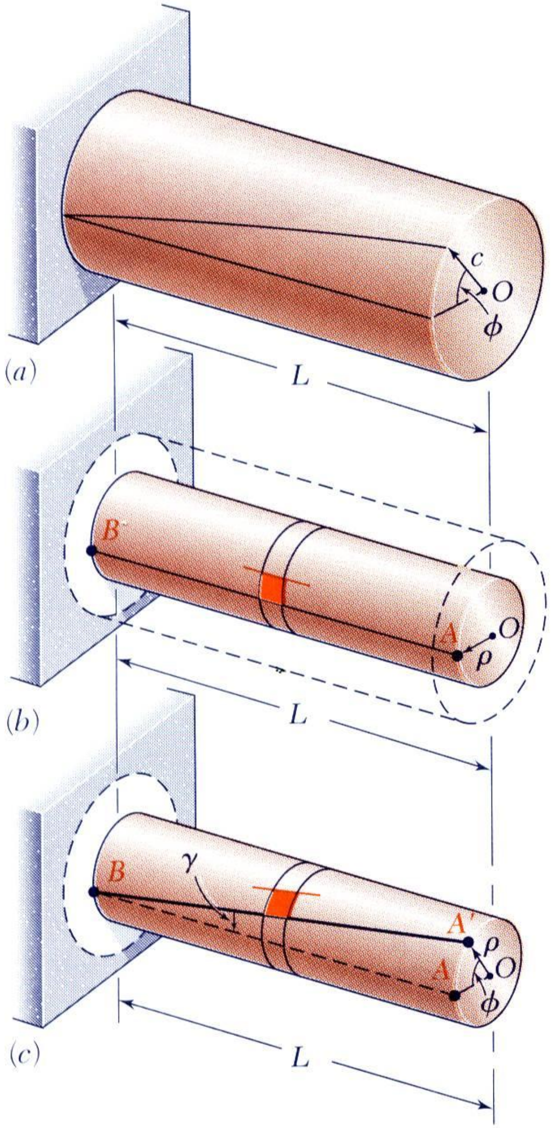
- Since the ends of the element remain planar, the shear strain is equal to angle of twist.

- It follows that

$$arch(AA') = L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

- Shear strain is proportional to twist and radius

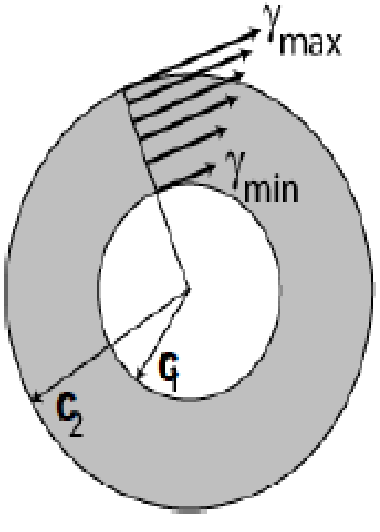
$$\gamma_{\max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{\max}$$



- We can also apply the equation for maximum surface shear strain to a hollow circular tube

$$\gamma_{\min} = \frac{c_1 \phi}{L}$$

$$\gamma_{\max} = \frac{c_2 \phi}{L}$$



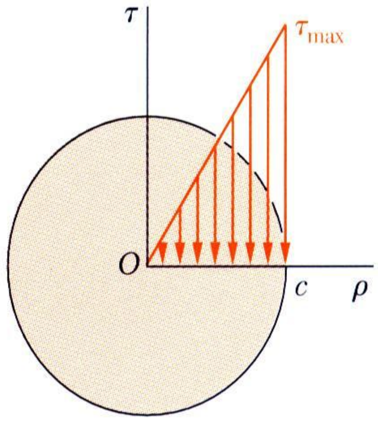
Stresses in Elastic Range

- Multiplying the previous equation by the shear modulus,

$$G\gamma = \frac{\rho}{c} G\gamma_{\max}$$ From Hooke's Law, $$\tau = G\gamma$$

$$\tau = \frac{\rho}{c}\tau_{\max}$$

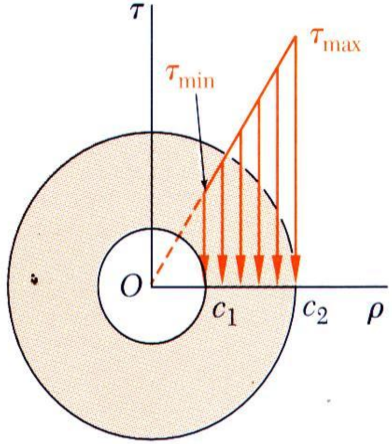
The shearing stress varies linearly with the radial position in the section.



$$J = \tfrac{1}{2}\pi c^4$$

- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the shaft at the section,

$$T = \int \rho\tau \, dA = \frac{\tau_{\max}}{c}\int \rho^2 \, dA = \frac{\tau_{\max}}{c} J$$



$$J = \tfrac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

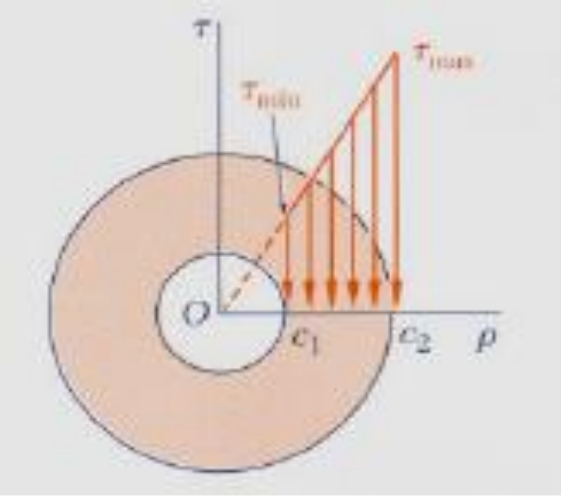
$$J = \int \rho^2 dA$$

$= moment\ of\ inertia\ center\ O$

- The results are known as the *elastic torsion formulas*,

$$\tau_{\max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

$$\tau_{min} = \frac{c_1}{c_2}\tau_{max}$$



J = polar moment of inertia

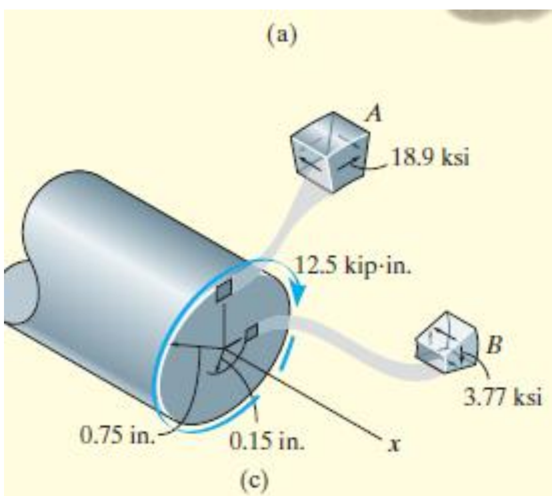
Solid shaft:

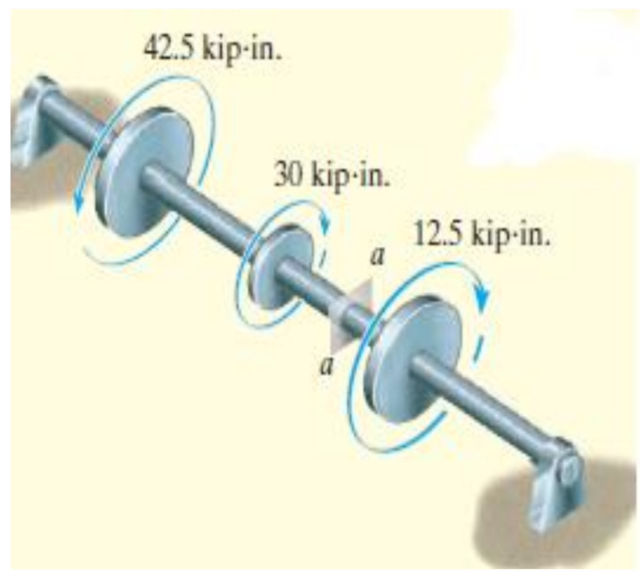
$$J = \frac{\pi}{2} c^4$$

- Hollow shaft:

$$J = \frac{\pi}{2}\left(c_o^4 - c_i^4\right)$$

Example: The shaft shown in Fig. 5–11a is supported by two bearings and is subjected to three torques. Determine the shear stress developed at points A and B, located at section a–a of the shaft, Fig. 5–11c





$\Sigma M_x = 0; \quad 42.5 \text{ kip} \cdot \text{in.} - 30 \text{ kip} \cdot \text{in.} - T = 0 \qquad T = 12.5 \text{ kip} \cdot \text{in.}$

Section Property. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.75 \text{ in.})^4 = 0.497 \text{ in}^4$$

Shear Stress. Since point A is at $\rho = c = 0.75$ in.,

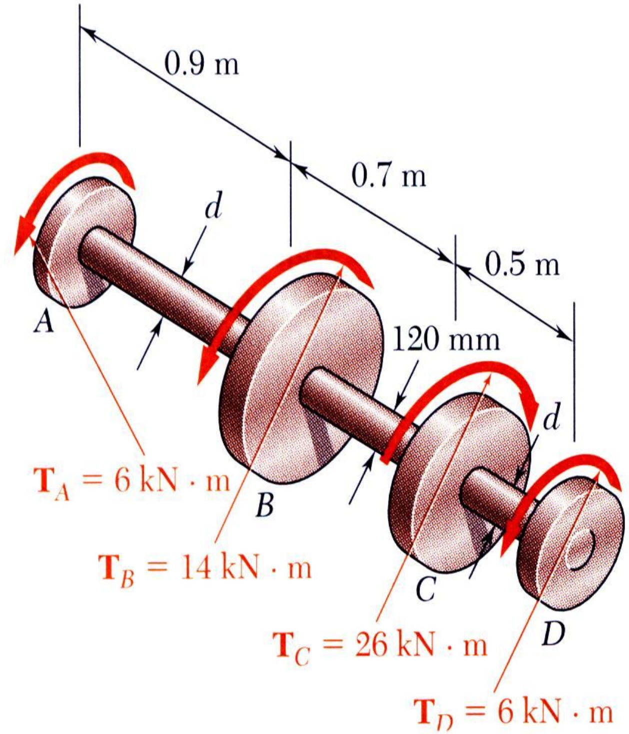
$$\tau_A = \frac{Tc}{J} = \frac{(12.5 \text{ kip} \cdot \text{in.})(0.75 \text{ in.})}{(0.497 \text{ in}^4)} = 18.9 \text{ ksi} \qquad \textit{Ans.}$$

Likewise for point B, at $\rho = 0.15$ in., we have

$$\tau_B = \frac{T\rho}{J} = \frac{(12.5 \text{ kip} \cdot \text{in.})(0.15 \text{ in.})}{(0.497 \text{ in}^4)} = 3.77 \text{ ksi} \qquad \textit{Ans.}$$

- Example:

Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine (*a*) the minimum and maximum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.

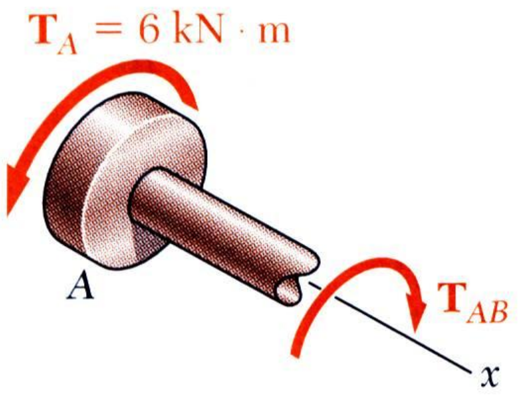


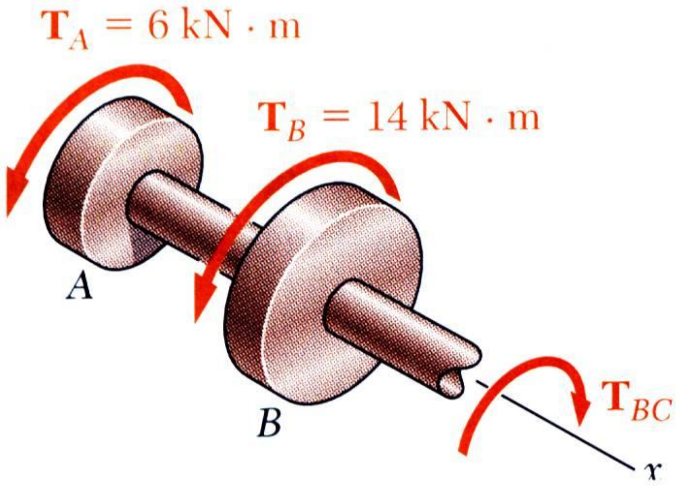
SOLUTION:

Cut sections through shafts *AB* and *BC* and perform static equilibrium analyses to find torque loadings.

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC*
- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

- Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings.





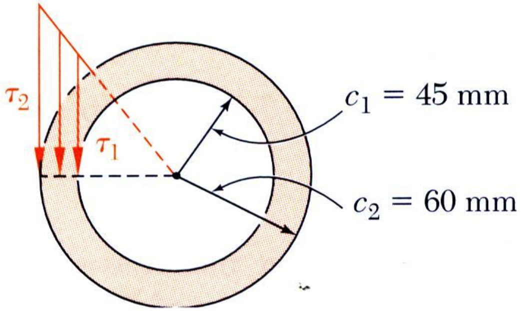
$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) - T_{AB}$$

$$T_{AB} = 6\,\text{kN}\cdot\text{m} = T_{CD}$$

$$\sum M_x = 0 = (6\,\text{kN}\cdot\text{m}) + (14\,\text{kN}\cdot\text{m}) - T_{BC}$$

$$T_{BC} = 20\,\text{kN}\cdot\text{m}$$

- Apply elastic torsion formulas to find minimum and maximum stress on shaft *BC.*



$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right]$$

$$= 13.92 \times 10^{-6}\,\mathrm{m}^4$$

$$\tau_{max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\,\mathrm{kN\cdot m})(0.060\,\mathrm{m})}{13.92\times10^{-6}\,\mathrm{m}^4}$$

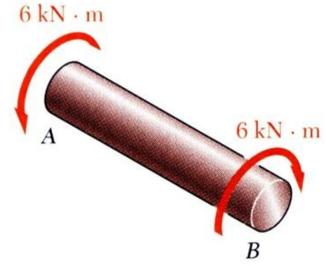
$$= 86.2\,\mathrm{MPa}$$

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{min}}{86.2\,\mathrm{MPa}} = \frac{45\,\mathrm{mm}}{60\,\mathrm{mm}}$$

$$\tau_{min} = 64.7\,\mathrm{MPa}$$

$$\tau_{max} = 86.2\,\mathrm{MPa}$$
$$\tau_{min} = 64.7\,\mathrm{MPa}$$

- Given allowable shearing stress and applied torque, invert the elastic torsion formula to find the required diameter.

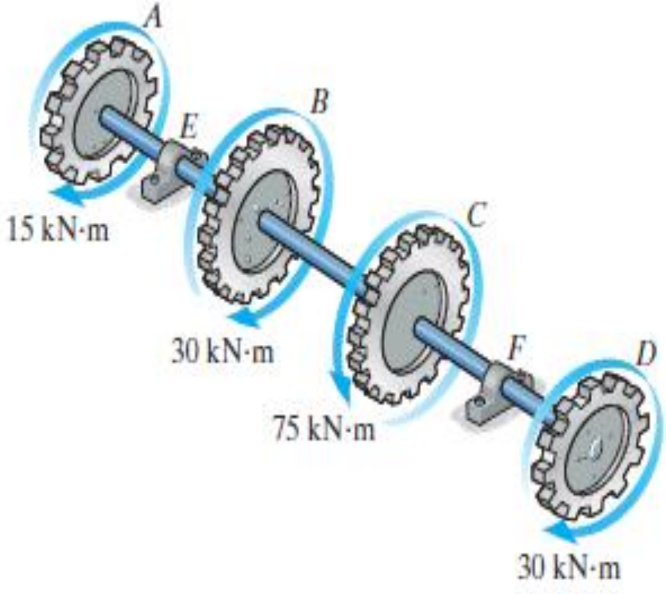


$$\tau_{max} = \frac{Tc}{J} = \frac{Tc}{\frac{\pi}{2}c^4} \qquad 65MPa = \frac{6\,\mathrm{kN\cdot m}}{\frac{\pi}{2}c^3}$$

$$c = 38.9\times10^{-3}\,\mathrm{m}$$

$$d = 2c = 77.8\,\mathrm{mm}$$

Example: If the tubular shaft is made from material having an allowable shear stress τ_{all}=85 Mpa of determine the required minimum wall thickness of the shaft to the nearest millimeter. The shaft has an outer diameter of 150 mm.



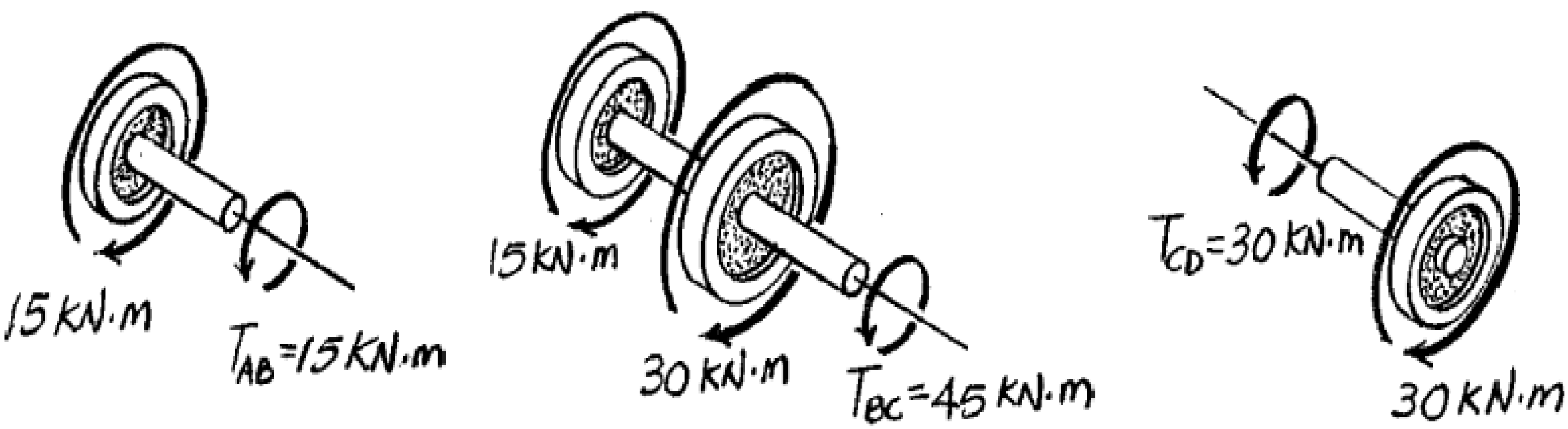
Allowable Shear Stress: Segment BC is critical since it is subjected to the greatest internal torque. The polar moment of inertia of the shaft is $J = \frac{\pi}{2}(0.075^4 - c_i^4)$.

$$\tau_{\text{allow}} = \frac{T_{BC}\,c}{J}; \qquad 85(10^6) = \frac{45(10^3)(0.075)}{\frac{\pi}{2}(0.075^4 - c_i^4)}$$

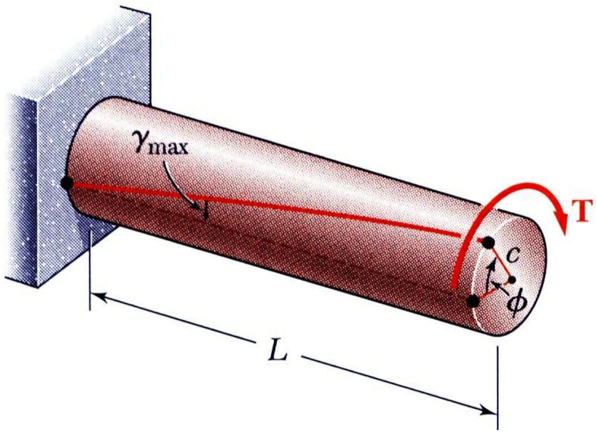
$$c_i = 0.05022 \text{ m} = 50.22 \text{ mm}$$

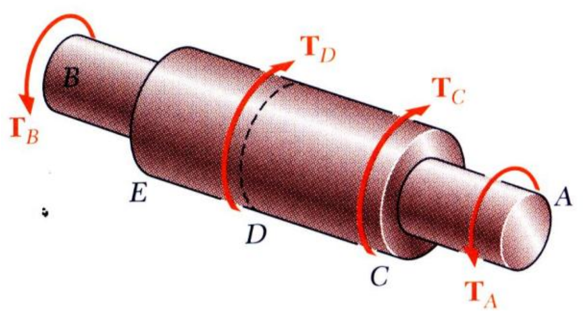
Thus, the minimum wall thickness is

$$t = c_o - c_i = 75 - 50.22 = 24.78 \text{ mm} = 25 \text{ mm}$$ **Ans.**



Angle of Twist in Elastic Range





Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

- Equating the expressions for shearing strain and solving for the angle of twist,

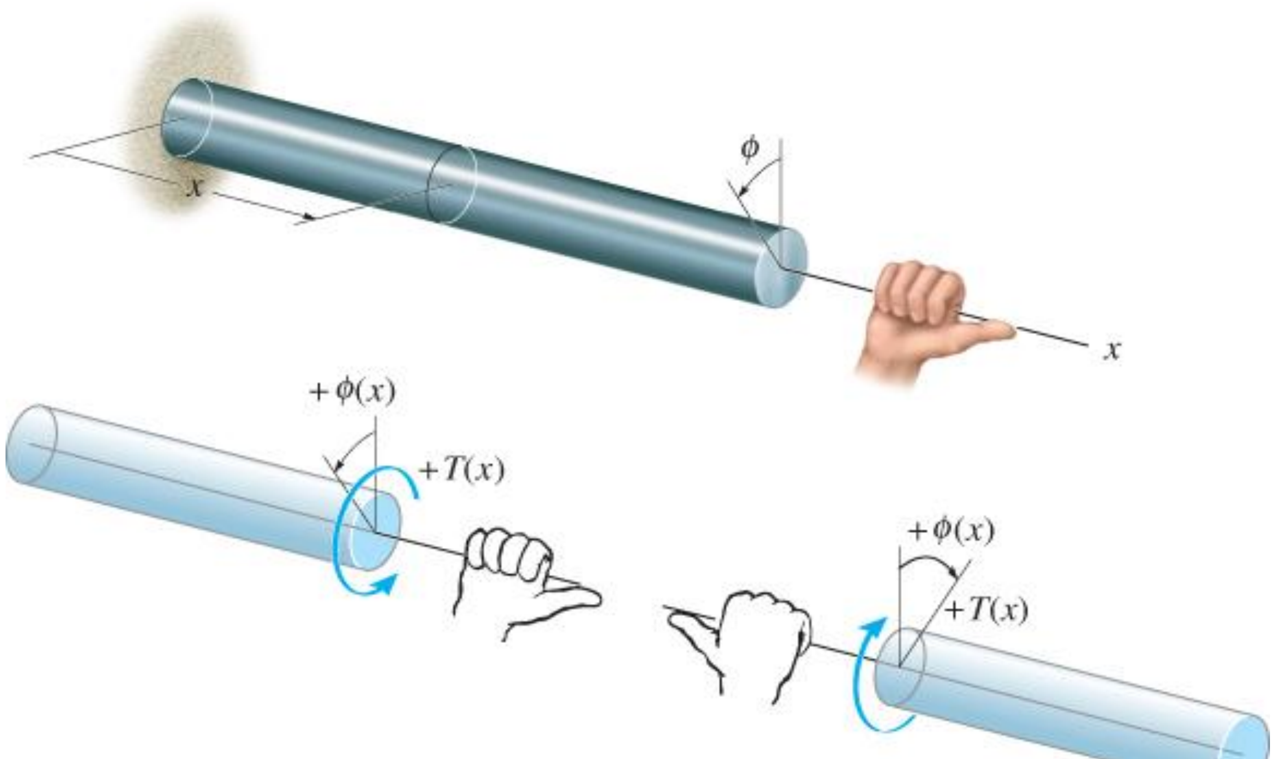
$$\phi = \frac{TL}{JG}$$

If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

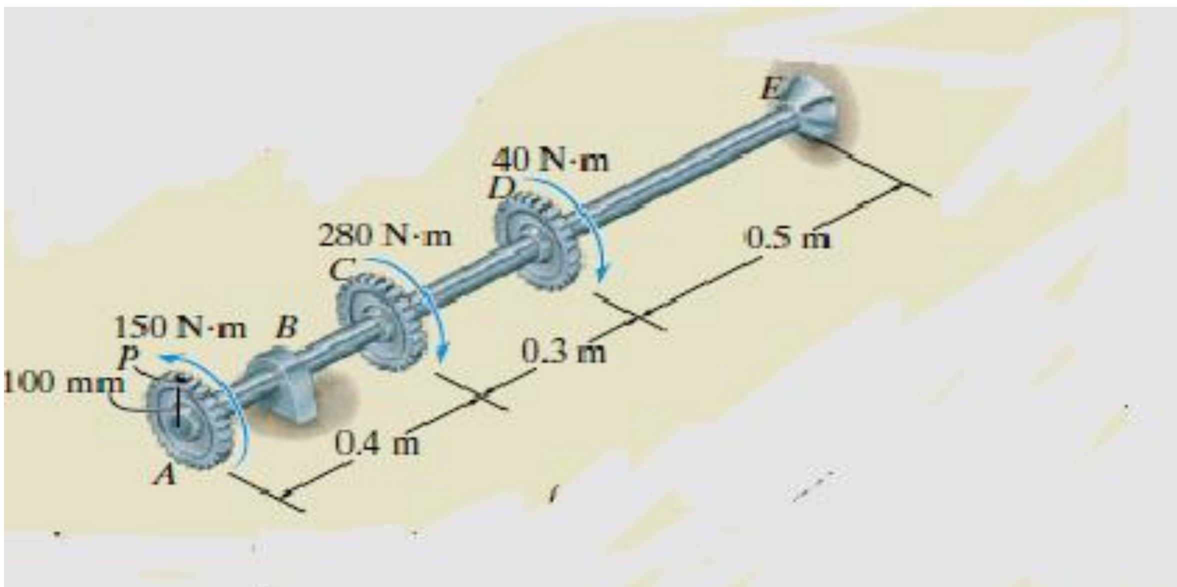
Sign convention

- Use right-hand rule: torque and angle of twist are positive when thumb is directed outward from the shaft

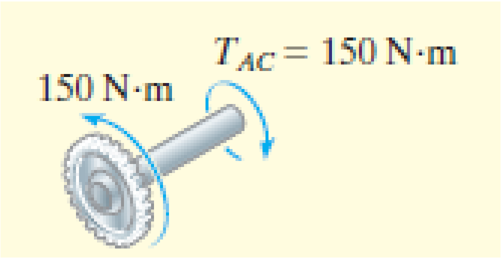


Positive sign convention for T and ϕ.

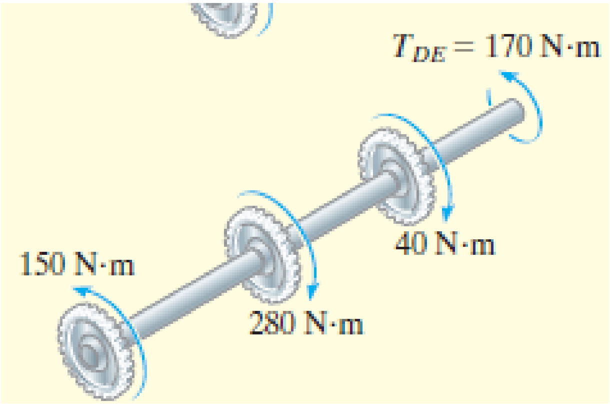
Example: The gears attached to the fixed-end steel shaft are subjected to the torques shown in Fig. 5–19a. If the shear modulus of elasticity is 80 Gpa and the shaft has a diameter of 14 mm, determine the displacement of the tooth P on gear A. The shaft turns freely within the bearing at B



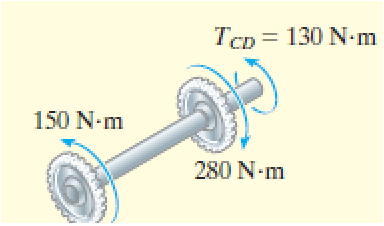
Solution



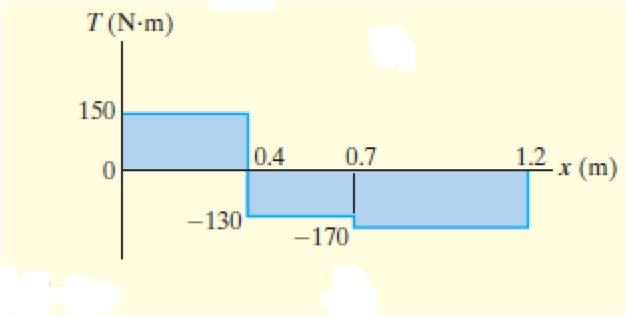
T_{AC} = 150 N·m
150 N·m



T_{DE} = 170 N·m
40 N·m
150 N·m
280 N·m



T_{CD} = 130 N·m
150 N·m
280 N·m



T (N·m)
150
0
0.4
0.7
1.2
x (m)
−130
−170

$$T_{AC} = +150\ \text{N}\cdot\text{m} \qquad T_{CD} = -130\ \text{N}\cdot\text{m} \qquad T_{DE} = -170\ \text{N}\cdot\text{m}$$

These results are also shown on the torque diagram, Fig. 5–19*c*.

Angle of Twist. The polar moment of inertia for the shaft is

$$J = \frac{\pi}{2}(0.007\ \text{m})^4 = 3.771(10^{-9})\ \text{m}^4$$

Applying Eq. 5–16 to each segment and adding the results algebraically, we have

$$\phi_A = \sum\frac{TL}{JG} = \frac{(+150\ \text{N}\cdot\text{m})(0.4\ \text{m})}{3.771(10^{-9})\ \text{m}^4\,[80(10^9)\ \text{N/m}^2]}$$

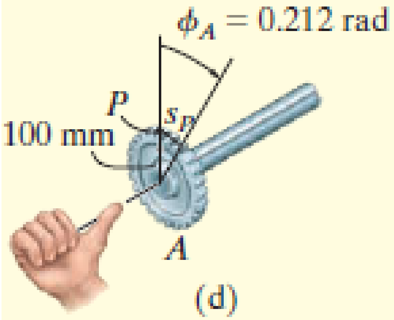
$$+ \frac{(-130\ \text{N}\cdot\text{m})(0.3\ \text{m})}{3.771(10^{-9})\ \text{m}^4\,[80(10^9)\ \text{N/m}^2)]}$$

$$+ \frac{(-170\ \text{N}\cdot\text{m})(0.5\ \text{m})}{3.771(10^{-9})\ \text{m}^4\,[80(10^9)\ \text{N/m}^2)]} = -0.2121\ \text{rad}$$

Since the answer is negative, by the right-hand rule the thumb is directed *toward* the end E of the shaft, and therefore gear A will rotate as shown in Fig. 5–19*d*.

The displacement of tooth P on gear A is

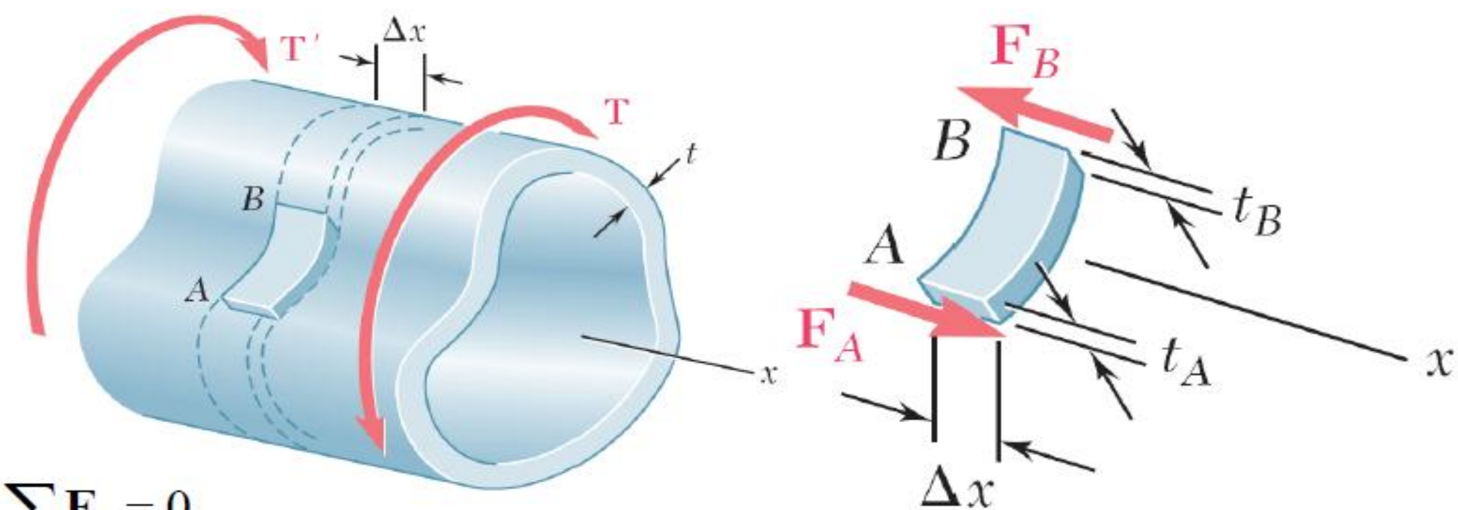
$$s_P = \phi_A r = (0.2121 \text{ rad})(100 \text{ mm}) = 21.2 \text{ mm} \qquad \textit{Ans.}$$



(d)

THIN WALLED HOLLOW SHAFTS

3.15 THIN WALLED HOLLOW SHAFTS



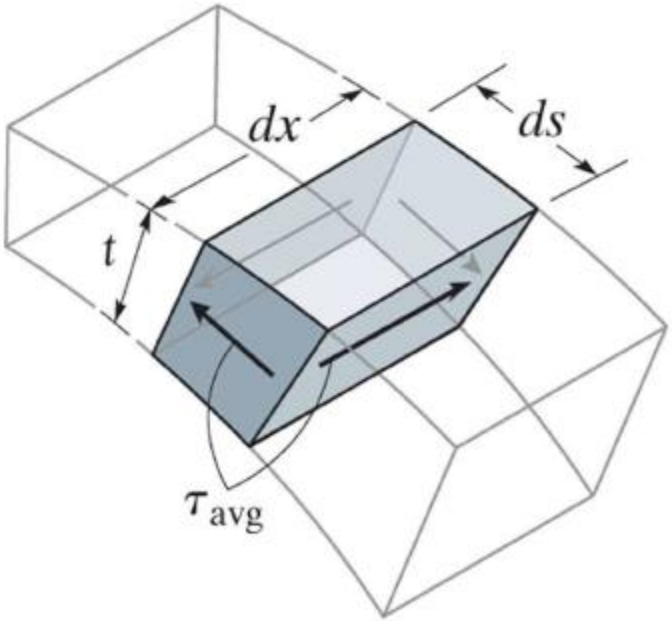
$$\sum \mathbf{F_x} = 0$$

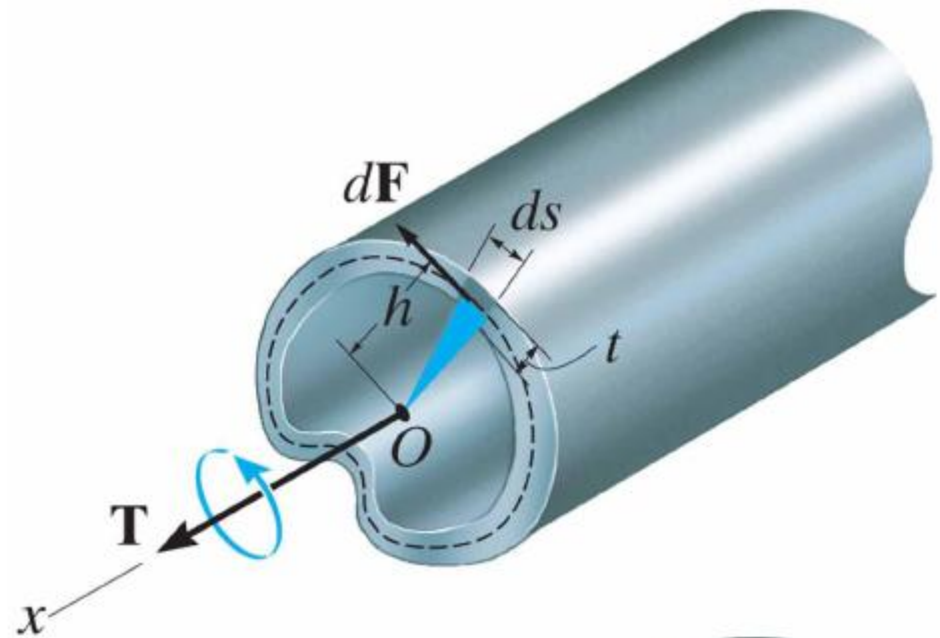
$$\mathbf{F_A} = \mathbf{F_B}$$

$$\tau_A(t_A \Delta x) = \tau_B(t_B \Delta x)$$

Thus,

$\tau t = q =$ constant (Shear flow)





$$d\mathbf{T} = h(dF) = h(\tau_{ave} t ds)$$

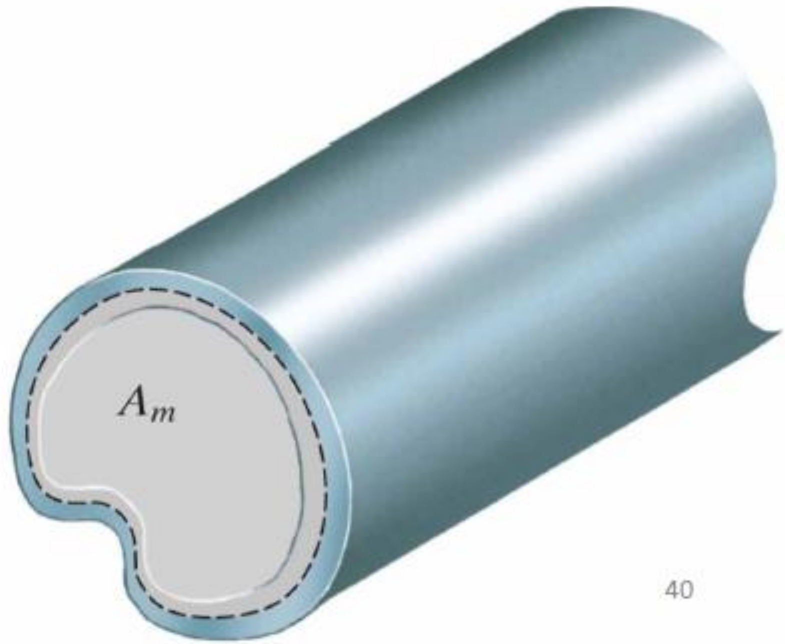
$$\mathbf{T} = \oint h\tau_{ave} t ds = q\oint h ds$$

but

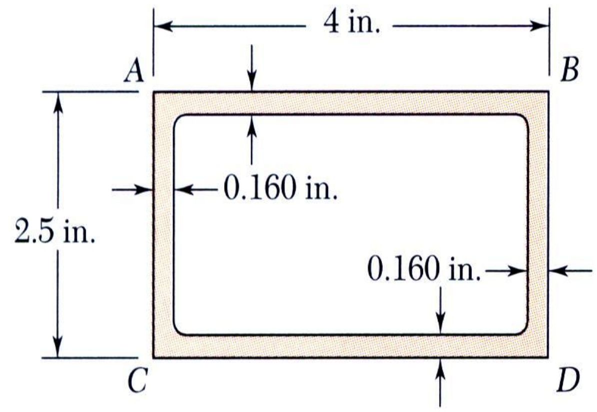
$$dA_m = \frac{1}{2} h ds$$

$$\mathbf{T} = 2\tau_{ave} t A_m$$

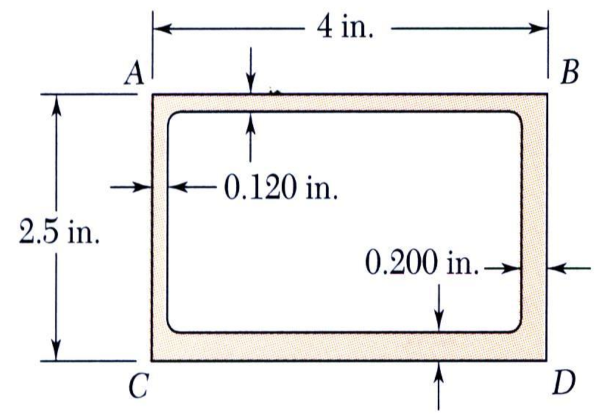
$$\tau_{ave} = \frac{T}{2tA_m}$$



- Example: Extruded aluminum tubing with a rectangular cross-section has a torque loading of 24 kip-in. Determine the shearing stress in each of the four walls with (a) uniform wall thickness of 0.160 in. and wall thicknesses of (b) 0.120 in. on *AB* and *CD* and 0.200 in. on *CD* and *BD*.



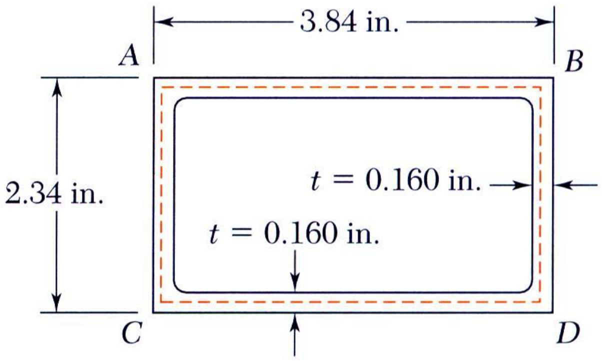
(a)



(b)

SOLUTION:

- Determine the shear flow through the tubing walls.



$$A = (3.84\,\text{in.})(2.34\,\text{in.}) = 8.986\,\text{in.}^2$$

$$q = \frac{T}{2A} = \frac{24\,\text{kip-in.}}{2\left(8.986\,\text{in.}^2\right)} = 1.335\frac{\text{kip}}{\text{in.}}$$

- Find the corresponding shearing stress with each wall thickness.

With a uniform wall thickness,

$$\tau = \frac{q}{t} = \frac{1.335\,\text{kip/in.}}{0.160\,\text{in.}}$$

$$\boxed{\tau = 8.34\,\text{ksi}}$$

With a variable wall thickness

$$\tau_{AB} = \tau_{AC} = \frac{1.335\,\text{kip/in.}}{0.120\,\text{in.}}$$

$$\boxed{\tau_{AB} = \tau_{BC} = 11.13\,\text{ksi}}$$

$$\tau_{BD} = \tau_{CD} = \frac{1.335\,\text{kip/in.}}{0.200\,\text{in.}}$$

$$\boxed{\tau_{BC} = \tau_{CD} = 6.68\,\text{ksi}}$$

3.139 A torque $T = 5\ \text{kN} \cdot \text{m}$ is applied to a hollow shaft having the cross section shown. Neglecting the effect of stress concentrations, determine the shearing stress at points a and b.

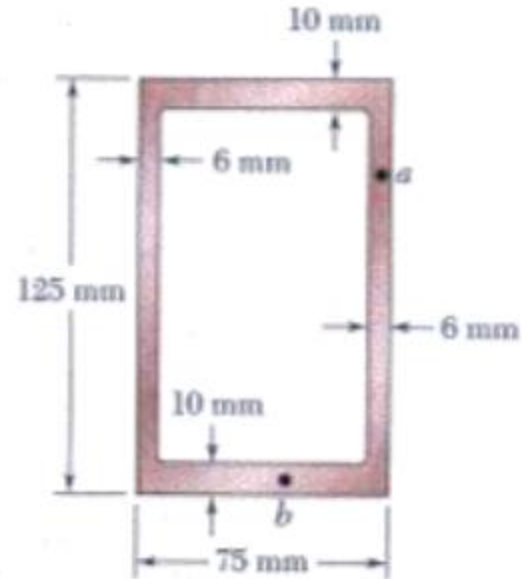


Fig. P3.139

$$\partial = (75 - 6) * (125 - 10) * 10^{-6}$$

$$= 7.935 * 10^{-3} \quad m^2$$

$$\tau = \frac{T}{2\,t\,\partial}$$

$$\tau_a = \frac{5 * 10^3}{2 * (6 * 10^{-3}) * (7.935 * 10^{-3})}$$

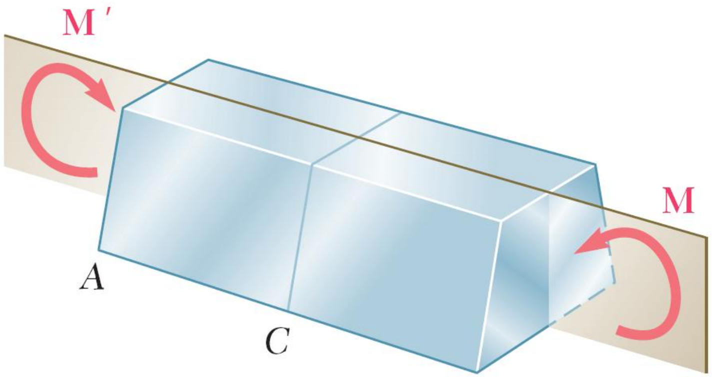
$$= 52.51 \quad MPa$$

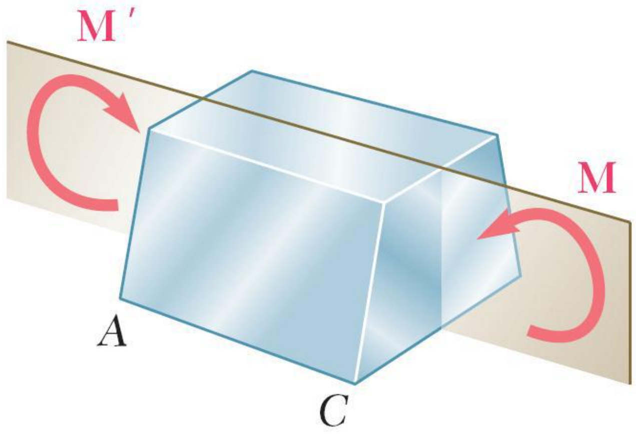
$$\tau_b = \frac{5 * 10^{-3}}{2 * (10 * 10^{-3}) * (7.935 * 10^{-3})}$$

$$= 31.51 \quad MPa$$

PURE BENDING

SYMMETRIC MEMBER IN PURE BENDING

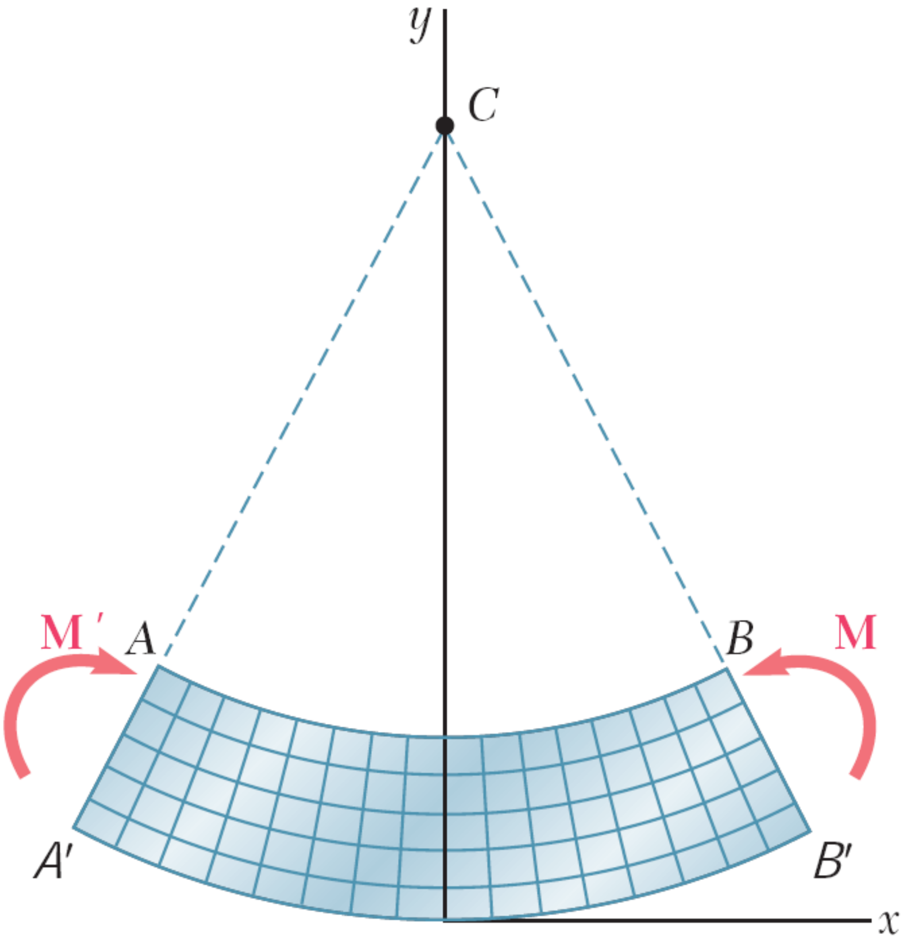


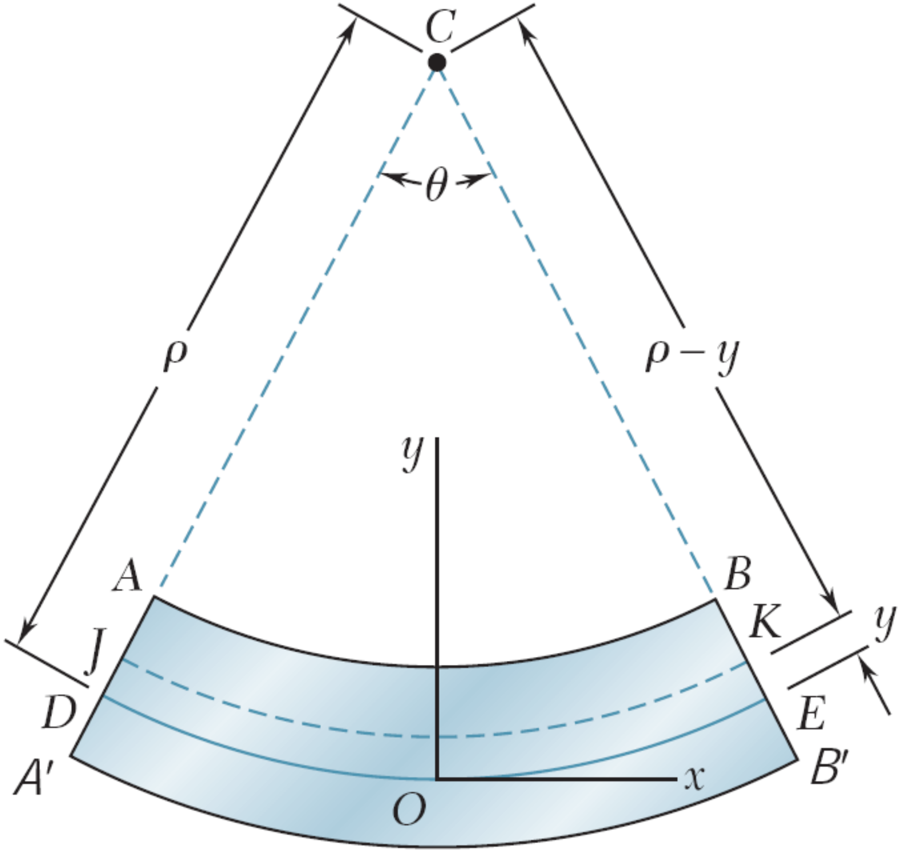


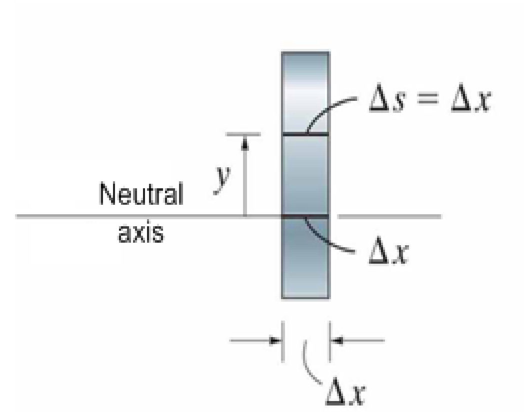
Any section will have same magnitude of moment with no other forces acting (Pure bending)

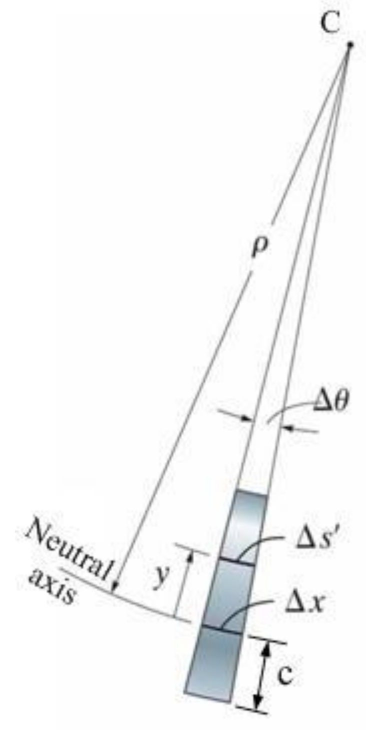
Deformation In A Symmetric Member In Pure Bending

❖Line AB will be transformed to circular arc centered at C.
❖Any cross-section perpendicular to the axis of the member remains plane.
❖Line AB decreased in length and line A’B’ increase in length; causing compression on the upper surface and tension on the lower surface.
❖There should be a surface in between where no tension or compression occurs; this called the neutral surface.









$$\varepsilon = \frac{\Delta s' - \Delta x}{\Delta x} = \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} = -\frac{y}{\rho}$$

$$\varepsilon_{\max} = \frac{c}{\rho}, \quad \text{then} \quad \varepsilon = -\left(\frac{y}{c}\right)\varepsilon_{\max}$$

Stresses And Deformations In The Elastic Range

From hook's law: linear variation of normal strain leads to linear variation in normal stress

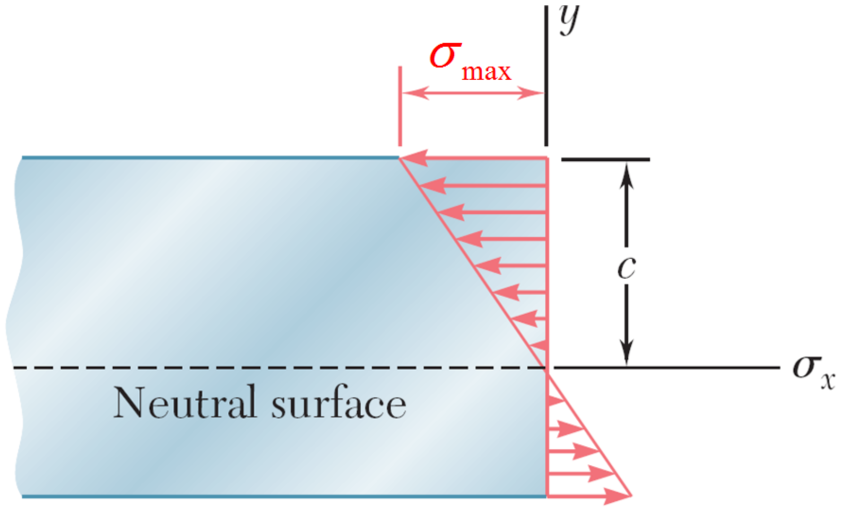
$$\sigma = -\left(\frac{y}{c}\right)\sigma_{\max}$$

$$\sum \mathbf{F}_x = 0$$

$$\int_A dF = \int_A \sigma dA = \int_A -\left(\frac{y}{c}\right)\sigma_{\max} dA$$

thus,

$$\int_A y dA = 0$$



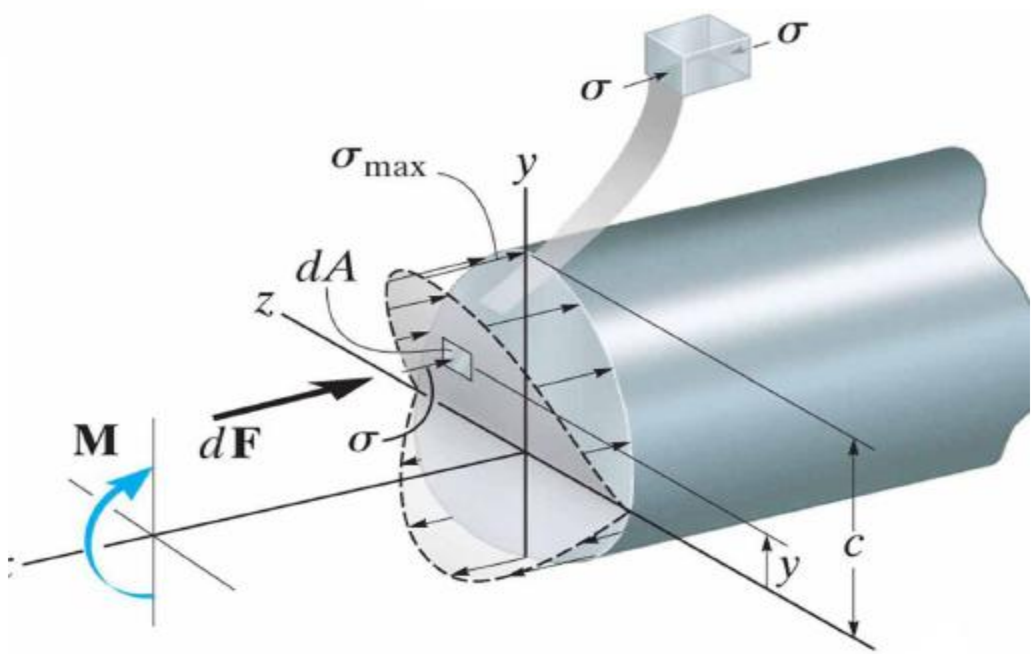
The neutral axis is the horizontal centroidal axis

$$\mathbf{M} = \int_A y\,dF = \int_A y\sigma\,dA = \int_A y\left(\frac{y}{c}\sigma_{max}\right)dA$$

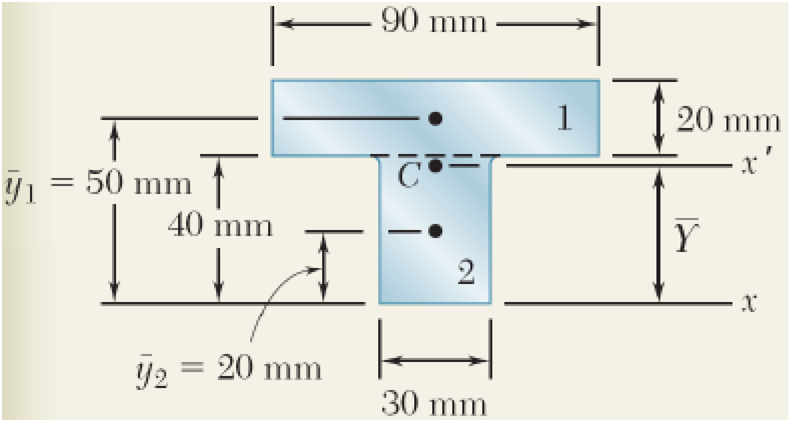
$$\mathbf{M} = \frac{\sigma_{max}}{c}\int_A y^2\,dA \longrightarrow \text{Moment of inertia (I)}$$

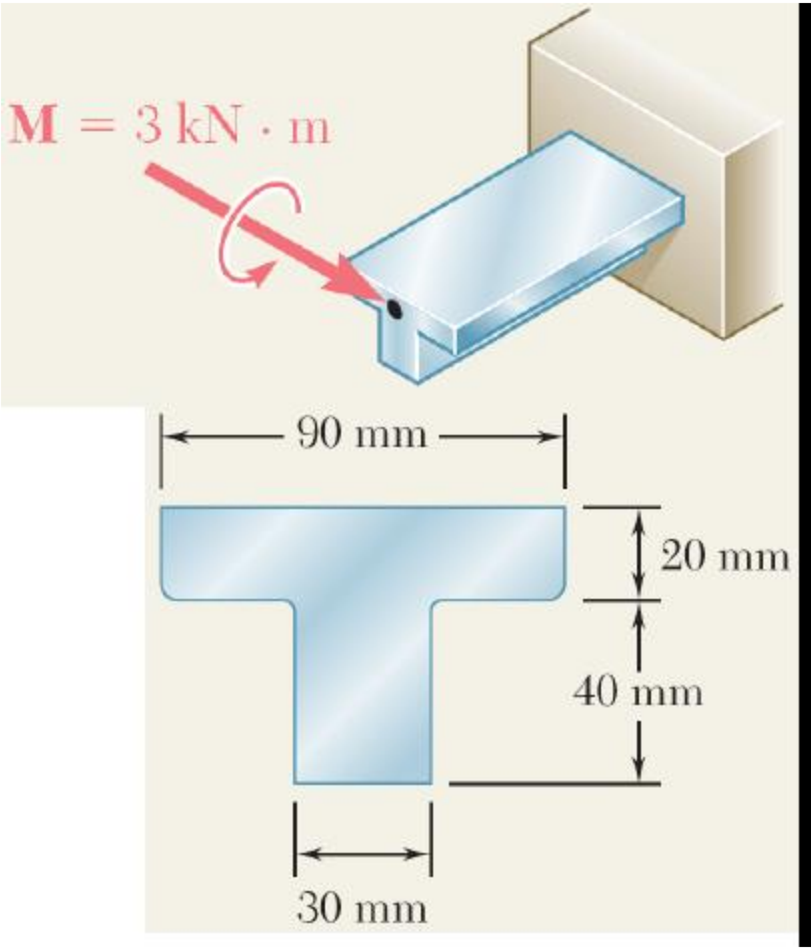
$$\sigma_{max} = \frac{\mathbf{M}c}{I} \quad \text{and} \quad \sigma = \frac{-\mathbf{M}y}{I}$$

$$\text{curveture} = \frac{1}{\rho} = \frac{\mathbf{M}}{EI}$$



Example: Find maximum tensile and compressive stresses. 1





$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2} = 38 \text{ mm}$$

$$I_1 = \frac{1}{12} \times 90 \times (20)^3 + 90 \times 20 \times (12)^2 \text{ mm}^4$$

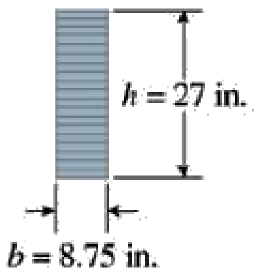
$$I_2 = \frac{1}{12} \times 30 \times (40)^3 + 30 \times 40 \times (18)^2 \text{ mm}^4$$

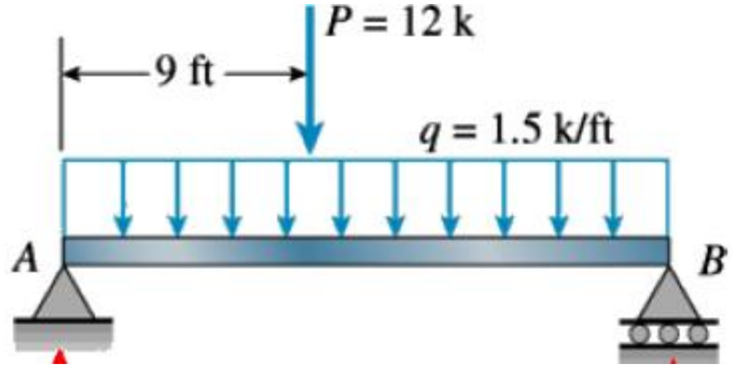
$$I = I_1 + I_2 = 868 \times 10^{-9} \text{ m}^4$$

$$(\sigma_t)_{\max} = \frac{3\times10^3 \times 22\times10^{-3}}{868\times10^{-9}} = 76 \text{ MPa}$$

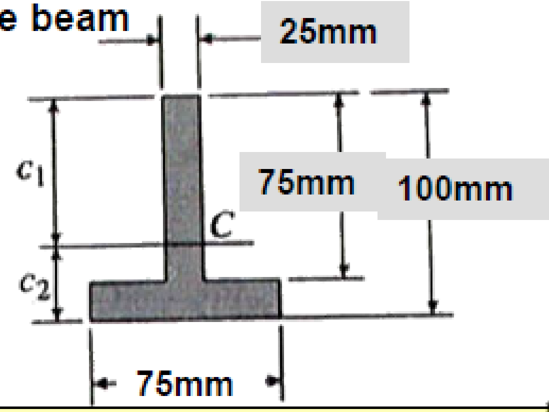
$$(\sigma_c)_{\max} = \frac{3\times10^3 \times 38\times10^{-3}}{868\times10^{-9}} = 131.3 \text{ MPa}$$

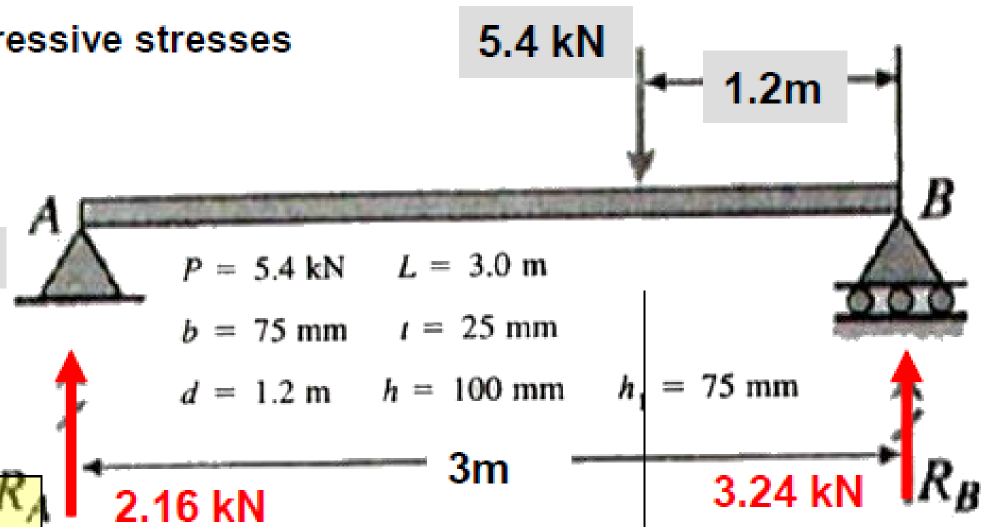
Example: The beam is constructed of a glued laminated wood. Determine the max. stress in the beam due to bending





Prob. 5.16 Max. Tensile and compressive stresses in the beam





PROPERTIES OF THE CROSS SECTION

$A = 3750 \text{ mm}^2$

$c_1 = 62.5 \text{ mm} \qquad c_2 = 37.5 \text{ mm}$

$I_C = 3.3203 \times 10^6 \text{ mm}^4$

MAXIMUM TENSILE STRESS

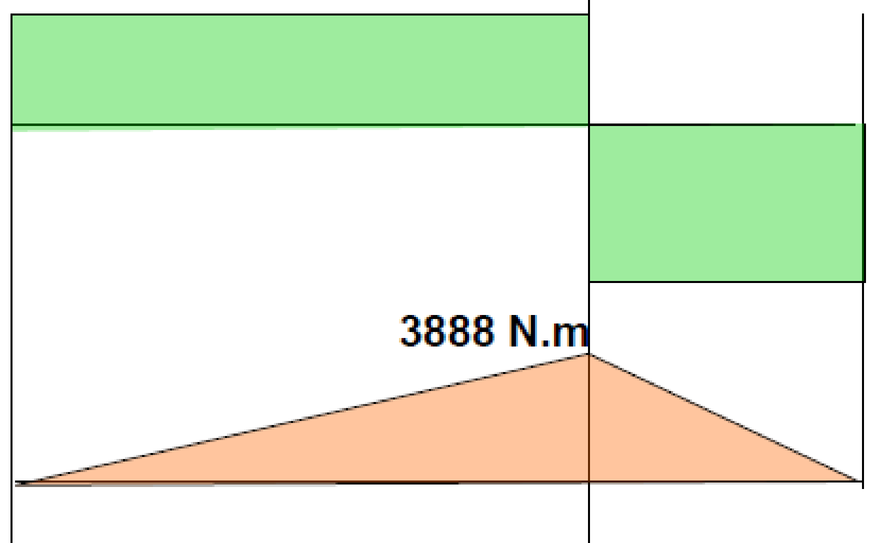
$$\sigma_t = \frac{M_{max} c_2}{I_C} = \frac{(3888 \text{ N} \cdot \text{m})(0.0375 \text{ m})}{3.3203 \times 10^6 \text{ mm}^4}$$

$$= 43.9 \text{ MPa} \longleftarrow$$

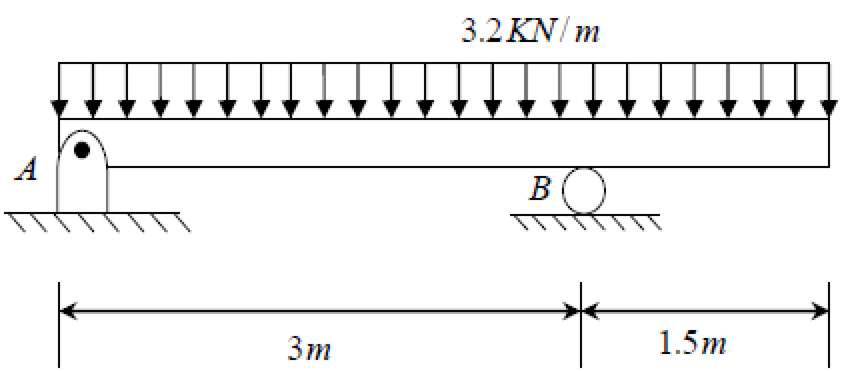
MAXIMUM COMPRESSIVE STRESS

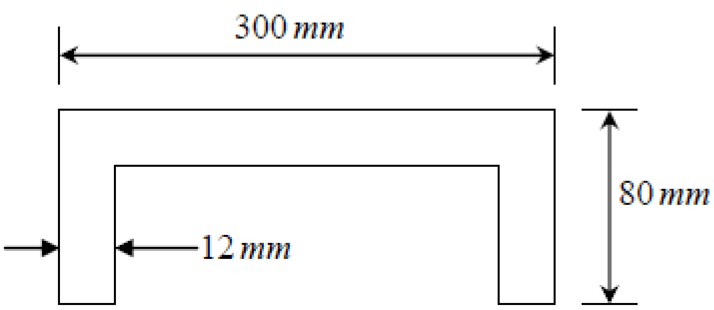
$$\sigma_c = \frac{M_{max} c_1}{I_C} = \frac{(3888 \text{ N} \cdot \text{m})(0.0625 \text{ m})}{3.3203 \times 10^6 \text{ mm}^4}$$

$$= 73.2 \text{ MPa} \longleftarrow$$



Example : The beam shown below has a cross section of channel shape with width b=300 mm and height h=80 mm, the web thickness is t=12 mm. Determine the maximum tensile and compressive stresses in the beam due to uniform load.





$$\sum M_A = 0$$

$$B_y \times 3 - 14.4 \times 2.25 = 0$$

$$B_y = 10.8\ KN$$

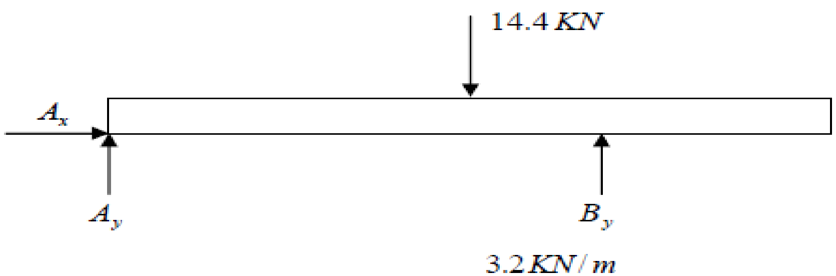
$$\sum F_y = 0$$

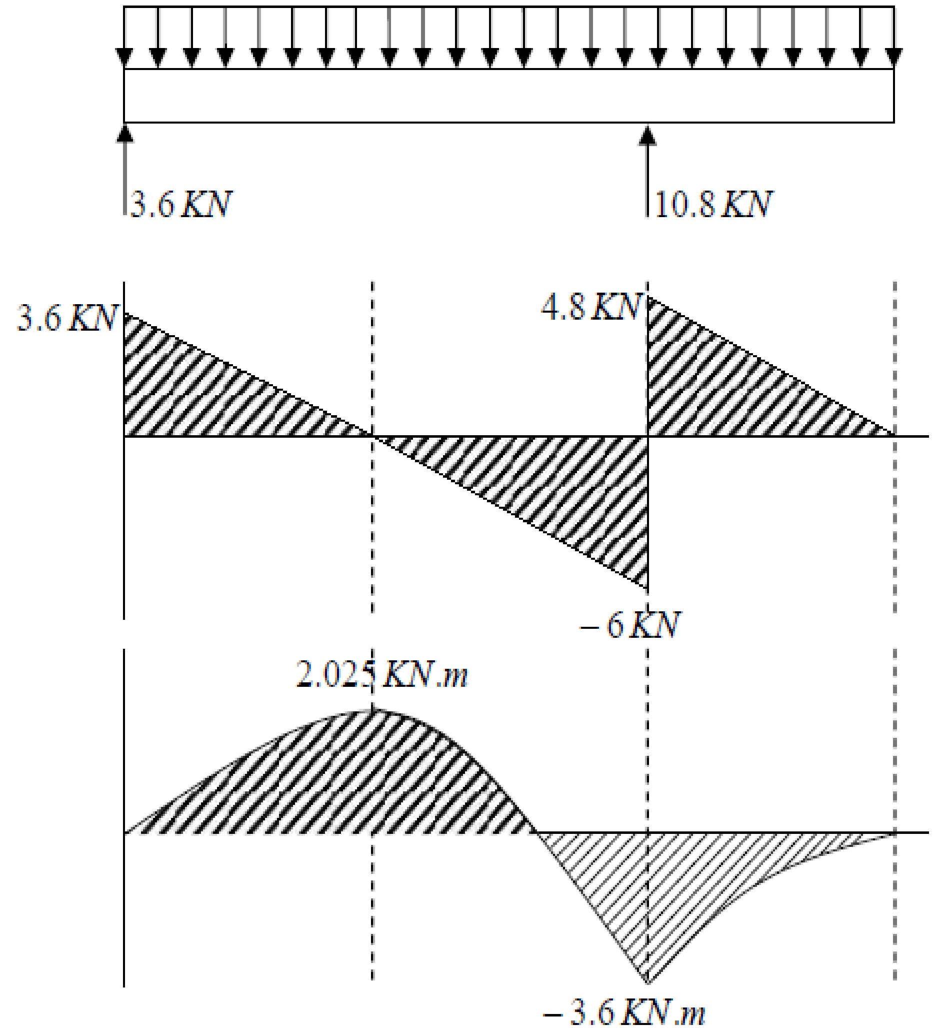
$$A_y + 10.8 - 14.4 = 0$$

$$A_y = 3.6\ KN$$

$$\sum F_x = 0$$

$$A_x = 0$$

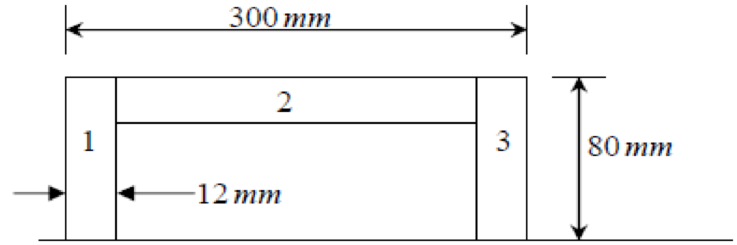




$M_1 = 2.025$ KN.m

$M_2 = 3.6$ KN.m

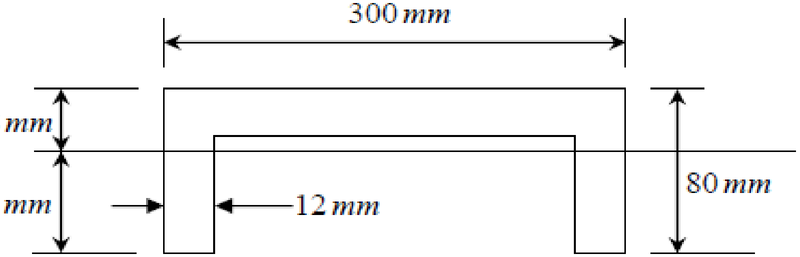
$$y_c = \frac{\sum \bar{y}A}{\sum A}$$



No. of Area	A(m²)	$\bar{y}$ (m)	$\bar{y}A$ (m³)
1	960×10^{-6}	40×10^{-3}	38400×10^{-9}
2	3312×10^{-6}	74×10^{-3}	245088×10^{-9}
3	960×10^{-6}	40×10^{-3}	38400×10^{-9}
	$\sum A=5232\times10^{-6}$		$\sum \bar{y}A=321888\times10^{-9}$

$$y_c = \frac{321888\times10^{-9}}{5232\times10^{-6}} = 61.52\times10^{-3}\ \text{m}$$

$y_c = 61.52$ mm



$$I_1 = \frac{bh^3}{12} + Ad^2$$

$$I_1 = \frac{12\times10^{-3}(80\times10^{-3})^3}{12} + 960\times10^{-6}\times(21.52\times10^{-3})^2 = 0.95658\times10^{-6}\ \text{m}^4$$

$$I_3 = I_1 = 0.95658\times10^{-6}\ \text{m}^4$$

$$I_2 = \frac{bh^3}{12} + Ad^2$$

$$I_2 = \frac{276\times10^{-3}(12\times10^{-3})^3}{12} + 3312\times10^{-6}\times(12.48\times10^{-3})^2 = 0.55558\times10^{-6}\ \text{m}^4$$

$$I = I_1 + I_2 + I_3 = 2.46874\times10^{-6}\ \text{m}^4$$

$$(\sigma_t)_1 = \frac{M_1 c_2}{I} = \frac{2.025\times10^3 \times 61.52\times10^{-3}}{2.46874\times10^{-6}} = 50.462179 \text{ MPa}$$

$$(\sigma_t)_2 = \frac{M_2 c_1}{I} = \frac{3.6\times10^3 \times 18.48\times10^{-3}}{2.46874\times10^{-6}} = 26.94815 \text{ MPa}$$

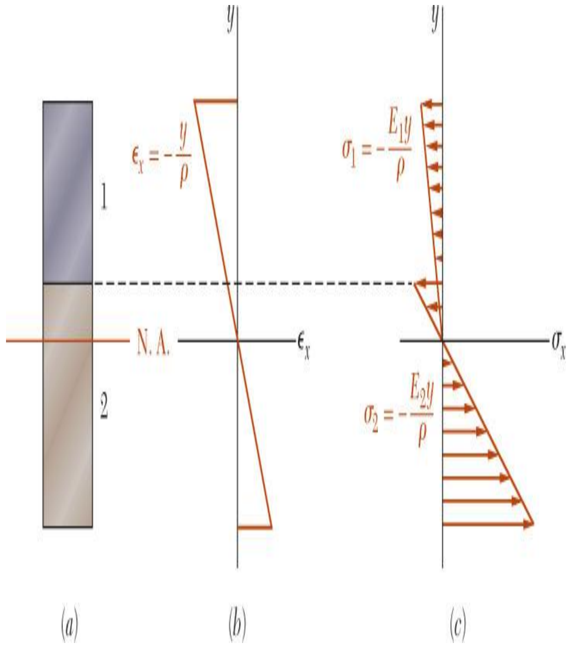
$(\sigma_t)_{max}$=50.462179 MPa

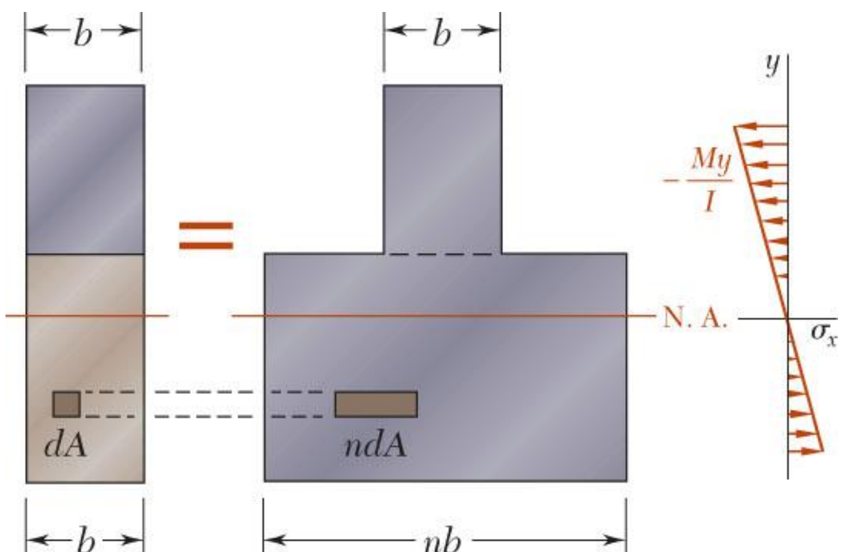
$$(\sigma_c)_1 = -\frac{M_1 c_1}{I} = -\frac{2.025\times10^3 \times 18.48\times10^{-3}}{2.46874\times10^{-6}} = -15.158339 \text{ MPa}$$

$$(\sigma_c)_2 = -\frac{M_2 c_2}{I} = -\frac{3.6\times10^3 \times 61.52\times10^{-3}}{2.46874\times10^{-6}} = -89.71054 \text{ MPa}$$

$(\sigma_c)_{max}$=-89.71054 MPa

Bending of Members Made of Several Materials





Consider a composite beam formed from two materials with E_1 and E_2.

Normal strain varies linearly.

$$\varepsilon_x = -\frac{y}{\rho}$$

Piecewise linear normal stress variation.

$$\sigma_1 = E_1\varepsilon_x = -\frac{E_1 y}{\rho} \qquad \sigma_2 = E_2\varepsilon_x = -\frac{E_2 y}{\rho}$$

Neutral axis does not pass through section centroid of composite section.

Elemental forces on the section are

$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \quad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

Define a transformed section such that

$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho}(n\,dA) \qquad n = \frac{E_2}{E_1}$$

Procedure

Assume E1 > E2 ,

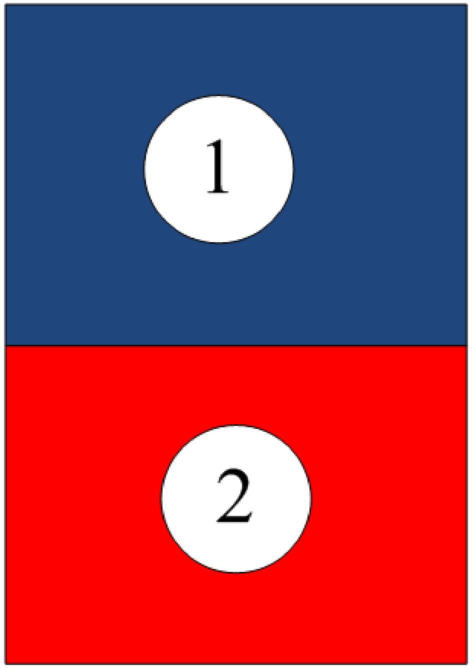
1- $n = \frac{E_1}{E_2}$

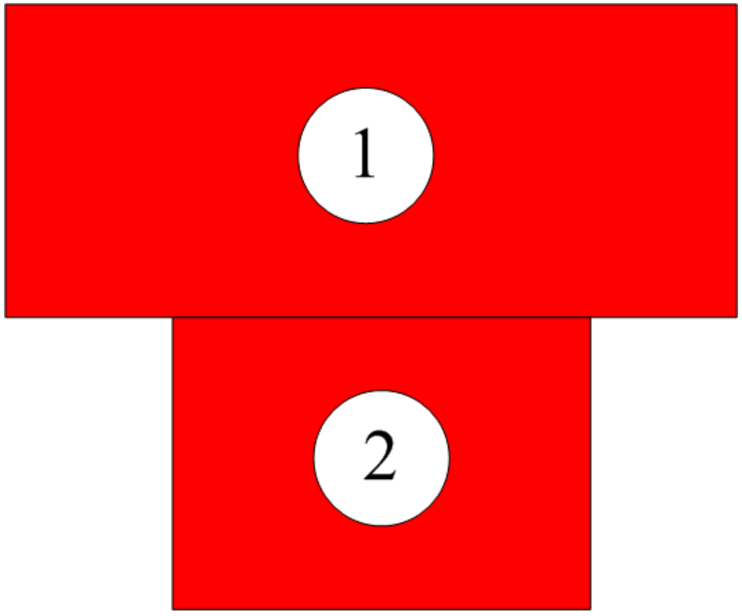
2- Multiply the width of material 1 by *n*.

3- Now consider all the section as made of material 2.

4- Find *I* and then the stresses at any point on the section.

5- the stress at any point located on material 1 should by multiplied by *n*.



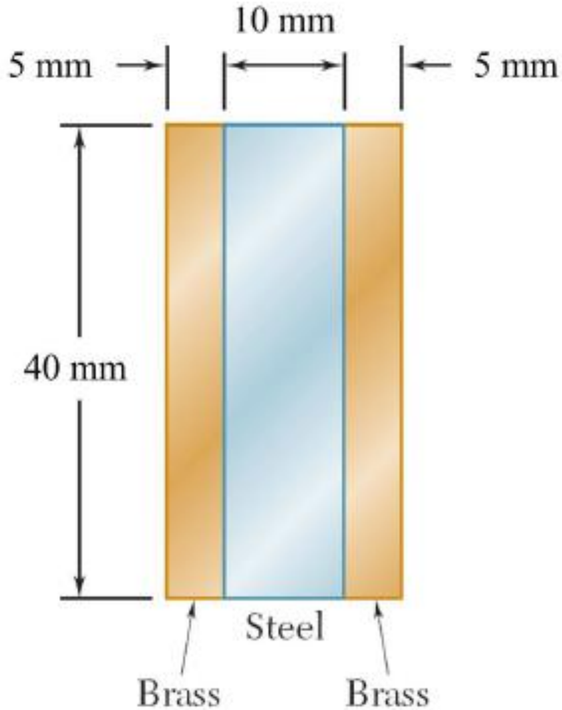


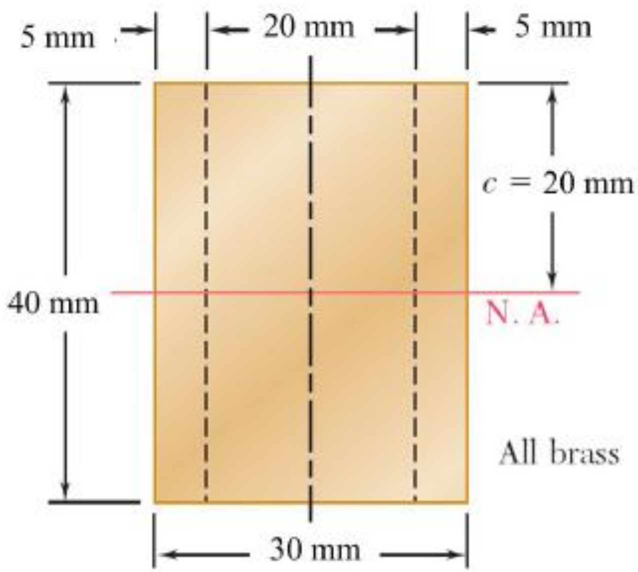
Example: find maximum stress in brass and steel

$\mathbf{M} = 2$ kN.m

$E_{br} = 100$ GPa

$E_{st} = 200$ GPa





$$n = \frac{200}{100} = 2$$

$$I = \frac{1}{12} \times 0.03 \times (0.04)^3 = 160 \times 10^{-9} \text{ m}^4$$

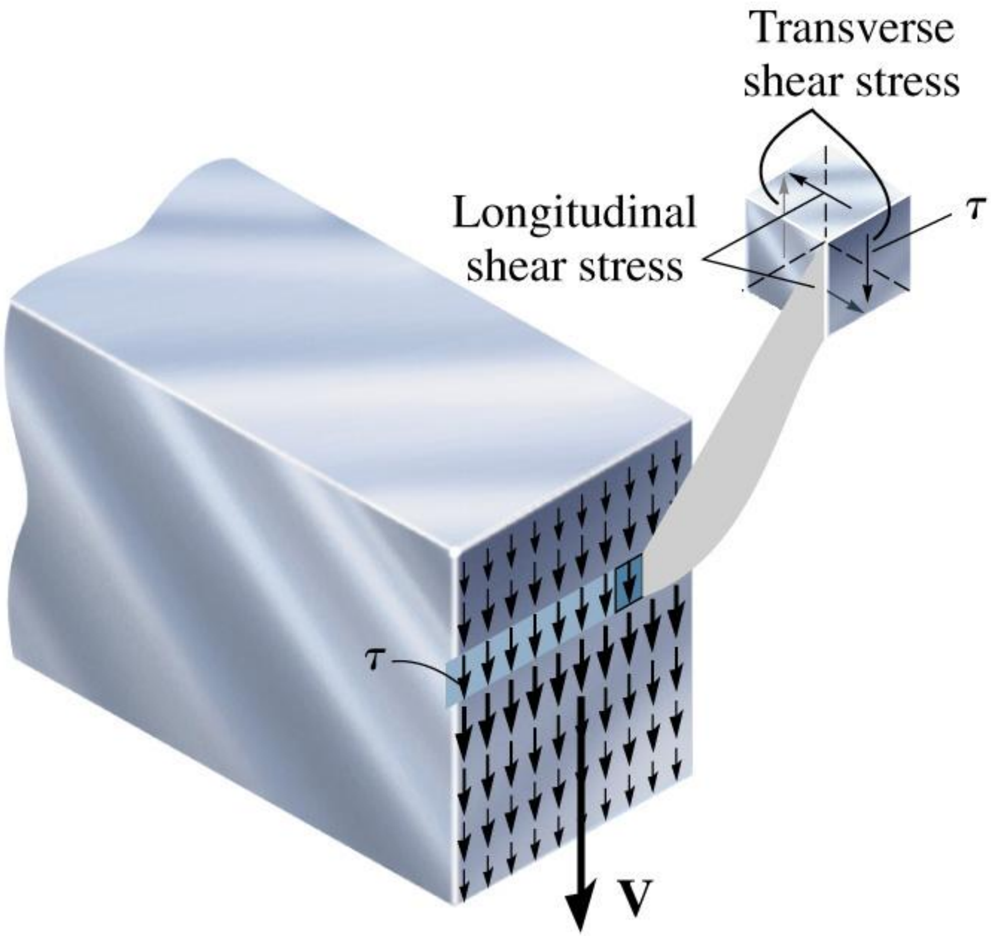
$$c = 0.02$$

$$\sigma_{br} = \frac{\mathbf{M} \cdot c}{I} = \frac{2 \times 10^3 \times 0.02}{160 \times 10^{-9}} = 250 \text{ MPa}$$

$$\sigma_{st} = n\frac{\mathbf{M} \cdot c}{I} = 2 \times \frac{2 \times 10^3 \times 0.02}{160 \times 10^{-9}} = 500 \text{ MPa}$$

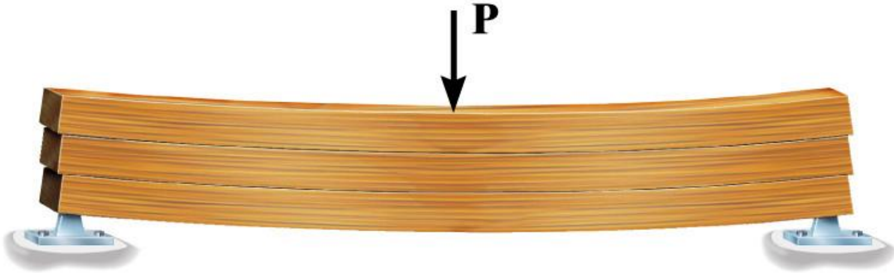
Shear stress in beam

7.1 Shear in Straight Members



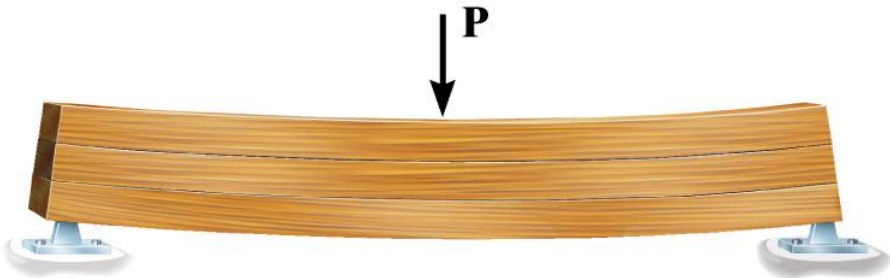
Internal shear force – creates shear deformation, strain and shear stress!

Note: due to nature of shear stress get transverse and longitudinal strain.



Boards not bonded together

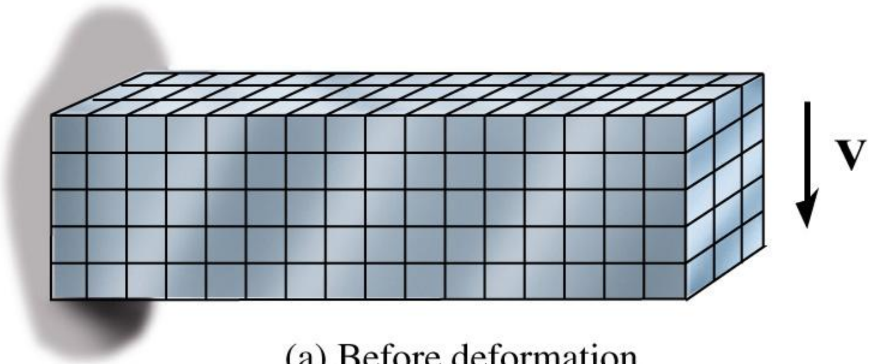
(a)



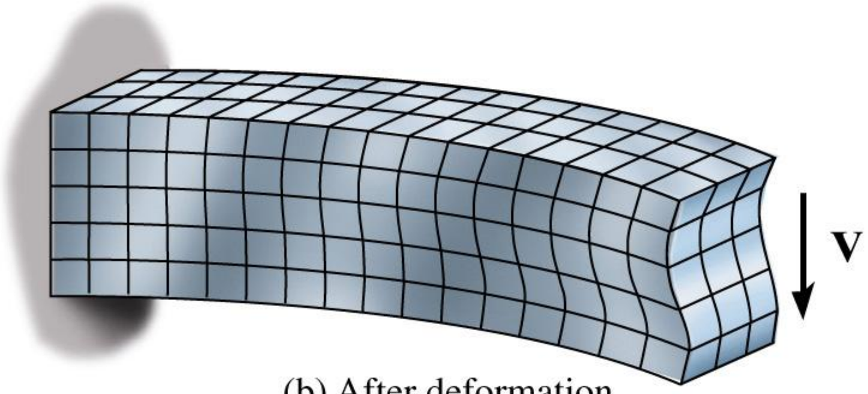
Boards bonded together

(b)

Physical example – when boards glued together, shear stress is developed at surfaces which prevents slippage.



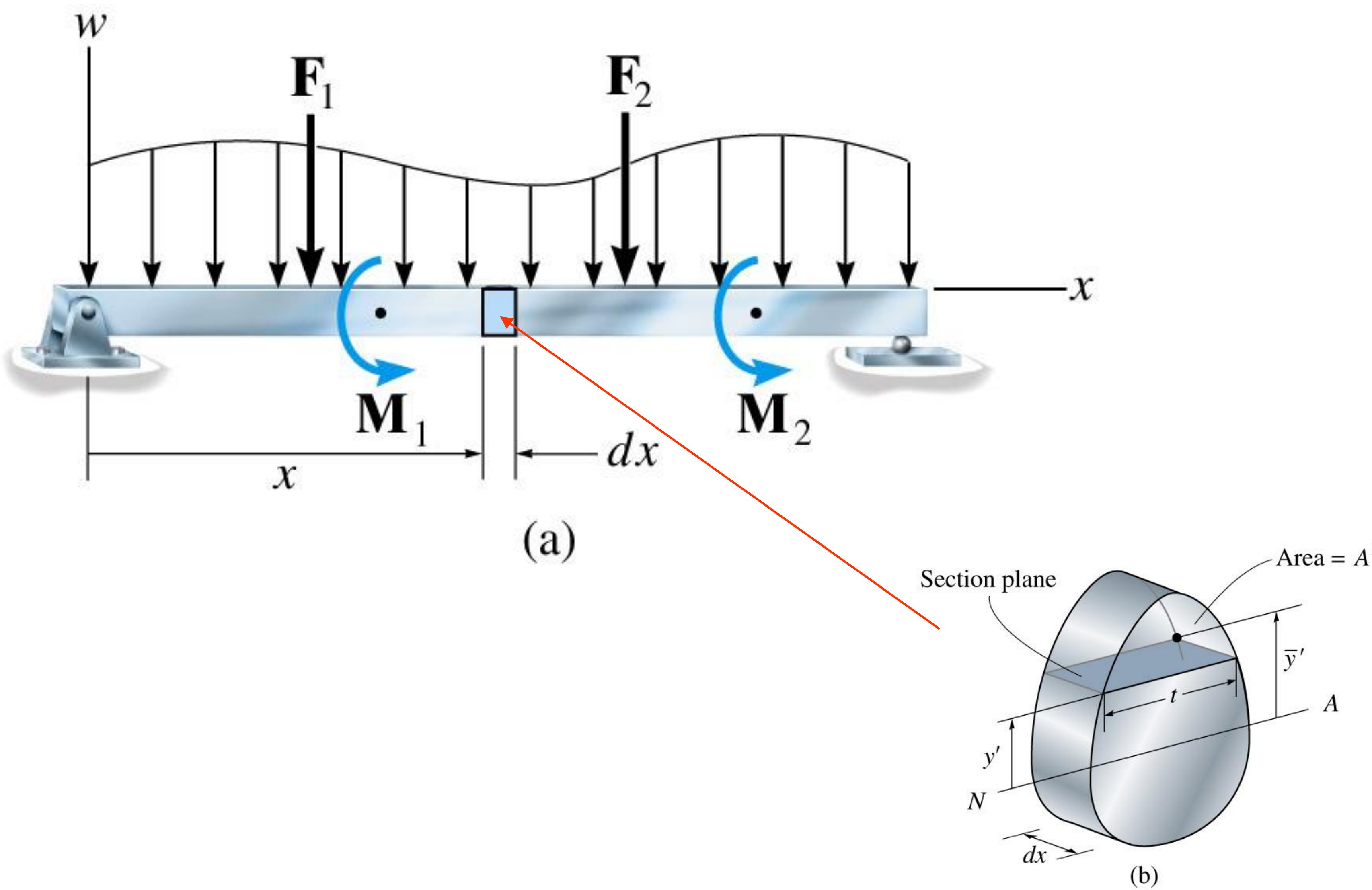
(a) Before deformation



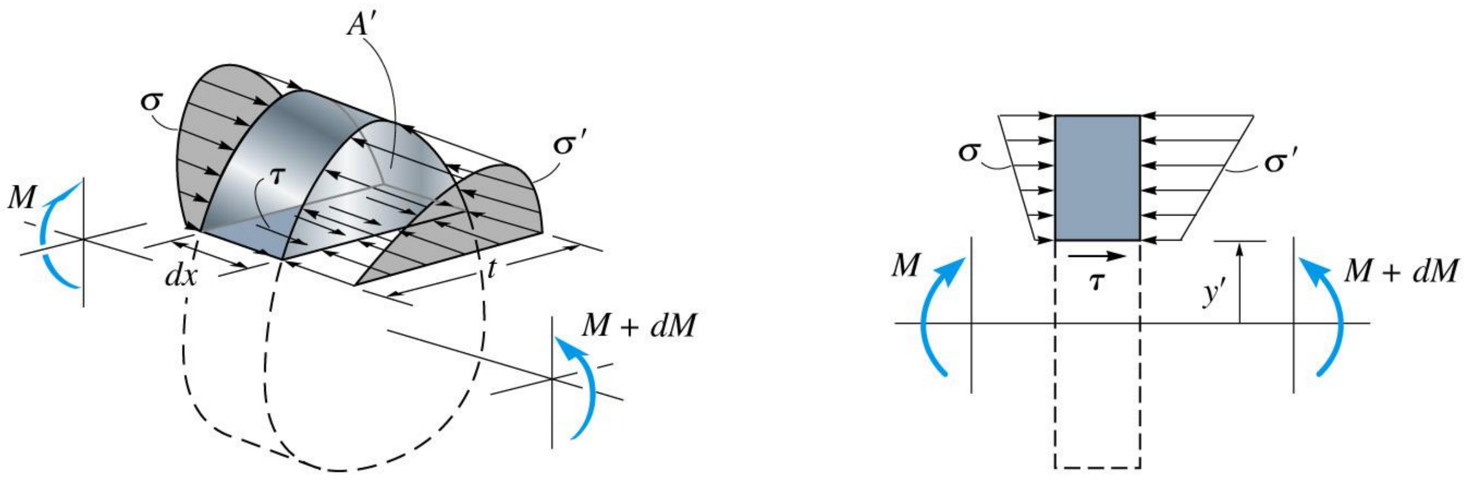
(b) After deformation

Notice deformation: key point, deformation not uniform!!

6.2 – Shear Stress Formula:



Derivation of Beam Shear Stress Equation:



Three-dimensional view (d) Profile view

$$\overset{+}{\leftarrow}\sum F_x = 0$$

$$\int_{A'} \sigma' dA - \int_{A'} \sigma dA - \tau(tdx) = 0$$

$$\int_{A'} \left(\frac{M + dM}{I}\right) y dA - \int_{A'} \left(\frac{M}{I}\right) y dA - \tau(tdx) = 0$$

Derivation of Beam Shear Stress Equation (cont'd):

$$\tau = \frac{1}{It}\left(\frac{dM}{dx}\right)\int_{A'} y\,dA$$

Recall, dM/dx = V

$\int_{A'} y\,dA$ = Q

$$\tau = \frac{VQ}{It}$$

V: Internal Shear (lb)

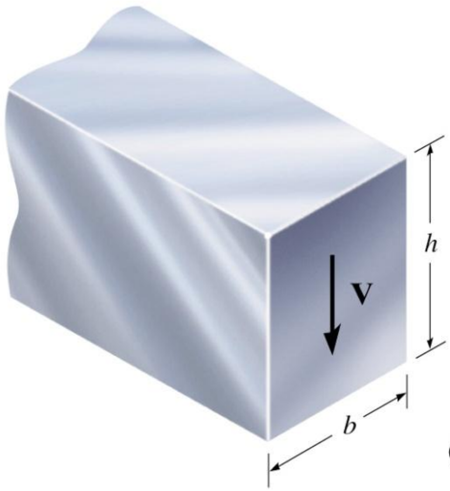
Q: First Moment of area (in^3) at point of interest

I: Moment of inertia of entire cross section (in^4)

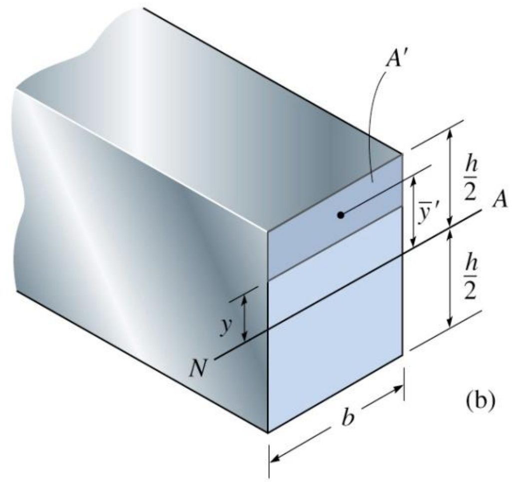
t: Thickness of cross-section at point of interest (in)

$$Q = \bar{y}' \cdot A'$$

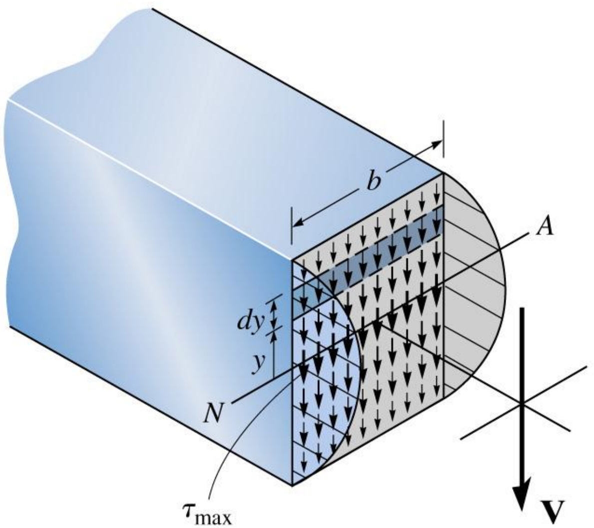
Example: Square Cross-section:



(a)

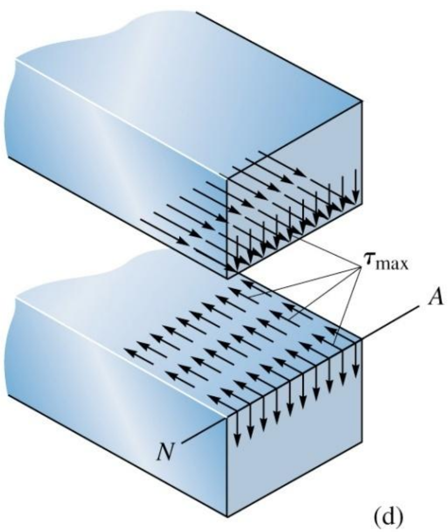


(b)



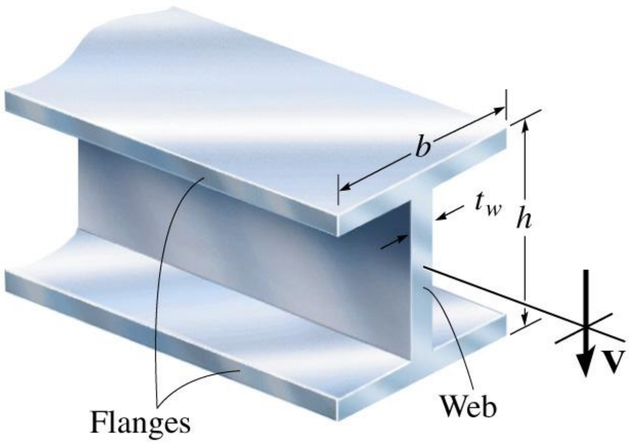
Shear–stress distribution

(c)

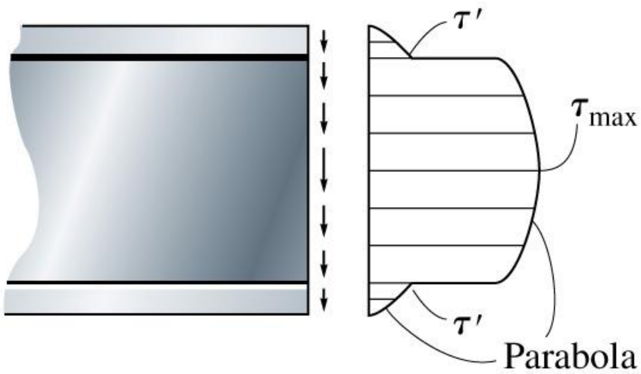


(d)

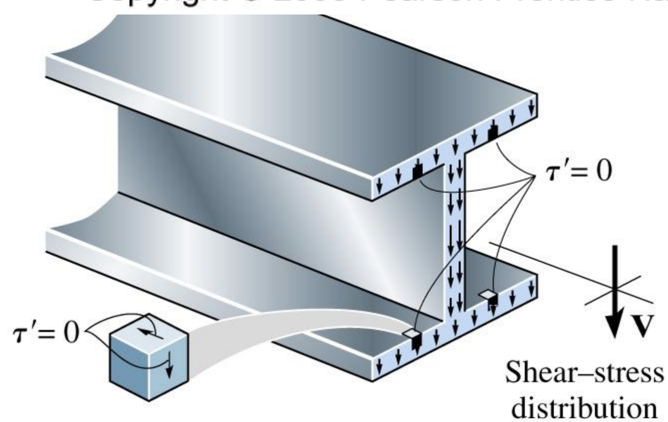
Example: I-Beam



(a)



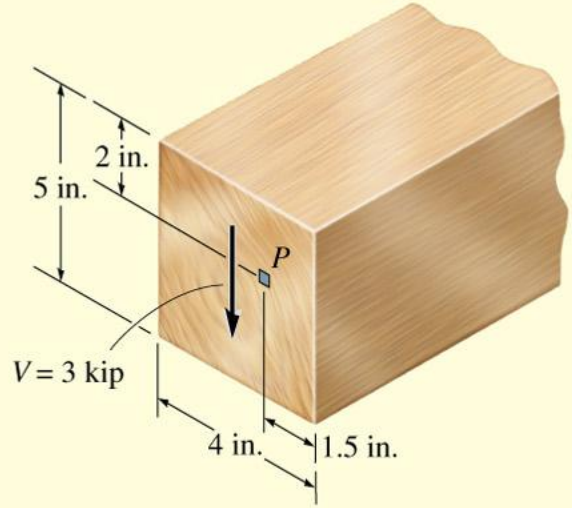
(c)



(b)

EXAMPLE 7.1

The beam shown in Fig. 7–10*a* is made of wood and is subjected to a resultant internal vertical shear force of $V = 3$ kip. (a) Determine the shear stress in the beam at point P, and (b) compute the maximum shear stress in the beam.



(a)

Solution

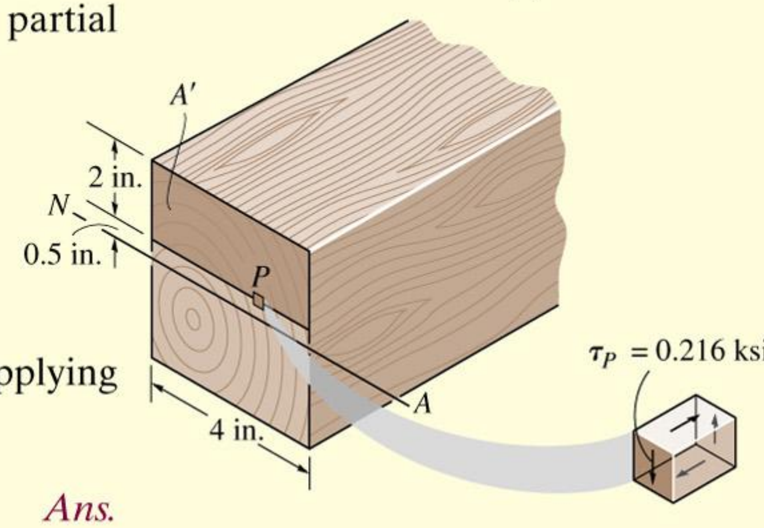
Part (a).

Section Properties. The moment of inertia of the cross-sectional area computed about the neutral axis is

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(4\text{ in.})(5\text{ in.})^3 = 41.7\text{ in}^4$$

A horizontal section line is drawn through point P and the partial area A' is shown shaded in Fig. 7–10*b*. Hence

$$Q = \bar{y}'A' = \left[0.5\text{ in.} + \frac{1}{2}(2\text{ in.})\right](2\text{ in.})(4\text{ in.}) = 12\text{ in}^3$$



(b) (c)

Shear Stress. The shear force at the section is $V = 3$ kip. Applying the shear formula, we have

$$\tau_P = \frac{VQ}{It} = \frac{(3\text{ kip})(12\text{ in}^3)}{(41.7\text{ in}^4)(4\text{ in.})} = 0.216\text{ ksi} \qquad \textit{Ans.}$$

Since τ_P contributes to V, it acts downward at P on the cross section. Consequently, a volume element of the material at this point would have shear stresses acting on it as shown in Fig. 7–10*c*.

Part (b).

Section Properties. Maximum shear stress occurs at the neutral axis, since t is constant throughout the cross section and Q is largest for this case. For the dark shaded area A' in Fig. 7–10*d*, we have

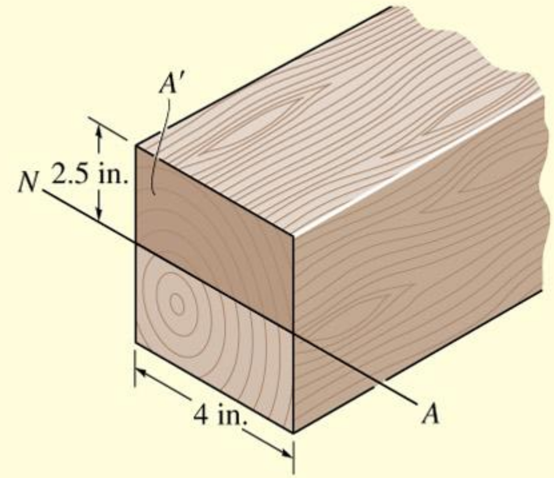
$$Q = \overline{y}'A' = \left[\frac{2.5 \text{ in.}}{2}\right](4 \text{ in.})(2.5 \text{ in.}) = 12.5 \text{ in}^3$$

Shear Stress. Applying the shear formula yields

$$\tau_{\max} = \frac{VQ}{It} = \frac{(3 \text{ kip})(12.5 \text{ in}^3)}{(41.7 \text{ in}^4)(4 \text{ in.})} = 0.225 \text{ ksi} \qquad \textit{Ans.}$$

Note that this is equivalent to

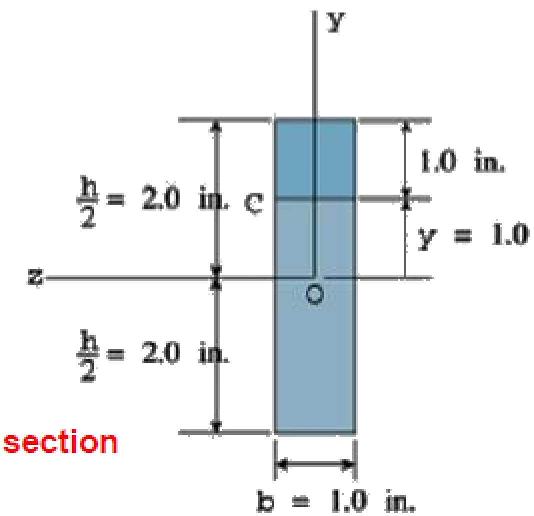
$$\tau_{\max} = 1.5\frac{V}{A} = 1.5\frac{3 \text{ kip}}{(4 \text{ in.})(5 \text{ in.})} = 0.225 \text{ ksi} \qquad \textit{Ans.}$$

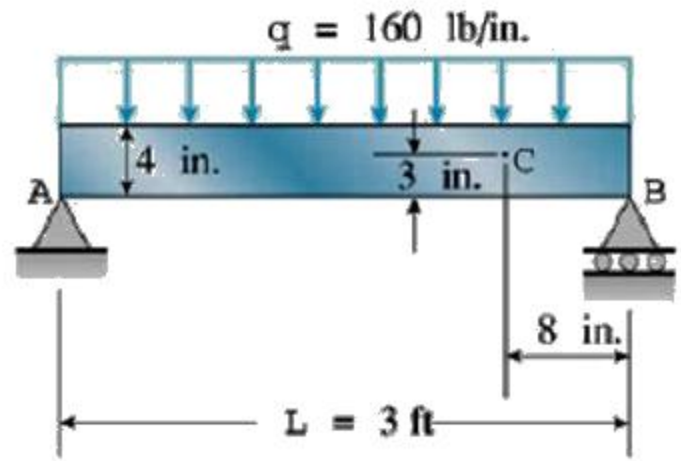


(d)

Fig. 7–10

Example: Determine the normal and shear stresses at Point C





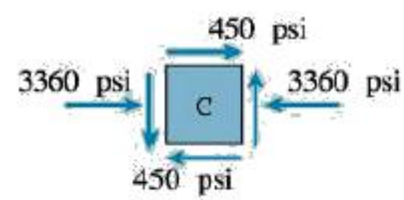
$$I = \frac{bh^3}{12} = 5.333\ \text{in}^4$$

$$Q_C = A_C \bar{y}_C$$
$$= (1.0)(1.5) = 1.5\ \text{in}^3$$

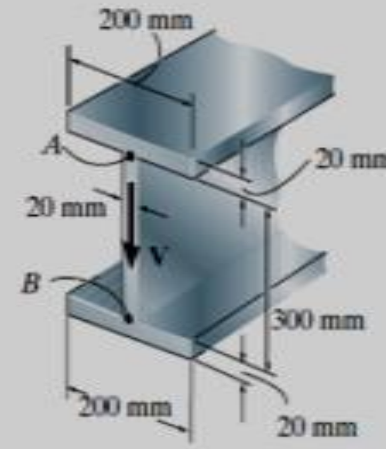
$$M_C = 17{,}920\ \text{lb-in}$$
$$V_C = -1{,}600\ \text{lb}$$

$$\sigma_C = -\frac{M_C y_C}{I}$$
$$= -\frac{(17{,}920)(1.0)}{5.333} = -3{,}360\ \text{psi}$$

$$\tau_C = \frac{V_C Q_C}{Ib} = \frac{(1{,}600)(1.5)}{(5.333)(1.0)} = -450\ \text{psi}$$



7–2. If the wide-flange beam is subjected to a shear of $V = 20$ kN, determine the maximum shear stress in the beam.



The moment of inertia of the cross-section about the neutral axis is

$$I = \frac{1}{12}(0.2)(0.34^3) - \frac{1}{12}(0.18)(0.3^3) = 0.2501(10^{-3})\ \text{m}^4$$

From Fig. *a*.

$$Q_{max} = \Sigma \bar{y}' A' = 0.16\,(0.02)(0.2) + 0.075\,(0.15)(0.02) = 0.865(10^{-3})\ \text{m}^3$$

The maximum shear stress occurs at the points along neutral axis since Q is maximum and thickness t is the smallest.

$$\tau_{max} = \frac{VQ_{max}}{It} = \frac{20(10^3)\,[0.865(10^{-3})]}{0.2501(10^{-3})\,(0.02)}$$

$$= 3.459(10^6)\ \text{Pa} = 3.46\ \text{MPa} \qquad \textbf{Ans.}$$

Example:

Find the shear stress distribution over the cross-section.

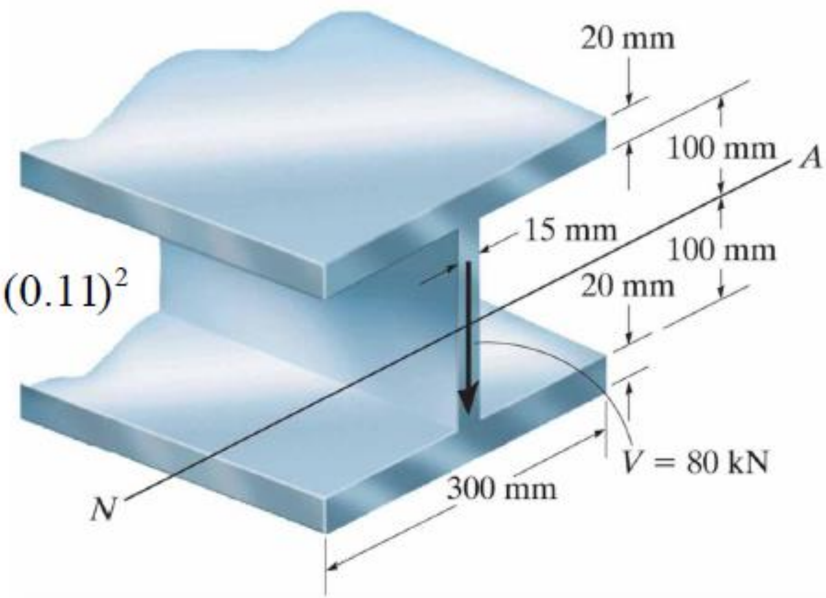
Solution :

$\mathbf{V} = 80\ \text{kN}$

$$I_1 = \frac{1}{12}\times 0.015\times(0.2)^3$$

$$I_2 = \frac{1}{12}\times 0.3\times(0.02)^3 + 0.3\times 0.02\times(0.11)^2$$

$$I = I_1 + 2I_2 = 155.6\times 10^{-6}\ \text{m}^4$$



$t_A = 0.3,$

$Q_A = 0$

$t_{B'} = 0.3 \text{ m}$

$Q_{B'} = 0.3 \times 0.02 \times 0.11 = 660 \times 10^{-6} \text{ m}^3$

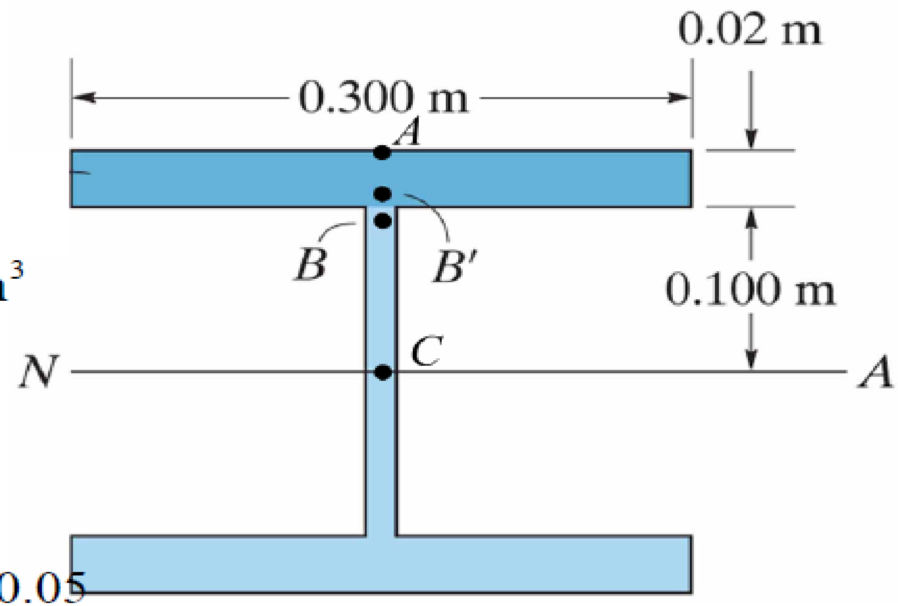
$t_B = 0.015 \text{ m}$

$Q_B = Q_{B'} = 660 \times 10^{-6} \text{ m}^3$

$t_C = 0.015 \text{ m}$

$Q_C = 0.3 \times 0.02 \times 0.11 + 0.015 \times 0.1 \times 0.05$

$= 735 \times 10^{-6} \text{ m}^3$



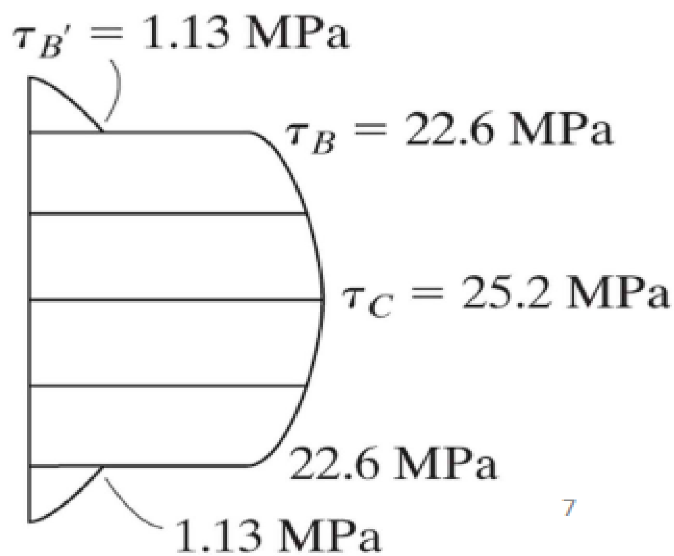
apply the shear formula, you get

$\tau_A = 0$

$\tau_{B'} = 1.13 \text{ MPa}$

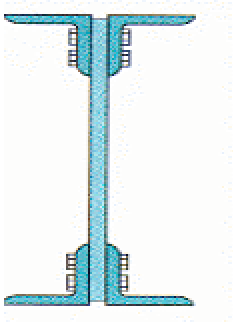
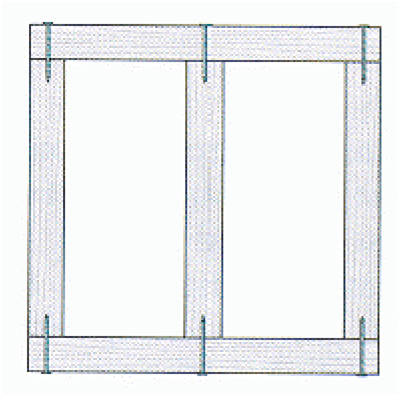
$\tau_B = 22.6 \text{ MPa}$

$\tau_C = 25.2 \text{ MPa}$

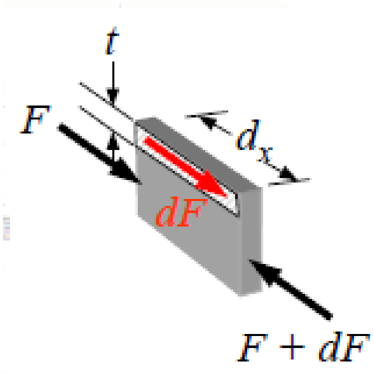


SHEAR FLOW IN BUILT-UP MEMBERS

Built-Up Beams

In many applications, beam sections consist of several pieces of material that are attached together in a number ways: bolts, rivets, nails, glue, weld, etc. In such so called built-up sections we are interested in knowing the amount of shear stress and the resulting shear force at the cross section of fasteners or over the glued surface .



$$dF = \frac{dM}{I} \int_{A'} y dA$$

$$q = \frac{dF}{dx} = \frac{dM}{I dx} \int_{A'} y dA$$

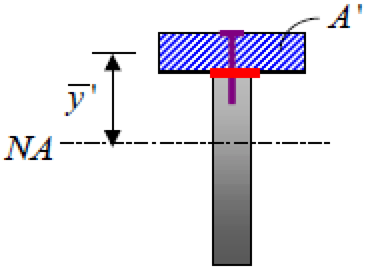
V

Q

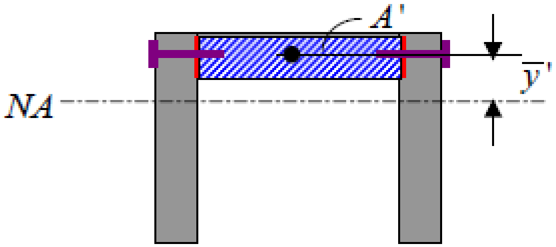
$$q = \frac{VQ}{I} = \tau t$$

$$F_{nail} = q\ s$$

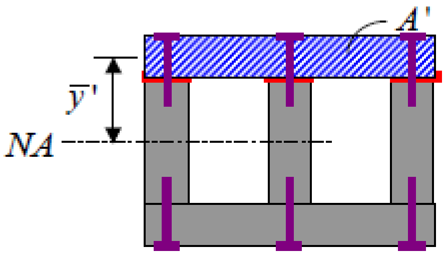
(a) The shear flow is resisted at *one surfaces*



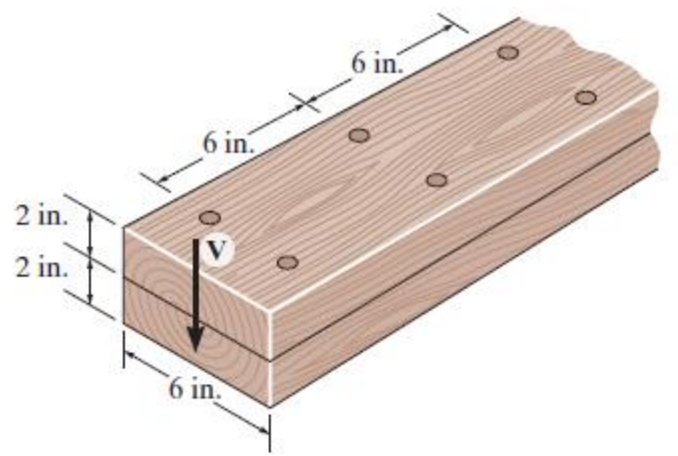
(b) The shear flow is resisted at *two surfaces*



c) The shear flow is resisted at *three surfaces*

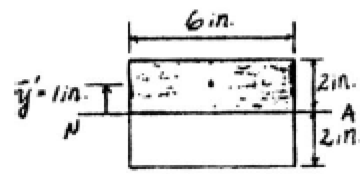


- Example: The beam is constructed from two boards fastened together at the top and bottom with two rows of nails spaced every 6 in. If an internal shear force of is applied to the boards, determine the shear force resisted by each nail.



Section Properties:

$$I = \frac{1}{12}(6)\left(4^3\right) = 32.0 \text{ in}^4$$



$$Q = \overline{y}' A' = 1(6)(2) = 12.0 \text{ in}^4$$

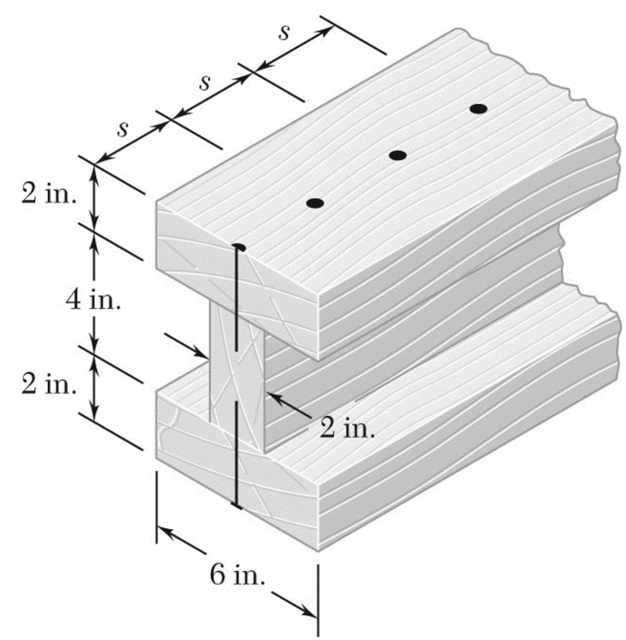
Shear Flow:

$$q = \frac{VQ}{I} = \frac{600(12.0)}{32.0} = 225 \text{ lb/in.}$$

There are two rows of nails. Hence, the shear force resisted by each nail is

$$F = \left(\frac{q}{2}\right)s = \left(\frac{225 \text{ lb/in.}}{2}\right)(6 \text{ in.}) = 675 \text{ lb} \qquad \textbf{Ans.}$$

Example:Three boards, each 2 in. thick, are nailed together to form a beam that is subjected to a vertical shear. Knowing that the allowable shearing force in each nail is 150 lb, determine the allowable shear if the spacing *s between the nails is 3 in*



SOLUTION

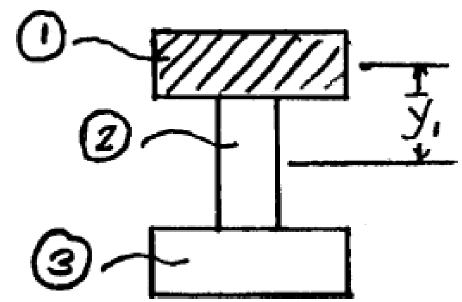
$$I_1 = \frac{1}{12}bh^3 + Ad^2$$

$$= \frac{1}{12}(6)(2)^3 + (6)(2)(3)^2 = 112 \text{ in}^4$$

$$I_2 = \frac{1}{12}bh^3 = \frac{1}{12}(2)(4)^3 = 10.667 \text{ in}^4$$

$$I_3 = I_1 = 112 \text{ in}^4$$

$$I = I_1 + I_2 + I_3 = 234.667 \text{ in}^4$$



$$Q = A_1\bar{y}_1 = (6)(2)(3) = 36 \text{ in}^3$$

$$qs = F_{\text{nail}} \quad (1)$$

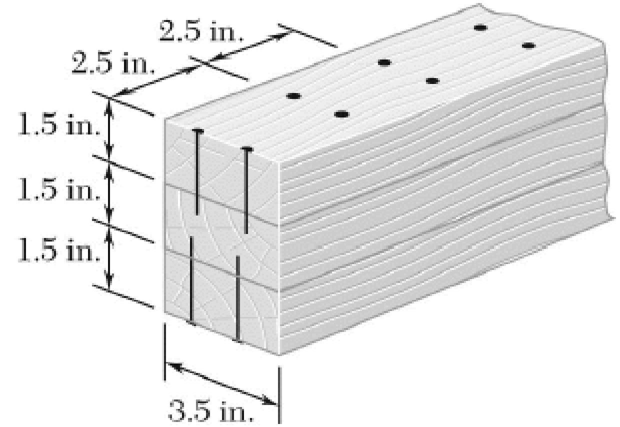
$$q = \frac{VQ}{I} \quad (2)$$

Dividing Eq. (2) by Eq. (1), $$\frac{1}{s} = \frac{VQ}{F_{\text{nail}}I}$$

$$V = \frac{F_{\text{nail}}I}{Qs} = \frac{(150)(234.667)}{(36)(3)}$$

$V = 326$ lb ◀

Three boards, each of 1.5 × 3.5-in. rectangular cross section, are nailed together to form a beam that is subjected to a vertical shear of 250 lb. Knowing that the spacing between each pair of nails is 2.5 in., determine the shearing force in each nail.



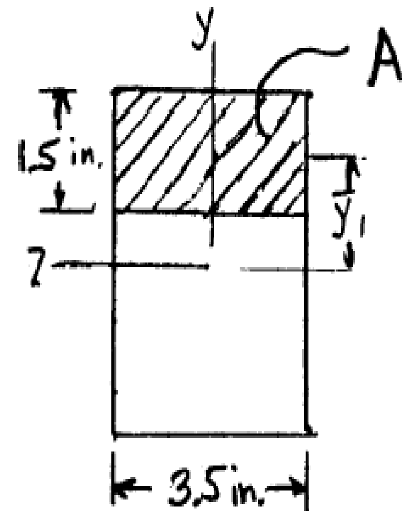
$$I = \frac{1}{12}bh^3 = \frac{1}{12}(3.5)(4.5)^3 = 26.578 \text{ in}^4$$

$$A = (3.5)(1.5) = 5.25 \text{ in}^2$$

$$\bar{y}_1 = 1.5 \text{ in.}$$

$$Q = A\bar{y}_1 = 7.875 \text{ in}^3$$

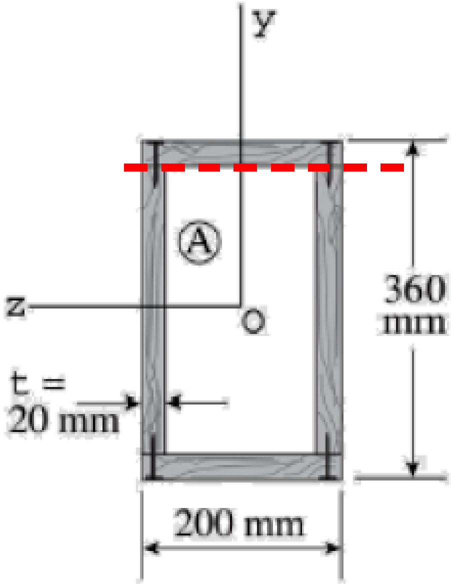
$$q = \frac{VQ}{I} = \frac{(250)(7.875)}{26.578} = 74.074 \text{ lb/in}$$

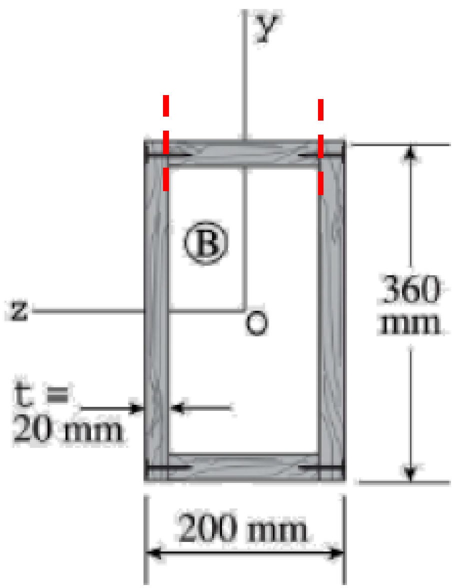


$$qs = 2F_{\text{nail}} \qquad F_{\text{nail}} = \frac{qs}{2} = \frac{(74.074)(2.5)}{2}$$

$F_{\text{nail}} = 92.6 \text{ lb}$ ◀

Find the spacing for each case





$V = 3.2\text{ kN} \qquad F = 250\text{ N}$

$$I = \frac{(200)(360)^3}{12} - \frac{(160)(320)^3}{12} = 340.69 \times 10^6\text{ mm}^4$$

$$Q = (200)(20)(180 - 10) = 680 \times 10^3\text{ mm}^3$$

$$s = \frac{2FI}{VQ} = 78.3\text{ mm}$$

$$Q = (160)(20)(180 - 10) = 544 \times 10^3\text{ mm}^3$$

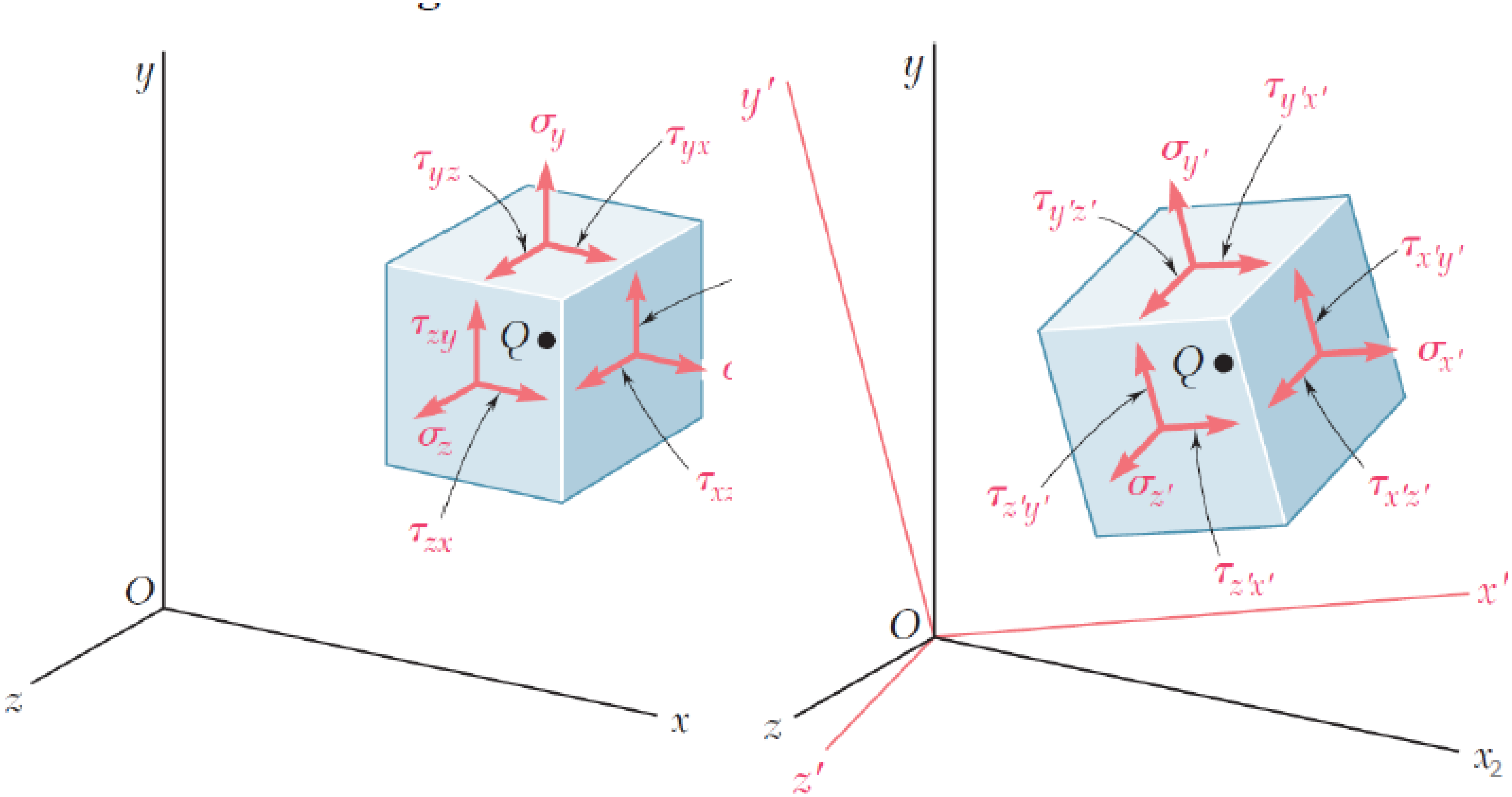
$$s = \frac{2FI}{VQ} = 97.9\text{ mm}$$

Stress transformation

Introduction

- Failure can occur in any angle.

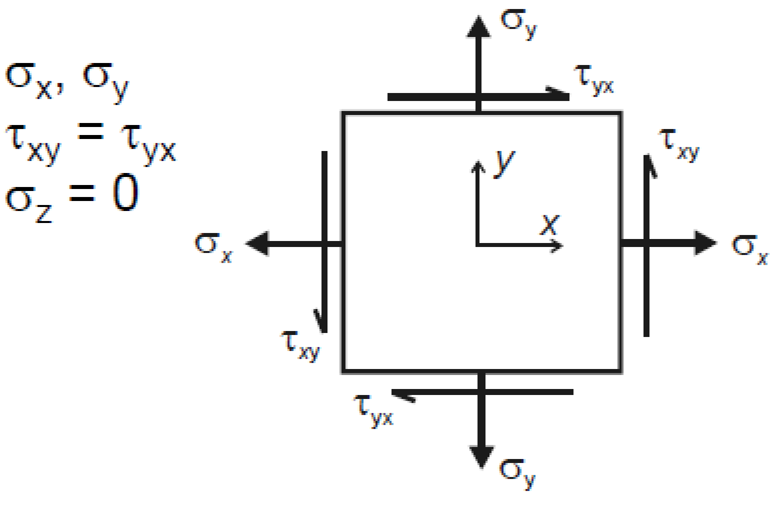
General loading condition is:



Plane stress

- When an element is in ***plane stress*** in the xy plane, only the x and y faces are subjected to stresses ($\sigma z = 0$ and $\tau zx = \tau xz = \tau zy = \tau yz = 0$).

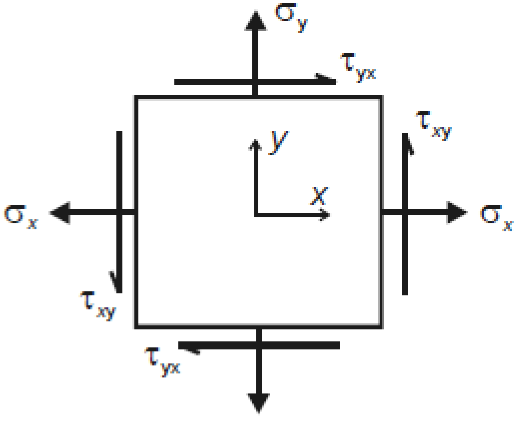
Plane stress element in 2D

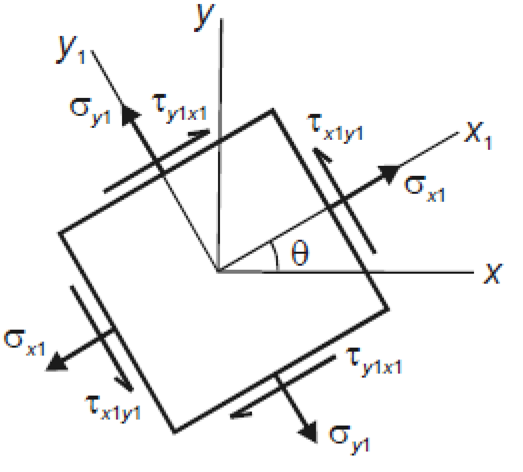


Stresses on Inclined Sections

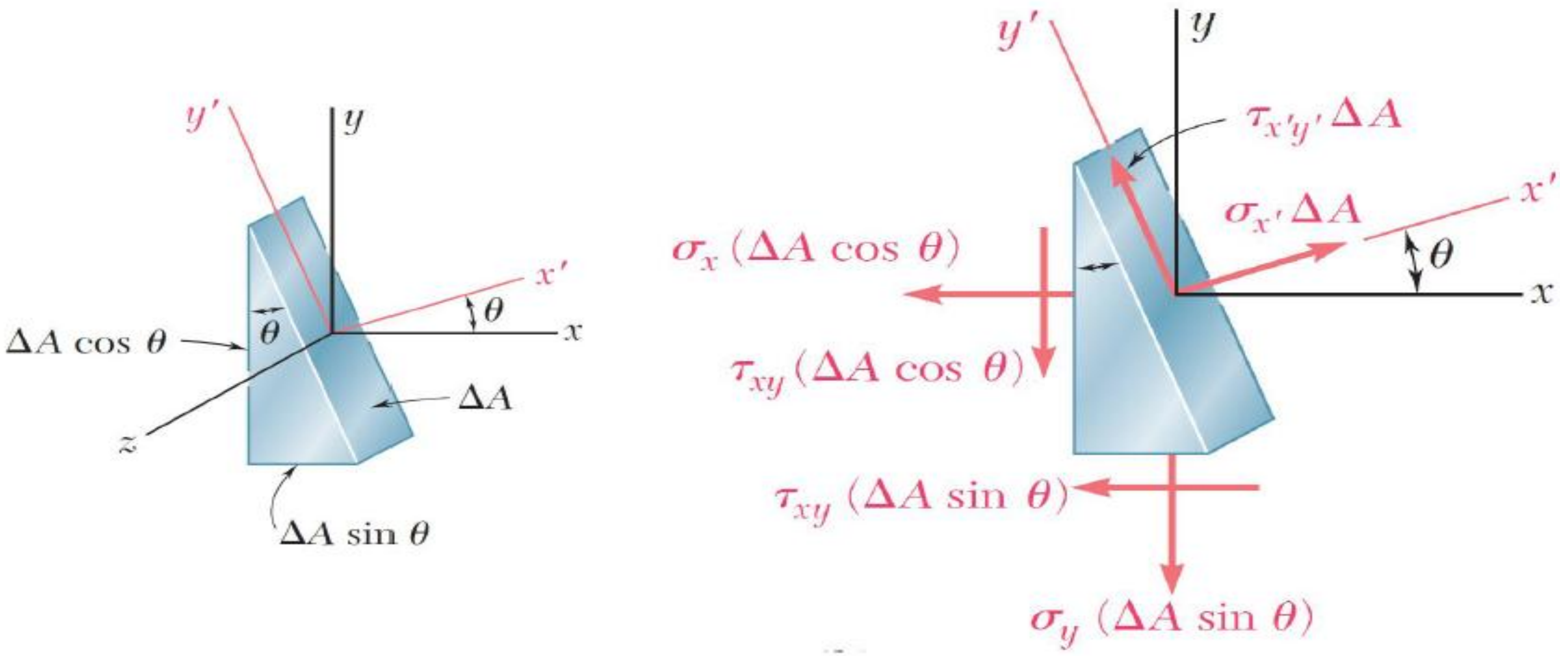
The stress system is known in terms of coordinate system *xy*. We want to find the stresses in terms of the rotated coordinate system X*1y1*.

Why? A material may yield or fail at the maximum value of σ or τ. This value may occur at some angle other than θ = 0.





Transformation Equations



$$\sum F_{x'} = 0: \ \sigma_{x'}\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta$$
$$-\sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta = 0$$

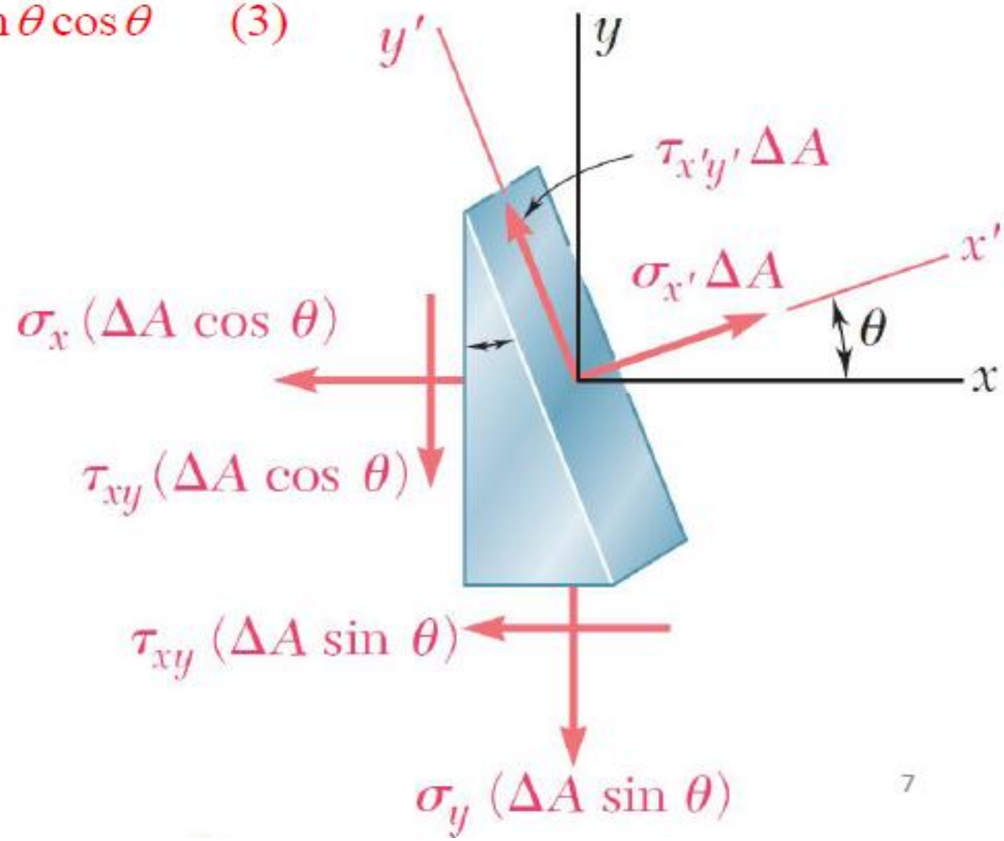
$$\sigma_{x'} = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \qquad (1)$$

$$\sum F_{y'} = 0: \ \tau_{x'y'}\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta$$
$$-\sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta = 0$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \quad (2)$$

in same manner, $\sigma_{y'}$ is obtained as

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \qquad (3)$$



Using the following trigonometric identities

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$

gives the transformation equations for plane stress :

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

For stresses on the y_1 face, substitute $\theta + 90^\circ$ for θ :

$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

TRANSFORMATION EQUATIONS SUMMARY

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta \qquad (1)$$

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta \qquad (2)$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \qquad (3)$$

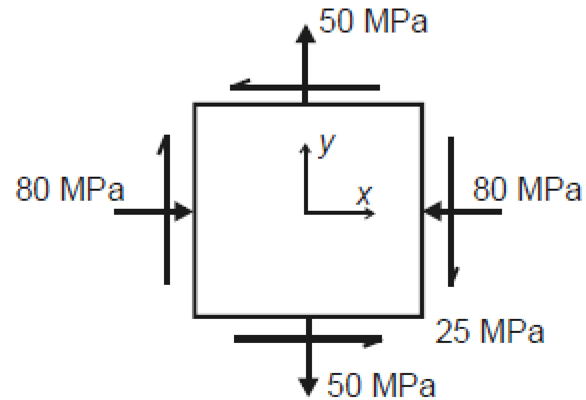
The equations can also be rewritten as:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (2)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (3)$$

Example: The state of plane stress at a point is represented by the stress element below. Determine the stresses acting on an element oriented 30° clockwise with respect to the original element.



Solution: Substitute numerical values into the transformation equations:

Define the stresses in terms of the established sign convention:

σ_x = -80 MPa σ_y = 50 MPa

τ_{xy} = -25 MPa

We need to find σ_{x1}, σ_{y1}, and τ_{x1y1} when θ = -30°.

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

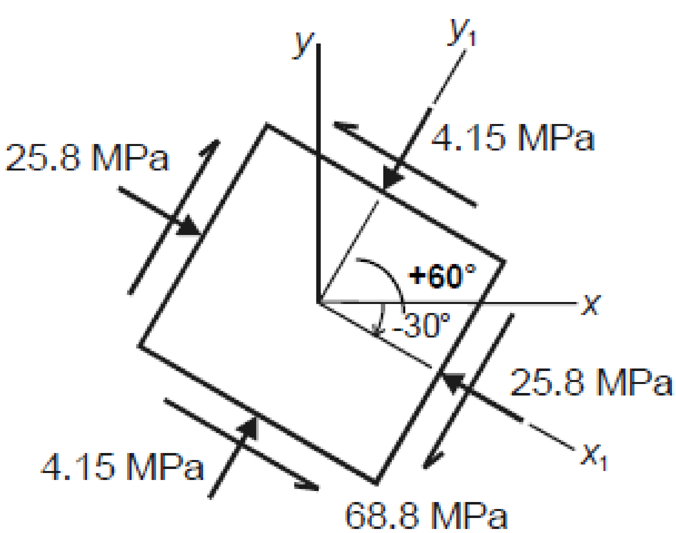
$$\sigma_{x1} = \frac{-80+50}{2} + \frac{-80-50}{2}\cos 2(-30°) + (-25)\sin 2(-30°) = -25.9\text{ MPa}$$

$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\sigma_{y1} = \frac{-80+50}{2} - \frac{-80-50}{2}\cos 2(-30°) - (-25)\sin 2(-30°) = -4.15 \text{ MPa}$$

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

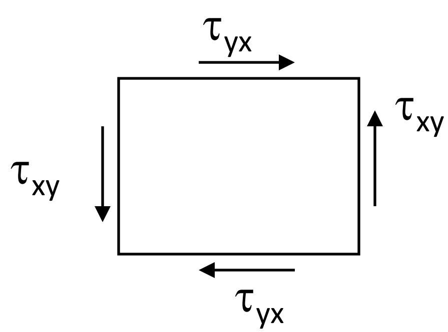
$$\tau_{x1y1} = -\frac{(-80-50)}{2}\sin 2(-30°) + (-25)\cos 2(-30°) = -68.8 \text{ MPa}$$



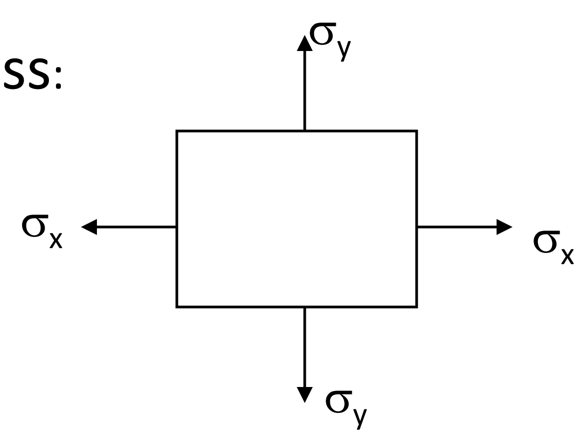
Plane Stress – Special Cases

Uniaxial Stress:

Pure Shear:



Biaxial Stress:



Principal Stresses

- The maximum and minimum normal stresses ($\sigma 1$ and $\sigma 2$) are known as the **principal stresses. To find the principal stresses,**we must differentiate the transformation equations

Principal Stresses

Principal stresses: maximum and minimum normal stresses.
Principal planes: the planes on which the principal stresses act

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\Downarrow$$

$$\frac{d\sigma_{x_1}}{d\theta} = -\frac{\sigma_x - \sigma_y}{2} 2\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\Downarrow$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

θ_p: The angle defines the orientation of the principal planes.

Principal Stresses

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \Rightarrow \cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}, \quad \sin 2\theta_p = \frac{\tau_{xy}}{2R}, \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cdot \frac{\sigma_x - \sigma_y}{2R} + \tau_{xy} \cdot \frac{\tau_{xy}}{2R}$$

$$\Downarrow$$

$$\boxed{\sigma_1 = \sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}}$$

OR

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \Rightarrow \cos 2\theta_p = -\frac{\sigma_x - \sigma_y}{2R}, \quad \sin 2\theta_p = -\frac{\tau_{xy}}{2R}, \quad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cdot \frac{-\sigma_x + \sigma_y}{2R} + \tau_{xy} \cdot \frac{-\tau_{xy}}{2R}$$

$$\Downarrow$$

$$\boxed{\sigma_2 = \sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}}$$

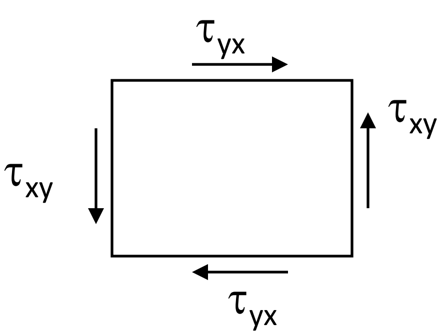
$$\sigma_1 \geq \sigma_2$$

Shear Stress

Shear stresses on the principal planes:

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta_p + \tau_{xy} \cos 2\theta_p = 0$$

Example 2: Principal stresses in pure shear case:



Maximum Shear Stresses

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \quad \Longrightarrow \quad \frac{d\tau_{x_1y_1}}{d\theta} = -\left(\sigma_x - \sigma_y\right)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\Longrightarrow \quad \tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \Rightarrow \tan 2\theta_s = -\frac{1}{\tan 2\theta_p}$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\theta_{s_1} = \theta_{p_1} - \frac{\pi}{4}$$

$$\theta_{s_2} = \theta_{p_1} + \frac{\pi}{4}$$

MOHR'S CIRCLE: PLANE STRESS

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$$

square both equations and sum them, we will get

$$\left[\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right]^2 + \left[\tau_{x'y'}\right]^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2 \qquad (*)$$

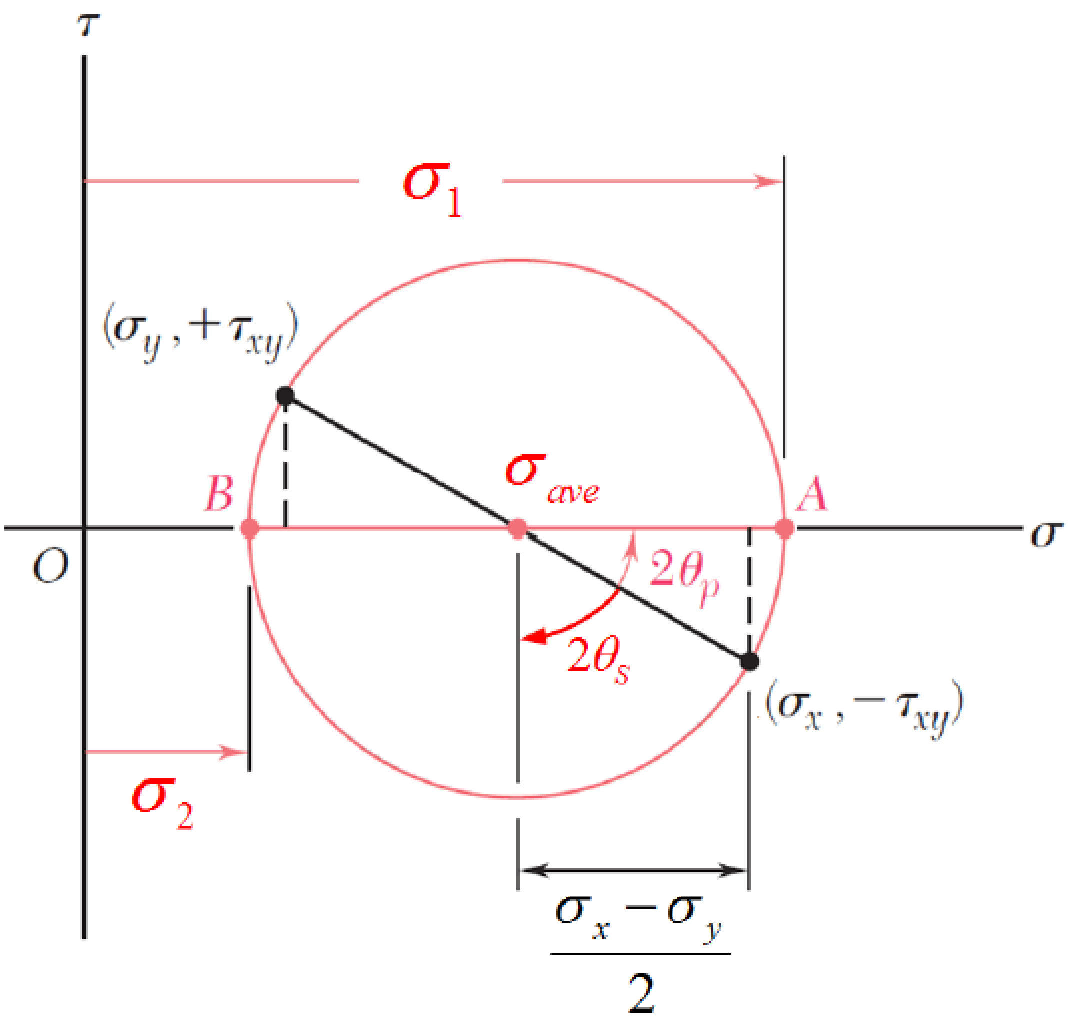
let

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

then, * will be rewritten as

$$\left(\sigma_{x'} - \sigma_{ave}\right)^2 + \left(\tau_{x'y'}\right)^2 = R^2 \qquad \text{(Equation of a circle)}$$



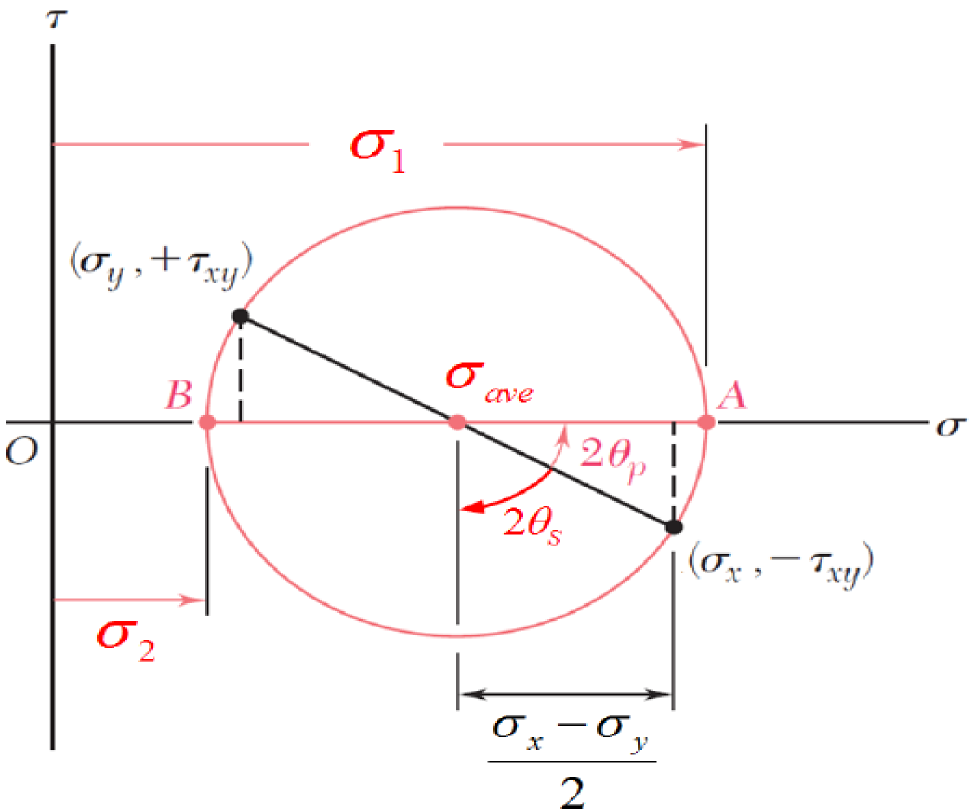
center $= (\sigma_{ave}, 0)$

points on the circle

$(\sigma_x, -\tau_{xy})$

(σ_y, τ_{xy})

- At the stress orientation represented by the black line; if you rotate the element ccw by θ_P you will get the principal stresses.
- If you rotate cw by θ_S you will get the maximum shear



center $= (\sigma_{ave}, 0)$

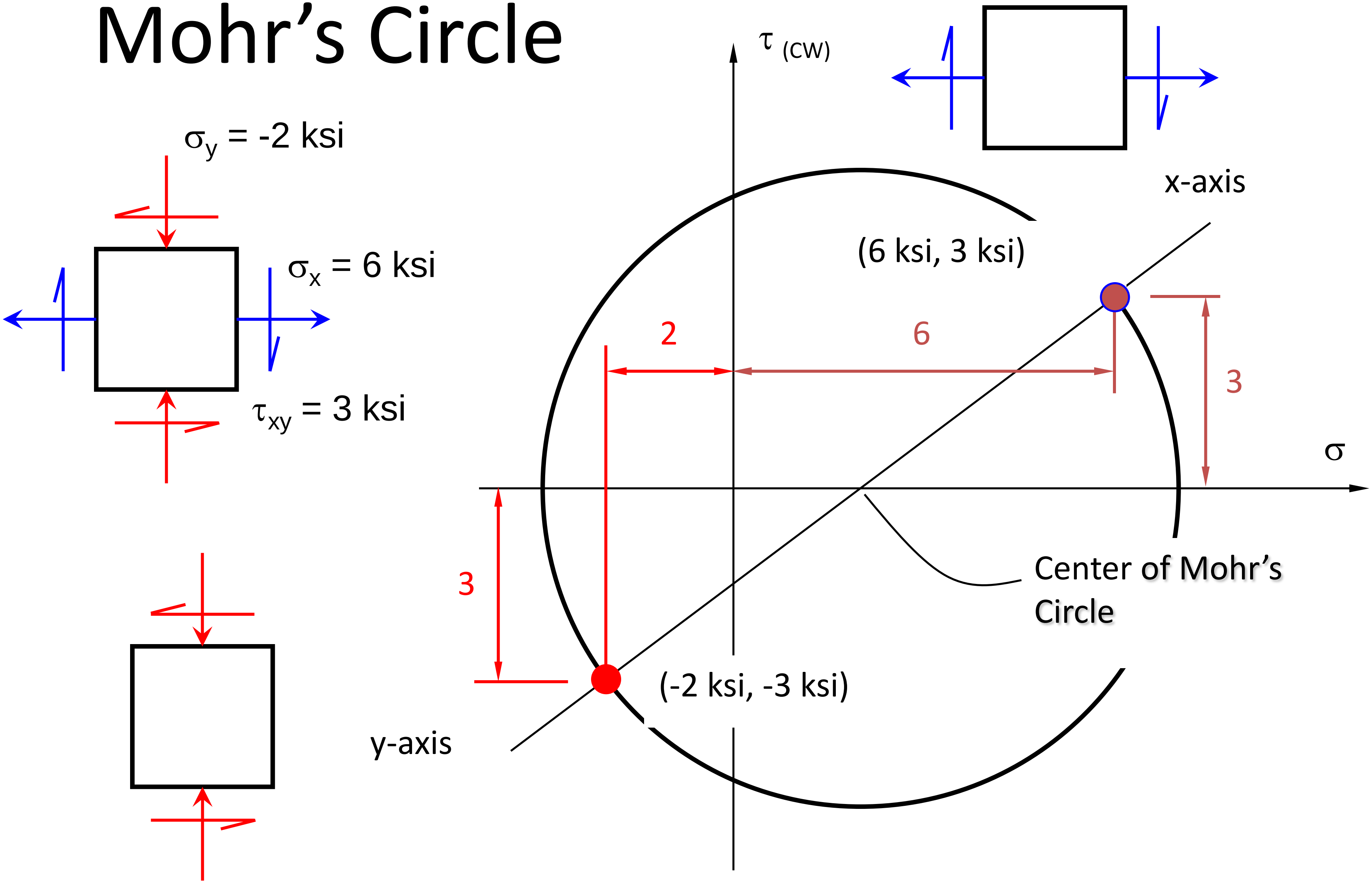
points on the circle

$(\sigma_x, -\tau_{xy})$

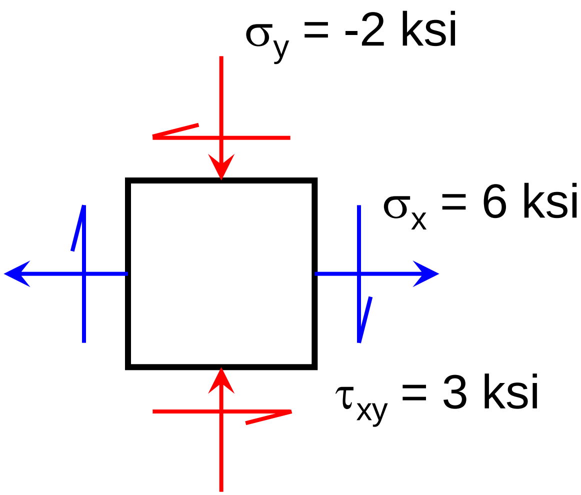
(σ_y, τ_{xy})

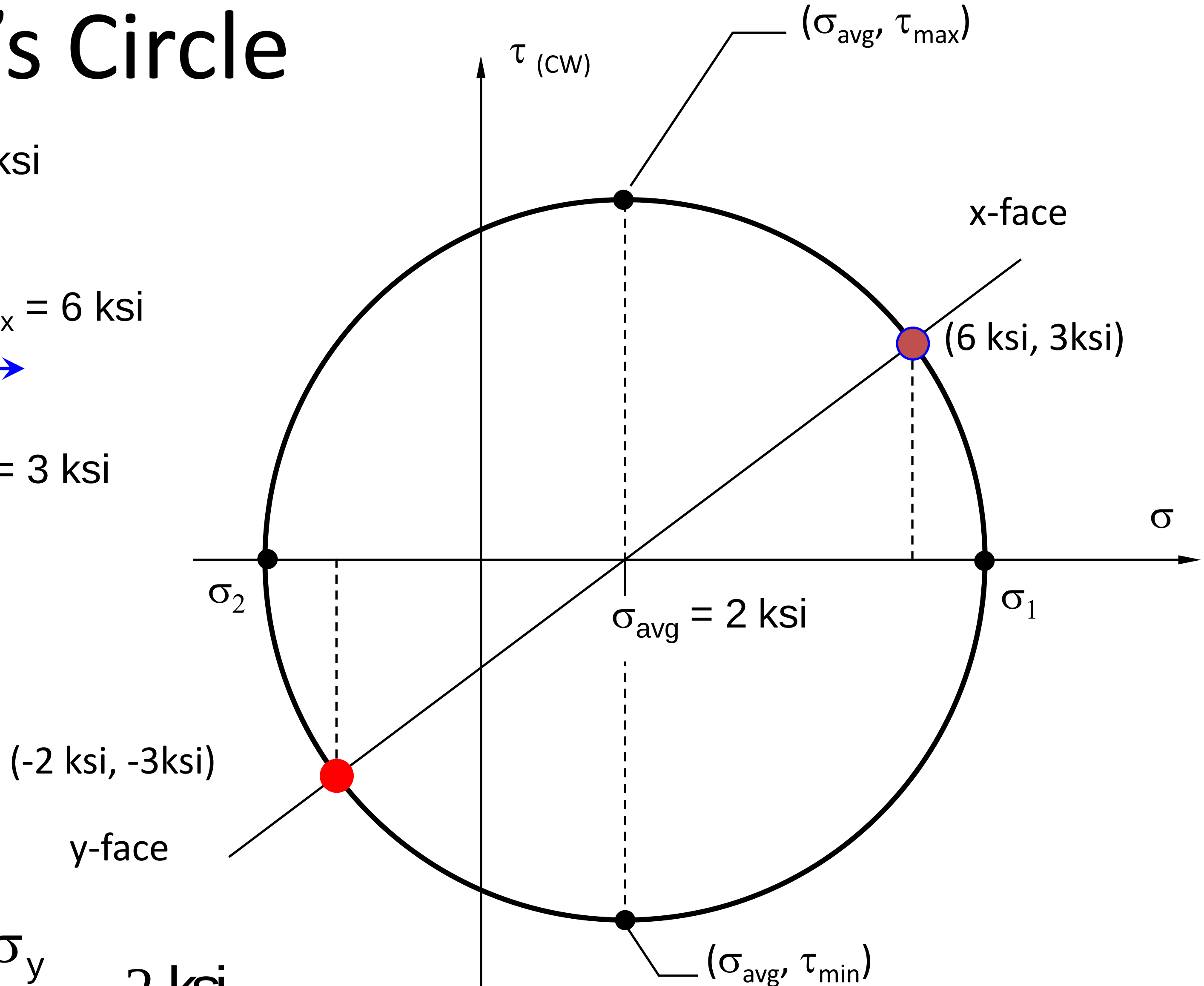
- At the stress orientation represented by the black line; if you rotate the element ccw by ϴP you will get the principal stresses.
- If you rotate cw by ϴS you will get the maximum shear

Mohr's Circle



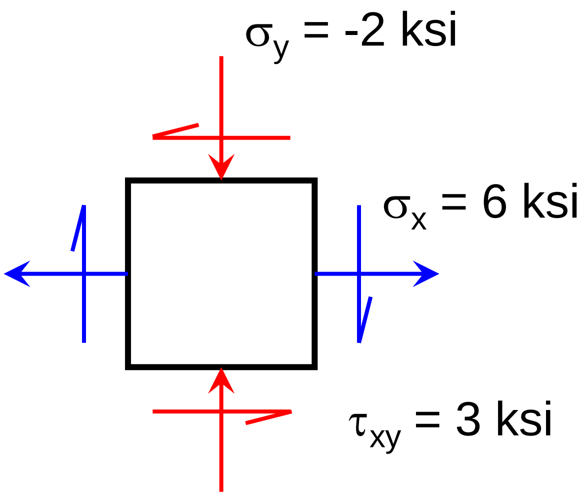
Mohr's Circle

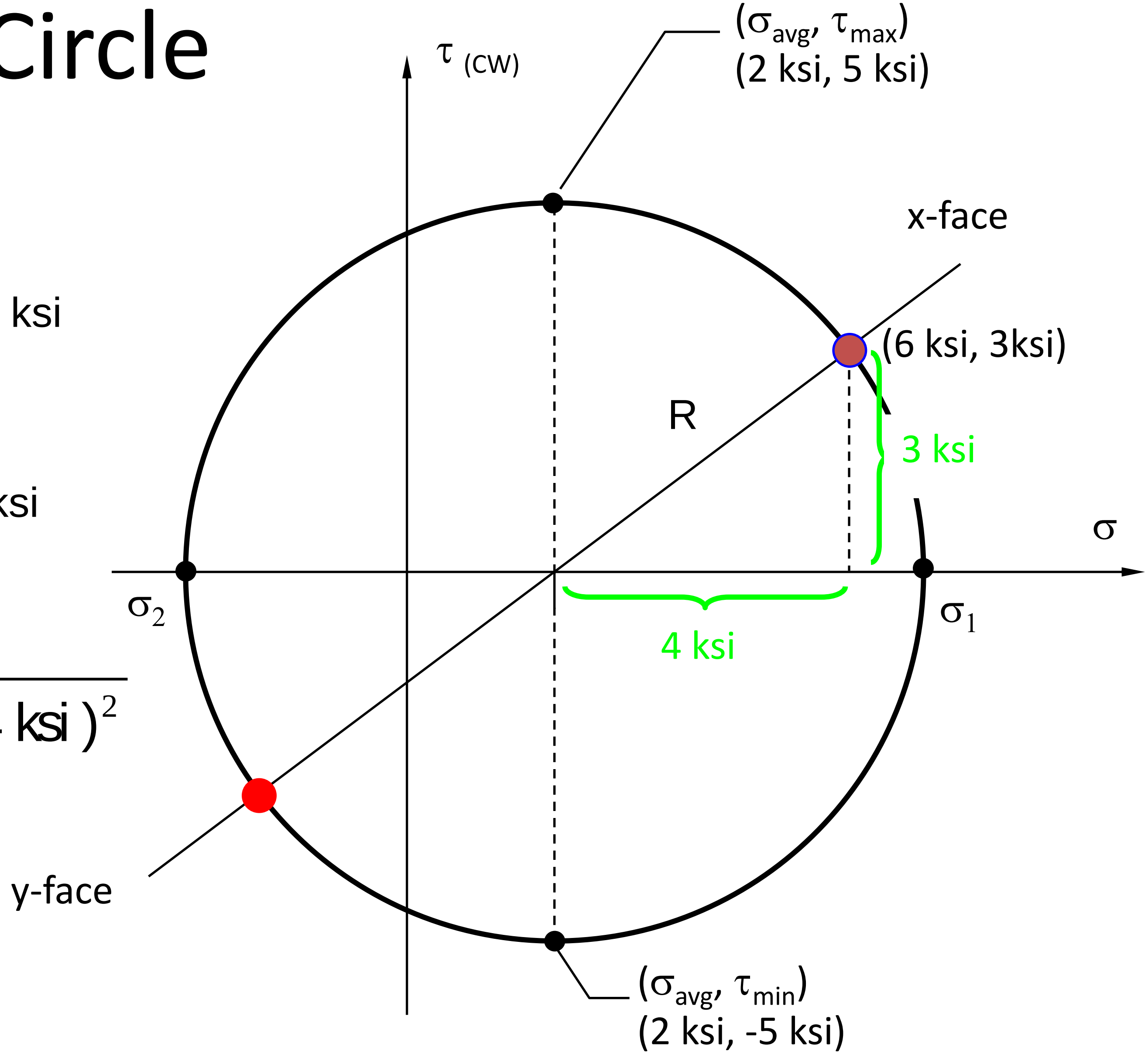




$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = 2 \text{ ksi}$$

Mohr’s Circle





$$R = \sqrt{(3\ \text{ksi})^2 + (4\ \text{ksi})^2}$$

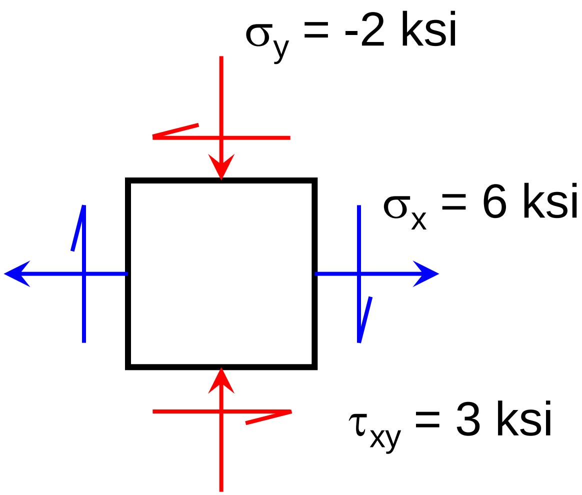
$$= 5\ \text{ksi}$$

$$R = \tau_{max}$$

$$\sigma_1 = \sigma_{avg} + R = 7\ \text{ksi}$$

$$\sigma_2 = \sigma_{avg} - R = -3\ \text{ksi}$$

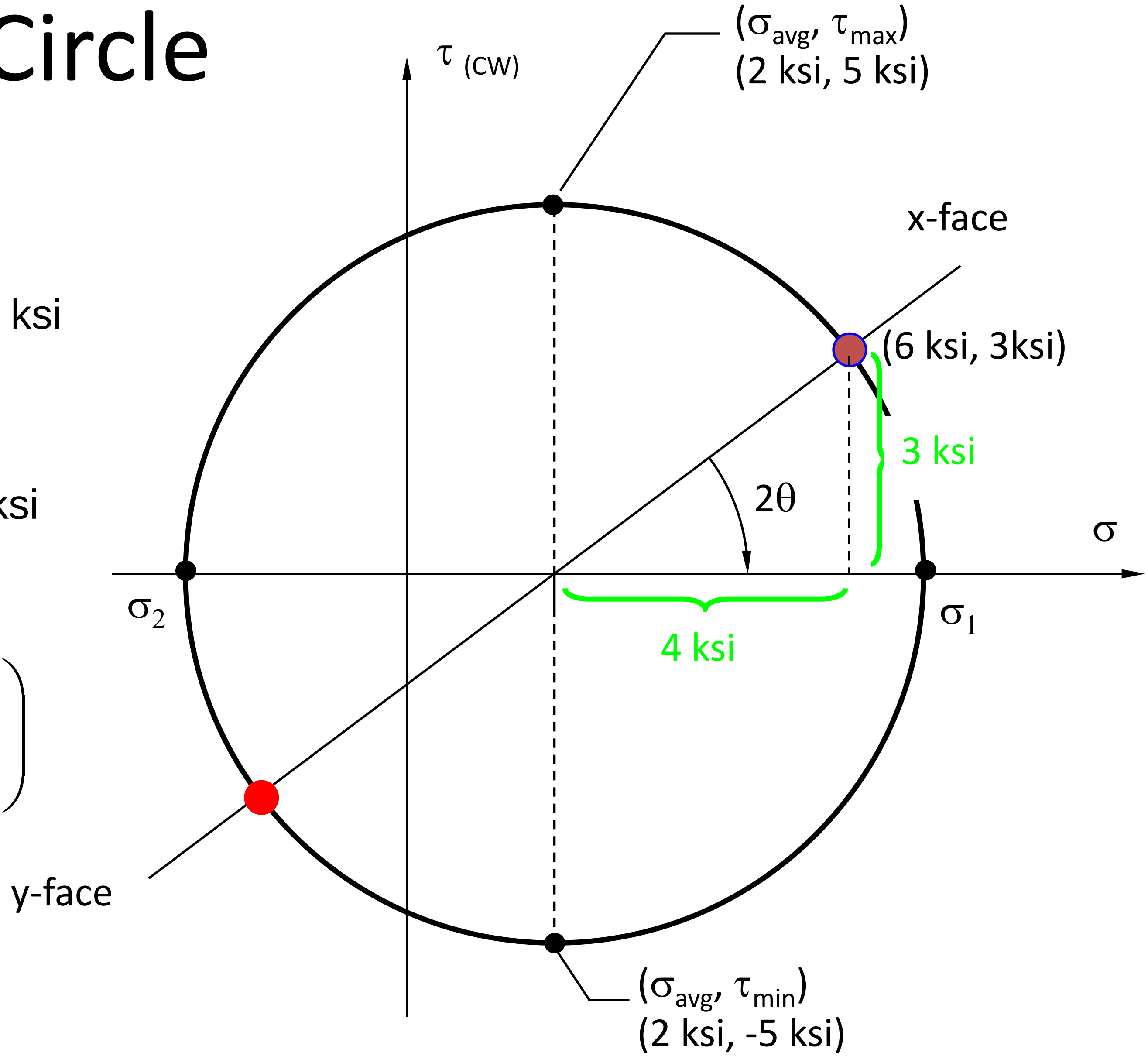
Mohr's Circle



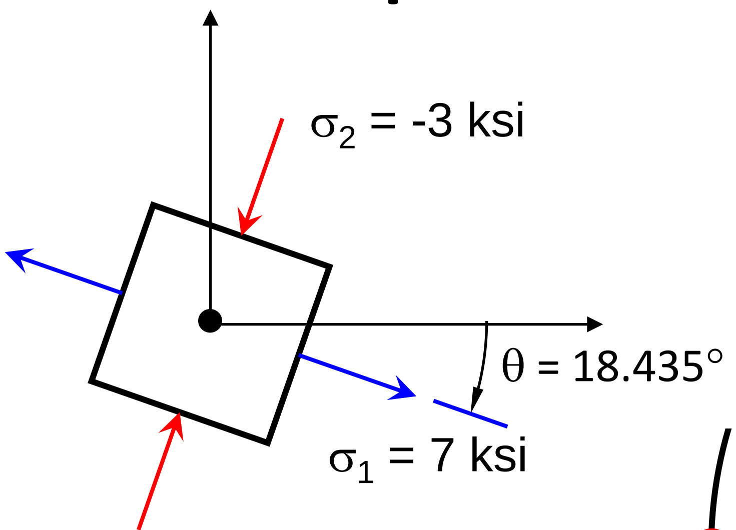
$$2\theta = \mathrm{Tan}^{-1}\left(\frac{3\ \mathrm{ksi}}{4\ \mathrm{ksi}}\right)$$

$$2\theta = 36.869^\circ$$

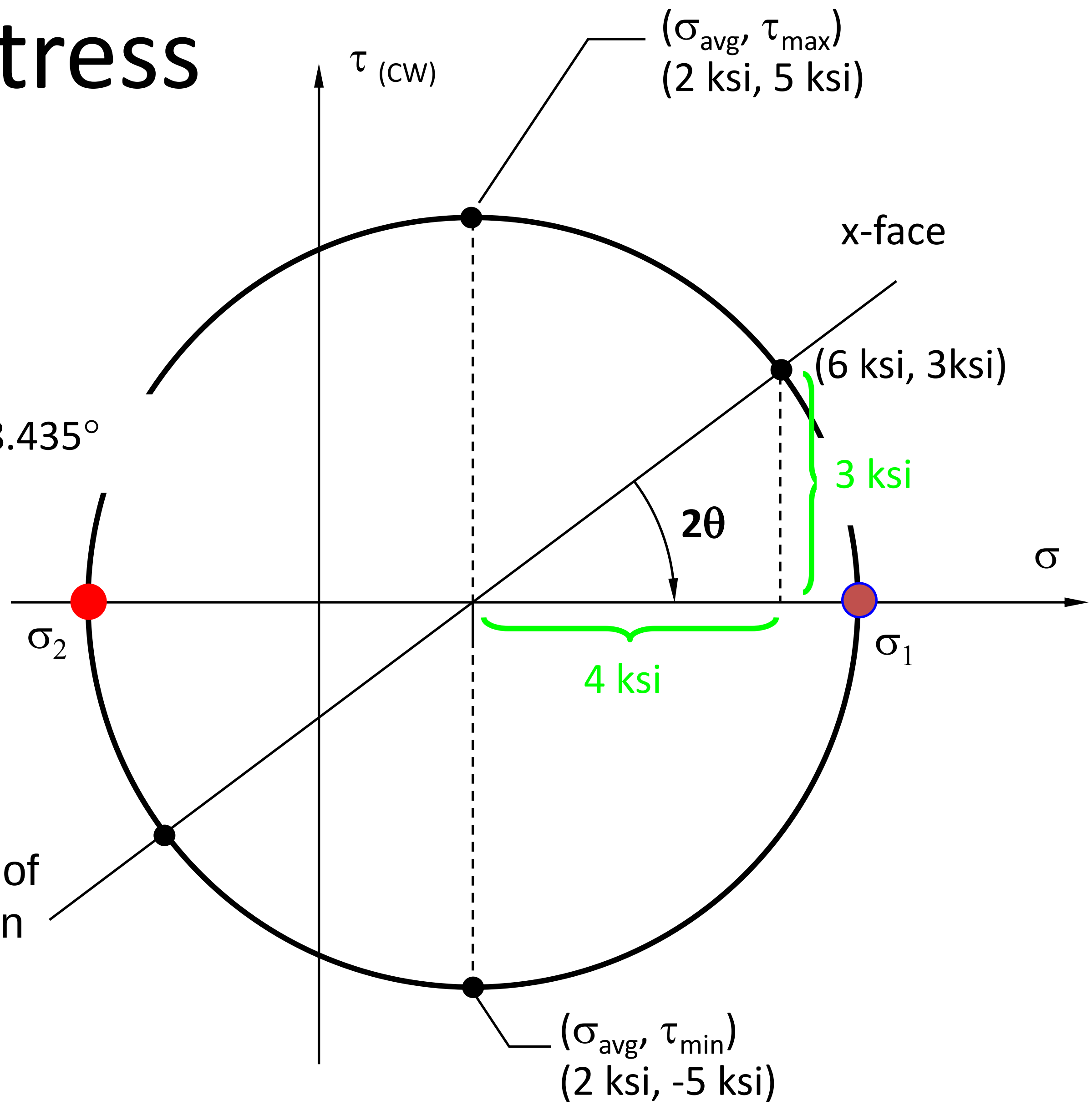
$$\theta = 18.435^\circ$$



Principle Stress

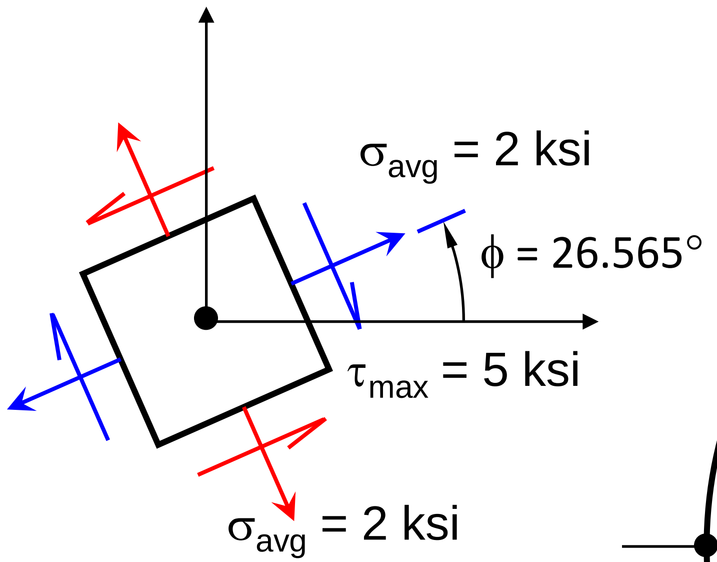


Principle Stress Element

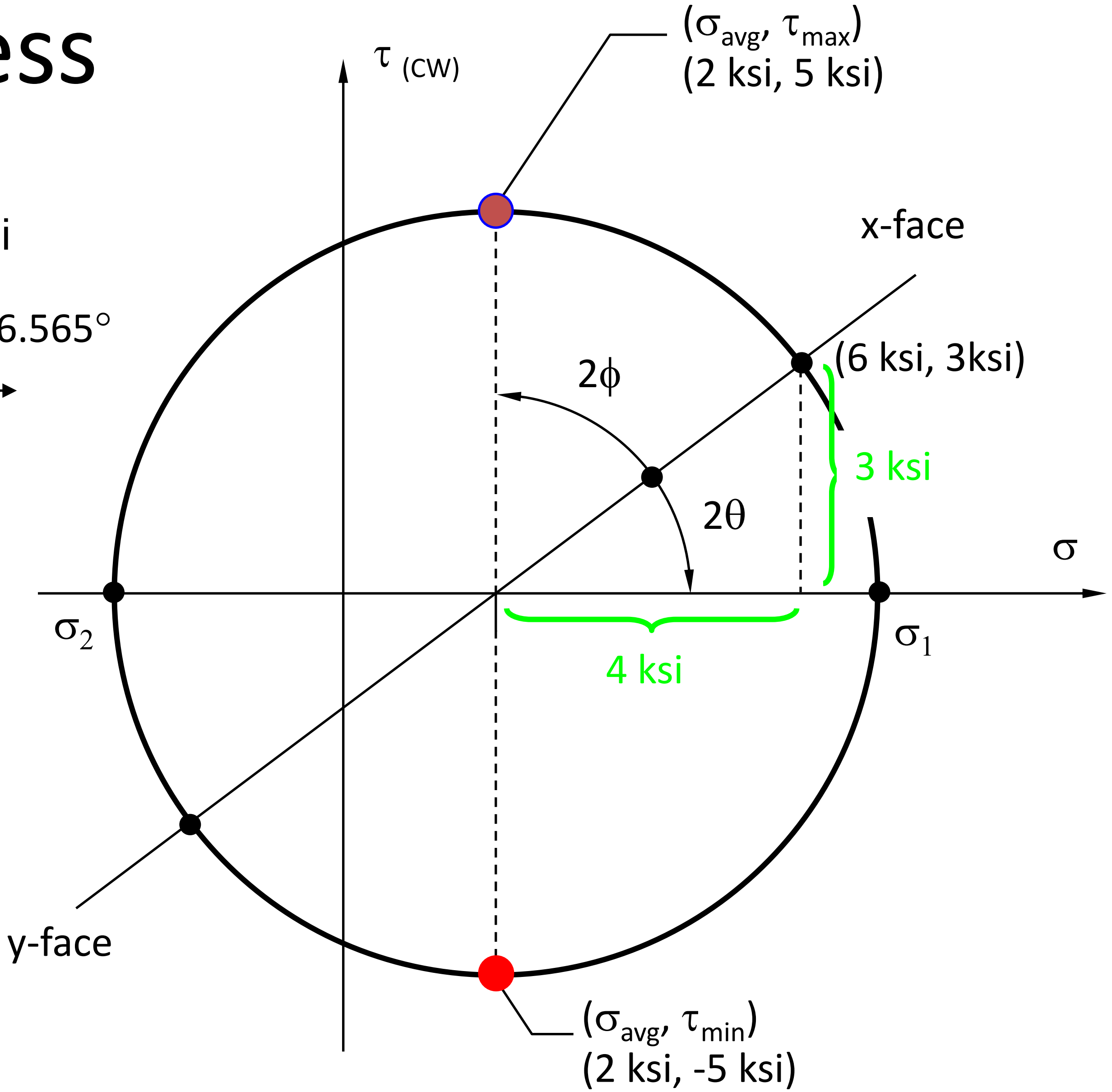


Rotation on element is half of the rotation from the circle in same direction from x-axis

Shear Stress



Maximum Shear Stress Element



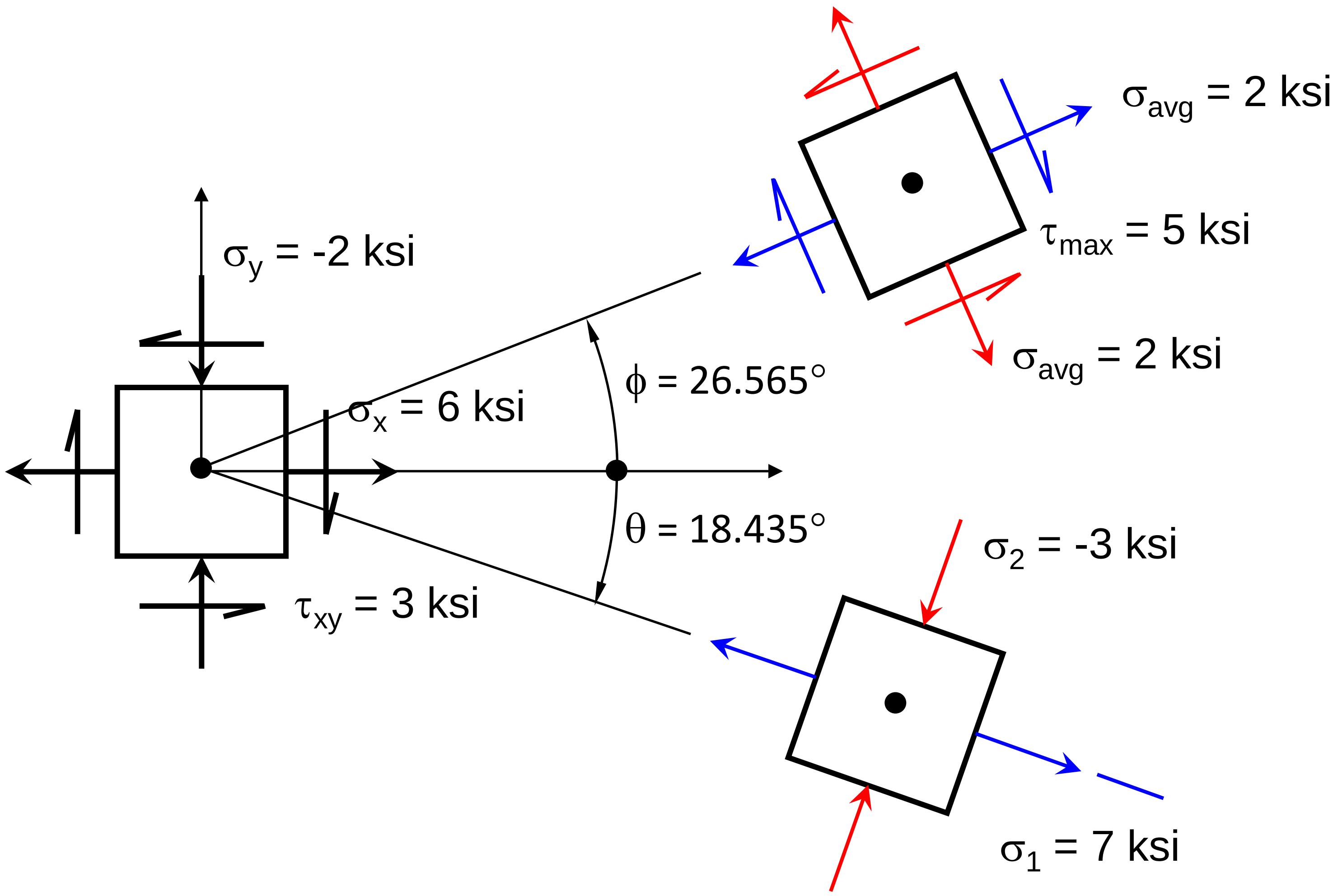
$$2\phi = 90° - 2\theta$$

$$2\phi = 90 - 36.869°$$

$$2\phi = 53.130°$$

$$\phi = 26.565°$$

Relationship Between Elements



$\theta + \phi = 18.435° + 26.565° = 45°$

Example: For the state of plane stress shown, construct Mohr's circle,

SOLUTION:

- Construction of Mohr's circle

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{(50)+(-10)}{2} = 20\,\text{MPa}$$

$$CF = 50 - 20 = 30\,\text{MPa} \qquad FX = 40\,\text{MPa}$$

$$R = CX = \sqrt{(30)^2 + (40)^2} = 50\,\text{MPa}$$

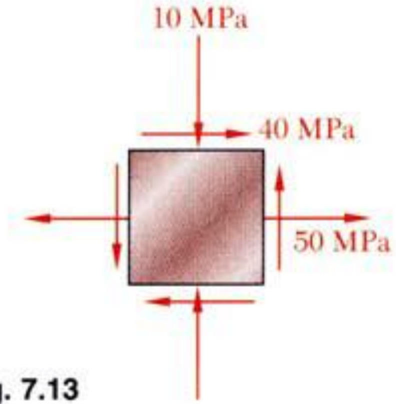
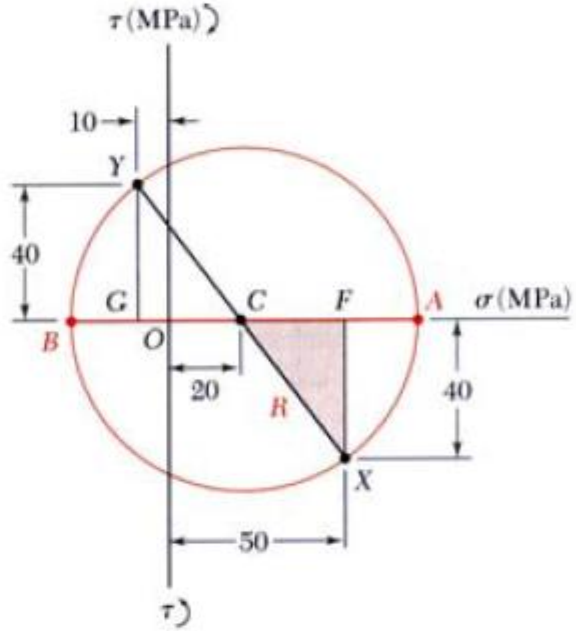


Fig. 7.13



- Principal planes and stresses

$$\sigma_{\max} = OA = OC + CA = 20 + 50$$

$$\boxed{\sigma_{\max} = 70\,\text{MPa}}$$

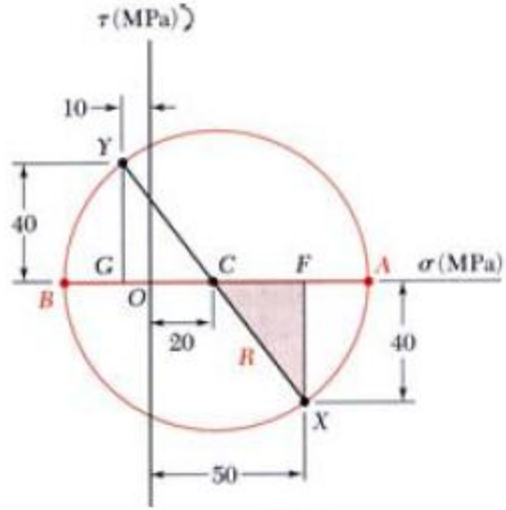
$$\sigma_{\max} = OB = OC - BC = 20 - 50$$

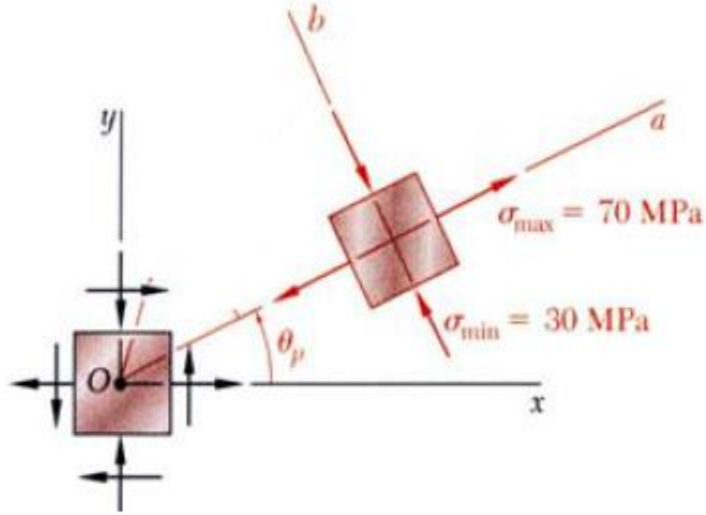
$$\boxed{\sigma_{\max} = -30\,\text{MPa}}$$

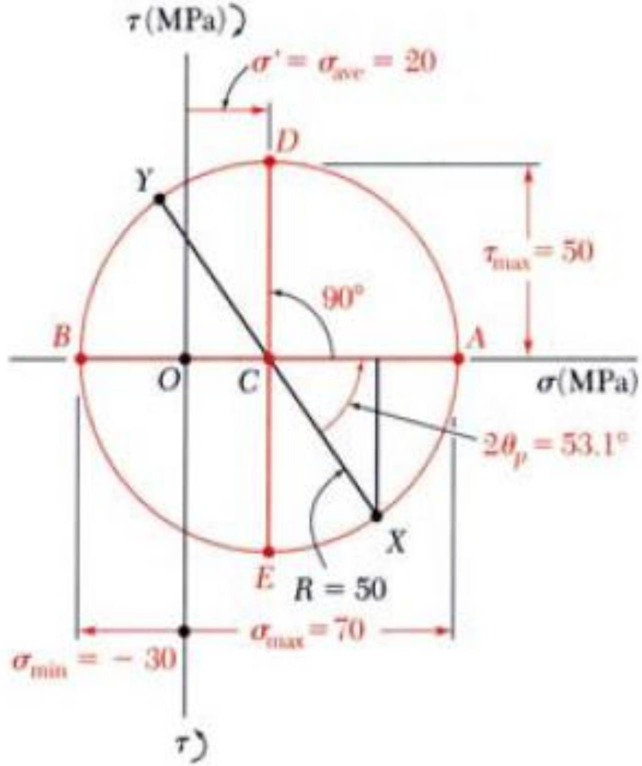
$$\tan 2\theta_p = \frac{FX}{CP} = \frac{40}{30}$$

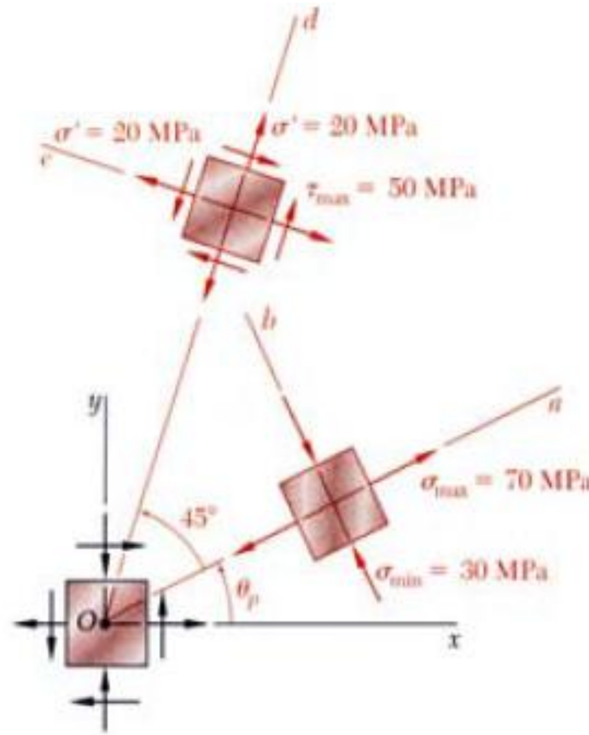
$$2\theta_p = 53.1^\circ$$

$$\boxed{\theta_p = 26.6^\circ}$$









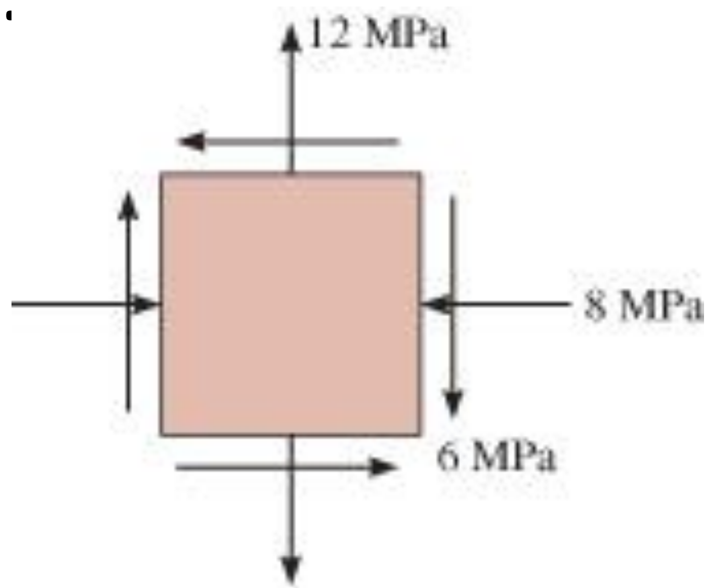
- Maximum shear stress

$$\theta_s = \theta_p + 45^\circ \qquad \tau_{\max} = R \qquad \sigma' = \sigma_{ave}$$

$$\boxed{\theta_s = 71.6^\circ} \qquad \boxed{\tau_{\max} = 50\ \text{MPa}} \qquad \boxed{\sigma' = 20\ \text{MPa}}$$

EXAMPLE

State of plane stress at a pt is shown on the element. Represent this state of stress on an element oriented 30° counterclockwise from position shown.



(a)

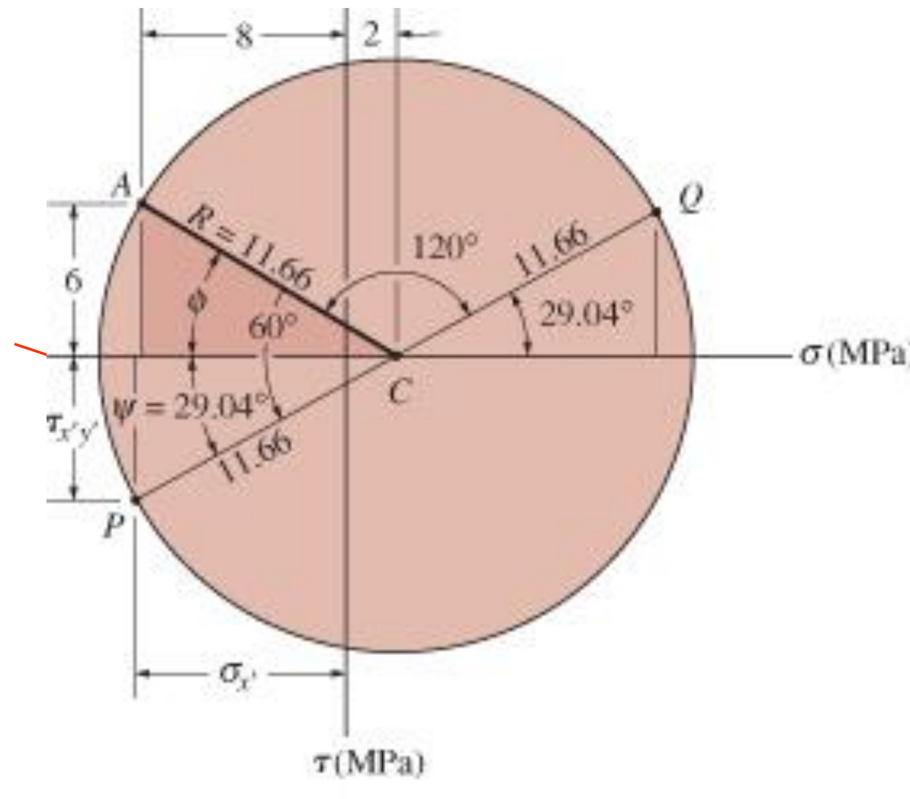
EXAMPLE (SOLN)

Construction of circle

$\sigma_x = -8\text{ MPa}$ $\sigma_y = 12\text{ MPa}$ $\tau_{xy} = -6\text{ MPa}$

- Establish the σ, τ axes as shown.
  Center of circle C located on the σ-axis, at the pt:

$$\sigma_{avg} = \frac{-8+12}{2} = 2\text{ MPa}$$



(b)

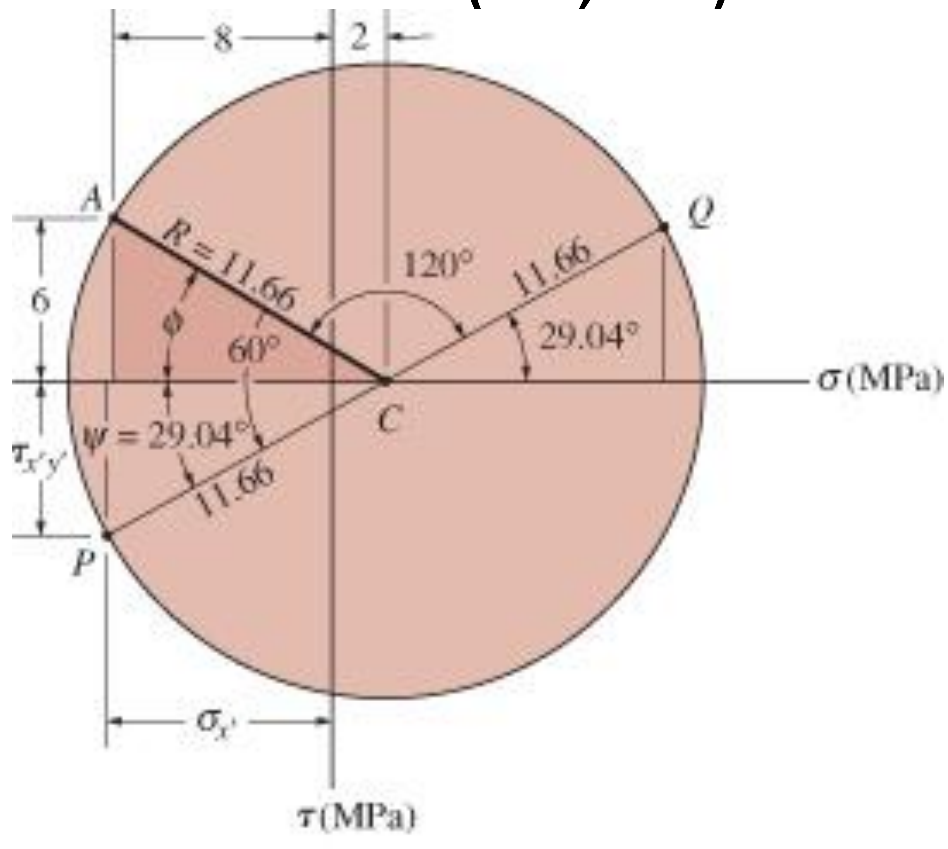
EXAMPLE (SOLN)

Construction of circle

- Initial pt for $\theta = 0°$ has coordinates A (–8, –6) are plotted. Apply Pythagoras theorem to shaded triangle to get circle's radius CA,

$$R = \sqrt{(10)^2 + (6)^2}$$

$$R = 11.66 \text{ MPa}$$



(b)

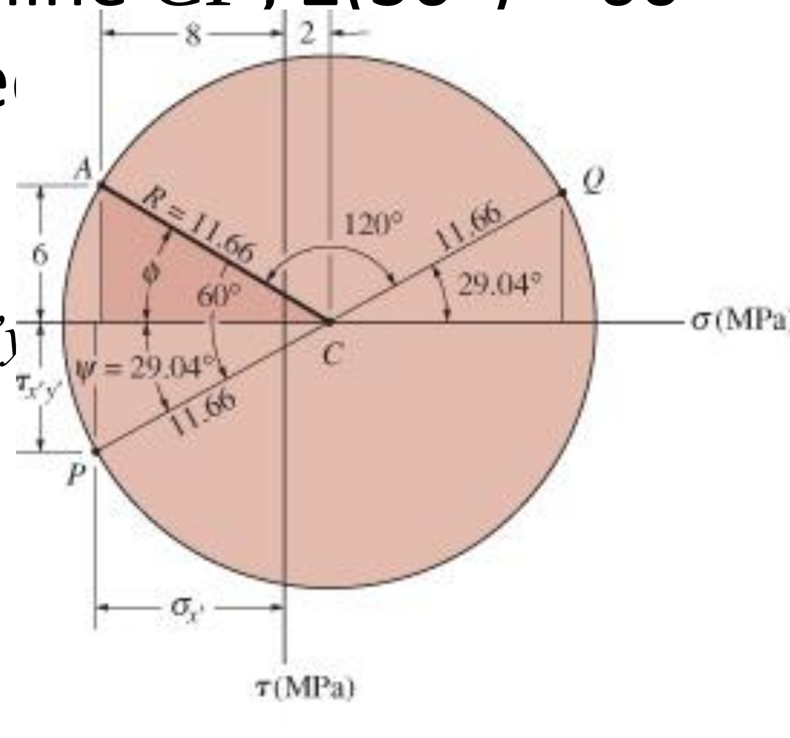
EXAMPLE 9.11 (SOLN)

Stresses on 30° element

- Since element is rotated 30° counterclockwise, we must construct a radial line CP, 2(30°) = 60° counterclockwise, measure from CA (θ = 0°).
- Coordinates of pt P ($\sigma_{x'}$, $\tau_{x'}$ must be obtained. From geometry of circle,

$$\phi = \tan^{-1}\frac{6}{10} = 30.96°$$

$$\psi = 60° - 30.96° = 29.04°$$



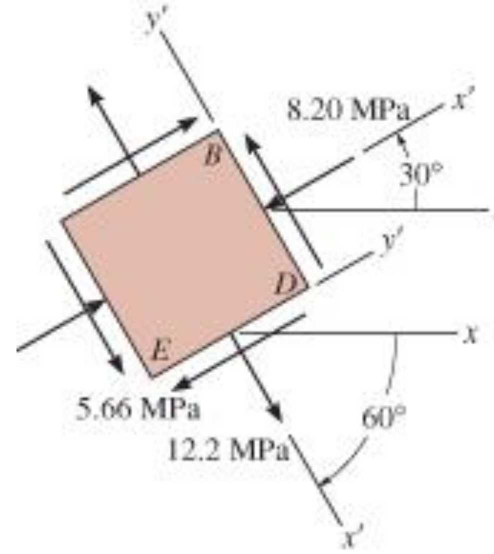
(b)

EXAMPLE 9.11 (SOLN)

Stresses on 30° element

$$\sigma_{x'} = 2 - 11.66 \cos 29.04° = -8.20 \text{ MPa}$$

$$\tau_{x'y'} = 11.66 \sin 29.04° = 5.66 \text{ MPa}$$



(c)

- The two stress components act on face BD of element shown, since the x' axis for this face if oriented 30° counterclockwise from the x-axis.
- Stress components acting on adjacent face DE of element, which is 60° clockwise from $+x$-axis, are represented by the coordinates of pt Q on the circle.
- This pt lies on the radial line CQ, which is 180° from CP.

EXAMPLE 9.11 (SOLN)

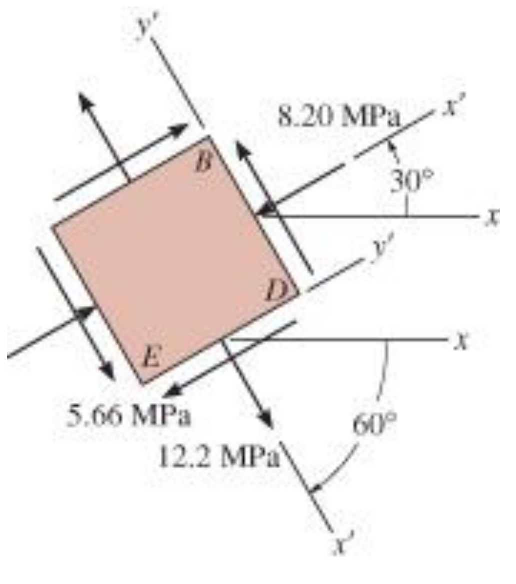
Stresses on 30° element

- The coordinates of pt Q are

$$\sigma_{x'} = 2 + 11.66 \cos 29.04° = 12.2 \text{ MPa}$$

$$\tau_{x'y'} = -(11.66 \sin 29.04°) = -5.66 \text{ MPa (Check!)}$$

- Note that here $\tau_{x'y'}$ acts in the $-y'$ direction.



(c)

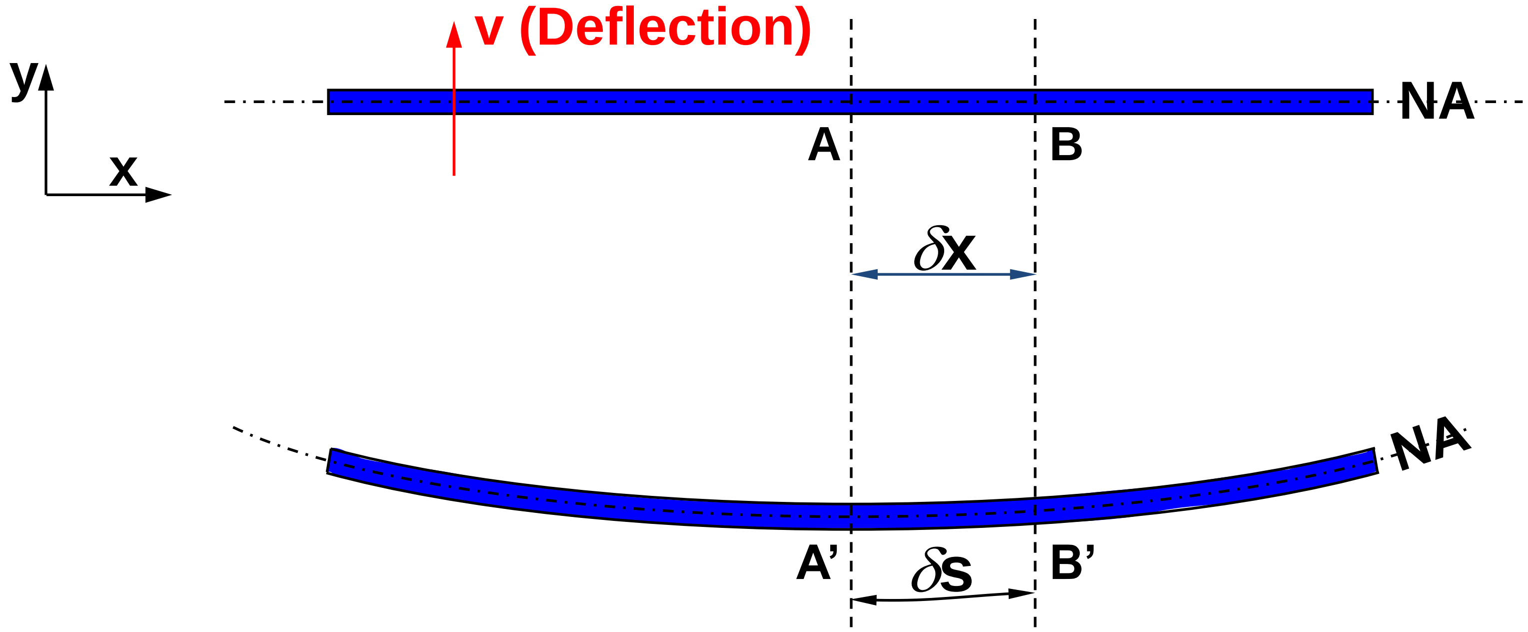
Deflections in Beams

Beam Deflection

Recall: THE ENGINEERING BEAM THEORY

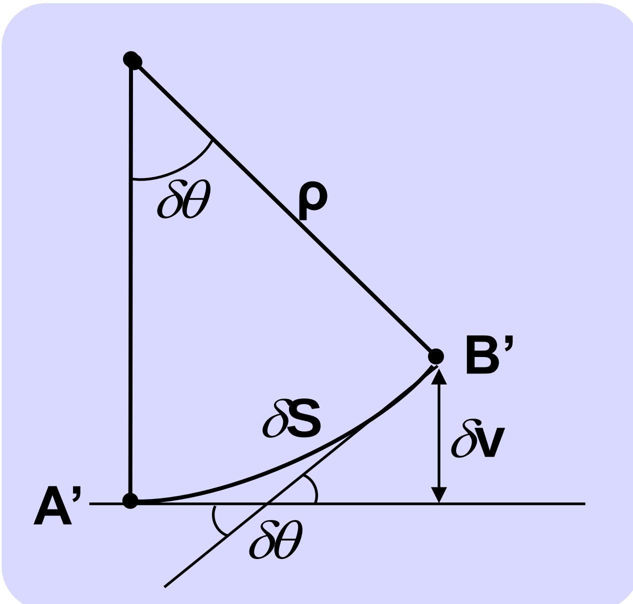
$$-\frac{\sigma_x}{y'} = \boxed{\frac{M_{xz}}{I_z} = \frac{E}{\rho}}$$

Moment-Curvature Equation



If deformation is small (i.e. slope is "flat"):

$$\delta s \approx \delta x$$

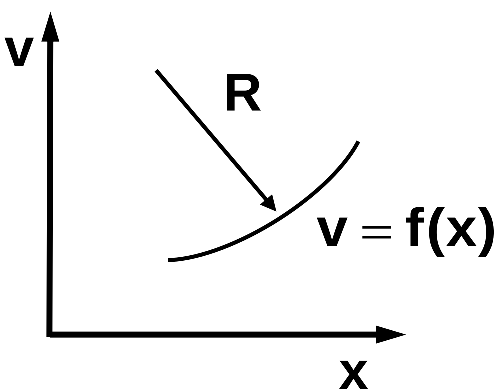


$$\rho.\delta\theta = \delta S \approx \delta x \qquad \therefore \frac{I}{\rho} \approx \frac{d\theta}{dx}$$

and $\delta\theta \approx \dfrac{\delta v}{\delta x}$ (slope is "flat")

$$\Rightarrow \frac{I}{\rho} \approx \frac{d^2 v}{dx^2}$$

<u>Alternatively: from Newton's Curvature Equation</u>



$$\frac{I}{\rho} = \frac{\left(\dfrac{d^2 v}{dx^2}\right)}{\left(1 + \left(\dfrac{dv}{dx}\right)^2\right)^{\frac{3}{2}}}$$

if $\left(\dfrac{dv}{dx}\right)^2 <<<< 1$

$$\Rightarrow \frac{I}{\rho} \approx \frac{d^2 v}{dx^2}$$

From the Engineering Beam Theory:

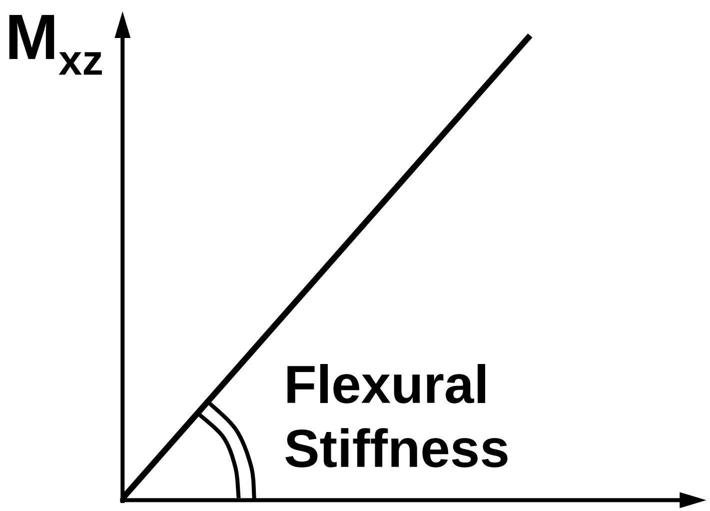
$$\frac{M_{xz}}{I_z} = \frac{E}{\rho} \qquad \frac{1}{\rho} = \frac{M_{xz}}{EI_z} = \frac{\mathbf{d^2v}}{\mathbf{dx^2}}$$

$$\Rightarrow \left(\mathbf{EI_z}\right)\frac{\mathbf{d^2v}}{\mathbf{dx^2}} = \mathbf{M_{xz}}$$

Flexural Stiffness → (EI_z)

Curvature → $\frac{d^2v}{dx^2}$

Bending Moment → M_{xz}



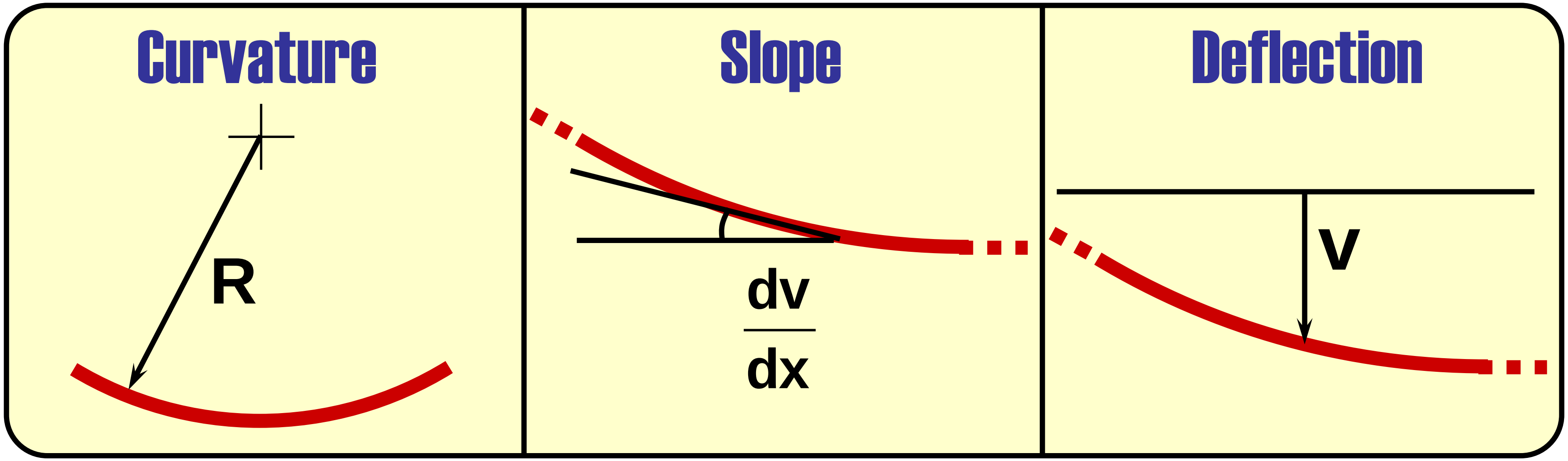
$$\frac{1}{\rho} = \frac{d^2v}{dx^2}$$

Since, $$\frac{d^2v}{dx^2} = \left(\frac{1}{EI_z}\right)M_{xz}$$ ← Curvature

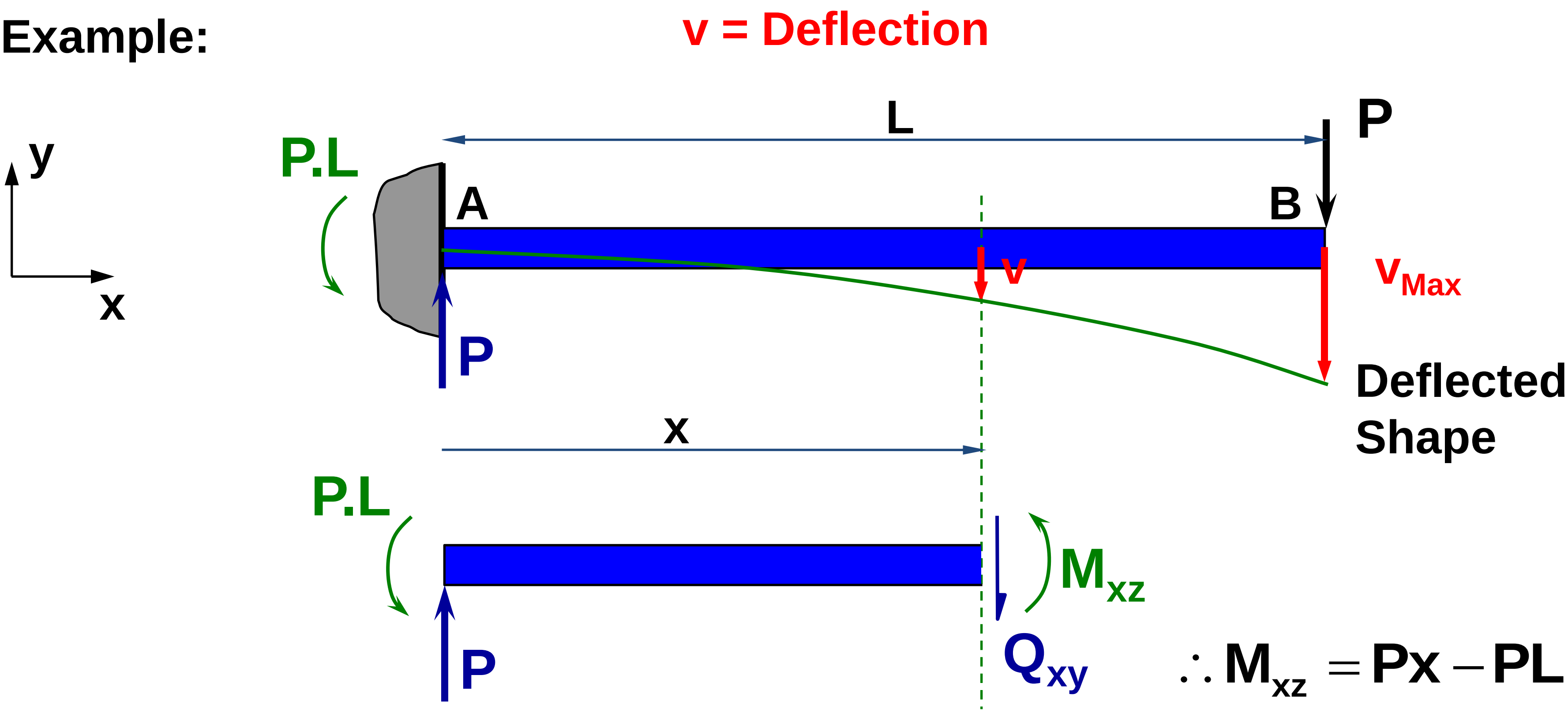
$$\Rightarrow \quad \frac{dv}{dx} = \left(\frac{1}{EI_z}\right)\int M_{xz}\cdot dx \; + \; C_1$$ ← Slope

$$\Rightarrow \quad v = \left(\frac{1}{EI_z}\right)\int\int M_{xz}\cdot dx\cdot dx \; + \; \int C_1\cdot dx \; + \; C_2$$ ← Deflection

Where C_1 and C_2 are found using the boundary conditions.



Example:



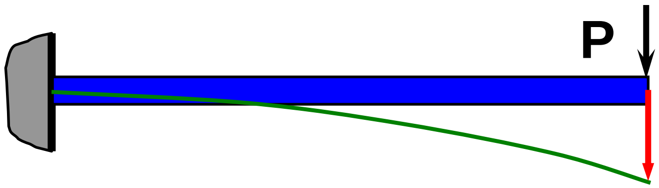
$$\therefore M_{xz} = Px - PL$$

$$(EI_z)\frac{d^2v}{dx^2} = M_{xz} = Px - PL$$

$$\Rightarrow \quad (EI_z)\frac{dv}{dx} = P\frac{x^2}{2} - PLx + C_1$$

$$\Rightarrow \quad (EI_z)v = P\frac{x^3}{6} - \frac{PLx^2}{2} + C_1x + C_2$$

$$\Rightarrow \quad (EI_z)v = P\frac{x^3}{6} - \frac{PLx^2}{2} + C_1x + C_2$$



To find C_1 and C_2:

Boundary conditions:

(i) @ x=0 $\quad \frac{dv}{dx} = 0$

(ii) @ x=0 $\quad v = 0$

$$\therefore C_1 = 0 \quad \& \quad C_2 = 0$$

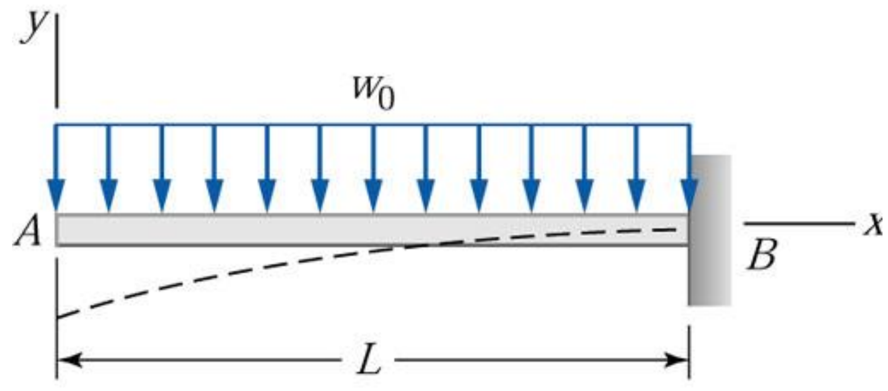
Equation of the deflected shape is:

$$(EI_z)v = P\frac{x^3}{6} - \frac{PLx^2}{2}$$

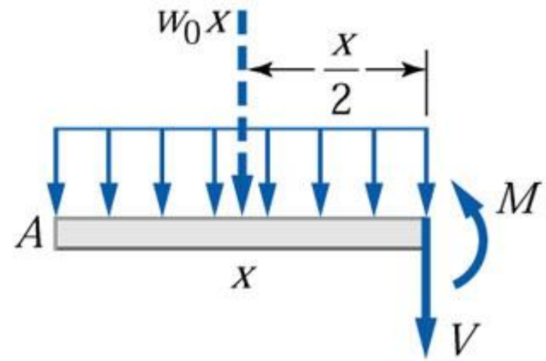
v_{Max} occurs at x=L

$$v_{Max} = -\frac{1}{3}\frac{PL^3}{EI_z}$$

Example: The cantilever beam AB of length L shown in Fig. carries a uniformly distributed load of intensity wo , which includes the weight of the beam. Derive the equation of the elastic curve.



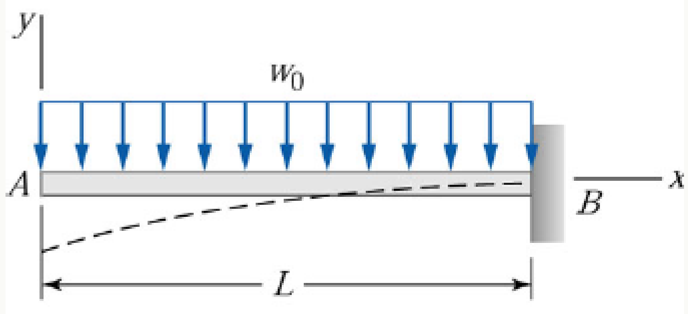
(a)



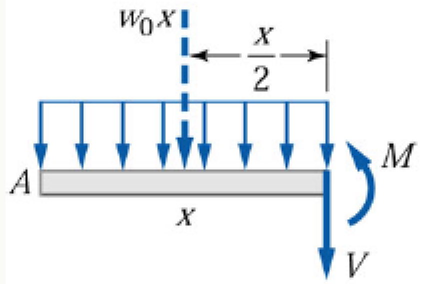
(b)

Solution

Patr1



(a)



(b)

The dashed line in Fig. (a) represents the elastic curve of the beam. The bending moment acting at the distance *x* from the left end can be obtained from the free-body diagram in Fig. (b) (note that *V* and *M* are shown acting in their positive directions):

$$M = -w_0 x \frac{x}{2} = -\frac{w_0 x^2}{2}$$

Substituting the expression for *M* into the differential equation EI υ "= *M*,

$$EI\upsilon'' = -\frac{w_0 x^2}{2}$$

Successive integrations yield

$$EI\upsilon' = -\frac{w_0 x^3}{6} + C_1 \qquad \text{(a)}$$

$$EI\upsilon = -\frac{w_0 x^4}{24} + C_1 x + C_2 \qquad \text{(b)}$$

The constants C_1 and C_2 are obtained from the boundary conditions at the built-in end *B*, which are :

1. $\upsilon'\big|_{x=L} = 0$ (support prevent rotation at B) . Substituting $\upsilon' = 0$ and $x = L$ into Eq. (a),

$$C_1 = \frac{w_0 L^3}{6}$$

2. $\upsilon\big|_{x=L} = 0$ (support prevent deflection at B) . With $\upsilon = 0$ and $x = L$, Eq.(b) becomes

$$0 = \frac{w_0 L^4}{24} + \left(\frac{w_0 L^3}{6}\right) L + C_2$$

$$C_2 = \frac{w_0 L^4}{8}$$

If we substitute C_1 and C_2 into Eq. (b), the equation of the elastic curve is

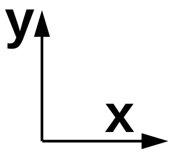
$$EI\upsilon = \frac{w_0 x^4}{24} + \frac{w_0 L^3}{6} x - \frac{w_0 L^4}{8}$$

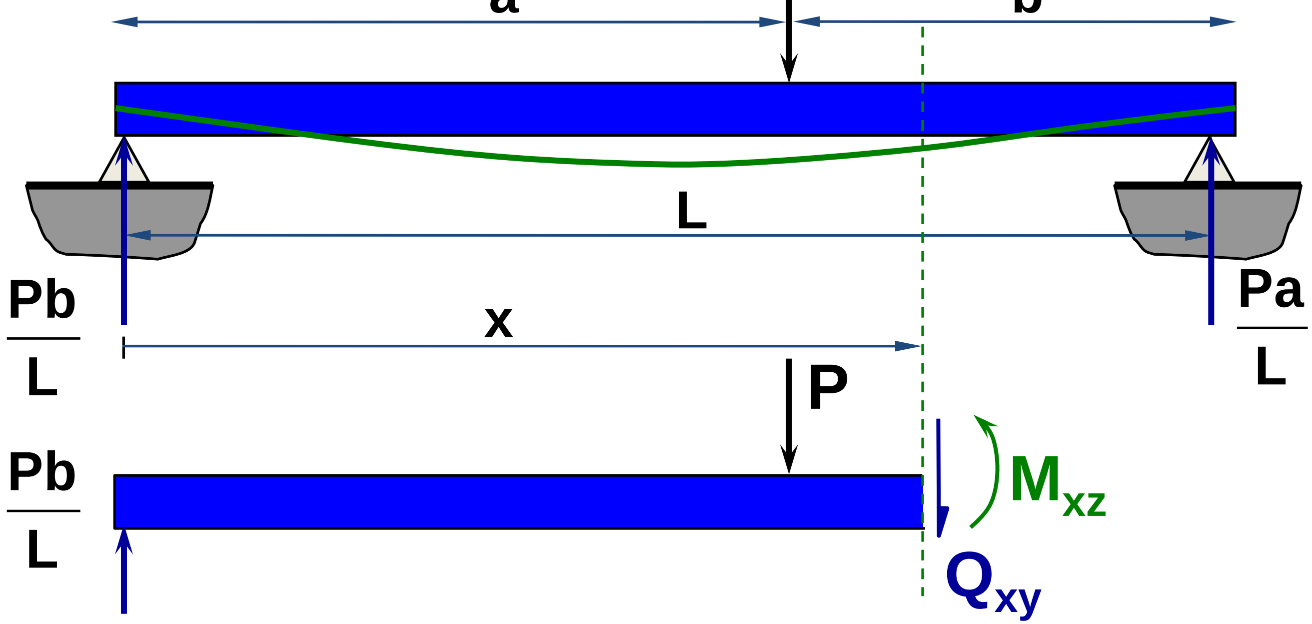
$$EI\upsilon = \frac{w_0}{24}\left(-x^4 + 4L^3 x - 3L^4\right)$$

Answer

2.4.2 Macaulay's Notation

Example:





$$\therefore M_{xz} = \frac{Pb}{L}(x) - P\langle (x-a) \rangle$$

$$\Rightarrow \quad (EI_z)\frac{d^2v}{dx^2} = M_{xz} = \frac{Pb}{L}(x) - P\langle (x-a) \rangle$$

$$\Rightarrow \quad (EI_z)\frac{dv}{dx} = \frac{Pb}{L}\left(\frac{x^2}{2}\right) - \frac{P}{2}\langle (x-a)^2 \rangle + C_1$$

$$\Rightarrow \quad (EI_z)v = \frac{Pb}{6L}(x^3) - \frac{P}{6}\langle (x-a)^3 \rangle + C_1(x) + C_2$$

$$\Rightarrow \quad (EI_z)v = \frac{Pb}{6L}(x^3) - \frac{P}{6}\langle (x-a)^3 \rangle + C_1(x) + C_2$$

Boundary conditions: (i) @ x=0 $v = 0$

(ii) @ x=L $v = 0$

From (i): $C_2 = 0$

From (ii): $0 = \frac{Pb}{6L}(L^3) - \frac{P}{6}\langle (L-a)^3 \rangle + C_1(L)$

$$\therefore C_1 = \frac{Pb}{6L}(b^2 - L^2)$$ **Since (L-a)=b**

Equation of the deflected shape is:

$$\Rightarrow \quad v = \frac{1}{EI_z}\left[\frac{Pb}{6L}(x^3) - \frac{P}{6}\langle (x-a)^3 \rangle + \frac{Pb}{6L}(b^2 - L^2)(x)\right]$$

To find v_{Max}:

v_{Max} occurs where $\frac{dv}{dx} = 0$ (i.e. slope=0)

$$\text{i.e.} \quad (EI_z)(0) = \frac{Pb}{L}\left(\frac{x^2}{2}\right) - \frac{P}{2}\left\langle (x-a)^2 \right\rangle + \frac{Pb}{6L}\left(b^2 - L^2\right)$$

Assuming v_{Max} will be at x<a, i.e. $\left\langle (x-a)^2 \right\rangle = 0$

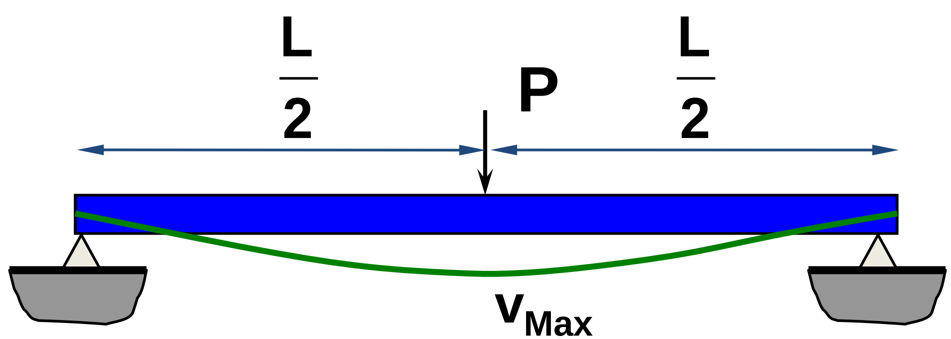
$$\therefore \frac{dv}{dx} = 0 \quad \text{when} \quad x^2 = -\frac{1}{3}\left(b^2 - L^2\right) = \frac{1}{3}\left(L^2 - b^2\right)$$

This value of x is then substituted into the above equation of the deflected shape in order to obtain v_{Max}.

Note:

$$\text{if} \quad a = b = \frac{L}{2}$$

$$\therefore v_{Max} = -\frac{PL^3}{48EI_z}$$



Summary

After considering stress caused by bending, we have now looked at the deflections generated. Keep in mind the relationships between Curvature, Slope, and Deflection, and understand what they are:

- **Curvature** $$\frac{d^2v}{dx^2} = \frac{1}{EI_z} M_{xz} \approx \frac{1}{R}$$
- **Slope** $$\frac{dv}{dx}$$
- **Deflection** $$v$$