Solutions Manual

Second Edition

Field and Wave Electromagnetics

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Chapter 2

Vector Analysis

P.2-1 a) $\bar{a}_A = \frac{\bar{A}}{A} = \frac{\bar{a}_x + \bar{a}_y 2 - \bar{a}_z 3}{\sqrt{1^2 + 2^2 + (-3)^2}} = \frac{1}{\sqrt{14}}(\bar{a}_x + \bar{a}_y 2 - \bar{a}_z 3).$

b) $|\bar{A} - \bar{B}| = |\bar{a}_x + \bar{a}_y 6 - \bar{a}_z 4| = \sqrt{1^2 + 6^2 + (-4)^2} = \sqrt{53}.$

c) $\bar{A} \cdot \bar{B} = 0 + 2(-4) + (-3) = -11.$

d) $\theta_{AB} = \cos^{-1}(\bar{A} \cdot \bar{B} / AB) = \cos^{-1}(-11/\sqrt{14}\sqrt{17}) = 135.5°.$

e) $\bar{A} \cdot \bar{a}_C = \bar{A} \cdot \frac{\bar{C}}{C} = \bar{A} \cdot \frac{1}{\sqrt{29}}(\bar{a}_x 5 - \bar{a}_z 2) = \frac{11}{\sqrt{29}}.$

f) $\bar{A} \times \bar{C} = -\bar{a}_x 4 - \bar{a}_y 13 - \bar{a}_z 10.$

g) $\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{A} \times \bar{B}) \cdot \bar{C} = -42$

h) $(\bar{A} \times \bar{B}) \times \bar{C} = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{A}(\bar{C} \cdot \bar{B}) = \bar{a}_x 2 - \bar{a}_y 40 + \bar{a}_z 5.$

$\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B}) = \bar{a}_x 55 - \bar{a}_y 44 - \bar{a}_z 11.$

P.2-2 Let $\bar{C} = \bar{a}_x C_x + \bar{a}_y C_y + \bar{a}_z C_z,$

where $C_x^2 + C_y^2 + C_z^2 = 1.$ ①

For $\bar{C} \perp \bar{A}$: $\bar{C} \cdot \bar{A} = 0 \longrightarrow C_x - 2C_y + 3C_z = 0.$ ②

For $\bar{C} \perp \bar{B}$: $\bar{C} \cdot \bar{B} = 0 \longrightarrow C_x + C_y - 2C_z = 0.$ ③

Solving ①, ②, and ③ simultaneously, we obtain

$$C_x = \frac{1}{\sqrt{35}}, \quad C_y = \frac{5}{\sqrt{35}}, \quad C_z = \frac{3}{\sqrt{35}},$$

and $$\bar{C} = \frac{1}{\sqrt{35}}(\bar{a}_x + \bar{a}_y 5 + \bar{a}_z 3).$$

P.2-3 For $\bar{A} \parallel \bar{B}$ everywhere, $\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = 0,$

which requires that $\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}.$

P.2-4 From $\bar{A}\cdot\bar{B} = \bar{A}\cdot\bar{C}$ we have $\bar{A}\cdot(\bar{B}-\bar{C}) = 0$. ①
From $\bar{A}\times\bar{B} = \bar{A}\times\bar{C}$ we have $\bar{A}\times(\bar{B}-\bar{C}) = 0$. ②
① implies $\bar{A} \perp (\bar{B}-\bar{C})$, and ② implies $\bar{A} \parallel (\bar{B}-\bar{C})$.

Since A is not a null vector, ① and ② cannot hold at the same time unless $(\bar{B}-\bar{C})$ is a null vector. Thus, $\bar{B}-\bar{C}=0$, or $\bar{B}=\bar{C}$.

P.2-5 Expand $\bar{A}\times(\bar{A}\times\bar{X}) = \bar{A}(\bar{A}\cdot\bar{X}) - \bar{X}(\bar{A}\cdot\bar{A})$,
or $\bar{A}\times\bar{B} = p\bar{A} - A^2\bar{X}$.
$\therefore \quad \bar{X} = \frac{1}{A^2}(p\bar{A} + \bar{B}\times\bar{A})$.

P.2-6 Position vectors of the three corners:
$\overrightarrow{OP_1} = \bar{a}_y - \bar{a}_z 2$, $\overrightarrow{OP_2} = \bar{a}_x 4 - \bar{a}_y - \bar{a}_z 3$, $\overrightarrow{OP_3} = \bar{a}_x 6 + \bar{a}_y 2 + \bar{a}_z 5$
Vectors representing the three sides of the triangle:
$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \bar{a}_x 4 - \bar{a}_z$, $\overrightarrow{P_2P_3} = \bar{a}_x 2 + \bar{a}_y + \bar{a}_z 8$, $\overrightarrow{P_3P_1} = -\bar{a}_x 6 - \bar{a}_y - \bar{a}_z 7$.
a) $\overrightarrow{P_1P_2}\cdot\overrightarrow{P_2P_3} = 0$. $\therefore \triangle P_1P_2P_3$ is a right triangle.
b) Area of triangle $= \frac{1}{2}|\overrightarrow{P_1P_2}\times\overrightarrow{P_2P_3}| = 17.1$

P.2-7

$\bar{D}_1 = \bar{B} + \bar{A}$, $\bar{D}_2 = \bar{B} - \bar{A}$.
$\bar{D}_1\cdot\bar{D}_2 = (\bar{B}+\bar{A})\cdot(\bar{B}-\bar{A})$
$= \bar{B}\cdot\bar{B} - \bar{A}\cdot\bar{A} = 0$
for a rhombus.
$\therefore \quad \bar{D}_1 \perp \bar{D}_2$.

P.2-8

Let A, B, and C denote the vertices of a triangle, and B' and C' be the midpoints of sides AB and AC, respectively. The following vector relations hold:

$$\overrightarrow{AB'} = \frac{1}{2}\overrightarrow{AB}, \quad \overrightarrow{AC'} = \frac{1}{2}\overrightarrow{AC}.$$

$$\overrightarrow{B'C'} = \overrightarrow{AC'} - \overrightarrow{AB'} = \frac{1}{2}(\overrightarrow{AC} - \overrightarrow{AB})$$
$$= \frac{1}{2}\overrightarrow{BC}. \qquad \text{Q.E.D.}$$

P.2-9

$$\bar{a}_A = \bar{a}_x \cos\alpha + \bar{a}_y \sin\alpha,$$
$$\bar{a}_B = \bar{a}_x \cos\beta + \bar{a}_y \sin\beta.$$

a) $\bar{a}_A \cdot \bar{a}_B = \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

b)
$$\bar{a}_B \times \bar{a}_A = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix} = \bar{a}_z(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= \bar{a}_z \sin(\alpha - \beta).$$

$$\therefore \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

P.2-10

$$\bar{A} + \bar{B} + \bar{C} = 0.$$
$$\bar{A}\times: \quad \bar{A}\times\bar{B} = \bar{C}\times\bar{A}.$$
$$\bar{C}\times: \quad \bar{C}\times\bar{A} = \bar{B}\times\bar{C}.$$
$$\bar{B}\times: \quad \bar{B}\times\bar{C} = \bar{A}\times\bar{B}.$$

Magnitude relations:

$$AB\sin\theta_{AB} = CA\sin\theta_{CA} = BC\sin\theta_{BC}.$$

Hence,
$$\frac{A}{\sin\theta_{BC}} = \frac{B}{\sin\theta_{CA}} = \frac{C}{\sin\theta_{AB}}. \quad \text{(Law of sines.)}$$

P.2-11

$$\bar{r}' = -\bar{r}, \quad r' = r = c.$$

$$(\bar{c}-\bar{r}')\cdot(\bar{c}-\bar{r}) = (\bar{c}+\bar{r})\cdot(\bar{c}-\bar{r}) = 0.$$

$$\therefore \quad (\bar{c}-\bar{r}') \perp (\bar{c}-\bar{r}).$$

P.2-13 Consider line L_1: $\quad b_1 x + b_2 y = c$, which has a slope equal to $-b_1/b_2$. Denote the shifted line passing through the origin and parallel to L_1 as L_1^o: $\quad b_1 x + b_2 y = 0$.
The position vector of a point (x, y) on L_1^o is

$$\bar{r}_1 = \bar{a}_x x + \bar{a}_y y.$$

If we introduce the vector $\bar{n} = \bar{a}_x b_1 + \bar{a}_y b_2$, we can write the equation of L_1^o as

$$\bar{n}\cdot\bar{r}_1 = 0.$$

Thus the vector $\bar{n}$ is $\perp$ to $\bar{r}_1$, and is normal to both L_1 and L_1^o. It follows that the two lines L_1 and L_2 are perpendicular to each other if and only if their normal vectors $\bar{n}$ and $\bar{n}' = \bar{a}_x b_1' + \bar{a}_y b_2'$ are orthogonal: $\bar{n}\cdot\bar{n}' = 0$, which implies

$$b_1 b_1' + b_2 b_2' = 0, \quad \text{or} \quad \frac{b_2'}{b_1'} = -\frac{b_1}{b_2};$$

that is, the slopes of lines L_2 and L_1 are the negative reciprocals of each other.

P.2-14 a) Letting the position vector of a point in the plane be

$$\bar{R} = \bar{a}_x x + \bar{a}_y y + \bar{a}_z z$$

and introducing the vector $\bar{N} = \bar{a}_x b_1 + \bar{a}_y b_2 + \bar{a}_z b_3$, we can write the given equation as

$$\bar{R}\cdot\bar{N} = c \text{ (a constant).}$$

This shows that the projection of the position vector to any point in the plane on $\bar{N}$ is a constant, and that $\bar{N}$ is a normal vector.

b) $$\bar{a}_N = \frac{\bar{N}}{|\bar{N}|} = \frac{\bar{a}_x b_1 + \bar{a}_y b_2 + \bar{a}_z b_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

c) The perpendicular distance from the origin to the plane is
$$\bar{a}_N \cdot \bar{R} = \frac{c}{|\bar{N}|}.$$
For our case, $c = 5$, $|\bar{N}| = \sqrt{3^2 + (-2)^2 + 6^2} = 7$, and $\bar{a}_N \cdot \bar{R} = 5/7$.

P.2-15 $\bar{A}_{P_1} = -\bar{a}_y 3 - \bar{a}_z 2$, $\quad \overrightarrow{OP_1} = -\bar{a}_y 2 + \bar{a}_z 3.$

$\overrightarrow{OP_2} = \bar{a}_x (r\cos\phi) + \bar{a}_y (r\sin\phi) + \bar{a}_z = \bar{a}_x \frac{\sqrt{3}}{2} - \bar{a}_y \frac{1}{2} + \bar{a}_z.$

$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \bar{a}_x \frac{\sqrt{3}}{2} + \bar{a}_y \frac{1}{2} - \bar{a}_z 2$, $\quad |\overrightarrow{P_1P_2}| = \sqrt{5}.$

$$\bar{A}_{P_1} \cdot \bar{a}_{P_1P_2} = \bar{A}_{P_1} \cdot \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{\sqrt{5}}{2} = 1.12.$$

P.2-16 a) $x = r\cos\phi = 4\cos(2\pi/3) = -2.$
$y = r\sin\phi = 4\sin(2\pi/3) = 2\sqrt{3}.$
$z = 3.$

b) $R = (r^2 + z^2)^{1/2} = (4^2 + 3^2)^{1/2} = 5.$
$\theta = \tan^{-1}(r/z) = \tan^{-1}(4/3) = 53.1°.$
$\phi = 2\pi/3 = 120°.$

P.2-17 a) $\bar{E}_P = \bar{a}_R \frac{25}{(-3)^2 + 4^2 + (-5)^2} = \bar{a}_R \frac{1}{2}.$

$(E_P)_x = \frac{1}{2}\left(\frac{-3}{\sqrt{(-3)^2 + 4^2 + (-5)^2}}\right) = -0.212.$

b) $\bar{a}_R = \frac{1}{\sqrt{50}}(-\bar{a}_x 3 + \bar{a}_y 4 - \bar{a}_z 5)$, $\bar{a}_B = \frac{\bar{B}}{|\bar{B}|} = \frac{1}{3}(\bar{a}_x 2 - \bar{a}_y 2 + \bar{a}_z).$

$\theta = \cos^{-1}(\bar{a}_R \cdot \bar{a}_B) = \cos^{-1}\left(-\frac{19}{3\sqrt{50}}\right) = 154°.$

P.2-18 $\bar{a}_R = \bar{a}_x \sin\theta\cos\phi + \bar{a}_y \sin\theta\sin\phi + \bar{a}_z\cos\theta = \dfrac{\bar{a}_x x + \bar{a}_y y + \bar{a}_z z}{\sqrt{x^2+y^2+z^2}}$,

$$\bar{a}_\theta = \bar{a}_x \cos\theta\cos\phi + \bar{a}_y\cos\theta\sin\phi - \bar{a}_z\sin\theta = \frac{\bar{a}_x xz + \bar{a}_y yz - \bar{a}_z(x^2+y^2)}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}},$$

$$\bar{a}_\phi = -\bar{a}_x \sin\phi + \bar{a}_y\cos\phi = \frac{-\bar{a}_x y + \bar{a}_y x}{\sqrt{x^2+y^2}}.$$

P.2-19

a) $\bar{a}_x \cdot \bar{a}_\phi = -\sin\phi$, b) $\bar{a}_\theta \cdot \bar{a}_y = \cos\theta\sin\phi$, c) $\bar{a}_r \times \bar{a}_x = -\bar{a}_z \sin\phi$,

d) $\bar{a}_R \cdot \bar{a}_r = \sin\theta$, e) $\bar{a}_y \cdot \bar{a}_R = \sin\theta\sin\phi$, f) $\bar{a}_R \cdot \bar{a}_z = \cos\theta$,

g) $\bar{a}_R \times \bar{a}_z = -\bar{a}_\phi \sin\theta$, h) $\bar{a}_\theta \cdot \bar{a}_z = -\sin\theta$, i) $\bar{a}_z \times \bar{a}_\theta = \bar{a}_\phi \cos\theta$.

P.2-20 $\bar{F}\cdot d\bar{\ell} = [\bar{a}_x xy + \bar{a}_y(3x-y^2)]\cdot(\bar{a}_x dx + \bar{a}_y dy)$
$= xy\,dx + (3x - y^2)\,dy$.

a) *Along direct path ①. The equation of P_1P_2 is*

$$y = \tfrac{3}{2}(x-1).$$

$$\underset{(\text{Path ①})}{\int_{P_1}^{P_2}} \bar{F}\cdot d\bar{\ell} = \underset{(\text{Path ①})}{\int_{P_1}^{P_2}} [xy\,dx + (3x-y^2)dy]$$

$$= \int_5^2 \frac{3}{2}x(x-1)dx + \int_6^3 (2y+3-y^2)\,dy$$

$$= -37 + 27 = -10.$$

b) *Along path ②. This path has two straight-line segments. From P_1 to A: $x=5$, $dx=0$, $\bar{F}\cdot d\bar{\ell} = (15-y^2)dy$. From A to P_2: $y=3$, $dy=0$, $\bar{F}\cdot d\bar{\ell} = 3x\,dx$. Hence,*

$$\underset{(\text{Path ②})}{\int_{P_1}^{P_2}} \bar{F}\cdot d\bar{\ell} = \int_6^3 (15-y^2)\,dy + \int_5^2 3x\,dx = 18 - 24 = -6.$$

$$\neq \underset{(\text{Path ①})}{\int_{P_1}^{P_2}} \bar{F}\cdot d\bar{\ell} \longrightarrow \text{Vector field } \bar{F} \text{ is not conservative}$$

P.2-21 $\int_{P_1}^{P_2} \bar{E}\cdot d\bar{\ell} = \int_{P_1}^{P_2} (y\,dx + x\,dy)$.

a) $x = 2y^2$, $dx = 4y\,dy$: $\int_{P_1}^{P_2} \bar{E}\cdot d\bar{\ell} = \int_1^2 (4y^2 dy + 2y^2 dy) = 14$

b) $x = 6y - 4$, $dx = 6\,dy$; $\int_{P_1}^{P_2} \bar{E} \cdot d\bar{\ell} = \int_1^2 [6y\,dy + (6y-4)]dy = 14.$

Equal line integrals along two specific paths do not necessarily imply a conservative field. $\bar{E}$ is a conservative field in this case because $\bar{E} = \nabla(xy + c)$.

P.2-22

$$\begin{bmatrix} E_r \\ E_\phi \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} r\sin\phi \\ r\cos\phi \end{bmatrix}.$$

$\bar{E} = \bar{a}_r\, r\sin 2\phi + \bar{a}_\phi\, r\cos 2\phi.$

$\bar{E} \cdot d\bar{\ell} = r\sin 2\phi\, dr + r^2 \cos 2\phi\, d\phi.$

$P_3(3, 4, -1) = P_3(5, 53.1°, -1)$; $P_4(4, -3, -1) = P_4(5, -36.9°, -1)$

There is no change in $r (=5)$ from P_3 to P_4.

$$\therefore \int_{P_3}^{P_4} \bar{E} \cdot d\bar{\ell} = 5^2 \int_{53.1°}^{-36.9°} \cos 2\phi\, d\phi = -24.$$

P.2-23 a) $\nabla V = [\bar{a}_x(\frac{\pi}{2}\cos\frac{\pi}{2}x)(\sin\frac{\pi}{3}y) + \bar{a}_y(\sin\frac{\pi}{2}x)(\frac{\pi}{3}\cos\frac{\pi}{3}y) - \bar{a}_z(\sin\frac{\pi}{2}x)(\sin\frac{\pi}{3}y)]e^{-z}.$

$(\nabla V)_P = -(\bar{a}_y \frac{\pi}{6} + \bar{a}_z \frac{\sqrt{3}}{2})e^{-3} = -(\bar{a}_y 0.026 + \bar{a}_z 0.043).$

b) $\overline{PO} = -\bar{a}_x - \bar{a}_y 2 - \bar{a}_z 3$; $\bar{a}_{PO} = -\frac{1}{\sqrt{14}}(\bar{a}_x + \bar{a}_y 2 + \bar{a}_z 3).$

$\therefore (\nabla V)_P \cdot \bar{a}_{PO} = \frac{1}{\sqrt{14}}(\frac{\pi}{3} + \frac{3\sqrt{3}}{2})e^{-3} = 0.0485.$

P.2-24 On the surface of the sphere, $R = 5$.

$$\oint_S (\bar{a}_R 3\sin\theta) \cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi} (\bar{a}_R 3\sin\theta) \cdot (\bar{a}_R 5^2 \sin\theta)\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} 75\sin^2\theta\, d\theta\, d\phi = 75\pi^2.$$

P.2-25 The first step is to find the expression for the unit normal $\bar{a}_n = \bar{a}_x \ell + \bar{a}_y m + \bar{a}_z p$ to the given surface. The given four corner points of the surface lead to the following four equations:

Corner (0,0,2):
$$\ell x + my + p(z-2) = 0 \qquad ①$$

Corner (2,0,2):
$$\ell(x-2) + my + p(z-2) = 0. \qquad ②$$

Corner (2,2,0):
$$\ell(x-2) + m(y-2) + pz = 0 \qquad ③$$

Corner (0,2,0):
$$\ell x + m(y-2) + pz = 0. \qquad ④$$

The direction cosines satisfy the condition:
$$\ell^2 + m^2 + p^2 = 1. \qquad ⑤$$

From ①-⑤ we obtain $\ell = 0$, and $m = p = 1/\sqrt{2}$.

Thus, $\bar{a}_n = \frac{1}{\sqrt{2}}(\bar{a}_y + \bar{a}_z)$, $\bar{F}\cdot\bar{a}_n = \frac{5}{\sqrt{2}}$ (a constant),

and $\int_S \bar{F}\cdot d\bar{s} = \frac{5}{\sqrt{2}} S = \frac{5}{\sqrt{2}}(2\times 2\sqrt{2}) = 20.$

P.2-26 In spherical coordinates, $\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R)$, if $\bar{A} = \bar{a}_R A_R$.

a) $\bar{A} = f_1(\bar{R}) = \bar{a}_R R^n$, $A_R = R^n$.
$$\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^{n+2}) = (n+2)R^{n-1}.$$

b) $\bar{A} = f_2(\bar{R}) = \bar{a}_R \frac{k}{R^2}$, $A_R = kR^{-2}$.
$$\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(k) = 0.$$

P.2-27 For radial vector $\bar{R} = \bar{a}_R R$, $\bar{\nabla}\cdot\bar{R} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 R) = 3$.
Using divergence theorem, we have
$$\frac{1}{3}\oint_S \bar{R}\cdot d\bar{s} = \frac{1}{3}\int_V \bar{\nabla}\cdot\bar{R}\, dv = \frac{1}{3}(3V) = V.$$

P.2-29 $\oint_S \bar{A}\cdot d\bar{s} = \left(\int_{\text{top face}} + \int_{\text{bottom face}} + \int_{\text{walls}}\right)\bar{A}\cdot d\bar{s}.$

Top face $(z=4)$: $\bar{A} = \bar{a}_r r^2 + \bar{a}_z 8$, $\quad d\bar{s} = \bar{a}_z ds$.

$$\int_{\substack{top\\face}} \bar{A}\cdot d\bar{s} = \int_{\substack{top\\face}} 8\, ds = 8(\pi 5^2) = 200\pi.$$

Bottom face $(z=0)$: $\bar{A} = \bar{a}_r r^2$; $\quad d\bar{s} = -\bar{a}_z ds$.

$$\int_{\substack{bottom\\face}} \bar{A}\cdot d\bar{s} = 0.$$

Walls $(r=5)$: $\bar{A} = \bar{a}_r 25 + \bar{a}_z 2z$, $\quad d\bar{s} = \bar{a}_r ds$.

$$\int_{walls} \bar{A}\cdot d\bar{s} = 25\int_{walls} ds = 25(2\pi 5\times 4) = 1000\pi.$$

$$\therefore \oint_S \bar{A}\cdot d\bar{s} = 200\pi + 0 + 1000\pi = 1{,}200\pi.$$

$$\bar{\nabla}\cdot\bar{A} = 3r+2, \quad \int_V \bar{\nabla}\cdot\bar{A}\, dv = \int_0^4\int_0^{2\pi}\int_0^5 \bar{\nabla}\cdot\bar{A}\; r\,dr\,d\phi\,dz = 1{,}200\pi = \oint_S \bar{A}\cdot d\bar{s}.$$

P.2-30 $\quad \bar{\nabla}\cdot\bar{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{\partial}{\partial z}F_z = k_2.$

$$\int_V \bar{\nabla}\cdot\bar{F}\, dv = k_2 V = k_2(\pi 2^2\times 6) = 24\pi k_2 \neq \oint_S \bar{A}\cdot d\bar{s}.$$

Divergence theorem fails here because $\bar{F}$ has a singularity inside the volume at $r=0$.

P.2-31 $\quad$ Eq. (2-98): $\quad \bar{\nabla}\cdot\bar{A} = \lim_{\Delta v\to 0}\dfrac{\oint_S \bar{A}\cdot d\bar{s}}{\Delta v}$. $\qquad$ ①

Referring to Fig. 2-15, we note that the areas on the opposite sides of a differential volume in cylindrical coordinates are the same in ϕ- and z-directions, but are different in the r-direction. Let us first evaluate the contributions to $\oint_S \bar{A}\cdot d\bar{s}$ of the inside and outside faces.

On the inside face:

$$\int_{\substack{inside\\face}} \bar{A}\cdot d\bar{s} = \bar{A}_{\substack{inside\\face}}\cdot\Delta\bar{s}_{\substack{inside\\face}} = -A_r\left(r_0-\frac{\Delta r}{2},\phi_0,z_0\right)\left(r_0-\frac{\Delta r}{2}\right)\Delta\phi\,\Delta z$$

$$= -\left[A_r(r_0,\phi_0,z_0) - \frac{\Delta r}{2}\frac{\partial A_r}{\partial r}\bigg|_{(r_0,\phi_0,z_0)} + H.O.T.\right]\left(r_0-\frac{\Delta r}{2}\right)\Delta\phi\,\Delta z. \qquad ②$$

On the outside face:

$$\int_{\substack{\text{outside}\\\text{face}}} \bar{A}\cdot d\bar{s} = A_r\left(r_0+\frac{\Delta r}{2},\phi_0,z_0\right)\left(r_0+\frac{\Delta r}{2}\right)\Delta\phi\,\Delta z$$

$$= \left[A_r(r_0,\phi_0,z_0)+\frac{\Delta r}{2}\frac{\partial A_r}{\partial r}\bigg|_{(r_0,\phi_0,z_0)}+H.O.T.\right]\left(r_0+\frac{\Delta r}{2}\right)\Delta\phi\,\Delta z. \quad ③$$

Adding ② and ③, we have

$$\left[\int_{\substack{\text{inside}\\\text{face}}}+\int_{\substack{\text{outside}\\\text{face}}}\right]\bar{A}\cdot d\bar{s} = \left(A_r+r_0\frac{\partial A_r}{\partial r}\right)\bigg|_{(r_0,\phi_0,z_0)}\Delta r\,\Delta\phi\,\Delta z+H.O.T.$$

$$= \frac{\partial}{\partial r}(rA_r)\bigg|_{(r_0,\phi_0,z_0)}\Delta r\,\Delta\phi\,\Delta z+H.O.T., \quad ④$$

where H.O.T. contain second and higher powers of Δr. The sum of the contributions of the front and back faces (differential area $=\Delta r\,\Delta z$) is

$$\left[\int_{\substack{\text{front}\\\text{face}}}+\int_{\substack{\text{back}\\\text{face}}}\right]\bar{A}\cdot d\bar{s} = \frac{\partial A_\phi}{\partial\phi}\bigg|_{(r_0,\phi_0,z_0)}\Delta r\,\Delta\phi\,\Delta z+H.O.T., \quad ⑤$$

where H.O.T. contain second and higher powers of $\Delta\phi$. Similarly, the sum of the contributions of the top and bottom faces (differential area $=r_0\,\Delta r\,\Delta\phi$) is

$$\left[\int_{\substack{\text{top}\\\text{face}}}+\int_{\substack{\text{bottom}\\\text{face}}}\right]\bar{A}\cdot d\bar{s} = \left(r\frac{\partial A_z}{\partial z}\right)\bigg|_{(r_0,\phi_0,z_0)}\Delta r\,\Delta\phi\,\Delta z+H.O.T., \quad ⑥$$

where H.O.T. contain second and higher powers of Δz.

Combining ④, ⑤, and ⑥ in ①, dividing by $\Delta v = r_0\,\Delta r\,\Delta\phi\,\Delta z$, and letting $\Delta r\,\Delta\phi\,\Delta z\to 0$, we get

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z},$$

where the subscript 0 has been dropped for simplicity.

P.2-33 a) $\bar{D}=\bar{a}_R\frac{\cos^2\phi}{R^3}$, $\quad ds=R^2\sin\theta\,d\theta\,d\phi$.

$$\oint\bar{D}\cdot d\bar{s}=\int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{2}-1\right)\sin\theta\,d\theta\,\cos^2\phi\,d\phi=-\pi.$$

b) $\bar{\nabla}\cdot\bar{D}=-\frac{\cos^2\phi}{R^4}$, $\quad dv=R^2\sin\theta\,dR\,d\theta\,d\phi$.

$$\int_V\bar{\nabla}\cdot\bar{D}\,dv=\int_0^{2\pi}\int_0^{\pi}\int_1^2\left(-\frac{\cos^2\phi}{R^2}\right)\sin\theta\,dR\,d\theta\,d\phi=-\pi.$$

P.2-34 $\bar{A}=\bar{a}_x 3x^2y^3-\bar{a}_y x^3y^2$; $d\bar{\ell}=\bar{a}_x dx+\bar{a}_y dy$.

a) $\bar{A}\cdot d\bar{\ell}=3x^2y^3dx-x^3y^2dy$

Path ①: $x=y$, $\int_{①}\bar{A}\cdot d\bar{\ell}=\int_1^2 2x^5dx=21$

Path ②: $x=2, dx=0$, $\int_{②}\bar{A}\cdot d\bar{\ell}=\int_2^1(-2^3y^2)dy=\frac{56}{3}$

Path ③: $y=1, dy=0$, $\int_{③}\bar{A}\cdot d\bar{\ell}=\int_2^1 3x^2dx=-7$

$$\therefore\oint\bar{A}\cdot d\bar{\ell}=21+\frac{56}{3}-7=\frac{98}{3}=32\frac{2}{3}.$$

b) $\bar{\nabla}\times\bar{A}=-\bar{a}_z 12x^2y^2$, $\quad d\bar{s}=-\bar{a}_z\,dx\,dy$.

$$\int_S(\bar{\nabla}\times\bar{A})\cdot d\bar{s}=12\int_1^2x^2dx\int_1^x y^2dy=32\frac{2}{3}.$$

P.2-35 Eq. (2-126): $(\bar{\nabla}\times\bar{A})_R=\lim_{\Delta s_R\to 0}\frac{1}{\Delta s_R}\left(\oint_{C_R}\bar{A}\cdot d\bar{\ell}\right)$. ①

where $\Delta s_R=R_0^2\sin\theta_0\Delta\theta\Delta\phi$ ② and the contour consists of the four sides 1, 2, 3, & 4.

Side 1: $d\bar{\ell}=\bar{a}_\theta(R_0\Delta\theta)$,

$\bar{A}\cdot d\bar{\ell}=A_\theta(R_0,\theta_0,\phi_0-\frac{\Delta\phi}{2})R_0\Delta\theta$,

where $A_\theta(R_0,\theta_0,\phi_0-\frac{\Delta\phi}{2})=A_\theta(R_0,\theta_0,\phi_0)-\frac{\Delta\phi}{2}\frac{\partial A_\theta}{\partial\phi}\Big|_{(R_0,\theta_0,\phi_0)}+\text{H.O.T.}$

$$\int_{\text{side 1}}\bar{A}\cdot d\bar{\ell}=\left\{\left[A_\theta-\frac{\Delta\phi}{2}\frac{\partial A_\theta}{\partial\phi}\right]_{(R_0,\theta_0,\phi_0)}+\text{H.O.T.}\right\}R_0\Delta\theta.\quad ③$$

Side 3: $d\bar{\ell} = -\bar{a}_\theta (R_0 \Delta\theta)$, $\bar{A}\cdot d\bar{\ell} = A_\theta (R_0, \theta_0, \phi_0 + \frac{\Delta\phi}{2}) R_0 \Delta\theta$,

$$\int_{side\,3} \bar{A}\cdot d\bar{\ell} = -\left\{ \left[A_\theta + \frac{\Delta\phi}{2} \frac{\partial A_\theta}{\partial\phi} \right]_{(R_0,\theta_0,\phi_0)} + H.O.T. \right\} R_0 \Delta\theta. \quad ④$$

Combining ③ and ④:

$$\int_{sides\,1\,\&\,3} \bar{A}\cdot d\bar{\ell} = \left(-\frac{\partial A_\theta}{\partial\phi} + H.O.T. \right)\Bigg|_{(R_0,\theta_0,\phi_0)} R_0 \Delta\theta \Delta\phi. \quad ⑤$$

Side 2: $d\bar{\ell} = \bar{a}_\phi R_0 \sin(\theta_0 + \frac{\Delta\theta}{2})\Delta\phi$, $\bar{A}\cdot d\bar{\ell} = A_\phi (R_0, \theta_0 + \frac{\Delta\theta}{2}, \phi_0) R_0 \sin(\theta_0 + \frac{\Delta\theta}{2})\Delta\phi$.

$$\int_{side\,2} \bar{A}\cdot d\bar{\ell} \cong \left\{ \left[A_\phi + \frac{\Delta\theta}{2} \frac{\partial A_\phi}{\partial\theta} \right]_{(R_0,\theta_0,\phi_0)} + H.O.T. \right\} R_0 \left(\sin\theta_0 + \frac{\Delta\theta}{2}\cos\theta_0 \right) \Delta\phi. \quad ⑥$$

Side 4:

$$\int_{side\,4} \bar{A}\cdot d\bar{\ell} \cong -\left\{ \left[A_\phi - \frac{\Delta\theta}{2} \frac{\partial A_\phi}{\partial\theta} \right]_{(R_0,\theta_0,\phi_0)} + H.O.T. \right\} R_0 \left(\sin\theta_0 - \frac{\Delta\theta}{2}\cos\theta_0 \right) \Delta\phi. \quad ⑦$$

Combining ⑥ and ⑦:

$$\int_{sides\,2\,\&\,4} \bar{A}\cdot d\bar{\ell} = \frac{\partial A_\phi}{\partial\theta}\Bigg|_{(R_0,\theta_0,\phi_0)} R_0 \sin\theta_0 \Delta\theta\Delta\phi + A_\phi R_0 \cos\theta_0 \Delta\theta\Delta\phi + H.O.T.$$

$$= \frac{\partial}{\partial\theta}(A_\phi \sin\theta)\Bigg|_{(R_0,\theta_0,\phi_0)} R_0 \Delta\theta\Delta\phi + H.O.T. \quad ⑧$$

Substituting ②, ⑤, and ⑧ in ①, we obtain

$$(\bar{\nabla}\times\bar{A})_R = \frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi} \right],$$

where the subscript 0 has been dropped for simplicity.

P.2-36 $\bar{\nabla}\times\bar{A} = \frac{1}{R\sin\theta}\left(\bar{a}_R \cos\theta \sin\frac{\phi}{2} - \bar{a}_\theta \sin\theta \sin\frac{\phi}{2} \right)$.

$$\int_S (\bar{\nabla}\times\bar{A})\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi/2} (\bar{\nabla}\times\bar{A})_{R=b} \cdot (\bar{a}_R b^2 \sin\theta \, d\theta \, d\phi) = 4b.$$

$$\oint_C \bar{A}\cdot d\bar{\ell} = \int_0^{2\pi} (\bar{A})_{\substack{R=b \\ \theta=\pi/2}} \cdot (\bar{a}_\phi b \, d\phi) = \int_0^{2\pi} b \sin\frac{\phi}{2} d\phi = 4b.$$

P.2-39. $\bar{F} = \bar{a}_x (x + c_1 z) + \bar{a}_y (c_2 x - 3z) + \bar{a}_z (x + c_4 y + c_5 z)$.

a) $\bar{F}$ irrotational $\longrightarrow$ $\bar{\nabla}\times\bar{F} = 0$,

or

$$\bar{a}_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \bar{a}_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \bar{a}_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial F_y} \right) = 0,$$

which gives three equations:

$$\frac{\partial}{\partial y}(x+c_3y+c_4z)-\frac{\partial}{\partial z}(c_2x-3z)=0 \longrightarrow c_3+3=0 \longrightarrow c_3=-3$$

$$\frac{\partial}{\partial z}(x+c_1z)-\frac{\partial}{\partial x}(x+c_3y+c_4z)=0 \longrightarrow c_1-1=0 \longrightarrow c_1=1.$$

$$\frac{\partial}{\partial x}(c_2x-3z)-\frac{\partial}{\partial y}(x+c_1z)=0 \longrightarrow c_2=0.$$

b) $\bar{F}$ also solenoidal $\longrightarrow \bar{\nabla}\cdot F=0$,

or

$$\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}=0.$$

or

$$\frac{\partial}{\partial x}(x+c_1z)+\frac{\partial}{\partial y}(c_2x-3z)+\frac{\partial}{\partial z}(x+c_3y+c_4z)=0,$$

or

$$1+c_4=0 \longrightarrow c_4=-1.$$

c) $\bar{F}=-\bar{\nabla}V \longrightarrow \bar{a}_x(x+z)-\bar{a}_y 3z+\bar{a}_z(x-3y-z)$

$$=-a_x\frac{\partial V}{\partial x}-a_y\frac{\partial V}{\partial y}-a_z\frac{\partial V}{\partial z}$$

$$\frac{\partial V}{\partial x}=-(x+z) \longrightarrow V=-\frac{x^2}{2}-xz+f_1(y,z)$$

$$\frac{\partial V}{\partial y}=3z \longrightarrow V=3yz+f_2(x,z).$$

$$\frac{\partial V}{\partial z}=-x+3y+z \longrightarrow V=-xz+3yz+\frac{z^2}{2}+f_3(x,y).$$

$$\therefore\ V=-\frac{x^2}{2}-xz+3yz+\frac{z^2}{2}.$$

Chapter 4

Solution of Electrostatic Problems

P.4-1 Use subscripts d and a to denote dielectric and air regions respectively. $\nabla^2 V = 0$ in both regions.

$$V_d = c_1 y + c_2,\quad \bar{E}_d = -\bar{a}_y c_1,\quad \bar{D}_d = -\bar{a}_y \epsilon_0 \epsilon_r c_1.$$

$$V_a = c_3 y + c_4,\quad \bar{E}_a = -\bar{a}_y c_3,\quad \bar{D}_a = -\bar{a}_y \epsilon_0 c_3.$$

B.C.: At $y=0$, $V_d = 0$; at $y=d$, $V_a = V_0$;
at $y = 0.8d$: $V_d = V_a$, $\bar{D}_d = \bar{D}_a$.

Solving: $c_1 = \dfrac{V_0}{(0.8+0.2\epsilon_r)d}$, $c_2 = 0$, $c_3 = \dfrac{\epsilon_r V_0}{(0.8+0.2\epsilon_r)d}$, $c_4 = \dfrac{(1-\epsilon_r)V_0}{1+0.25\epsilon_r}$.

a) $V_d = \dfrac{5yV_0}{(4+\epsilon_r)d}$, $\quad \bar{E}_d = -\bar{a}_y \dfrac{5V_0}{(4+\epsilon_r)d}$.

b) $V_a = \dfrac{5\epsilon_r y - 4(\epsilon_r - 1)d}{(4+\epsilon_r)d} V_0$, $\quad \bar{E}_a = -\bar{a}_y \dfrac{5\epsilon_r V_0}{(4+\epsilon_r)d}$.

c) $(\rho_s)_{y=d} = -(D_a)_{y=d} = \dfrac{5\epsilon_0\epsilon_r V_0}{(4+\epsilon_r)d}$.

$(\rho_s)_{y=0} = (D_d)_{y=0} = -\dfrac{5\epsilon_0\epsilon_r V_0}{(4+\epsilon_r)d}$.

P.4-3 At a point where V is a maximum (minimum) the second derivatives of V with respect to x, y, and z would all be negative (positive); their sum could not vanish, as required by Laplace's equation.

P.4-6 Poisson's eq. $\nabla^2 V = -\dfrac{A}{\epsilon r} \longrightarrow \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial V}{\partial r}\right) = -\dfrac{A}{\epsilon r}$.

Solution: $V = -\dfrac{A}{\epsilon} r + c_1 \ln r + c_2$.

B.C.: $\begin{cases} \text{At } r=a, & V_0 = -\frac{A}{\epsilon}a + c_1 \ln a + c_2. \\ \text{At } r=b, & 0 = -\frac{A}{\epsilon}b + c_1 \ln b + c_2. \end{cases}$ $\quad c_1 = \dfrac{\frac{A}{\epsilon}(b-a) - V_0}{\ln(b/a)}$, $\quad c_2 = \dfrac{V_0 \ln b + \frac{A}{\epsilon}(a\ln b - b\ln a)}{\ln(b/a)}$.

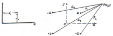
P.4-7

$$\bar{E}\Big|_{y=0} = -\bar{a}_y \frac{Q}{4\pi\epsilon R^2}(2\sin\theta) = -\bar{a}_y \frac{Qd}{2\pi\epsilon(d^2+r^2)^{3/2}}.$$

a) $\rho_s = \bar{a}_y \cdot \epsilon \bar{E}\Big|_{y=0} = -\dfrac{Qd}{2\pi(d^2+r^2)^{3/2}}$

b) $\int_0^\infty \rho_s\, 2\pi r\, dr = -Q$.

P.4-8



Consider the conditions in the xy-plane ($z=0$).

a) $V_P = \frac{Q}{4\pi\epsilon}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right)$, where

$R_1 = [(x-d_1)^2 + (y-d_2)^2]^{1/2}$, $\quad R_2 = [(x-d_1)^2 + (y+d_2)^2]^{1/2}$,

$R_3 = [(x+d_1)^2 + (y+d_2)^2]^{1/2}$, $\quad R_4 = [(x+d_1)^2 + (y-d_2)^2]^{1/2}$.

$$\bar{E}_P = -\bar{\nabla} V_P = -\bar{a}_x \frac{\partial V_P}{\partial x} - \bar{a}_y \frac{\partial V_P}{\partial y}$$

$$= \bar{a}_x \frac{Q}{4\pi\epsilon}\left[-\frac{x-d_1}{R_1^3} + \frac{x-d_1}{R_2^3} - \frac{x+d_1}{R_3^3} + \frac{x+d_1}{R_4^3}\right]$$

$$+ \bar{a}_y \frac{Q}{4\pi\epsilon}\left[-\frac{y-d_2}{R_1^3} + \frac{y+d_2}{R_2^3} - \frac{y+d_2}{R_3^3} + \frac{y-d_2}{R_4^3}\right].$$

E_P will have a z-component if the point P does not lie in the xy-plane.

b) On the conducting half-planes, $\rho_s = D_n = \epsilon E_n$.

Along the x-axis, $y=0$: $R_1 = [(x-d_1)^2 + d_2^2]^{1/2} = R_2$,

and $R_3 = [(x+d_1)^2 + d_2^2]^{1/2} = R_4$.

$E_x = 0$, $E_y = \frac{Q}{2\pi\epsilon}\left[\frac{d_2}{R_1^3} - \frac{d_2}{R_3^3}\right]$.

$$\therefore \rho_s(y=0) = \frac{Qd_2}{2\pi}\left\{\frac{1}{[(x-d_1)^2+d_2^2]^{3/2}} - \frac{1}{[(x+d_1)^2+d_2^2]^{3/2}}\right\}$$

$$= \begin{cases} 0, & \text{at } x=0. \\ \text{max.}, & \text{at } x=d_1. \end{cases}$$

Similarly for $\rho_s(x=0)$ on the vertical conducting conducting plane by changing x to y and $d_1 \leftrightarrow d_2$.

P.4-10 Refer to Example 4-4.

$$C' = \frac{2\pi\epsilon_0}{\ln\left[(h/a) + \sqrt{(h/a)^2 - 1}\right]} = \frac{2\pi\epsilon_0}{\cosh^{-1}(h/a)} \quad (\text{F/m}).$$

P.4-11 Same as C_{12} in problem P.3-38.

P.4-12 a) From Eqs. (4-48) and (4-49):

$$V_P = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{\sqrt{(x+b)^2+y^2}}{\sqrt{(x-b)^2+y^2}}.$$

$$\bar{E}_P = -\bar{a}_x \frac{\partial V_P}{\partial x} - \bar{a}_y \frac{\partial V_P}{\partial y} = \frac{\rho_\ell}{2\pi\epsilon_0} \left\{ \frac{\bar{a}_x 2b(y^2+b^2-x^2) - \bar{a}_y 4bxy}{[(x+b)^2+y^2][y^2+(x-b)^2]} \right\}.$$

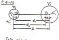
b) Equation for lines everywhere tangent to the electric field lines is obtained by requiring

$$\frac{dy}{dx} = \frac{E_y}{E_x} = -\frac{2xy}{y^2+b^2-x^2}.$$

Which reduces to $\dfrac{d(x^2+y^2)}{(x^2+y^2)-b^2} = \dfrac{dy}{y}$.

Integrating, we obtain $x^2+y^2-2Ky=b^2$,
or $x^2+(y-K)^2=b^2+K^2$;
where K is a constant. — Circles of radii $\sqrt{b^2+K^2}$ having centers at $(0, K)$.

P.4-13



$V_1 = -\frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{a_1}{d_1}$, $V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{a_2}{d_2}$.

Capacitance per unit length

$$C' = \frac{\rho_\ell}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\ln \frac{d_1 d_2}{a_1 a_2}}.$$

Four equations:

$a_1^2 = d_{i1} d_1$, $a_2^2 = d_{i2} d_2$,

$d_1 + d_{i2} = D$, $d_2 + d_{i1} = D$.

We obtain

$$\frac{d_1 d_2}{a_1 a_2} = \frac{a_1 a_2}{d_{i1} d_{i2}} \quad \text{and} \quad a_1^2 + a_2^2 + d_1 d_2 + d_{i1} d_{i2} = D^2.$$

$$\frac{d_1 d_2}{a_1 a_2} = \frac{D^2}{2a_1 a_2} - \frac{a_1}{2a_2} - \frac{a_2}{2a_1} + \sqrt{\left(\frac{D^2}{2a_1 a_2} - \frac{a_1}{2a_2} - \frac{a_2}{2a_1}\right)^2 - 1}.$$

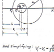
$$\therefore \quad C' = \frac{2\pi\epsilon_0}{\ln\left[\frac{1}{2}\left(\frac{D^2}{a_1 a_2} - \frac{a_1}{a_2} - \frac{a_2}{a_1}\right) + \sqrt{\frac{1}{4}\left(\frac{D^2}{a_1 a_2} - \frac{a_1}{a_2} - \frac{a_2}{a_1}\right)^2 - 1}\right]}$$

$$= \frac{2\pi\epsilon_0}{\cosh^{-1}\left[\frac{1}{2}\left(\frac{D^2}{a_1 a_2} - \frac{a_1}{a_2} - \frac{a_2}{a_1}\right)\right]} \quad \text{(F/m)}.$$

P. 4-14 Eq. (4-61): $c_1 = \frac{1}{2D}(a_2^2 - a_1^2 - D^2)$; Eq. (4-62): $c_2 = \frac{1}{2D}(a_2^2 - a_1^2 + D^2)$.

Eq. (4-55): $b^2 = c_1^2 - a_1^2$;

Eq. (4-56): $b^2 = c_2^2 - a_2^2$.



a) $V = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{r_2}{r_1}$ ← Distance to $-\rho_\ell$ / ← Distance to $+\rho_\ell$.

At P_1: $r_2 = b + (c_1 - a_1)$, $r_1 = b - (c_1 - a_1)$.

At P_2: $r_2 = b + (c_2 - a_2)$, $r_1 = b - (c_2 - a_2)$.

$$V_1 - V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \left[\frac{b + (c_1 - a_1)}{b - (c_1 - a_1)} \, \frac{b - (c_2 - a_2)}{b + (c_2 - a_2)} \right].$$

Expressing b, c_1 & c_2 in terms of D, a_1 & a_2 and simplifying:

$$V_1 - V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \left\{ \left(\frac{a_1^2 + a_2^2 - D^2}{2a_1 a_2} \right) + \left[\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1 a_2} \right)^2 - 1 \right]^{1/2} \right\}$$

$$C' = \frac{\rho_\ell}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\ln\left\{ \left(\frac{a_1^2 + a_2^2 - D^2}{2a_1 a_2} \right) + \left[\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1 a_2} \right)^2 - 1 \right]^{1/2} \right\}} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1 a_2} \right)} \quad (F/m).$$

b) Force per unit length $F' = \frac{\rho_\ell^2}{2\pi\epsilon_0 (4b^2)} = \frac{D^2 \rho_\ell^2}{2\pi\epsilon_0 [(a_1^2 + a_2^2 - D^2)^2 - 4a_1^2 D^2]}$ (N/m).

P. 4-15

$Q_i = -\frac{b}{d} Q$, $\quad d_i = \frac{b^2}{d}$.

a) $V_P = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{R_Q} - \frac{b}{d R_i} \right)$;

$R_Q = (R^2 + d^2 - 2Rd\cos\theta)^{1/2}$,

$R_i = (R^2 + d_i^2 - 2Rd_i\cos\theta)^{1/2}$.

b) $\rho_s = -\epsilon_0 \frac{\partial V}{\partial R}\Big|_{R=b} = -\frac{Q(b^2 - d^2)}{4\pi b (b^2 + d^2 - 2bd\cos\theta)^{3/2}}$.

P. 4-16 (See next page.)

P. 4-17 Required boundary conditions at $x=0$: $V_1 = V_2$, and $\epsilon_1 \frac{\partial V_1}{\partial x} = \epsilon_2 \frac{\partial V_2}{\partial x}$.

From Fig. 4-23 and the hypotheses in parts a) and b):

$$V_1 = \frac{Q}{4\pi\epsilon_1\sqrt{(x-d)^2 + y^2 + z^2}} + \frac{Q_1}{4\pi\epsilon_1\sqrt{(x+d)^2 + y^2 + z^2}},$$

$$V_2 = \frac{Q + Q_2}{4\pi\epsilon_2\sqrt{(d-x)^2 + y^2 + z^2}}.$$

In order to satisfy the b.c.'s at $x=0$, we require

$$\frac{Q - Q_1}{\epsilon_1} = \frac{Q + Q_2}{\epsilon_2} \quad \text{and} \quad Q + Q_1 = Q + Q_2 \longrightarrow Q_1 = Q_2 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_2 + \epsilon_1} Q.$$

P.4-16

a)

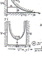
Q_0 and system of image charges:

In left sphere	In right sphere
Q_0 at $d_0 = 0$.	$-Q_1 = -\frac{a}{D}Q_0$ at d_1.
$Q_2 = \frac{a^2}{D(D-d_1)}Q_0$ at d_2.	$-Q_3 = -\frac{a}{D-d_2}Q_2 = -\frac{a^3}{D(D-d_1)(D-d_2)}Q_0$ at d_3.
$Q_4 = \frac{a^4}{D(D-d_1)(D-d_2)(D-d_3)}Q_0$ at d_4.	⋮
⋮	
$Q_{2n} = Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}}$ at d_{2n} $(n = 2, 4, 6, \cdots)$	$-Q_{2n} = -Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}}$ at d_n. $(n=1, 3, 5, \cdots)$

$$d_m = \frac{a^2}{D-d_{m-1}}, \quad m = 1, 2, 3, \cdots; \; d_0 = 0.$$

b) $C = \frac{Q_0 + \sum Q_{2n}}{V_0} = 4\pi\epsilon_0 a \left[1 + \sum_{n=2,4,\cdots} \left(\prod_{m=1}^{n} \frac{a}{D-d_{m-1}}\right)\right].$

P.4-18 a) $V(x,y) = c_1 xy$. Required boundary conditions:



① $V(0,y) = 0$ → Grounded conducting plane at $x=0$.

② $V(x,0) = 0$ → Grounded conducting plane at $y=0$.

If $V(a,b) = c_1 ab = V_0$ → $c_1 = \frac{V_0}{ab}$ → $V(x,y) = \frac{V_0}{ab}xy$

③ Shape of curved boundary defined by:
$V(x,y) = V_0 = \left(\frac{V_0}{ab}\right)xy$, or $xy = ab$ (a hyperbola).

b) $V(x,y) = c_2 \sin kx \sinh ky$. Required boundary conditions:

① $V(0,y) = 0$ → Grounded plane at $x=0$.

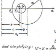
② $V(\pi/k, y) = 0$ → Grounded plane at $x = \pi/k = b$.

③ $V(x,0) = 0$ → Grounded plane at $y=0$.

④ If $V(b/2, a) = c_2 \sin\left(\frac{\pi}{b}\right)\left(\frac{b}{2}\right)\sinh\left(\frac{\pi}{b}\right)a = c_2 \sinh(a\pi/b) = V_0$,

→ $V(x,y) = \frac{V_0}{\sinh(a\pi/b)} \sin(\pi x/b)\sinh(\pi y/b)$. Curved boundary: $\sinh\left(\frac{\pi}{b}y\right) = \frac{\sinh(a\pi/b)}{\sin(\pi x/b)}$

P.4-14 Eq. (4-61): $c_1 = \frac{1}{2D}(a_2^2 - a_1^2 - D^2)$; Eq. (4-62): $c_2 = \frac{1}{2D}(a_2^2 - a_1^2 + D^2)$.



Eq. (4-55): $b^2 = c_1^2 - a_1^2$;

Eq. (4-56): $b^2 = c_2^2 - a_2^2$.

a) $V = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{r_2}{r_1}$ ← Distance to $-\rho_\ell$ / ← Distance to $+\rho_\ell$.

At P_1: $r_2 = b + (c_1 - a_1)$, $r_1 = b - (c_1 - a_1)$.

At P_2: $r_2 = b + (c_2 - a_2)$, $r_1 = b - (c_2 - a_2)$.

$$V_1 - V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln\left[\frac{b+(c_1-a_1)}{b-(c_1-a_1)} \cdot \frac{b-(c_2-a_2)}{b+(c_2-a_2)}\right].$$

Expressing b, c_1 & c_2 in terms of D, a_1 & a_2 and simplifying:

$$V_1 - V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln\left\{\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right) + \left[\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right)^2 - 1\right]^{1/2}\right\}$$

$$C' = \frac{\rho_\ell}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\ln\left\{\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right) + \left[\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right)^2 - 1\right]^{1/2}\right\}} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right)} \quad \text{(F/m)}.$$

b) Force per unit length $F' = \frac{\rho_\ell^2}{2\pi\epsilon_0(4b^2)} = \frac{D^2\rho_\ell^2}{2\pi\epsilon_0\left[(a_1^2+a_2^2-D^2)^2 - 4a_1^2D^2\right]}$ (N/m).

P.4-15

$Q_i = -\frac{b}{d}Q$, $\quad d_i = \frac{b^2}{d}$.

a) $V_P = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_Q} - \frac{b}{dR_i}\right)$;

$R_Q = (R^2 + d^2 - 2Rd\cos\theta)^{1/2}$,

$R_i = (R^2 + d_i^2 - 2Rd_i\cos\theta)^{1/2}$.

b) $\rho_s = -\epsilon_0 \frac{\partial V}{\partial R}\Big|_{R=b} = -\frac{Q(b^2-d^2)}{4\pi b(b^2+d^2-2bd\cos\theta)^{3/2}}$.

P.4-16 (See next page.)

P.4-17 Required boundary conditions at $x=0$: $V_1 = V_2$, and $\epsilon_1\frac{\partial V_1}{\partial x} = \epsilon_2\frac{\partial V_2}{\partial x}$.

From Fig. 4-23 and the hypotheses in parts a) and b:

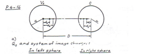
$$V_1 = \frac{Q}{4\pi\epsilon_1\sqrt{(x-d)^2+y^2+z^2}} - \frac{Q_1}{4\pi\epsilon_1\sqrt{(x+d)^2+y^2+z^2}},$$

$$V_2 = \frac{Q+Q_2}{4\pi\epsilon_2\sqrt{(d-x)^2+y^2+z^2}}.$$

In order to satisfy the b.c.'s at $x=0$, we require

$$\frac{Q-Q_1}{\epsilon_1} = \frac{Q+Q_2}{\epsilon_2} \text{ and } Q+Q_1 = Q+Q_2 \longrightarrow Q_1 = Q_2 = \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}Q.$$

P.4-16



a)

Q_0 and system of image charges:

In left sphere

Q_0 at $d_0 = 0$.

$Q_2 = \frac{a^2}{D(D-d_1)} Q_0$ at d_2.

$Q_4 = \frac{a^4}{D(D-d_1)(D-d_2)(D-d_3)} Q_0$ at d_4.

⋮

$Q_{2n} = Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}}$ at d_n. $(n = 2, 4, 6, \cdots)$

In right sphere

$-Q_1 = -\frac{a}{D} Q_0$ at d_1.

$-Q_3 = -\frac{a}{D-d_2} Q_2 = -\frac{a^3}{D(D-d_1)(D-d_2)} Q_0$ at d_3.

⋮

$-Q_{2n} = -Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}}$ at d_n. $(n = 1, 3, 5, \cdots)$

$d_m = \frac{a^2}{D-d_{m-1}}$, $m = 1, 2, 3, \cdots$; $d_0 = 0$.

b) $C = \frac{Q_0 + \sum Q_{2n}}{V_0} = 4\pi\epsilon_0 a \left[1 + \sum_{n=2,4,\cdots} \left(\prod_{m=1}^{n} \frac{a}{D-d_{m-1}}\right)\right]$.

P.4-18 a) $V(x,y) = c_1 xy$. Required boundary conditions:

① $V(0,y) = 0 \to$ Grounded conducting plane at $x=0$.

② $V(x,0) = 0 \to$ Grounded conducting plane at $y=0$.

If $V(a,b) = c_1 ab = V_0 \to c_1 = \frac{V_0}{ab} \to V(x,y) = \frac{V_0}{ab} xy$

③ Shape of curved boundary defined by: $V(x,y) = V_0 = \left(\frac{V_0}{ab}\right) xy$, or $xy = ab$ (a hyperbola).

b) $V(x,y) = c_2 \sin kx \sinh y$. Required boundary conditions:

① $V(0,y) = 0 \to$ Grounded plane at $x=0$.

② $V(\pi/k, y) = 0 \to$ Grounded plane at $x = \pi/k = b$.

③ $V(x,0) = 0 \to$ Grounded plane at $y=0$.

④ If $V(b/2, a) = c_2 \sin\left(\frac{\pi}{b}\right)\left(\frac{b}{2}\right) \sinh\left(\frac{\pi}{b}\right) a = c_2 \sinh(a\pi/b) = V_0$,

$\to V(x,y) = \frac{V_0}{\sinh(a\pi/b)} \sin(\pi x/b) \sinh(\pi y/b)$. Curved boundary: $\sinh\left(\frac{\pi}{b} y\right) = \frac{\sinh(a\pi/b)}{\sin(\pi x/b)}$

P.4-19 $V_n(x,y) = C_n' \cosh \frac{n\pi}{b}(x-a) \cos \frac{n\pi}{b} y.$

P.4-20 $V_n(x,y) = C_n \sinh \frac{n\pi}{b} x \sin \frac{n\pi}{b} y$

B.C. at $x=a$: $V_0 = \sum_{n=1}^{\infty} V_n(a,y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi}{b} a \sin \frac{n\pi}{b} y$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sinh(n\pi x/b)}{n \sinh(n\pi a/b)} \sin \frac{n\pi}{b} y.$$

P.4-21 $V(x,y) = \sum_n \sin \frac{n\pi}{a} x \left[A_n \sinh \frac{n\pi}{a} y + B_n \cosh \frac{n\pi}{a} y \right].$

At $y=0$, $V(x,0) = V_2 = \sum_n B_n \sin \frac{n\pi}{a} x \longrightarrow B_n = \begin{cases} \frac{4V_2}{n\pi}, & n=\text{odd}. \\ 0, & n=\text{even}. \end{cases}$

At $y=b$, $V(x,b) = V_1 = \sum_n \sin \frac{n\pi}{a} x \left[A_n \sinh \frac{n\pi}{a} b + B_n \cosh \frac{n\pi}{a} b \right]$

$\longrightarrow A_n \sinh \frac{n\pi}{a} b + B_n \cosh \frac{n\pi}{a} b = \begin{cases} \frac{4V_1}{n\pi}, & n=\text{odd}. \\ 0, & n=\text{even}. \end{cases}$

$$\therefore A_n = \begin{cases} \frac{4}{n\pi \sinh(n\pi b/a)} \left(V_1 - V_2 \cosh \frac{n\pi}{a} b \right), & n=\text{odd}. \\ 0, & n=\text{even}. \end{cases}$$

P.4-22 $V(x,y,z) = \sum_m \sum_n C_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \sinh k_{mn} z,$

where $k_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}.$

At $z=c$, $V(x,y,c) = V_0 = \sum_m \sum_n C_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \sinh k_{mn} c$

$$\longrightarrow C_{mn} = \begin{cases} \frac{16V_0}{mn\pi^2 \sinh k_{mn} c}; & m,n=\text{odd}. \\ 0, & m,n=\text{even}. \end{cases}$$

P.4-23 Solution: $V(\phi) = A_0 \phi + B_0.$

a) B.C. ①: $V(0) = 0 \longrightarrow B_0 = 0.$
B.C. ②: $V(\alpha) = V_0 = A_0 \alpha \longrightarrow A_0 = \frac{V_0}{\alpha}.$
$\therefore V(\phi) = \frac{V_0}{\alpha} \phi, \quad 0 \le \phi \le \alpha.$

b) B.C. ①: $V(\alpha) = V_0 = A_1 \alpha + B_1$
B.C. ②: $V(2\pi) = 0 = 2\pi A_1 + B$
$\longrightarrow A_1 = -\frac{V_0}{2\pi - \alpha}, \; B_1 = \frac{2\pi V_0}{2\pi - \alpha}.$

$$\therefore V(\phi) = \frac{V_0}{2\pi - \alpha}(2\pi - \phi), \quad \alpha \le \phi \le 2\pi.$$

Chapter 5

Steady Electric Currents

P.5-1 a) Integrating Eq. (5-16): $V(y)=\left[\frac{9J}{4\epsilon_0}\sqrt{\frac{m}{2e}}\right]^{2/3}y^{4/3}=V_0\left(\frac{y}{d}\right)^{4/3}$. ①

$E(y)=-\bar{a}_y\frac{dV(y)}{dy}=-\bar{a}_y\frac{4V_0}{3d}\left(\frac{y}{d}\right)^{1/3}$.

b) $\rho(y)=\epsilon_0\frac{dE(y)}{dy}=-\frac{4\epsilon_0V_0}{9d^2}\left(\frac{y}{d}\right)^{-2/3}$.

$Q=\int_0^d\rho(y)S\,dy=-\frac{4\epsilon_0V_0S}{9d^{4/3}}\int_0^d y^{-2/3}dy=-\frac{4\epsilon_0V_0}{3d}S$.

c) On the anode, $y=d$. $\rho_s=-\epsilon_0E(d)=\frac{4\epsilon_0V_0}{3d}$.

Total surface charge on anode $=\rho_sS=\frac{4\epsilon_0V_0}{3d}S$.

Total charge on cathode $=0$.

d) Substituting ① in Eq. (5-12):

$u=\frac{dy}{dt}=\left(\frac{2e}{m}V_0\right)^{1/2}\left(\frac{y}{d}\right)^{2/3}$, $\quad y^{-2/3}dy=\left(\frac{2eV_0}{md^{4/3}}\right)^{1/2}dt$.

Integrating: $y=\left(\frac{2eV_0}{9md^{4/3}}\right)^{3/2}t^3$.

$\therefore$ Transit time $\underset{(y=d)}{T_t}=3d\left(\frac{m}{2eV_0}\right)^{1/2}$.

For $V_0=200$ (V), $d=10^{-3}$ (m), $m=9.1\times10^{-31}$ (kg),

and $e=1.6\times10^{-19}$ (C), $T_t=3.58\times10^{-9}$ (s) $=3.58$ (ns).

P.5-3 R_1 = Resistance per unit length of core $=\frac{1}{\sigma S_1}=\frac{1}{\sigma\pi a^2}$.

R_2 = Resistance per unit length of coating $=\frac{1}{0.1\sigma S_2}$.

Let b = Thickness of coating. $\longrightarrow S_2=\pi(a+b)^2-\pi a^2=\pi(2ab+b^2)$.

a) $R_1=R_2 \longrightarrow b=(\sqrt{11}-1)a=2.32a$.

b) $I_1=I_2=\frac{I}{2}$. $\quad J_1=\frac{I}{2\pi a^2}$, $\quad J_2=\frac{I}{2S_2}=\frac{I}{20S_1}=\frac{I}{20\pi a^2}$.

$E_1=\frac{J_1}{\sigma}=\frac{I}{2\pi a^2\sigma}$, $E_2=\frac{J_2}{0.1\sigma}=\frac{I}{2\pi a^2\sigma}$.

Thus, $J_1=10J_2$ and $E_1=E_2$.

P.5-4 $I_1 = 0.1\,(A),\ P_{R1} = 3.33\,(mW)$; $I_2 = 0.02\,(A),\ P_{R2} = 8.00\,(mW)$;

$I_3 = 0.0133\,(A),\ P_{R3} = 5.31\,(mW)$; $I_4 = 0.0333\,(A),\ P_{R4} = 2.87\,(mW)$;

$I_5 = 0.0667\,(A),\ P_{R5} = 44.5\,(mW)$. $\sum_n P_{Rn} = V_0 I_1 = 70\,(mW)$.

P.5-5 $I_1 = 0.700\,(A),\ P_{R1} = 0.163\,(W)$; $I_2 = 0.140\,(A),\ P_{R2} = 0.392\,(W)$;

$I_3 = 0.093\,(A),\ P_{R3} = 0.261\,(W)$; $I_4 = 0.233\,(A),\ P_{R4} = 0.436\,(W)$;

$I_5 = 0.467\,(A),\ P_{R5} = 2.178\,(W)$.

P.5-6 $\rho_0 = \dfrac{Q_0}{(4\pi/3)b^3} = \dfrac{10^{-3}}{(4\pi/3)(0.1)^3} = 0.239\,(C/m^3)$, $\rho = \rho_0 e^{-(\sigma/\epsilon)t}$.

a) $R < b$: $\bar{E}_i = \bar{a}_R \dfrac{(4\pi/3)R^3\rho}{4\pi\epsilon R^2} = \bar{a}_R \dfrac{\rho_0 R}{3\epsilon} e^{-(\sigma/\epsilon)t} = \bar{a}_R\, 7.5\times10^9 R\, e^{-9.42\times10^{12}t}$ (V/m)

$R > b$: $\bar{E}_o = \bar{a}_R \dfrac{Q_0}{4\pi\epsilon_0 R^2} = \bar{a}_R \dfrac{9}{R^2}\times 10^6$ (V/m).

b) $R < b$: $\bar{J}_i = \sigma \bar{E}_i = \bar{a}_R\, 7.5\times10^{10} R\, e^{-9.42\times10^{12}t}$

$R > b$: $\bar{J}_o = 0$.

P.5-7 a) $e^{-(\sigma/\epsilon)t} = \dfrac{\rho}{\rho_0} = 0.01 \longrightarrow t = \dfrac{\ln 100}{(\sigma/\epsilon)} = 4.88\times10^{-13}(s) = 4.88\,(ps)$.

b) $W_i = \dfrac{\epsilon}{2}\int_{V'} E_i^2\, dv' = \dfrac{2\pi\rho_0^2 b^5}{45\epsilon} e^{-2(\sigma/\epsilon)t} = (W_i)_0 \left[e^{-(\sigma/\epsilon)t}\right]^2$.

$\therefore \dfrac{W_i}{(W_i)_0} = \left[e^{-(\sigma/\epsilon)t}\right]^2 = 0.01^2 = 10^{-4}$. Energy dissipated as heat loss.

c) Electrostatic energy stored outside the sphere $W_o = \dfrac{\epsilon_0}{2}\int_b^\infty E_o^2\, 4\pi R^2 dR = \dfrac{Q_0^2}{8\pi\epsilon_0 b} = 45\,(kJ)$

— constant.

P.5-8 a) $R = \dfrac{\ell}{\sigma S} = \dfrac{V}{I} \longrightarrow \sigma = \dfrac{\ell I}{SV} = 3.54\times10^7\,(S/m)$.

b) $E = \dfrac{V}{\ell} = 6\times10^{-3}\,(V/m)$.

c) $P = VI = 1\,(W)$.

d) $\rho_e = -\dfrac{\sigma}{\mu_e}$ The given electron mobility $1.4\times10^{-3}\,(m^2/V{\cdot}s)$ is that of a good conductor.

$u = \left|\dfrac{J}{\rho_e}\right| = \left|\dfrac{\mu_e J}{\sigma}\right| = |\mu_e E| = 1.4\times10^{-3}\times(6\times10^{-3})$

$= 8.4\times10^{-6}\,(m/s)$.

P.5-9 a) Eq. (3-118): $E_{1t} = E_{2t} \longrightarrow E_2 \sin\alpha_2 = E_1 \sin\alpha_1$.

Eq. (5-58): $J_{1n} = J_{2n} \longrightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n} \longrightarrow \sigma_2 E_2 \cos\alpha_2 = \sigma_1 E_1 \cos\alpha_1$.

$$\therefore \quad E_2 = E_1 \sqrt{\sin^2\alpha_1 + \left(\frac{\sigma_1}{\sigma_2}\cos\alpha_1\right)^2}. \qquad ①$$

$$\tan\alpha_2 = \frac{\sigma_2}{\sigma_1}\tan\alpha_1 \longrightarrow \alpha_2 = \tan^{-1}\left(\frac{\sigma_2}{\sigma_1}\tan\alpha_1\right). \qquad ②$$

b) Eq. (3-121b): $D_{2n} - D_{1n} = \rho_s \longrightarrow \epsilon_2 E_{2n} - \epsilon_1 E_{1n} = \rho_s$.

$$\rho_s = \left(\frac{\sigma_1}{\sigma_2}\epsilon_2 - \epsilon_1\right)E_{1n} = \left(\frac{\sigma_1}{\sigma_2}\epsilon_2 - \epsilon_1\right)E_1 \cos\alpha_1.$$

c) If both media are perfect dielectrics, $\sigma_1 = \sigma_2 = 0$, Eqs. ① and ② revert to Eqs. (3-130) and (3-129) respectively and $\rho_s = 0$.

P.5-10

$$\sigma(y) = \sigma_1 + (\sigma_2 - \sigma_1)\frac{y}{d}$$

a) Neglecting fringing effect and assuming a current density

$$\vec{J} = -\vec{a}_y J_0 \longrightarrow \vec{E} = \frac{\vec{J}}{\sigma} = -\vec{a}_y \frac{J_0}{\sigma(y)}.$$

$$V_0 = -\int_0^d \vec{E}\cdot\vec{a}_y\,dy = \int_0^d \frac{J_0\,dy}{\sigma_1 + (\sigma_2-\sigma_1)\frac{y}{d}} = \frac{J_0 d}{\sigma_2 - \sigma_1}\ln\frac{\sigma_2}{\sigma_1}.$$

$$R = \frac{V_0}{I} = \frac{V_0}{J_0 S} = \frac{d}{(\sigma_2 - \sigma_1)S}\ln\frac{\sigma_2}{\sigma_1}.$$

b) $(\rho_s)_u = \epsilon_0 E_y(d) = \frac{\epsilon_0 J_0}{\sigma_2} = \frac{\epsilon_0(\sigma_2-\sigma_1)V_0}{\sigma_2 d \ln(\sigma_2/\sigma_1)}$ on upper plate.

$(\rho_s)_\ell = -\epsilon_0 E_y(0) = -\frac{\epsilon_0 J_0}{\sigma_1} = -\frac{\epsilon_0(\sigma_2-\sigma_1)V_0}{\sigma_1 d \ln(\sigma_2/\sigma_1)}$ on lower plate.

c) $$\rho = \vec{\nabla}\cdot\vec{D} = \frac{d}{dy}(\epsilon_0 E) = -\epsilon_0 J_0 \frac{d}{dy}\left[\frac{1}{\sigma_1 + (\sigma_2-\sigma_1)y/d}\right] = \epsilon_0 J_0 \frac{(\sigma_2-\sigma_1)/d}{[\sigma_1 + (\sigma_2-\sigma_1)y/d]^2}$$

P.5-11

a) $C_1 = \frac{\epsilon_1 S}{d_1}$, $\quad G_1 = \frac{\sigma_1 S}{d_1}$;

$C_2 = \frac{\epsilon_2 S}{d_2}$, $\quad G_2 = \frac{\sigma_2 S}{d_2}$.

b) $$P = V^2 G = V^2 \frac{G_1 G_2}{G_1 + G_2} = V^2 S \frac{\sigma_1\sigma_2}{\sigma_1 d_2 + \sigma_2 d_1}.$$

P.5-12 Refer to Fig. 5-6. In the transient state, the equation of continuity must be satisfied at the interface.

$$-\frac{\partial \rho_{si}}{\partial t} = J_2 - J_1 = \sigma_2 E_2 - \sigma_1 E_1 \qquad ①$$

Now

$$E_1 d_1 + E_2 d_2 = V \qquad ②$$

$$\epsilon_2 E_2 - \epsilon_1 E_1 = \rho_{si} \qquad ③$$

Solving ③ and ③ for E_1 and E_2 in terms of $\mathcal{V}$ and ρ_{si}:

$$E_1 = \frac{\epsilon_2 \mathcal{V} - d_2 \rho_{si}}{\epsilon_2 d_1 + \epsilon_1 d_2} \quad ④ ; \qquad E_2 = \frac{\epsilon_1 \mathcal{V} + d_1 \rho_{si}}{\epsilon_2 d_1 + \epsilon_1 d_2} \quad ⑤ .$$

a) Substituting ④ and ⑤ in ①:

$$-\frac{\partial \rho_{si}}{\partial t} = \frac{\sigma_2 d_1 + \sigma_1 d_2}{\epsilon_2 d_1 + \epsilon_1 d_2} \rho_{si} + \frac{\epsilon_1 \sigma_2 - \epsilon_2 \sigma_1}{\epsilon_2 d_1 + \epsilon_1 d_2} \mathcal{V}. \qquad ⑥$$

Solution of ⑥:

$$\rho_{si} = \left(\frac{\epsilon_2 \sigma_1 - \epsilon_1 \sigma_2}{\sigma_2 d_1 + \sigma_1 d_2} \right) \mathcal{V} \left[1 - e^{-t/\tau} \right]. \qquad ⑦$$

where $\tau = \text{Relaxation time} = \frac{\epsilon_2 d_1 + \epsilon_1 d_2}{\sigma_2 d_1 + \sigma_1 d_2}$. ⑧

b) Using ④ and ⑤:

$$E_1 = \frac{\sigma_2 \mathcal{V}}{\sigma_2 d_1 + \sigma_1 d_2} (1 - e^{-t/\tau}) + \frac{\epsilon_2 \mathcal{V}}{\epsilon_2 d_1 + \epsilon_1 d_2} e^{-t/\tau};$$

$$E_2 = \frac{\sigma_1 \mathcal{V}}{\sigma_2 d_1 + \sigma_1 d_2} (1 - e^{-t/\tau}) + \frac{\epsilon_1 \mathcal{V}}{\epsilon_2 d_1 + \epsilon_1 d_2} e^{-t/\tau}.$$

P.5-13 a) $G_1 = \frac{2\pi\sigma_1 L}{\ln(c/a)}, \quad G_2 = \frac{2\pi\sigma_2 L}{\ln(b/c)}$

$$I = V_0 G = V_0 \frac{G_1 G_2}{G_1 + G_2} = \frac{2\pi \sigma_1 \sigma_2 L V_0}{\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)}.$$

$$J_1 = J_2 = \frac{I}{2\pi r L} = \frac{\sigma_1 \sigma_2 V_0}{r[\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)]}.$$

b) $\rho_{sa} = \epsilon_1 E_1 \Big|_{r=a} = \frac{\epsilon_1 \sigma_2 V_0}{a[\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)]}.$

$$\rho_{sb} = -\epsilon_2 E_2 \Big|_{r=b} = -\frac{\epsilon_2 \sigma_1 V_0}{b[\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)]}.$$

$$\rho_{sc} = -(\epsilon_1 E_1 - \epsilon_2 E_2)\Big|_{r=c} = \frac{(\epsilon_2 \sigma_1 - \epsilon_1 \sigma_2) V_0}{c[\sigma_1 \ln(b/c) + \sigma_2 \ln(c/a)]}.$$

P.5-14 $\nabla^2 V = 0 \longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r} \right) = 0.$

Solution: $V(r) = c_1 \ln r + c_2$.

Boundary conditions: $V(a) = V_0$; $\quad V(b) = 0$.

$$\longrightarrow V(r) = V_0 \frac{\ln(b/r)}{\ln(b/a)}.$$

$$\vec{E}(r) = -\vec{a}_r \frac{\partial V}{\partial r} = \vec{a}_r \frac{V_0}{r \ln(b/a)}.$$

$$\vec{J}(r) = \sigma \vec{E}(r).$$

$$I = \int_S \vec{J} \cdot d\vec{s} = \int_0^{\pi/2} \vec{J} \cdot (\vec{a}_r h r \, d\phi) = \frac{\pi \sigma h V_0}{2 \ln(b/a)}.$$

$$R = \frac{V_0}{I} = \frac{2 \ln(b/a)}{\pi \sigma h}.$$

P.5-15 Assume a potential difference V_0 between the inner and outer spheres.

$$\nabla^2 V = 0 \longrightarrow \frac{1}{R^2}\frac{d}{dR}(R^2 V') = 0 \longrightarrow V = \frac{k}{R} \longrightarrow E_R = \frac{k}{R^2}.$$

$$V_0 = -\int_{R_2}^{R_1} E_R\, dR = -k\int_{R_2}^{R_1} \frac{1}{R^2}dR = k\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

$$\longrightarrow k = \frac{V_0}{\frac{1}{R_1} - \frac{1}{R_2}}. \qquad J_R = \sigma E_R = \frac{\sigma V_0}{\frac{1}{R_1} - \frac{1}{R_2}}.$$

$$I = \int_0^{2\pi}\int_0^{\pi} J_R R^2 \sin\theta\, d\theta\, d\phi = \frac{4\pi\sigma V_0}{\frac{1}{R_1} - \frac{1}{R_2}}.$$

$$R = \frac{V_0}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{R_1} - \frac{1}{R_2}\right),$$ which can be obtained by combining Eqs. (3-140) and (5-81).

P.5-16 Assume a current I between the spherical surfaces.

$$\bar{J} = \bar{a}_R \frac{I}{4\pi R^2} = \sigma \bar{E}.$$

$$V_0 = -\int_{R_2}^{R_1} \bar{E}\cdot d\bar{R} = \int_{R_1}^{R_2} \frac{I\, dR}{4\pi\sigma R^2} = \frac{I}{4\pi\sigma_0}\int_{R_1}^{R_2} \frac{dR}{R^2(1+k/R)}$$

$$= \frac{I}{4\pi\sigma_0}\int_{R_1}^{R_2} \frac{1}{k}\left(\frac{1}{R} - \frac{1}{R+k}\right)dR = \frac{I}{4\pi\sigma_0 k}\ln\frac{R_2(R_1+k)}{R_1(R_2+k)}.$$

$$R = \frac{V_0}{I} = \frac{1}{4\pi\sigma_0 k}\ln\frac{R_2(R_1+k)}{R_1(R_2+k)}.$$

P.5-17 Assume I. $\qquad \bar{J}(R) = \bar{a}_R \frac{I}{S(R)}.$

$$S(R) = \int_0^{2\pi}\int_0^{\theta_0} R^2 \sin\theta\, d\theta\, d\phi = 2\pi R^2(1-\cos\theta_0).$$

$$\bar{E}(R) = \frac{1}{\sigma}\bar{J}(R) = \bar{a}_R \frac{I}{2\pi\sigma R^2(1-\cos\theta_0)}.$$

$$V_0 = -\int_{R_2}^{R_1} E(R)\, dR = \frac{I(R_2 - R_1)}{2\pi\sigma R_1 R_2(1-\cos\theta_0)}$$

$$R = \frac{V_0}{I} = \frac{1}{2\pi(1-\cos\theta_0)}\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

P.5-18 $\nabla\cdot\bar{J} = 0 = \nabla\cdot(\sigma\bar{E}) = \sigma\nabla\cdot\bar{E} + (\nabla\sigma)\cdot\bar{E} = 0.$

$$\bar{E} = \bar{a}_R E, \quad \nabla\cdot\bar{E} = \frac{1}{R^2}\frac{d}{dR}(R^2 E); \qquad \nabla\sigma = \bar{a}_R\frac{d\sigma}{dR} = -\bar{a}_R\frac{\sigma_0 R_1}{R^2}.$$

Substituting back : $R\frac{dE}{dR} = E \longrightarrow \bar{E} = \bar{a}_R \frac{c}{R}$

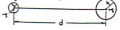
$$V = -\int_{R_2}^{R_1} \bar{E}\cdot d\bar{R} = c \ln\frac{R_2}{R_1} \longrightarrow c = \frac{V_0}{\ln(R_2/R_1)}, \quad \bar{E} = \bar{a}_R \frac{V_0}{R\ln(R_2/R_1)}$$

$$I = \int_S \bar{J}\cdot d\bar{s} = \int_S \sigma\bar{E}\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\theta_0}\left(\frac{\sigma_0 R_1}{R}\right)\left[\frac{V_0}{R\ln(R_2/R_1)}\right]R^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{2\pi\sigma_0 R_1 V_0(1-\cos\theta_0)}{\ln(R_2/R_1)}.$$

$$R = \frac{V_0}{I} = \frac{\ln(R_2/R_1)}{2\pi\sigma_0 R_1(1-\cos\theta_0)}.$$

P.5-19 Assume charges $+q$ and $-q$ to concentrate at the centers of spheres 1 and respectively. $d \gg b_1$, $d \gg b_2$.



$$V_1 \cong \frac{q}{4\pi\epsilon}\left(\frac{1}{b_1} - \frac{1}{d-b_1}\right).$$

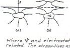
$$V_2 \cong \frac{q}{4\pi\epsilon}\left(\frac{1}{d-b_2} - \frac{1}{b_2}\right).$$

$$C = \frac{q}{V_1 - V_2} = \frac{4\pi\epsilon}{\frac{1}{b_1}+\frac{1}{b_2}-\frac{1}{d-b_1}-\frac{1}{d-b_2}}$$

$$= G\frac{\epsilon}{\sigma} = \frac{\epsilon}{R\sigma}.$$

$$R = \frac{1}{4\pi\sigma}\left(\frac{1}{b_1}+\frac{1}{b_2}-\frac{1}{d-b_1}-\frac{1}{d-b_2}\right) \cong \frac{1}{4\pi\sigma}\left(\frac{1}{b_1}+\frac{1}{b_2}-\frac{2}{d}\right).$$

P.5-20



The curved flow pattern of the lower half of Fig.(b), if both the conductor and its image are fed with the same current, is exactly the same as that of Fig.(a). All boundary conditions are satisfied.

$$\bar{\nabla}\times\left(\frac{\bar{J}}{\sigma}\right) = 0 \longrightarrow \bar{\nabla}\times\bar{J} = 0.$$

We can write $\bar{J} = -\bar{\nabla}\psi$, where ψ and electrostatic potential V are simply related. The streamlines are similar to the E-lines of a conductor and its image, both carrying a charge $+Q$ in the electrostatic case.

P.5-21 According to problem P.5-20, the current flow pattern would be the same as that of a whole sphere in an unbounded earth medium. Hence the current lines would be radial. Assume a current I.

$$\bar{J} = \bar{a}_R \frac{I}{2\pi R^2}, \qquad \bar{E} = \bar{a}_R \frac{I}{2\pi\sigma R^2}.$$

$$V_0 = -\int_\infty^b E\,dR = -\frac{I}{2\pi\sigma}\int_\infty^b \frac{dR}{R^2} = \frac{I}{2\pi\sigma b}.$$

$$R = \frac{V_0}{I} = \frac{1}{2\pi\sigma b} = \frac{1}{2\pi(10^{-6})(25\times10^{-3})} = 6.36\times10^6\ (\Omega).$$

P.5-22 Specified boundary conditions can be satisfied by solutions of Laplace's equation with zero separation constants: $k_x = k_y = 0$. $X(x) = A_0 x + B_0$, $Y(y) = C_0 y + D_0$.

$B_0 = C_0 = 0$. $\quad V(x) = A_0 D_0 x$

a) At $x = a$, $V(a) = V_0 = A_0 D_0 a \longrightarrow A_0 D_0 = \frac{V_0}{a}$

$$\therefore V = \frac{V_0}{a} x$$

b) $\bar{E} = -\bar{\nabla} V = -\bar{a}_x \frac{V_0}{a} \longrightarrow \bar{J} = \sigma \bar{E} = -\bar{a}_x \frac{\sigma V_0}{a}.$

P.5-23 $V(r,\phi) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n})(C_n \cos n\phi + D_n \sin n\phi).$

B.C.: $V(r,\phi) = V(r,-\phi) \longrightarrow D_n = 0.$

$r \to \infty$, $V = -\frac{J_0}{\sigma} r\cos\phi \longrightarrow A_n = C_n = 0$ for $n \neq 1$.

Write $V(r,\phi) = \left(K_1 r + \frac{K_2}{r}\right)\cos\phi$. $\quad K_1 = A_1 C_1 = -\frac{J_0}{\sigma}$, $K_2 = B_1 C_1$.

B.C.: $\left.\frac{\partial V}{\partial r}\right|_{r=b} = 0 \longrightarrow K_1 - \frac{K_2}{b^2} = 0$, $\quad K_2 = b^2 K_1 = -\frac{J_0}{\sigma} b^2.$

$$\therefore V(r,\phi) = -\frac{J_0}{\sigma}\left(r + \frac{b^2}{r}\right)\cos\phi.$$

$$\bar{J} = -\sigma\bar{\nabla}V = -\sigma\left(\bar{a}_r \frac{\partial V}{\partial r} + \bar{a}_\phi \frac{\partial V}{r\,\partial\phi}\right)$$

$$= \bar{a}_r J_0\left(1 - \frac{b^2}{r^2}\right)\cos\phi - \bar{a}_\phi J_0\left(1 + \frac{b^2}{r^2}\right)\sin\phi$$

$$= J_0(\bar{a}_r\cos\phi - \bar{a}_\phi\sin\phi) - \frac{J_0 b^2}{r^2}(\bar{a}_r\cos\phi + \bar{a}_\phi\sin\phi)$$

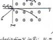
$$= \bar{a}_x J_0 - \frac{J_0 b^2}{r^2}(\bar{a}_r\cos\phi + \bar{a}_\phi\sin\phi), \quad r > b;$$

$\bar{J} = 0, \quad r < b.$

Chapter 6

Static Magnetic Fields

P.6-1



$$\frac{du_y}{dt} = \frac{qB_0}{m}u_z = \omega_0 u_z, \qquad (1)$$

$$\frac{du_z}{dt} = -\frac{qB_0}{m}u_y = -\omega_0 u_y, \qquad (2)$$

$$\omega_0 = qB_0/m.$$

Combining (1) and (2):

$$\frac{d^2u_z}{dt^2} + \omega_0^2 u_z = 0$$

$$\longrightarrow u_z = A\cos\omega_0 t + B\sin\omega_0 t.$$

At $t=0$, $u_z = 0 \longrightarrow A = 0$; $u_z = B\sin\omega_0 t$.

Substituting u_z in (2): $u_y = -B\cos\omega_0 t$. At $t=0$, $u_y = u_0 \longrightarrow B = -u_0$.

$\therefore\ u_y = u_0\cos\omega_0 t \longrightarrow y = \frac{u_0}{\omega_0}\sin\omega_0 t, \quad (t=0,\ y=0);$ (3)

$u_z = -u_0\sin\omega_0 t \longrightarrow z = \frac{u_0}{\omega_0}\cos\omega_0 t + c_1 \quad (t=0, z=0 \rightarrow c_1 = -\frac{u_0}{\omega_0})$

$= -\frac{u_0}{\omega_0}(1-\cos\omega_0 t).$ (4)

From (3) and (4): $y^2 + (z - \frac{u_0}{\omega_0})^2 = \left(\frac{u_0}{\omega_0}\right)^2$ --- Eq. of a shifted circle.

P.6-2 $\frac{\partial \vec{u}}{\partial t} = -\frac{e}{m}(\vec{E} + \vec{u}\times\vec{B})$.

a) $\vec{E} = \vec{a}_z E_0$, $\vec{B} = \vec{a}_x B_0$.

$$\left.\begin{aligned} &\frac{\partial u_x}{\partial t} = 0, \\ &\frac{\partial u_y}{\partial t} = -\frac{e}{m}B_0 u_z, \\ &\frac{\partial u_z}{\partial t} = -\frac{e}{m}(E_0 - B_0 u_y). \end{aligned}\right. \longrightarrow \left\{\begin{aligned} &u_x = 0, \\ &u_y = (u_0 - \frac{E_0}{B_0})\cos\omega_0 t + \frac{E_0}{B_0}, \\ &u_z = (\frac{E_0}{B_0} - u_0)\sin\omega_0 t; \quad \omega_0 = \frac{e}{m}B_0. \end{aligned}\right.$$

If the electron is injected at the origin ($x=y=z=0$) at $t=0$:

$x = 0$, $y = \frac{c_1}{\omega_0}\sin\omega_0 t + \frac{E_0}{B_0}t$, $z = -\frac{c_1}{\omega_0}(1-\cos\omega_0 t)$; $c_1 = u_0 - \frac{E_0}{B_0}$.

Eq. of motion: $(y - \frac{E_0}{B_0}t)^2 + (z + \frac{c_1}{\omega_0})^2 = (\frac{c_1}{\omega_0})^2$.

If $\frac{E_0}{B_0} = u_0$, $u_x = u_z = 0$, $u_y = u_0$;

$x = z = 0$, and $y = u_0 t$

b) $\bar{E} = -\bar{a}_z E_0$, B

$$\frac{\partial u_x}{\partial t} = \frac{e}{m} B_0 u_y = \omega_0 u_y,$$
$$\frac{\partial u_y}{\partial t} = -\omega_0 u_x,$$
$$\frac{\partial u_z}{\partial t} = \frac{e}{m} E_0.$$

(first two equations) Circular motion (See P.6-1); (all three) Helical motion with constant acceleration eE_0/m in z-direction.

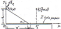
P.6-3 Application of Ampère's circuital law.

$0 \le r \le a$, $\quad \bar{B} = \bar{a}_\phi \dfrac{\mu r I}{2\pi a^2}$.

$a \le r \le b$, $\quad \bar{B} = \bar{a}_\phi \dfrac{\mu I}{2\pi r}$.

$b \le r \le c$, $\quad \bar{B} = \bar{a}_\phi \left(\dfrac{c^2 - r^2}{c^2 - b^2}\right) \dfrac{\mu I}{2\pi r}$.

P.6-4



a) Using Eq. (6-33a):

$$d\bar{B}_{P_1} = \bar{a}_x dB_x + \bar{a}_y dB_y$$
$$= \bar{a}_x (dB_{P_1}) \sin\alpha + \bar{a}_y (dB_{P_1}) \cos\alpha,$$
$$dB_{P_1} = \frac{\mu_0 (I/w) dx'}{2\pi (x'^2 + d^2)^{1/2}},$$
$$\sin\alpha = \frac{d}{(x'^2 + d^2)^{1/2}}, \quad \cos\alpha = \frac{x'}{(x'^2 + d^2)^{1/2}}.$$
$$\therefore \bar{B}_{P_1} = \bar{a}_x B_x + \bar{a}_y B_y,$$

where $B_x = \dfrac{\mu_0 I d}{2\pi w} \displaystyle\int_0^w \frac{dx'}{x'^2 + d^2} = \frac{\mu_0 I}{2\pi w} \tan^{-1}\left(\frac{w}{d}\right)$,

$$B_y = \frac{\mu_0 I}{2\pi w} \int_0^w \frac{x' dx'}{x'^2 + d^2} = \frac{\mu_0 I}{4\pi w} \ln\left(1 + \frac{w^2}{d^2}\right).$$

b) To find $\bar{B}$ at $P_2(\frac{2}{3}w, d)$, we add vectorially the contributions of the current strips to the right and to the left of point P_2' using the result in part (a).

$$\bar{B}_{P_2} = \bar{B}_{2R} + \bar{B}_{2L}.$$
$$\bar{B}_{2R} = \frac{\mu_0 I}{2\pi w}\left[\bar{a}_x \tan^{-1}\left(\frac{w}{3d}\right) + \bar{a}_y \frac{1}{2} \ln\left(1 + \frac{w^2}{9d^2}\right)\right].$$
$$\bar{B}_{2L} = \frac{\mu_0 I}{2\pi w}\left[\bar{a}_x \tan^{-1}\left(\frac{2w}{3d}\right) - \bar{a}_y \frac{1}{2} \ln\left(1 + \frac{4w^2}{9d^2}\right)\right].$$
$$\therefore \bar{B}_{P_2} = \frac{\mu_0 I}{2\pi w}\left[\bar{a}_x \left(\tan^{-1}\frac{w}{3d} + \tan^{-1}\frac{2w}{3d}\right) - \bar{a}_y \ln\sqrt{\frac{1 + (2w/3d)^2}{1 + (w/3d)^2}}\right].$$

P.6-5

We first find $\bar{B}_{P_1}$ at P_1 flush with one end of a wire carrying a current I and making an angle α with the other end as shown.

$$d\bar{B}_{P_1} = \frac{\mu_0 I}{4\pi R^2} dz' \times \bar{a}_R \qquad z' = b\cot\theta,\ dz' = -b\csc^2\theta\, d\theta,$$

$$= \frac{\mu_0 I}{4\pi b}(-\bar{a}_\phi \sin\theta\, d\theta). \qquad R = b\csc\theta. \quad \bar{a}_z \times \bar{a}_R = \bar{a}_\phi \sin\theta.$$

$$\bar{B}_{P_1} = -\bar{a}_\phi \frac{\mu_0 I}{4\pi b}\int_{\pi/2}^{\alpha} \sin\theta\, d\theta$$

$$= \bar{a}_\phi \frac{\mu_0 I}{4\pi b}\cos\alpha.$$

Applying the above result to the four-sided loop at left, we have

$$\bar{B}_P = \bar{a}_z \frac{\mu_0 I}{4\pi}\Big(\frac{1}{a}\cos\alpha_a + \frac{1}{b}\sin\alpha_a + \frac{1}{b}\cos\alpha_b + \frac{1}{c}\sin\alpha_b + \frac{1}{c}\cos\alpha_c + \frac{1}{d}\sin\alpha_c + \frac{1}{d}\cos\alpha_d + \frac{1}{a}\sin\alpha_d\Big).$$

For this problem, $a=c=\frac{w}{2}$, $b=\frac{w}{4}$, $d=\frac{3}{4}w$.

$\alpha_a = \tan^{-1} 2 = 63.4°$, $\alpha_b = 90° - 63.4° = 26.6°$, $\alpha_c = \tan^{-1}\frac{2}{3} = 33.7°$, $\alpha_d = 56.3°$

$$\therefore \bar{B}_P = \bar{a}_z\, 3.44\, \frac{\mu_0 I}{\pi w}.$$

P.6-6 The problem can be decomposed into two subproblems (assuming b = radius of solenoid):

1. A cylindrical tube carrying a uniformly distributed longitudinal surface current $2\pi b n I \sin\alpha$.

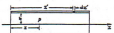
$$\longrightarrow \bar{B}_1 = \begin{cases} 0, & 0<r<b, \\ \bar{a}_\phi \frac{bnI}{r}\sin\alpha, & r>b. \end{cases}$$

2. A solenoid with n turns per unit length carrying a current $I\cos\alpha$.

$$\longrightarrow \bar{B}_2 = \begin{cases} \bar{a}_z \mu_0 n I\cos\alpha, & 0<r<b, \\ 0, & r>b. \end{cases}$$

Total $\bar{B} = \bar{B}_1 + \bar{B}_2$.

P.6-7 From Example 6-6, Eq. (6-35):



$$dB = \frac{\mu_0 I b^2}{2[(z'-z)^2+b^2]^{3/2}}\left(\frac{N}{L}\right)dz'.$$

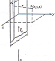
$$B = \frac{\mu_0 N I b^2}{2L}\int_0^L \frac{dz'}{[(z-z')^2+b^2]^{3/2}}$$

$$= \frac{\mu_0 N I}{2L}\left[\frac{L-z}{\sqrt{(L-z)^2+b^2}} + \frac{z}{\sqrt{z^2+b^2}}\right]$$

$$\longrightarrow \mu_0\left(\frac{N}{L}\right)I \quad \text{as } L\to\infty.$$

Direction of $\bar{B}$ is determined by the right-hand rule

P.6-10



Top view

$\bar{J}_s = \bar{a}_z J_{s0}$.

At $P(x,y,z)$ the magnetic flux density due to an infinitely long strip of width dx' is

$$d\bar{B} = \frac{\mu_0 J_{s0}\,dx'}{2\pi r}\left(-\bar{a}_x\frac{y}{r} + \bar{a}_y\frac{x-x'}{r}\right),$$

$$r = \sqrt{(x-x')^2+y^2}.$$

$$\therefore\ \bar{B} = \int d\bar{B} = \bar{a}_x B_x + \bar{a}_y B_y,$$

where

$$B_x = -\frac{\mu_0 J_{s0} y}{2\pi}\int_{-w/2}^{w/2}\frac{dx'}{(x-x')^2+y^2}$$

$$= \frac{\mu_0 J_{s0}}{2\pi}\left[\tan^{-1}\left(\frac{x-\frac{w}{2}}{y}\right) - \tan^{-1}\left(\frac{x+\frac{w}{2}}{y}\right)\right],$$

$$B_y = \frac{\mu_0 J_{s0}}{2\pi}\int_{-w/2}^{w/2}\frac{(x-x')\,dx'}{(x-x')^2+y^2}$$

$$= \frac{\mu_0 J_{s0}}{4\pi}\ln\frac{(x+\frac{w}{2})^2+y^2}{(x-\frac{w}{2})^2+y^2}.$$

P.6-11 This problem is a superposition of two problems:

$$\bar{B} = \bar{B}_1 + \bar{B}_2,$$

where

1. $\bar{B}_1$ is the magnetic flux density at P due to two

Semi-infinite wires carrying equal and opposite currents. Assuming $\bar{a}_z$ points out of paper:

$$\bar{B}_1 = \bar{a}_z \frac{\mu_0 I}{2\pi b}.$$

2. $\bar{B}_2$ is the magnetic flux density at P due to a half-circle. Taking one-half of the result in Eq. (6-38) for $z=0$:

$$\bar{B}_2 = \bar{a}_z \frac{\mu_0 I}{4b}.$$

$$\therefore \quad \bar{B} = \bar{a}_z \frac{\mu_0 I}{2b}\left(\frac{1}{\pi}+\frac{1}{2}\right).$$

P.6-12 Use Eq. (6-38) $B_x = \frac{N\mu_0 I b^2}{2}\left\{\frac{1}{[(\frac{d}{2}+x)^2+b^2]^{3/2}} - \frac{1}{[(\frac{d}{2}-x)^2+b^2]^{3/2}}\right\}.$

a) At $x=0$, $B_x = \frac{N\mu_0 I b^2}{[(\frac{d}{2})^2+b^2]^{3/2}}$.

b) $$\frac{dB_x}{dx} = \frac{N\mu_0 I b^2}{2}\left\{-\frac{3(\frac{d}{2}+x)}{[(\frac{d}{2}+x)^2+b^2]^{5/2}} + \frac{3(\frac{d}{2}-x)}{[(\frac{d}{2}-x)^2+b^2]^{5/2}}\right\}.$$

At the midpoint, $x=0$, $\frac{dB_x}{dx}=0$.

c) $$\frac{d^2B_x}{dx^2} = -\frac{3N\mu_0 I b^2}{2}\left\{\frac{1}{[(\frac{d}{2}+x)^2+b^2]^{5/2}} - \frac{5(\frac{d}{2}+x)^2}{[(\frac{d}{2}+x)^2+b^2]^{7/2}} + \frac{1}{[(\frac{d}{2}-x)^2+b^2]^{5/2}} - \frac{5(\frac{d}{2}-x)^2}{[(\frac{d}{2}-x)^2+b^2]^{7/2}}\right\}.$$

At $x=0$, $\frac{d^2B_x}{dx^2} = -3N\mu_0 I b^2\left\{\frac{b^2-4(d/2)^2}{[(d/2)^2+b^2]^{7/2}}\right\} \to 0$, if $b=d$.

P.6-13 Use Eq. (6-35) for a wire of length $2L$.

$$\bar{B} = \bar{a}_\phi \frac{\mu_0 I L}{2\pi r\sqrt{L^2+r^2}}.$$

In this problem, $\alpha = \frac{\pi}{N}$, $\frac{L}{r} = \tan\alpha = \tan\frac{\pi}{N}$.

$$\bar{B} = \bar{a}_n N\left(\frac{\mu_0 I L}{2\pi r b}\right) = \bar{a}_n \frac{\mu_0 N I}{2\pi b}\tan\frac{\pi}{N}.$$

When N is very large, $\tan\frac{\pi}{N} \cong \frac{\pi}{N}$. $\bar{B} \to \bar{a}_n \frac{\mu_0 I}{2b}$, which is the same as Eq. (6-38) with $z=0$.

P.6-14 $B_\phi = \frac{\mu_0 NI}{2\pi r}$, $\quad \Phi = \int_S B_\phi \, ds = \frac{\mu_0 NI}{2\pi}\int_a^b \frac{h}{r}dr = \frac{\mu_0 NIh}{2\pi}\ln\frac{b}{a}$.

If B_ϕ at $r = \frac{a+b}{2}$ is used, $\Phi' = \frac{\mu_0 NIh}{\pi}\left(\frac{b-a}{b+a}\right)$.

$$\%\ \text{error} = \frac{\Phi' - \Phi}{\Phi}\times 100\% = \left[\frac{2(b-a)}{(b+a)\ln(b/a)} - 1\right]\times 100\%.$$

P.6-15 $\bar{J} = \bar{a}_z J$. $\quad \oint \bar{B}\cdot d\bar{\ell} = \mu_0 I$.

If there is no hole,

$2\pi r_1 B_{\phi 1} = \mu_0 \pi r_1^2 J$

$$\longrightarrow B_{\phi 1} = \frac{\mu_0 r_1}{2}J \longrightarrow \begin{cases} B_{x1} = -\frac{\mu_0 J}{2}y_1, \\ B_{y1} = +\frac{\mu_0 J}{2}x_1. \end{cases}$$

For $-\bar{J}$ in the hole portion:

$$B_{\phi 2} = -\frac{\mu_0 r_2}{2}J \longrightarrow \begin{cases} B_{x2} = +\frac{\mu_0 J}{2}y_2, \\ B_{y2} = -\frac{\mu_0 J}{2}x_2. \end{cases}$$

Superposing $B_{\phi 1}$ and $B_{\phi 2}$ and noting that $y_1 = y_2$ and $x_1 = x_2 + d$, we have $B_x = B_{x1} + B_{x2} = 0$, and $B_y = B_{y1} + B_{y2} = \frac{\mu_0 J}{2}d$.

P.6-17 $\bar{B} = \bar{\nabla}\times\bar{A} \longrightarrow \bar{B} = \bar{a}_\phi B = \bar{a}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) = -\bar{a}_\phi\frac{\partial A_z}{\partial r}$.

For $0 \le r \le b$, Eq. (6-10) gives $\bar{B}_1 = \bar{a}_\phi \frac{\mu_0 I}{2\pi b^2}r$.

For $r \ge b$, Eq. (6-11) gives $\bar{B}_2 = \bar{a}_\phi \frac{\mu_0 I}{2\pi r}$.

Integrating, $\quad \bar{A}_1 = \bar{a}_z\left[-\frac{\mu_0 I}{4\pi}\left(\frac{r}{b}\right)^2 + c_1\right], \quad 0 \le r \le b.$

$\bar{A}_2 = \bar{a}_z\left[-\frac{\mu_0 I}{2\pi}\ln r + c_2\right], \quad r \ge b.$

At $r = b$, $\bar{A}_1 = \bar{A}_2 \longrightarrow c_2 = \frac{\mu_0 I}{4\pi}(2\ln b - 1) + c_1$

$\therefore \bar{A}_2 = \bar{a}_z\left\{-\frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{r}{b}\right)^2 + 1\right] + c_1\right\}, \quad r \ge b.$

P.6-18 Eq. (6-34) for one wire: $\bar{A} = \bar{a}_z\frac{\mu_0 I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}$.

For two wires carrying equal and opposite currents:

a) $\bar{A} = \bar{a}_z\frac{\mu_0 I}{4\pi}\ln\left[\frac{\sqrt{L^2+r_2^2}+L}{\sqrt{L^2+r_2^2}-L}\,\frac{\sqrt{L^2+r_1^2}-L}{\sqrt{L^2+r_1^2}+L}\right] = \bar{a}_z\frac{\mu_0 I}{2\pi}\ln\left[\frac{r_1}{r_2}\,\frac{\sqrt{L^2+r_2^2}-L}{\sqrt{L^2+r_1^2}+L}\right]$.

b) For a very long two-wire transmission line, $L \to \infty$:

$$\bar{A} = \bar{a}_z \frac{\mu_0 I}{2\pi} \ln\left(\frac{r_1}{r_2}\right) = \bar{a}_z \frac{\mu_0 I}{4\pi} \ln \frac{(\frac{d}{2}+y)^2+x^2}{(\frac{d}{2}-y)^2+x^2}.$$

c) $\bar{B} = \bar{\nabla}\times\bar{A} = \bar{a}_x \frac{\partial A_z}{\partial y} - \bar{a}_y \frac{\partial A_z}{\partial x}$

$$= \bar{a}_x \frac{\mu_0 I}{2\pi}\left[\frac{\frac{d}{2}+y}{(\frac{d}{2}+y)^2+x^2} - \frac{\frac{d}{2}-y}{(\frac{d}{2}-y)^2+x^2}\right] - \bar{a}_y \frac{\mu_0 I}{2\pi}\left[\frac{x}{(\frac{d}{2}+y)^2+x^2} - \frac{x}{(\frac{d}{2}-y)^2+x^2}\right]$$

$$= \frac{\mu_0 I}{2\pi}\left[\bar{a}_{\phi_1}\frac{1}{r_1} - \bar{a}_{\phi_2}\frac{1}{r_2}\right].$$

d) To find the equation for magnetic flux lines:

$$\frac{dx}{B_x} = \frac{dy}{B_y} \longrightarrow \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = 0$$

$$\longrightarrow dA = 0 \longrightarrow A = \text{constant}.$$

Thus, $$\frac{r_1^2}{r_2^2} = \frac{(\frac{d}{2}+y)^2+x^2}{(\frac{d}{2}-y)^2+x^2} = K.$$

P.6-20 Apply divergence theorem to $(\bar{F}\times\bar{C})$, where $\bar{C}$ is a constant vector.

$$\int_V \bar{\nabla}\cdot(\bar{F}\times\bar{C})\,dv = \oint_S (\bar{F}\times\bar{C})\cdot d\bar{s}. \qquad ①$$

Now, from problem P.2-23: $\bar{\nabla}\cdot(\bar{F}\times\bar{C}) = \bar{C}\cdot(\bar{\nabla}\times\bar{F}) - \bar{F}\cdot(\bar{\nabla}\times\bar{C})$

$$= \bar{C}\cdot(\bar{\nabla}\times\bar{F}); \qquad ②$$

from Eq. (2-19): $(\bar{F}\times\bar{C})\cdot d\bar{s} = -\bar{C}\cdot(\bar{F}\times d\bar{s})$ ③

Substituting ② and ③ in ①:

$$\bar{C}\cdot\int(\bar{\nabla}\times\bar{F})dv = -C\cdot\oint_S(\bar{F}\times d\bar{s}) \longrightarrow \int_V(\bar{\nabla}\times\bar{F})dv = -\oint \bar{F}\times d\bar{s}.$$

P.6-21

a) Given $\bar{B}_2 = \mu_2 \bar{H}_2$

$B_{2z} = B_{1z} \longrightarrow \mu H_2 = \mu_0 H_0 \longrightarrow \bar{H}_2 = \bar{a}_z H_2 = \bar{a}_z \frac{\mu_0}{\mu} H_0.$

b) Given $\bar{B}_2 = \mu_0(\bar{H}_2 + \bar{M}_i)$.

$B_{2z} = B_{1z} \longrightarrow \mu_0(H_2 + M_i) = \mu_0 H_0 \longrightarrow \bar{H}_2 = \bar{a}_z(H_0 - M_i).$

P.6-22 a) $r<a$: $\left.\begin{array}{l}\bar{H} = \bar{a}_z nI, \\ \bar{B} = \bar{a}_z \mu nI\end{array}\right\}$ Eq. (6-14).

$\bar{M} = \frac{\bar{B}}{\mu_0} - \bar{H} = \bar{a}_z\left(\frac{\mu}{\mu_0}-1\right)nI.$

$a<r<b$: $\bar{H} = \bar{a}_z nI,$
$\bar{B} = \bar{a}_z \mu_0 nI,$
$\bar{M} = 0.$

b) $\bar{J}_m = \bar{\nabla}\times\bar{M} = 0$; $\bar{J}_{ms} = \bar{M}\times\bar{a}_n = (\bar{a}_z\times\bar{a}_r)\left(\frac{\mu}{\mu_0}-1\right)nI = \bar{a}_\phi\left(\frac{\mu}{\mu_0}-1\right)nI.$

P.6-24

a) $V_m = \frac{I}{4\pi}\int \frac{d\bar{s}\cdot\bar{a}_R}{R^2} = \frac{I}{4\pi}\Omega.$

$d\bar{s}\cdot\bar{a}_R = (\cos\alpha)\rho\, d\rho\, d\phi = \frac{z}{\sqrt{z^2+\rho^2}}\rho\, d\rho\, d\phi,$

$R = \sqrt{z^2+\rho^2}.$

$\therefore V_m = \frac{I}{4\pi}\int_0^{2\pi}\int_0^{b}\frac{z}{(z^2+\rho^2)^{3/2}}\rho\, d\rho\, d\phi$

$= \frac{I}{2}\left(1 - \frac{z}{\sqrt{z^2+b^2}}\right).$

b) $\bar{B} = -\mu_0\bar{\nabla}V_m = -\bar{a}_z\mu_0\frac{\partial V_m}{\partial z} = \bar{a}_z\frac{\mu_0 I b^2}{2(z^2+b^2)^{3/2}}$, which is the same as Eq. (6-38).

P.6-25 A cylindrical bar magnet having a uniform magnetization $\bar{M} = \bar{a}_z M_0$ is equivalent to a $\bar{J}_m = \bar{\nabla}\times\bar{M} = 0$ and a $\bar{J}_{ms} = \bar{M}\times\bar{a}_n = (\bar{a}_z M_0)\times\bar{a}_r = \bar{a}_\phi M_0$ on the cylinder wall. At a distant point, $\bar{B}$ due to this $\bar{J}_{ms}$ flowing on a cylindrical wall of length L and radius b is the same as that due to a circular loop of radius b carrying a current $I = M_0 L$. It is given by Eq. (6-44), which is the same as Eq. (6-73) obtained in Example 6-9 where the total dipole moment of the cylindrical magnet is $M_T = I\pi b^2 = M_0 L\pi b^2$.

P.6-26

$\bar{M} = \bar{a}_z M_0$

a) $\bar{J}_m = \bar{\nabla}\times\bar{M} = 0.$

$\bar{J}_{ms} = (\bar{a}_R\cos\theta - \bar{a}_\theta\sin\theta)M\times\bar{a}_R$

$= \bar{a}_\phi M_0\sin\theta.$

b) Apply Eq. (6-38) to a loop of radius $b\sin\theta$ carrying a current $J_{ms}\, b\, d\theta$:

$d\bar{B} = \bar{a}_z\frac{\mu_0(J_{ms}b\,d\theta)(b\sin\theta)^2}{2(b^2)^{3/2}}$

$= \bar{a}_z\frac{\mu_0 M_0}{2}\sin^3\theta.$

$\bar{B} = \int d\bar{B} = \bar{a}_z\frac{\mu_0 M_0}{2}\int_0^{\pi}\sin^3\theta\, d\theta = \bar{a}_z\frac{2}{3}\mu_0 M_0 = \frac{2}{3}\mu_0\bar{M}.$

P.6-27 a) $\mathcal{R}_g = \frac{\ell_g}{\mu_0 S} = \frac{3\times10^{-3}}{4\pi\times10^{-7}\times(\pi\times0.025)^2} = 1.21\times10^6\ (H^{-1})$,

$\mathcal{R}_c = \frac{2\pi\times0.08 - 0.003}{3000\times(4\pi\times10^{-7})\times(\pi\times0.025)^2} = 6.75\times10^4\ (H^{-1})$.

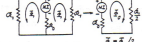
b) $\bar{B}_g = \bar{B}_c = \bar{a}_\phi \frac{10^{-5}}{\pi\times0.025^2} = \bar{a}_\phi\, 5.09\times10^{-3}\ (T)$,

$\bar{H}_g = \frac{1}{\mu_0}\bar{B}_g = \bar{a}_\phi \frac{5.09\times10^{-3}}{4\pi\times10^{-7}} = \bar{a}_\phi\, 4.05\times10^3\ (A/m)$,

$\bar{H}_c = \frac{1}{\mu_0\mu_r}\bar{B}_c = \bar{a}_\phi \frac{4.05\times10^3}{3000} = \bar{a}_\phi\, 1.35\ (A/m)$.

c) $NI = \Phi(\mathcal{R}_c + \mathcal{R}_g)$, $I = \frac{1}{N}\Phi(\mathcal{R}_c + \mathcal{R}_g) = 0.0256\ (A) = 25.6\ (mA)$

P.6-28 Magnetic circuit:



$\Phi_1 = \Phi_0/2$

$\frac{1}{\mu_0 S} = \frac{1}{(4\pi 10^{-7})\times10^{-3}} = 7.95\times10^8$

Neglecting leakage flux and assuming constant flux density over S:

$\mathcal{R}_0 = \frac{0.002}{\mu_0 S} + \frac{0.24 - 0.002}{\mu_0\mu_r S} = 1.60\times10^6\ (H^{-1})$,

$\mathcal{R}_1 = \frac{0.24 + 2\times0.2}{\mu_0\mu_r S} = 0.102\times10^6\ (H^{-1})$.

a) $\Phi_0 = \frac{NI}{\mathcal{R}_0 + \mathcal{R}_1/2} = 3.63\times10^{-4}\ (Wb)$; $\Phi_1 = \frac{\Phi_0}{2} = 1.82\times10^{-4}\ (Wb)$.

b) $H_1 = \frac{\Phi_1}{\mu_0\mu_r S} = 28.9\ (A/m)$,

$(H_0)_g = \frac{1}{\mu_0 S}\Phi_0 = 28.9\times10^4\ (A/m)$ in air gap,

$(H_0)_c = (H_0)_g/\mu_r = 57.8\ (A/m)$.

P.6-29 a) Work required per unit length in time dt:

$$P_1\, dt = nI\, d\Phi.$$

Work per unit volume in dt: $dW = \frac{1}{S}P_1\, dt = nI\, dB = H\, dB$.

Thus. $W_1 = \int_0^{B_f} H\, dB$.

P.6-31 $\bar{H}_1 = -\bar{\nabla}V_{m1}$, $\bar{H}_2 = -\bar{\nabla}V_{m2}$.

Boundary conditions: $\mu_1 H_{1n} = \mu_2 H_{2n} \longrightarrow \mu_1 \frac{\partial V_{m1}}{\partial n} = \mu_2 \frac{\partial V_{m2}}{\partial n}$,

$H_{1t} = H_{2t} \longrightarrow V_{m1} = V_{m2}$ (assuming absence of current)

P.6-32

a) $\bar{B}_1 = \bar{a}_x 0.5 - \bar{a}_y 10$ (mT).

$\bar{B}_2 = \bar{a}_x B_{2x} - \bar{a}_y B_{2y}$.

$H_{2x} = \frac{B_{2x}}{5000\mu_0} = H_{1x} = \frac{0.5}{\mu_0}$

$\longrightarrow B_{2x} = 2{,}500$ (mT),

$B_{2y} = B_{1y} = -10$ (mT).

$\therefore \bar{B}_2 = \bar{a}_x 2500 - \bar{a}_y 10$ (mT).

$\tan\alpha_2 = \frac{\mu_2}{\mu_1}\tan\alpha_1 = 5000\left(\frac{B_{1x}}{B_{1y}}\right) = 250 \longrightarrow \alpha_2 = 89.8°, \alpha_2' = 0.2°.$

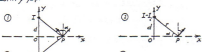
b) If $\bar{B}_2 = \bar{a}_x 10 + \bar{a}_y 0.5$ (mT), $\quad \bar{B}_1 = \bar{a}_x B_{1x} + \bar{a}_y B_{1y}$.

$H_{1x} = \frac{B_{1x}}{\mu_1} = H_{2x} = \frac{B_{2x}}{\mu_2} \longrightarrow B_{1x} = \frac{1}{\mu_{r2}} B_{2x} = \frac{10}{5000} = 0.002.$

$B_{1y} = B_{2y} = 0.5$. $\quad \therefore \bar{B}_1 = \bar{a}_x 0.002 + \bar{a}_y 0.5$ (mT)

$\alpha_1 = \tan^{-1}\frac{B_{1x}}{B_{1y}} \cong \frac{0.002}{0.5} = 0.004$ (rad) $= 0.23°$

P.6-33 a) Consider two situations: ① I and I_i both in air; and ② I and $-I_i$ both in magnetic medium with relative permeability μ_r.



Find B_{1y} and H_{1x} at P (y=0).	Find B_{2y} and H_{2x} at P (x=0).
$B_{1y} = \frac{\mu_0}{2\pi r}(I+I_i)\cos\alpha = \frac{\mu_0\mu_r}{\pi(\mu_r+1)}\frac{x}{r^2}I$,	$B_{2y} = \frac{\mu_0\mu_r}{2\pi r}(I-I_i)\cos\alpha = \frac{\mu_0\mu_r}{\pi(\mu_r+1)}\frac{x}{r^2}I$,
$B_{1x} = \frac{\mu_0}{2\pi r}(I-I_i)\sin\alpha = \frac{\mu_0}{\pi(\mu_r+1)}\frac{d}{r^2}I$,	$B_{2x} = \frac{\mu_0\mu_r}{2\pi r}(I-I_i)\sin\alpha = \frac{\mu_0\mu_r}{\pi(\mu_r+1)}\frac{d}{r^2}I$,
$H_{1x} = \frac{B_{1x}}{\mu_0} = \frac{I}{\pi(\mu_r+1)}\frac{d}{r^2}$.	$H_{2x} = \frac{B_{2x}}{\mu_0\mu_r} = \frac{I}{\pi(\mu_r+1)}\frac{d}{r^2}$.

$\therefore B_{1y} = B_{2y}$ and $H_{1x} = H_{2x}$ (Boundary conditions satisfied)

b) For $\mu_r >> 1$, $I_i = \frac{\mu_r - 1}{\mu_r + 1} I \cong I$.

Refer to the following figure.

$$\bar{B}_I = \frac{\mu_0 I}{2\pi r_1}\left(-\bar{a}_x \frac{y-d}{r_1} + \bar{a}_y \frac{x}{r_1}\right),$$

$$\bar{B}_{I_i} = \frac{\mu_0 I}{2\pi r_2}\left(-\bar{a}_x \frac{y+d}{r_2} + \bar{a}_y \frac{x}{r_2}\right).$$

$$\therefore\ \bar{B} = \bar{B}_I + \bar{B}_{I_i}$$
$$= -\bar{a}_x \frac{\mu_0 I}{2\pi}\left[\frac{y-d}{(y-d)^2+x^2} + \frac{y+d}{(y+d)^2+x^2}\right]$$
$$+ \bar{a}_y \frac{\mu_0 I}{2\pi}\left[\frac{1}{(y-d)^2+x^2} + \frac{1}{(y+d)^2+x^2}\right]$$

P.6-34

a) (i) If $\sigma_2 \to \infty$, $\bar{E}_2 = \bar{H}_2 = 0$.

B_n continuous $\longrightarrow B_{1n} = H_{2n} = 0$;

$\bar{a}_y \times \bar{H}_1 = \bar{J}_s \longrightarrow \bar{J}_s = -\bar{a}_z H_{1x}$.

Image $I_i (=-I)$ flowing out of the paper.

(ii) If $\mu_2 \to \infty$, $\bar{H}_2 = 0$, but $\bar{B}_2$ is finite.

No surface current $\longrightarrow H_{1t} = H_{2t} = 0$;

B_n continuous $\longrightarrow B_{1n} = B_{2n}$.

Image $I_i (=I)$ flowing into the paper.

b) (i) $\bar{H}_P = \bar{H}_1 + (\bar{H}_2)_i$, where $\bar{H}_1 = \frac{I}{2\pi}\left[\bar{a}_x \frac{y-d}{x^2+(y-d)^2} - \bar{a}_y \frac{x}{x^2+(y-d)^2}\right]$,

$(\bar{H}_2)_i = \frac{I}{2\pi}\left[-\bar{a}_x \frac{y+d}{x^2+(y+d)^2} + \bar{a}_y \frac{x}{x^2+(y+d)^2}\right]$.

(ii) $\bar{H}'_P = \bar{H}_1 + (\bar{H}_2)_{i_2} = \bar{H}_1 - (\bar{H}_2)_i$.

c) (i) $\bar{J}_s = -\bar{a}_z (H_P)_x \Big|_{y=0} = \bar{a}_z \left(\frac{Id}{x^2+d^2}\right)$.

(ii) $\bar{J}_s = 0$.

P.6-35

$$\bar{B} = \bar{a}_\phi B_\phi = \bar{a}_\phi \frac{\mu_0 N I}{2\pi r}, \quad r = r_0 - \rho\cos\alpha$$

$$\Phi = \frac{\mu_0 N I}{2\pi}\int_0^b \int_0^{2\pi} \frac{\rho\, d\alpha\, d\rho}{r_0 - \rho\cos\alpha} = \mu_0 N I\left(r_0 - \sqrt{r_0^2 - b^2}\right).$$

$$\therefore\ L = \frac{N\Phi}{I} = \mu_0 N^2\left(r_0 - \sqrt{r_0^2 - b^2}\right).$$

If $r_0 \gg b$, $B_\phi \cong \frac{\mu_0 N I}{2\pi r_0}$ (constant).

$$\Phi \cong B_\phi S = B_\phi(\pi b^2) = \frac{\mu_0 N b^2 I}{2 r_0} \longrightarrow L \cong \frac{\mu_0 N^2 b^2}{2 r_0}$$

P.6-36 For $b \le r \le (b+d)$, $\bar{B}_3 = \bar{a}_\phi B_{3\phi} = \bar{a}_\phi \frac{\mu_0 I}{2\pi r}\left[1 - \frac{\pi(r^2-b^2)}{\pi(b+d)^2-\pi b^2}\right]$

$$= \bar{a}_\phi \frac{\mu_0 I}{2\pi r}\left[\frac{(b+d)^2-r^2}{(b+d)^2-b^2}\right].$$

Magnetic energy per unit length stored in the outer conductor:

$$W'_m = \frac{1}{2\mu_0}\int_b^{b+d} B_{3\phi}^2\, 2\pi r\, dr$$

$$= \frac{\mu_0 I^2}{4\pi}\left\{\frac{(b+d)^4}{[(b+d)^2-b^2]^2}\ln\left(1+\frac{d}{b}\right) + \frac{b^2-3(b+d)^2}{4[(b+d)^2-b^2]}\right\}$$

From Eqs. (6-175), (6-176a), and (6-176b) we have

$$L' = \frac{2}{I^2}\left(W'_{m1} + W'_{m2} + W'_{m3}\right)$$

$$= \frac{\mu_0}{2\pi}\left\{\frac{1}{4} + \ln\frac{b}{a} + \frac{(b+d)^4}{[(b+d)^2-b^2]^2}\ln\left(1+\frac{d}{b}\right) - \frac{b^2-3(b+d)^2}{4[(b+d)^2-b^2]}\right\} \text{ (H)}$$

P.6-37

$\bar{B}$ at a distance r from an infinitely long line carrying a current I: $\bar{B} = \bar{a}_\phi \frac{\mu_0 I}{2\pi r}$.

For a unit length the flux due to I in line A that links with the second line pair B-B' is

$$\Phi'_A = \frac{\mu_0 I}{2\pi}\int_{AB}^{AC}\frac{dr}{r} = \frac{\mu_0 I}{2\pi}\ln\frac{AC}{AB}.$$

That due to I in line A' is

$$\Phi'_{A'} = \frac{\mu_0 I}{2\pi}\ln\frac{A'C'}{A'B'}.$$

Total flux linkage per unit length

$$\Lambda'_{12} = \Phi'_A + \Phi'_{A'} = \frac{\mu_0 I}{2\pi}\ln\frac{(AC)(A'C')}{(AB)(A'B')}$$

$$= \frac{\mu_0 I}{2\pi}\ln\frac{(AB')(A'B)}{(AB)(A'B')} = \frac{\mu_0 I}{2\pi}\ln\frac{D^2+d^2}{D^2}$$

$$\therefore M'_{12} = \frac{\Lambda'_{12}}{I} = \frac{\mu_0}{2\pi}\ln\left(1+\frac{d^2}{D^2}\right).$$

P.6-38 For I in the long straight wire, $\bar{B} = \bar{a}_\phi \frac{\mu_0 I}{2\pi r}$.

$$\Lambda_{12} = \int_S \bar{B}\cdot d\bar{s} = \int B_\phi \frac{2}{\sqrt{3}}(r-d)\,dr = \frac{\mu_0 I}{\pi\sqrt{3}}\int_d^{d+\frac{\sqrt{3}}{2}b}\left(\frac{r-d}{r}\right)dr$$

$$= \frac{\mu_0 I}{\pi\sqrt{3}}\left[\frac{\sqrt{3}}{2}b - d\ln\left(1+\frac{\sqrt{3}b}{2d}\right)\right] \longrightarrow L_{12} = \frac{\mu_0}{\pi}\left[\frac{b}{2} - \frac{d}{\sqrt{3}}\ln\left(1+\frac{\sqrt{3}b}{2d}\right)\right].$$

P.6-39

Assume a current I.

B at $P(r, \theta)$ is $\bar{a}_\phi \dfrac{\mu_0 I}{2\pi(d + r\cos\theta)}$.

$$\Lambda_{12} = \frac{\mu_0 I}{2\pi}\int_0^b\int_0^{2\pi} \frac{r\,dr\,d\theta}{d + r\cos\theta}$$

$$= \frac{\mu_0 I}{2\pi}\int_0^b \frac{2\pi r\,dr}{\sqrt{d^2 - r^2}} = \mu_0 I\left(d - \sqrt{d^2 - b^2}\right).$$

$$L_{12} = \mu_0\left(d - \sqrt{d^2 - b^2}\right).$$

P.6-40 Approximate the magnetic flux due to the long loop linking with the small loop by that due to two infinitely long wires carrying equal and opposite current I.

$$\Lambda_{12} = \frac{\mu_0 h_2 I}{2\pi}\int_0^{w_2}\left(\frac{1}{d+x} - \frac{1}{w_1 + d + x}\right)dx = \frac{\mu_0 h_2 I}{2\pi}\ln\left(\frac{w_2 + d}{d}\cdot\frac{w_1 + d}{w_1 + w_2 + d}\right).$$

$$L_{12} = \frac{\Lambda_{12}}{I} = \frac{\mu_0 h_2}{2\pi}\ln\frac{(w_1 + d)(w_2 + d)}{d(w_1 + w_2 + d)}.$$

P.6-41 Eq. (6-161): $W_2 = \frac{1}{2}L_1 I_1^2 + M I_1 I_2 + \frac{1}{2}L_2 I_2^2$.

a) $W_2 = \dfrac{I_2^2}{2}\left[L_1\left(\dfrac{I_1}{I_2}\right)^2 + 2M\left(\dfrac{I_1}{I_2}\right) + L_2\right] = \dfrac{I_2^2}{2}(L_1 x^2 + 2Mx + L_2)$, $x = \dfrac{I_1}{I_2}$.

$\dfrac{dW_2}{dx} = \dfrac{I_2^2}{2}(2L_1 x + 2M) = 0$, $\quad \dfrac{d^2 W_2}{dx^2} = I_2^2 L_1 > 0$.

$\therefore\ x = \dfrac{I_1}{I_2} = -\dfrac{M}{L_1}$ for minimum W_2.

b) $(W_2)_{min} = \dfrac{I_2^2}{2}\left(-\dfrac{M^2}{L_1} + L_2\right) \geq 0 \longrightarrow M \leq \sqrt{L_1 L_2}$.

P.6-42

$I_1 = I_2 = I_3 = 25$ (A) ; $\quad d = 0.15$ (m).

$\bar{B}_2 = \bar{a}_x 2B_{12}\cos 30° = \bar{a}_x \dfrac{\sqrt{3}\mu_0 I}{2\pi d}$.

Force per unit length on wire 2:

$\bar{F}_2 = -\bar{a}_y I B_2 = -\bar{a}_y \dfrac{\sqrt{3}\mu_0 I^2}{2\pi d}$

$= -\bar{a}_y 1150\mu_0 = -\bar{a}_y 1.44 \times 10^{-3}$ (N/m).

Forces on all three wires are of equal magnitude and toward the center of the triangle.

P.6-43 Magnetic field intensity at the wire due to the current $dI = \frac{I}{w}dy$ in an elemental dy is

$$|d\bar{H}| = \frac{dI}{2\pi r} = \frac{I\,dy}{2\pi w\sqrt{D^2+y^2}}.$$

Symmetry ⟶ H at the wire has only a y-component.

$$\bar{H} = \bar{a}_y \int (dH)\cdot\left(\frac{D}{r}\right) = \bar{a}_y 2\int_0^{w/2} \frac{ID\,dy}{2\pi w(D^2+y^2)}$$

$$= \bar{a}_y \frac{I}{\pi w}\tan^{-1}\left(\frac{w}{2D}\right).$$

$$\bar{f}' = \bar{I}\times\bar{B} = (-\bar{a}_z I)\times(\mu_0\bar{H}) = \bar{a}_x \frac{\mu_0 I^2}{\pi w}\tan^{-1}\left(\frac{w}{2D}\right) \quad \text{(N/m)}.$$

P.6-44 From Problem P.6-4 we have the y-component of the magnetic flux density at an arbitrary point $P(d, y)$ on the right-hand strip due to I_1 in the left-hand strip

$$B_{Py} = -\frac{\mu_0 I_1}{2\pi w}\left[\tan^{-1}\left(\frac{y}{d}\right) + \tan^{-1}\left(\frac{w-y}{d}\right)\right].$$

The x-component of the force on a strip of width dy due to I_2 in the right-hand conductor is

$$dF'_{2x} = \left(\frac{I_2}{w}dy\right)B_{Py} \quad \text{(in the +x direction, a repulsive force).}$$

$$\bar{F}'_2 = \bar{a}_x \int dF_{2x} = \bar{a}_x \frac{\mu_0 I_1 I_2}{2\pi w^2}\int_0^{w}\left[\tan^{-1}\left(\frac{y}{d}\right) + \tan^{-1}\left(\frac{w-y}{d}\right)\right]dy$$

$$= \bar{a}_x \frac{\mu_0 I_1 I_2}{2\pi w^2}\left[2w\tan^{-1}\left(\frac{w}{d}\right) - d\cdot\ln\left(1+\frac{w^2}{d^2}\right)\right] \quad \text{per unit length}$$

There is no net force in the y-direction.

P.6-45 $\bar{B}$ due to I_1 in the straight wire in the z-direction at an elemental arc $b\,d\theta$ on the circular loop is

$$\bar{B} = \bar{a}_\phi \frac{\mu_0 I_1}{2\pi(d+b\cos\theta)}.$$

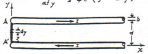
$$\bar{F}_{\text{on loop}} = -\bar{a}_x 2\int_0^{\pi}(I_2 b\,d\theta)\cos\theta$$

$$= -\bar{a}_x \frac{\mu_0 I_1 I_2 b}{\pi}\int_0^{\pi}\frac{\cos\theta}{d+b\cos\theta}d\theta$$

$$= \bar{a}_x \mu_0 I_1 I_2\left[\frac{1}{\sqrt{1-(b/d)^2}} - 1\right] \quad \text{(Repulsive force).}$$

$\bar{F}$ has no net y-component.

P.6-46 $\bar{B}_{\text{at } y} = \bar{a}_z \frac{\mu_0 I}{4\pi}\left(\frac{1}{y}+\frac{1}{d-y}\right)$. $d\bar{\ell} = \bar{a}_y dy$



$$d\bar{F} = I\,d\bar{\ell}\times\bar{B}$$
$$= -\bar{a}_x \frac{\mu_0 I^2}{4\pi}\left(\frac{1}{y}+\frac{1}{d-y}\right)dy$$
$$\bar{F} = -\bar{a}_x \frac{\mu_0 I^2}{4\pi}\int_b^{d-b}\left(\frac{1}{y}+\frac{1}{d-y}\right)dy$$
$$= -\bar{a}_x \frac{\mu_0 I^2}{2\pi}\ln\left(\frac{d}{b}-1\right).$$

(A rail-gun problem.)

P.6-47

$\bar{B}=\bar{a}_y 0.5$ (T) $\bar{B}\times$	$I(\overrightarrow{AB})$	$I(\overrightarrow{CA})$	$I(\overrightarrow{BC})$
	$(\bar{a}_y 0.5)\times 10(-\bar{a}_x 0.1-\bar{a}_y 0.2)$,	$10(-\bar{a}_x 0.1+\bar{a}_y 0.2)$,	$10\,\bar{a}_x 0.2$
Force:	$\bar{a}_z 0.5$	$\bar{a}_z 0.5$	$\bar{a}_z 1.0$ (N)

Torque on loop:

$$\bar{T} = \bar{m}\times\bar{B} = (\bar{a}_z IS)\times\bar{B}$$
$$= (\bar{a}_z 10\times\tfrac{1}{2}\times 0.2\times 0.2)\times(\bar{a}_y 0.5) = -\bar{a}_x 0.1 \text{ (N-m)}.$$

P.6-48 Let x-axis be the center line of the coaxial cable. The magnetic energy stored in a section of length x is

$$W_m = \tfrac{1}{2}LI^2.$$
$$L = \frac{\Phi}{I} = \frac{x}{I}\int_a^b B_\phi\,dr = \frac{x}{I}\int_a^b \frac{\mu_0 I}{2\pi r}dr = \frac{\mu_0 x}{2\pi}\ln\frac{b}{a}.$$
$$\bar{F}_I = \bar{a}_x\frac{\partial W_m}{\partial x} = \bar{a}_x\left(\frac{I^2}{2}\right)\frac{\partial L}{\partial x} = \bar{a}_x\frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a}.$$

P.6-49 Resolve the circular loop into many small loops, each with a magnetic dipole moment $d\bar{m} = I_2 d\bar{s}$, $d\bar{T} = d\bar{m}\times\bar{B}$.

$$\bar{T} = \int d\bar{T} = I_2\int d\bar{s}\times\bar{B} = -\bar{a}_x I_2 \sin\alpha\int B\,ds = -\bar{a}_x \mu_0 I_1 I_2\left(d-\sqrt{d^2-b^2}\right)\sin\alpha$$

From Problem P.6-39

—— This torque is in the direction of aligning the flux produced by I_2 in the loop with that of $\bar{B}$ due to I_1 in the straight wire.

P.6-50 $\bar{B}_2$ at the of the large circular turn of wire carrying a current I_2 is (by setting z=0 in Eq. 6-38):

$$\bar{B}_2 = \bar{a}_{z2}\frac{\mu_0 I_2}{2r_2}.$$

Torque on the small circular wire: $\bar{T} = \bar{m}_1\times\bar{B}_2 = (\bar{a}_{z1} I_1\pi r_1^2)\times(\bar{a}_{z2}\frac{\mu_0 I_2}{2r_2}) = (\bar{a}_{z1}\times\bar{a}_{z2})\frac{\mu_0 I_2 I_1\pi r_1^2}{2r_2}$

$\longrightarrow$ Magnitude $= \frac{\mu_0 I_1 I_2 \pi r_1^2}{2r_2}\sin\theta$ in a direction to align the magnetic fluxes produced by I_1 & I_2.

P.6-51

$\bar{B}_m$ (magnetized compass needle)

$$= \frac{\mu_0 m}{4\pi R^3}(\bar{a}_R 2\cos\theta + \bar{a}_\theta\sin\theta) \quad \text{from Eq. (6-49)}$$

$$= \frac{(4\pi\times10^{-7})\times2}{4\pi(0.15)^3}(\bar{a}_R 2\cos\theta + \bar{a}_\theta\sin\theta)$$

$$= 0.59\times10^{-4}(\bar{a}_R 2\cos\theta + \bar{a}_\theta\sin\theta)\ (T).$$

$\bar{B}_e(\text{earth}) = -\bar{a}_\theta 10^{-4}$ (T).

Max. deflection occurs when $\left|\frac{B_R}{B_\theta}\right|$ is max. or when

$$\left|\frac{B_\theta}{B_R}\right| = \left|\frac{(0.59\sin\theta-1)\times10^{-4}}{1.18\times10^{-4}\cos\theta}\right| \text{ is min.}$$

Set $\frac{d}{d\theta}\left(\frac{1-0.59\sin\theta}{1.18\cos\theta}\right)=0$ —— $\sin\theta = 0.59$, or $\theta = 36.2°$

At $\theta = 36.2°$, $|B_R/B_\theta| = 1.47$, and $\alpha = \tan^{-1}1.47 = 55.8°$.

P.6-52

$$F = \frac{\Phi^2}{\mu_0 S} = \frac{(NI)^2}{\mu_0 S\left(\frac{2\ell_g}{\mu_0 S}+\frac{\ell_i}{\mu_0\mu_r S}\right)^2} = \frac{(NI)^2\mu_0 S}{\left(2\ell_g - \frac{\ell_i}{\mu_r}\right)^2}$$

$F = 100\times9.8 = 980$ (N), $S = 0.01$ (m²), $\ell_g = 2\times10^{-3}$ (m), $\ell_i = 3$ (m), $\mu_r = 4000$.

Solving: mmf $= NI = 1.33\times10^3$ (A-t).

P.6-53 $W_m = \frac{1}{2}\int \mu H^2\,dv$

Assume a virtual displacement, Δx, of the iron core.

$$W_m(x+\Delta x) = W_m(x) + \frac{1}{2}\int_{S\Delta x}(\mu-\mu_0)H^2\,dv$$

$$= W_m(x) + \frac{1}{2}\mu_0(\mu_r-1)n^2I^2S\Delta x.$$

$(F_I)_x = \frac{\partial W_m}{\partial x} = \frac{\mu_0}{2}(\mu_r-1)n^2I^2S$, in the direction of increasing x.

Chapter 7

Time-Varying Field and Maxwell's Equations

P.7-1 $\mathcal{V} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} = -\int_S \frac{\partial}{\partial t}(\nabla \times \bar{A}) \cdot d\bar{s} = -\oint_C \frac{\partial \bar{A}}{\partial t} \cdot d\bar{\ell}$.

P.7-2. $\bar{B} = \bar{a}_z 3 \cos(5\pi 10^7 t - \frac{2}{3}\pi x) \cdot 10^{-6}$ (T),

$$\int_S \bar{B} \cdot d\bar{s} = \int_0^{0.6} \bar{a}_z 3 \cos(5\pi 10^7 t - \tfrac{2}{3}\pi x) 10^{-6} \cdot (\bar{a}_z 0.2\, dx)$$

$$= -\frac{0.18}{2\pi}\left[\sin(5\pi 10^7 t - 0.4\pi) - \sin 5\pi 10^7 t\right] \cdot 10^{-6} \text{ (Wb)},$$

$$\mathcal{V} = -\frac{d}{dt}\int \bar{B} \cdot d\bar{s} = 45\left[\cos(5\pi 10^7 t - 0.4\pi) - \cos 5\pi 10^7 t\right] \text{ (V)},$$

$$i = \frac{\mathcal{V}}{2R} = 1.5\left[\cos(5\pi 10^7 t - 0.4\pi) - \cos 5\pi 10^7 t\right]$$

$$= 1.76 \sin(5\pi 10^7 t - 0.2\pi) \text{ (A)}.$$

P.7-3 In the rectangur loop with the assigned direction for i_2

$$L_{12}\frac{di_1}{dt} = L\frac{di_2}{dt} + Ri_2. \qquad ①$$

where $L_{12} = \frac{\Phi_{12}}{i_1} = \frac{h}{i_1}\int_d^{d+w} B_{12}\, dr = \frac{h}{i_1}\int_d^{d+w} \frac{\mu_0 i_1}{2\pi r}\, dr$

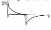
$$= \frac{\mu_0 h}{2\pi} \ln\left(1 + \frac{w}{d}\right). \qquad ②$$

a) At $t = 0$, $i_1(t) = I_1 U(t)$ is applied and ① becomes

$$L\frac{di_2}{dt} + Ri_2 = L_{12} I_1 \delta(t). \qquad ③$$

Solution of ③: $i_2 = \frac{L_{12}}{L} I_1 e^{-(R/L)t}$, $\quad 0 < t < T \qquad ④$

At $t = T$, $i_2 = \frac{L_{12}}{L} I_1 e^{-RT/L}$ when a negative step function $-I_1 U(t)$ is applied. If $T \gg L/R$, then i_2 for $t > T$ is the reverse of i_2 for $0 < t < T$; i.e., $i_2 = -\frac{L_{12}}{L} I_1 e^{-(R/L)(t-T)}$, $t > T$.



b). Energy dissipated in R.

$$W \cong 2\left(\frac{L_{12}}{L} I_1\right)^2 R \int_0^\infty e^{-(2R/L)t}\, dt$$

$$= \frac{1}{L}(L_{12} I_1)^2.$$

P.7-4 $\bar{B} = \bar{a}_\phi \frac{\mu_0 I \sin\omega t}{2\pi r}$, $\Phi = \int_S \bar{B}\cdot d\bar{s}$, $d\bar{s} = \bar{a}_\phi 2z\,dr$, $z = \frac{\sqrt{3}}{3}(r-d)$.

a) $\Phi = \frac{\sqrt{3}}{3}\frac{\mu_0 I \sin\omega t}{\pi}\int_d^{\frac{\sqrt{3}}{2}b+d}\left(1-\frac{d}{r}\right)dr = \frac{\sqrt{3}\mu_0 I \sin\omega t}{3\pi}\left[\frac{\sqrt{3}}{2}b - d\ln\left(\frac{\frac{\sqrt{3}}{2}b+d}{d}\right)\right]$.

$d = \frac{b}{2}$, $\mathcal{V} = -\frac{\partial\Phi}{\partial t} = -\frac{\sqrt{3}\mu_0 I\omega b}{3\pi}\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\ln(\sqrt{3}+1)\right]\cos\omega t$

$= V_m \cos\omega t$.

$V_{rms} = \frac{\sqrt{2}}{2}|V_m| = \frac{\sqrt{6}\mu_0 I\omega b}{12\pi}\left[\sqrt{3} - \ln(\sqrt{3}+1)\right]$

$= 0.0472\,\mu_0 I\omega b$ (V).

b) $z = \frac{1}{\sqrt{3}}\left[\frac{b}{2}\left(1+\frac{4}{\sqrt{3}}\right) - r\right]$.

$\int \bar{B}\cdot d\bar{s} = \frac{\mu_0 I \sin\omega t}{\sqrt{3}\pi}\int_{\frac{b}{2}(1+\frac{1}{\sqrt{3}})}^{\frac{b}{2}(1+\frac{4}{\sqrt{3}})}\left[\frac{b}{2}\left(1+\frac{4}{\sqrt{3}}\right)\frac{1}{r} - 1\right]dr$

$= \frac{\mu_0 I \sin\omega t}{\sqrt{3}\pi}\left[\frac{b}{2}\left(1+\frac{4}{\sqrt{3}}\right)\ln\left(\frac{4+\sqrt{3}}{1+\sqrt{3}}\right) - \frac{\sqrt{3}}{2}b\right]$

$V_{rms} = \frac{1}{\sqrt{2}}\frac{\mu_0 I\omega}{\sqrt{3}\pi}\frac{b}{2}\left|\left(1+\frac{4}{\sqrt{3}}\right)\ln\left(\frac{4+\sqrt{3}}{1+\sqrt{3}}\right) - \sqrt{3}\right|$

$= 0.0469\,\mu_0 I\omega b$ (V).

P.7-5 From Problem P.6-40: $\Phi_{12} = \mu_0 I(\sin\omega t)(d - \sqrt{d^2-b^2})$.

a) $\mathcal{V} = -\frac{d\Phi}{dt} = -\mu_0 I\omega(\cos\omega t)(d-\sqrt{d^2-b^2}) = V_m \cos\omega t$,

$3\times 10^{-6} = \frac{|V_m|}{\sqrt{2}R} = \frac{\mu_0 I\omega(d-\sqrt{d^2-b^2})}{\sqrt{2}R}$.

$I = \frac{\sqrt{2}R\times 3\times 10^{-6}}{\mu_0\omega(d-\sqrt{d^2-b^2})} = \frac{3\sqrt{2}\times 10^{-6}}{4\pi 10^{-7}(2\pi 60)\times 0.0382} = 0.234$ (A).

b) $\alpha = \cos^{-1}(0.2/0.3) = 48.2°$.

P.7-6 a) Flux enclosed in the ring in Fig. 7-12(a): $\Phi = \pi r^2 B(t)$. ①

The induced emf in the ring referring to the assigned direction for current: $\mathcal{V} = iR_r = \frac{d\Phi}{dt} = \pi r^2\frac{dB(t)}{dt}$. ②

Resistance of differential circular ring: $R_r = \frac{2\pi r}{\sigma h\,dr}$. ③

Combining ② and ③: $i = \frac{\pi r^2}{R_r}\frac{dB(t)}{dt} = \frac{\sigma h}{2}r\,dr\left(\frac{dB}{dt}\right)$. ④

$dp = i^2 R_r = \frac{\pi\sigma h}{2}r^3 dr\left(\frac{dB}{dt}\right)^2$. ⑤

$P = \int dp = \frac{\pi\sigma h}{8}a^4\left(\frac{dB}{dt}\right)^2 = \frac{\pi\sigma h}{8}a^4\omega^2 B_0^2\cos^2\omega t$. ⑥

$P_{av} = \frac{\pi\sigma h}{16}a^4\omega^2 B_0^2$. ⑦

b) For N insulated filamentary parts, each with an area

$$S = \frac{0.95\pi a^2}{N} = \pi b^2 \longrightarrow b = \sqrt{\frac{S}{N}} = a\sqrt{\frac{0.95}{N}}.$$

Power loss in N filaments in Fig. 7-12(b) from ⑥: $P' = N\left(\frac{\pi\sigma h}{8}\right)\left(a\sqrt{\frac{0.95}{N}}\right)^4 \omega^2 B_0^2 \cos^2\omega t = \frac{0.95^2}{N}P$,

$$P'_{av} = \frac{0.95^2}{N}P_{av}.$$

P.7-7 $\Phi(t) = \vec{B}(t)\cdot\vec{S}(t) = -(5\cos\omega t)\times 0.2\,(0.7-x)$

$= -0.35\cos\omega t\,(1+\cos\omega t)$ (mT),

$i = -\frac{1}{R}\frac{d\Phi}{dt} = -\frac{1}{R}\,0.35\,\omega(\sin\omega t + \sin 2\omega t)$

$= -1.75\,\omega\sin\omega t\,(1+2\cos\omega t)$ (mA).

P.7-8

Assuming the loop to have N turns each with an area a×b, the torque on the loop is $\vec{T} = \vec{m}\times\vec{B} = -\vec{a}_x NabIB\sin\alpha$.

Mechanical work done by the motor in rotating through an angle $(-\Delta\alpha)$ is

$$W_m = T(-\Delta\alpha) = NabIB(\Delta\alpha)\sin\alpha.$$

Flux linking with the loop, $\Phi = NabB\cos\alpha$.

Emf induced in the loop, $\mathcal{V} = -\frac{d\Phi}{dt} = NabB\left(\frac{\Delta\alpha}{\Delta t}\right)\sin\alpha$.

Electric energy required to send current I against this emf in time Δt, $W_e = \mathcal{V}I(\Delta t) = NabB(\Delta\alpha)\sin\alpha = W_m$.

P.7-9 $i = \frac{\mathcal{V}}{R} = -\frac{1}{R}\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s} = -\frac{1}{R}\frac{d}{dt}(B_0 hw\cos\omega t)$

$= \frac{\omega B_0 hw}{R}\sin\omega t,$

$p_d = i^2R = \frac{(\omega B_0 hw)^2}{R}\sin^2\omega t$ (Power dissipated in R).

On the other hand,

for side 1-2: $\vec{F}_{12} = \vec{a}_z ihB_0$, $\vec{u}_{12} = \frac{\omega w}{2}(\vec{a}_y\cos\omega t - \vec{a}_z\sin\omega t)$;

for side 4-3: $\vec{F}_{43} = -\vec{a}_z ihB_0$, $\vec{u}_{43} = \frac{\omega w}{2}(-\vec{a}_y\cos\omega t + \vec{a}_z\sin\omega t)$.

Mechanical power required to rotate coil:

$p_m = -(\vec{F}_{12}\cdot\vec{u}_{12} + \vec{F}_{43}\cdot\vec{u}_{43}) = \omega B_0 hwi\sin\omega t = p_d$.

(Alternatively, $p_m = \vec{T}\cdot\vec{\omega}$; where $\vec{T} = -\vec{a}_x(ihB_0)w\sin\omega t$, and $\vec{\omega} = -\vec{a}_x\omega$)

P.7-10 a) $\mu_r = 1 + \chi_m$, $\chi_m = 5000 - 1 = 4999$.

$\bar{H} = \frac{\bar{M}}{\chi_m} = \bar{a}_z \frac{M_0}{4999}$; $\bar{B} = \mu_0\mu_r\bar{H} = \bar{a}_z \frac{5000}{4999}\mu_0 M_0$.

b) $V_0 = \oint \bar{u}\times\bar{B}\cdot d\bar{\ell} = \int_b^a (\bar{a}_\phi \omega r)\times(\bar{a}_z B)\cdot\bar{a}_r\, dr$

$= -\frac{\omega B}{2}(b^2-a^2) = -\frac{2500}{4999}\mu_0 M_0 \omega (b^2-a^2)$.

c) $E_r = \bar{E}_r' - \bar{u}\times\bar{B} = \frac{J_r}{\sigma} - (\bar{a}_\phi u)\times(\bar{a}_z B) = \bar{a}_r\left(\frac{i}{2\pi r h\sigma} - \omega r B\right)$ (at r)

Induced voltage $V = \int_a^b E_r\, dr = \frac{i}{2\pi h\sigma}\ln\frac{b}{a} - \frac{\omega B}{2}(b^2-a^2)$

$= iR + V_0$.

Short circuit: $V_0 = 0$, $i_{sc} = \frac{\omega B}{2R}(b^2-a^2)$, where $R = \frac{\ln(b/a)}{2\pi h\sigma}$.

P.7-13 a) $\bar{E} = -\nabla V - \frac{\partial\bar{A}}{\partial t} = -\nabla\left(V - \frac{\partial\psi}{\partial t}\right) - \frac{\partial\bar{A}'}{\partial t} \longrightarrow V' = V - \frac{\partial\psi}{\partial t}$.

b) Eq. (7-62): $\nabla\cdot\bar{A}' + \mu\epsilon\frac{\partial V'}{\partial t} = 0$.

$\longrightarrow \nabla\cdot(\bar{A} + \nabla\psi) + \mu\epsilon\frac{\partial}{\partial t}\left(V - \frac{\partial\psi}{\partial t}\right) = 0$.

$\longrightarrow \nabla^2\psi - \mu\epsilon\frac{\partial^2\psi}{\partial t^2} = 0$.

P.7-14 Eq. (7-53b): $\nabla\times\bar{H} = \bar{J} + \frac{\partial\bar{D}}{\partial t} \longrightarrow \nabla\times\left(\frac{\bar{B}}{\mu}\right) = \bar{J} + \epsilon\frac{\partial\bar{E}}{\partial t}$ ①

Eqs. (7-55) & (7-57): $\bar{B} = \nabla\times\bar{A}$, $\bar{E} = -\nabla V - \frac{\partial\bar{A}}{\partial t}$ ②, ③

Substituting ② & ③ in ①: $\mu\nabla\times\left(\frac{1}{\mu}\nabla\times\bar{A}\right) = \mu\bar{J} - \mu\epsilon\frac{\partial^2\bar{A}}{\partial t^2} - \mu\epsilon\nabla\left(\frac{\partial V}{\partial t}\right)$.

Using gauge condition for potentials in an inhomogeneous medium:

$\nabla\cdot(\epsilon\bar{A}) + \mu\epsilon^2\frac{\partial V}{\partial t} = 0$.

We obtain the wave equation for vector potential: $-\nabla\times\left(\frac{1}{\mu}\nabla\times\bar{A}\right) + \epsilon\nabla\left[\frac{1}{\mu\epsilon^2}\nabla\cdot(\epsilon\bar{A})\right] - \epsilon\frac{\partial^2\bar{A}}{\partial t^2} = -\bar{J}$

Eq. (7-53c): $\nabla\cdot\bar{D} = \rho \longrightarrow \nabla\cdot(\epsilon\nabla V) + \frac{\partial}{\partial t}\nabla\cdot(\epsilon\bar{A}) = -\rho$.

Wave equation for scalar potential: $\frac{1}{\epsilon}\nabla\cdot(\epsilon\nabla V) - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$.

P.7-15 a) Eq. (3-89): $\nabla\cdot\bar{P} = -\rho_p \longrightarrow \bar{a}_{n2}\cdot(\bar{P}_1 - \bar{P}_2) = -\rho_{ps}$.

b) $\epsilon_0\bar{E} = \bar{D} - \bar{P}$ $\begin{cases} \nabla\cdot\bar{D} = \rho_f \longrightarrow \bar{a}_{n2}\cdot(\bar{D}_1 - \bar{D}_2) = \rho_{fs} & ① \\ -\nabla\cdot\bar{P} = \rho_p \longrightarrow \bar{a}_{n2}\cdot(\bar{P}_1 - \bar{P}_2) = -\rho_{ps} & ② \end{cases}$

Combining ① & ②: $\bar{a}_{n2}\cdot(\bar{E}_1 - \bar{E}_2) = \frac{1}{\epsilon_0}(\rho_{fs} + \rho_{ps})$. Subscript f signifies free charge.

P.7-19 Medium 1: Free space.

Medium 2: $\mu_2 \to \infty$. H_2 must be zero so that B_2 is not infinite.

Boundary Conditions: $\bar{a}_{n2} \times \bar{H}_1 = \bar{J}_s$, $B_{1n} = B_{2n}$.

$E_{1t} = E_{2t}$, $\bar{a}_{n2} \cdot (\bar{D}_1 - \bar{D}_2) = \rho_s$.

P.7-21 We wish to prove $\nabla^2 V - \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$. ①

where $V(R,t) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho(t-R/u)}{R}\, dv'$ ②

We need: $\nabla^2\left(\frac{\rho}{R}\right) = \frac{1}{R}\nabla^2\rho + \rho\nabla^2\left(\frac{1}{R}\right) + 2(\nabla\rho)\cdot\nabla\left(\frac{1}{R}\right)$. ③

(Formula: $\nabla^2(fg) = \nabla\cdot\nabla(fg) = g\nabla^2 f + f\nabla^2 g + 2(\nabla f)\cdot(\nabla g)$)

Let $\xi = t - R/u$. $\nabla^2\rho(\xi) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\rho}{\partial R}\right) = \frac{1}{u^2}\frac{d^2\rho}{d\xi^2} - \frac{2}{uR}\frac{d\rho}{d\xi}$. ④

$\nabla^2\left(\frac{1}{R}\right) = -4\pi\delta(R)$ ⑤, $(\nabla\rho)\cdot\left(\nabla\frac{1}{R}\right) = \frac{\partial\rho}{\partial R}\left(-\frac{1}{R^2}\right) = \frac{1}{uR^2}\frac{d\rho}{d\xi}$. ⑥

Substituting ④, ⑤ & ⑥ in ③: $\nabla^2\left(\frac{\rho}{R}\right) = \frac{1}{u^2 R}\frac{d^2\rho}{d\xi^2} - 4\pi\rho\,\delta(R)$. ⑦

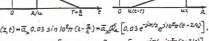
From ②: $\nabla^2 V = \frac{1}{4\pi\epsilon}\nabla^2\int_{V'}\frac{\rho}{R}\,dv' = \frac{1}{4\pi\epsilon}\int_{V'}\left[\frac{1}{u^2 R}\frac{d^2\rho}{d\xi^2} - 4\pi\rho\,\delta(R)\right]dv'$. ⑧

$\frac{\partial^2 V}{\partial t^2} = \frac{1}{4\pi\epsilon}\int_{V'}\frac{1}{R}\frac{d^2\rho}{d\xi^2}\,dv'$

$\therefore\quad \nabla^2 V - \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2} = \frac{1}{4\pi\epsilon}\int_{V'}\left[\frac{1}{u^2 R}\frac{d^2\rho}{d\xi^2} - 4\pi\rho\,\delta(R) - \frac{1}{u^2 R}\frac{d^2\rho}{d\xi^2}\right]dv'$

$= -\frac{\rho}{\epsilon}$ Q.E.D.

P.7-22 a) b)



P.7-23 $\bar{E}_1(z,t) = \bar{a}_x 0.03\sin 10^8\pi(t - \frac{z}{c}) = \bar{a}_x \mathcal{R}_e\left[0.03 e^{-j\pi/2} e^{j10^8\pi(t-z/c)}\right]$,

$\bar{E}_2(z,t) = \bar{a}_x 0.04\cos\left[10^8\pi(t-\frac{z}{c}) - \frac{\pi}{3}\right] = \bar{a}_x \mathcal{R}_e\left[0.04 e^{-j\pi/3} e^{j10^8\pi(t-z/c)}\right]$.

Phasors: $\bar{E} = \bar{E}_1 + \bar{E}_2 = \bar{a}_x\left[0.03 e^{-j\pi/2} + 0.04\, e^{-j\pi/3}\right]$

$= \bar{a}_x\left[-j0.03 + (0.02 - j0.02\sqrt{3})\right] = \bar{a}_x(0.065 e^{-j1.27}) = \bar{a}_x E_0 e^{j\theta}$.

$\therefore\ E_0 = 0.065$, $\theta = -1.27$ (rad), or $-72.8°$

P.7-24 $\quad \nabla\times\bar{E} = -\mu\frac{\partial\bar{H}}{\partial t}$ ① $\qquad \nabla\times\bar{H} = \bar{J} + \epsilon\frac{\partial\bar{E}}{\partial t}$ ②

$\nabla\cdot\bar{E} = \frac{\rho}{\epsilon}$ ③ $\qquad \nabla\cdot\bar{H} = 0$ ④

$\nabla\times$①: $\nabla\times\nabla\times\bar{E} = -\mu\frac{\partial}{\partial t}(\nabla\times\bar{H}) = -\mu\frac{\partial}{\partial t}\left(\bar{J}+\epsilon\frac{\partial\bar{E}}{\partial t}\right) = \nabla(\nabla\cdot\bar{E}) - \nabla^2\bar{E}.$

Wave equation for $\bar{E}$: $\quad \nabla^2\bar{E} - \mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} = \mu\frac{\partial\bar{J}}{\partial t} + \frac{1}{\epsilon}\nabla\rho.$

$\nabla\times$②: $\nabla\times\nabla\times\bar{H} = \nabla\times\bar{J} + \epsilon\frac{\partial}{\partial t}(\nabla\times\bar{E}) = \nabla\times\bar{J} - \mu\epsilon\frac{\partial^2\bar{H}}{\partial t^2} = \nabla(\nabla\cdot\bar{H}) - \nabla^2\bar{H}.$

Wave equation for $\bar{H}$: $\quad \nabla^2\bar{H} - \mu\epsilon\frac{\partial^2\bar{H}}{\partial t^2} = -\nabla\times\bar{J}.$

For sinusoidal time dependence: $\frac{\partial}{\partial t}\longrightarrow j\omega,\ \frac{\partial^2}{\partial t^2}\longrightarrow -\omega^2.$

Helmholtz's equations: $\quad \nabla^2\bar{E} + \omega^2\mu\epsilon\bar{E} = j\omega\mu\bar{J} + \frac{1}{\epsilon}\nabla\rho;$
(for phasors)
$\qquad \nabla^2\bar{H} + \omega^2\mu\epsilon\bar{H} = -\nabla\times\bar{J}.$

P.7-25 $\quad \bar{E} = \bar{a}_y 0.1\sin(10\pi x)\cos(6\pi 10^9 t - \beta z) \quad$ (V/m).

Use phasors: $\quad \bar{H} = -\frac{1}{j\omega\mu_0}\nabla\times\bar{E} = \frac{j}{\omega\mu_0}\left[\bar{a}_x j0.1\beta\sin(10\pi x) + \bar{a}_z 0.1(10\pi)\cos(10\pi x)\right]\cdot e^{-j\beta z}$, ①

$\bar{E} = \frac{1}{j\omega\epsilon_0}\nabla\times\bar{H} = \bar{a}_y\frac{0.1}{\omega^2\mu_0\epsilon_0}\left[(10\pi)^2+\beta^2\right]\sin(10\pi x)e^{-j\beta z}.$ ②

Phasor form for given $\bar{E}$: $\quad \bar{E} = \bar{a}_y\, 0.1\sin(10\pi x)e^{-j\beta z}.$ ③

Equating ② and ③: $\quad (10\pi)^2 + \beta^2 = \omega^2\mu_0\epsilon_0 = 400\pi^2.$

$\longrightarrow \beta = \sqrt{300}\,\pi = 54.4$ (rad/m).

From ①: $\quad \bar{H}(x,z;t) = \mathcal{R}_e(\bar{H}e^{j\omega t})$

$= -\bar{a}_x 2.30\times10^{-4}\sin(10\pi x)\cos(6\pi 10^9 t - 54.4z)$

$-\bar{a}_z 1.33\times10^{-4}\cos(10\pi x)\sin(6\pi 10^9 t - 54.4z)$ (A/m).

P.7-26 $\quad \bar{H}(x,z;t) = \bar{a}_y 2\cos(15\pi x)\sin(6\pi 10^9 t - \beta z) \quad$ (A/m).

Phasor: $\quad \bar{H} = \bar{a}_y 2\cos(15\pi x)e^{-j\beta z}.$

Similar to Problem P.7-25: $\quad (15\pi)^2 + \beta^2 = \omega^2\mu_0\epsilon_0 = \frac{(6\pi 10^9)^2}{(3\times10^8)^2} = 400\pi^2.$

$\longrightarrow \beta = 13.2\pi = 41.6$ (rad/m).

$\bar{E} = \frac{1}{j\omega\epsilon_0}\nabla\times\bar{H} = \left[\bar{a}_x 158\pi\cos(15\pi x) + \bar{a}_z j180\pi\sin(15\pi x)\right]e^{-j\beta z}.$

$\bar{E}(x,z;t) = \mathcal{I}_m(\bar{E}e^{j\omega t}) = \bar{a}_x 496\cos(15\pi x)\sin(6\pi 10^9 t - 41.6z)$

$+ \bar{a}_z 565\sin(15\pi x)\cos(6\pi 10^9 t - 41.6z)$ (V/m).

P.7-27 Use phasors: $\bar{E} = \bar{a}_\phi \frac{E_0}{R} \sin\theta \cdot e^{-jkR}$

$$\bar{\nabla} \times \bar{E} = \bar{a}_\phi \frac{1}{R} \frac{\partial}{\partial R}(RE_\phi) = \bar{a}_\phi (-jk) \frac{E_0}{R} \sin\theta \cdot e^{-jkR}$$

$$= -j\omega\mu_0 \bar{H} \longrightarrow \bar{H} = \bar{a}_\phi \frac{kE_0}{\omega\mu_0 R} \sin\theta \cdot e^{-jkR}$$

In free space, $k = \omega\sqrt{\mu_0\epsilon_0} \longrightarrow \bar{H} = \bar{a}_\phi \frac{E_0}{R}\sqrt{\frac{\epsilon_0}{\mu_0}} \sin\theta \cdot e^{-j\omega\sqrt{\mu_0\epsilon_0}R}$,

$$\bar{H}(R,\theta;t) = \bar{a}_\phi \frac{E_0}{R}\sqrt{\frac{\epsilon_0}{\mu_0}} \sin\theta \cos\omega(t - \sqrt{\mu_0\epsilon_0}R).$$

P.7-28 Maxwell's curl eqs: $\bar{\nabla} \times \bar{E} = -j\omega\mu\bar{H}$, ①

$\bar{\nabla} \times \bar{H} = j\omega\epsilon\bar{E}$. ②

From $\bar{\nabla}\cdot\bar{E} = 0$, define $\bar{A}_e$ such that $\bar{E} = \bar{\nabla} \times \bar{A}_e$. ③

From ①, $\bar{H} = \frac{j}{\omega\mu} \bar{\nabla} \times \bar{E} = \frac{j}{\omega\mu} \bar{\nabla} \times \bar{\nabla} \times \bar{A}_e$

$$= \frac{j}{\omega\mu}\left[\bar{\nabla}(\bar{\nabla}\cdot\bar{A}_e) - \bar{\nabla}^2\bar{A}_e\right]. \quad ④$$

From ②, $\bar{\nabla} \times (\bar{H} - j\omega\epsilon\bar{A}_e) = 0$. Let $\bar{H} - j\omega\epsilon\bar{A}_e = -\bar{\nabla}V_m$. ⑤

Subtracting ⑤ from ④: $\omega\epsilon\bar{A}_e = \frac{1}{\omega\mu}\left[\bar{\nabla}(\bar{\nabla}\cdot\bar{A}_e) - \bar{\nabla}^2\bar{A}_e\right] - j\bar{\nabla}V_m$.

Choose $\bar{\nabla}\cdot\bar{A}_e = j\omega\mu V_m$. ⑥

a) Eq. ⑤ becomes $\bar{H} = j\omega\epsilon\bar{A}_e + \frac{j}{\omega\mu}\bar{\nabla}(\bar{\nabla}\cdot\bar{A}_e)$.

b) Eq. ⑥ becomes $\bar{\nabla}^2\bar{A}_e + \omega^2\mu\epsilon\bar{A}_e = 0$, a homogeneous Helmholtz's eq.

P.7-29 $\bar{H} = j\omega\epsilon_0 \bar{\nabla} \times \bar{\pi}_e$. ①

$\bar{\nabla} \times \bar{E} = -j\omega\mu_0\bar{H} = \omega^2\mu_0\epsilon_0 \bar{\nabla} \times \bar{\pi}_e$.

$\longrightarrow \bar{\nabla} \times (\bar{E} - k_0^2\bar{\pi}_e) = 0$. Let $\bar{E} - k_0^2\bar{\pi}_e = \bar{\nabla}V_e$. ②

$\bar{\nabla} \times \bar{H} = j\omega\bar{D} = j\omega(\epsilon_0\bar{E} + \bar{P}) = j\omega\epsilon_0(\bar{E} + \frac{\bar{P}}{\epsilon_0})$. ③

Substituting ① and ② in ③:

$$j\omega\epsilon_0 \bar{\nabla} \times \bar{\nabla} \times \bar{\pi}_e = j\omega\epsilon_0 (k_0^2\bar{\pi}_e + \bar{\nabla}V_e + \frac{\bar{P}}{\epsilon_0})$$

$$= j\omega\epsilon_0 (\bar{\nabla}\bar{\nabla}\cdot\bar{\pi}_e - \bar{\nabla}^2\bar{\pi}_e). \quad ④$$

Choose $\bar{\nabla}\cdot\bar{\pi}_e = V_e$. Eq. ④ becomes

b) $$\bar{\nabla}^2\bar{\pi}_e + k_0^2\bar{\pi}_e = -\frac{\bar{P}}{\epsilon_0} \quad (7\text{-}119)$$

a) Eq. ② becomes

$$\bar{E} = k_0^2 \bar{\Pi}_e + \bar{\nabla}\bar{\nabla}\cdot\bar{\Pi}_e$$

$$= k_0^2 \bar{\Pi}_e + (\bar{\nabla}^2\bar{\Pi}_e + \bar{\nabla}\times\bar{\nabla}\times\bar{\Pi}_e). \qquad ⑤$$

Combination of Eqs. (7-119) and ⑤ gives

$$\bar{E} = \bar{\nabla}\times\bar{\nabla}\times\bar{\Pi}_e - \frac{\bar{P}}{\epsilon_0}.$$

P.7-30

a) $$\left|\frac{\text{Displacement current}}{\text{Conduction current}}\right| = \frac{\omega\epsilon}{\sigma} = \frac{(2\pi\times 100\times 10^6)\times\frac{1}{36\pi}\times 10^{-9}}{5.70\times 10^7}$$

$$= 9.75\times 10^{-8}.$$

b) In a source-free conductor:

$$\bar{\nabla}\times\bar{H} = \sigma\bar{E}, \qquad ①$$

$$\bar{\nabla}\times\bar{E} = -j\omega\mu\bar{H}. \qquad ②$$

$$\bar{\nabla}\times ①:\ \bar{\nabla}\times\bar{\nabla}\times\bar{H} = \bar{\nabla}(\bar{\nabla}\cdot\bar{H}) - \bar{\nabla}^2\bar{H} = \sigma\bar{\nabla}\times\bar{E}. \qquad ③$$

But $\bar{\nabla}\cdot\bar{H} = 0$, Eq. ③ becomes

$$\bar{\nabla}^2\bar{H} + \sigma\bar{\nabla}\times\bar{E} = 0. \qquad ④$$

Combining ② and ④:

$$\bar{\nabla}^2\bar{H} - j\omega\mu\sigma\bar{H} = 0.$$

Chapter 8

Plane Electromagnetic Waves

P.8-1 In a source-free simple medium,

Eq. (7-53b): $\bar{\nabla}\times\bar{H}=\bar{J}+\frac{\partial\bar{D}}{\partial t}=\sigma\bar{E}+\epsilon\frac{\partial\bar{E}}{\partial t}$. ①

Eq. (7-53a): $\bar{\nabla}\times\bar{\nabla}\times\bar{E}=\bar{\nabla}\bar{\nabla}\cdot\bar{E}-\bar{\nabla}^2\bar{E}=-\mu\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{H})$. ②

Substituting ① in ② and noting that $\bar{\nabla}\cdot\bar{E}=0$:

$$\bar{\nabla}^2\bar{E}-\mu\sigma\frac{\partial\bar{E}}{\partial t}-\mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2}=0.$$

Similarly for $\bar{H}$.

P.8-3 Assume that the vehicle moves with a velocity u in the $+z$ direction, which is the direction of propagation of the incident wave.

a) $\bar{E}_i=\bar{a}_x E_0 e^{j(\omega t-kz)}$, $\quad \bar{E}_r=-\bar{a}_x E_0 e^{j(\omega' t+k'z)}$

$\bar{E}_i+\bar{E}_r=0$ must be satisfied on reflecting surface for all t, $z=ut$:

$(\omega-ku)t=(\omega'+k'u)t$.

$\longrightarrow \omega'-\omega=-(k+k')u=-\left(\frac{\omega}{c}+\frac{\omega'}{c}\right)u=-(\omega+\omega')\frac{u}{c}$.

$\longrightarrow \frac{\omega'}{\omega}=1-\frac{u}{c}\left(1+\frac{\omega'}{\omega}\right)$.

$\longrightarrow \frac{\omega'}{\omega}=\frac{f'}{f}=\frac{1-u/c}{1+u/c}\cong 1-\frac{2u}{c}$, for $u \ll c$.

$\longrightarrow \Delta f=f'-f=-\frac{2u}{c}f$.

b) For $\Delta f=-2.33\times10^3$ (Hz) and $f=10.5\times10^9$ (Hz): $u=120$ (km/hr) $=74.6$ (mi/hr).

P.8-4 Harmonic time dependence: $e^{j\omega t}$; $\frac{\partial}{\partial t}\to j\omega$.

Phasors: $\bar{E}=\bar{E}_0 e^{-j\bar{k}\cdot\bar{R}}$, $\quad \bar{H}=\bar{H}_0 e^{-j\bar{k}\cdot\bar{R}}$; where $\bar{E}_0$ and $\bar{H}_0$ are constant vectors.

Now: $\bar{\nabla}(e^{-j\bar{k}\cdot\bar{R}})=e^{-j\bar{k}\cdot\bar{R}}\bar{\nabla}(-j\bar{k}\cdot\bar{R})=e^{-j\bar{k}\cdot\bar{R}}\left[-j\bar{\nabla}(k_x x+k_y y+k_z z)\right]$

$=-j(\bar{a}_x k_x+\bar{a}_y k_y+\bar{a}_z k_z)e^{-j\bar{k}\cdot\bar{R}}=-j\bar{k}e^{-j\bar{k}\cdot\bar{R}}$.

Maxwell's equations: $\bar{\nabla}\times\bar{E}=\bar{\nabla}(e^{-j\bar{k}\cdot\bar{R}})\times\bar{E}_0=-j\omega\mu\bar{H} \longrightarrow \bar{k}\times\bar{E}=\omega\mu\bar{H}$;

$\bar{\nabla}\times\bar{H}=\bar{\nabla}(e^{-j\bar{k}\cdot\bar{R}})\times\bar{H}_0=j\omega\epsilon\bar{E} \longrightarrow \bar{k}\times\bar{H}=-\omega\epsilon\bar{E}$;

$\bar{\nabla}\cdot\bar{E}=\bar{\nabla}(e^{-j\bar{k}\cdot\bar{R}})\cdot\bar{E}_0=0 \longrightarrow \bar{k}\cdot\bar{E}=0$; $\quad \bar{\nabla}\cdot\bar{H}=\bar{\nabla}(e^{-j\bar{k}\cdot\bar{R}})\cdot\bar{H}_0=0 \longrightarrow \bar{k}\cdot\bar{H}=0$.

P.8-5 $\bar{H} = \bar{a}_z 4\times10^{-6}\cos(10^7\pi t - k_0 y + \frac{\pi}{4})$ (A/m).

a) $k_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{10^7\pi}{3\times10^8} = \frac{\pi}{30} = 0.105$ (rad/m).

$\lambda = 2\pi/k_0 = 60$ (m).

At $t = 3\times10^{-3}$ (s), we require the argument of cosine in $\bar{H}$:

$$10^7\pi(3\times10^{-3}) - \frac{\pi}{30}y + \frac{\pi}{4} = \pm n\pi + \frac{\pi}{2}, \quad n = 0, 1, 2, \cdots$$

$$\longrightarrow y = \pm 30n - 7.5 \text{ (m)} = 22.5 \pm n\lambda/2 \text{ (m)}.$$

P.8-6 Phasor: $\bar{E} = \bar{a}_x 2e^{-jz/\sqrt{3}} + \bar{a}_y j e^{-jz/\sqrt{3}}$ (V/m).

a) $\omega = 10^8$ (rad/s) $\longrightarrow f = 10^8/2\pi = 1.59\times10^7$ (Hz),

$\beta = 1/\sqrt{3}$ (rad/m) $\longrightarrow \lambda = 2\pi/\beta = 2\sqrt{3}\pi$ (m).

b) $u = \frac{c}{\sqrt{\epsilon_r}} = \frac{\omega}{\beta} \longrightarrow \epsilon_r = \left(\frac{\beta c}{\omega}\right)^2 = 3.$

c) Left-hand elliptically polarized.

d) $\eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{120\pi}{\sqrt{\epsilon_r}} = \frac{120\pi}{\sqrt{3}}$ (Ω).

$$\bar{H} = \frac{1}{\eta}\bar{a}_z \times \bar{E} = \frac{\sqrt{3}}{120\pi}(\bar{a}_y 2e^{-jz/\sqrt{3}} - \bar{a}_x j e^{-jz/\sqrt{3}}),$$

$$\bar{H}(z,t) = \frac{\sqrt{3}}{120\pi}\left[\bar{a}_x \sin(10^8 t - z/\sqrt{3}) + \bar{a}_y \cos(10^8 t - z/\sqrt{3})\right] \text{ (A/m)}.$$

P.8-7 Let $\alpha = \omega t - kz$. $\bar{E} = \bar{a}_x E_{10}\sin\alpha + \bar{a}_y E_{20}\sin(\alpha + \psi) = \bar{a}_x E_x + \bar{a}_y E_y$.

$$\frac{E_x}{E_{10}} = \sin\alpha, \quad \frac{E_y}{E_{20}} = \sin(\alpha+\psi) = \sin\alpha\cos\psi + \cos\alpha\sin\psi$$

$$= \frac{E_x}{E_{10}}\cos\psi + \sqrt{1 - \left(\frac{E_x}{E_{10}}\right)^2}\sin\psi.$$

$$\left(\frac{E_y}{E_{20}} - \frac{E_x}{E_{10}}\cos\psi\right)^2 = \left(1 - \frac{E_x^2}{E_{10}^2}\right)\sin^2\psi.$$

$$\left(\frac{E_y}{E_{20}\sin\psi}\right)^2 + \left(\frac{E_x}{E_{10}\sin\psi}\right)^2 - 2\frac{E_x E_y}{E_{10}E_{20}}\frac{\cos\psi}{\sin^2\psi} = 1, \qquad (1)$$

which is the equation of an ellipse. In order to find the parameters of the polarization ellipse, rotate the coordinate axes x-y counterclockwise by an angle θ to x'-y'. Assume the equation of the ellipse in terms of the new coordinates to be

$$\left(\frac{E_x'}{a}\right)^2 + \left(\frac{E_y'}{b}\right)^2 = 1. \qquad (2)$$

where $E_x' = E_x\cos\theta + E_y\sin\theta$, ③

and $E_y' = -E_x\sin\theta + E_y\cos\theta$. ④

Substituting ③ and ④ in ② and rearranging:

$$E_x^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)+E_y^2\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}\right)-2E_xE_y\sin\theta\cos\theta\left(\frac{1}{b^2}-\frac{1}{a^2}\right)=1. \quad ⑤$$

Comparing ① and ⑤, we obtain

$$\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{E_{10}^2\sin^2\psi}, \quad ⑥$$

$$\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}=\frac{1}{E_{20}^2\sin^2\psi}, \quad ⑦$$

$$\sin\theta\cos\theta\left(\frac{1}{b^2}-\frac{1}{a^2}\right)=\frac{\cos\psi}{E_{10}E_{20}\sin^2\psi}. \quad ⑧$$

Eqs. ⑥, ⑦, and ⑧ can be solved for three unknowns:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2E_{10}E_{20}\cos\psi}{E_{10}^2-E_{20}^2}\right),$$

$$a=\sqrt{\frac{2}{\frac{1}{E_{10}^2}(1-\sec 2\theta)+\frac{1}{E_{20}^2}(1-\sec 2\theta)}}\ \sin\psi,$$

$$b=\sqrt{\frac{2}{\frac{1}{E_{10}^2}(1-\sec 2\theta)+\frac{1}{E_{20}^2}(1+\sec 2\theta)}}\ \sin\psi.$$

In particular, if $E_{10}=E_{20}=E_0$: $\theta=45°$, $a=\sqrt{2}E_0\cos\frac{\psi}{2}$, $b=\sqrt{2}E_0\sin\frac{\psi}{2}$.

P.8-8 Let an elliptically polarized plane wave be represented by the phasor (with propagation factor e^{-jkz} omitted):

a) $$\bar{E} = \bar{a}_x E_1 \pm \bar{a}_y E_2 e^{j\alpha},$$

where E_1, E_2, and α are arbitrary constants.

Right-hand circularly polarized wave: $\bar{E}_{rc}=E_{rc}(\bar{a}_x-\bar{a}_y j)$;

Left-hand circularly polarized wave: $\bar{E}_{lc}=E_{lc}(\bar{a}_x+\bar{a}_y j)$.

If $E_{rc}=\frac{1}{2}(E_1 \pm jE_2e^{j\alpha})$ and $E_{lc}=\frac{1}{2}(E_1 \mp jE_2e^{j\alpha})$,

then $$\bar{E}=\bar{E}_{rc}+\bar{E}_{lc}.$$

b) Right-hand circularly polarized wave: $\bar{E}_{rc}=E(\bar{a}_x-\bar{a}_y j)$

$$=E(\bar{a}_x\tfrac{1}{2}-\bar{a}_y j2)+E(\bar{a}_x\tfrac{1}{2}+\bar{a}_y j)$$

$$=\bar{E}_{e+}+\bar{E}_{e-},$$ where $\bar{E}_{e+}$ and $\bar{E}_{e-}$ are right-hand and left-hand elliptically polarized waves respectively.

Similarly for a left-hand circularly polarized wave: $\bar{E}_{lc}=E(\bar{a}_x+\bar{a}_y j)=E(\bar{a}_x\tfrac{1}{2}+\bar{a}_y j2)+E(\bar{a}_x\tfrac{1}{2}-\bar{a}_y j)$

$$=\bar{E}_{e-}'+\bar{E}_{e+}'.$$

P.8-9 For conducting media: $k_c = \beta - j\alpha$.

$$k_c^2 = \beta^2 - \alpha^2 - 2j\alpha\beta$$
$$= \omega^2\mu\epsilon_c = \omega^2\mu\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right).$$
$$\therefore \quad \beta^2 - \alpha^2 = \mathcal{R}_e(k_c^2) = \omega^2\mu\epsilon, \qquad (1)$$
$$\beta^2 + \alpha^2 = |k_c^2| = \omega^2\mu\epsilon\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2}. \qquad (2)$$

From (1) and (2) we obtain

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]^{1/2}, \quad \beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]^{1/2}.$$

P.8-10 All three metals are good conductors, $\left(\frac{\sigma}{\omega\epsilon}\right)^2 \gg 1$.

$$\alpha = \sqrt{\pi f \mu \sigma}, \quad \delta = \frac{1}{\alpha}, \quad \eta_c = (1+j)\frac{\alpha}{\sigma}.$$

a) $f = 60$ (Hz)

	η_c (Ω)	α (Np/m)	α (dB/m)	δ (m)
Copper	$2.02(1+j)\times10^{-6}$	0.117×10^{3}	1.02×10^{3}	8.53×10^{-3}
Silver	$2.08(1+j)\times10^{-6}$	0.121×10^{3}	1.05×10^{3}	8.29×10^{-3}
Brass	$3.86(1+j)\times10^{-6}$	0.061×10^{3}	0.53×10^{3}	16.3×10^{-3}

b) $f = 1$ (MHz)

	η_c (Ω)	α (Np/m)	α (dB/m)	δ (m)
Copper	$2.61(1+j)\times10^{-4}$	1.51×10^{4}	1.31×10^{5}	6.61×10^{-5}
Silver	$2.57(1+j)\times10^{-4}$	1.58×10^{4}	1.36×10^{5}	6.32×10^{-5}
Brass	$4.98(1+j)\times10^{-4}$	0.79×10^{4}	0.69×10^{5}	12.6×10^{-5}

c) $f = 1$ (GHz)

	η_c (Ω)	α (Np/m)	α (dB/m)	δ (m)
Copper	$8.25(1+j)\times10^{-3}$	4.79×10^{5}	4.16×10^{6}	2.09×10^{-6}
Silver	$8.01(1+j)\times10^{-3}$	4.93×10^{5}	4.28×10^{6}	2.03×10^{-6}
Brass	$15.9(1+j)\times10^{-3}$	2.51×10^{5}	2.18×10^{6}	3.98×10^{-6}

P.8-11 $f = 3\times10^9$ (Hz), $\epsilon_r = 2.5$, $\tan\delta_c = \frac{\epsilon''}{\epsilon'} = 10^{-2}$.

a) Eq. (8-48): $\alpha = \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\omega}{2}\left(\frac{\epsilon''}{\epsilon'}\right)\frac{\sqrt{\epsilon_r}}{c} = 0.497$ (Np/m).

$$e^{-\alpha x} = \frac{1}{2} \longrightarrow x = \frac{1}{\alpha}\ln 2 = 1.395 \text{ (m)}.$$

b) Eq. (8-50): $\eta_c = \frac{1}{\sqrt{\epsilon_r}}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right) = 238(1 + j0.005) = 238\underline{/0.29°}$ (Ω).

Eq. (8-49): $\beta = \omega\sqrt{\mu\epsilon}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right] = 31.6\pi$ (rad/m).

$\lambda = \frac{2\pi}{\beta} = 0.063$ (m),

$u_p = \frac{\omega}{\beta} = 1.8973\times10^8$ (m/s).

$u_g = \frac{1}{\frac{d\beta}{d\omega}} = \frac{c}{\sqrt{\epsilon_r}}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right] = 1.8975\times10^8$ (m/s).

c) At $x = 0$, $\bar{E} = \bar{a}_y e^{j\pi/2}$.

$\bar{H} = \frac{1}{\eta_c}\bar{a}_x\times\bar{E} = \bar{a}_z 0.210 e^{j(\frac{\pi}{2} - 0.0016\pi)}$.

$\bar{H}(x,t) = \bar{a}_z 0.210 e^{-0.497x}\sin(6\pi10^9 t - 31.6\pi x + 0.332\pi)$ (A/m).

P.8-12 $\sigma/\omega\epsilon = 4/10^{10}\pi\times80\times\left(\frac{1}{36}\times10^{-9}\right) = 0.18$.

a) $\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]^{1/2} = 84$ (Np/m),

$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]^{1/2} = 300\pi$ (rad/m),

$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \frac{120\pi}{\sqrt{\epsilon_r}\left[1 + (\sigma/\omega\epsilon)^2\right]^{1/4}} e^{j\frac{1}{2}\tan^{-1}(\sigma/\omega\epsilon)} = 41.8 e^{j0.0283\pi}$ (Ω),

$u_p = \omega/\beta = 33.3\times10^6$ (m/s), $\lambda = 2\pi/\beta = 0.67$ (cm), $\delta = \frac{1}{\alpha} = 1.19$ (cm).

b) $e^{-\alpha y} = \frac{1}{10}$, $y = \frac{1}{\alpha}\ln 10 = 2.74$ (cm).

c) $\bar{H}(y,t) = \bar{a}_x 0.1 e^{-84\times0.5}\sin(10^{10}\pi t - 300\pi\times0.5 - \pi/3)$

$= \bar{a}_x 5.75\times10^{-20}\sin(10^{10}\pi t - \pi/3)$ (A/m).

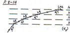
$\bar{E}(y,t) = \mathcal{I}m\left[\eta_c \bar{H}(y)\times\bar{a}_y\right]e^{j\omega t} = \bar{a}_z 2.41\times10^{-18}\sin\left(10^{10}\pi t - \frac{\pi}{3} + 0.0283\pi\right)$ (V/m).

P.8-13 a) $\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} \longrightarrow \sigma = \frac{1}{\pi f\mu\delta^2} = 0.99\times10^8$ (S/m).

b) At $f = 10^9$ (Hz), $\alpha = \sqrt{\pi f\mu\sigma} = 1.98\times10^4$ (Np/m).

$20\log_{10} e^{-\alpha z} = -30$ (dB) $\longrightarrow z = \frac{1.5}{\alpha\log_{10}e} = 1.75\times10^{-4}$ (m) $= 0.175$ (mm).

P.8-14



Assume the ionosphere to be stratified into layers having electron densities

$$N_1 < N_2 < N_3 < \cdots < N_{max}.$$

The corresponding equivalent permittivities of the layers are:

$$\epsilon_n = \epsilon_0\left(1 - \frac{f_{pn}^2}{f^2}\right) \quad \text{with} \quad f_{pn} = \frac{1}{2\pi}\sqrt{\frac{N_n e^2}{m\epsilon_0}}$$

and $\epsilon_i = \epsilon_0 > \epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots > \epsilon_{min}$ (Corresponding to N_{max}).

From Snell's law of reflection:

$$\sin\theta_1 = \sin\theta_i\sqrt{\epsilon_0/\epsilon_1} = \sin\theta_i\Big/\sqrt{1-(f_{p1}/f)^2},$$

$$\sin\theta_2 = \sin\theta_1\sqrt{\epsilon_1/\epsilon_2} = \sin\theta_i\sqrt{\epsilon_0/\epsilon_2},$$

$$\sin\theta_3 = \sin\theta_2\sqrt{\epsilon_2/\epsilon_3} = \sin\theta_i\sqrt{\epsilon_0/\epsilon_3},$$

$$\vdots$$

For total reflection at the layer with ϵ_{min}, the angle of refraction $\theta_{max} = \pi/2$, and $\sin\theta_{max} = 1 = \sin\theta_i\sqrt{\epsilon_0/\epsilon_{min}}$.

$$\therefore \quad \epsilon_{min} = \epsilon_0(1 - f_{p,max}^2/f^2) = \epsilon_0\sin^2\theta_i.$$

$$\longrightarrow f = f_{p,max}/\cos\theta_i = 9\sqrt{N_{max}}/\cos\theta_i.$$

P.8-15 a) From Eq. (8-72): $u_g = \dfrac{d\omega}{d\beta} = \dfrac{d}{d\beta}(\beta u_p) = u_p + \beta\dfrac{du_p}{d\beta}.$

b) $\lambda = \dfrac{2\pi}{\beta}$, $\dfrac{d\lambda}{d\beta} = -\dfrac{2\pi}{\beta^2} = -\dfrac{\lambda}{\beta}$.

$$u_g = u_p + \beta\left(\frac{du_p}{d\lambda}\frac{d\lambda}{d\beta}\right) = u_p - \lambda\frac{du_p}{d\lambda}.$$

P.8-16 $\mathcal{P}_{av} = |E|^2/2\eta_0 = 10^{-2}$ (W/cm²).

a) $|E| = \sqrt{0.02\eta_0} = 2.75$ (V/cm) $= 275$ (V/m).
$|H| = |E|/\eta_0 = 7.25\times10^{-3}$ (A/cm) $= 0.725$ (A/m).

b) $\mathcal{P}_{av} = |E|^2/2\eta_0 = 1300$ (W/m²).
$|E| = 990$ (V/m), $|H| = 2.63$ (A/m).

P.8-17 Assume circularly polarized plane wave:

$$\bar{E}(z,t) = \bar{a}_x E_0 \cos(\omega t - kz + \phi) + \bar{a}_y E_0 \sin(\omega t - kz + \phi),$$

$$\bar{H}(z,t) = \bar{a}_y \frac{E_0}{\eta} \cos(\omega t - kz + \phi) - \bar{a}_x \frac{E_0}{\eta} \sin(\omega t - kz + \phi).$$

Poynting vector, $\bar{\mathcal{P}} = \bar{E} \times \bar{H} = \bar{a}_z \frac{E_0^2}{\eta}\left[\cos^2(\omega t - kz + \phi) + \sin^2(\omega t - kz + \phi)\right]$

$= \bar{a}_z \frac{E_0^2}{\eta}$, a constant independent of t and z.

P.8-18

$$\bar{E} = \bar{a}_\theta E_\theta + \bar{a}_\phi E_\phi.$$

$$\bar{H} = \frac{1}{\eta}\bar{a}_R \times \bar{E} = \frac{1}{\eta}(\bar{a}_\phi E_\theta - \bar{a}_\theta E_\phi).$$

$$\bar{\mathcal{P}}_{av} = \frac{1}{2}\mathcal{R}e(\bar{E} \times \bar{H}^*) = \bar{a}_R \frac{1}{2\eta}(|E_\theta|^2 + |E_\phi|^2).$$

P.8-19 From Gauss's law, $\bar{E} = \bar{a}_r \frac{\rho}{2\pi\epsilon r}$, where ρ is the line charge density on the inner conductor.

$$V_0 = -\int_b^a \bar{E} \cdot d\bar{r} = \frac{\rho}{2\pi\epsilon}\ln\left(\frac{b}{a}\right) \longrightarrow \bar{E} = \bar{a}_r \frac{V_0}{r\ln(b/a)}.$$

From Ampère's circuital law, $\bar{H} = \bar{a}_\phi \frac{I}{2\pi r}$.

Poynting vector, $\bar{\mathcal{P}} = \bar{E} \times \bar{H} = \bar{a}_z \frac{V_0 I}{2\pi r^2 \ln(b/a)}$.

Power transmitted over cross-sectional area:

$$P = \int_S \bar{\mathcal{P}} \cdot d\bar{s} = \frac{V_0 I}{2\pi \ln(b/a)} \int_0^{2\pi}\int_a^b \left(\frac{1}{r^2}\right) r\,dr\,d\phi = V_0 I.$$

P.8-20 a) $\delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma}}$, $f = 10^4/2\pi$.

b) $\bar{H}(x,t) = \bar{a}_y H_0 e^{-x/\delta}\cos(10^4 t - \frac{x}{\delta})$.

$$\eta_c = (1+j)\frac{\alpha}{\sigma} = (1+j)\frac{1}{\sigma\delta} = \frac{\sqrt{2}}{\sigma\delta}e^{j\pi/4}.$$

$$\bar{E}(x,t) = \bar{a}_x \frac{\sqrt{2}}{\sigma\delta} H_0 e^{-x/\delta}\cos(10^4 t - \frac{x}{\delta} + \frac{\pi}{4}).$$

c) $\bar{\mathcal{P}}_{av} = \frac{1}{2}\mathcal{R}e(\bar{E} \times \bar{H}^*) = \bar{a}_x \frac{1}{2}\frac{\sqrt{2}}{\sigma\delta} H_0^2 \cos\frac{\pi}{4}$

$= \bar{a}_x \frac{1}{2}\left(\frac{H_0^2}{\sigma\delta}\right) \quad (W/m^2).$

P.8-21 Given $\bar{E}_i = E_0(\bar{a}_x - j\bar{a}_y)e^{-j\beta z}$.

a) Assume reflected $\bar{E}_r(z) = (\bar{a}_x E_{rx} + \bar{a}_y E_{ry})e^{j\beta z}$.

Boundary condition at $z=0$: $\bar{E}_i(0) + \bar{E}_r(0) = 0$.

$\longrightarrow \bar{E}_r(z) = E_0(-\bar{a}_x + j\bar{a}_y)e^{j\beta z}$, a left-hand circularly polarized wave in $-z$ direction.

b) $\bar{a}_{n2} \times (\bar{H}_1 - \bar{H}_2) = \bar{J}_s \longrightarrow -\bar{a}_z \times [\bar{H}_i(0) + \bar{H}_r(0)] = \bar{J}_s$. ($\bar{H}_2 = 0$ in perfect conductor)

$\bar{H}_i(0) = \frac{1}{\eta_0}\bar{a}_z \times \bar{E}_i(0) = \frac{E_0}{\eta_0}(j\bar{a}_x + \bar{a}_y)$, $\bar{H}_r(0) = \frac{1}{\eta_0}(-\bar{a}_z) \times \bar{E}_r(0) = \frac{E_0}{\eta_0}(j\bar{a}_x + \bar{a}_y)$.

$\bar{H}_1(0) = \bar{H}_i(0) + \bar{H}_r(0) = \frac{2E_0}{\eta_0}(j\bar{a}_x + \bar{a}_y)$,

$\bar{J}_s = -\bar{a}_z \times \bar{H}_1(0) = \frac{2E_0}{\eta_0}(\bar{a}_x - j\bar{a}_y)$.

c) $\bar{E}_1(z,t) = \mathcal{Re}[\bar{E}_i(z) + \bar{E}_r(z)]e^{j\omega t}$

$= \mathcal{Re}\, E_0[(\bar{a}_x - j\bar{a}_y)e^{-j\beta z} + (-\bar{a}_x + j\bar{a}_y)e^{j\beta z}]e^{j\omega t}$

$= \mathcal{Re}\, E_0[-2j(\bar{a}_x - j\bar{a}_y)\sin\beta z]e^{j\omega t}$

$= 2E_0 \sin\beta z(\bar{a}_x \sin\omega t - \bar{a}_y \cos\omega t)$.

P.8-22 Given $\bar{E}_i(x,z) = \bar{a}_y 10 e^{-j(6x+8z)}$ (V/m).

a) $k_x = 6$, $k_z = 8 \longrightarrow k = \beta = \sqrt{k_x^2 + k_z^2} = 10$ (rad/m).

$\lambda = 2\pi/k = 2\pi/10 = 0.628$ (m); $f = c/\lambda = 4.78 \times 10^8$ (Hz); $\omega = kc = 3 \times 10^9$ (rad/s)

b) $\bar{E}_i(x,z;t) = \bar{a}_y 10\cos(3\times 10^9 t - 6x - 8z)$ (V/m).

$\bar{H}_i(x,z) = \frac{1}{\eta_0}\bar{a}_{ni} \times \bar{E}_i$ $\quad \left(\bar{a}_{ni} = \frac{\bar{k}}{k} = \bar{a}_x 0.6 + \bar{a}_z 0.8\right)$

$= \frac{1}{120\pi}(\bar{a}_x 0.6 + \bar{a}_z 0.8) \times \bar{a}_y 10 e^{-j(6x+8z)} = \left(-\bar{a}_x \frac{1}{15\pi} + \bar{a}_z \frac{1}{20\pi}\right)e^{-j(6x+8z)}$

$\bar{H}_i(x,z;t) = \left(-\bar{a}_x \frac{1}{15\pi} + \bar{a}_z \frac{1}{20\pi}\right)\cos(3\times 10^9 t - 6x - 8z)$ (A/m).

c) $\cos\theta_i = \bar{a}_{ni} \cdot \bar{a}_z = \left(\frac{\bar{k}}{k}\right) \cdot \bar{a}_z = 0.8 \longrightarrow \theta_i = \cos^{-1} 0.8 = 36.9°$.

d) $\bar{E}_i(x,0) + \bar{E}_r(x,0) = 0 \longrightarrow \bar{E}_r(x,z) = -\bar{a}_y 10 e^{-j(6x-8z)}$

$\bar{H}_r(x,z) = \frac{1}{\eta_0}\bar{a}_{nr} \times \bar{E}_r(x,z)$ $\quad (\bar{a}_{nr} = \bar{a}_x 0.6 - \bar{a}_z 0.8)$

$= -\left(\bar{a}_x \frac{1}{15\pi} + \bar{a}_z \frac{1}{20\pi}\right)e^{-j(6x-8z)}$

e) $\bar{E}_1(x,z) = \bar{E}_i(x,z) + \bar{E}_r(x,z) = \bar{a}_y 10(e^{-j8z} - e^{j8z})e^{-j6x}$

$= -\bar{a}_y j20 e^{-j6x}\sin 8z$ (V/m).

$\bar{H}_1(x,z) = \bar{H}_i(x,z) + \bar{H}_r(x,z) = -\left(\bar{a}_x \frac{2}{15\pi}\cos 8z + \bar{a}_z \frac{j}{10\pi}\sin 8z\right)e^{-j6x}$ (A/m).

P.8-23 Given $\bar{E}_i(y,z) = 5(\bar{a}_y + \bar{a}_z\sqrt{3})e^{j6(\sqrt{3}y-z)}$ (V/m).

a) $k_y = -6\sqrt{3}$, $k_z = 6 \longrightarrow k = \sqrt{k_y^2+k_z^2} = 12$ (rad/m).

$\lambda = 2\pi/k = \pi/6 = 0.524$ (m); $f = c/\lambda = 5.73\times10^8$ (Hz); $\omega = kc = 3.60\times10^9$ (rad/s).

b) $\bar{E}_i(y,z;t) = 5(\bar{a}_y + \bar{a}_z\sqrt{3})\cos(3.60\times10^9 t + 6\sqrt{3}y - 6z)$ (V/m).

$\bar{H}_i(y,z) = \frac{1}{\eta_0}\bar{a}_{ni}\times\bar{E}_i = \frac{1}{120\pi}(-\bar{a}_y\frac{\sqrt{3}}{2} + \bar{a}_z\frac{1}{2})\times 5(\bar{a}_y + \bar{a}_z\sqrt{3})e^{j6(\sqrt{3}y-z)}$

$= \bar{a}_x(-\frac{1}{12\pi})e^{j6(\sqrt{3}y-z)}$

$\bar{H}_i(y,z;t) = \bar{a}_x(-\frac{1}{12\pi})\cos(3.60\times10^9 t + 6\sqrt{3}y - 6z)$ (A/m).

c) $\cos\theta_i = \bar{a}_{ni}\cdot\bar{a}_z = \frac{1}{2} \longrightarrow \theta_i = \cos^{-1}(\frac{1}{2}) = 60°$.

d) Conditions $\bar{a}_{nr}\cdot\bar{E}_r(y,z) = 0$ and $E_{iy}(y,0) + E_{ry}(y,0) = 0$ lead to:

$\bar{E}_r(y,z) = 5(-\bar{a}_y + \bar{a}_z\sqrt{3})e^{j6(\sqrt{3}y+z)}$ (V/m),

$\bar{H}_r(y,z) = \frac{1}{\eta_0}\bar{a}_{nr}\times\bar{E}_r(y,z) = \frac{1}{120\pi}(-\bar{a}_y\frac{\sqrt{3}}{2} - \bar{a}_z\frac{1}{2})\times 5(-\bar{a}_y + \bar{a}_z\sqrt{3})e^{j6(\sqrt{3}y+z)}$

$= \bar{a}_x(-\frac{1}{12\pi})e^{j6(\sqrt{3}y+z)}$ (A/m).

e) $\bar{E}_1(y,z) = \bar{E}_i(y,z) + \bar{E}_r(y,z) = (-\bar{a}_y j10\sin 6z + \bar{a}_z 10\sqrt{3}\cos 6z)e^{j6\sqrt{3}y}$ (V/m),

$\bar{H}_1(y,z) = \bar{H}_i(y,z) + \bar{H}_r(y,z) = \bar{a}_x(-\frac{1}{6\pi})\cos 6z\cdot e^{j6\sqrt{3}y}$ (A/m).

P.8-24 a) From Eqs. (8-113) and (8-114):

$\bar{E}_1(x,z;t) = \bar{a}_y 2E_{i0}\sin(\beta_1 z\cos\theta_i)\sin(\omega t - \beta_1 x\sin\theta_i)$,

$\bar{H}_1(x,z;t) = \frac{2E_{i0}}{\eta_1}\Big[-\bar{a}_x\cos\theta_i\cos(\beta_1 z\cos\theta_i)\cos(\omega t - \beta_1 x\sin\theta_i)$
$+ \bar{a}_z\sin\theta_i\sin(\beta_1 z\cos\theta_i)\sin(\omega t - \beta_1 x\sin\theta_i)\Big]$.

b) $\bar{\mathscr{P}}_{av} = \frac{1}{2}\mathcal{R}e(\bar{E}\times\bar{H}^*) = \bar{a}_x\frac{2E_{i0}^2}{\eta_1}\sin\theta_i\sin^2(\beta_1 z\cos\theta_i)$.

P.8-25 a) From Eqs. (8-128) and (8-129):

$\bar{E}_1(x,z;t) = -2E_{i0}\Big[\bar{a}_x\cos\theta_i\sin(\beta_1 z\cos\theta_i)\cos(\omega t - \beta_1 x\sin\theta_i)$
$+ \bar{a}_z\sin\theta_i\cos(\beta_1 z\cos\theta_i)\sin(\omega t - \beta_1 x\sin\theta_i)\Big]$,

$\bar{H}_1(x,z;t) = \bar{a}_y\frac{2E_{i0}}{\eta_1}\cos(\beta_1 z\cos\theta_i)\sin(\omega t - \beta_1 x\sin\theta_i)$.

b) $\bar{\mathscr{P}}_{av} = \frac{1}{2}\mathcal{R}e(\bar{E}\times\bar{H}^*) = \bar{a}_x\frac{2E_{i0}^2}{\eta_1}\sin\theta_i\cos^2(\beta_1 z\cos\theta_i)$.

P.8-26 For normal incidence: $1 + \Gamma = \tau$, where $|\Gamma| \le 1$.

If $|\tau| = |\Gamma|$: $\Gamma < 0$ and $\eta_1 - \eta_2 = 2\eta_1 \longrightarrow \eta_1 = 3\eta_2 \longrightarrow |\Gamma| = \frac{1}{2}$.

$\therefore\ S = \frac{1+|\Gamma|}{1-|\Gamma|} = 3 \longrightarrow S_{dB} = 20\log_{10}3 = 9.54$ (dB).

P.8-27 a) In the lossy medium (medium 2):

$$\bar{E}_t = \bar{a}_x E_{t0} e^{-\alpha_2 z} e^{-j\beta_2 z},$$

where, from Problem P.8-9, $\alpha_2 = \omega\sqrt{\frac{\mu_2\epsilon_2}{2}}\left[\sqrt{1+\left(\frac{\sigma_2}{\omega\epsilon_2}\right)^2}-1\right]^{\frac{1}{2}}$, $\beta_2 = \omega\sqrt{\frac{\mu_2\epsilon_2}{2}}\left[\sqrt{1+\left(\frac{\sigma_2}{\omega\epsilon_2}\right)^2}+1\right]^{\frac{1}{2}}$.

Given: $\beta_1 = 6$ (rad/m) $\longrightarrow \omega = \beta_1 c = 1.8\times10^9$ (rad/s).

$\tan\delta_c = \frac{\sigma_2}{\omega\epsilon_2} = 0.5 \longrightarrow \alpha_2 = 2.30$ (Np/m), $\beta_2 = 9.76$ (rad/m).

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_c}} = \frac{120\pi}{\sqrt{\epsilon_{r2}}\,(1+\tan^2\delta_c)^{1/4}}\, e^{j\frac{1}{2}\tan^{-1}(\sigma_2/\omega\epsilon_2)} = 225\, e^{j13.3^\circ}.$$

$\bar{E}_t = \bar{a}_x E_{t0} e^{-2.30z} e^{-j9.76z}$, $\quad \bar{H}_t = \bar{a}_z \times \frac{\bar{E}_t}{\eta_2} = \bar{a}_y \frac{E_{t0}}{225} e^{-j13.3^\circ} e^{-2.30z} e^{-j9.76z}$

Let $\bar{E}_r = \bar{a}_x E_{r0} e^{j6z} \longrightarrow \bar{H}_r = -\bar{a}_y \frac{E_{r0}}{120\pi} e^{j6z}$. $\quad \bar{H}_i = \bar{a}_y \frac{10}{120\pi} e^{-j6z}$.

Boundary conditions for $\bar{E}$ and $\bar{H}$ at $z=0$: $\begin{cases} 10 + E_{r0} = E_{t0}, \\ 10 - E_{r0} = E_{t0}\sqrt{\epsilon_{r2}}\,(1+\tan^2\delta_c)^{1/4} e^{-j13.3^\circ}. \end{cases}$

$\longrightarrow E_{r0} = 2.77\, e^{j157^\circ}$; $\quad E_{t0} = 7.53\, e^{-j172^\circ}$

$\therefore$

$\bar{E}_r(z,t) = \bar{a}_x 2.77\cos(1.8\times10^9 t + 6z + 157^\circ)$ (V/m),

$\bar{H}_r(z,t) = -\bar{a}_y 0.073\cos(1.8\times10^9 t + 6z + 157^\circ)$ (A/m),

$\bar{E}_t(z,t) = \bar{a}_x 7.53\, e^{-2.30z}\cos(1.8\times10^9 t - 9.76z - 172^\circ)$ (V/m),

$\bar{H}_t(z,t) = \bar{a}_y 0.033\, e^{-2.30z}\cos(1.8\times10^9 t - 9.76z + 174.7^\circ)$ (A/m).

b) $(\bar{\mathcal{P}}_{av})_1 = \bar{a}_z\left(\frac{10^2}{2\times120\pi} - \frac{2.77^2}{2\times120\pi}\right) = \bar{a}_z 0.122.$ (W/m²),

$(\bar{\mathcal{P}}_{av})_2 = \bar{a}_z \frac{7.53^2}{2\times225}(\cos 13.3^\circ)\, e^{-4.60z} = \bar{a}_z 0.122\, e^{-4.60z}$ (W/m²).

P.8-28 a) $\Gamma = \frac{E_r}{E_i} = \frac{\eta_c - \eta_0}{\eta_c + \eta_0}$; $\quad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi$, $\eta_c = \sqrt{j\omega\mu/\sigma}$.

$|\eta_c| \ll \eta_0$.

b) $|\Gamma|^2 = \left|\frac{\eta_c - \eta_0}{\eta_c + \eta_0}\right|^2 = \left|\frac{1-\eta_c/\eta_0}{1+\eta_c/\eta_0}\right|^2 \cong \left|1 - 2\eta_c/\eta_0\right|^2$

$= (1-2\eta_c/\eta_0)(1-2\eta_c^*/\eta_0) \cong 1 - 4\mathcal{R}_e(\eta_c)/\eta_0$.

Fraction of power absorbed, $F = 1 - |\Gamma|^2 = \frac{4}{\eta_0}\mathcal{R}_e\sqrt{\frac{j\omega\mu}{\sigma}}$

$= \frac{4}{\eta_0}\sqrt{\frac{\omega\mu}{2\sigma}}$.

c) $\omega = 2\pi\times10^6$ (Hz). For iron: $\mu = 4{,}000\times(4\pi10^{-7})$ (H/m), $\sigma = 10^7$ (S/m).

$F = 4.21\times10^{-4}$, or 0.0421%.

P.8-29 From Eqs. (8-156) through (8-161):

$$\bar{E}_1 = \bar{a}_x\left(E_{i0}e^{-j\beta_0 z} + E_{r0}e^{j\beta_0 z}\right), \qquad \bar{H}_1 = \bar{a}_y\frac{1}{\eta_0}\left(E_{i0}e^{-j\beta_0 z} - E_{r0}e^{j\beta_0 z}\right),$$

$$\bar{E}_2 = \bar{a}_x\left(E_2^+e^{-j\beta_2 z} + E_2^-e^{j\beta_2 z}\right), \qquad \bar{H}_2 = \bar{a}_y\frac{1}{\eta_2}\left(E_2^+e^{-j\beta_2 z} - E_2^-e^{j\beta_2 z}\right),$$

$$\bar{E}_t = \bar{a}_x E_{t0}e^{-j\beta_0 z}, \qquad \bar{H}_t = \bar{a}_y\frac{1}{\eta_0}e^{-j\beta_0 z}.$$

Boundary conditions:
- at $z=0$: $\bar{E}_1(0) = \bar{E}_2(0)$, $\quad \bar{H}_1(0) = \bar{H}_2(0)$.
- at $z=d$: $\bar{E}_2(d) = \bar{E}_t(d)$, $\quad \bar{H}_2(d) = \bar{H}_t(d)$.

Four equations to solve four unknowns E_2^+, E_2^-, E_{r0}, and E_{t0} in terms of E_{i0}:

a) $$E_{r0} = -\frac{j(\eta_2^2-\eta_0^2)\tan\beta_2 d}{\eta_0\eta_2 + j(\eta_0^2+\eta_2^2)\tan\beta_2 d}E_{i0},$$

$$E_2^+ = \frac{\eta_2(\eta_0+\eta_2)e^{j\beta_2 d}}{\eta_0\eta_2\cos\beta_2 d + j(\eta_0^2+\eta_2^2)\sin\beta_2 d}E_{i0},$$

$$E_2^- = \frac{\eta_2(\eta_0-\eta_2)e^{-j\beta_2 d}}{\eta_0\eta_2\cos\beta_2 d + j(\eta_0^2+\eta_2^2)\sin\beta_2 d}E_{i0},$$

$$E_{t0} = \frac{2\eta_0\eta_2 e^{j\beta_0 d}}{\eta_0\eta_2\cos\beta_2 d + j(\eta_0^2+\eta_2^2)\sin\beta_2 d}E_{i0},$$

where
$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi$,
$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$,
$\beta_0 = \frac{\omega}{c}$,
$\beta_2 = \omega\sqrt{\mu_2\epsilon_2}$.

b) If $d = \lambda_2/4$, $\beta_2 d = \pi/2$, $\quad E_{r0} = -\frac{\eta_2^2-\eta_0^2}{\eta_0^2+\eta_2^2}E_{i0}$.

$\therefore\ \Gamma = -\frac{\eta_2^2-\eta_0^2}{\eta_0^2+\eta_2^2} \neq 0$ unless $\eta_2 = \eta_0$ (a trivial case).

If $d = n\lambda_2/2$ $(n = 1, 2, 3, \cdots)$, $\tan\beta_2 d = 0 \longrightarrow \Gamma = 0$.

P.8-30 From Example 8-12: $\eta_2 = \sqrt{\eta_1\eta_3} \longrightarrow \epsilon_{2r} = \sqrt{\epsilon_{r1}\epsilon_{r3}} = 2$.

a) Wavelength of red light in dielectric coating: $\lambda_2 = \frac{0.75}{\sqrt{\epsilon_{r2}}} = 0.530\ (\mu m) \longrightarrow d = \frac{\lambda_2}{4} = 0.133\ (\mu m)$

b) For violet light: $\lambda_2' = \frac{0.42}{\sqrt{2}} = 0.297\ (\mu m)$.

$\frac{d}{\lambda_2'} = 0.447 \longrightarrow \beta_2 d = 0.894\pi$.

From Eq. (8-173) and using impedances normalized with respect to $\eta_1 = \eta_0$:
$$Z_2(0) = \eta_2\frac{\eta_3 + j\eta_2\tan\beta_2 d}{\eta_2 + j\eta_3\tan\beta_2 d} = \frac{1}{\sqrt{2}}\,\frac{\frac{1}{2}+j\frac{1}{\sqrt{2}}\tan\beta_2 d}{\frac{1}{\sqrt{2}}+j\frac{1}{2}\tan\beta_2 d} = \frac{0.5 - j0.247}{1.0 - j0.247}.$$

$$\Gamma = \frac{Z_2(0)-1}{Z_2(0)+1} = 0.317e^{j195°}.$$

Percentage of power reflected $= |\Gamma|^2 \times 100\%$
$= (0.317)^2 \times 100\% = 10\%$.

P.8-31

$$\Gamma_0 = \frac{Z_2(0)-\eta_1}{Z_2(0)+\eta_1}, \quad Z_2(0) = \eta_2\,\frac{\eta_3 + j\eta_2 \tan\beta_2 d}{\eta_2 + j\eta_3 \tan\beta_2 d}.$$

$$\Gamma_{12} = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} \longrightarrow \frac{\eta_1}{\eta_2} = \frac{1-\Gamma_{12}}{1+\Gamma_{12}}.$$

$$\Gamma_{23} = \frac{\eta_3-\eta_2}{\eta_3+\eta_2} \longrightarrow \frac{\eta_2}{\eta_3} = \frac{1-\Gamma_{23}}{1+\Gamma_{23}}.$$

$$\therefore \Gamma_0 = \frac{1 + j\frac{\eta_2}{\eta_3}\tan\beta_2 d - \frac{\eta_1}{\eta_2}\left(\frac{\eta_2}{\eta_3} + j\tan\beta_2 d\right)}{1 + j\frac{\eta_2}{\eta_3}\tan\beta_2 d + \frac{\eta_1}{\eta_2}\left(\frac{\eta_2}{\eta_3} + j\tan\beta_2 d\right)}$$

$$= \frac{1 + j\frac{1-\Gamma_{23}}{1+\Gamma_{23}}\tan\beta_2 d - \frac{1-\Gamma_{12}}{1+\Gamma_{12}}\left(\frac{1-\Gamma_{23}}{1+\Gamma_{23}} + j\tan\beta_2 d\right)}{1 + j\frac{1-\Gamma_{23}}{1+\Gamma_{23}}\tan\beta_2 d + \frac{1-\Gamma_{12}}{1+\Gamma_{12}}\left(\frac{1-\Gamma_{23}}{1+\Gamma_{23}} + j\tan\beta_2 d\right)}$$

$$= \frac{(\Gamma_{12}+\Gamma_{23}) + j(\Gamma_{12}-\Gamma_{23})\tan\beta_2 d}{(1+\Gamma_{12}\Gamma_{23}) + j(1-\Gamma_{12}\Gamma_{23})\tan\beta_2 d}.$$

P.8-32

$$\bar{E}_1 = \bar{a}_x\left(E_{i0}e^{-j\beta_1 z} + E_{r0}e^{j\beta_1 z}\right),$$

$$\bar{H}_1 = \bar{a}_y\frac{1}{\eta_1}\left(E_{i0}e^{-j\beta_1 z} - E_{r0}e^{j\beta_1 z}\right).$$

$$\bar{E}_2 = \bar{a}_x\left(E_2^+ e^{-j\beta_2 z} + E_2^- e^{j\beta_2 z}\right).$$

$$\bar{H}_2 = \bar{a}_y\frac{1}{\eta_2}\left(E_2^+ e^{-j\beta_2 z} - E_2^- e^{j\beta_2 z}\right).$$

At $z=d$, $\bar{E}_2 = 0 \longrightarrow E_2^- = -E_2^+ e^{-j2\beta_2 d}$.

$$\bar{E}_2 = \bar{a}_x E_2^+\left[e^{-j\beta_2 z} - e^{j\beta_2(z-2d)}\right].$$

$$\bar{H}_2 = \bar{a}_y\frac{E_2^+}{\eta_2}\left[e^{-j\beta_2 z} + e^{j\beta_2(z-2d)}\right].$$

Boundary conditions at $z=0$: $E_1(0)=E_2(0) \longrightarrow E_{i0}+E_{r0} = E_2^+\left(1-e^{-j2\beta_2 d}\right)$,

$H_1(0)=H_2(0) \longrightarrow E_{i0}-E_{r0} = E_2^+\frac{\eta_1}{\eta_2}\left(1+e^{-j2\beta_2 d}\right)$.

$$E_2^+ = \frac{2\eta_2 E_{i0}}{(\eta_2+\eta_1)+(\eta_1-\eta_2)e^{-j2\beta_2 d}}.$$

$$E_{r0} = -\left(\frac{\eta_1 - j\eta_2\tan\beta_2 d}{\eta_1 + j\eta_2\tan\beta_2 d}\right)E_{i0}.$$

a) $\bar{E}_r(z,t) = \bar{a}_x E_{i0}\cos\left[\omega\left(t-\frac{z}{u_p}\right)+\theta\right]$, $\theta = \pi - 2\tan^{-1}\left(\frac{\eta_2}{\eta_0}\tan\beta_2 d\right)$.

b) $\bar{E}_1(z,t) = \bar{a}_x E_{i0}\left\{\cos\omega\left(t-\frac{z}{u_p}\right)+\cos\left[\omega\left(t-\frac{z}{u_p}\right)+\theta\right]\right\}$.

c) $$\bar{E}_2(z,t) = \bar{a}_x \frac{2\eta_2 E_{i0}}{\sqrt{2[(\eta_0^2+\eta_2^2)+(\eta_0^2-\eta_2^2)\cos 2\beta_2 d]}}\left\{\cos\left[\omega\left(t-\frac{z}{u_{p2}}\right)+\psi\right] - \cos\left[\omega\left(t+\frac{z}{u_{p2}}\right)-\frac{2\omega d}{u_{p2}}+\psi\right]\right\},$$

$$\psi = \tan^{-1}\left[\frac{(\eta_0-\eta_2)\sin 2\beta_2 d}{(\eta_0+\eta_2)+(\eta_0-\eta_2)\cos 2\beta_2 d}\right].$$

d) $(\bar{\mathcal{P}}_{av})_1 = \frac{1}{2}\mathcal{R}e(\bar{E}_1\times\bar{H}_1^*) = 0$.

e) $(\bar{\mathcal{P}}_{av})_2 = 0$.

f) Let $E_r = -E_{i0} \longrightarrow \tan\beta_2 d = 0 \longrightarrow d = n\lambda_2/2$, $n = 0, 1, 2, \cdots$

P.8-33 $\quad k_2 = \beta_2 - j\alpha_2 = (1-j)\frac{1}{\delta}$, $\quad \alpha_2 = \beta_2 = \frac{1}{\delta} = \sqrt{\pi f\mu_0\sigma}$.

$\eta_2 = (1+j)\frac{\alpha_2}{\sigma_2} \ll \eta_0$ at 10 (MHz).

a) From Problem P.8-29,

$$E_2^+ = \eta_2 H_2^+ \cong -j\left(\frac{\eta_2}{\eta_0}\right)\frac{e^{\alpha_2 d}e^{j\beta_2 d}E_{i0}}{\sin(\beta_2-j\alpha_2)d}.$$

b) $$E_2^- = -\eta_2 H_2^- \cong -j\left(\frac{\eta_2}{\eta_0}\right)\frac{e^{-\alpha_2 d}e^{-j\beta_2 d}E_{i0}}{\sin(\beta_2-j\alpha_2)d}.$$

c) $$E_{30} = E_{t0} = \eta_0 H_{30} \cong -j\left(\frac{\eta_2}{\eta_0}\right)\frac{2e^{j\beta_0 d}E_{i0}}{\sin(\beta_2-j\alpha_2)d}.$$

d) $$E_{r0} \cong -\frac{E_{i0}}{1-j\frac{\eta_2}{\eta_0}\cot(\beta_2-j\alpha_2)d}$$

$$\cong -\left(1+j\frac{\eta_2}{\eta_0}\,\frac{1+j\tan\beta_2 d\tanh\alpha_2 d}{\tan\beta_2 d - j\tanh\alpha_2 d}\right)E_{i0}.$$

$$(\bar{\mathcal{P}}_{av})_1 = \frac{1}{2}\mathcal{R}e\left[(\bar{E}_{i0}\times\bar{H}_{i0}^*)-(\bar{E}_{r0}\times\bar{H}_{r0}^*)\right]$$
$$= \bar{a}_z\frac{\alpha}{\eta_0^2\sigma}(A+B),$$

where $$\frac{1+j\tan\beta_2 d\tanh\alpha_2 d}{\tan\beta_2 d - j\tanh\alpha_2 d} = A+jB.$$

$$(\mathcal{P}_{av})_1 = \frac{\alpha}{\eta_0^2\sigma}\,\frac{\sin\beta_2 d\cos\beta_2 d+\sinh\alpha_2 d\cosh\alpha_2 d}{(\sin\beta_2 d\cosh\alpha_2 d)^2+(\cos\beta_2 d\sinh\alpha_2 d)^2}.$$

$$(\mathcal{P}_{av})_3 = \frac{1}{2\eta_0}|E_{30}|^2 = \frac{1}{\eta_0^2}\left(\frac{\alpha}{\sigma}\right)^2\frac{4E_{i0}^2}{(\sin\beta_2 d\cosh\alpha_2 d)^2+(\cos\beta_2 d\sinh\alpha_2 d)^2}.$$

$$\therefore \quad \frac{(\mathcal{P}_{av})_2}{(\mathcal{P}_{av})_i} = \frac{4}{\eta_0}\left(\frac{\alpha}{\sigma}\right)\frac{1}{\sin\beta_2 d\cos\beta_2 d + \sinh\alpha_2 d\cosh\alpha_2 d}.$$

$$(\mathcal{P}_{av})_i = \frac{1}{2\eta_0}E_{i0}^2.$$

$$\frac{(\mathcal{P}_{av})_3}{(\mathcal{P}_{av})_i} = \frac{8}{\eta_0^2}\left(\frac{\alpha}{\sigma}\right)^2\frac{1}{(\sin\beta_2 d\cosh\alpha_2 d)^2 + (\cos\beta_2 d\sinh\alpha_2 d)^2}.$$

At $f = 10^9$ (Hz), $\sigma = 5.80\times10^7$ (S/m), $\alpha_2 = \beta_2 = 4.785\times10^5$, $d = \delta = \frac{1}{\alpha_2}$.

$$\frac{(\mathcal{P}_{av})_3}{(\mathcal{P}_{av})_i} = 1.839\times10^{-11}.$$

P.8-34 Given $f = f_p/2$ and $\theta_i = 60°$.

$$\longrightarrow \eta_p = \eta_0\Big/\sqrt{1-(f_p/f)^2} = -j\eta_0/\sqrt{3}, \quad \eta_2/\eta_0 = -j/\sqrt{3}.$$

From Eq.(8-185): $\sin\theta_t = \frac{\eta_2}{\eta_0}\sin\theta_i = -j/2$, $\cos\theta_t = \sqrt{5}/2$, $\cos\theta_i = 1/2$.

a) From Eq. (8-206): $\Gamma_\perp = \frac{(\eta_p/\eta_0)\cos\theta_i - \cos\theta_t}{(\eta_p/\eta_0)\cos\theta_i + \cos\theta_t} = e^{j209°}$.

From Eq. (8-207): $\tau_\perp = \frac{2(\eta_p/\eta_0)\cos\theta_i}{(\eta_p/\eta_0)\cos\theta_i + \cos\theta_t} = 0.5\,e^{-j75.5°}$.

b) From Eq. (8-221): $\Gamma_\parallel = \frac{(\eta_p/\eta_0)\cos\theta_t - \cos\theta_i}{(\eta_p/\eta_0)\cos\theta_t + \cos\theta_i} = e^{j76°}$.

From Eq. (8-222): $\tau_\parallel = \frac{2(\eta_p/\eta_0)\cos\theta_i}{(\eta_p/\eta_0)\cos\theta_t + \cos\theta_i} = 0.177\,e^{-j38°}$.

$|\Gamma_\perp| = |\Gamma_\parallel| = 1$, but the phase shift of the reflected wave depends on the polarization of the incident wave. There are standing waves in the air and exponentially decaying transmitted waves in the ionosphere.

P.8-35 $k_{2x}^2 + k_{2z}^2 = k_2^2 = \omega^2\mu_0\epsilon_2 - j\omega\mu_0\sigma_2$. ①

Continuity conditions at $z=0$ for all x and y require:

$$k_{2x} = k_{1x} = \omega\sqrt{\mu_0\epsilon_0}\sin\theta_i = \beta_x = 2.09\times10^{-4}. \quad ②$$

$$k_{2z} = \beta_{2z} - j\alpha_{2z}. \quad ③$$

Combining ①, ② and ③, we can solve for α_{2z} and β_{2z} in terms of ω, μ_0, ϵ_2, σ_2, and β_x. But, since

$$\beta_x^2 << \omega^2 \mu_0 \epsilon_2 .$$

we have $\alpha_2 = \alpha_{2x} = \beta_{2x} \cong \frac{1}{\delta} = \sqrt{\pi f \mu_0 \sigma_2} = 0.3974 \ (m^{-1})$.

a) $\theta_t = \tan^{-1}\frac{\beta_x}{\beta_{2x}} = \tan^{-1}\frac{2.09}{0.3974}\times 10^{-6} \cong 5.26\times 10^{-6} \ (rad)$
$= 0.03°$.

b) $\Gamma_\parallel = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_0 \cos\theta_i}$ $\qquad \eta_2 = \frac{\alpha_2}{\sigma_2}(1+j) = 0.0993(1+j)$.

$= \frac{2\times 0.0993(1+j)}{0.0993(1+j) + 377\cos 35°}$ $\qquad \cos\theta_t = \cos 0.03° \cong 1$.

$\cong 0.0151(1+j) = 0.0214\, e^{j\pi/4}$.

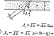
c) $(\mathcal{P}_{av})_i = \frac{E_{i0}^2}{2\eta_0}$.

$E_{t0} \cong 2E_{i0}\frac{\eta_2}{\eta_0}, \quad H_{t0} \cong \frac{2E_{i0}}{\eta_0} \longrightarrow (\mathcal{P}_{av})_t = 2\frac{E_{i0}^2 \alpha_2}{\eta_0^2 \sigma_2} e^{-2\alpha_2 z}$.

$\therefore \frac{(\mathcal{P}_{av})_t}{(\mathcal{P}_{av})_i} = \frac{4\alpha_2}{\eta_0 \sigma_2} e^{-2\alpha_2 z} = 1.054\times 10^{-3} e^{-0.795 z}$

d) $20\log_{10} e^{-\alpha_2 z} = -30. \longrightarrow z = \frac{1.5}{\alpha_2 \log_{10} e} = 8.69 \ (m)$.

P.8-36



a) Snell's law:

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{1}{n},$$

$$\theta_t = \sin^{-1}\left(\frac{1}{n}\sin\theta_i\right).$$

b) $\cos\theta_t = \sqrt{1 - \left(\frac{1}{n}\sin\theta_i\right)^2}$.

$$\ell_1 = \overline{BC} = \overline{AC}\tan\theta_t = d\frac{\sin\theta_t}{\cos\theta_t} = \frac{d\sin\theta_i}{\sqrt{n^2 - \sin^2\theta_i}}.$$

c) $\ell_2 = \overline{BD} = \overline{AC}\sin(\theta_i - \theta_t) = \frac{d}{\cos\theta_t}(\sin\theta_i\cos\theta_t - \cos\theta_i\sin\theta_t)$

$$= d\sin\theta_i\left[1 - \frac{\cos\theta_i}{\sqrt{n^2 - \sin^2\theta_i}}\right].$$

P.8-37

a) $\sin\theta_c = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \longrightarrow \sin\theta_t = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i > 1$ for $\theta_i > \theta_c$.

$$\cos\theta_t = -j\sqrt{\left(\frac{\epsilon_1}{\epsilon_2}\right)\sin^2\theta_i - 1}.$$

From Eqs. (8-200) and (8-201):

$$\bar{E}_t(x,z) = \bar{a}_y E_{t0} e^{-\alpha_2 z} e^{-j\beta_{2x} x},$$

$$\bar{H}_t(x,z) = \frac{E_{t0}}{\eta_2}\left(\bar{a}_x j\alpha_2 + \bar{a}_z \sqrt{\frac{\epsilon_1}{\epsilon_2}} \sin\theta_i\right) e^{-\alpha_2 z} e^{-j\beta_{2x} x},$$

where $\beta_{2x} = \beta_2 \sin\theta_t = \beta_2 \sqrt{\frac{\epsilon_1}{\epsilon_2}} \sin\theta_i$,

$$\alpha_2 = \beta_2 \sqrt{\left(\frac{\epsilon_1}{\epsilon_2}\right)\sin^2\theta_i - 1},$$

$$E_{t0} = \frac{2\eta_2 \cos\theta_i \, E_{i0}}{\eta_2 \cos\theta_i - j\eta_1 \sqrt{\left(\frac{\epsilon_1}{\epsilon_2}\right)\sin^2\theta_i - 1}} \quad \text{from Eq. (8-207).}$$

b) $(\mathcal{P}_{av})_{2z} = \frac{1}{2}\mathcal{Re}\,(E_{ty} H_{tx}^{*}) = 0.$

P. 8-38 Given $\theta_i = \theta_c \longrightarrow \theta_t = \pi/2$, $\cos\theta_t = 0$.

a) From Eq. (8-207): $(E_{t0}/E_{i0})_\perp = 2$.

b) From Eq. (8-221): $(E_{t0}/E_{i0})_\parallel = 2\eta_2/\eta_1$.

c) $\bar{E}_i(x,z;t) = \bar{a}_y E_{i0} \cos\omega\left[t - \frac{n_1}{c}(x\sin\theta_i + z\cos\theta_i)\right],$

$$\bar{E}_t(x,z;t) = \bar{a}_y 2E_{i0} e^{-\alpha z} \cos\omega\left(t - \frac{n_2}{c} x \sin\theta_t\right)$$

$$= \bar{a}_y 2E_{i0} e^{-\alpha z} \cos\omega\left(t - \frac{n_1}{c} x \sin\theta_i\right),$$

where $\alpha = \frac{n_2\omega}{c}\sqrt{\left(\frac{n_1}{n_2}\sin\theta_i\right)^2 - 1} = 0$ when $\theta = \theta_c$.

P. 8-39. a) $\theta_c = \sin^{-1}\sqrt{\epsilon_{r2}/\epsilon_{r1}} = \sin^{-1}\sqrt{1/81} = 6.38°$.

b) $\theta_i = 20° > \theta_c$. $\sin\theta_t = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i = 3.08$, $\cos\theta_t = -j2.91$.

$$\Gamma_\perp = \frac{\sqrt{\epsilon_{r1}}\cos\theta_i - \cos\theta_t}{\sqrt{\epsilon_{r1}}\cos\theta_i + \cos\theta_t} = e^{j\,38°} = e^{j0.66}$$

c) $$\tau_\perp = \frac{2\sqrt{\epsilon_{r1}}\cos\theta_i}{\sqrt{\epsilon_{r1}}\cos\theta_i + \cos\theta_t} = 1.89\, e^{j19°} = 1.89\, e^{j0.33}$$

d) The transmitted wave in air varies as $e^{-\alpha_2 z} e^{-j\beta_{2x} x}$, where $\alpha_2 = \beta_2\sqrt{\left(\frac{\epsilon_1}{\epsilon_2}\right)\sin^2\theta_i - 1} = \frac{2\pi}{\lambda_0}(2.91)$.

Attenuation in air for each wavelength

$$= 20\log_{10} e^{-\alpha_2\lambda_0} = 159 \text{ (dB)}.$$

P.8-40 When the incident light first strikes the hypotenuse surface, $\theta_i = \theta_t = 0$, $\tau_1 = \frac{2\eta_2}{\eta_2 + \eta_0}$.

$$\frac{(\mathcal{P}_{av})_{t1}}{(\mathcal{P}_{av})_i} = \frac{\eta_0}{\eta_2}\tau_1^2 = \frac{4\eta_0\eta_2}{(\eta_2+\eta_0)^2}.$$

Total reflections occur inside the prism at both slanting surfaces because

$$\theta_i = 45° > \theta_c = \sin^{-1}\left(\tfrac{1}{2}\right) = 30°.$$

On exit from the prism, $\tau_2 = \frac{2\eta_0}{\eta_2 + \eta_0}$.

$$\frac{(\mathcal{P}_{av})_o}{(\mathcal{P}_{av})_{t1}} = \frac{\eta_2}{\eta_0}\tau_2^2 = \frac{4\eta_0\eta_2}{(\eta_2+\eta_0)^2}.$$

$$\therefore \frac{(\mathcal{P}_{av})_o}{(\mathcal{P}_{av})_i} = \left[\frac{4\eta_0\eta_2}{(\eta_2+\eta_0)^2}\right]^2 = \left[\frac{4\sqrt{\epsilon_r}}{(1+\sqrt{\epsilon_r})^2}\right]^2 = 0.79.$$

P.8-41 a) $n_0 \sin\theta_a = n_1 \sin(90° - \theta_t) = n_1 \cos\theta_t$

$$= n_1\sqrt{1-\sin^2\theta_t} = n_1\sqrt{1-(n_2/n_1)^2} = \sqrt{n_1^2 - n_2^2}.$$

$$\sin\theta_a = \frac{1}{n_0}\sqrt{n_1^2-n_2^2} = \sqrt{n_1^2-n_2^2}. \qquad (n_0 = 1)$$

b) $N.A. = \sin\theta_a = \sqrt{2^2 - 1.74^2} = 0.9861$,

$$\theta_a = \sin^{-1}0.9861 = 80.4°.$$

P.8-42 Eq. (8-185): $\frac{\eta_2}{\eta_1} = \frac{\sin\theta_t}{\sin\theta_i}$.

a) Eq. (8-206): $$\Gamma_\perp = \frac{(\eta_2/\eta_1)\cos\theta_i - \cos\theta_t}{(\eta_2/\eta_1)\cos\theta_i + \cos\theta_t} = \frac{\sin\theta_t\cos\theta_i - \cos\theta_t\sin\theta_i}{\sin\theta_t\cos\theta_i + \cos\theta_t\sin\theta_i} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}.$$

Eq. (8-207): $$\tau_\perp = \frac{2(\eta_2/\eta_1)\cos\theta_i}{(\eta_2/\eta_1)\cos\theta_i + \cos\theta_t} = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_t+\theta_i)}.$$

b) Eq. (8-221): $$\Gamma_\parallel = \frac{(\eta_2/\eta_1)\cos\theta_t - \cos\theta_i}{(\eta_2/\eta_1)\cos\theta_t + \cos\theta_i} = \frac{\sin\theta_t\cos\theta_t - \sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i}.$$

Eq. (8-222): $$\tau_\parallel = \frac{2(\eta_2/\eta_1)\cos\theta_i}{(\eta_2/\eta_1)\cos\theta_t + \cos\theta_i} = \frac{4\sin\theta_t\cos\theta_i}{\sin 2\theta_t + \sin 2\theta_i}.$$

P.8-43 a) For perpendicular polarization and $\mu_1 \neq \mu_2$:

$$\sin\theta_{B\perp} = \frac{1}{\sqrt{1+(\frac{\mu_1}{\mu_2})}}.$$

Under condition of no reflection:

$$\cos\theta_t = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_{B\perp}} = \frac{1}{\sqrt{1+(\frac{\mu_1}{\mu_2})}}$$

$$= \sin\theta_{B\perp} \longrightarrow \theta_t + \theta_{B\perp} = \pi/2.$$

b) For parallel polarization and $\epsilon_1 \neq \epsilon_2$:

$$\sin\theta_{B\parallel} = \frac{1}{\sqrt{1+(\frac{\epsilon_1}{\epsilon_2})}}.$$

$$\cos\theta_t = \sqrt{1-\frac{n_1^2}{n_2^2}\sin^2\theta_{B\parallel}} = \frac{1}{\sqrt{1+(\frac{\epsilon_2}{\epsilon_1})}}$$

$$= \sin\theta_{B\parallel} \longrightarrow \theta_t + \theta_{B\parallel} = \pi/2.$$

P.8-44 a) $\sin\theta_c = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$; $\sin\theta_{B\parallel} = \frac{1}{\sqrt{1+(\frac{\epsilon_1}{\epsilon_2})}}$

$\longrightarrow \tan\theta_{B\parallel} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}.$

$\therefore \sin\theta_c = \tan\theta_{B\parallel}.$ $(\theta_c > \theta_{B\parallel})$

b) Let $\epsilon_1/\epsilon_2 = x$.

P.8-45 a) For perpendicular polarization:

$$\Gamma_\perp = \frac{\sqrt{\epsilon_{r1}}\cos\theta_i - \sqrt{\epsilon_{r2}}\cos\theta_t}{\sqrt{\epsilon_{r2}}\cos\theta_i + \sqrt{\epsilon_{r1}}\cos\theta_t},$$

$$\sin\theta_t = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}}\sin\theta_i, \quad \cos\theta_t = \sqrt{1-\left(\frac{\epsilon_{r1}}{\epsilon_{r2}}\right)\sin^2\theta_i}.$$

$$\Gamma_\perp = \frac{\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}\cos\theta_i - \sqrt{1-\frac{\epsilon_{r1}}{\epsilon_{r2}}\sin^2\theta_i}}{\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}\cos\theta_i + \sqrt{1-\frac{\epsilon_{r1}}{\epsilon_{r2}}\sin^2\theta_i}}.$$

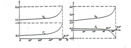
$$\tau_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{2\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}\cos\theta_i}{\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}\cos\theta_i + \sqrt{1-\frac{\epsilon_{r1}}{\epsilon_{r2}}\sin^2\theta_i}}.$$

For parallel polarization:

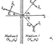
$$\Gamma_{\parallel} = \frac{\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}\sqrt{1-\left(\frac{\epsilon_{r1}}{\epsilon_{r2}}\right)\sin^2\theta_i} - \cos\theta_i}{\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}\sqrt{1-\left(\frac{\epsilon_{r1}}{\epsilon_{r2}}\right)\sin^2\theta_i} + \cos\theta_i},$$

$$\tau_{\parallel} = \frac{2\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}\cos\theta_i}{\sqrt{\frac{\epsilon_{r2}}{\epsilon_{r1}}}\sqrt{1-\left(\frac{\epsilon_{r1}}{\epsilon_{r2}}\right)\sin^2\theta_i} + \cos\theta_i}.$$

b) $\epsilon_{r1}/\epsilon_{r2} = 2.25$, $\sqrt{\epsilon_{r1}/\epsilon_{r2}} = 1.5 \longrightarrow \theta_c = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = 41.8°$.



P.8-46



Given: $\bar{E}_i(x,z) = \bar{a}_y E_{i0} e^{-jk_0(x\sin\theta_i - z\cos\theta_i)}$,

$\bar{a}_{ni} = \bar{a}_x \sin\theta_i - \bar{a}_z \cos\theta_i$,

$\bar{H}_i(x,z) = \frac{1}{\eta_0}\bar{a}_{ni} \times \bar{E}_i(x,z)$

$= \frac{1}{\eta_0}(\bar{a}_x\cos\theta_i + \bar{a}_z\sin\theta_i)e^{-jk_0(x\sin\theta_i - z\cos\theta_i)}$.

$\epsilon_2 = \epsilon' - j\epsilon''$, $k_2 = \omega\sqrt{\mu_0\epsilon_2} = \omega\sqrt{\mu_0\epsilon_0\left(\frac{\epsilon'}{\epsilon_0} - j\frac{\epsilon''}{\epsilon_0}\right)}$

$= k_0\sqrt{\epsilon_r' - j\epsilon_r''}$.

a) From Eq. (8-207):

$$\tau_{\perp} = \frac{2(\eta_2/\eta_0)\cos\theta_i}{(\eta_2/\eta_0)\cos\theta_i + \cos\theta_t},$$

where $(\eta_2/\eta_0) = \sqrt{\epsilon_0/\epsilon_2} = 1/\sqrt{\epsilon_r' - j\epsilon_r''}$.

$\bar{E}_t(x,z) = \bar{a}_y \tau_{\perp} E_{i0} e^{-jk_2(x\sin\theta_t - z\cos\theta_t)}$,

$\bar{H}_t(x,z) = \frac{1}{\eta_2}\bar{a}_{nt} \times \bar{E}_t(x,z) = \frac{1}{\eta_2}(\bar{a}_x\cos\theta_t + \bar{a}_z\sin\theta_t)\tau_{\perp}\bar{E}_t(x,z)$.

b) From Eq. (8-185): $\sin\theta_t = \frac{\sin\theta_i}{\sqrt{\epsilon_r' - j\epsilon_r''}}$ (complex).

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} \quad \text{(complex).}$$

The x- and z-components of $\bar{H}_t(x,z)$ in part a) have different amplitudes and are out of phase, indicating that it is elliptically polarized.

P.8-47

a) $$\Gamma_{\parallel}' = \left.\frac{(E_r)_{tan}}{(E_i)_{tan}}\right|_{z=0} = \frac{E_{r0}\cos\theta_i}{E_{i0}\cos\theta_i} = \frac{E_{r0}}{E_{i0}} = \Gamma_{\parallel} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i},$$

$$\tau_{\parallel}' = \left.\frac{(E_t)_{tan}}{(E_i)_{tan}}\right|_{z=0} = \frac{E_{t0}\cos\theta_t}{E_{i0}\cos\theta_i} = \tau_{\parallel}\left(\frac{\cos\theta_t}{\cos\theta_i}\right) = \frac{2\eta_2\cos\theta_t}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}.$$

b) From part a) we have

$$1 + \Gamma_{\parallel}' = \tau_{\parallel}'$$

This compares with

$$1 - \Gamma_{\parallel} = \tau_{\parallel}\left(\frac{\cos\theta_t}{\cos\theta_i}\right) \quad \text{in Eq. (8-223).}$$

Chapter 9

Theory and Applications of Transmission Lines

P.9-1

$$\bar{\nabla}\times\bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -j\beta \\ H_x & 0 & 0 \end{vmatrix} = \bar{a}_y j\omega\epsilon E_y \longrightarrow \frac{\partial H_x}{\partial y} = 0.$$

$$\bar{\nabla}\times\bar{E} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -j\beta \\ 0 & E_y & 0 \end{vmatrix} = -\bar{a}_x j\omega\mu H_x \longrightarrow \frac{\partial E_y}{\partial x} = 0.$$

P.9-2 a) $\nabla\times(\bar{a}_x E_x + \bar{a}_y E_y) = -j\omega\mu(\bar{a}_x H_x + \bar{a}_y H_y)$.

$$\longrightarrow \begin{cases} \beta E_y = -\omega\mu H_x, & ① \\ \beta E_x = \omega\mu H_y, & ② \\ \frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y}. & ③ \end{cases}$$

$\nabla\times(a_x H_x + a_y H_y) = j\omega\epsilon(a_x E_x + a_y E_y)$.

$$\longrightarrow \begin{cases} \beta H_y = \omega\epsilon E_x, & ④ \\ \beta H_x = -\omega\epsilon E_y, & ⑤ \\ \frac{\partial H_y}{\partial x} = \frac{\partial H_x}{\partial y}. & ⑥ \end{cases}$$

From ② and ④: $\beta = \omega\sqrt{\mu\epsilon}$. ⑦

From ② or ④: $\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\epsilon}} = \eta$. ⑧

From ① or ⑤: $\frac{E_y}{H_x} = -\sqrt{\frac{\mu}{\epsilon}} = -\eta$. ⑨

b) From ③: $\frac{\partial^2 E_x}{\partial y\,\partial x} = \frac{\partial^2 E_y}{\partial x^2}$. ⑩

From ⑦, ①, and ③: $\frac{\partial E_x}{\partial x} = -\frac{\partial E_y}{\partial y} \longrightarrow \frac{\partial^2 E_y}{\partial y^2} = -\frac{\partial^2 E_x}{\partial x\,\partial y}$. ⑪

Combining ⑩ and ⑪, we have $\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} = 0$.

Similarly, $\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} = 0$.

P.9-3 Eq. (9-20): $Z_0 = \frac{d}{w}\sqrt{\frac{\mu}{\epsilon}}$.

a) $Z_0 = \frac{d'}{w}\sqrt{\frac{\mu}{2\epsilon}} = \frac{d}{w}\sqrt{\frac{\mu}{\epsilon}} \longrightarrow d' = \sqrt{2}\,d$.

b) $Z_0 = \frac{d}{w'}\sqrt{\frac{\mu}{2\epsilon}} = \frac{d}{w}\sqrt{\frac{\mu}{\epsilon}} \longrightarrow w' = \frac{1}{\sqrt{2}}w$.

c) $Z_0 = \frac{2d}{w}\sqrt{\frac{\mu}{\epsilon}} = \frac{d}{w}\sqrt{\frac{\mu}{\epsilon}} \longrightarrow w' = 2w.$

d) $u_p = \frac{1}{\sqrt{\mu\epsilon}}$ —— $u_{pa} = u_p/\sqrt{2}$ for case a.
$u_{pb} = u_p/\sqrt{2}$ for case b.
$u_{pc} = u_p$ for case c.

<u>P.9-4</u> Given: $\sigma_c = 1.6\times10^7$ (S/m), $w = 0.02$ (m), $d = 2.5\times10^{-3}$(m)
Lossy dielectric slab: $\mu = \mu_0$, $\epsilon_r = 3$, $\sigma = 10^{-3}$ (S/m).
$f = 5\times10^8$ (Hz).

a) $R = \frac{2}{w}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} = 1.11$ (Ω/m)

$L = \mu\frac{d}{w} = 0.157$ (μH/m)

$G = \sigma\frac{w}{d} = 0.008$ (S/m)

$C = \epsilon\frac{w}{d} = 0.212$ (nF/m).

b) $\frac{|E_z|}{|E_y|} = \sqrt{\frac{\omega\epsilon}{\sigma_c}} = 4.167\times10^{-5}.$

c) $\omega L = 493.5 \gg R$, $\omega C = 0.667 \gg G$

$\gamma = j\omega\sqrt{LC}\left[1+\frac{1}{2j}\left(\frac{R}{\omega L}+\frac{G}{\omega C}\right)\right] = 0.129 + j18.14$ (m^{-1})

$Z_0 = \sqrt{\frac{L}{C}}\left[1+\frac{1}{2j}\left(\frac{R}{\omega L}-\frac{G}{\omega C}\right)\right] = 27.21 + j0.13$ (Ω).

<u>P.9-5</u> Substituting Eqs. (9-38a) and (9-38b)

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

into Eq. (9-35a), we have

$$[\gamma V_0^+ - (R+j\omega L)I_0^+]e^{-\gamma z} - [\gamma V_0^- + (R+j\omega L)I_0^-]e^{\gamma z} = 0,$$

which requires

$$\gamma V_0^+ - (R+j\omega L)I_0^+ = 0$$
$$\gamma V_0^- + (R+j\omega L)I_0^- = 0.$$
$$\therefore \quad \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R+j\omega L}{\gamma}.$$

<u>P.9-6</u> $\epsilon = \epsilon' - j\epsilon'' = \epsilon'\left(1-j\frac{\epsilon''}{\epsilon'}\right) = \epsilon'\left(1-j\frac{\sigma}{\omega\epsilon'}\right).$

From Eq. (9-70), $\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon'}\left(1-j\frac{\epsilon''}{\epsilon'}\right)^{1/2}.$

Squaring both sides, we obtain two equations from the real and imaginary parts:

$$\alpha^2-\beta^2=-\omega^2\mu\epsilon'$$
$$2\alpha\beta=\omega^2\mu\epsilon'\left(\frac{\epsilon''}{\epsilon'}\right),$$

from which Eqs. (9-214) and (9-215) follow.

P.9-7 $\gamma=j\omega\sqrt{LC}\left(1-j\frac{R}{\omega L}\right)^{1/2}\left(1-j\frac{G}{\omega C}\right)^{1/2}$

$$=j\omega\sqrt{LC}\left[1-j\frac{R}{2\omega L}+\frac{1}{8}\left(\frac{R}{\omega L}\right)^2+j\frac{R^3}{16\omega^3L^3}\right]$$
$$\times\left[1-j\frac{G}{2\omega C}+\frac{1}{8}\left(\frac{G}{\omega C}\right)^2+j\frac{G^3}{16\omega^3C^3}\right]=\alpha+j\beta.$$

$$\longrightarrow\ \alpha=\frac{\sqrt{LC}}{2}\left(\frac{R}{L}+\frac{G}{C}\right)\left[1-\frac{1}{8\omega^2}\left(\frac{R}{L}-\frac{G}{C}\right)^2\right].$$
$$\beta=\omega\sqrt{LC}\left[1+\frac{1}{8\omega^2}\left(\frac{R}{L}-\frac{G}{C}\right)^2\right].$$
$$Z_0=\sqrt{\frac{L}{C}}\left(1-j\frac{R}{\omega L}\right)^{1/2}\left(1-j\frac{G}{\omega C}\right)^{-1/2}$$
$$=\sqrt{\frac{L}{C}}\left[1-j\frac{R}{2\omega L}+\frac{1}{8}\left(\frac{R}{\omega L}\right)^2\right]\left[1-j\frac{G}{2\omega C}+\frac{3}{8}\left(\frac{G}{\omega C}\right)^2\right]=R_0+jX_0.$$
$$\longrightarrow\ R_0=\sqrt{\frac{L}{C}}\left[1+\frac{1}{8\omega^2}\left(\frac{R}{L}-\frac{G}{C}\right)\left(\frac{R}{L}+\frac{3G}{C}\right)\right].$$
$$X_0=-\frac{1}{2\omega}\sqrt{\frac{L}{C}}\left(\frac{R}{L}-\frac{G}{C}\right).$$
$$u_p=\frac{\omega}{\beta}\cong\frac{1}{\sqrt{LC}}\left[1-\frac{1}{8\omega^2}\left(\frac{R}{L}-\frac{G}{C}\right)^2\right].$$

P.9-8 $\gamma=\sqrt{(R+j\omega L)(G+j\omega C)}=\sqrt{RG}\left(1+\frac{j\omega L}{R}\right)^{1/2}\left(1+\frac{j\omega C}{G}\right)^{1/2}=\alpha+j\beta.$

$$\longrightarrow\ \alpha\cong\sqrt{RG}\left[1+\frac{1}{8}\left(\frac{\omega L}{R}-\frac{\omega C}{G}\right)^2\right],\quad \beta\cong\frac{\omega}{2}\sqrt{RG}\left(\frac{L}{R}+\frac{C}{G}\right).$$
$$Z_0=\sqrt{\frac{R+j\omega L}{G+j\omega C}}=\sqrt{\frac{R}{G}}\left(1+j\frac{\omega L}{R}\right)^{1/2}\left(1+j\frac{\omega C}{G}\right)^{-1/2}=R_0+jX_0.$$
$$\longrightarrow\ R_0\cong\sqrt{\frac{R}{G}}\left\{1+\frac{\omega^2}{8}\left[\left(\frac{L}{R}\right)^2+\frac{2LC}{RG}-3\left(\frac{C}{G}\right)^2\right]\right\}.$$
$$X_0\cong\frac{\omega}{2}\sqrt{\frac{R}{G}}\left(\frac{L}{R}-\frac{C}{G}\right).$$

P.9-9 $Z_0=\sqrt{\frac{R+j\omega L}{G+j\omega C}}.\quad \mathcal{I}m(Z_0)=0\longrightarrow\ \frac{R}{L}=\frac{G}{C}=k.$

From Eqs. (9-63), (9-64) and (9-66): $\alpha=R\sqrt{\frac{C}{L}},\quad \beta=\omega\sqrt{LC},\quad Z_0=\sqrt{\frac{L}{C}}.$

Given: $Z_0 = 50 + j0$ (Ω),
$\alpha = 0.01$ (dB/m) $= 0.00115$ (Np/m),
$\beta = 0.8\pi$ (rad/m),
$f = 10^8$ (Hz).

$R = \alpha Z_0 = 0.0576$ (Ω/m), $\quad L = \frac{\beta Z_0}{2\pi f} = 0.20$ $(\mu H/m)$,

$G = \frac{\alpha^2}{R} = 23.$ $(\mu S/m)$, $\quad C = \frac{L}{Z_0^2} = 80$ (pF/m).

P.9-10 a) For two-wire transmission line:

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon}}\cosh^{-1}\left(\frac{D}{2a}\right) = \frac{120}{\sqrt{\epsilon_r}}\ln\left[\frac{D}{2a} + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right] = 300\ (\Omega).$$

$$\frac{D}{2a} = 21.27 \longrightarrow D = 25.5 \times 10^{-3}\ (m).$$

b) For coaxial transmission line:

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ln\left(\frac{b}{a}\right) = \frac{40}{\sqrt{\epsilon_r}}\ln\left(\frac{b}{a}\right) = 75.$$

$$\frac{b}{a} = 6.52 \longrightarrow b = 3.91 \times 10^{-3}\ (m).$$

P.9-11 $(P_{av})_L = (P_{av})_i = \frac{1}{2}\mathcal{R}e[V_i I_i^*]$ $\qquad V_i = \frac{Z_i}{Z_g + Z_i}V_g$.

$$= \frac{|V_g|^2 R_i}{(R_g + R_i)^2 + (X_g + X_i)^2}. \qquad I_i = \frac{V_g}{Z_g + Z_i}.$$

To maximize $(P_{av})_L$, set $\frac{\partial (P_{av})_L}{\partial R_i} = 0$. and $\frac{\partial (P_{av})_L}{\partial X_i} = 0$. $\Big\}$ $R_i = R_g$, $X_i = -X_g$ or $Z_i = Z_g^*$.

$$\text{Max. } (P_{av})_L = \frac{|V_g|^2}{4R_g} = (P_{av})_{Z_g}.$$

$\longrightarrow$ Max. power-transfer efficiency = 50%.

P.9-12 $V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$.

$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$.

At $z = 0$: $V(0) = V_i = V_0^+ + V_0^-$; $\quad I(0) = I_i = I_0^+ + I_0^- = \frac{1}{Z_0}(V_0^+ - V_0^-)$.

$$\longrightarrow V_0^+ = \frac{1}{2}(V_i + I_i Z_0), \quad V_0^- = \frac{1}{2}(V_i - I_i Z_0).$$

a) $V(z) = \frac{1}{2}(V_i + I_i Z_0)e^{-\gamma z} + \frac{1}{2}(V_i - I_i Z_0)e^{\gamma z}$.

$I(z) = \frac{1}{2Z_0}(V_i + I_i Z_0)e^{-\gamma z} - \frac{1}{2Z_0}(V_i - I_i Z_0)e^{\gamma z}$.

b) $V(z) = V_i \cosh\gamma z - I_i Z_0 \sinh\gamma z$,

$I(z) = I_i \cosh\gamma z - \frac{V_i}{Z_0}\sinh\gamma z$.

P.9-13 From Fig. 9-44(b): $V_1 = \frac{Z_1}{2} I_1 + \frac{1}{Y_2}(I_1 - I_2)$

$$= \left(\frac{Z_1}{2} + \frac{1}{Y_2}\right) I_1 - \frac{1}{Y_2} I_2. \qquad (1)$$

Also: $I_1 = I_2 + \left(V_2 + \frac{Z_1}{2} I_2\right) Y_2$

$$= Y_2 V_2 + \left(1 + \frac{Z_1 Y_2}{2}\right) I_2. \qquad (2)$$

Substituting (2) in (1):

$$V_1 = \left(1 + \frac{Z_1 Y_2}{2}\right) V_2 + Z_1 \left(1 + \frac{Z_1 Y_2}{4}\right) I_2. \qquad (3)$$

a) Setting $z' = \ell$, $I_L = I_2$, and $I_L Z_L = V_2$ in Eqs. (9-100a) and (9-100b) on p. 451:

$$V_1 = V(\ell) = (\cosh \gamma \ell) V_2 + (Z_0 \sinh \gamma \ell) I_2. \qquad (4)$$

$$I_1 = I(\ell) = \left(\frac{1}{Z_0} \sinh \gamma \ell\right) V_2 + (\cosh \gamma \ell) I_2. \qquad (5)$$

Both Eqs. (3) & (2) and Eqs. (4) & (5) are of the following forms:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}, \qquad (6)$$

where $A = D = 1 + \frac{Z_1 Y_2}{2} = \cosh \gamma \ell$, (7)

$B = Z_1 \left(1 + \frac{Z_1 Y_2}{4}\right) = Z_0 \sinh \gamma \ell$, (8)

and $C = Y_2 = \frac{1}{Z_0} \sinh \gamma \ell$. (9)

$\longrightarrow AD - BC = \cosh^2 \gamma \ell - \sinh^2 \gamma \ell = 1.$

b) (9) is the required result in Eq. (9-220).

Eq. (9-219) can be obtained by using (9) in (7):

$$Z_1 = \frac{1}{Y_2}(2A - 1) = 2 Z_0 \tanh \frac{\gamma \ell}{2}.$$

P.9-14 a) $-\frac{dV}{dz} = RI$, $\quad -\frac{dI}{dz} = GV$.

$$\begin{cases} \frac{d^2 V}{dz^2} = RGV, \\ \frac{d^2 I}{dz^2} = RGI. \end{cases}$$

b) $V(z) = V_0^+ e^{-\alpha z} + V_0^- e^{\alpha z}$

$I(z) = I_0^+ e^{-\alpha z} + I_0^- e^{\alpha z}$, $\quad \alpha = \sqrt{RG}$

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = R_0 = \sqrt{\frac{R}{G}}.$$

We have $V(z) = \frac{1}{2}(V_i + I_i R_0)e^{-\alpha z} + \frac{1}{2}(V_i - I_i R_0)e^{\alpha z}$

$$I(z) = \frac{1}{2}\left(\frac{V_i}{R_0} + I_i\right)e^{-\alpha z} - \frac{1}{2}\left(\frac{V_i}{R_0} - I_i\right)e^{\alpha z},$$

where $V_i = \frac{R_i}{R_g + R_i} V_g$ and $I_i = \frac{V_g}{R_g + R_i}$.

c) For an infinite line, $R_i = R_0$:

$$V(z) = \frac{R_0}{R_g + R_0} V_g e^{-\alpha z}, \qquad I(z) = \frac{V_g}{R_g + R_0} e^{-\alpha z}.$$

d) For a finite line of length ℓ terminated in R_L:

$$R_i = R_0 \frac{R_L + R_0 \tanh \alpha \ell}{R_0 + R_L \tanh \alpha \ell}.$$

P.9-15 Distortionless line: $R_0 = \sqrt{\frac{L}{C}} = 50\ (\Omega)$, $R = 0.5\ (\Omega/m)$

$$\tan\left(\frac{R}{\omega L}\right) = \tan\left(\frac{G}{\omega C}\right) = 0.0018.$$

$$\longrightarrow \frac{G}{\omega C} = 0.0018;\ \frac{G}{C} = 8000\pi \times 0.0018 = 45.2 = \frac{R}{L}.$$

$$L = \frac{R}{G/C} = 0.011\ (H/m), \quad C = \frac{L}{R_0^2} = 4.02\ (\mu F/m).$$

$$\alpha = \frac{R}{R_0} = 0.010\ (Np/m), \quad \beta = \omega\sqrt{LC} = 5.55\ (rad/m).$$

a) $V(z) = \frac{V_{0g} R_0}{R_0 + R_g} e^{-\alpha z} e^{-j\beta z} = \frac{50}{9 + j3} e^{-0.01z} e^{-j5.55z}$; $I(z) = \frac{V(z)}{50}$.

$$\therefore V(z,t) = 5.27\, e^{-0.01z} \sin(8000\pi t - 5.55z - 0.322) \quad (V),$$

$$I(z,t) = 0.105\, e^{-0.01z} \sin(8000\pi t - 5.55z - 0.322) \quad (A).$$

b) At $z = 50$ (m): $V(50,t) = 3.20 \sin(8000\pi t - 0.432\pi)$ (V),

$$I(50,t) = 0.064 \sin(8000\pi t - 0.432\pi) \quad (A).$$

c) $(P_{av})_L = \frac{1}{2}\mathcal{Re}|V_L I_L^*| = \frac{1}{2}(3.20 \times 0.064) = 0.102$ (W).

P.9-16 a) From Eq.(9-117): $Z_{is} = Z_0 \tanh \gamma\ell \cong Z_0(\gamma\ell)$.

From Eqs.(9-37) and (9-41): $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$, $Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$.

$$\therefore Z_{is} = (R + j\omega L)\ell.$$

b) From Eq.(9-116): $Z_{io} = Z_0 \coth \gamma\ell \cong \frac{Z_0}{\gamma\ell} = \frac{G - j\omega C}{[G^2 + \omega^2 C^2]\ell}$.

P.9-17 a) From Eq. (9-117): $Z_{is} = Z_0 \tanh \gamma \ell = Z_0 \dfrac{1 - e^{-2\gamma\ell}}{1 + e^{-2\gamma\ell}}$.

For $\ell = \lambda/4$, $\beta\ell = \pi/2$, $\alpha\lambda/2 \ll 1$.

$$Z_{is} = Z_0 \frac{1 - e^{-2\alpha(\lambda/4)} e^{-j\pi}}{1 + e^{-2\alpha(\lambda/4)} e^{-j\pi}} \cong Z_0 \frac{1 + (1 - \alpha\lambda/2)}{1 - (1 - \alpha\lambda/2)}$$

$$\cong 4Z_0/\alpha\lambda.$$

b) From Eq. (9-116): $Z_{io} = Z_0 \coth \gamma\ell = Z_0 \dfrac{1 + e^{-2\gamma\ell}}{1 - e^{-2\gamma\ell}}$.

For $\ell = \lambda/4$, $Z_{io} = Z_0 \dfrac{1 + e^{-(\alpha\lambda/2)} e^{-j\pi}}{1 - e^{-(\alpha\lambda/2)} e^{-j\pi}} \cong Z_0 \dfrac{1 - (1 - \alpha\lambda/2)}{1 + (1 - \alpha\lambda/2)}$

$$\cong Z_0 \alpha\lambda/4.$$

P.9-18 $\beta\ell = \dfrac{2\pi f}{c}\ell = \dfrac{8\pi}{3} = 480°$.

$\tan\beta\ell = \tan 480° = -1.732$.

$$Z_i = Z_0 \frac{Z_L + jZ_0 \tan\beta\ell}{Z_0 + jZ_L \tan\beta\ell} = 50 \frac{(40 + j30) + j50(-1.732)}{50 + j(40 + j30)(-1.732)}$$

$$= 26.3 - j9.87 \quad (\Omega).$$

P.9-19 Given: $Z_{io} = Z_0 \coth \gamma\ell = 250\angle{-50°} \quad (\Omega)$.

$Z_{is} = Z_0 \tan \gamma\ell = 360\angle{20°} \quad (\Omega)$.

a) $Z_0 = \sqrt{Z_{io} Z_{is}} = 300\angle{-15°} = 289.8 - j77.6 \ (\Omega)$.

$\tanh \gamma\ell = \sqrt{\dfrac{Z_{is}}{Z_{io}}} = 1.2\angle{35°} = 0.983 + j688 = \tanh(\alpha\ell + j\beta\ell)$.

$\ell = 4\ (m) \longrightarrow \alpha = 0.139 \ (Np/m)$,

$\beta = 0.235 \ (rad/m)$.

b) $Z_0 = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}}$, $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$.

$\longrightarrow R + j\omega L = \gamma Z_0$; $G + j\omega C = \dfrac{\gamma}{Z_0}$.

$\omega = \beta c = 0.235 \times 3 \times 10^8 = 0.705 \times 10^8 \ (rad/m)$.

We obtain: $R = 58.6\ (\Omega)$, $L = 0.812 \ (\mu H/m)$,

$G = 0.246 \ (mS/m)$, $C = 12.4 \ (pF/m)$.

P.9-20 a) Since the line is very short compared to a wavelength, we may use Eqs. (9-111) and (9-112).

$$\therefore\ C = \frac{54\times10^{-12}}{0.6} = 9\times10^{-11}\ (F/m), \quad L = \frac{0.3\times10^{-6}}{0.6} = 5\times10^{-7}\ (H/m). \quad \Big\}\ R_0 = \sqrt{\frac{L}{C}} = 74.5\ (\Omega).$$

$$\mu\epsilon = LC \longrightarrow \epsilon_r = \frac{LC}{\mu_0\epsilon_0} = 4.05.$$

b) $\beta = \frac{\omega}{u_p} = 2\pi\times10^{8}\sqrt{LC} = 0.42\ (rad/m);\quad \beta\ell = 0.42\times0.6 = 0.25\ (rad) = 14.4°$

$$\therefore\ X_{io} = -R_0\cot\beta\ell = -\frac{1}{\omega C\ell} = -290\ (\Omega),$$

$$X_{is} = R_0\tan\beta\ell = \omega L\ell = 19.2\ (\Omega).$$

P.9-21 From Eq. (9-116): $Z_{io} = Z_0\coth\gamma\ell = Z_0\frac{\cosh(\alpha+j\beta)\ell}{\sinh(\alpha+j\beta)\ell}$

$$= Z_0\frac{\cosh\alpha\ell\cos\beta\ell + j\sinh\alpha\ell\sin\beta\ell}{\sinh\alpha\ell\cos\beta\ell + j\cosh\alpha\ell\sin\beta\ell}. \quad (1)$$

For a low-loss line, $\alpha\ell \ll 1$, $\cosh\alpha\ell \cong 1$, $\sinh\alpha\ell \cong \alpha\ell$. $\quad Z_{io} \cong Z_0\frac{\cos\beta\ell + j\alpha\ell\sin\beta\ell}{\alpha\ell\cos\beta\ell + j\sin\beta\ell}. \quad (2)$

At $\ell = n\lambda/2$, $\beta_0\ell = n\pi$. $\quad (Z_{io})_{max} \cong \frac{Z_0}{\alpha\ell}$ (antiresonance). $\quad (3)$

When the frequency is slightly off antiresonance:

$f = f_0 + \Delta f$ $(\Delta f \ll f_0)$, $\beta\ell = n\pi(1+\frac{\Delta f}{f_0})$, $\cos\beta\ell \cong \cos n\pi = (-1)^n$

and $\sin\beta\ell = \cos n\pi\,\sin(n\pi\Delta f/f_0) \cong (-1)^n n\pi(\Delta f/f_0)$.

(2) becomes, after leaving out second-order small terms:

$$Z_{io} \cong \frac{Z_0}{\alpha\ell + jn\pi(\Delta f/f_0)}. \quad (4)$$

Combining (3) & (4): $\quad \frac{|Z_{io}|^2}{(Z_{io})^2_{max}} = \frac{1}{1+[n\pi(\Delta f/f_0\alpha\ell)]^2}. \quad (5)$

Half-power points at: $n\pi\left(\frac{\Delta f}{f_0\alpha\ell}\right) = \pm1$; or $\Delta f = \pm\frac{\alpha f_0\ell}{n\pi}$. (6)

For $\ell = n\lambda/2$, $f_0\ell = \frac{n}{2}f_0\lambda = \frac{n}{2\sqrt{LC}}$, and $\Delta f = \pm\frac{\alpha}{2\pi\sqrt{LC}}$,

which, for a low-loss transmission line, becomes (see Eq. 9-55): $\Delta f = \pm\frac{1}{4\pi}\left(\frac{R}{L}+\frac{G}{C}\right) \longrightarrow$ Half-power bandwidth $2\Delta f = \frac{1}{2\pi}\left(\frac{R}{L}+\frac{G}{C}\right)$

$$Q = \frac{f_0}{2\Delta f} = \frac{1}{[(R/\omega_0 L)+(G/\omega_0 C)]}.$$

P.9-22 a) For a lossless quarter-wave line section:

$$Z_i = \frac{R_0^2}{Z_L} = \frac{R_0^2}{R_L + jX_L} = \frac{R_0^2 R_L}{R_L^2 + X_L^2} - j\frac{R_0^2 X_L}{R_L^2 + X_L^2} = R_i' + jX_i'.$$

$$\longrightarrow R_i' = \frac{R_0^2 R_L}{R_L^2 + X_L^2} \quad ① ; \quad X_i' = -\frac{R_0^2 X_L}{R_L^2 + X_L^2}. \quad ②$$

(Resistance R_i' and capacitive reactance X_i' in series.)

Input impedance Z_i can also be expressed in terms of a resistance R_i and a capacitive reactance X_i in parallel:

$$Z_i = \frac{jX_i R_i}{R_i + jX_i} = \frac{R_i X_i^2}{R_i^2 + X_i^2} + j\frac{R_i^2 X_i}{R_i^2 + X_i^2} = R_i' + jX_i'. \quad ③$$

Combining Eqs. ①, ②, and ③, we find

$$R_i = \frac{R_0^2}{R_L} \quad \text{and} \quad X_i = -\frac{R_0^2}{X_L},$$

both of which are reminiscent of Eq. (9-114).

b) From Eq. (9-100a): $V(z') = I_L(Z_L \cos\beta z' + R_0 \sin\beta z')$.

At the input, $z' = \lambda/4$, $\beta z' = \pi/2$, we have

$$V_i = V(\lambda/4) = I_L R_0.$$

At the load, $z' = 0$, $\beta z' = 0$, and $V_L = V(0) = I_L Z_L$.

$$\therefore \frac{|V_i|}{|V_L|} = \frac{R_0}{|Z_L|} = \frac{R_0}{\sqrt{R_L^2 + X_L^2}}.$$

P.9-23 a) $$|\Gamma| = \frac{S-1}{S+1} = \frac{\left|\frac{Z_L}{Z_0} - 1\right|}{\left|\frac{Z_L}{Z_0} + 1\right|} = \frac{\sqrt{(r_L - 1)^2 + x_L^2}}{\sqrt{(r_L + 1)^2 + x_L^2}},$$

where $r_L = R_L/Z_0$ and $x_L = X_L/Z_0$.

$$\longrightarrow x_L = \pm\left[\frac{\left(\frac{S-1}{S+1}\right)^2 (r_L + 1)^2 - (r_L - 1)^2}{1 - \left(\frac{S-1}{S+1}\right)^2}\right]^{1/2}.$$

When $S = 3$, $x_L = \pm\sqrt{(10r_L - 3r_L^2 - 3)/3}$.

b) $S = 3$ and $r_L = 150/75 = 2 \longrightarrow x_L = \pm\sqrt{5/3}$.

$X_L = x_L Z_0 = \pm 96.8\ (\Omega)$.

c) From Eq. (9-147): $r_L + jx_L = \dfrac{r_m + jt}{1 + jr_m t}$,

where $r_m = R_m/Z_0$, and $t = \tan\beta\ell_m$.

$$\longrightarrow r_m = \frac{(1+r_L^2+x_L^2) \pm \sqrt{(1+r_L^2+x_L^2)^2 - 4r_L^2}}{2r_L}$$

$= 3$ or $\frac{1}{3}$, for $r_L = 2$ and $x_L^2 = 5/3$.

Also, $x_L = \dfrac{(1-r_m^2)t}{1+r_m^2 t^2} \longrightarrow t = \dfrac{1}{2x_L r_m^2}\left[(1-r_m^2) \pm \sqrt{(1-r_m^2)^2 - 4x_L^2 r_m^2}\right]$

$r_m = 3$ yields negative t (discard).

For $r_m = \frac{1}{3}$, $t = \begin{cases} 3\sqrt{3/5} \longrightarrow \ell_m = 0.1865\lambda \\ \text{or } \sqrt{15} \longrightarrow \ell_m = 0.2098\lambda \end{cases}$

Use $\ell_m = 0.2095\lambda$ to obtain V_{min} nearest to the load at $(0.5 - 0.2098)\lambda = 0.2902\lambda$.

P.9-24 a) $|\Gamma|^2 = \left|\dfrac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}\right| = \dfrac{(R_L - Z_0)^2 + X_L^2}{(R_L + Z_0)^2 + X_L^2}$

$\dfrac{\partial|\Gamma|^2}{\partial Z_0} = 0 \longrightarrow Z_0 = \sqrt{R_L^2 + X_L^2}$.

If $Z_L = 40 + j30$ (Ω), $Z_0 = 50$ (Ω).

b) Min. $|\Gamma| = \sqrt{\dfrac{Z_0 - R_L}{Z_0 + R_L}} = \sqrt{\dfrac{50-40}{50+40}} = \dfrac{1}{3}$.

Min. $S = \dfrac{1+\frac{1}{3}}{1-\frac{1}{3}} = 2$.

c) From Eq. (9-147): $r_L + jx_L = \dfrac{r_m + jt}{1 + jr_m t} = 0.8 + j0.6$.

$\longrightarrow t = \dfrac{1}{2x_L r_m^2}\left[(1-r_m^2) \pm \sqrt{(1-r_m^2)^2 - 4x_L^2 r_m^2}\right]$ (See problem P.9-23.)

At voltage minimum, $r_m = \frac{1}{S} = \frac{1}{2}$.

$t = 1$ (Use negative sign.)

$\tan\beta\ell_m = \tan(2\pi\ell_m/\lambda) = 1 \longrightarrow \ell_m = \frac{\lambda}{8}$.

∴ Voltage minimum nearest to the load is $(\frac{\lambda}{2} - \frac{\lambda}{8})$ or $3\lambda/8$ from the load.

P.9-25 a) From Eqs. (9-133a) and (9-134):

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left[1 + |\Gamma| e^{-2\alpha z'} e^{j\phi}\right],$$

where $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta_\Gamma}$, $\phi = \theta_\Gamma - 2\beta z'$.

$$Max|V(z')| = \left|\frac{I_L}{2}(Z_L + Z_0)e^{\alpha z'}\right|\left[1 + |\Gamma| e^{-2\alpha z'}\right] \text{ for } \phi = 0,$$

$$min.|V(z')| = \left|\frac{I_L}{2}(Z_L + Z_0)e^{\alpha z'}\right|\left[1 - |\Gamma| e^{-2\alpha z'}\right] \text{ for } \phi = \pi.$$

$$S(z') = \frac{Max.|V(z')|}{min.|V(z')|} = \frac{1 + |\Gamma| e^{-2\alpha z'}}{1 - |\Gamma| e^{-2\alpha z'}}.$$ (Slight ambiguity in z' — insignificant for small α)

b) From Eq. (9-193): $Z_i(z') = \frac{1 + |\Gamma| e^{-2\alpha z'} e^{j\phi}}{1 - |\Gamma| e^{-2\alpha z'} e^{j\phi}} Z_0.$

At a voltage max., $\phi = 0$, $Z_i(z') = S(z') Z_0$.

c) At a voltage min., $\phi = \pi$, $Z_i(z') = \frac{Z_0}{S(z')}$.

P.9-26 From Eq. (9-147): $Z_i = R_0' \frac{Z_L + jR_0' t}{R_0' + jZ_L t} \longrightarrow Z_L = R_0' \frac{Z_i - jR_0' t}{R_0' - jZ_i t}$, $t = \tan\beta\ell'$

$$Z_i = R_0' \frac{Z_L + jR_0' t}{R_0' + jZ_L t} \longrightarrow Z_L = R_0' \frac{Z_i - jR_0' t}{R_0' - jZ_i t}.$$

With $Z_i = 50\ (\Omega)$ and $Z_L = 40 + j10\ (\Omega)$, we have

$$40 + j10 = R_0' \frac{50 - jR_0' t}{R_0' - j50t} \longrightarrow \begin{cases} 40R_0' + 500t = 50R_0', \\ 10R_0' - 2000t = -R_0'^2 t. \end{cases}$$

$\therefore R_0' = 38.7\ (\Omega)$; $t = \tan\beta\ell' = 0.775 \longrightarrow \ell' = 0.105\lambda$.

P.9-27 a) $|\Gamma| = \frac{S-1}{S+1} = \frac{2-1}{2+1} = \frac{1}{3}$.

Eq. (9-133a): $V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{j\beta z'}\left[1 + |\Gamma| e^{j\phi}\right]$.

Eq. (9-134): $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta_\Gamma}$, $\phi = \theta_\Gamma - 2\beta z'$.

Voltage is a minimum when $\phi = \pm\pi \longrightarrow \theta_\Gamma = 2\left(\frac{2\pi}{\lambda}\right) \times 0.3\lambda - \pi = 0.2\pi$.

$\therefore \Gamma = \frac{1}{3} e^{j0.2\pi}$.

b) $Z_L = Z_0\left(\frac{1+\Gamma}{1-\Gamma}\right) = 466 + j206\ (\Omega)$.

c) Terminating resistance $R_m = \frac{R_0}{S} = \frac{300}{2} = 150\ (\Omega)$.

$\ell_m = \frac{\lambda}{2} - z_m' = (0.5 - 0.3)\lambda = 0.2\lambda$.

Another set of solutions is: $R_m' = SR_0 = 600\ (\Omega)$ & $\ell_m' = 0.45\lambda$.

P.9-28 Eq. (9-147): $R_i + jX_i = R_0 \dfrac{R_m + jR_0 \tan\beta l_m}{R_0 + jR_m \tan\beta l_m}$.

Let $r_i = \frac{R_i}{R_0}$, $x_i = \frac{X_i}{R_0}$, $r_m = \frac{R_m}{R_0}$, and $t = \tan\beta l_m$.

$$r_i + jx_i = \frac{r_m + jt}{1 + jr_m t} \longrightarrow \begin{cases} r_m(1 + x_i t) = r_i, \\ t(1 - r_m r_i) = x_i. \end{cases}$$

We have

$$r_m = \frac{1}{2r_i}\left[(1 + r_i^2 + x_i^2) \pm \sqrt{(1 + r_i^2 + x_i^2)^2 - 4r_i^2}\right]$$

$$t = \frac{1}{2x_i}\left\{-\left[1 - (r_i^2 + x_i^2)\right] \pm \sqrt{\left[1 - (r_i^2 + x_i^2)\right]^2 + 4x_i^2}\right\},$$

$$l_m = \frac{\lambda}{2\pi}\tan^{-1} t.$$

P.9-29 $Z_L = Z_0 \dfrac{1 + \Gamma}{1 - \Gamma}$

$\Gamma = |\Gamma| e^{j\theta_\Gamma}$, $|\Gamma| = \dfrac{S - 1}{S + 1}$, $\theta_\Gamma = \frac{4\pi}{\lambda} z_m' \pm \pi$.

$$\therefore Z_L = Z_0 \frac{(S+1) - (S-1)e^{j(4\pi z_m'/\lambda)}}{(S+1) + (S-1)e^{j(4\pi z_m'/\lambda)}}$$

$$= Z_0 \frac{(S+1)e^{-j(2\pi z_m'/\lambda)} - (S-1)e^{j(2\pi z_m'/\lambda)}}{(S+1)e^{-j(2\pi z_m'/\lambda)} + (S-1)e^{j(2\pi z_m'/\lambda)}}$$

$$= Z_0 \frac{1 - jS\tan(2\pi z_m'/\lambda)}{S - j\tan(2\pi z_m'/\lambda)}.$$

P.9-30 a) Given: $V_g = 0.1\angle 0°$ (V), $Z_g = Z_0 = 50$ (Ω), $R_L = 25$ (Ω) $= 0.5 Z_0$.

$$V_i = \frac{Z_i}{Z_g + Z_i} V_g, \qquad I_i = \frac{V_g}{Z_g + Z_i},$$

where $Z_i = Z_0 \dfrac{0.5Z_0 + jZ_0\tan\beta l}{Z_0 + j0.5Z_0\tan\beta l} = Z_0 \dfrac{1 + j2\tan\beta l}{2 + j\tan\beta l}$.

$$\therefore V_i = \frac{1 + j2\tan\beta l}{3(1 + j\tan\beta l)} V_g = \frac{1}{30}\left(\frac{1 + j2\tan\beta l}{1 + j\tan\beta l}\right) \text{ (V)},$$

$$I_i = \frac{2 + j\tan\beta l}{3Z_0(1 + j\tan\beta l)} V_g = \frac{2}{3}\left(\frac{2 + j\tan\beta l}{1 + j\tan\beta l}\right) \text{ (mA)}.$$

Setting $Z_g = Z_0$ and $\Gamma_g = 0$ in Eqs. (9-120a) and (9-120b), we have $V_L = V(z'=0) = \dfrac{V_g Z_0}{Z_0 + Z_g} e^{-j\beta l}(1 + \Gamma)$ $\left\{\Gamma = \frac{R_L - Z_0}{R_L + Z_0} = -\frac{1}{3}\right\}$

$$= \frac{1}{30} e^{-j\beta l} \text{ (V)}.$$

$$I_L = I(z'=0) = \frac{V_g}{Z_0 + Z_g} e^{-j\beta l}(1 - \Gamma) = \frac{4}{3} e^{-j\beta l} \text{ (mA)}.$$

b) $S = \frac{1+|\Gamma|}{1-|\Gamma|} = 2.$

c) $(P_{av})_L = \frac{1}{2}\mathcal{R}_e(V_L I_L^*) = \frac{1}{2}\left(\frac{1}{30}\right)\left(\frac{4}{3}\times 10^{-3}\right) = 2.22\times 10^{-5}$ (W) $= 0.0222$ (mW).

If $R_L = 50\ (\Omega) = Z_0$, $V_L = \frac{V_g}{2}e^{-j\beta\ell}$, $I_L = \frac{V_g}{2Z_0}e^{-j\beta\ell}$

$\longrightarrow \text{Max.}(P_{av})_L = \frac{V_g^2}{8Z_0} = 2.50\times 10^{-5}$ (W).

P.9-31 $V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$, $\quad I(z) = \frac{1}{R_0}(V_0^+ e^{-j\beta z} - V_0^- e^{j\beta z})$.

$V_{inc.}(z=0) = V_0^+ = \frac{V_g}{2}$, $\quad I_{inc.}(z=0) = I_0^+ = \frac{V_g}{2R_0}$.

a) $P_{inc.} = \frac{1}{2}\mathcal{R}_e\left[V_0^+(I_0^+)^*\right] = |V_0^+|^2/2R_0 = V_g^2/8R_0$.

b) $P_L = \frac{1}{2}\mathcal{R}_e\left[V(z)I^*(z)\right] = \frac{1}{2R_0}\mathcal{R}_e\left[(V_0^+e^{-j\beta z}+V_0^-e^{j\beta z})(V_0^{+*}e^{j\beta z}-V_0^{-*}e^{-j\beta z})\right]$

$= \frac{1}{2R_0}\left\{|V_0^+|^2 - |V_0^-|^2\right\} = \frac{|V_0^+|^2}{2R_0}\left\{1-|\Gamma|^2\right\} = \frac{V_g^2}{2R_0}\left\{1-|\Gamma|^2\right\}$.

c) $\frac{P_L}{P_{inc.}} = 1-|\Gamma|^2 = 1-\left(\frac{S-1}{S+1}\right)^2 = \frac{4S}{(S+1)^2}$.

d) $P_{inc.} = \frac{100^2}{8\times 50} = 25$ (W), $\quad \Gamma = \frac{Z_L-R_0}{Z_L+R_0} = \frac{-j25}{100-j25} = 0.243\angle{-76°}$.

$S = \frac{1+|\Gamma|}{1-|\Gamma|} = 1.64$, $\quad P_L = 25(1-0.243^2) = 23.5$ (W).

$P_L = \frac{1}{2}|I_L|^2 R_L \longrightarrow |I_L| = \sqrt{\frac{2P_L}{R_L}} = 0.97$ (A), $\quad |V_L| = |I_L Z_L| = 54.2$ (V).

P.9-32 $\Gamma_g = 0$, $\Gamma = \frac{Z_L-R_0}{Z_L+R_0} = \frac{j-1}{j+1} = j$, $\quad z = \ell - z'$, $\quad \beta = \omega\sqrt{LC}$.

$\ell = \lambda/4$, $e^{-j\beta z} = e^{-j\beta(\lambda/4-z')} = e^{-j(\pi/2-\beta z')} = -j\,e^{j\beta z'}$.

a) From Eq.(9-153a): $V(z') = -j\frac{V_g}{2}e^{j\beta z'}(1+j\,e^{-j2\beta z'}) = 55\,(e^{-j\beta z'} - j\,e^{j\beta z'})$.

From Eq.(9-153b): $I(z') = -j\frac{V_g}{2Z_0}e^{j\beta z'}(1-j\,e^{-j2\beta z'}) = -1.1\,(e^{-j\beta z'} + j\,e^{j\beta z'})$.

b) $v(z',t) = \mathcal{I}_m\left[V(z')e^{j\omega t}\right] = 55\left[\sin(\omega t-\beta z') - \cos(\omega t+\beta z')\right]$ (V),

$i(z',t) = -1.1\left[\sin(\omega t-\beta z') + \cos(\omega t+\beta z')\right]$ (A).

c) At the load, $z'=0$,

$p_L(t) = v(0,t)\,i(0,t)$

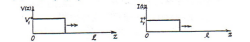
$= 60.5\,(\cos^2\omega t - \sin^2\omega t) = 60.5\cos(2\omega t)$ (W).

$V_L = \frac{V_g}{2}(1-j)$, $\quad I_L = -\frac{V_g}{2Z_0}(1+j)$.

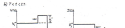
$(P_{av})_L = \frac{1}{2}\mathcal{R}_e(V_L I_L^*) = \frac{V_g^2}{4Z_0}\mathcal{R}_e(2j) = 0$.

P.9-33 $\Gamma_g = -1, \quad \Gamma_L = +1.$

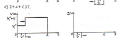
a) $0 < t < T \ (= \ell/u)$.



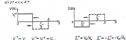
b) $T < t < 2T$.



c) $2T < t < 3T$.



d) $3T < t < 4T$.



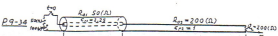
$V_1^+ = V_0, \quad V_1^- = V_1^+ = V_0,$ $\qquad$ $I_1^+ = V_0/R_0, \quad I_1^- = -I_1^+ = -V_0/R_0.$

$V_2^+ = -V_1^- = -V_0,$ $\qquad$ $I_2^+ = I_1^- = -V_0/R_0,$

$V_2^- = V_2^+ = -V_0.$ $\qquad$ $I_2^- = -I_2^+ = V_0/R_0.$

At $t = 4T$, both $V(z)$ and $I(z)$ revert back to the conditions at $t=0$, and the cycle repeats itself with a period $4T$.

P9-34



At the connecting points of two transmission lines with different characteristic impedances Z_{01} and Z_{02}:

$$V_1 = V_2 \longrightarrow V_1^+ + V_1^- = V_2^+$$

$$I_1 = I_2 \longrightarrow I_1^+ + I_1^- = \frac{1}{Z_{01}}(V_1^+ - V_1^-) = \frac{V_2^+}{Z_{02}}$$

Solving $V_1^- = \frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}} V_1^+$, $\quad V_2^+ = \frac{2Z_{02}}{Z_{02}+Z_{01}} V_1^+$;

$$I_1^- = -\frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}} I_1^+, \quad I_2^+ = \frac{2Z_{01}}{Z_{02}+Z_{01}} I_1^+.$$

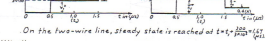
a) $V_1^+ = \frac{V_0}{2} = 50\,(V)$, $\quad I_1^+ = \frac{V_0}{50+50} = 1\,(A)$;

$V_1^- = \frac{200-50}{200+50} V_1^+ = 30\,(V)$, $\quad I_1^- = -\frac{200-50}{200+50} I_1^+ = -0.6\,(A)$;

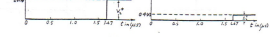
$V_2^+ = \frac{2\times 200}{200+50} V_1^+ = 80\,(V)$, $\quad I_2^+ = \frac{2\times 50}{200+50} I_1^+ = 0.4\,(A)$.

No transient waves on the coaxial cable after V_1^- and I_1^- reach the input terminals (at $2t_1 = 2\ell_1/u_1 = 400/2\times10^8 = 2\,\mu s$), and no transient waves on the two-wire line after V_2^+ and I_2^+ reach the load R_L (at $t_2 = t_1 + \ell_2/u_2 = 1\times10^{-6} + 400/3\times10^8 = 2.33\times10^{-6}\,s = 2.33\,\mu s$).

b) On the coaxial cable it takes $t_1/2 = 0.5\,(\mu s)$ for V_1^+ and I_1^+ to reach the midpoint ($x = 100\,m$). The reflected waves V_1^- and I_1^- arrive at the midpoint at $t = 3t_1/2 = 1.5\,(\mu s)$. There are no changes after that.

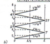


On the two-wire line, steady state is reached at $t = t_1 + \frac{200}{3\times10^8} = 1.67\,(\mu s)$.



P.9-35 $R_g = R_0/2 \longrightarrow \Gamma_g = -\frac{1}{3}$, $R_L = \infty \longrightarrow \Gamma_L = 1$; $T = \ell/u$

a) Voltage reflection diagram



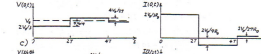
$$V_1^+ = \frac{R_0}{R_0 + \frac{R_0}{2}} V_0 = \frac{2}{3} V_0$$

$$I_1^+ = \frac{V_1^+}{R_0} = \frac{2V_0}{3R_0}$$

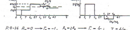
$$t_1 = \frac{\ell}{2u} = \frac{T}{2}$$

Current reflection diagram

b)

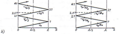


c)

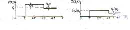


P.9-36 $R_g = 0 \longrightarrow \Gamma_g = -1$, $R_L = 2R_0 \longrightarrow \Gamma_L = \frac{1}{3}$; $T = \ell/u$.

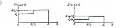
a) Voltage reflection diagram Current reflection diagram



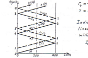
b)



c)



P.9-37 The current reflection diagram for Example 9-11 is:

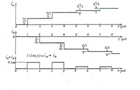


$\Gamma_g = -\frac{1}{3}, \quad \Gamma_L = -1$

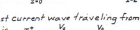
$T = 2\ (\mu s)$.

Indices on the directed lines are normalized with respect to

$$I_1^+ = \frac{V_1^+}{R_0} = \frac{10}{50} = 0.2\ (A).$$



P.9-38 Use the equivalent circuit in Fig. 9-25(d) to study transient voltages and currents:



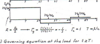
a) Amplitude of first current wave traveling from $z=0$ to $z=\ell$ is $I_1^+ = -\frac{V_0}{2R_0+R_0} = -\frac{V_0}{3R_0}$.

$I_g = -I_1$, $V_R(t) = R\,I_g(t)$.

Refer to Fig. 9-19.

$R = 2R_0 \longrightarrow \Gamma_g = \frac{2R_0 - R_0}{2R_0+R_0} = \frac{1}{3}$, $\Gamma_L = 1$ $T = \ell/c$

b)



$R = \frac{R_0}{2} \longrightarrow \Gamma_g = \frac{(1/2)-1}{(1/2)+1} = -\frac{1}{3}$. $\Gamma_L = 1$ $T = \ell/c$.

P.9-39 a) Governing equation at the load for $t \ge T$:

$$L_L \frac{di_L(t)}{dt} + (R_0 + R_L)\,i_L(t) = 2V_1^+.$$

Solution: $i(t) = \frac{2V_1^+}{R_0+R_L}\left[1 - e^{-\frac{R_0+R_L}{L_L}(t-T)}\right]$, $t \ge T$

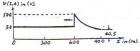
For the present problem, $V_1^+ = V_0/2 = 50\,(V)$, $R_0 + R_L = 60\,(\Omega)$,

$L_L = 48\times10^{-6}\,(H)$; $T = \ell/c = 900/3\times10^8 = 3\times10^{-6}\,(s)$.

$i_L(t) = \frac{5}{3}\left[1 - e^{-1.25\times10^6(t-3\times10^{-6})}\right]$, $t \ge 3\times10^{-6}\,(s)$.

$V_L(t) = R_L i_L + L_L \frac{di_L}{dt} = \frac{50}{3}\left[1 + 5e^{-1.25\times10^6(t-3\times10^{-6})}\right]$, $t \ge 3\times10^{-6}$

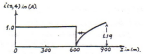
b)



At $t_1 = 4\,(\mu s)$,

$V_L(4) = \frac{50}{3}\left[1 + 5e^{-1.25(4-3)}\right]$

$= 40.5\,(V)$.



At $t_1 = 4\,(\mu s)$,

$$i_L(4) = \frac{5}{3}\left[1 - e^{-1.25(4-3)}\right] = 1.19\ (A).$$

P.9-40 From Eq.(9-171): $v_L(t) = 2V_1^+ - R_0 i_L(t)$. ①

At the load: $i_L(t) = \frac{v_L(t)}{R_L} + C_L \frac{dv_L(t)}{dt}$ ②

Substituting ② in ① $C_L \frac{dv_L(t)}{dt} + \left(\frac{1}{R_L} + \frac{1}{R_0}\right) v_L(t) = \frac{2}{R_0} V_1^+$. ③

a) Solution of ③: $v_L(t) = 2V_1^+ \frac{R_{\parallel}}{R_0}\left[1 - e^{-(t-T)/R_{\parallel}C_L}\right]$, $t \geq T$.

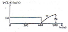
For this problem: $V_1^+ = \frac{V_0}{2} = 50\ (V)$, $R_{\parallel} = \frac{R_L R_0}{R_L + R_0}$.

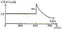
$T = 3\,(\mu s)$, $R_{\parallel} = 47.6\,(\Omega)$, $R_{\parallel}C_L = 6.66 \times 10^{-7}\ (s)$.

$v_L(t) = 95.2\left[1 - e^{-(t-3)/0.666}\right]\ (V)$, t in (μs).

From ②: $i_L(t) = 0.095 + 1.905\, e^{-(t-3)/0.666}\ (A)$.

b) At $t_1 = 4\,(\mu s)$, $v_L(4) = 74\ (V)$, $i_L(4) = 0.519\ (A)$.





P.9-41 a) $Z_L = R_L + jX_L = |Z_L| e^{j\theta_z} \longrightarrow$ Since $R_L \geq 0$, θ_z lies between $\pm\pi/2$.

$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = |Z_0| e^{j\theta_0} \longrightarrow$ Since R, L, G, C all ≥ 0, θ_0 lies between $\pm\frac{\pi}{4}$.

$\therefore\ z_L = \frac{Z_L}{Z_0} = \left|\frac{Z_L}{Z_0}\right| e^{j(\theta_z - \theta_0)} = |z_L| e^{j\theta_L} \longrightarrow \theta_L$ must lie between $\pm 3\pi/4$.

b) $|\Gamma|^2 = \left|\frac{z_L - 1}{z_L + 1}\right|^2 = \frac{(|z_L|e^{j\theta_L} - 1)(|z_L|e^{-j\theta_L} - 1)}{(|z_L|e^{j\theta_L} + 1)(|z_L|e^{-j\theta_L} + 1)} = \frac{|z_L|^2 + 1 - 2|z_L|\cos\theta_L}{|z_L|^2 + 1 + 2|z_L|\cos\theta_L} > 1$ when $\cos\theta_L < 0$.

c) $\frac{\partial|\Gamma|^2}{\partial|z_L|} = 0 \longrightarrow |z_L| = 1$; at $\theta_L = 3\pi/4$,

$\text{Max}|\Gamma| = 1 + \sqrt{2} = 2.414$.

P.9-42 $f = 2\times10^8$ (Hz), $\lambda = \frac{c}{f} = 1.5$ (m)

a) Open-circuited line, $\ell = 1$ (m), $\ell/\lambda = 0.667$.
Smith chart: Start from P_{oc} on the extreme right, rotate clockwise one complete revolution ($\Delta z' = \lambda/2$) and continue on for an additional 0.167λ to 0.417λ on the "wavelength toward generator" scale. Read $x = -j0.575$ ⟶ $Z_i = 75\times(-j0.575) = -j43.1$ (Ω).
Draw a straight line from the $(0-j0.575)$ point through the center and intersect at $(0+j1.74)$ on the opposite side of the chart. ⟶ $Y_i = \frac{1}{75}\times(j1.74) = j0.0232$ (S).

b) Short-circuited line, $\ell = 0.8$ (m), $\ell/\lambda = 0.533$.
Start from the extreme left point P_{sc}, rotate clockwise one complete revolution and continue on for an additional 0.033λ to read $x = j0.21$ ⟶ $Z_i = 75\times j0.21 = j15.8$ (Ω).
Draw a straight line from the $(0+j0.21)$ point through the center and intersect at $(0-j4.75)$ on the opposite side of the chart. ⟶ $Y_i = \frac{1}{75}\times(-j4.75) = -j0.063$ (S).

P.9-43

$z_L = \frac{1}{50}(30+j10) = 0.6+j0.2$.

a) 1. Locate $z_L = 0.6+j0.2$ on Smith chart (Point P_1).
2. With center at O draw a $|\Gamma|$-circle through P_1, intersecting OP_{oc} at 1.77. ⟶ $S = 1.77$.

b) $\Gamma = \frac{1.77-1}{1.77+1}e^{j146°} = 0.28\,e^{j146°}$.

c) 1. Draw line OP_1, intersecting the periphery at P_1'. Read 0.046 on "wavelengths toward generator" scale.
2. Move clockwise by 0.101λ to 0.147 (Point P_2').
3. Join O and P_2', intersecting the $|\Gamma|$-circle at P_2.
4. Read $z_i = 1+j0.59$ at P_2.
$Z_i = 50z_i = 50+j29.5$ (Ω).

d) Extend line $P_2'P_2O$ to P_3. Read $y_i = 0.75 - j0.43$.

$$Y_i = \frac{1}{50} y_i = 0.015 - j0.009 \text{ (S)}.$$

e) There is no voltage minimum on the line, but $V_L < V_i$.

P. 9-44

$$z_L = \frac{1}{50}(30 - j10) = 0.6 - j0.2$$

a) Locate $z_L = 0.6 - j0.2$ on Smith chart (Point P_1). With center at 0 draw a $|\Gamma|$-circle through P_1, intersecting line OP_{sc} at 1.77. ⟶ $S = 1.77$.

b) $\Gamma = 0.28 e^{j216°}$.

c) 1. Draw line OP_1, intersecting the periphery at P_1'. Read 0.454 on "wavelengths toward generator" scale.

2. Move clockwise by 0.101λ to 0.055 (Point P_2').

3. Join O and P_2', intersecting the $|\Gamma|$-circle at P_2.

4. Read $z_i = 0.61 + j0.23$ at P_2.

$$Z_i = 50 z_i = 30.5 + j11.5 \ (\Omega).$$

d) Extend line $P_2'P_2O$ to P_3. Read $y_i = 1.42 - j0.54$.

$$Y_i = \frac{1}{50} y_i = 0.0284 - j0.0108 \text{ (S)}.$$

e) There is a voltage minimum at $z_m' = 0.046\lambda$.

P. 9-45 $\lambda/2 = 25$, $\lambda = 50$ (cm).

First voltage minimum occurs at $z_m' = \frac{5}{50} = 0.1\lambda$.

a) 1. Start from P_{sc} and rotate counterclockwise 0.10λ toward the load to P_1'.

2. Draw the $|\Gamma|$-circle, intersecting line OP_{sc} at 2 $(S = 2)$.

3. Join OP_1', intersecting the $|\Gamma|$-circle at P_1.

4. Read $z_L = 0.675 - j0.475$.

$$Z_L = 50 z_L = 33.75 - j23.75 \ (\Omega).$$

b) $\Gamma = \frac{2-1}{2+1} \, \epsilon^{j\theta_r} = \frac{1}{3} e^{j252.5°}$

c) If $Z_L = 0$, the first voltage minimum would be at $z'_m = \lambda/2 = 25$ (cm) from the short-circuit.

P9-46

a) $z_i = \frac{1}{100}(40 - j280)$
$= 0.40 - j2.80$.

1. Enter z_i on Smith chart (Point P).
2. Join O and P, and extend to P'.
3. Read on "wavelengths toward generator" scale: 0.303.

$$\beta\ell = 0.606\pi, \quad \ell = 1.5 \text{ (m)} \longrightarrow \beta = 1.269 \text{ (rad/m)}.$$

$$\frac{\overline{OP}}{\overline{OP'}} = 0.915 \longrightarrow \alpha = \frac{1}{2\ell} \ln \frac{1}{0.915} = 0.0297 \text{ (Np/m)}.$$

b) 1. Enter $z_L = 0.5 + j0.5$ on Smith chart (Point P_1).

2. Draw line from O through P_1 to P_1'. Read on "wavelengths toward generator" scale: 0.088.

3. Move clockwise by 0.302λ to 0.391 (Point P'').

4. Join OP'', intersecting the $|\Gamma|$-circle through P_1 at P'.

5. Mark point P on line OP' such that $\frac{\overline{OP}}{\overline{OP'}} = 0.915$.

6. Read at P: $z_i = 0.625 - j1.59 \longrightarrow Z_i = 62.5 - j159.0 \ (\Omega)$

c) 1. Move clockwise from P_m on "wavelengths toward generator" scale to 0.15, say P_i'.

2. Join OP_i'.

3. Mark point P_i on line OP_i' such that $\overline{OP_i} = e^{-2\alpha\ell}\,\overline{OP_i'} = 0.967\,\overline{OP_i'}$.

4. Read at P_i: $z_i = 0.065 + j1.38 \longrightarrow Z_i = 6.5 + j138\ (\Omega)$.

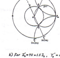
P.9-47 $f = 2\times10^8\ (\mathrm{Hz})$, $\lambda = 1.5\ (\mathrm{m}) \longrightarrow \ell = \frac{\lambda}{4} = 0.375\ (\mathrm{m})$.

$$Z_0 = \sqrt{73\times100} = 142\ (\Omega).$$

For two-wire transmission line: $Z_0 = 120\cosh^{-1}\left(\frac{D}{2a}\right)$.

$D = 2\ (\mathrm{cm}) \longrightarrow a = 0.56\ (\mathrm{cm})$.

P.9-48



$z_L = 0.5 + j0.5$.

$Y_L = 1 - j$.

a) See construction.

P_1 : $z_L = 0.5 + j0.5$.

P_2 : $Y_L = 1 - j1 = Y_1 \longrightarrow d_1 = 0$.

P_3 : $Y_2 = 1 + j1$.

$\longrightarrow d_2 = 0.162\lambda + (0.5 - 0.338)\lambda = 0.324\lambda$.

P_4'' : $b_1 = j1 \longrightarrow \ell_1 = (0.5 + 0.125)\lambda = 0.375\lambda$.

P_3'' : $b_2 = -j1 \longrightarrow \ell_2 = (0.375 - 0.25)\lambda = 0.125\lambda$.

b) For $Z_0' = 75 = 1.5\,Z_0$, $Y_0' = 0.667\,Y_0$.

The required normalized stub admittances are $b_2' = -b_1' = \frac{j}{0.667} = j1.5$.

	$(Z_0)_{stub} = (Z_0)_{line}$	$(Z_0)_{stub} = 1.5(Z_0)_{line}$
$z_L = 0.5 + j0.5$	$d_1 = 0$, $\ell_1 = 0.375\lambda$	$d_1' = 0$, $\ell_1' = 0.406\lambda$
$Y_L = 1 - j1$	$d_2 = 0.324\lambda$, $\ell_2 = 0.125\lambda$	$d_2' = 0.324\lambda$, $\ell_2' = 0.0936\lambda$

P.9-49 $\quad z_L = 0.5 + j0.5$

Use Smith chart as an impedance chart. Same construction as that in problem P. 9-34 except P_{sc} would be on the extreme left (marked by a x), and g=1 circle becomes r=1 circle.

$P_1: \; z_L = 0.5 + j0.5 \,; \; P_2: z_{i1} = 1 + j1$ with $d_1 = (0.162 - 0.088)\lambda = 0.074\lambda$.

$P_3: \; z_{i2} = 1 - j1$ with $d_2 = (0.338 - 0.088)\lambda = 0.250\lambda$.

To achieve a match with a series stub having $R_0' = \frac{35}{50} R_0$, we need a normalized stub susceptance $-j\frac{50}{35} = -j1.43$ for solution corresponding to P_2. From Smith chart we obtain the required stub length $\ell_2 = 0.347\lambda$.

Similarly for solution corresponding to P_3, a stub with a normalized susceptance $+j1.43$ is needed, which requires a stub length $\ell_3 = 0.153\lambda$.

P.9-50

$z_L = 0.33 + j0.33$.

$P_L: \; y_L = 1.50 - j1.50 \; (0.306\lambda \text{ at } P_L')$,

$P_{A1}: \; y_{A1} = 1.50 - j1.80 \; (0.304\lambda \text{ at } P_{A1}')$,

$P_{A2}: \; y_{A2} = 1.50 - j0.14 \; (0.269\lambda \text{ at } P_{A2}')$,

$P_{B1}: \; y_{B1} = 1.00 + j1.60 \; (0.179\lambda \text{ at } P_{B1}')$,

$P_{B2}: \; y_{B2} = 1.00 + j0.40 \; (0.144\lambda \text{ at } P_{B2}')$.

	a) Short-circuited stubs	b) Open-circuited stubs
$(y_{sA})_1 = y_{A1} - y_L = -j0.30$	$\ell_{A1} = 0.203\lambda$	$\ell_{A1} = 0.453\lambda$
$(y_{sA})_2 = y_{A2} - y_L = j1.36$	$\ell_{A2} = 0.399\lambda$	$\ell_{A2} = 0.149\lambda$
$(y_{sB})_1 = -j1.60$	$\ell_{B1} = 0.089\lambda$	$\ell_{B1} = 0.339\lambda$
$(y_{sB})_2 = -j0.40$	$\ell_{B2} = 0.189\lambda$	$\ell_{B2} = 0.439\lambda$

P.9-51

$y_L = \frac{300}{100+j50} = 2.4 - j1.2.$

Point P_L on Smith chart. $(0.280\lambda \text{ at } P_L')$

Since the rotated $g=1.0$ circle is tangent to the $g=2.0$ circle, an added line length d_L is needed to convert g_L (2.4) to 2.0, moving from P_L along the $|\Gamma|$-circle to P_{L1} (not shown) on the $g=2.0$ circle $(0.291\lambda$ at $P_{L1}')$. Note that P_{L1} is different from P_A, the point of tangency between the $g=2.0$ and rotated $g=1.0$ circles.

a) Min. $d_L = 0.291\lambda - 0.280\lambda = 0.011\lambda.$

b) P_A : $y_A = 2 - j1$ $(0.287\lambda$ at $P_A')$.

P_B : $y_B = 1 + j1$ $(0.162\lambda$ at $P_B')$.

$y_{sA} = y_A - y_{L1} = (2-j1) - (2-j1.35) = j0.35 \longrightarrow \ell_A = 0.304\lambda.$

$y_{sB} = -j1 \longrightarrow \ell_B = 0.125\lambda.$

P.9-52 Let $\theta = \beta d_0 = \frac{2\pi}{\lambda} d_0$.

Require: $g_L \le \frac{1}{\sin^2\theta}$ (Analytical solution†)

d_0	θ	No solution if g_L is
$\lambda/16$	22.5°	>6.83
$\lambda/8$	45°	>2.0
$\lambda/4$	90°	>1.0
$3\lambda/8$	135°	>2.0
$7\lambda/16$	157.5°	>6.83

† See D.K. Cheng and C.H. Liang, "Computer Solution of Double-Stub Impedance-Matching Problems," IEEE Transactions on Education, vol. E-25, pp. 120-123, November 1982.

Chapter 10

Waveguides and Cavity Resonators

P.10-1

$\nabla \times \vec{E} = -j\omega\mu\vec{H}$. ①

$\nabla \times \vec{H} = j\omega\epsilon\vec{E}$. ②

From ①, $(\bar{\nabla}_T + \vec{a}_z \frac{\partial}{\partial z}) \times (\vec{E}_T + \vec{a}_z E_z) = -j\omega\mu(\vec{H}_T + \vec{a}_z H_z)$.

$\longrightarrow \bar{\nabla}_T \times (\vec{a}_z E_z) + \gamma \vec{E}_T \times \vec{a}_z = -j\omega\mu\vec{H}_T$

$\longrightarrow \bar{\nabla}_T E_z \times \vec{a}_z + \gamma \vec{E}_T \times \vec{a}_z = -j\omega\mu\vec{H}_T$. ③

$(\because \bar{\nabla}_T \times (\vec{a}_z E_z) = \bar{\nabla}_T E_z \times \vec{a}_z)$

Similarly from ② we obtain

$\bar{\nabla}_T H_z \times \vec{a}_z + \gamma \vec{H}_T \times \vec{a}_z = j\omega\epsilon\vec{E}_T$. ④

Combining ③ and ④, we have

$\omega^2\mu\epsilon\vec{E}_T = \gamma[\bar{\nabla}_T E_z \times \vec{a}_z + \gamma\vec{E}_T \times \vec{a}_z] \times \vec{a}_z - j\omega\mu\bar{\nabla}_T H_z \times \vec{a}_z$.

$\longrightarrow \vec{E}_T = -\frac{1}{h^2}(\gamma\bar{\nabla}_T E_z - \vec{a}_z j\omega\mu \times \bar{\nabla}_T H_z)$, $h^2 = \gamma^2 + \omega^2\mu\epsilon$.

Similarly, $\vec{H}_T = -\frac{1}{h^2}(\gamma\bar{\nabla}_T H_z + \vec{a}_z j\omega\epsilon \times \bar{\nabla}_T E_z)$.

P.10-2 a)

From Eq. (10-38):

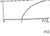
$$\left(\frac{\beta}{k}\right)^2 + \left(\frac{f_c}{f}\right)^2 = 1.$$

From Eq. (10-37):

$$\left(\frac{u_g}{u}\right)^2 + \left(\frac{f_c}{f}\right)^2 = 1.$$

Both are equations of a unit circle.

b)

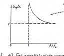


From Eq. (10-38):

$$\left(\frac{\beta}{k}\right)^2 = 1 - \frac{1}{(f/f_c)^2}.$$

From Eq. (10-42):

$$\left(\frac{u}{u_p}\right)^2 = 1 - \frac{1}{(f/f_c)^2}.$$



From Eq. (10-39):

$$\left(\frac{\lambda_g}{\lambda}\right)^2 = \frac{(f/f_c)^2}{(f/f_c)^2 - 1}.$$

c) At $f/f_c = 1.25$:

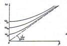
$u_p/u = 1.67.$

$u_g/u = 0.60,$

$\beta/k = 0.60,$

$\lambda_g/\lambda = 1.67.$

P.10-3 a) For parallel-plate waveguides:



$$\omega^2\mu\epsilon - \beta^2 = \left(\frac{n\pi}{b}\right)^2.$$

$$\omega_{c1} = \frac{\pi}{b\sqrt{\mu\epsilon}},$$

$$\omega_{c2} = \frac{2\pi}{b\sqrt{\mu\epsilon}},$$

$$\omega_{c3} = \frac{3\pi}{b\sqrt{\mu\epsilon}}.$$

a) Constitutive parameters ϵ and μ affect both ω_c and the slope of the ω-β curves; b affects ω_c but not the slope at high frequencies. b) Yes.

P.10-4 Field expressions for TM_n modes, from Eqs. (10-63, 64 & 65):

$$E_z^0(y) = A_n \sin(n\pi y/b),$$

$$H_x^0(y) = \frac{j\omega\epsilon}{h} A_n \cos(n\pi y/b),$$

$$E_y^0(y) = -\frac{\gamma}{h} A_n \cos(n\pi y/b).$$

Surface charge densities:

$$\rho_{sl} = \bar{a}_n \cdot \bar{D}\Big|_{y=0} = \epsilon E_y^0(0) = -\frac{\gamma\epsilon}{h} A_n,$$

$$\rho_{su} = \bar{a}_n \cdot \bar{D}\Big|_{y=b} = -\epsilon E^0(b) = (-1)^n \frac{\gamma\epsilon}{h} A_n.$$

Surface current densities:

$$\bar{J}_{sl} = \bar{a}_n \times \bar{H}\Big|_{y=0} = \bar{a}_y \times \bar{H}(0) = -\bar{a}_z \frac{j\omega\epsilon}{h} A_n,$$

$$\bar{J}_{su} = \bar{a}_n \times \bar{H}\Big|_{y=b} = -\bar{a}_y \times \bar{H}(b) = \bar{a}_z (-1)^n \frac{j\omega\epsilon}{h} A_n = \begin{cases} \bar{J}_{sl} & \text{for } n \text{ odd,} \\ -\bar{J}_{sl} & \text{for } n \text{ even.} \end{cases}$$

P.10-5 Field expressions for TE_n modes, from Eqs.(10-83, 84 & 85):

$$H_z^\circ(y) = B_n \cos(n\pi y/b),$$
$$H_y^\circ(y) = \frac{\gamma}{h} B_n \sin(n\pi y/b),$$
$$E_x^\circ(y) = \frac{j\omega\mu}{h} B_n \sin(n\pi y/b).$$
$$\bar{J}_{sl} = \bar{a}_y \times \bar{H}(0) = \bar{a}_x B_n .$$
$$\bar{J}_{su} = -\bar{a}_y \times \bar{H}(b) = \bar{a}_x (-1)^{n+1} B_n = \begin{cases} \bar{J}_{sl} \text{ for } n \text{ odd}, \\ -\bar{J}_{sl} \text{ for } n \text{ even}. \end{cases}$$

P.10-6 a) Set $n=2$ in the field expressions in problem P.10-4.

——— Electric field lines
× · Magnectic field lines
Equation for electric field lines: $\cos\beta z = \frac{\cos(2\pi y_0/b)}{\cos(2\pi y/b)}$.

b) Set $n=2$ in the field expressions in problem P.10-5.

× · Electric field lines
——— Magnetic field lines
Equation for magnetic field lines: $\cos\beta z = \frac{\sin(2\pi y_0/b)}{\sin(2\pi y/b)}$.

P.10-7 Using the field expressions in problem P.10-5 above:

$$\bar{\mathcal{P}}_{av} = \frac{1}{2}\mathcal{R}e(\bar{E}\times\bar{H}^*) = \frac{1}{2}\mathcal{R}e(\bar{a}_z E_x^\circ H_y^{\circ *} - \bar{a}_y E_x^\circ H_z^{\circ *}).$$
$$\bar{\mathcal{P}}_{av}\cdot\bar{a}_z = \frac{1}{2}\mathcal{R}e(E_x^\circ H_y^{\circ *}) = \frac{\omega\mu\beta}{2h^2} B_n^2 \sin^2\left(\frac{n\pi y}{b}\right).$$
$$(P_z)_{av} = \int_0^b \bar{\mathcal{P}}_{av}\cdot\bar{a}_z\, dy = \frac{\omega\mu\beta b}{4h^2} B_n^2 \quad \text{(per unit guide width)}.$$
$$(w_e)_{av} = \frac{\epsilon}{4}\mathcal{R}e(\bar{E}^\circ\cdot\bar{E}^{\circ *}) = \frac{\epsilon\,\omega^2\mu^2}{4h^2} B_n^2 \sin^2\left(\frac{n\pi y}{b}\right).$$
$$(W_e)'_{av} = \int_0^b (w_e)_{av}\, dy = \frac{\epsilon\,\omega^2\mu^2 b}{8h^2} B_n^2 = (W_m)'_{av} \quad \text{(per unit guide width)}$$

From Eq.(10-92): $u_{en} = \frac{(P_z)_{av}}{(W_e)'_{av}+(W_m)'_{av}} = \frac{\omega\beta}{k^2} = u\sqrt{1-\left(\frac{f_c}{f}\right)^2}$,

which is the same as Eq.(10-104).

P.10-8 Given: $\sigma_c = 5.80\times10^{7}$ (S/m), $\epsilon_r = 2.25$, $\mu_r = 1$,
$\sigma = 10^{-10}$ (S/m), $b = 5\times10^{-2}$ (m), $f = 10^{10}$ (Hz).

a) TEM mode

$\beta = \omega\sqrt{\mu\epsilon} = 314.2$ (rad/m),

$\alpha_d = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} = 1.257\times10^{-8}$ (Np/m),

$\alpha_c = \frac{1}{b}\sqrt{\frac{\pi f\epsilon}{\sigma_c}} = 2.076\times10^{-3}$ (Np/m),

$u_p = u_g = u = \frac{1}{\sqrt{\mu\epsilon}} = 2\times10^{8}$ (m/s),

$\lambda_g = \lambda = \frac{u}{f} = 2\times10^{-2}$ (m).

b) TM$_1$ mode — $(f_c)_{TM_1} = \frac{1}{2b\sqrt{\mu\epsilon}} = 2\times10^{9}$ (Hz) $< f$.

$$F_1 = \sqrt{1-(f_c/f)^2} = 0.9798.$$

$\beta = \omega\sqrt{\mu\epsilon}\cdot F_1 = 307.8$ (rad/m),

$\alpha_d = \frac{\sigma\eta}{2F_1} = 1.283\times10^{-8}$ (Np/m),

$\alpha_c = \frac{2R_s}{\eta b F_1} = \frac{2}{bF_1}\sqrt{\frac{\pi f\epsilon}{\sigma_c}} = 4.238\times10^{-3}$ (Np/m),

$u_p = u/F_1 = 2.041\times10^{8}$ (m/s),

$u_g = u\cdot F_1 = 1.960\times10^{8}$ (m/s),

$\lambda_g = \lambda/F_1 = 2.041\times10^{-2}$ (m).

c) TM$_2$ mode — $(f_c)_{TM_2} = \frac{1}{b\sqrt{\mu\epsilon}} = 4\times10^{9}$ (Hz) $< f$.

$$F_2 = \sqrt{1-(f_c/f)^2} = 0.9165.$$

$\beta = \omega\sqrt{\mu\epsilon}\cdot F_2 = 287.9$ (rad/m),

$\alpha_d = \frac{\sigma\eta}{2F_2} = 1.371\times10^{-8}$ (Np/m),

$\alpha_c = \frac{2}{bF_2}\sqrt{\frac{\pi f\epsilon}{\sigma_c}} = 4.530\times10^{-3}$ (Np/m),

$u_p = u/F_2 = 2.182\times10^{8}$ (m/s),

$u_g = u\cdot F_2 = 1.833\times10^{8}$ (m/s),

$\lambda_g = \lambda/F_2 = 2.182\times10^{-2}$ (m).

P.10-9 a) TE$_1$ mode — $(f_c)_{TE_1} = (f_c)_{TM_1} = 2\times10^{9}$ (Hz) $< f$.

All required quantities are the same as those for the TM$_1$ mode in problem P.10-8 (b), except α_c. Using Eq. (10-83), we have

$$\alpha_c = \frac{2}{bF_1}\sqrt{\frac{\pi f\epsilon}{\sigma_c}}\left(\frac{f_c}{f}\right)^2 = 1.695\times10^{-4} \text{ (Np/m)}.$$

b) <u>TE_1 mode</u> —— $(f_c)_{TE_1} = (f_c)_{TM_1} = 4\times10^{9}$ (Hz) $< f$.

All required quantities are the same as those for the TM_1 mode in problem P.10-8 (c), except α_c.

$$\alpha_c = \frac{2}{b F_1}\sqrt{\frac{\pi f \epsilon}{\sigma_c}}\left(\frac{f_c}{f}\right)^2 = 7.249\times10^{-4}\ (\text{Np/m}).$$

P.10-10 For TM_n modes in a parallel-plate waveguide,

$$\alpha_c = \frac{2}{\eta b}\sqrt{\frac{\pi\mu_c f_c}{\sigma_c}}\ \frac{1}{\sqrt{(f_c/f)[1-(f_c/f)^2]}}$$

$$= \frac{2}{\eta b}\sqrt{\frac{\pi\mu_c f_c}{\sigma_c}}\ \frac{1}{\sqrt{F(x)}},$$

where $F(x) = x - x^3$, $x = f_c/f$.

a) To find minimum α_c, set

$$\frac{\partial F(x)}{\partial x} = 1 - 3x^2 = 0 \longrightarrow x = \frac{1}{\sqrt{3}}.$$

$$\therefore\ f = \sqrt{3}\, f_c.$$

b) At $f_c/f = 1/\sqrt{3}$, $\frac{1}{\sqrt{F(x)}} = 1.612$,

and
$$\text{min. } \alpha_c = \frac{3.224}{\eta b}\sqrt{\frac{\pi\mu_c f_c}{\sigma_c}}.$$

c) For $\sigma_c = 5.80\times10^{7}$ (S/m), $b = 5\times10^{-2}$ (m), $\eta = 120\pi$ (Ω), and $\mu_c = 4\pi\times10^{-7}$ (H/m),

$$(f_c)_{TM_1} = \frac{1}{2b\sqrt{\mu_0\epsilon_0}} = 3\times10^{9}\ (\text{Hz}),$$

$$\text{min. } \alpha_c = 2.444\times10^{-3}\ (\text{Np/m}).$$

P.10-11 Parallel-plate waveguide: $b = 0.03$ (m), $f = 10^{10}$ (Hz).

a) <u>TEM mode</u>

From Eqs. (9-1a) and (9-1b):

$$\begin{cases} E_y^0 = E_0, \\ H_x^0 = -\frac{E_0}{\eta_0}. \end{cases}$$

$$P_{av} = \frac{w}{2}\int_0^b -E_y^0 H_x^0\, dy = \frac{wb}{2\eta_0}E_0^2.$$

Dielectric strength of air: Max. $E_0 = 3\times10^{6}$ (V/m).

$$\text{Max.}\left(\frac{P_{av}}{w}\right) = \frac{b}{2\eta_0}(3\times10^{6})^2 = 358\times10^{6}\ (\text{W/m}) = 358\ (\text{MW/m}).$$

b) TM$_1$ mode

From Eqs. (10-64) and (10-65):

$$\begin{cases} E_y^0(y) = E_0 \cos\left(\frac{\pi y}{b}\right). \\ H_x^0(y) = -\frac{E_0}{\eta_0\sqrt{1-(f_c/f)^2}} \cos\left(\frac{\pi y}{b}\right). \end{cases}$$

$$f_c = \frac{1}{2b\sqrt{\mu_0\epsilon_0}} = 5\times 10^9 \text{ (Hz)}.$$

$$P_{av} = \frac{w}{2}\int_0^b - E_y^0(y) H_x^0(y)\, dy = \frac{wb E_0^2}{4\eta_0\sqrt{1-(f_c/f)^2}}.$$

$$\text{Max.}\left(\frac{P_{av}}{w}\right) = \frac{b(3\times10^4)^2}{4\eta_0\sqrt{1-(f_c/f)^2}} = 2.07\times10^8 \text{(W/m)} = 207 \text{(MW/m)}$$

c) TE$_1$ mode

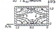
From Eqs. (10-84) and (10-85):

$$\begin{cases} E_x^0(y) = E_0 \sin\left(\frac{\pi y}{b}\right). \\ H_y^0(y) = \frac{E_0}{\eta_0}\sqrt{1-\left(\frac{f_c}{f}\right)^2} \sin\left(\frac{\pi y}{b}\right). \end{cases}$$

$$P_{av} = \frac{w}{2}\int_0^b E_x^0(y) H_y^0(y)\, dy = \frac{wb E_0^2}{4\eta_0}\sqrt{1-(f_c/f)^2}.$$

$$\text{Max.}\left(\frac{P_{av}}{w}\right) = \frac{b(3\times10^4)^2}{4\eta_0}\sqrt{1-(f_c/f)^2} = 1.55\times10^8 \text{(W/m)} = 155 \text{(MW/m)}.$$

P.10-12 a) TM$_{11}$ mode b) TE$_{11}$ mode



—— Electric field lines
- - - - Magnetic field lines

P.10-13 Equations (10-134) through (10-137) for TM$_{11}$ mode:

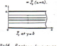
$$E_x^0(x,y) = \frac{-j\beta_{11}}{h^2}\left(\frac{\pi}{a}\right) E_0 \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right).$$

$$E_y^0(x,y) = \frac{-j\beta_{11}}{h^2}\left(\frac{\pi}{b}\right) E_0 \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right).$$

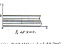
$$E_z^0(x,y) = E_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right).$$
$$H_x^0(x,y) = \frac{j\omega\epsilon}{h^2}\left(\frac{\pi}{b}\right)E_0 \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right).$$
$$H_y^0(x,y) = \frac{-j\omega\epsilon}{h^2}\left(\frac{\pi}{a}\right)E_0 \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right).$$

a) Surface current densities:

$$\bar{J}_s(y=0) = \bar{a}_n \times \bar{H}\big|_{y=0} = \bar{a}_y \times \left[\bar{a}_x H_x^0(x,0) + \bar{a}_y H_y^0(x,0)\right]$$
$$= -\bar{a}_z H_x^0(x,0) = -\bar{a}_z \frac{j\omega\epsilon}{h^2}\left(\frac{\pi}{b}\right)E_0 \sin\left(\frac{\pi x}{a}\right)e^{-j\beta_{11}z}$$
$$= \bar{J}_s(y=b).$$
$$\bar{J}_s(x=0) = \bar{a}_n \times \bar{H}\big|_{x=0} = \bar{a}_x \times \left[\bar{a}_x H_x^0(0,y) + \bar{a}_y H_y^0(0,y)\right]$$
$$= \bar{a}_z H_y^0(0,y) = -\bar{a}_z \frac{j\omega\epsilon}{h^2}\left(\frac{\pi}{a}\right)E_0 \sin\left(\frac{\pi y}{b}\right)e^{-j\beta_{11}z}$$
$$= \bar{J}_s(x=a).$$



$\bar{J}_s$ at $y=b$



$\bar{J}_s$ at $x=0$.

P.10-14 Rectangular waveguide: $a = 7.21$ (cm), $b = 3.40$ (cm).

Eq. (10-140): $$(\lambda_c)_{mn} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}.$$

Modes with the shortest $\lambda_c < 5$ (cm) are:

Mode	TE_{10}	TE_{20}	TE_{01}	TE_{11}/TM_{11}
λ (cm)	14.4	7.20	6.80	6.15

a) For $\lambda = 10$ (cm), the only propagating mode is TE_{10}.

b) For $\lambda = 5$ (cm), the propagating modes are: TE_{10}, TE_{20}, TE_{01}, TE_{11}, and TM_{11}.

P.10-15 $u_{en} = u\sqrt{1-\left(\frac{f_c}{f}\right)^2}$.

For the TE_{10} mode, $f_{10} = \frac{u}{2a}$.

$\therefore\ u_{en} = u\sqrt{1-\left(\frac{u}{2af}\right)^2} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1-\frac{1}{\mu\epsilon(2af)^2}}$.

P.10-16 $(f_c)_{mn} = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2} = \frac{1}{2a\sqrt{\mu\epsilon}}F(m,n)$.

a) $a=2b$, $F(m,n)=\sqrt{m^2+4n^2}$

Modes	$F(m,n)$
TE_{10}	1
TE_{01}, TE_{20}	2
TE_{11}, TM_{11}	$\sqrt{5}$
TE_{21}	4
TM_{12}	$\sqrt{17}$
TM_{21}	$\sqrt{20}$

b) $a=b$, $F(m,n)=\sqrt{m^2+n^2}$.

Modes	$F(m,n)$
TE_{10}, TE_{01}	1
TE_{11}, TM_{11}	$\sqrt{2}$
TE_{02}, TE_{20}	2
TM_{12}	$\sqrt{5}$
TM_{22}	$2\sqrt{2}$

P.10-17 $f = 3\times10^9$ (Hz), $\lambda = c/f = 0.1$ (m).

Let $a = kb$, $1<k<2$. $(f_c)_{mn} = \frac{3\times10^8}{2a}\sqrt{m^2+k^2n^2}$.

a) $(f_c)_{10} = \frac{1.5\times10^8}{a}$ for the dominant TE_{10} mode.

For $f > 1.2\,(f_c)_{10}$: $a > 0.06$ (m).

The next higher-order mode is TE_{01} with $(f_c)_{01} = \frac{1.5\times10^8}{b}$.

For $f < 0.8\,(f_c)_{01}$: $b < 0.04$ (m).

We choose $a = 6.5$ (cm) and $b = 3.5$ (cm).

b) $u_p = \frac{c}{\sqrt{1-(\lambda/2a)^2}} = 4.70\times10^8$ (m/s),

$\lambda_g = \frac{\lambda}{\sqrt{1-(\lambda/2a)^2}} = 0.157$ (m) $= 15.7$ (cm),

$\beta = \frac{2\pi}{\lambda_g} = 40.1$ (rad/m),

$(Z_{TE})_{10} = \frac{\eta_0}{\sqrt{1-(\lambda/2a)^2}} = 590\ (\Omega)$.

P.10-18 Given: $a = 2.5\times10^{-2}$ (m), $b = 1.5\times10^{-2}$ (m), $f = 7.5\times10^{9}$ (Hz).

a) $\lambda = \frac{c}{f} = \frac{3\times10^{8}}{7.5\times10^{9}} = 0.04$ (m).

$F_1 = \sqrt{1-(\lambda/2a)^2} = 0.60.$

$\lambda_g = \lambda/F_1 = 0.0667$ (m) $= 6.67$ (cm),

$\beta = 2\pi/\lambda_g = 94.2$ (rad/m),

$u_p = c/F_1 = 5\times10^{8}$ (m/s),

$u_g = c\cdot F_1 = 1.8\times10^{8}$ (m/s),

$(Z_{TE})_{10} = \eta_0/F_1 = 200\pi = 628$ (Ω).

b) $\lambda' = \frac{u}{f} = \frac{\lambda}{\sqrt{2}} = 0.0283$ (m),

$F_2 = \sqrt{1-(\lambda'/2a)^2} = 0.825,$

$\lambda_g' = \lambda'/F_2 = 0.0343$ (m) $= 3.43$ (cm),

$\beta' = 2\pi/\lambda_g' = 183.2$ (rad/m),

$u_p' = u/F_2 = 2.57\times10^{8}$ (m/s),

$u_g' = u\cdot F_2 = 1.75\times10^{8}$ (m/s),

$(Z_{TE})_{10} = \frac{\eta_0}{\sqrt{2}\,F_2} = 323$ (Ω).

P.10-19 Given: $a = 7.20\times10^{-2}$ (m), $b = 3.40\times10^{-2}$ (m), $f = 3\times10^{9}$ (Hz).

a) $\lambda_c = 2a = 14.40\times10^{-2}$ (m).

$f_c = \frac{c}{\lambda_c} = 2.08\times10^{9}$ (Hz).

b) $\lambda = \frac{c}{f} = 0.1$ (m), $\sqrt{1-\left(\frac{\lambda}{2a}\right)^2} = 0.720,$

$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}} = 0.139$ (m).

c) $R_s = \sqrt{\frac{\pi f \mu_c}{\sigma}} = 1.429\times10^{-2}$ (Ω) $\longrightarrow (\alpha_c)_{TE_{10}} = \frac{R_s\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right]}{\eta_0 b\sqrt{1-(f_c/f)^2}} = 2.26\times10^{-3}$ (Np/m).

d) $e^{-\alpha_c z} = \frac{1}{2} \longrightarrow z = \frac{1}{\alpha_c}\ln 2 = 307$ (m).

P.10-20 Given: $a = 2.25\times10^{-2}$ (m), $b = 1.00\times10^{-2}$ (m), $f = 10^{10}$ (Hz).

a) $\lambda = \frac{c}{f} = 3\times10^{-2}$ (m), $\lambda_c = 2a = 4.50\times10^{-2}$ (m).

$\sqrt{1-(f_c/f)^2} = \sqrt{1-(\lambda/\lambda_c)^2} = 0.745.$

Eq. (10-187): $(\alpha_c)_{TE_{10}} = \frac{1}{\eta_0 b}\sqrt{\frac{\pi f \mu_c}{\sigma_c[1-(f_c/f)^2]}}\left[1+\frac{2b}{a}\left(\frac{f_c}{f}\right)^2\right]$

$= 1.295\times10^{-2}$ (Np/m).

b) From Eqs. (10-159), (10-160), and (10-158):

$$E_y^0 = E_0 \sin\left(\frac{\pi x}{a}\right).$$

$$H_x^0 = -\frac{E_0}{\eta_0}\sqrt{1-\left(\frac{f_c}{f}\right)^2}\,\sin\left(\frac{\pi x}{a}\right).$$

$$H_z^0 = j\left(\frac{f_c}{f}\right)\frac{E_0}{\eta_0}\cos\left(\frac{\pi x}{a}\right).$$

$$P_{av} = \frac{1}{2}\int_0^b\int_0^a (-E_y^0 H_x^0)\,dx\,dy = \frac{E_0^2 ab}{4\eta_0}\sqrt{1-\left(\frac{f_c}{f}\right)^2}.$$

For $P_{av} = 10^3$ (W) at the load (antenna), assuming under matched conditions:

$|E_y^0| = E_0 = 94{,}800$ (V/m), $|H_x^0| = 187.4$ (A/m), $|H_z^0| = 167.6$ (A/m).

The waveguide is 1 (m) long. —— The field intensities are higher at the sending end by a factor of $e^{\alpha_c \cdot 1} = 1.138$.

$\therefore$ Max. $|E_y^0| = 10{,}788$ (V/m).

Max. $|H_x^0| = 213.3$ (A/m).

Max. $|H_z^0| = 190.7$ (A/m).

c) $\bar{J}(x=0) = \bar{a}_x \times (\bar{a}_x H_x^0 + \bar{a}_z H_z^0)\Big|_{x=0} = -\bar{a}_y H_z^0(0,y) = -\bar{a}_y\, j\left(\frac{f_c}{f}\right)\frac{E^0}{\eta_0}$

$|\bar{J}(x=0)| = |H_z^0| = 167.6$ (A/m).

$\bar{J}(y=0) = \bar{a}_y \times (\bar{a}_x H_x^0 + \bar{a}_z H_z^0)\Big|_{y=0} = -\bar{a}_z H_x^0(x,0) + \bar{a}_x H_z^0(x,0).$

$|\bar{J}(y=0)| = \left[(H_x^0)^2 + (H_z^0)^2\right]^{1/2} = \frac{E_0}{\eta_0}\left\{\left(\frac{f_c}{f}\right)^2 + \left[1 - 2\left(\frac{f_c}{f}\right)^2 \sin^2\left(\frac{\pi x}{a}\right)\right]\right\}^{1/2},$

which is maximum at $x = a/2$.

At the sending end: Max. $|\bar{J}| = \frac{E_0}{\eta_0}\sqrt{1-\left(\frac{f_c}{f}\right)^2} \times 1.138 = 213.3$ (A/m).

d) Total amount of average power dissipated in 1 (m) of waveguide:

$$P_d = 1000\,(e^{2\alpha_c} - 1) = 1000\,(e^{0.0259} - 1) = 26.2 \text{ (W)}.$$

<u>P.10-21</u> From problem P.10-20, we have

$$P_{av} = \frac{E_0^2 ab}{4\eta_0}\sqrt{1-\left(\frac{f_c}{f}\right)^2}, \quad \sqrt{1-\left(\frac{f_c}{f}\right)^2} = 0.745.$$

$$\therefore \text{Max. } P_{av} = \frac{(3\times10^6)^2 \times (2.25\times10^{-4})}{4\times120\pi} \times 0.745 = 10^6 \text{ (W)} = 1 \text{ (MW)}.$$

P.10-22 Let $A = \frac{1}{\eta_0 b}\sqrt{\frac{\pi f_c \mu_c}{\sigma_c}}$ and $x = \frac{f_c}{f}$ in Eq. (10-157).

We write $\quad (\alpha_c)_{TE_{10}} = A F(x)$, where $F(x) = \frac{1 + \frac{2b}{a}x^2}{\sqrt{x(1-x^2)}}$.

For min. $(\alpha)_{TE_{10}}$, set $\frac{dF(x)}{dx} = 0$.

$$\longrightarrow x = \frac{f_c}{f} = \sqrt{\frac{3}{2}}\left[\left(1+\frac{a}{2b}\right) - \sqrt{\left(1+\frac{a}{2b}\right)^2 - \frac{2a}{9b}}\right]^{1/2}.$$

P.10-23 Field expressions for TM_{11} mode from Eqs. (10-132) and (10-136) through (10-137):

$$E_x^0(x,y) = -\frac{j\beta_{11}}{h^2}\left(\frac{\pi}{a}\right)E_0\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right),$$

$$E_y^0(x,y) = -\frac{j\beta_{11}}{h^2}\left(\frac{\pi}{b}\right)E_0\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right),$$

$$E_z^0(x,y) = E_0\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right),$$

$$H_x^0(x,y) = \frac{j\omega\epsilon}{h^2}\left(\frac{\pi}{b}\right)E_0\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right),$$

$$H_y^0(x,y) = -\frac{j\omega\epsilon}{h^2}\left(\frac{\pi}{a}\right)E_0\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right).$$

Calculate α_c from Eq. (10-112): $\alpha_c = \frac{P_L(z)}{2P(z)}$.

$$P(z) = \frac{1}{2}\int_0^b\int_0^a\left[E_x^0 H_y^{0*} - E_y^0 H_x^{0*}\right]dx\,dy = \frac{\omega\epsilon\beta E_0^2 ab}{8\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]}.$$

From problem P.10-13:

$$\bar{J}_s(y=0) = \bar{J}_s(y=b) = -\bar{a}_z\frac{j\omega\epsilon}{h^2}\left(\frac{\pi}{b}\right)E_0\sin\left(\frac{\pi x}{a}\right)e^{-j\beta_{11}z},$$

$$\bar{J}_s(x=0) = \bar{J}_s(x=a) = -\bar{a}_z\frac{j\omega\epsilon}{h^2}\left(\frac{\pi}{a}\right)E_0\sin\left(\frac{\pi y}{b}\right)e^{-j\beta_{11}z}.$$

$$P_L(z) = 2\left[P_L(z)\right]_{x=0} + 2\left[P_L(z)\right]_{y=0}.$$

$$\left[P_L(z)\right]_{x=0} = \frac{1}{2}\int_0^b\left|\bar{J}_s(x=0)\right|^2R_s\,dy = \frac{(\omega\epsilon)^2R_s}{4h^4}\left(\frac{\pi}{a}\right)^2E_0^2 b,$$

$$\left[P_L(z)\right]_{y=0} = \frac{1}{2}\int_0^a\left|\bar{J}_s(y=0)\right|^2R_s\,dx = \frac{(\omega\epsilon)^2R_s}{4h^4}\left(\frac{\pi}{b}\right)^2E_0^2 a.$$

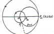
$$P_L(z) = \frac{(\omega\epsilon)^2R_sE_0^2}{2\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}\left[\left(\frac{\pi}{a}\right)^2b+\left(\frac{\pi}{b}\right)^2a\right].$$

$$\therefore\ (\alpha_c)_{TM_{11}} = \frac{2R_s\left(b/a^2+a/b^2\right)}{\eta ab\sqrt{1-(f_c/f)^2}\left(1/a^2+1/b^2\right)}.$$

Min. $(\alpha_c)_{TM_{11}}$ occurs at $(f_c/f) = \sqrt{3}$.

P.10-24 $f = 10^{10}$ (Hz), $f_c = \frac{c}{2a} = \frac{3\times10^8}{2\times0.0229} = 6.55\times10^9$ (Hz)

$\lambda_g = \frac{\lambda}{\sqrt{1-(f_c/f)^2}} = 0.0397$ (m) $= 3.97$ (cm) for TE_{10} mode.



Min.$|E|$ at $\frac{6}{3.97} = 1.511\lambda_g$ from the load, which is represented by the point P_m at $S = 1.80$.

Use Smith admittance chart:

Draw circle centered at O through P_m, intersecting the g=1 circle at P_1. Read at P_1: $y_1 = 1 - j0.60$.

Draw a straight line from O through P_1, intersecting the perimeter at P_1'. Read 0.152 on the "wavelengths toward load" scale. ∴ P_1' is at $(0.250+0.152)\lambda_g = 0.402\lambda_g$ from P_m, the position of a min.$|E|$. In other words, P_1', the desired location of the iris, should be at $(1.511-0.402)\lambda_g$, or 4.40 (cm), from the load.

From Eq.(10-159): Capacitive iris to supply $b = +0.60$,

$$0.60 = \frac{4b}{\lambda_g}\ln\left[\csc\left(\frac{\pi d}{2b}\right)\right] \longrightarrow d = 0.384 \text{ (cm)}.$$

P.10-26 TM modes: $E_z = E_z^0 e^{-\gamma z}$, $H_z = 0$.

$\nabla\times\bar{E} = -j\omega\mu\bar{H}$	$\nabla\times\bar{H} = j\omega\epsilon\bar{E}$
$\frac{\partial E_z^0}{r\partial\phi} + \gamma E_\phi^0 = -j\omega\mu H_r^0$. ①	$\gamma H_\phi^0 = j\omega\epsilon E_r^0$. ④
$-\gamma E_r^0 - \frac{\partial E_z^0}{\partial r} = -j\omega\mu H_\phi^0$. ②	$-\gamma H_r^0 = j\omega\epsilon E_\phi^0$. ⑤
$\frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\phi^0) - \frac{\partial E_r^0}{\partial\phi}\right] = 0$. ③	$\frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\phi^0) - \frac{\partial H_r^0}{\partial\phi}\right] = j\omega\epsilon E_z^0$ ⑥

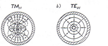
Eliminating H_ϕ^0 from ② and ④: $E_r^0 = -\frac{\gamma}{h^2}\frac{\partial E_z^0}{\partial r}$. ⑦

Eliminating H_r^0 from ① and ⑤: $E_\phi^0 = -\frac{\gamma}{h^2}\frac{\partial E_z^0}{r\partial\phi}$. ⑧

Combining ⑦ and ⑧: $(E_T^0)_{TM} = \bar{a}_r E_r^0 + \bar{a}_\phi E_\phi^0 = -\frac{\gamma}{h^2}\left(\bar{a}_r\frac{\partial E_z^0}{\partial r} + \bar{a}_\phi\frac{\partial E_z^0}{r\partial\phi}\right)$

$= -\frac{\gamma}{h^2}\nabla_T E_z^0$.

P.10-27 a) TM_{11} b) TE_{01}



——— Electric field lines.
- - - - Magnetic field lines

c) Eq. (10-35): $f_c = \frac{h}{2\pi\sqrt{\mu\epsilon}} = \frac{1.5\times10^8}{\pi}h$

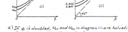
For TM_{11} mode, $(h)_{TM_{11}} = \frac{3.832}{a} \longrightarrow (f_c)_{TM_{11}} = \frac{1.83\times10^8}{a}$ (Hz).

For TE_{01} mode, $(h)_{TE_{01}} = \frac{3.832}{a} \longrightarrow (f_c)_{TE_{01}} = (f_c)_{TM_{11}}$.
(Degenerate mode)

P.10-28 $\beta^2 = k^2 - h^2 = \omega^2\mu\epsilon - h^2$

For TE_{11} mode, $h = x'_{11}/a = 1.841/a$,
$\beta = \sqrt{(\omega/c)^2 - (1.841/a)^2}$, $\omega_{c1} = 1.841c/a$.

For TM_{01} mode, $h = x_{01}/a = 2.405/a$,
$\beta = \sqrt{(\omega/c)^2 - (2.405/a)^2}$, $\omega_{c2} = 2.405c/a$.



a) If a is doubled, ω_{c1} and ω_{c2} in diagram (i) are halved; but diagram (ii) will remain the same.

b) If the waveguide medium is changed from (ϵ_0, μ_0) to $(\epsilon_0\epsilon_r, \mu_0)$, both ω_{c1} and ω_{c2} are reduced by a factor $\sqrt{\epsilon_r}$ and the slope of the asymptotic line is changed from c to $u_p = c/\sqrt{\epsilon_r}$; diagram (ii) remains unchanged.

P.10-29 a) TM modes: $E_z^0 = C_n J_n(hr)\sin n\phi$.

Boundary conditions $E_z^0 = 0$ at both $\phi = 0$ and $\phi = \pi$ are satisfied when n is an integer.

There are no TM_{0p} modes.

b) TE modes: $H_z^0 = C_n J_n(hr)\cos n\phi$, where n is an integer.

TE_{0p} modes can exist.

c) For TM modes, b.c. at $r=a$ requires that $J_n(ha) = 0$.

$\longrightarrow$ Eigenvalues $(h)_{TM_{np}} = x_{np}/a$, $n = 1, 2, 3, \cdots$

For TE modes, b.c. at $r=a$ requires that $J_n'(ha) = 0$.

$\longrightarrow$ Eigenvalues $(h)_{TE_{np}} = x'_{np}/a$, $n = 0, 1, 2, \cdots$

P.10-30 From Eqs. (10-245) and (10-249):

Inside the slab: $\beta^2 = \omega^2\mu_d\epsilon_d - k_y^2 < \omega^2\mu_d\epsilon_d$.

Outside the slab: $\beta^2 = \omega^2\mu_0\epsilon_0 + \alpha^2 > \omega^2\mu_0\epsilon_0$.

$$\therefore\quad \omega\sqrt{\mu_0\epsilon_0} < \beta < \omega\sqrt{\mu_d\epsilon_d},$$

and
$$\frac{1}{\sqrt{\mu_0\epsilon_0}} > u_p = \frac{\omega}{\beta} > \frac{1}{\sqrt{\mu_d\epsilon_d}}.$$

P.10-31 From Eqs. (10-288) and (10-287):

$$\left(\frac{\alpha d}{2}\right)^2 + \left(\frac{k_y d}{2}\right)^2 = \left(\frac{k_0 d}{2}\right)^2\left(\frac{\mu_d\epsilon_d}{\mu_0\epsilon_0} - 1\right). \quad ①$$

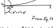
$$\frac{\epsilon_d}{\epsilon_0}\left(\frac{\alpha d}{2}\right) = \left(\frac{k_y d}{2}\right)\tan\left(\frac{k_y d}{2}\right). \quad ②$$

Let $X = k_y d/2$, $Y = \alpha d/2$, $A = \epsilon_0/\epsilon_d$, and $R = \frac{k_0 d}{2}\sqrt{\frac{\mu_d\epsilon_d}{\mu_0\epsilon_0} - 1}$.

Eqs. ① and ② become
$$\begin{cases} X^2 + Y^2 = R^2, & ③ \\ Y = AX\tan X. & ④ \end{cases}$$

a) $f = 2\times10^8$ (Hz), $\lambda = c/f = 1.5$ (m).

$k_0 d/2 = \pi d/\lambda = 0.0209$, $A = \epsilon_0/\epsilon_d = 0.308$, $R = 0.0314$.



Graphical solution:

$X_0 = 0.0314$, $Y_0 = 3.038\times10^{-4}$.

$\alpha = 2Y_0/d = 0.061$ (Np/m).

$k_y = 2X_0/d = 6.28$ (rad/m).

From Eq. (10-245): $\beta = \sqrt{\omega^2\mu_d\epsilon_d - k_y^2} = 4.19$ (rad/m).

b) $f = 5\times10^8$ (Hz), $\lambda = c/f = 0.60$ (m), $k_0 d/2 = 0.0524$.

$A = 0.308$, $R = 0.0785$,

$X_0 = 0.0785$, $Y_0 = 1.901\times10^{-3}$

We obtain $\alpha = 0.380$ (Np/m),

$\beta = 10.48$ (rad/m).

P.10-32 From Eq. (10-263):

$$\left(\frac{\alpha d}{2}\right) = -\frac{\epsilon_0}{\epsilon_d}\left(\frac{k_y d}{2}\right)\cot\left(\frac{k_y d}{2}\right). \qquad (3)$$

Using the notations in problem P.10-31, we obtain two equations from (1) in P.10-31 and (3) above:

$$\begin{cases} X^2 + Y^2 = R^2, & (4) \\ Y = -AX\cot X. & (5) \end{cases}$$

a) $f = 2\times10^8$ (Hz), $\lambda = 1.5$ (m).

$A = 0.308$,

$R = 0.0314$.

b) $f = 5\times10^8$ (Hz), $\lambda = 0.60$ (m).

$A = 0.308$,

$R = 0.075$.

There are no intersections for curves representing Eqs. (4) and (5); hence even TM modes do not exist at the given frequencies.

P.10-33 Use Eqs. (10-28) and (10-29):

$$E_y^0 = -\frac{j\beta}{h^2}\frac{\partial E_z^0}{\partial y}, \qquad H_x^0 = \frac{j\omega\epsilon}{h^2}\frac{\partial E_z^0}{\partial y}.$$

$$\bar{E}(y,z;t) = \mathcal{R}e\left[\bar{E}^0(y)\,e^{j(\omega t-\beta z)}\right].$$

$$\bar{H}(y,z;t) = \mathcal{R}e\left[\bar{H}^0(y)\,e^{j(\omega t-\beta z)}\right].$$

$|y| \leq d/2$:

$E_z^0(y) = E_e \cos k_y y \longrightarrow E_z(y,z;t) = E_e \cos k_y y \cos(\omega t - \beta z)$,

$E_y^0(y) = \frac{j\beta}{k_y} E_e \sin k_y y \longrightarrow E_y(y,z;t) = -\frac{\beta}{k_y} E_e \sin k_y y \sin(\omega t - \beta z)$,

$H_x^0(y) = -\frac{j\omega\epsilon_d}{k_y} E_e \sin k_y y \longrightarrow H_x(y,z;t) = \frac{\omega\epsilon_d}{k_y} E_e \sin k_y y \sin(\omega t - \beta z)$.

$y \geq d/2$:

$E_z^0(y) = E_e \cos\left(\frac{k_y d}{2}\right) e^{-\alpha(y-\frac{d}{2})} \longrightarrow E_z(y,z;t) = E_e \cos\left(\frac{k_y d}{2}\right) e^{-\alpha(y-\frac{d}{2})} \cos(\omega t - \beta z)$,

$E_y^0(y) = -\frac{j\beta}{\alpha} E_e \cos\left(\frac{k_y d}{2}\right) e^{-\alpha(y-\frac{d}{2})} \longrightarrow E_y(y,z;t) = \frac{\beta}{\alpha} E_e \cos\left(\frac{k_y d}{2}\right) e^{-\alpha(y-\frac{d}{2})} \sin(\omega t - \beta z)$,

$H_x^0(y) = \frac{j\omega\epsilon_0}{\alpha} E_e \cos\left(\frac{k_y d}{2}\right) e^{-\alpha(y-\frac{d}{2})} \longrightarrow H_x(y,z;t) = -\frac{\omega\epsilon_0}{\alpha} E_e \cos\left(\frac{k_y d}{2}\right) e^{-\alpha(y-\frac{d}{2})} \sin(\omega t - \beta z)$.

$y \leq -d/2$:

$E_z^0(y) = E_e \cos\left(\frac{k_y d}{2}\right) e^{\alpha(y+\frac{d}{2})} \longrightarrow E_z(y,z;t) = E_e \cos\left(\frac{k_y d}{2}\right) e^{\alpha(y+\frac{d}{2})} \cos(\omega t - \beta z)$,

$E_y^0(y) = \frac{j\beta}{\alpha} E_e \cos\left(\frac{k_y d}{2}\right) e^{\alpha(y+\frac{d}{2})} \longrightarrow E_y(y,z;t) = -\frac{\beta}{\alpha} E_e \cos\left(\frac{k_y d}{2}\right) e^{\alpha(y+\frac{d}{2})} \sin(\omega t - \beta z)$,

$H_x^0(y) = -\frac{j\omega\epsilon_0}{\alpha} E_e \cos\left(\frac{k_y d}{2}\right) e^{\alpha(y+\frac{d}{2})} \longrightarrow H_x(y,z;t) = \frac{\omega\epsilon_0}{\alpha} E_e \cos\left(\frac{k_y d}{2}\right) e^{\alpha(y+\frac{d}{2})} \sin(\omega t - \beta z)$.

P.10-34 a) From Table 10-4 on p. 572 it is seen that $f_{c0} = 0$ for TE_1 mode, which is the dominant mode.

From Eq. (10-275):

$$\alpha = \frac{\mu_0}{\mu_d} k_y \tan\frac{k_y d}{2} \cong \frac{\mu_0 d}{2\mu_d} k_y^2, \text{ for } k_y d \ll 1.$$

Neglecting the α^2 term in Eq. (10-267):

$$\beta^2 - \omega^2\mu_0\epsilon_0 = \alpha^2 \cong 0 \longrightarrow \beta \cong \omega\sqrt{\mu_0\epsilon_0} = k_0.$$

From Eq. (10-265): $k_y^2 = \omega^2\mu_d\epsilon_d - \beta^2 = k_d^2 - k_0^2$.

$$\therefore \quad \alpha \cong \frac{\mu_0 d}{2\mu_d}(k_d^2 - k_0^2).$$

b) $d = 5\times10^{-3}$ (m), $\epsilon_d = 3\epsilon_0$, $\mu_d = \mu_0$, $f = 3\times10^8$ (Hz), $k_0 = 2\pi$.

$$\alpha = \frac{d}{2} k_0^2 (\epsilon_r - 1) = 0.197 \quad \text{(Np/m)}.$$

$$e^{-\alpha(y-\frac{d}{2})} = 0.368, \quad \alpha\left(y - \frac{d}{2}\right) = 1.$$

$$\longrightarrow \left(y - \frac{d}{2}\right) = 5.066 \text{ (m)}.$$

P.10-35 Use Eqs. (10-50) and (10-51):

$$H_y^0 = -\frac{j\beta}{h^2}\frac{\partial H_z^0}{\partial y}, \qquad E_x^0 = -\frac{j\omega\mu}{h^2}\frac{\partial H_z^0}{\partial y}.$$

$$\bar{H}(y,z;t) = \mathcal{R}_e\left[\bar{H}^0(y)\, e^{j(\omega t-\beta z)}\right]$$

$$\bar{E}(y,z;t) = \mathcal{R}_e\left[\bar{E}^0(y)\, e^{j(\omega t-\beta z)}\right].$$

$|y| \le d/2$:

$$H_z^0(y) = H_0\cos k_y y \longrightarrow H_z(y,z;t) = H_0\cos k_y y\cos(\omega t-\beta z),$$

$$H_y^0(y) = \frac{j\beta}{k_y}H_0\sin k_y y \longrightarrow H_y(y,z;t) = -\frac{\beta}{k_y}H_0\sin k_y y\sin(\omega t-\beta z),$$

$$E_x^0(y) = \frac{j\omega\mu_d}{k_y}H_0\sin k_y y \longrightarrow E_x(y,z;t) = -\frac{\omega\mu_d}{k_y}H_0\sin k_y y\sin(\omega t-\beta z).$$

$y \ge d/2$:

$$H_z^0(y) = H_0\cos\left(\frac{k_y d}{2}\right)e^{-\alpha(y-\frac{d}{2})} \longrightarrow H_z(y,z;t) = H_0\cos\left(\frac{k_y d}{2}\right)e^{-\alpha(y-\frac{d}{2})}\cos(\omega t-\beta z),$$

$$H_y^0(y) = -\frac{j\beta}{\alpha}H_0\cos\left(\frac{k_y d}{2}\right)e^{-\alpha(y-\frac{d}{2})} \longrightarrow H_y(y,z;t) = \frac{\beta}{\alpha}H_0\cos\left(\frac{k_y d}{2}\right)e^{-\alpha(y-\frac{d}{2})}\sin(\omega t-\beta z),$$

$$E_x^0(y) = -\frac{j\omega\mu_0}{\alpha}H_0\cos\left(\frac{k_y d}{2}\right)e^{-\alpha(y-\frac{d}{2})} \longrightarrow E_x(y,z;t) = \frac{\omega\mu_0}{\alpha}H_0\cos\left(\frac{k_y d}{2}\right)e^{-\alpha(y-\frac{d}{2})}\sin(\omega t-\beta z).$$

$y \le -d/2$:

$$H_z^0(y) = H_0\cos\left(\frac{k_y d}{2}\right)e^{\alpha(y+\frac{d}{2})} \longrightarrow H_z(y,z;t) = H_0\cos\left(\frac{k_y d}{2}\right)e^{\alpha(y+\frac{d}{2})}\cos(\omega t-\beta z),$$

$$H_y^0(y) = \frac{j\beta}{\alpha}H_0\cos\left(\frac{k_y d}{2}\right)e^{\alpha(y+\frac{d}{2})} \longrightarrow H_y(y,z;t) = -\frac{\beta}{\alpha}H_0\cos\left(\frac{k_y d}{2}\right)e^{\alpha(y+\frac{d}{2})}\sin(\omega t-\beta z),$$

$$E_x^0(y) = \frac{j\omega\mu_0}{\alpha}H_0\cos\left(\frac{k_y d}{2}\right)e^{\alpha(y+\frac{d}{2})} \longrightarrow E_x(y,z;t) = -\frac{\omega\mu_0}{\alpha}H_0\cos\left(\frac{k_y d}{2}\right)e^{\alpha(y+\frac{d}{2})}\sin(\omega t-\beta z).$$

Setting $y = d/2$ in $E_x^0(y) = \frac{j\omega\mu_d}{k_y}H_0\sin k_y y$ and

in $E_x^0(y) = -\frac{j\omega\mu_0}{\alpha}H_0\cos\left(\frac{k_y d}{2}\right)e^{-\alpha(y-\frac{d}{2})}$,

and equating, we obtain

$$E_x^0\left(\frac{d}{2}\right) = \frac{j\omega\mu_d}{k_y}H_0\sin\left(\frac{k_y d}{2}\right) = -\frac{j\omega\mu_0}{\alpha}H_0\cos\left(\frac{k_y d}{2}\right).$$

$$\longrightarrow \quad \frac{\alpha}{k_y} = -\frac{\mu_0}{\mu_d}\cot\left(\frac{k_y d}{2}\right).$$

P.10-36 a) Odd TM and even TE modes are the propagating modes. Using 2d for d in the formulas in Table 10-3, p.578, we have

$$f_{cn} = \frac{n-1}{2d\sqrt{\mu_d\epsilon_d-\mu_0\epsilon_0}} \quad \text{for odd TM modes,}$$

$$f_{cn} = \frac{n-\frac{1}{2}}{2d\sqrt{\mu_d\epsilon_d-\mu_0\epsilon_0}} \quad \text{for even TE modes.}$$

b) <u>For odd TM modes</u> — From Eqs. (10-127 b and c):

$|y| \le d/2.$ $\quad E_y^o(y) = -\frac{j\beta}{k_y} E_o \cos k_y y$

$H_x^o(y) = \frac{j\omega\epsilon_d}{k_y} E_o \cos k_y y.$

Surf. current density on conductor $\bar{J}_s = \bar{a}_n \times \bar{H}\big|_{y=0}$

$\bar{J}_s = -\bar{a}_z H_x^o(0) = -\bar{a}_z \frac{j\omega\epsilon_d}{k_y} E_o.$

Surf. charge density on conductor $\rho_s = \bar{a}_n \cdot \bar{D}\big|_{y=0}$

$\rho_s = \epsilon_d E^o(0) = -\frac{j\beta\epsilon_d}{k_y} E_o.$

<u>For even TE modes</u> — From problem P.10-27:

$|y| \le d.$ $\quad H_y^o(y) = \frac{j\beta}{k_y} H_o \sin k_y y$

$E_x^o(y) = \frac{j\omega\mu_d}{k_y} H_o \sin k_y y.$

$H_z^o(y) = H_o \cos k_y y.$

$\therefore \bar{J}_s = \bar{a}_y \times [\bar{a}_y H_y^o(0) + \bar{a}_z H_z^o(0)] = \bar{a}_x H_o$

$\rho_s = \bar{a}_y \cdot \epsilon_d \bar{E}(0) = 0.$

<u>P.10-37</u> For circularly symmetrical TE modes, $E_z^0 = 0$, $\frac{\partial}{\partial\phi} \to 0.$

a) For $r \le a$: $E_{zi}^0 = 0$, $H_{zi}^0 = C_0 J_0(hr)$. $\quad h^2 = \omega^2\mu_1\epsilon_1 - \beta^2.$

$$\begin{cases} \bar{E}_T^0 = \frac{1}{h^2}(\bar{a}_z j\omega\mu_1 \times \bar{\nabla}_T H_{zi}^0) = \frac{j\omega\mu_1}{h^2}\bar{a}_z \times (\bar{a}_r \frac{\partial H_{zi}}{\partial r}) = \bar{a}_\phi \frac{j\omega\mu_1}{h} C_0 J_0'(hr). & ① \\ \bar{H}_T^0 = -\frac{1}{h^2}(\gamma \bar{\nabla}_T H_{zi}^0) = -\frac{\gamma}{h^2}(\bar{a}_r \frac{\partial H_{zi}^0}{\partial r}) = -\bar{a}_r \frac{\gamma}{h} C_0 J_0'(hr). & ② \end{cases}$$

From ① & ②: $r \le a$ $\begin{cases} E_{\phi i}^0 = \frac{j\omega\mu_1}{h} C_0 J_0'(hr), \\ E_{ri}^0 = 0. \end{cases}$ $\quad H_{ri}^0 = -\frac{\gamma}{h} C_0 J_0'(hr),$ $\quad H_{\phi i}^0 = 0,$ $\quad H_{zi}^0 = C_0 J_0(hr).$

Similarly, for $r \ge a$:

$H_{zo}^0 = D_0 K_0(\xi r)$, where $\xi^2 = -(\omega^2\mu_0\epsilon_0 - \beta^2).$

$\longrightarrow E_{\phi o}^0 = -\frac{j\omega\mu_0}{\xi} D_0 K_0'(\xi r),$ $\quad H_{ro}^0 = \frac{\gamma}{\xi} D_0 K'(\xi r).$

b) Boundary conditions at $r = a$: $\begin{cases} \text{for } H_z^0: & C_0 J_0(ha) = D_0 K_0(\xi a); & ③ \\ \text{for } E_\phi^0: & \frac{\mu_1}{h} C_0 J_0'(ha) = -\frac{\mu_0}{\xi} D_0 K_0'(\xi a). & ④ \end{cases}$

$\frac{③}{④} \longrightarrow$ Characteristic equation for symmetrical TE modes: $\quad \frac{J_0(ha)}{J_0'(ha)} = -\frac{\mu_1 \xi}{\mu_0 h} \frac{K_0(\xi a)}{K_0'(\xi a)}.$

P.10-38 From Eq. (10-301): $f_{mnp} = \frac{u}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{d}\right)^2}$.

$$f_{mnp} = 1.5\times10^{10}F(m,n,p), \quad F(m,n,p)=\sqrt{\left(\frac{m}{8}\right)^2+\left(\frac{n}{6}\right)^2+\left(\frac{p}{5}\right)^2}.$$

Lowest-order modes and resonant frequencies:

Modes	$F(m,n,p)$	$(f_r)_{mnp}$ in (Hz)
TM_{110}	0.2083	3.125×10^9
TE_{101}	0.2358	3.538×10^9
TE_{011}	0.2603	3.905×10^9
TE_{111}, TM_{111}	0.2888	4.332×10^9
TM_{210}	0.3005	4.507×10^9
TE_{201}	0.3202	4.802×10^9
TM_{120}	0.3560	5.340×10^9
TE_{211}, TM_{211}	0.3609	5.414×10^9
TE_{021}	0.3887	5.831×10^9
TE_{121}, TM_{121}	0.4083	6.125×10^9

P.10-39 a) Since $d>a>b$, the lowest-order resonant mode is TE_{101} mode.

$$f_{101} = \frac{c}{2}\sqrt{\frac{1}{a^2}+\frac{1}{d^2}} = 4.802\times10^9 \text{ (Hz)}.$$

b) From Eq. (10-322):

$$Q_{101} = \frac{\pi f_{101}\mu_0 abd(a^2+d^2)}{R_s[2b(a^3+d^3)+ad(a^2+d^2)]} \qquad \left(R_s=\sqrt{\frac{\pi f_{101}\mu_0}{\sigma}}\right)$$
$$= \frac{\sqrt{\pi f_{101}\mu_0\sigma}\; abd(a^2+d^2)}{2b(a^3+d^3)+ad(a^2+d^2)} = 6869.$$

From Eqs. (10-316) and (10-317):

$$W_e = \frac{1}{4}\epsilon_0\mu_0^2 a^3bd f_{101}^2 H_0^2 = 0.07728\times10^{-12} \text{ (J)},$$
$$W_m = \frac{\mu_0}{16}abd\left(\frac{a^2}{d^2}+1\right)H_0^2 = 0.07728\times10^{-12}\text{ (J)} = W_e.$$

P.10-40 a) $(f_{101})_{\epsilon_d} = \frac{u}{2}\sqrt{\frac{1}{a^2}+\frac{1}{d^2}} = \frac{1}{\sqrt{\epsilon_r}}(f_{101})_{\epsilon_0} = 3.037\times10^9$ (Hz).

b) $(Q_{101})_{\epsilon_d} = \frac{1}{(\epsilon_r)^{1/4}}(Q_{101})_{\epsilon_0} = 5462.$

c) $(W_e)_{\epsilon_d} = (W_e)_{\epsilon_0} = 0.07728\times10^{-12}$ (J) $= 0.07728$ (pJ)

$= (W_m)_{\epsilon_d}.$

P.10-41 a) Combining Eqs. (10-318) and (10-322):

$$Q_{101} = \frac{\pi\eta}{2R_s}\,\frac{b(a^2+d^2)^{3/2}}{[2b(a^3+d^3)+ad(a^2+d^2)]}.$$

⟶ Q_{101} has a symmetrical dependence on a and d. It will be maximum when $a=d$, which gives a max. volume-to-surface ratio.

b) When $a=d$, $Q_{101} = \frac{\pi\eta}{2R_s}\,\frac{1}{\sqrt{2}\,(1+b/2a)}.$

P.10-42 a) $Q_{101} = \frac{\sqrt{\pi f_{101}\mu_0\sigma}\; abd(a^2+d^2)}{2b(a^3+d^3)+ad(a^2+d^2)}.$

For $a=d=1.8b$, $f_{101} = \frac{1}{2\sqrt{\mu_0\epsilon_0}}\sqrt{\frac{1}{a^2}+\frac{1}{d^2}} = 1.179\times10^8\left(\frac{1}{b}\right).$

$Q_{101} = 10.22\sqrt{\sigma b}.$

b) For $Q'_{101} = 1.20\,Q_{101}$, $b' = 1.20^2 b = 1.44\,b.$

P.10-43 (I) From the field configurations in the cavity we see that the TM_{110} mode with respect to z is the same as the TE_{101} mode with respect to y. Thus, $(Q_{110})_{TM}$ can be obtained from $(Q_{101})_{TE}$ in Eq. (10-322) by changing b to d and d to b.

or, (II) Q for the TM_{110} mode can be derived from the field expressions in Eqs. (10-295, 298, & 299) by setting $m=n=1$, and using Eq. (10-315).

$W = 2W_m = \frac{\mu_0}{8}\left(\frac{\omega^2\epsilon_0^2}{h^4}\right)abd\,E_0^2$ at f_{110}.

$P_L = \oint \frac{1}{2}|\bar{J}_s|^2 R_s\, ds = \oint \frac{1}{2}|\bar{H}_t|^2 R_s\, ds.$

$$P_L = R_s\Big\{\int_0^d\int_0^b |H_y(x=0)|^2\,dy\,dz + \int_0^d\int_0^a |H_x(y=0)|^2\,dx\,dz$$
$$+\int_0^b\int_0^a\left[|H_x(z=0)|^2+|H_y(z=0)|^2\right]dx\,dy\Big\}$$
$$= \frac{R_s}{2}\left(\frac{\omega^2\epsilon^2}{h^4}\right)E_0^2\left\{\frac{1}{h^2}\left(\frac{\pi}{a}\right)^2 bd + \frac{1}{h^2}\left(\frac{\pi}{b}\right)^2 ad + \frac{1}{2}ab\right\},$$
$$h^2 = \left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2.$$

$$Q_{110} = \frac{\omega_{110}W}{P_L} = \frac{\pi f_{110}\mu_0 abd(a^2+b^2)}{R_s\left[2d(a^3+b^3)+ab(a^2+b^2)\right]}, \quad R_s = \sqrt{\frac{\pi f_{110}\mu_c}{\sigma}}.$$

<u>P.10-44</u> a) TM_{mnp} mode:

$$E_z^0 = C_m J_m\left(\frac{x_{mn}}{a}r\right)\cos m\phi\,\cos\left(\frac{p\pi}{d}z\right),$$
$$m = 0,1,2,\cdots;\quad n = 1,2,3,\cdots;\quad p = 0,1,2,\cdots$$
$$(f_{TM})_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{p\pi}{d}\right)^2},$$
$$(\lambda_{TM})_{mnp} = \frac{2\pi}{\sqrt{\left(\frac{x_{mn}}{a}\right)^2+\left(\frac{p\pi}{d}\right)^2}}.$$

TE_{mnp} mode: $H_z^0 = C_m J_m\left(\frac{x'_{mn}}{a}r\right)\cos m\phi\,\sin\left(\frac{p\pi}{d}z\right),$
$$m = 0,1,2,\cdots;\quad n = 1,2,3,\cdots;\quad p = 1,2,3,\cdots$$
$$(f_{TE})_{mnp} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x'_{mn}}{a}\right)^2+\left(\frac{p\pi}{d}\right)^2},$$
$$(\lambda_{TE})_{mnp} = \frac{2\pi}{\sqrt{\left(\frac{x'_{mn}}{a}\right)^2+\left(\frac{p\pi}{d}\right)^2}}.$$

b) For $d=a$, the dominant mode is TM_{010}: $(f_{TM})_{010} = \frac{2.405}{2\pi a\sqrt{\mu\epsilon}} = \frac{0.38}{a\sqrt{\mu\epsilon}}$

The first seven modes with the lowest resonant freqs:

Mode	TM_{010}	TE_{111}	TM_{110}	TM_{011}	TE_{211}	TM_{111}	TE_{011}
$f_r/(f_r)_{TM010}$	1.00	1.51	1.59	1.65	1.82	2.06	2.06

<u>P.10-45</u> $C = \frac{\epsilon S}{d} = \frac{\epsilon\pi a^2}{d}, \qquad L = \frac{\mu h}{2\pi}\ln\left(\frac{b}{a}\right).$

a) $f_r = \frac{1}{2\pi\sqrt{LC}} \cong \frac{1}{\pi a\sqrt{\mu\epsilon}\sqrt{\frac{2h}{d}\ln\left(\frac{b}{a}\right)}}.$

b) $\lambda_r = \frac{1}{f_r\sqrt{\mu\epsilon}} \cong \pi a\sqrt{\frac{2h}{d}\ln\left(\frac{b}{a}\right)}.$

Chapter 11

Antennas and Radiating Systems

P.11-1 Maxwell's equations for simple media:

$$\bar{\nabla}\times\bar{E} = -\mu\frac{\partial\bar{H}}{\partial t}. \quad ①$$

$$\bar{\nabla}\times\bar{H} = \bar{J} + \epsilon\frac{\partial\bar{E}}{\partial t}. \quad ②$$

$$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\epsilon}. \quad ③$$

$$\bar{\nabla}\cdot\bar{H} = 0. \quad ④$$

a) $\bar{\nabla}\times①$: $\bar{\nabla}\times\bar{\nabla}\times\bar{E} = -\mu\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{H})$

$$= -\mu\frac{\partial\bar{J}}{\partial t} - \mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2}. \quad ⑤$$

But $\bar{\nabla}\times\bar{\nabla}\times\bar{E} = \bar{\nabla}(\bar{\nabla}\cdot\bar{E}) - \bar{\nabla}^2\bar{E}$

$$= \frac{1}{\epsilon}\bar{\nabla}\rho - \bar{\nabla}^2\bar{E}. \quad ⑥$$

Combining ⑤ and ⑥, we obtain

$$\bar{\nabla}^2\bar{E} - \mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} = \frac{1}{\epsilon}\bar{\nabla}\rho + \mu\frac{\partial\bar{J}}{\partial t}.$$

b) Similarly, we have $\bar{\nabla}^2\bar{H} - \mu\epsilon\frac{\partial^2\bar{H}}{\partial t^2} = -\bar{\nabla}\times\bar{J}$.

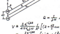
P.11-2 Eq. (11-2): $\bar{E} = -\bar{\nabla}V - j\omega\bar{A} = \bar{a}_R E_R + \bar{a}_\theta E_\theta + \bar{a}_\phi E_\phi$.

$$E_R = -\frac{\partial V}{\partial R} - j\omega A_R$$

$$E_\theta = -\frac{\partial V}{R\partial\theta} - j\omega A_\theta$$

$$E_\phi = -\frac{\partial V}{R\sin\theta\,\partial\phi} - j\omega A_\phi$$

The expressions of A_R, A_θ, and A_ϕ are given in Eqs. (11-14a, b, and c).



$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{e^{-j\beta R_+}}{R_+} - \frac{e^{-j\beta R_-}}{R_-}\right];$$

$$R_+ \cong R - \frac{1}{2}d\ell\cos\theta,$$

$$R_- \cong R + \frac{1}{2}d\ell\cos\theta.$$

$$Q = \frac{I}{j\omega}, \quad (d\ell)^2 << R^2.$$

$$V \cong \frac{Ie^{-j\beta R}}{4\pi\epsilon_0 j\omega}\frac{1}{R^2}\left[\left(R+\frac{d\ell}{2}\cos\theta\right)e^{j\beta(d\ell\cos\theta)/2} - \left(R-\frac{d\ell}{2}\cos\theta\right)e^{-j\beta(d\ell\cos\theta)/2}\right]$$

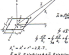
$$= \frac{Ie^{-j\beta R}}{4\pi\epsilon_0 j\omega R^2}\left[2jR\sin\left(\frac{\beta\,d\ell\cos\theta}{2}\right) + 2\left(\frac{d\ell}{2}\cos\theta\right)\cos\left(\frac{\beta\,d\ell\cos\theta}{2}\right)\right]$$

$$V \cong \frac{I e^{-j\beta R}}{4\pi\epsilon_0 j\omega R^2}\left[2j R\left(\frac{\beta\, d\ell\cos\theta}{2}\right)+d\ell\cos\theta\right]$$

$$= \frac{I\, d\ell\cos\theta}{4\pi R^2}\eta_0\left(R+\frac{1}{j\beta}\right)e^{-j\beta R}.$$

Using A_R, A_θ, A_ϕ, and V in E_R, E_θ, and E_ϕ, we obtain the same results as given in Eqs. (11-16a,b,&c).

P.11-3



a) $$\bar{A}=\frac{\mu_0 I}{4\pi}\oint\frac{e^{-j\beta R_1}}{R_1}d\bar{\ell}'$$
$$=\frac{\mu_0 I}{4\pi}e^{-j\beta R}\oint\frac{e^{-j\beta(R_1-R)}}{R_1}d\bar{\ell}'.$$

$$\bar{A}\cong\frac{\mu_0 I}{4\pi}e^{-j\beta R}\left[(1+j\beta R)\oint\frac{d\bar{\ell}'}{R_1}-j\beta\oint d\bar{\ell}'\right].$$

$$\oint d\bar{\ell}'=0.$$

$$\oint\frac{d\bar{\ell}'}{R_1}=\int_{AB}\frac{d\bar{\ell}'}{R_1}+\int_{CD}\frac{d\bar{\ell}'}{R_1}+\int_{BC}\frac{d\bar{\ell}'}{R_1}+\int_{DA}\frac{d\bar{\ell}'}{R_1}.$$

$$R_1^2=R^2+r^2-2\bar{R}\cdot\bar{r}.$$

$$\bar{R}=\bar{a}_x R\sin\theta\cos\phi+\bar{a}_y R\sin\theta\sin\phi+\bar{a}_z R\cos\theta.$$

$$\bar{r}=\bar{a}_x x+\bar{a}_y\frac{L_y}{2},\quad \bar{R}\cdot\bar{r}=Rx\sin\theta\cos\phi+R\frac{L_y}{2}\sin\theta\sin\phi.$$

$$\frac{1}{R_1}\cong\frac{1}{R}\left[1+\frac{\bar{R}\cdot\bar{r}}{R^2}\right]=\frac{1}{R}\left(1+\frac{x}{R}\sin\theta\cos\phi+\frac{L_y}{2R}\sin\theta\sin\phi\right).$$

$$\frac{\mu_0 I}{4\pi}e^{-j\beta R}(1+j\beta R)\int_{AB}\frac{d\bar{\ell}'}{R_1}=\bar{a}_x\frac{\mu_0 I}{4\pi}e^{-j\beta R}(1+j\beta R)\frac{1}{R}\int_{L_x/2}^{-L_x/2}\left(1+\frac{x}{R}\sin\theta\cos\phi+\frac{L_y}{2R}\sin\theta\sin\phi\right)dx$$

$$=\bar{a}_x\frac{\mu_0 I}{4\pi}e^{-j\beta R}(1+j\beta R)\frac{1}{R}\left(-L_x-\frac{L_x L_y}{2R}\sin\theta\sin\phi\right).$$

In the same manner, we have

$$\frac{\mu_0 I}{4\pi}e^{-j\beta R}(1+j\beta R)\int_{CD}\frac{d\bar{\ell}'}{R_1}=\bar{a}_x\frac{\mu_0 I}{4\pi}e^{-j\beta R}(1+j\beta R)\frac{1}{R}\left(L_x-\frac{L_x L_y}{2R}\sin\theta\sin\phi\right).$$

$$\therefore\ \frac{\mu_0 I}{4\pi}e^{-j\beta R}(1+j\beta R)\int_{AB+CD}\frac{d\bar{\ell}'}{R_1}=-\bar{a}_x\frac{\mu_0 I}{4\pi R^2}e^{-j\beta R}(1+j\beta R)L_x L_y\sin\theta\sin\phi,$$

and

$$\frac{\mu_0 I}{4\pi}e^{-j\beta R}(1+j\beta R)\int_{BC+DA}\frac{d\bar{\ell}'}{R_1}=\bar{a}_y\frac{\mu_0 I}{4\pi R^2}e^{-j\beta R}(1+j\beta R)L_x L_y\sin\theta\cos\phi.$$

Let $m=IL_xL_y=IS$.

$$\bar{A}=\frac{\mu_0 m}{4\pi R^2}e^{-j\beta R}(1+j\beta R)\sin\theta(-\bar{a}_x\sin\phi+\bar{a}_y\cos\phi)$$

$$=\bar{a}_\phi\frac{\mu_0 m}{4\pi R^2}e^{-j\beta R}(1+j\beta R)\sin\theta.$$

c) $\bar{H} = \frac{1}{\mu_0}\nabla\times\bar{A} = \bar{a}_R H_R + \bar{a}_\theta H_\theta$. Expressions for H_R, H_θ,
b) $\bar{E} = \frac{1}{j\omega\epsilon_0}\nabla\times\bar{H} = \bar{a}_\phi E_\phi$. and E_ϕ same as those given in Eqs. (11-26a, b, c).

In the far zone, $\beta R >> 1$, $1/(j\beta R)^2$ and $1/(j\beta R)^3$ terms can be neglected. We have the following instantaneous expressions; assuming $i(t) = I\cos\omega t$:

$$\bar{A}(R,\theta;t) = -\bar{a}_\phi \frac{\mu_0 m}{4\pi R}\beta\sin\theta\sin(\omega t-\beta R).$$
$$\bar{E}(R,\theta;t) = \bar{a}_\phi \frac{\omega\mu_0 m}{4\pi R}\beta\sin\theta\cos(\omega t-\beta R).$$
$$\bar{H}(R,\theta;t) = -\bar{a}_\theta \frac{m}{4\pi R}\beta^2\sin\theta\cos(\omega t-\beta R).$$

P.11-4 Far-zone electric field of elemental electric dipole:
$$E_\theta(R) = j\frac{I_0 dl}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\eta_0\beta\sin\theta \longrightarrow E_\theta(R,t) = -\frac{I_0\eta_0\beta\sin\theta}{4\pi R}(L)\sin(\omega t-\beta R).$$
For the elemental magnetic dipole:
$$E_\phi(R) = \frac{\omega\mu_0 m}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)\beta\sin\theta \longrightarrow E_\phi(R,t) = \frac{I_0\eta_0\beta\sin\theta}{4\pi R}\left(\frac{2\pi S}{\lambda}\right)\cos(\omega t-\beta R).$$
a) Thus, $$\frac{E_\theta^2(R,t)}{\left(\frac{I_0\eta_0\beta\sin\theta}{4\pi R}\right)^2(L)^2} + \frac{E_\phi^2(R,t)}{\left(\frac{I_0\eta_0\beta\sin\theta}{4\pi R}\right)^2\left(\frac{2\pi S}{\lambda}\right)^2} = 1.$$
——— Elliptic polarization.

b) Circular polarization if $L = 2\pi S/\lambda$.

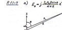
P.11-5 Equation of continuity: $\nabla\cdot\bar{J} = -j\omega\rho \longrightarrow \rho_\ell = \frac{j}{\omega}\frac{dI(z)}{dz}$.
a) $I(z) = I_0\cos\beta z \longrightarrow \rho_\ell = -j\frac{\beta}{\omega}I_0\sin\beta z = -j\frac{I_0}{c}\sin\beta z$.
b) $I(z) = I_0\left(1-\frac{4}{\lambda}|z|\right) \longrightarrow \rho_\ell = \begin{cases} -j\frac{2I_0}{\pi c} & \text{for } z>0 \\ +j\frac{2I_0}{\pi c} & \text{for } z<0. \end{cases}$

P.11-6 $\lambda = \frac{3\times10^8}{10^6} = 300\,(m)$, $\frac{d\ell}{\lambda} = \frac{15}{300} = \frac{1}{20} << 1$ (Hertzian dipole).
a) Radiation resistance, $R_r = 80\pi^2\left(\frac{d\ell}{\lambda}\right)^2 = 1.97\,(\Omega)$.
b) Eq. (11-49): $R_s = \sqrt{\frac{\pi f\mu_0}{\sigma}} = \sqrt{\frac{\pi 10^6(4\pi\times10^{-7})}{5.80\times10^7}} = 2.61\times10^{-4}\,(\Omega)$.
Eq. (11-48): $R_\ell = R_s\left(\frac{d\ell}{2\pi a}\right) = 0.031\,(\Omega) \longrightarrow \eta_r = \frac{R_r}{R_r+R_\ell} = 98.5\%$.
c) Eq. (11-43): $P_r = \frac{I^2(d\ell)^2}{12\pi}\eta_0\beta^2$; Eq. (11-19b): $|E_\theta|_{max.} = \left(\frac{I\,d\ell}{4\pi}\right)^2\frac{\eta_0^2\beta^2}{R^2}$ $\Big\} \longrightarrow |E_\theta|_{max.} = \frac{1}{R}\sqrt{90P_r} = 19\,(mV/m)$.

P.11-7 a) $E_\theta = j\frac{I_0\eta_0\beta\sin\theta}{4\pi R}e^{-j\beta R}\int_{-h}^{h}\left(1-\frac{|z|}{h}\right)e^{j\beta z\cos\theta}dz$

$= j\frac{I_0\eta_0\beta\sin\theta}{2\pi R}e^{-j\beta R}\int_0^h\left(1-\frac{z}{h}\right)\cos(\beta z\cos\theta)dz$

$= \frac{j60I_0}{(\beta h)R}e^{-j\beta R}F(\theta)$



$(R_1 \cong R - z\cos\theta)$

$H_\phi = \frac{E_\theta}{\eta_0} = \frac{jI_0}{(\beta h)2\pi R}e^{-j\beta R}F(\theta).$

$F(\theta) = \frac{\sin\theta[1-\cos(\beta h\cos\theta)]}{\cos^2\theta}.$

In case $\beta h \ll 1$, $\cos(\beta h\cos\theta) \cong 1 - \frac{1}{2!}(\beta h\cos\theta)^2$, and

$F(\theta) \cong \frac{1}{2}(\beta h)^2\sin\theta.$

$\therefore\ E_\theta = \frac{j60I_0}{R}e^{-j\beta R}\left(\frac{1}{2}\beta h\sin\theta\right) = \frac{j30\beta h}{R}I_0e^{-j\beta R}\sin\theta.$

$H_\phi = \frac{jI_0}{2\pi R}e^{-j\beta R}\left(\frac{1}{2}\beta h\sin\theta\right) = \frac{j\beta h}{4\pi R}I_0e^{-j\beta R}\sin\theta.$

b) $P_r = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}E_\theta H_\phi^* R^2\sin\theta\,d\theta d\phi = \frac{I_0^2}{2}\left[80\pi^2\left(\frac{h}{\lambda}\right)^2\right],$

$R_r = P_r/\left(\frac{1}{2}I_0^2\right) = 20\pi^2\left(\frac{2h}{\lambda}\right)^2.$

c) $D = \frac{4\pi|E_{max}|^2}{\int_0^{2\pi}\int_0^{\pi}|E_\theta(\theta)|^2\sin\theta\,d\theta d\phi} = \frac{2}{\int_0^{\pi}\sin^3\theta\,d\theta} = 1.5$ — $10\log_{10}D = 1.76$ (dB).

P.11-8 $I(z) = I_0\left(1-\frac{z}{h}\right).$

a) $\ell_e = \int_0^h\left(1-\frac{z}{h}\right)dz = \frac{h}{2} = 20$ (m), over conducting earth.

b) From Eq. (11-19b): $|E_\theta|_{max} = \frac{I_0}{4\pi R}\eta_0\beta\times 2\ell_e = \frac{20I_0\beta h}{R} = 2.52$ (m

c) $P_r = \frac{1}{2}\int_0^{\pi/2}\frac{|E_\theta|^2}{\eta_0}2\pi R^2\sin\theta\,d\theta = \frac{2\pi}{3}\eta_0\left(\frac{I_0\beta h}{4\pi}\right)^2 = 1.14$ (kW).

d) $R_r = 2P_r/I_0^2 = 0.227$ (Ω).

P.11-9 a) $R_r = 80\pi^2\left(\frac{\beta\pi b^2}{\lambda}\right)^2 = 20(\beta^2\pi b^2)^2 = 320\pi^6\left(\frac{b}{\lambda}\right)^4.$

b) $\eta_r = \frac{R_r}{R_r+R_\ell}$; $R_\ell = \frac{b}{a}R_s$, $R_s = \sqrt{\pi f\mu_0/\sigma}$, $\sigma = 5.80\times10^7$ (S/m).

c) $R_r = 320\pi^6\left(\frac{0.8}{300}\right)^4 = 2.35\times10^{-6}$ (Ω).

From problem P.11-6, $R_s = 2.61\times10^{-4}$ ⟶ $R_\ell = 0.0435$, $\eta_r = 0.0055\%$

d) For $N = 10$, $R_r(\propto N^2) = 2.35\times10^{-4}$ (Ω), $R_\ell(\propto N) = 0.435$ (Ω)

⟶ $\eta_r = 0.055\%$ (increases approximately $\propto N$).

P.11-10 a) $R_r = 80\pi^2\left(\frac{\beta L_x L_y}{\lambda}\right)^2 = 20\,(\beta^2 L_x L_y)^2$

b) $R_\ell = \frac{L_x + L_y}{\pi b} R_s$, $\quad \eta_r = \frac{R_r}{R_r + R_\ell}$.

c) For $f = 1$ (MHz), $L_x = L_y = 1$ (m), and $a = 0.003$ (m)

$\beta = 2\pi/300$ (rad/m), $R_s = 2.61\times10^{-4}$ (Ω) from P.11-6.

$R_r = 3.85\times10^{-6}$ (Ω), $\quad R_\ell = 0.055$ (Ω).

$\longrightarrow \eta_r = 0.007\,\%$

P.11-11 $P_r = \oint \bar{\mathscr{P}}_{av}\cdot d\bar{s} = \frac{1}{2}\mathscr{Re}\int_0^{2\pi}\!\!\int_0^{\pi} E_\theta H_\phi^* R^2 \sin\theta\, d\theta\, d\phi$

$$= \frac{(I\,d\ell)^2}{16\pi}\beta^4 R^2 \eta_0\, \mathscr{Re}\left\{\left[\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2}\right]\left[-\frac{1}{j\beta R} + \frac{1}{(j\beta R)^2} - \frac{1}{(j\beta R)^3}\right]\right\} \cdot \int_0^{\pi} \sin^3\theta\, d\theta$$

$= \frac{I^2}{2}\left[80\pi^2\left(\frac{d\ell}{\lambda}\right)^2\right]$, which is the same as Eq.(11-43).

P.11-12 $F(\theta) = \frac{\cos(\beta h\cos\theta) - \cos\beta h}{\sin\theta}$

For $2h = 1.25\lambda$,

$$|F(\theta)| = \left|\frac{\cos(1.25\pi\cos\theta) - \cos(1.25\pi)}{\sin\theta}\right|$$

Width of main beam between the first nulls $= 2\times 53.13° = 106.26°$.

P.11-13 $I(z) = I_0\left(1 - \frac{|z|}{h}\right)$.

From Eq.(11-70): $\ell_e = \frac{\sin\theta}{I_0}\int_{-h}^{h} I(z)\, e^{j\beta z\cos\theta}\, dz$

$$= \frac{\sin\theta}{I_0}\, 2\int_0^h I_0\left(1 - \frac{z}{h}\right)\cos(\beta z\cos\theta)\, dz$$

$$= \frac{2\sin\theta}{h(\beta^2\cos^2\theta)}\left[1 - \cos(\beta h\cos\theta)\right].$$

Maximum ℓ_e occurs at $\theta = \frac{\pi}{2}$, where

$$\ell_e\left(\frac{\pi}{2}\right) = h = \frac{\lambda}{12}.$$

P.11-14 a) $V_{oc} = -E_\theta \ell_e = -\frac{\lambda E_\theta}{\pi}\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right].$

b) $P_L = \frac{1}{2}\left|\frac{V_{oc}}{R_r + R_L}\right|^2 R_L = \frac{|V_{oc}|^2}{8R_r} = \frac{\lambda^2 E_\theta^2}{8\pi^2 R_r}\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right]^2,$

which has a maximum value $(\lambda^2 E_\theta^2/8\pi^2 R_r)$ at $\theta = \frac{\pi}{2}$.

c) For $\lambda = \frac{3\times10^8}{1.5\times10^6} = 200\ (m)$ and $E_\theta = 5\times10^{-2}\ (V/m)$:

$\theta = \frac{\pi}{2} \longrightarrow \ell_e(\frac{\pi}{2}) = \frac{\lambda}{\pi} = 63.7\ (m),\ V_{oc} = -3.15\ (V),$

$P_L = \frac{3.15^2}{8\times73.1} = 0.017\ (W) = 17\ (mW).$

$\theta = \frac{\pi}{4} \longrightarrow \ell_e(\frac{\pi}{4}) = \frac{\lambda}{\pi}\left[\frac{\cos(\frac{\pi}{2}\cos45°)}{\sin45°}\right] = 40\ (m),$

$V_{oc} = -2\ (V),\quad P_L = 6.8\ (mW).$

P.11-15

$R' \cong R - d\cos\theta$

a) $E_\theta = \frac{jI(2h)}{4\pi}\eta_0\beta\sin\theta\cdot e^{-j\beta R}\cdot(1+e^{j\beta d\cos\theta})$

$= \frac{j60Ih}{R}2\beta e^{-j\beta(R-\frac{d}{2}\cos\theta)}F(\theta),$

where $F(\theta) = \sin\theta\cos(\frac{\beta d}{2}\cos\theta)$.

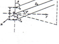
b) $d = \lambda/2$.

$|F(\theta)| = \left|\sin\theta\cos(\frac{\pi}{2}\cos\theta)\right|.$

c) $d = \lambda$.

$|F(\theta)| = \left|\sin\theta\cos(\pi\cos\theta)\right|.$

P.11-16



From Eq. (11-19b):

$$E_{\psi_1} = \frac{jI_0 d\ell \eta_0 \beta}{4\pi R_0} e^{-j\beta(R_0 - d\cos\theta)} \sin\psi,$$

$$E_{\psi_2} = -\frac{jI_0 d\ell \eta_0 \beta}{4\pi R_0} e^{-j\beta(R_0 + d\cos\theta)} \sin\psi,$$

where

$$\sin\psi = |\bar{a}_R \times \bar{a}_y|$$
$$= |(\bar{a}_x \sin\theta\cos\phi + \bar{a}_y \sin\theta\sin\phi + \bar{a}_z \cos\theta) \times \bar{a}_y|$$
$$= |\bar{a}_z \sin\theta\cos\phi - \bar{a}_x \cos\theta|$$
$$= \sqrt{1-\sin^2\theta\sin^2\phi}.$$

$$E_\psi = E_{\psi_1} + E_{\psi_2} = j\frac{I_0 d\ell}{2\pi}\left(\frac{e^{-j\beta R_0}}{R_0}\right)\eta_0\beta \sin(\beta d\cos\theta)\sqrt{1-\sin^2\theta\sin^2\phi}.$$

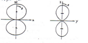
Pattern function $F(\theta,\phi) = |\sin(\beta d\cos\theta)|\sqrt{1-\sin^2\theta\sin^2\phi}$.

a) In the xy-plane: $\theta = 90°$, $F_{xy}(90°,\phi) = 0$.

b) In the xz-plane: $\phi = 0°$, $E_\phi = 0$, $F_{xz}(\theta,0°) = |\sin(\beta d\cos\theta)|$.

c) In the yz-plane: $\phi = 90°$, $E_\phi = 0$, $F_{yz}(\theta,90°) = |\sin(\beta d\cos\theta)\cdot\cos\theta|$.

d) $d = \lambda/4$, $\beta d = \pi/2$:



$F_{xz}(\theta,0°) = |\sin(\frac{\pi}{2}\cos\theta)|$. $F_{yz}(\theta,90°) = |\sin(\frac{\pi}{2}\cos\theta)\cos\theta|$.

P.11-17 From Eq. (11-83): $|E| = \frac{2E_m}{R_0}\left|F(\theta,\phi)\cos\frac{\psi}{2}\right|$, where

$$\psi = \beta d\sin\theta\cos\phi + \xi.$$

In the H-plane of a dipole, $\theta = \pi/2$, $F(\frac{\pi}{2},\phi) = 1$.

a) $d = \frac{\lambda}{4}$, $\xi = \frac{\pi}{2}$.

$|f(\phi)| = \left|\cos\left[\frac{\pi}{4}(1+\cos\phi)\right]\right|$.

b) $d = \frac{3\lambda}{4}$, $\xi = \frac{\pi}{2}$.

$|f(\phi)| = \left|\cos\left(\frac{3\pi}{4}\cos\phi + \frac{\pi}{4}\right)\right|$.

P.11-18 a) Relative excitation amplitudes: 1:4:6:4:1.

b) Array factor: $|A(\phi)| = \left|\cos\left(\frac{\pi}{2}\cos\phi\right)\right|^4$.

c) $\cos\left(\frac{\pi}{2}\cos\phi\right) = (\sqrt{2})^{-1/4}$

$\longrightarrow \phi = 74.86°$.

Half-power beamwidth

$= 2(90° - 74.86°)$

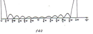
$= 30.28°$.

For uniform array, from Eq. (11-89):

$$\frac{1}{5}\left|\frac{\sin(\frac{5\pi}{2}\cos\phi)}{\sin(\frac{\pi}{2}\cos\phi)}\right| = \frac{1}{\sqrt{2}} \longrightarrow \phi = 79.61°$$

Half-power beamwidth for 5-element uniform array with $\lambda/2$ spacing $= 2(90° - 79.61°) = 20.78°$.

P.11-19 a) From Eq. (11-89) for $N=12$: $|A(\psi)| = \frac{1}{12}\left|\frac{\sin 6\psi}{\sin(\psi/2)}\right|$.



b) Broadside Operation. $\psi = \beta d \cos \phi$.

$$|A(\psi)| = \frac{1}{N}\left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right| \cong \left|\frac{\sin X}{X}\right| \quad \text{for } \psi \ll 1,$$

where $X = N\psi/2$.

At half-power points: $\left|\frac{\sin X}{X}\right| = \frac{1}{\sqrt{2}} \longrightarrow X = 1.391$

(for both broadside & endfire operations).

For broadside operation, the half-power beamwidth is $(2\Delta\phi)_{1/2} = 0.886\left(\frac{\lambda}{Nd}\right)$ (rad)

$= 50.75\left(\frac{\lambda}{Nd}\right)$ (deg).

For $N=12$, $(2\Delta\phi)_{1/2} = 4.23\left(\frac{\lambda}{d}\right)$ (deg).

From Eq. (11-92): $(2\Delta\phi)_0 = 9.55\left(\frac{\lambda}{d}\right)$ (deg).

c) Endfire Operation. $\psi = \beta d(\cos\phi - 1)$.

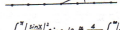
$(2\Delta\phi)_{1/2} = 1.882\sqrt{\frac{\lambda}{Nd}}$ (rad.) $= 107.8\sqrt{\frac{\lambda}{Nd}}$ (deg).

For $N=12$, $(2\Delta\phi)_{1/2} = 31.13\sqrt{\frac{\lambda}{d}}$ (deg).

From Eq. (11-93): $(2\Delta\phi)_0 = 46.78\sqrt{\frac{\lambda}{d}}$ (deg).

P.11-20 $|A(\psi)| = \frac{1}{N}\left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right| \cong \left|\frac{\sin X}{X}\right|$, where $X = \frac{-N\psi}{2}$.

Assume broadside operation: $\psi = \beta d \cos\theta$.



$$D = \frac{4\pi |A(\psi)_{max}|^2}{\int_0^{2\pi}\int_0^{\pi}\left|\frac{\sin X}{X}\right|^2 \sin\theta \, d\theta \, d\phi},$$

$|A(\psi)_{max}| = 1$,

$$\int_0^{\pi}\left|\frac{\sin X}{X}\right|^2 \sin\theta \, d\theta \cong \frac{4}{N\beta d}\int_0^{\infty}\left|\frac{\sin X}{X}\right|^2 dX = \frac{4}{N\beta d}\left(\frac{\pi}{2}\right) = \frac{\lambda}{Nd}.$$

$\therefore \; D = \frac{2Nd}{\lambda} \cong \frac{2L}{\lambda}$, where L = array length.

P.11-21 Construction follows the steps outlined on pp. 629-630.

P_0 is at $\xi = -\frac{\pi}{2}$.

Radius of circle is $\beta d = \frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right) = \frac{\pi}{2}$.

P.11-22 a) $A(\psi) = 1+\xi$, $\xi = e^{j\psi}$. ψ_0 at π

b) $A(\psi) = (1+\xi)^2$. A double-zero $\psi_{01} = \psi_{02} = \pi$.

c) $A(\psi) = \frac{\xi^5 - 1}{\xi - 1}$. Zeros at ψ_{0m} at $\frac{2\pi}{5}m$, $m = 1, 2, 3, 4$.

d) $A(\psi) = (1+\xi+\xi^2)^2 = \left(\frac{\xi^3-1}{\xi-1}\right)^2$.

Double zeros $\begin{cases} \psi_{01} = \psi_{02} = +2\pi/3, \\ \psi_{03} = \psi_{04} = -2\pi/3. \end{cases}$

e) The zeros of an array polynomial specify the nulls in the array pattern as ϕ changes from 0 to π. $\xi = 1$ ($\psi = 0$) gives the main beam. The regions between the nulls (except the main-beam region) are sidelobes. The double-zero of the array in part (d) are more widely apar leading to a wider beamwidth and lower sidelobes (which are 3 dB lower than those of a three-element uniform array).

P.11-23 From Eqs. (11-58) and (11-89):

$$|E_\theta| = \frac{j60 I_m N_1 N_2}{R} e^{-j\beta R} \left| \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta} A_x(\psi_x) A_y(\psi_y) \right|$$

where $A_x(\psi_x) = \frac{1}{N_1} \frac{\sin(N_1\psi_x/2)}{\sin(\psi_x/2)}$, $\psi_x = \frac{\beta d_1}{2}\sin\theta\cos\phi$;

$A_y(\psi_y) = \frac{1}{N_2} \frac{\sin(N_2\psi_y/2)}{\sin(\psi_y/2)}$, $\psi_y = \frac{\beta d_2}{2}\sin\theta\cos\phi$.

$$|F(\theta,\phi)| = \frac{1}{N_1 N_2} \left| \left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right] \frac{\sin(\frac{N_1\psi_x}{2})\sin(\frac{N_2\psi_y}{2})}{\sin(\frac{\psi_x}{2})\sin(\frac{\psi_y}{2})} \right|.$$

P.11-24 From Eqs. (11-32) and (11-34): $G_D(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{W_r} = \frac{4\pi R^2 \mathcal{P}_{av}}{W_r}$ ①

Using Eqs. (11-60) & (11-62) in ①: $G_D(\theta) = 1.64 \left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right]^2$. ②

a) Substituting ② in Eq. (11-121): $A_e(\theta) = \frac{\lambda^2}{4\pi} G_D(\theta) = 0.13\lambda^2 \left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right]^2$.

b) Max. value of $A_e(\theta)$ in a) is $A_e(\pi/2) = 0.13\lambda^2$.

For $f = 10^8$ (Hz), $\lambda = c/f = 3$ (m), $A_e(\pi/2) = 1.17$ (m²).

P.11-25 a) Eq. (11-127): $E_s = -\beta^2 b^3 \left(\frac{e^{-j\beta r}}{r}\right)\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) E_i \sin\theta$.

$$P_s = \oint_S \bar{\mathcal{P}}_{av} \cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi} \frac{E_s^2}{2\eta_0} r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{8\pi}{3}\beta^4 b^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 \left(\frac{E_i^2}{2\eta_0}\right) \quad (W).$$

b) $\sigma_s = \frac{P_s}{\mathcal{P}_i} = \frac{8\pi}{3}\beta^4 b^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 = \frac{2}{3}\sigma_{bs}$.

P.11-26 From Eq. (11-132): $\frac{P_L}{P_t} = \left(\frac{\lambda}{4\pi r}\right)^2 G_{D1} G_{D2}$.

a) For half-wave dipoles: $G_{D1} = G_{D2} = 1.64$.

$f = 3 \times 10^{10}$ (Hz), $\lambda = c/f = 1$ (cm), $r = 1.5 \times 10^3$ (m).

$P_L = 7.57 \times 10^{-9} P_t = 7.57 \times 10^{-7}$ (W) $= 0.757$ (μW).

b) For Hertzian dipoles: $G_{D1} = G_{D2} = 1.50$.

$P_L = 0.633$ (μW).

P.11-27 From p.641: Earth radius = 6,380 (km),
Geosynchronous orbit radius = 42,300 (km).

$\psi = \sin^{-1}\left(\frac{6,380}{42,300}\right) = \sin^{-1} 0.151 = 8.7°$

$\psi' = 90° - 8.7° = 81.3°$

a) Two satellites would cover only $2 \times (2\psi') = 325.2° < 360°$.

Use three satellites in the equatorial plane:
$3 \times (2\psi') = 487.8° > 360°$.

$\psi' < 90°$ ⟶ cannot cover the polar regions.

b) Let P_t = Power transmitted by satellite antenna.

p_c = Power density within the cone $= \frac{G_D}{4\pi r^2} P_t$.

Area of cone cap on earth $= \int_0^{2\pi}\int_0^{\psi} r^2 \sin\theta \, d\theta \, d\phi$

$= 2\pi r^2 (1 - \cos\psi) \cong 2\pi r^2 (\psi^2/2) = \pi (r\psi)^2$.

$\therefore P_t = \pi (r\psi)^2 p_c$ ⟶ $\psi = \frac{1}{r}\sqrt{P_t/\pi p_c} = 2/\sqrt{G_D}$.

Main-lobe beamwidth $= 2\psi = 4/\sqrt{G_D}$.

P.11-28 a) From Eq. (11-132): $P_t = \frac{(4\pi r)^2}{G_t G_r \lambda_t^2} P_L$.

$$\lambda_t = \frac{3\times10^8}{14\times10^9} = 2.14\times10^{-2}\,(m),\quad G_t = 10^{(55/10)} = 3.16\times10^5;$$
$$\lambda_s = \frac{3\times10^8}{12\times10^9} = 2.50\times10^{-2}\,(m),\quad G_s = 10^{(35/10)} = 3.16\times10^3;$$
$$r = 3.65\times10^7\,(m),\quad P_L = 8\times10^{-12}\,(W).$$
$$\longrightarrow P_t = 2.7\,(W).$$

b) From Eq. (11-136): $P_t = \frac{4\pi}{\sigma_{bs}}\left(\frac{\lambda_t r^2}{A_e}\right)^2 P_L$.

$$A_e = \frac{\lambda_t^2}{4\pi} G_t = 15.7\,(m^2). \longrightarrow P_t = 1.13\,(GW).$$

P.11-29 a) From Eq. (11-132): $P_L = G^2\left(\frac{\lambda}{4\pi r}\right)^2 P_t'$, ①

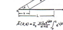
where, from Eq. (11-143), $P_t' = 4\left|\sin\left(\frac{2\pi h_1 h_2}{\lambda d}\right)\right|^2 P_t$
$= 0.0225\,P_t = 9\,(W)$.

Using ①: $P_L = 5.5\times10^{-10}\,(W) = 0.55\,(nW)$.

b) Null field at $\frac{2\pi h_1 h_2}{\lambda d} = n\pi,\quad n = 1, 2, 3,$

$$\longrightarrow h_2 = 1.25n\ \ (km).$$

P.11-30



$$I(z) = I_0 e^{-j\beta z}$$

a) $\bar{A} = \bar{a}_z \frac{\mu}{4\pi}\int_0^L \frac{I_0 e^{-j\beta z} e^{-j\beta R'}}{R'}\,dz$.

In the far-zone, $R' \cong R - z\cos\theta$.

$$\bar{A}(R,\theta) = \bar{a}_z \frac{\mu I_0 e^{-j\beta R}}{4\pi R}\int_0^L e^{-j\beta z(1-\cos\theta)}\,dz$$
$$= \bar{a}_z \frac{\mu I_0}{2\pi R\beta} e^{-j\beta R} e^{-j\beta\frac{L}{2}(1-\cos\theta)} F(\theta),$$

where $F(\theta) = \frac{\sin\left[\frac{\beta L}{2}(1-\cos\theta)\right]}{1-\cos\theta}$.

b) $A_R = A_z\cos\theta,\quad A_\theta = -A_z\sin\theta,\quad A_\phi = 0.$

$$\bar{H} = \frac{1}{\mu}\bar{\nabla}\times\bar{A} = \bar{a}_\phi \frac{1}{\mu R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right].$$

In the far-zone, $\frac{1}{R}\frac{\partial A_R}{\partial \theta} \propto \frac{1}{R^2} \longrightarrow \bar{H} = \bar{a}_\phi \frac{1}{\mu R}\frac{\partial}{\partial R}(RA_\theta)$

$$\bar{H}(R,\theta) = \bar{a}_\phi \frac{jI_0}{2\pi R} e^{-j\beta[R+L(1-\cos\theta)/2]} F(\theta)\sin\theta.$$

$$\bar{E}(R,\theta) = \bar{a}_\theta \eta_0 H_\phi(R,\theta).$$

c)

Radiation pattern for $L = \lambda/2$: Plot of $|F(\theta)\sin\theta|$.

P.11-31 From Eq. (11-74): $\bar{\ell}_e = \frac{\lambda}{\pi}(\bar{a}_x - \bar{a}_y j)$.

a) $\bar{E}_i = E_0(\bar{a}_x + \bar{a}_y jp)e^{jkz}$

From Eq. (11-76): $|V_{oc}| = |\bar{\ell}_e \cdot \bar{E}_i| = \frac{\lambda E_0}{\pi}(1+p) \longrightarrow 2\frac{\lambda E_0}{\pi}$, if $p=1$.

b) For $\bar{E}_i = E_0(\bar{a}_x - \bar{a}_y jp)e^{jkz}$:

$$|V_{oc}| = \frac{\lambda E_0}{\pi}(1-p) \longrightarrow 0, \text{ if } p=1.$$

c) For $\bar{E}_i = \bar{a}_x E_0 e^{jkz}$, $|V_{oc}| = \frac{\lambda E_0}{\pi}$.

P.11-32 Eq. (11-148): $\bar{E} = \frac{N\omega\mu_0 I}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)[\bar{a}_\theta js + \bar{a}_\phi \beta\pi b^2]\sin\theta.$

$$\bar{H} = \frac{1}{\eta_0}\bar{a}_R \times \bar{E} = \frac{N\beta I}{4\pi}\left(\frac{e^{-j\beta R}}{R}\right)[-\bar{a}_\theta \beta\pi b^2 + \bar{a}_\phi js]\sin\theta.$$

For circular polarization, $s = \beta\pi b^2$.

a) $U = R^2 \mathscr{P}_{av} = R^2\,\bar{a}_R \cdot \bar{\mathscr{P}}_{av} = R^2 \bar{a}_R \cdot \frac{1}{2}\mathcal{Re}(\bar{E}\times\bar{H}^*)$

$= \frac{\beta^2\eta_0}{16\pi^2}(NIs)^2\sin^2\theta.$

$P_r = \int_0^{2\pi}\int_0^{\pi} U \sin\theta\, d\theta\, d\phi = \frac{\beta^2\eta_0}{6\pi}(NIs)^2.$

$\therefore G_D = \frac{4\pi U}{P_r} = \frac{3}{2}\sin^2\theta$; $D = G_D(\pi/2) = 1.5$

b) $R_r = 2P_r/I^2 = \eta_0(N\beta s)^2/3\pi = \frac{4\pi}{3}\eta_0\left(\frac{Ns}{\lambda}\right)^2 = 40(N\beta^2\pi b^2)^2.$

P.11-33 Assume $\bar{E}_a(x',y') = \bar{a}_x 1$.

$$F(\theta,\phi) = \int_{-a/2}^{a/2} e^{j\beta x'\sin\theta\cos\phi}dx' \int_{-b/2}^{b/2} e^{j\beta y'\sin\theta\cos\phi}dy'$$

$$= ab\left[\frac{\sin(\frac{\beta a}{2}\sin\theta\cos\phi)}{\frac{\beta a}{2}\sin\theta\cos\phi}\right]\left[\frac{\sin(\frac{\beta b}{2}\sin\theta\cos\phi)}{\frac{\beta b}{2}\sin\theta\cos\phi}\right].$$

$$\bar{E}_p(\theta,\phi) = \frac{jab}{\lambda R_0} e^{-j\beta R_0}\left[\frac{\sin(\frac{\pi a}{\lambda}\sin\theta\cos\phi)}{\frac{\pi a}{\lambda}\sin\theta\cos\phi}\right]\left[\frac{\sin(\frac{\pi b}{\lambda}\sin\theta\cos\phi)}{\frac{\pi b}{\lambda}\sin\theta\cos\phi}\right](\bar{a}_\theta\cos\phi - \bar{a}_\phi\sin\phi)$$

P.11-34 From Eq. (11-161):

$$F(\theta,\phi) = \iint_{aper.} E_a(x',y')\, e^{j\beta\sin\theta(x'\cos\phi + y'\sin\phi)}dx'dy'$$

a) In the xz-plane, $\phi = 0°$:

$$F_{xz}(\theta) = b\int_{-a/2}^{a/2} f(x')\, e^{j\beta x'\sin\theta}dx'$$

$$= 2b\int_0^{a/2}\left(1 - \frac{2}{a}x'\right)\cos(\beta x'\sin\theta)\,dx'$$

$$= ab\,\frac{1-\cos(\frac{\beta a}{2}\sin\theta)}{(\frac{\beta a}{2}\sin\theta)^2}.$$

Let $\psi = \frac{\beta a}{2}\sin\theta = \frac{\pi a}{\lambda}\sin\theta$

$$F_{xz}(\psi) = \frac{ab}{2}\left[\frac{\sin(\psi/2)}{(\psi/2)}\right]^2.$$

b) Set $\left[\frac{\sin(\psi/2)}{(\psi/2)}\right]^2 = \frac{1}{\sqrt{2}} \longrightarrow \frac{\psi}{2} = 1.005.$

Half-power beamwidth $(2\Delta\theta)_{1/2} = 2\sin^{-1}(0.640\frac{\lambda}{a})$.

For $\lambda/a \ll 1$, $(2\Delta\theta)_{1/2} \cong 1.280\frac{\lambda}{a}$ (rad)

$= 73.3\frac{\lambda}{a}$ (deg.).

c) Set $\frac{\psi}{2} = \pi \longrightarrow \theta_{01} = \sin^{-1}(\frac{2\lambda}{a}) \cong \frac{2\lambda}{a}$ (rad)

$= 114.6\frac{\lambda}{a}$ (deg).

d) First sidelobes occur at $\psi/2 = 3\pi/2$,

where $F_{xz} = \frac{ab}{2}\left[\frac{1}{3\pi/2}\right]^2$.

$\therefore$ Level of first sidelobes, $L_1 = -20\log_{10}\left(\frac{1}{3\pi/2}\right)^2 = 26.9$ (dB).

Comparison of Results:

	Uniform Distr.	Triangular Distr.
Pattern function	$ab\left(\frac{\sin\psi}{\psi}\right)$	$\frac{ab}{2}\left(\frac{\sin\frac{\psi}{2}}{\frac{\psi}{2}}\right)^2$
Half-power beamwidth	$50\frac{\lambda}{a}$ (deg)	$73.3\frac{\lambda}{a}$ (deg)
Location of first null	$57.3\frac{\lambda}{a}$ (deg)	$114.6\frac{\lambda}{a}$ (deg)
First-side-lobe level	13.3 (dB)	26.9 (dB)

P.11-35 a) In the xz-plane, $\phi = 0°$:

$$F_{xz}(\theta) = 2b\int_0^{a/2} \cos\left(\frac{\pi x}{a}\right)\cos(\beta x'\sin\theta)\,dx'$$

$$= \frac{2ab}{\pi}\left[\frac{(\pi/2)^2\cos\psi}{(\pi/2)^2-\psi^2}\right], \quad \psi = \frac{\beta a}{2}\sin\theta = \frac{\pi a}{\lambda}\sin\theta.$$

b) Set $\frac{(\frac{\pi}{2})^2\cos\psi}{(\frac{\pi}{2})^2-\psi^2} = \frac{1}{\sqrt{2}} \longrightarrow \psi = 1.868.$

Half-power beamwidth $(2\Delta\theta)_{1/2} = 2\sin^{-1}(0.594\frac{\lambda}{a})$.

For $\lambda/a \ll 1$, $(2\Delta\theta)_{1/2} \cong 1.188\frac{\lambda}{a}$ (rad)

$= 68.1\frac{\lambda}{a}$ (deg).

c) Set $\psi = \frac{3\pi}{2} \longrightarrow \theta_{n1} = \sin^{-1}\left(\frac{3\lambda}{2a}\right) \cong \frac{3\lambda}{2a}$ (rad)

$= 85.9\frac{\lambda}{a}$ (deg).

d) At first sidelobe, $\psi \cong 2\pi$:

$$L_1 = -20\log_{10}\frac{(\frac{\pi}{2})^2}{(\frac{\pi}{2})^2-(2\pi)^2} = 20\log_{10}15 = 23.5 \text{ (dB)}.$$

	Uniform Distr.	Cosine Distr.
Pattern function	$ab\left(\frac{\sin\psi}{\psi}\right)^2$	$\frac{2ab}{\pi}\frac{\cos\psi}{1-(\frac{2}{\pi}\psi)^2}$
Half-power beamwidth	$50\frac{\lambda}{a}$ (deg)	$68.1\frac{\lambda}{a}$ (deg)
Location of first null	$57.3\frac{\lambda}{a}$ (deg)	$85.9\frac{\lambda}{a}$ (deg)
First-side-lobe level	13.3 (dB)	23.5 (dB)

The following corrections should be made to Field and Wave Electromagnetics by David K. Cheng. We apologize for any inconvienience that may result.

ERRATA SHEET

"Field and Wave Electromagnetics" by David K. Cheng (2nd. Ed.)

P. vii, 2nd paragraph, 7th line: filters ⟶ fibers.

P. 9, Eq. (1-9): Add (in free space) after the boxed equation—as in Eq.(1-8).

P. 29, Fig. 2-15: Two dashed lines for the bottom face of the volume are left out. The dot for $d\mathbf{s}_1$ should be put at the center of the bottom face. The dot for $d\mathbf{s}_2$ should be put at the center of the inner face.

P. 44, Eq. (2-96): $\mathbf{a}_z \longrightarrow \mathbf{a}_R$.

P. 70, problem P.2-34: $3x^2y^2 \longrightarrow 3x^2y^3$.

P. 125, Fig. 3-30: Add c after comma.

P. 146, problem P.3-10: density ⟶ intensity.

P. 257, Eq. (6-106), in the numerator: $- \longrightarrow +$.

P. 297, problem 6-6, 1st. line: Add 'a radius b and 'after 'having'.

p. 297, problem 6-9: (6-32) ⟶ (6-33).

p. 300, Eq. (6-224): Insert '-' sign after the '=' sign.

P. 345, Table 7-5, 5th line: The letter K in Ku is not sharp.

P. 374, Eq. (8-66): $c \longrightarrow c_0$; Eq. (8-68): $n_1 \longrightarrow n_2$; 5th line from bottom: $n_1 \longrightarrow n_2$

P. 408, line 5: Insert (transmission) after the word 'refraction'.

P. 422, problem P.8-27: Insert (V/m) before the word 'is'.

P. 423, problem P.8-29, b): $\lambda/2 \longrightarrow \lambda_1/2$.

P. 424, problem 8-34, line 2: Change 'the wave frequency to be' to 'a constant electron density and a wave frequency equal to'.

P. 426, line 2: $\theta_i \longrightarrow \theta_t$ problem P.8-42, line 3: reflection ⟶ refraction.

P. 476, Fig. 9-18(b): Color the segment between 0 and f_p. (See Fig.9-20, p.477.)

P. 481, 3rd. line from bottom: Fig.9-24(d), ⟶ Fig.9-25(d).

P. 517, problem P.9-38: 3rd. line: $R_0 \longrightarrow R$; R missing in Fig.9-46.

P. 537, Fig.10-6: Insert colored —— before 'Electric field lines' and ⊙ ⊗ before 'Magnetic field lines'.

P. 542, Eq. (10-95): [illegible] → [illegible] (2 places); Eq. (10-96): [illegible] → [illegible].

P. 598, problem P.10-41, 1st line: Change "An air-filled' to 'A';
2nd line: Delete 'made of copper'.

P. 599, problem P.10-44(b): Change 'cutoff' to 'resonant'.

P. 625, paragraph 3, line 3: $\binom{N}{n} \rightarrow \binom{N-1}{n}$.

P. 649, 1st. table, line 2: 0.451λ → 0.510λ.

P. 685, line 4: $z \geq b \rightarrow z \geq c$.

P. 687, problem P.8-27 b): $\mathscr{P}_{av}$. (more space.)

P. 688, problem P.9-15 b): 0.114π → 0.432π.

P. 690, problem P.11-9: 32.0 → 320.

Back endpaper, left: $\oint_s \rightarrow \oint_s$ (See Eq.2-115, p. 50 and Eq.2-143, p. 59.) Larger circles.

Figures in which the registers of color overlays are off:

P.2, Fig. 1-1.(Colored line should be centered on the circle.)

P.16, Fig. 2-6.

P.46, Fig.25(b). (q should be at the center.)

P.68, Fig.2-34.

P.76, Fig. 3-1. (P_2 should be on the curve.)

P.83, Fig. 3-5. Point +q should be colored; the arrow should be moved up to touch +q; the point at P should be at P.) See Fig.3-14 on p.95.

P.179, Fig.4-16 (2nd vertical colored line should coincide with the V(0,y) axis. See Fig.4-20 on p.186.

P.261, Fig.6-18: The arrows should pass through the centers of the circles.

P.297, Fig.6-37; P.480, Fig.9-24; P.550, Fig.10-11; P.556, Fig.10-14.