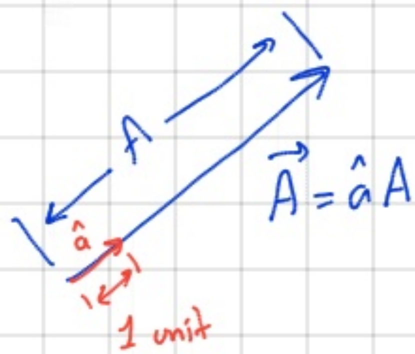
3.1 Basic Laws of Vector Algebra

- A vector has a magnitude and a direction
- Vector $\vec{A}$ has magnitude (or length)

$$A = |\vec{A}|$$

and a unit vector $\hat{a}$:

$$\vec{A} = \hat{a}|\vec{A}| = \hat{a}A$$



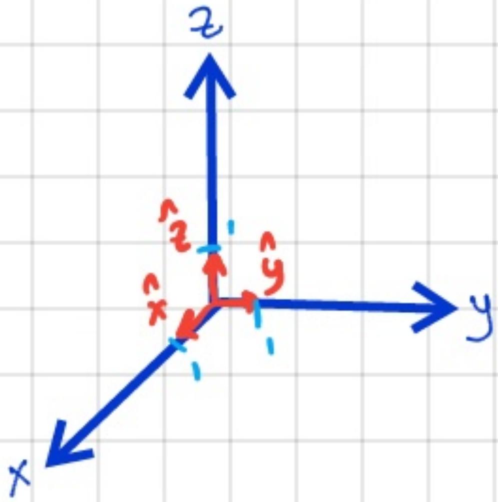
- The unit vector has a magnitude of one and a direction parallel to the vector. (i.e. same direction). e.g. 1m or 1 cm

Since $\vec{A} = \hat{a}A$

$$\Rightarrow \frac{\vec{A}}{A} = \hat{a}\frac{A}{A}$$

$$\Rightarrow \hat{a} = \frac{\vec{A}}{A}$$

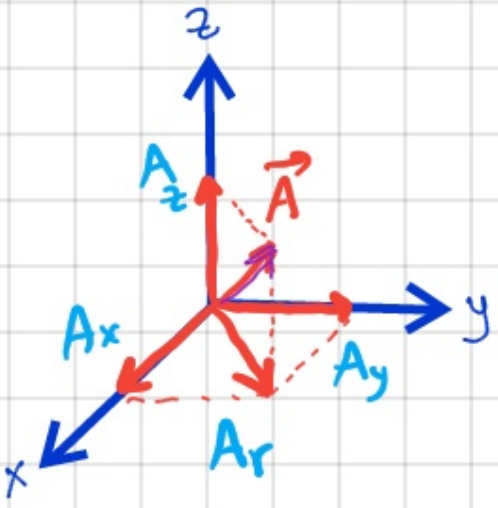
- In the Cartesian (or rectangular) coordinate system the unit vectors in the x, y, and z axes are $\hat{x}$, $\hat{y}$, and $\hat{z}$, respectively, also called base vectors.



- For a vector $\vec{A}$ in the below figure

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

where A_x, A_y, and A_z are $\vec{A}$'s scalar components along the x, y, and z axes, respectively.



A_z: is the perpendicular projection of $\vec{A}$ onto z-axis

A_r: " " " " " $\vec{A}$ onto xy-plane

- Then the magnitude of $\vec{A}$ is :

$$A = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

note : only the positive root since A is a nonnegative scalar (distance)

- The unit vector $\hat{a}$ is

$$\hat{a} = \frac{\vec{A}}{A} := \frac{\vec{A}}{|\vec{A}|}$$

$$= \frac{A_x \hat{x} + A_y \hat{y} + A_z \hat{z}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

- $\vec{A}$ could be shortened as

$$\vec{A} = (A_x , A_y , A_z)$$

- Two vectors $\vec{A}$ & $\vec{B}$ are equal if they have equal magnitudes & identical unit vectors. Thus, if

$$\vec{A} = A\,\hat{a} = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}$$

$$\vec{B} = B\,\hat{b} = B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}$$

then $\vec{A} = \vec{B}$ if and only if

$$A = B \quad \text{and} \quad \hat{a} = \hat{b},$$

which requires that

$$A_x = B_x, \quad A_y = B_y, \quad A_z = B_z.$$

Note: Equal vectors does not imply identical vectors, since parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if the lie on top of one another.

 : equal  : identical

- The sum of two vectors $\vec{A}$ & $\vec{B}$ is a vector.

- Let the sum be denoted by a vector $\vec{C}$,

then: $\vec{C} = \vec{A} + \vec{B}$

$$= (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) + (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

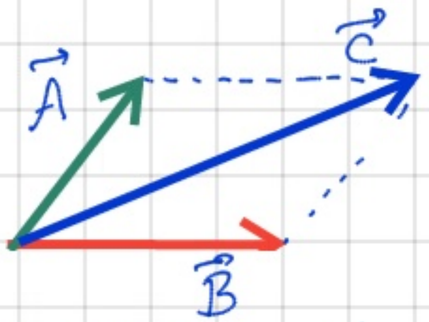
$$= (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$$

$$= C_x\hat{x} + C_y\hat{y} + C_z\hat{z}$$

- Thus, vector addition is commutative (تبديلي):

$\vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$

Graphically:

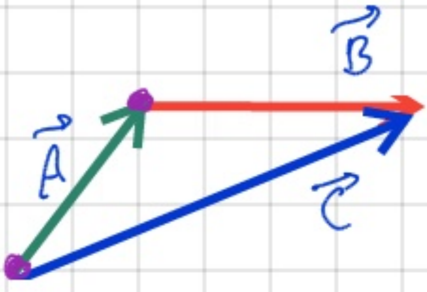


Parallelogram rule

$\vec{A} + \vec{B}$ or
$\vec{B} + \vec{A}$

head to tail rule

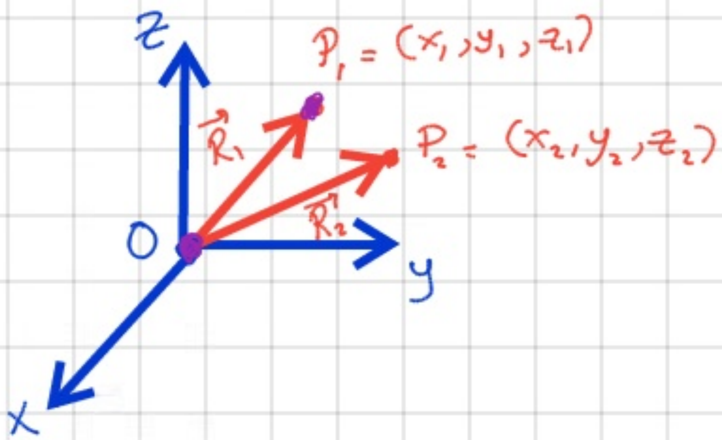
$\vec{B} + \vec{A}$



head to tail rule

$\vec{A} + \vec{B}$

- The position vector of a point $\underline{P}$ in space is the vector from the origin to $\underline{P}$.
- Let points P_1 & P_2 be at (x_1, y_1, z_1) & (x_2, y_2, z_2), respectively. For $\underline{O}$ denoting the origin, then their position vectors are:



$$\vec{R}_1 = \overrightarrow{OP_1} = x_1\hat{x} + y_1\hat{y} + z_1\hat{z}$$
$$\vec{R}_2 = \overrightarrow{OP_2} = x_2\hat{x} + y_2\hat{y} + z_2\hat{z}$$

- The displacement vector from P_1 to P_2 is defined as:

$$\vec{R}_{12} = \overrightarrow{P_1P_2} = \vec{R}_2 - \vec{R}_1 \quad \text{: (final position - initial position)}$$
$$= (x_2 - x_1)\hat{x} + (y_2 - y_1)\hat{y} + (z_2 - z_1)\hat{z}$$

and the distance d between P_1 & P_2 equals the magnitude of $\vec{R}_{12}$ as :

$$d = |\vec{R}_{12}|$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Simple Product : the multiplication of a vector by a scalar.
- For a vector $\vec{A} = A\,\hat{a}$ and a scalar k

Let $\vec{B} = k\vec{A}$, then

$$\vec{B} = k\vec{A}$$

$$= k\,A\,\hat{a}$$

$$= kA_x\hat{x} + kA_y\hat{y} + kA_z\hat{z}$$

$$= B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

- Scalar or dot product:

$\vec{A} \cdot \vec{B}$ ("A dot B") is defined geometrically as the product of the magnitude of $\vec{A}$ and the scalar component of $\vec{B}$ along $\vec{A}$, or vice versa.

Thus $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta_{AB}$

$= A\ B \cos\theta_{AB}$

where θ_{AB} is the angle between $\vec{A}$ & $\vec{B}$ measured from the tail of $\vec{A}$ to the tail of $\vec{B}$.

- The range of θ_{AB} is $0 \leq \theta_{AB} \leq 180$.

- $\vec{A} \cdot \vec{B} \leq |\vec{A}|\,|\vec{B}|$:

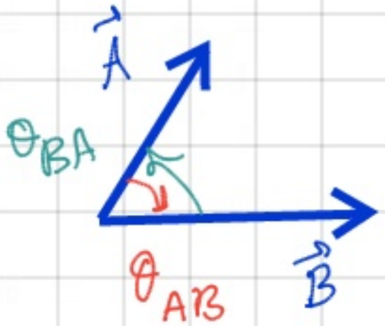
where $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|$ when $\theta_{AB} = 0$ $(\cos 0 = 1)$

and $\vec{A} \cdot \vec{B} = 0$ when $\theta_{AB} = 90$ $(\cos 90 = 0)$
(i.e. $\vec{A}$ & $\vec{B}$ are orthogonal)

and $\vec{A} \cdot \vec{B} < 0$ when $90 < \theta_{AB} < 180$

and $\vec{A} \cdot \vec{B} > 0$ when $0 < \theta_{AB} < 90$

- Remember



- In $\vec{A}\cdot\vec{B}$ the quantity $A\cos\theta_{AB}$ is the scalar component of $\vec{A}$ along $\vec{B}$.

  Similarly $B\cos\theta_{BA}$ is the salar component of $\vec{B}$ along $\vec{A}$.

- $\hat{x}\cdot\hat{x} = |\hat{x}||\hat{x}|\cos\theta = (1)(1)\cos(0) = 1$

  $\hat{y}\cdot\hat{y} = 1 \quad , \quad \hat{z}\cdot\hat{z} = 1$

- $\hat{x}\cdot\hat{y} = |\hat{x}||\hat{y}|\cos\theta = (1)(1)\cos(90) = 0$

  $\hat{y}\cdot\hat{x} = 0 \,,\ \hat{y}\cdot\hat{z} = 0 \,,\ \hat{z}\cdot y = 0 \,,\ \hat{x}\cdot\hat{z} = 0 \,,\ \hat{z}\cdot\hat{x} = 0$

- The dot product is commutative & distributive:

- Commutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

Proof: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

$$\begin{cases} (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) = \\ (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})\cdot(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \end{cases}$$

$$\begin{cases} A_x\hat{x}\cdot(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) + A_y\hat{y}\cdot(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) \\ \quad + A_z\hat{z}\cdot(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) = \\ B_x\hat{x}\cdot(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) + B_y\hat{y}\cdot(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \\ \quad + B_z\hat{z}\cdot(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \end{cases}$$

$$\begin{cases} A_xB_x + A_yB_y + A_zB_z = B_xA_x + B_yA_y + B_zA_z \end{cases}$$

#

- Distributive :

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

Proof: $\vec{A}\cdot(\vec{B}+\vec{C}) = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot$

$$\left[B_x\hat{x} + B_y\hat{y} + B_z\hat{z} + C_x\hat{x} + C_y\hat{y} + C_z\hat{z}\right]$$

$$= A_xB_x + A_xC_x + A_yB_y + A_yC_y + A_zB_z + A_zC_z$$

while $\vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C} =$

$$\begin{cases} (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) + \\ (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot(C_x\hat{x} + C_y\hat{y} + C_z\hat{z}) \end{cases}$$

$$= A_xB_x + A_yB_y + A_zB_z + A_xC_x + A_yC_y + A_zC_z$$

#

- The dot product of a vector with itself is:

$$\vec{A}\cdot\vec{A} = |\vec{A}|\,|\vec{A}|\cos(0) = |\vec{A}|^2 = A^2$$

thus

$$A = |\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$$

- For $\vec{A} = (A_x, A_y, A_z)$ & $\vec{B} = (B_x, B_y, B_z)$

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

- The angle θ_{AB} can be determined from

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta_{AB}$$

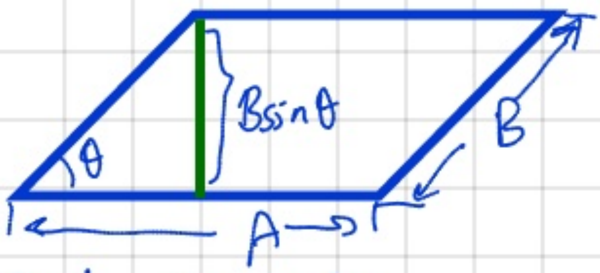
$$\Rightarrow \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{|\vec{A}|\,|\vec{B}|\cos\theta_{AB}}{|\vec{A}|\,|\vec{B}|}$$

$$\Rightarrow \cos\theta_{AB} = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$

$$\theta_{AB} = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}\right]$$

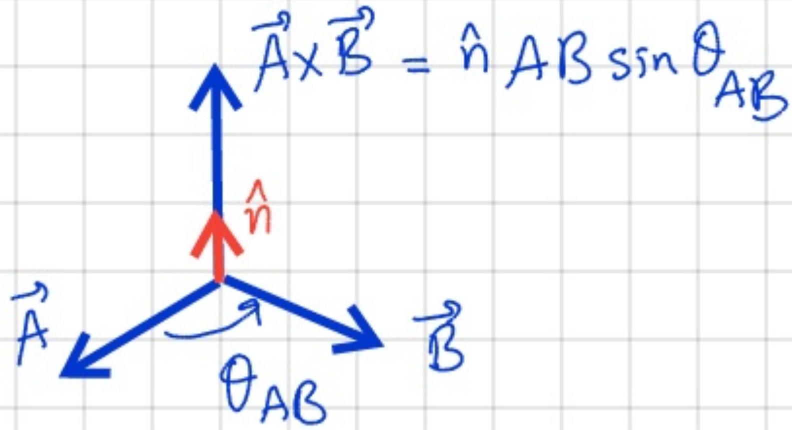
- Vector product or cross product of two vectors $\vec{A}$ & $\vec{B}$ denoted by $\vec{A} \times \vec{B}$ and prononced "A cross B" is

$$\vec{A} \times \vec{B} = \hat{n}\, AB \sin\theta_{AB}$$



where $\hat{n}$: unit vector normal to the plane containing $\vec{A}$ & $\vec{B}$.

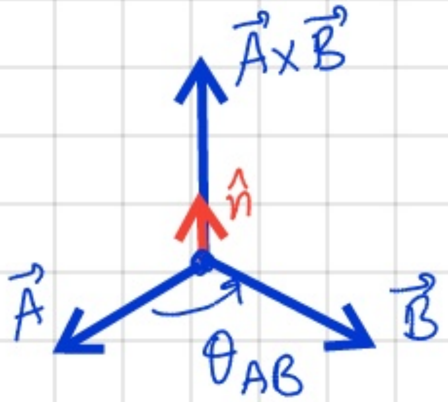
- The magnitude of the cross product, $|\vec{A} \times \vec{B}|$, is equal to the area of the parallelogram defined by the two vectors.
- The direction of $\hat{n}$ is governed by the right-hand rule.
- Here the right-hand rule states that: using the right hand, $\hat{n}$ points in the direction of the thumb when the fingers rotate from $\vec{A}$ to $\vec{B}$ through the angle θ_{AB} (14

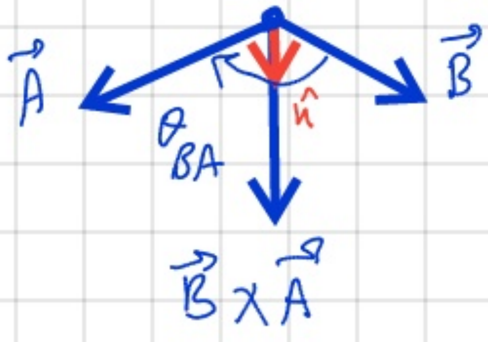


- The cross product is anti-commutative and distributive:

- Anti-commutative: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

It follows from the right-hand rule as shown bellow





- Distributive :

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

proof :

Homework

- The cross product of a vector with itself is zero: $\vec{A} \times \vec{A} = 0$

Since $\vec{A} \times \vec{A} = |\vec{A}|\,|\vec{A}| \sin(0) = 0$

- For the base vectors $\hat{x}, \hat{y}, \hat{z}$:

$$\hat{x} \times \hat{x} = 0 \quad , \quad \hat{y} \times \hat{y} = 0 \quad , \quad \hat{z} \times \hat{z} = 0$$

$$\hat{x} \times \hat{y} = \hat{z}$$

$$\hat{y} \times \hat{z} = \hat{x}$$

$$\hat{z} \times \hat{x} = \hat{y}$$

$$\hat{y} \times \hat{x} = -\hat{z}$$

$$\hat{z} \times \hat{y} = -\hat{x}$$

$$\hat{x} \times \hat{z} = -\hat{y}$$

- for $\vec{A} = (A_x, A_y, A_z)$ & $\vec{B} = (B_x, B_y, B_z)$

$$\vec{A} \times \vec{B} = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$$

$$= A_x \hat{x} \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$$

$$+ A_y \hat{y} \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$$

$$+ A_z \hat{z} \times (B_x \hat{x} + B_y \hat{y} + B_z \hat{z})$$

$$= 0 + A_x B_y \hat{z} - A_x B_z \hat{y}$$

$$- A_y B_x \hat{z} + 0 + A_y B_z \hat{x}$$

$$+ A_z B_x \hat{y} - A_z B_y \hat{x} + 0$$

$$= (A_y B_z - A_z B_y) \hat{x}$$

$$- (A_x B_z - A_z B_x) \hat{y}$$

$$+ (A_x B_y - A_y B_x) \hat{z}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

- Exercise 3-5 : If $\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$, does it follow that $\vec{B} = \vec{C}$.

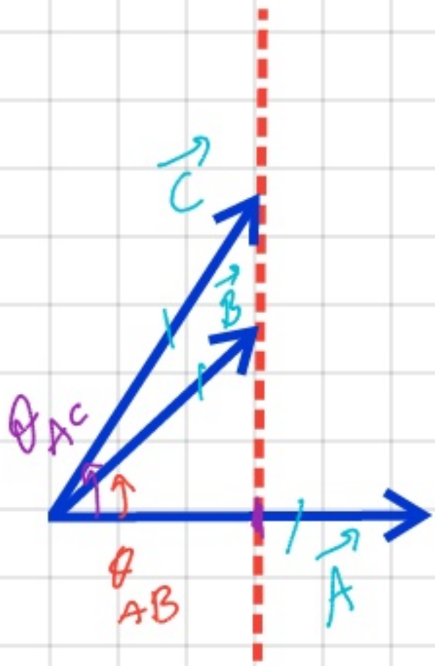
Sol.: $$\vec{A}\cdot\vec{B} = \vec{A}\cdot\vec{C}$$

$$AB\cos\theta_{AB} = AC\cos\theta_{AC}$$

$$B\cos\theta_{AB} = C\cos\theta_{AC}$$

Thus, $\vec{B} \neq \vec{C}$, however, the scalar component of $\vec{B}$ along $\vec{A}$ equals the scalar component of $\vec{C}$ along $\vec{A}$.

As an example:



- The scalar triple product is the dot product of a vector with the cross product of two other vectors, and the result is a scalar, as follows:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

where the equalities hold as long as the cyclic order (ABCABCA ...) is preserved.

- The scalar triple product of vectors $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$, and $\vec{C} = (C_x, C_y, C_z)$ can be expressed in the form of a 3x3 determinant:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Proof: LHS :

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot \Big[(B_yC_z - B_zC_y)\hat{x} - (B_xC_z - B_zC_x)\hat{y} + (B_xC_y - B_yC_z)\hat{z} \Big]$$

$$= A_x(B_yC_z - B_zC_y) - A_y(B_xC_z - B_zC_x) + A_z(B_xC_y - B_yC_x)$$

while the RHS :

$$\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = A_x(B_yC_z - B_zC_y) - A_y(B_xC_z - B_zC_x) + A_z(B_xC_y - B_yC_x)$$

#

- The vector triple product is the cross product of a vector with the cross product of two others, as follows:

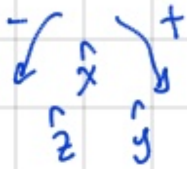
$$\vec{A} \times (\vec{B} \times \vec{C})$$

and the result is a vector.

- $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$ (not associative)

- The "bac - cab rule":

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$



Proof: LHS:

$$\vec{A} \times (\vec{B} \times \vec{C}) = (A_x \hat{x} + A_y \hat{y} + A_z \hat{z}) \times$$

$$\left[(B_y C_z - B_z C_y)\hat{x} - (B_x C_z - B_z C_x)\hat{y} + (B_x C_y - B_y C_x)\hat{z}\right]$$

$$= -A_x (B_x C_z - B_z C_x)\hat{z} - A_x (B_x C_y - B_y C_x)\hat{y}$$

$$-A_y (B_y C_z - B_z C_y)\hat{z} + A_y (B_x C_y - B_y C_x)\hat{x}$$

$$+A_z (B_y C_z - B_z C_y)\hat{y} + A_z (B_x C_z - B_z C_x)\hat{x}$$

$$LHS = \hat{x}\left[A_y(B_x C_y - B_y C_x) + A_z(B_x C_z - B_z C_x)\right]$$

$$+\hat{y}\left[A_z(B_y C_z - B_z C_y) - A_x(B_x C_y - B_y C_x)\right]$$

$$-\hat{z}\left[A_x(B_x C_z - B_z C_x) + A_y(B_y C_z - B_z C_y)\right]$$

while the RHS:

$$\vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) =$$

$$(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})(A_x C_x + A_y C_y + A_z C_z)$$

$$-(C_x\hat{x} + C_y\hat{y} + C_z\hat{z})(A_x B_x + A_y B_y + A_z B_z)$$

$$= \hat{x}\left[B_x(\cancel{A_x C_x} + A_y C_y + A_z C_z) - C_x(\cancel{A_x B_x} + A_y B_y + A_z B_z)\right]$$

$$+\hat{y}\left[B_y(A_x C_x + \cancel{A_y C_y} + A_z C_z) - C_y(A_x B_x + \cancel{A_y B_y} + A_z B_z)\right]$$

$$+\hat{z}\left[B_z(A_x C_x + A_y C_y + \cancel{A_z C_z}) - C_z(A_x B_x + A_y B_y + \cancel{A_z B_z})\right]$$

$$RHS = \hat{x}\left[A_y\left(B_x C_y - B_y C_x\right) + A_z\left(B_x C_z - B_z C_x\right)\right]$$
$$+ \hat{y}\left[A_z\left(B_y C_z - B_z C_y\right) - A_x\left(B_x C_y - B_y C_x\right)\right]$$
$$- \hat{z}\left[A_x\left(B_x C_z - B_z C_x + A_y\left(B_y C_z - B_z C_y\right)\right]$$

#

3.2 Orthogonal Coordinate Systems is when coordinates are measured along locally mutually perpendicular axes (i.e. any base vector has no scalar component along any other base vector)

(e.g. $\hat{x} \cdot \hat{y} = |\hat{x}||\hat{y}| \cos(90) = 0$: scalar component of $\hat{y}$ along $\hat{x}$, or vice versa

_ Here we will use the below three:

(1) The Cartesian (also called rectangular) coordinate system.

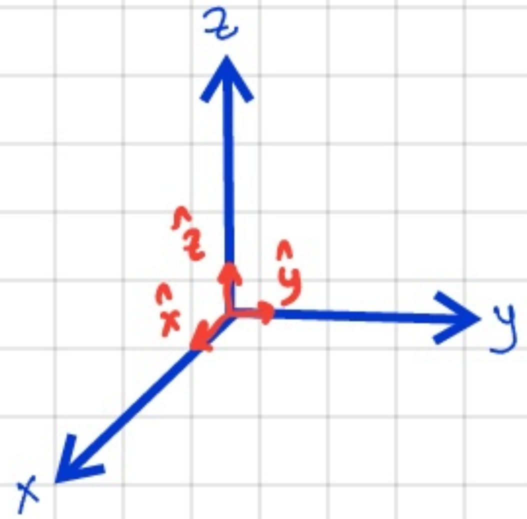
(2) The cylindrical coordinate system

(3) The spherical coordinate system

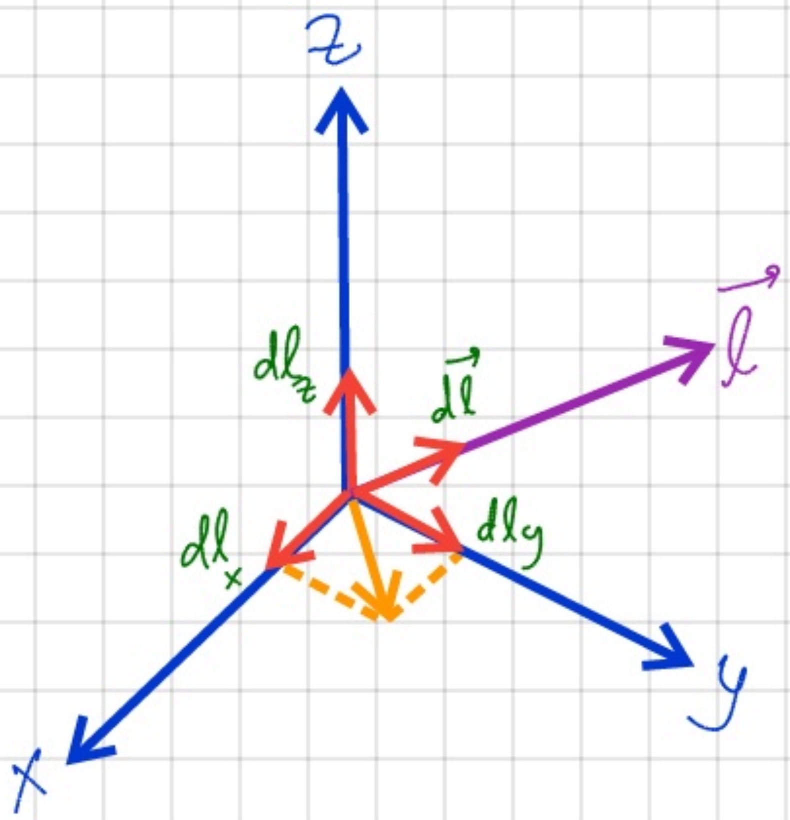
3.2.1 Cartesian Coordinates:

This coordinate system was introduced in Section 3.1 to illustrate the laws of vector Algebra.

- The coordinate variables are : x, y, z
- The unit vectors (i.e. base vectors) are $\hat{x}, \hat{y}, \hat{z}$



- The differential length vector



$$\vec{dl} = dl_x\,\hat{x} + dl_y\,\hat{y} + dl_z\,\hat{z}$$

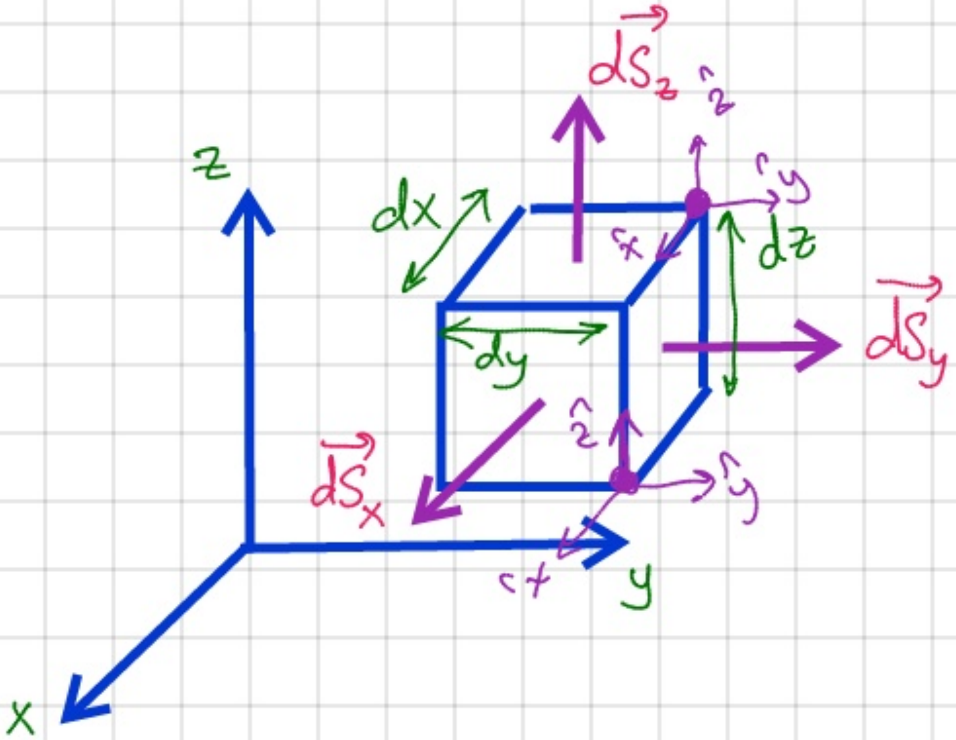
$$= dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

where $dl_x = dx$, is a differential length along $\hat{x}$

$dl_y = dy$, " " " " $\hat{y}$

$dl_z = dz$, " " " " $\hat{z}$

- The differential area vector $d\vec{S}$ is a vector with magnitude dS equal to the product of two differential lengths, and a direction normal to such area.



* A differential area vector in the :

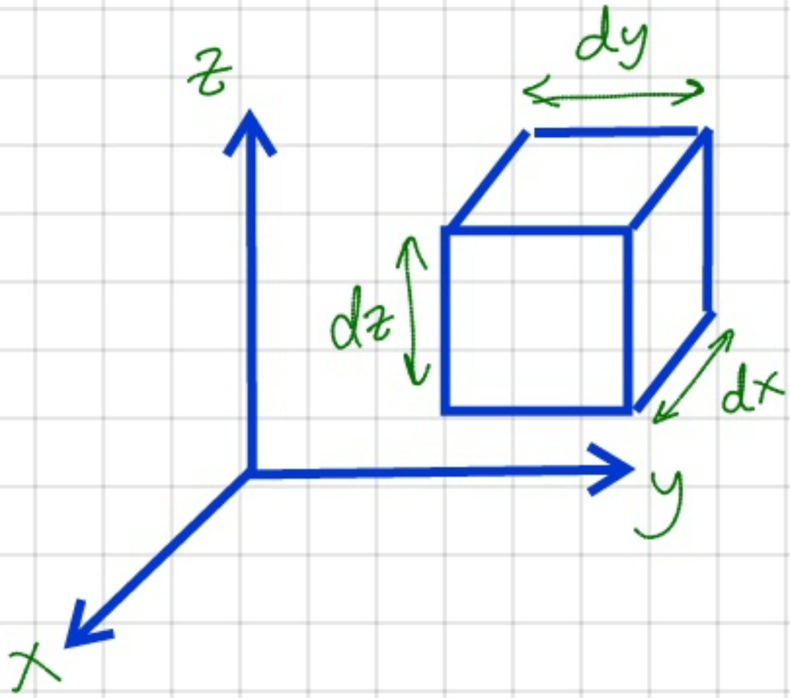
(1) yz-plane : $d\vec{S}_x = dy\,dz\,\hat{x}$

(2) xz-plane : $d\vec{S}_y = dx\,dz\,\hat{y}$

(3) xy-plane : $d\vec{S}_z = dx\,dy\,\hat{z}$

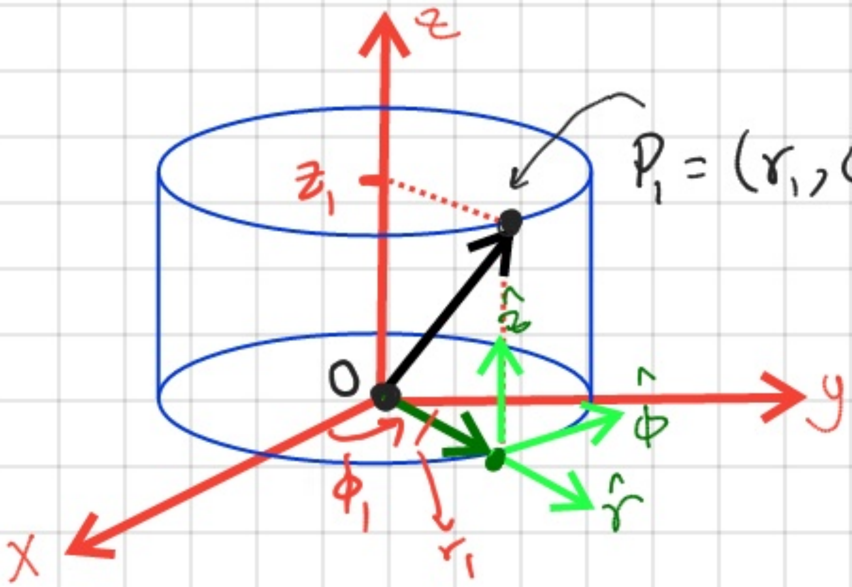
- The differential volume dv equals the product of all three differential lengths:

$$dv = dx\,dy\,dz$$



3.2.2 Cylindrical Coordinates:

- In this coordinate system, the location of a point in space is defined by three variables (r, ϕ, z).
- The coordinate r is the radial distance in the xy-plane
- The coordinate ϕ is the azimuth (السمت) angle measured from the positive x-axis
- The z is the same as in the Cartesian coordinate system.



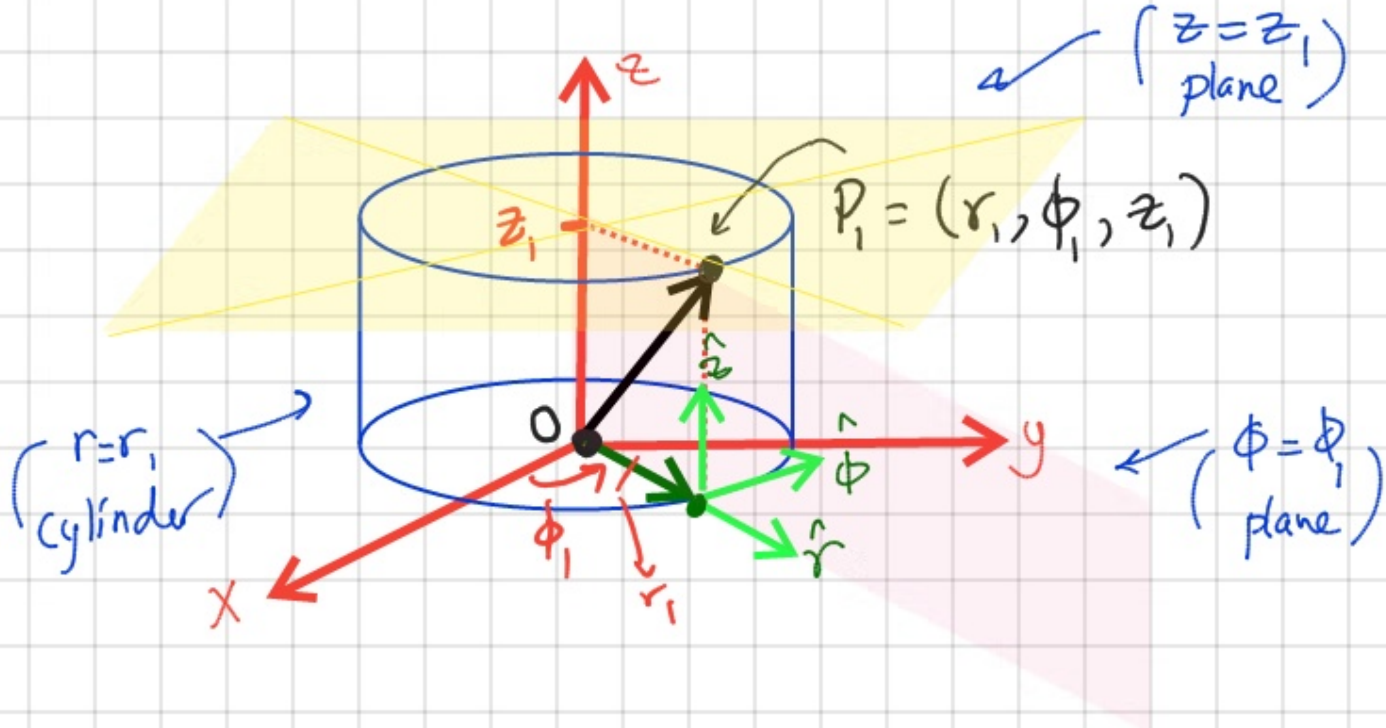
- To uniquely represent a point in the cylindrical coordinates, the range of the coordinates are:

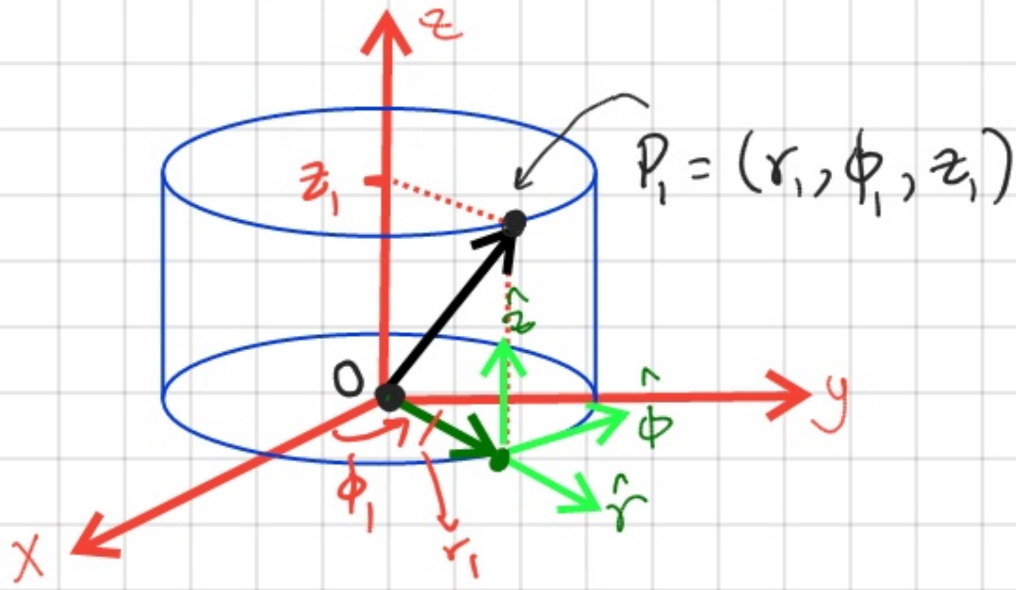
$$0 \le r < \infty$$

$$0 \le \phi < 2\pi \quad \leftarrow \text{not equal to } 2\pi$$

$$-\infty < z < \infty$$

- Point $P_1 = (r_1, \phi_1, z_1)$ is located at the intersection of three surfaces. these are the cylindrical surface defind by $r = r_1$, the vertical half-plane defined by $\phi = \phi_1$ (which extends outwardly from the z-axis), and the horizontal plane defined by $z = z_1$.





- The base unit vectors obey the right-hand cyclic rule.

$\hat{r} \times \hat{\phi} = \hat{z}$, $\hat{\phi} \times \hat{r} = -\hat{z}$

$\hat{\phi} \times \hat{z} = \hat{r}$, $\hat{z} \times \hat{\phi} = -\hat{r}$

$\hat{z} \times \hat{r} = \hat{\phi}$, $\hat{r} \times \hat{z} = -\hat{\phi}$

$$\begin{array}{ccc} & \hat{r} & \\ \hat{z} & & \hat{\phi} \end{array}$$
$-$ $+$

and $\hat{r} \times \hat{r} = 0$, $\hat{\phi} \times \hat{\phi} = 0$, $\hat{z} \times \hat{z} = 0$

and $\hat{r} \cdot \hat{r} = 1$, $\hat{\phi} \cdot \hat{\phi} = 1$, $\hat{z} \cdot \hat{z} = 1$

and $\hat{r} \cdot \hat{\phi} = 0$, $\hat{\phi} \cdot \hat{z} = 0$, $\hat{r} \cdot \hat{z} = 0$

- The vector representation in cylindrical coordinates is: $\vec{A} = |\vec{A}|\, \hat{a} = A_r \hat{r} + A_\phi \hat{\phi} + A_z \hat{z}$

where A_r, A_ϕ, and A_z are the componets of $\vec{A}$ along $\hat{r}$, $\hat{\phi}$, and $\hat{z}$ directions, respectively.

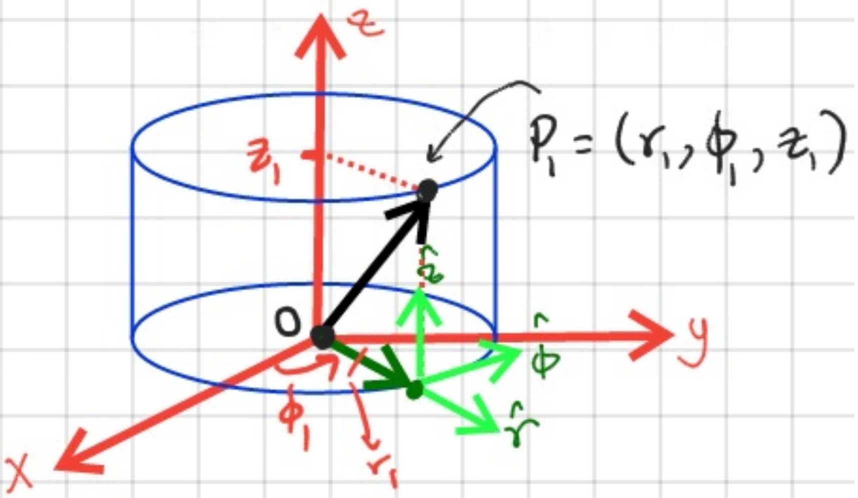
- The magnitude of $\vec{A}$ is obtained by applying

$$|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$$
$$= \sqrt{A_r^2 + A_\phi^2 + A_z^2}$$

- The position vector $\overrightarrow{OP_1}$, where $P_1 = (r_1, \phi_1, z_1)$ is: $\vec{R}_1 = \overrightarrow{OP_1} = r_1 \hat{r} + z_1 \hat{z}$ with $\hat{r}$ at ϕ_1.

(you have to specify that $\hat{r}$ is at ϕ_1)

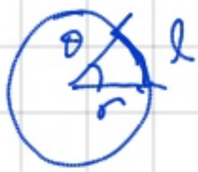
- Since the position vector is radial, it has no compont in the $\hat{\phi}$ direction.



- The differential lengths along $\hat{r}$, $\hat{\phi}$, and $\hat{z}$ are:
  - Along $\hat{r}$: $dl_r = dr$
  - Along $\hat{\phi}$: $dl_\phi = r d\phi$

Note:

circumfrance : $C = 2\pi r$

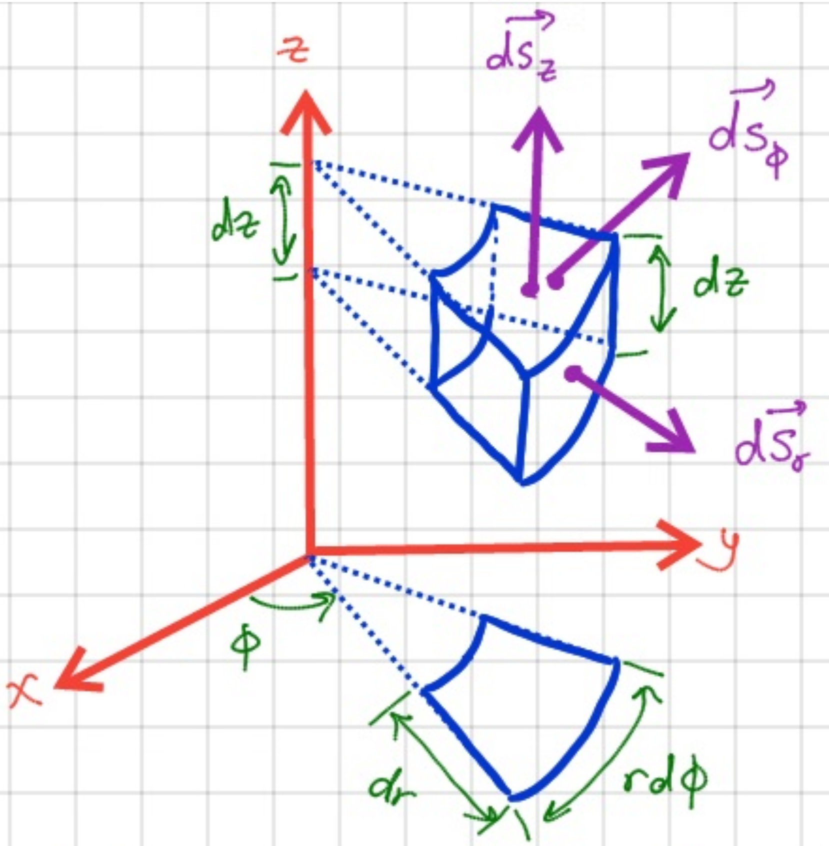


arc : $l = \theta r$

  - Along $\hat{z}$: $dl_z = dz$

- The differential length vector in the cylindrical coordinates is :

$$\vec{dl} = dl_r \hat{r} + dl_\phi \hat{\phi} + dl_z \hat{z}$$
$$= dr \hat{r} + r d\phi \hat{\phi} + dz \hat{z}$$



- The differential area vectors in the cylindrical coordinates are as follows:

* In the ϕz-cylindrical surface:

$$\vec{dS_r} = dl_\phi \, dl_z \, \hat{r} = r\,d\phi\,dz\,\hat{r}$$

* In the rz-plane:

$$d\vec{S_\phi} = dl_r \, dl_z \, \hat{\phi} = dr\,dz\,\hat{\phi}$$

* In the $r\phi$-plane:

$$d\vec{S_z} = dl_r \, dl_\phi \, \hat{z} = dr\,(r\,d\phi)\,\hat{z} = r\,dr\,d\phi\,\hat{z}$$

- The differential volume is the product of the three differential lengths:

$$dv = dl_r \, dl_\phi \, dl_z$$
$$= (dr)(r d\phi)(dz)$$
$$= r \, dr \, d\phi \, dz$$

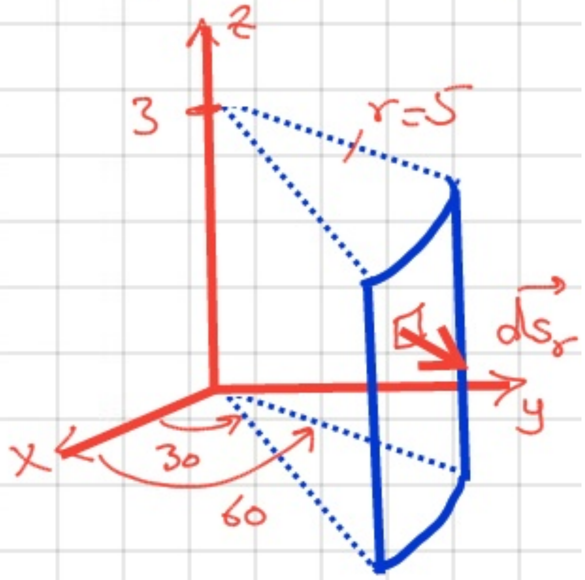
- The dot product in cylindrical coordinates is as follows; for $\vec{A} = (A_r, A_\phi, A_z)$ & $\vec{B} = (B_r, B_\phi, B_z)$

$$\vec{A} \cdot \vec{B} = A_r B_r + A_\phi B_\phi + A_z B_z$$

- The cross product in cylindrical coordinates is as follows; for $\vec{A} = (A_r, A_\phi, A_z)$ & $\vec{B} = (B_r, B_\phi, B_z)$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$$

Example: Find the area of a cylindrical surface described by $r = 5$, $30^\circ \leq \phi \leq 60^\circ$, and $0 \leq z \leq 3$.



$$ds_r = r\,d\phi\,dz$$

$$S_r = r\int_{\phi=\frac{\pi}{6}}^{\pi/3} d\phi \int_{z=0}^{3} dz$$

$$= (5)\ \phi\Big|_{\frac{\pi}{6}}^{\pi/3}\ z\Big|_{0}^{3}$$

$$= (5)\left(\frac{\pi}{3} - \frac{\pi}{6}\right)(3-0)$$

$$= 5\left(\frac{\pi}{\cancel{6}_2}\right)\cancel{3}^{1}$$

$$S_r = 5\frac{\pi}{2}$$

(Notes, ϕ has to be in radians)

Example: A circular cylinder of radius $r=5$ is concentric with the z-axis and extends between $z=-3$ and $z=3$. Find the cylinder's volume.

$$dv = r\,dr\,d\phi\,dz$$

$$v = \int_{r=0}^{5} r\,dr \int_{\phi=0}^{2\pi} d\phi \int_{z=-3}^{3} dz$$

$$= \frac{1}{2} r^2 \Big|_{r=0}^{5} \; \phi \Big|_{\phi=0}^{2\pi} \; z \Big|_{z=-3}^{3}$$

$$= \frac{1}{\cancel{2}}(25-0)(\cancel{2}\pi - 0)(3+3)$$

$$= 25(\pi)(9)$$

$$v = 150\pi$$

3.2.3 Spherical Coordinates:

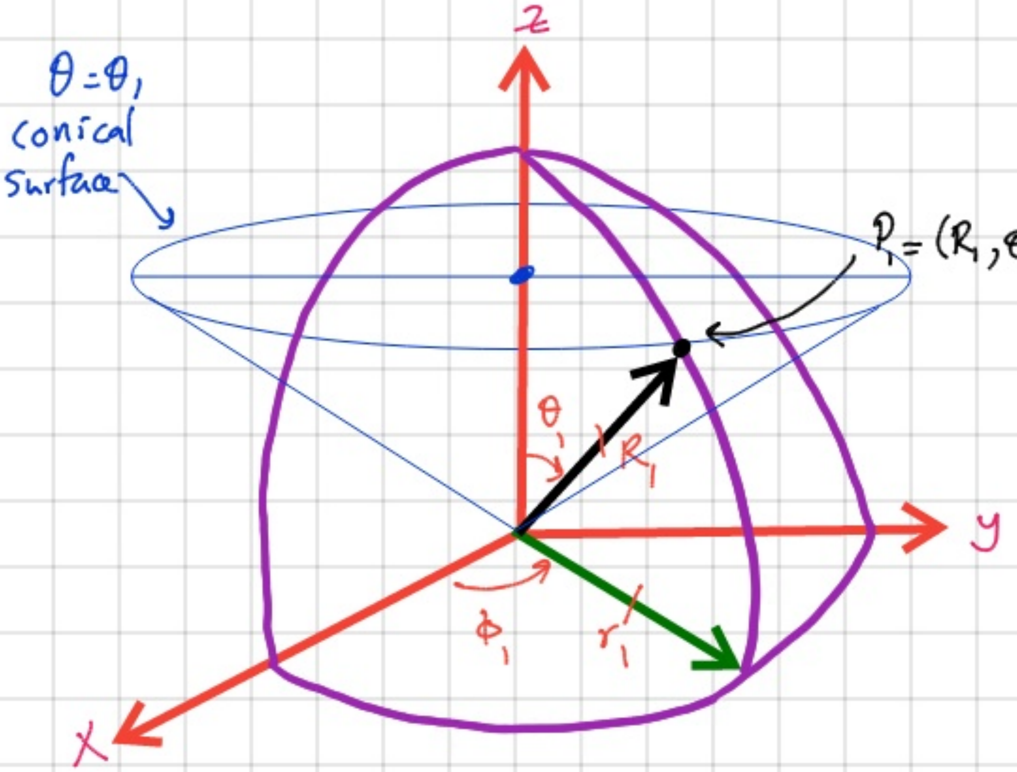
- In this coordinate system, the location of a point in space is defined by three variables (R, θ, ϕ).
- The coordinate R measures the distance from the origin to the point, it describes a sphere of radius R centered at the origin.
- The coordinate θ is the zenith angle measured from positive z-axis, it describes a conical surface with its tip at the origin.
- The coordinate ϕ is the same as in the cylindrical coordinates, which is the azimuth angle measured from the positive x-axis.

- To uniquely represent a point in the spherical coordinates, the ranges of the coordinates are:

$$0 \le R < \infty$$

$$0 \le \theta \le \pi \leftarrow \text{(only in this range)}$$

$$0 \le \phi < 2\pi$$

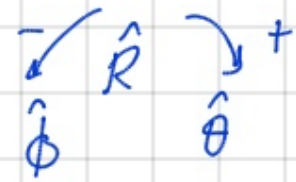


$R = R_1$ is the sphercal surface

$\phi = \phi_1$ is the vertical half-plane which extends outwardly from the z-axis

- The base unit vectors are $\hat{R}$, $\hat{\theta}$, and $\hat{\phi}$.

They obey the right-hand cyclic rule:



$$\hat{R}\times\hat{\theta} = \hat{\phi} \quad , \quad \hat{\theta}\times\hat{R} = -\hat{\phi}$$

$$\hat{\theta}\times\hat{\phi} = \hat{R} \quad , \quad \hat{\phi}\times\hat{\theta} = -\hat{R}$$

$$\hat{\phi}\times\hat{R} = \hat{\theta} \quad , \quad \hat{R}\times\hat{\phi} = -\hat{\theta}$$

and $\hat{R}\times\hat{R} = 0$, $\hat{\theta}\times\hat{\theta} = 0$, $\hat{\phi}\times\hat{\phi} = 0$

and $\hat{R}\cdot\hat{R} = 1$, $\hat{\theta}\cdot\hat{\theta} = 1$, $\hat{\phi}\cdot\hat{\phi} = 1$

and $\hat{R}\cdot\hat{\theta} = 0$, $\hat{\theta}\cdot\hat{\phi} = 0$, $\hat{\phi}\cdot\hat{R} = 0$

- The vector representation in spherical coordinates is:

$$\vec{A} = |\vec{A}|\,\hat{a} = A_R\hat{R} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

where A_R, A_θ, and A_ϕ are the components of $\vec{A}$ along $\hat{R}$, $\hat{\theta}$, and $\hat{\phi}$, respectively.

- The magnitude of $\vec{A}$ is obtained by

$$|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$$

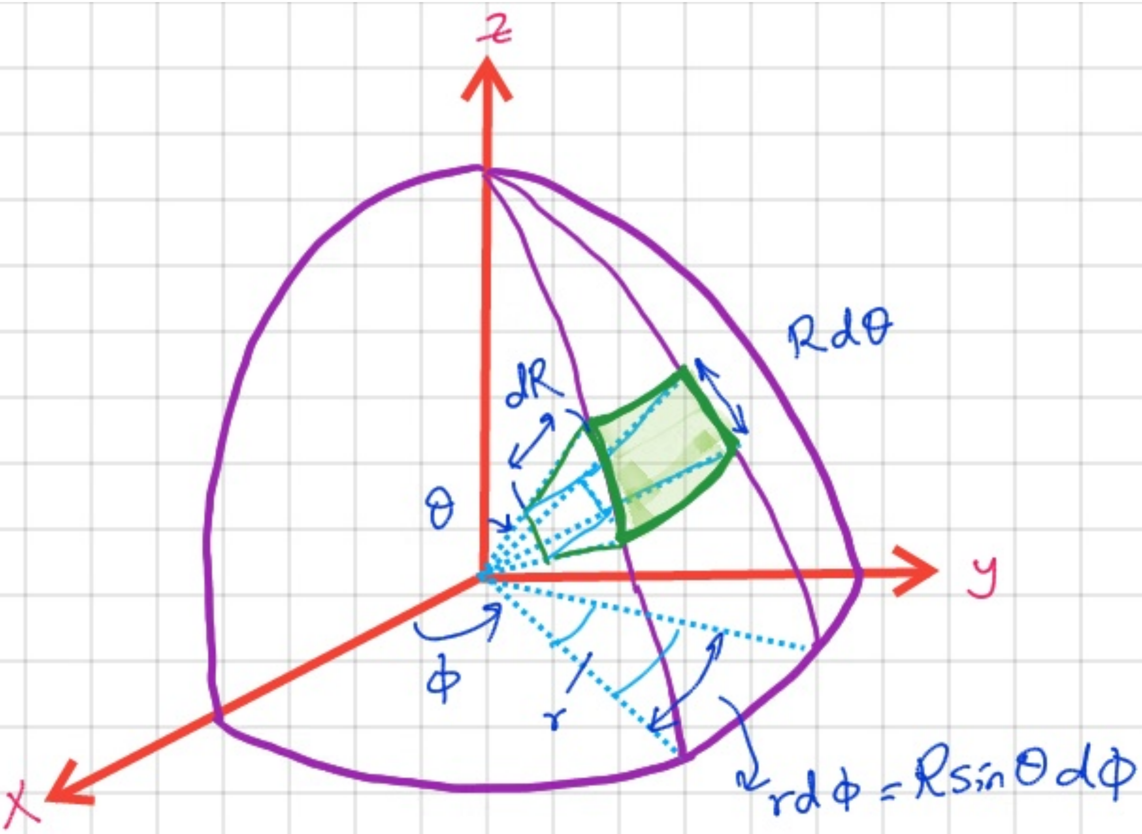
$$= \sqrt{A_R^2 + A_\theta^2 + A_\phi^2}$$

- The position vector $\vec{OP_1}$, where $P_1 = (R_1, \theta_1, \phi_1)$ is

$$\vec{R_1} = \vec{OP_1} = R_1 \hat{R},$$

where $\hat{R}$ depends on θ_1 & ϕ_1

- Since the position vector is radial, it has no components in the $\hat{\theta}$ direction & the $\hat{\phi}$ direction. It has only a component in the $\hat{R}$ direction.



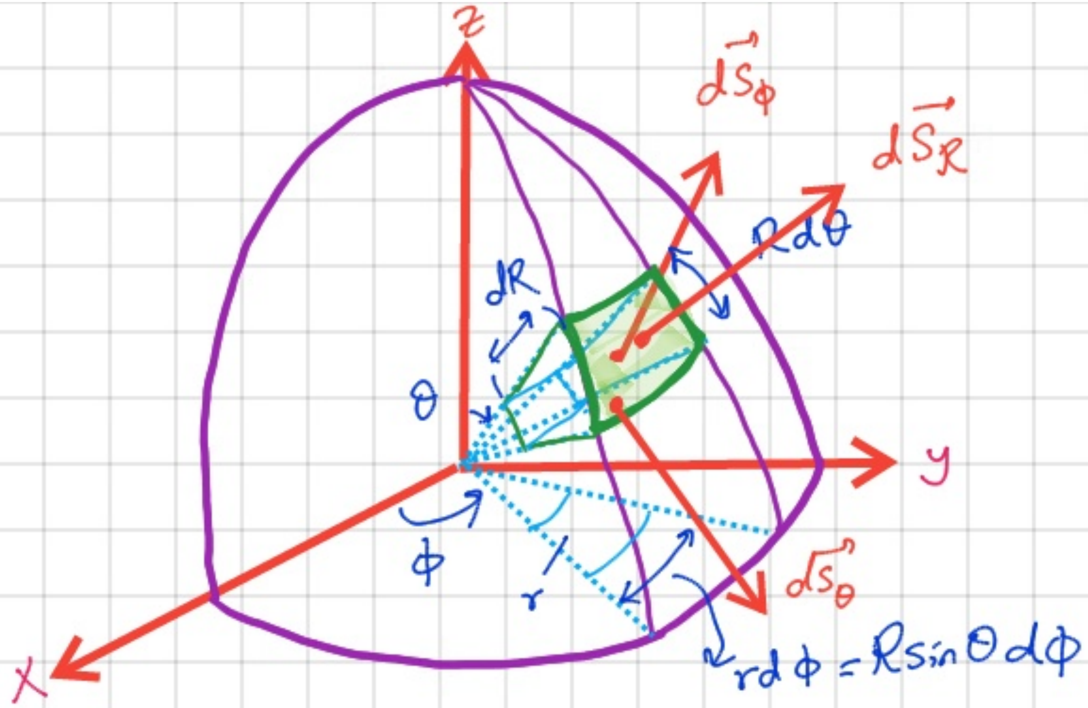
- The differential lengths along $\hat{R}$, $\hat{\theta}$, and $\hat{\phi}$ are:

* along $\hat{R}$: $dl_R = dR$
* along $\hat{\theta}$: $dl_\theta = R d\theta$
* along $\hat{\phi}$: $dl_\phi = r d\phi$ (r: in cylindrical)

$$dl_\phi = R \sin\theta \, d\phi$$

- The differential length vector in the spherical coord. is:

$$\vec{dl} = dl_R \hat{R} + dl_\theta \hat{\theta} + dl_\phi \hat{\phi}$$
$$= dR \hat{R} + R d\theta \hat{\theta} + R \sin\theta \, d\phi \hat{\phi}$$



- The differential area vectors in the spherical coord.s are;

* In the $\theta\phi$- spherical surface (هو سطح الكرة)

$$\vec{dS_R} = dl_\theta\, dl_\phi\, \hat{R} = (R d\theta)(R\sin\theta\, d\phi)\,\hat{R}$$
$$= R^2 \sin\theta\, d\theta\, d\phi\, \hat{R}$$

* In the $R\phi$- conical surface (سطح المخروط)

$$\vec{dS_\theta} = dl_R\, dl_\phi\, \hat{\theta} = dR\,(R\sin\theta\, d\phi)\,\hat{\theta}$$
$$= R\sin\theta\, dR\, d\phi\, \hat{\theta}$$

* In the $R\theta$- plane (R الزاوية التي تقطع)

$$\vec{dS_\phi} = dl_R\, dl_\theta\, \hat{\phi} = dR\,(R d\theta)\,\hat{\phi} = R\, dR\, d\theta\, \hat{\phi}$$

- The differential volume is the product of the three differential lengths:

$$dv = dl_R \, dl_\theta \, dl_\phi$$
$$= (dR)(R\,d\theta)(R\sin\theta\,d\phi)$$
$$= R^2 \sin\theta \, dR \, d\theta \, d\phi$$

- The dot product in spherical coord. is as follows; for $\vec{A} = (A_R, A_\theta, A_\phi)$ & $\vec{B} = (B_R, B_\theta, B_\phi)$

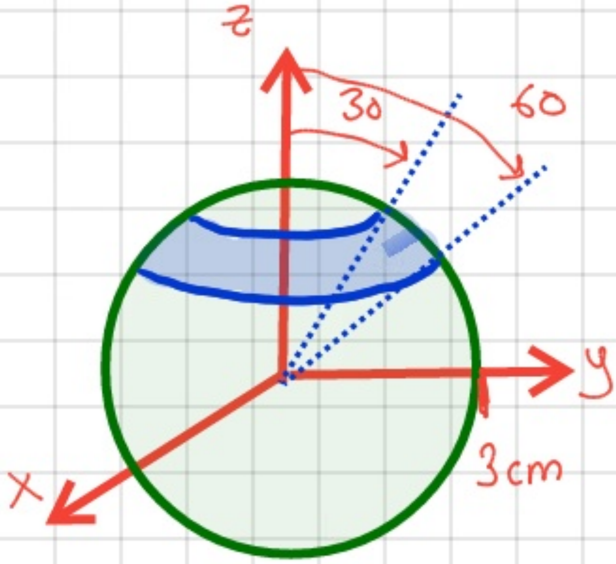
$$\vec{A} \cdot \vec{B} = A_R B_R + A_\theta B_\theta + A_\phi B_\phi$$

- The cross product in spherical coord. is as follows; for $\vec{A} = (A_R, A_\theta, A_\phi)$ & $\vec{B} = (B_R, B_\theta, B_\phi)$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{R} & \hat{\theta} & \hat{\phi} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$$

Example:

for a spherical strip as shown, it's a section of a sphere of radius 3 cm. Find the area of the strip



$$dS_R = dl_\theta \, dl_\phi$$

$$dS_R = R^2 \sin\theta \, d\theta \, d\phi$$

$$S_R = R^2 \int_{\theta=\frac{\pi}{6}}^{\pi/3} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= (3)^2 \left(-\cos\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}\right) \left(\phi \Big|_0^{2\pi}\right)$$

$$= 9 \left(-\cos\frac{\pi}{3} + \cos\frac{\pi}{6}\right)(2\pi - 0)$$

$$= 9(-0.5 + 0.866)(2\pi)$$

$$S_R = 20.7 \text{ cm}^2$$

Example:

A sphere of radius 2 cm contains a volume charge density $\rho_v = 4\cos^2\theta \; (C/m^2)$

Find the total charge Q contained in the sphere.

$R = 2\,cm = 0.02\,m$

$dQ = \rho_v \, dv$

$dQ = (4\cos^2\theta) R^2 \sin\theta \, dR \, d\theta \, d\phi$

$$Q = 4 \int_{R=0}^{0.02} R^2 dR \int_{\theta=0}^{\pi} \sin\theta \cos^2\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 4 \left(\frac{R^3}{3} \Big|_0^{0.02} \right) \left(-\frac{\cos^3\theta}{3} \Big|_{\theta=0}^{\pi} \right) \left(\phi \Big|_{\phi=0}^{2\pi} \right)$$

$$= \frac{4}{9} \left((0.02)^3 - 0 \right) \left(-(\cos\pi)^3 + (\cos 0)^3 \right) (2\pi - 0)$$

$$= \frac{4}{9} (8 \times 10^{-6}) (1 + 1) (2\pi)$$

$$Q = 44.68 \times 10^{-6} \, C$$

3.3 Transformation between coordinate systems:

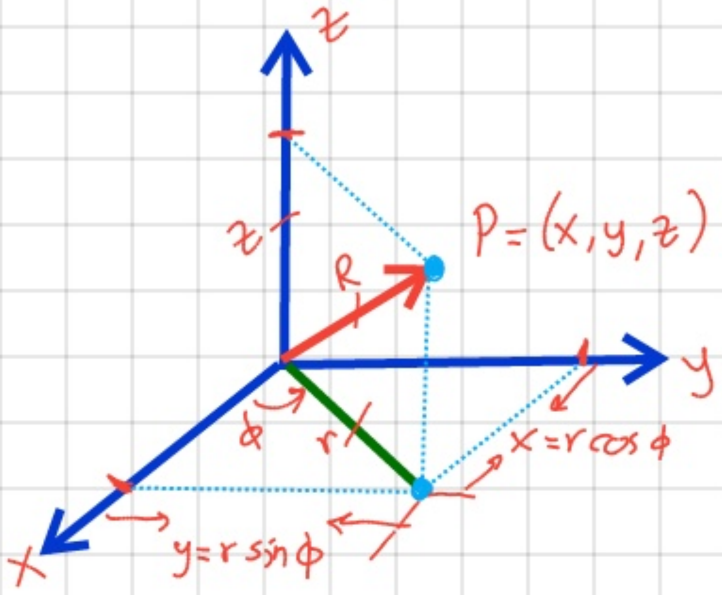
The position of a given point in space is the same regardless of which coordinate system is used to represent it. Certain coordinate systems are more usefull than others in solving a given problem, thus, the need for transforming between coordinate systems.

A. Cartesian Vs. Cylindrical

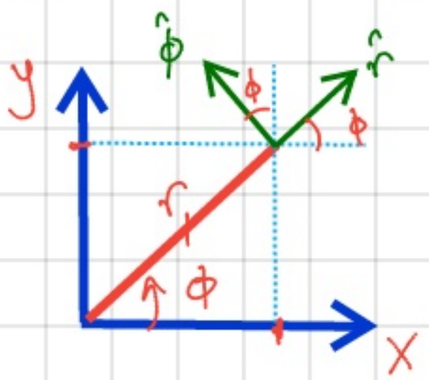
B. Cartesian Vs. Spherical

C. Cylindrical Vs. Spherical

A. Cartesian vs. Cylindrical :



Cartesian to Cylindrical	Cylindrical to Cartesian
$(x, y, z) \to (r, \phi, z)$	$(r, \phi, z) \to (x, y, z)$



$$r = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$z = z$$

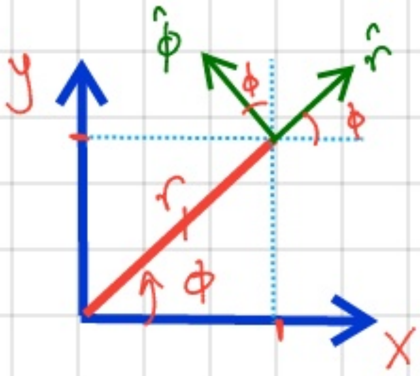
$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$z = z$$

Cartesian to Cylindrical

$(x, y, z) \to (r, \phi, z)$



$\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$

$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$

$\hat{z} = \hat{z}$

Given $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$, then

$\vec{A} = A_r\hat{r} + A_\phi\hat{\phi} + A_z\hat{z}$

where

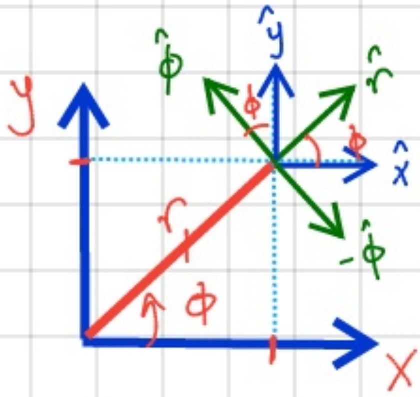
$A_r = A_x\cos\phi + A_y\sin\phi$

$A_\phi = -A_x\sin\phi + A_y\cos\phi$

$A_z = A_z$

Cylindrical to Cartesian

$(r, \phi, z) \to (x, y, z)$



$\hat{x} = \hat{r}\cos\phi - \hat{\phi}\sin\phi$

$\hat{y} = \hat{r}\sin\phi + \hat{\phi}\cos\phi$

$\hat{z} = \hat{z}$

Given $\vec{A} = A_r\hat{r} + A_\phi\hat{\phi} + A_z\hat{z}$, then

$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

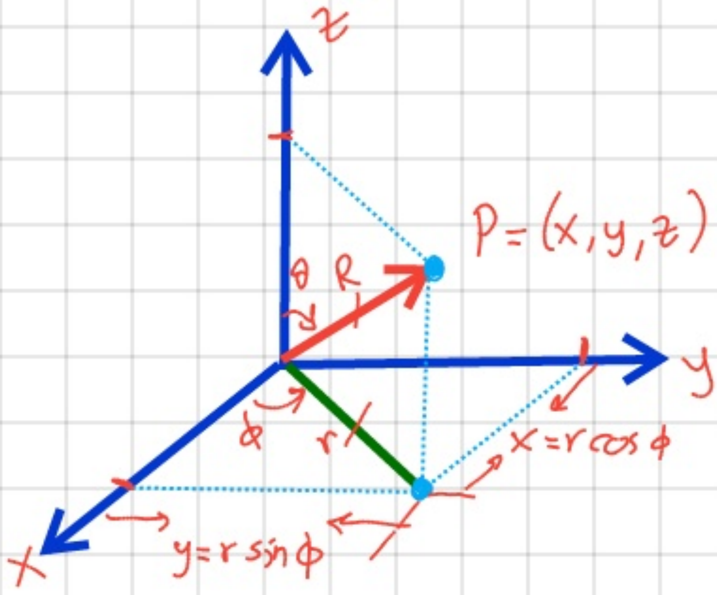
where

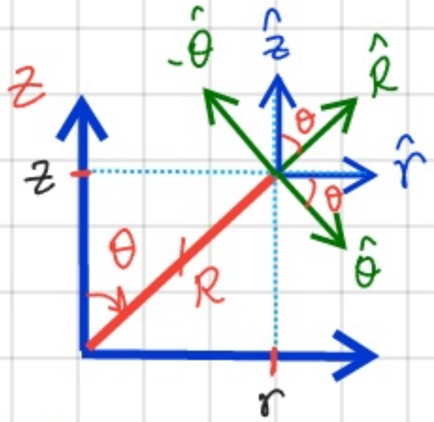
$A_x = A_r\cos\phi - A_\phi\sin\phi$

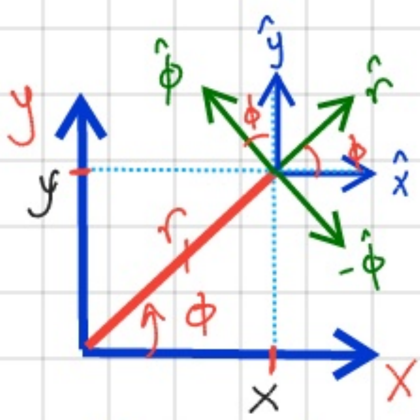
$A_y = A_r\sin\phi + A_\phi\cos\phi$

$A_z = A_z$

B. Cartesian vs. spherical:







Cartesian to spherical

$(x, y, z) \rightarrow (R, \theta, \phi)$

$R = \sqrt{x^2 + y^2 + z^2}$

$\theta = \tan^{-1}\left(\frac{r}{z}\right)$, r: cylindrical

$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

Spherical to Cartesian

$(R, \theta, \phi) \rightarrow (x, y, z)$

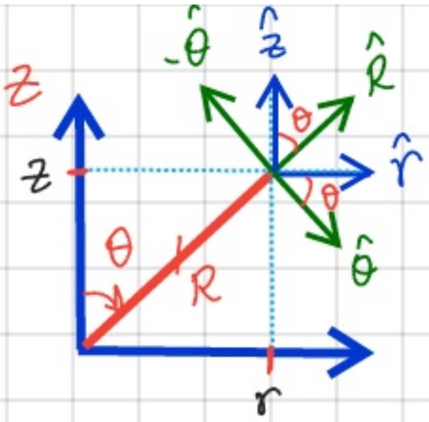
$x = r\cos\phi$

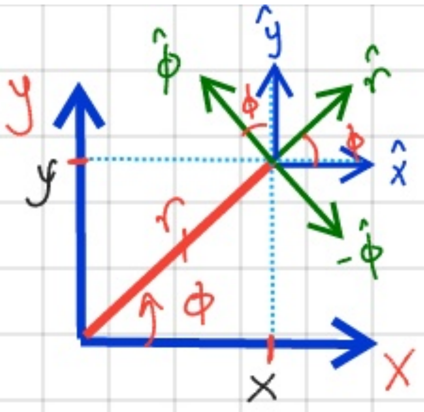
$x = R\sin\theta\cos\phi$

$y = r\sin\phi$

$y = R\sin\theta\sin\phi$

$z = R\cos\theta$





Cartesian to spherical

$$(x, y, z) \longrightarrow (R, \theta, \phi)$$

$$\hat{R} = \underbrace{\hat{r}}_{\text{cyl}} \sin\theta + \hat{z}\cos\theta$$

$$= [\hat{x}\cos\phi + \hat{y}\sin\phi]\sin\theta + \hat{z}\cos\theta$$

$$\hat{R} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

$$\hat{\theta} = \hat{r}\cos\theta - \hat{z}\sin\theta$$

$$= [\hat{x}\cos\phi + \hat{y}\sin\phi]\cos\theta - \hat{z}\sin\theta$$

$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

Spherical to Cartesian

$$(R, \theta, \phi) \longrightarrow (x, y, z)$$

$$\hat{x} = \underbrace{\hat{r}}_{\text{cyl}} \cos\phi - \hat{\phi}\sin\phi$$

$$= [\hat{R}\sin\theta + \hat{\theta}\cos\theta]\cos\phi - \hat{\phi}\sin\phi$$

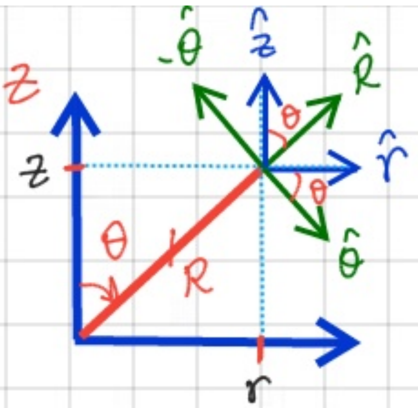
$$\hat{x} = \hat{R}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$

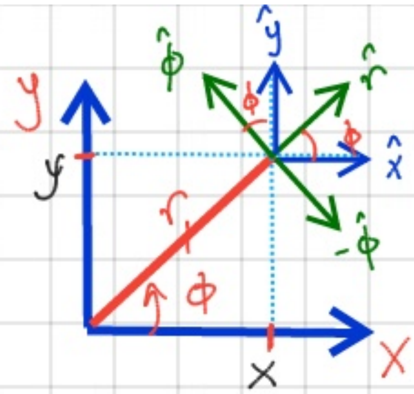
$$\hat{y} = \hat{r}\sin\phi + \hat{\phi}\cos\phi$$

$$= [\hat{R}\sin\theta + \hat{\theta}\cos\theta]\sin\phi + \hat{\phi}\cos\phi$$

$$\hat{y} = \hat{R}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$

$$\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$$





Cartesian to spherical

$(x, y, z) \rightarrow (R, \theta, \phi)$

$$\hat{R} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

$$\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$$

$$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$$

Given $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$,

$$\vec{A} = A_R\hat{R} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

where

$$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$$

$$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$$

$$A_\phi = -A_x\sin\phi + A_y\cos\phi$$

Spherical to Cartesian

$(R, \theta, \phi) \rightarrow (x, y, z)$

$$\hat{x} = \hat{R}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$

$$\hat{y} = \hat{R}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$

$$\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$$

Given $\vec{A} = A_R\hat{R} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

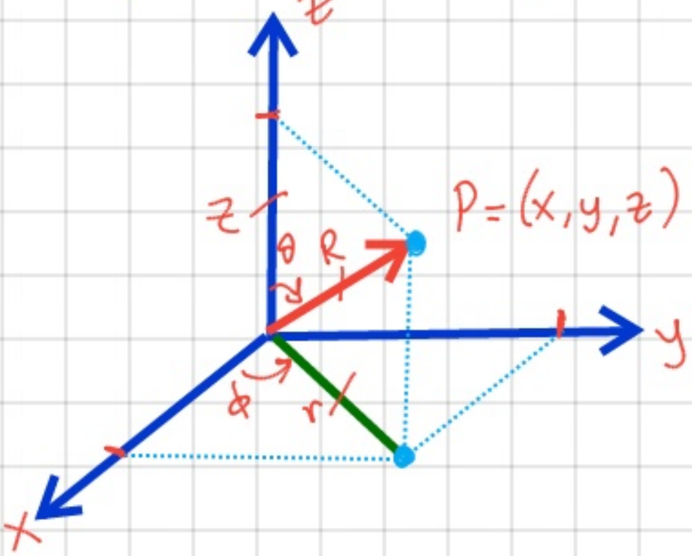
where

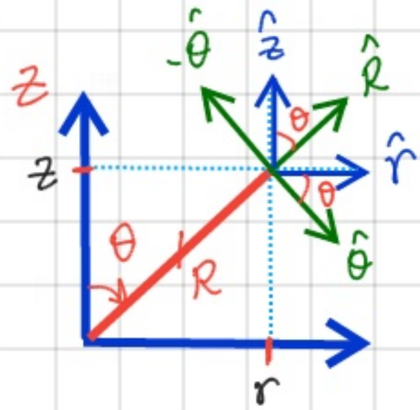
$$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$$

$$A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$$

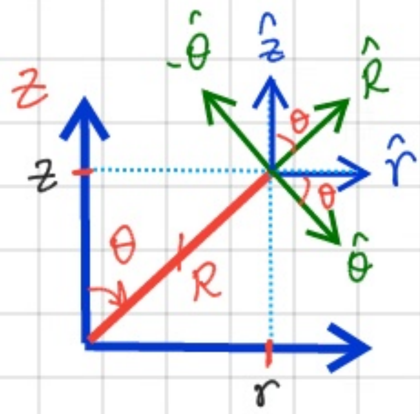
$$A_z = A_R\cos\theta - A_\theta\sin\theta$$

c. Cylindrical to spherical:





Cylindrical to spherical	Spherical to cylindrical
$(r, \phi, z) \to (R, \theta, \phi)$	$(R, \theta, \phi) \to (r, \phi, z)$
$R = \sqrt{r^2 + z^2}$	$r = R\sin\theta$
$\theta = \tan^{-1}\left(\frac{r}{z}\right)$	$\phi = \phi$
$\phi = \phi$	$z = R\cos\theta$
$\hat{R} = \hat{r}\sin\theta + \hat{z}\cos\theta$	$\hat{r} = \hat{R}\sin\theta + \hat{\theta}\cos\theta$
$\hat{\theta} = \hat{r}\cos\theta - \hat{z}\sin\theta$	$\hat{\phi} = \hat{\phi}$
$\hat{\phi} = \hat{\phi}$	$\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$



Cylindrical to spherical

$(r, \phi, z) \to (R, \theta, \phi)$

$\hat{R} = \hat{r}\sin\theta + \hat{z}\cos\theta$

$\hat{\theta} = \hat{r}\cos\theta - \hat{z}\sin\theta$

$\hat{\phi} = \hat{\phi}$

Given $\vec{A} = A_r\hat{r} + A_\phi\hat{\phi} + A_z\hat{z}$,

$\vec{A} = A_R\hat{R} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

where

$A_R = A_r\sin\theta + A_z\cos\theta$

$A_\theta = A_r\cos\theta - A_z\sin\theta$

$A_\phi = A_\phi$

Spherical to cylindrical

$(R, \theta, \phi) \to (r, \phi, z)$

$\hat{r} = \hat{R}\sin\theta + \hat{\theta}\cos\theta$

$\hat{\phi} = \hat{\phi}$

$\hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$

Given $\vec{A} = A_R\hat{R} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

$\vec{A} = A_r\hat{r} + A_\phi\hat{\phi} + A_z\hat{z}$

where

$A_r = A_R\sin\theta + A_\theta\cos\theta$

$A_\phi = A_\phi$

$A_z = A_R\cos\theta - R_\theta\sin\theta$

Example: Given $P_1 = (x, y, z) = (3, -4, 3)$

and $\vec{A} = (A_x, A_y, A_z) = (2, -3, 4)$

1. Express P_1 in cylindrical coordinates
2. " $\vec{A}$ " " "
3. Evaluate $\vec{A}$ at P_1

---

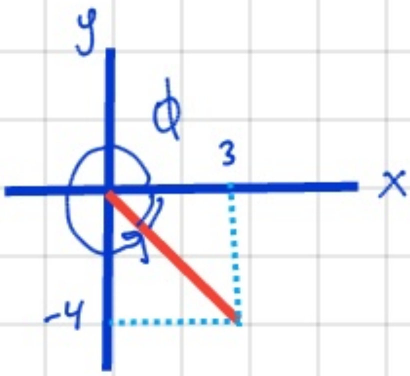
1. From Cartesian to cylindrical:

$r = \sqrt{x^2 + y^2}$

$= \sqrt{3^2 + (-4)^2}$

$= \sqrt{9 + 16}$

$r = 5$



$\phi = \tan^{-1}\left[\frac{y}{x}\right]$

$= 2\pi - \tan^{-1}\frac{4}{3}$

$= 2\pi - 0.927 \text{ rad}$

$\phi = 5.356 \text{ rad} = 306.87°$

$z = z$

$= 3$

2. Vector $\vec{A} = (A_x, A_y, A_z) = (2, -3, 4)$

is in Cartesian. In cylindrical form, it's as follows:

$\vec{A} = (A_r, A_\phi, A_z)$:

$A_r = A_x \cos\phi + A_y \sin\phi$

$= 2\cos\phi - 3\sin\phi$

$A_\phi = -A_x \sin\phi + A_y \cos\phi$

$= -2\sin\phi - 3\cos\phi$

$A_z = A_z$

$= 4$

Then $\vec{A} = A_r \hat{r} + A_\phi \hat{\phi} + A_z \hat{z}$

$= (2\cos\phi - 3\sin\phi)\hat{r} + (-2\sin\phi - 3\cos\phi)\hat{\phi}$

$+ 4\hat{z}$

3. To evaluate $\vec{A}$ at point P_1 :

since

$$\vec{A} = (2\cos\phi - 3\sin\phi)\hat{r} + (-2\sin\phi - 3\cos\phi)\hat{\phi} + 4\hat{z}$$

$$\vec{A}\Big|_{P_1} = \left[2\cos(306.87) - 3\sin(306.87)\right]\hat{r} + \left(-2\sin(306.87) - 3\cos(306.87)\right]\hat{\phi} + 4\hat{z}$$

$P_1 = (r, \phi, z)$
$= (5, 306.87, 3)$

$$\vec{A}\Big|_{P_1} = 3.6\,\hat{r} - 0.2\,\hat{\phi} + 4\hat{z}$$

Example: Express $\vec{A} = \hat{x}(x+y) + \hat{y}(y-x) + \hat{z}z$ in spherical coordinates.

In spherical $\vec{A} = A_R\hat{R} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$ where

$$A_R = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta$$

$$A_\theta = A_x \cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$$

$$A_\phi = -A_x\sin\phi + A_y\cos\phi$$

and from $\vec{A}$ in Cartesian:

$$A_x = x+y \quad , \quad A_y = y-x \quad , \quad A_z = z$$

Since $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, $z = R\cos\theta$

Then $A_x = x+y$

$$= R\sin\theta\cos\phi + R\sin\theta\sin\phi$$

$$= R\sin\theta(\cos\phi + \sin\phi)$$

$$A_y = y - x$$

$$= R\sin\theta\sin\phi - R\sin\theta\cos\phi$$

$$= R\sin\theta(\sin\phi - \cos\phi)$$

$$A_z = z$$

$$= R\cos\theta$$

$$A_R = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta$$

$$= R\sin\theta(\cos\phi+\sin\phi)\sin\theta\cos\phi$$

$$+ R\sin\theta(\sin\phi - \cos\phi)\sin\theta\sin\phi$$

$$+ R\cos\theta\cos\theta$$

$$= R\sin^2\theta\left[\cos^2\phi + \sin\phi\cos\phi + \sin^2\phi - \cos\phi\sin\phi\right]$$

$$+ R\cos^2\theta$$

$$= R\sin^2\theta\left[\cos^2\phi + \sin^2\phi\right] + R\cos^2\theta$$

$$= R(\sin^2\theta + \cos^2\theta)$$

$$\boxed{A_R = R}$$

$$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$$

$$= R\sin\theta(\cos\phi+\sin\phi)\cos\theta\cos\phi$$

$$+ R\sin\theta(\sin\phi-\cos\phi)\cos\theta\sin\phi - R\cos\theta\sin\theta$$

$$= R\sin\theta\cos\theta\left[\cos^2\phi + \sin\phi\cos\phi + \sin^2\phi - \cos\phi\sin\phi\right] - R\cos\theta\sin\theta$$

$$= R\sin\theta\cos\theta\left[1-1\right]$$

$$\boxed{A_\theta = 0}$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$= -R\sin\theta\left(\cos\phi + \sin\phi\right)\sin\phi$$

$$+ R\sin\theta\left(\sin\phi - \cos\phi\right)\cos\phi$$

$$= R\sin\theta\left[-\cos\phi\sin\phi - \sin^2\phi + \sin\phi\cos\phi - \cos^2\phi\right]$$

$$= -R\sin\theta\left[\sin^2\phi + \cos^2\phi\right]$$

$$\boxed{A_\phi = -R\sin\theta}$$

Then

$$\vec{A} = A_R\hat{R} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

becomes:

$$\vec{A} = R\hat{R} + (0)\hat{\theta} + (-R\sin\theta)\hat{\phi}$$

$$\boxed{\vec{A} = R\hat{R} - R\sin\theta\,\hat{\phi}}$$

Example: For $P = (r, \phi, z) = (2\sqrt{3}, \frac{\pi}{3}, -2)$, express P in spherical coordinates.

From cylindrical to spherical:

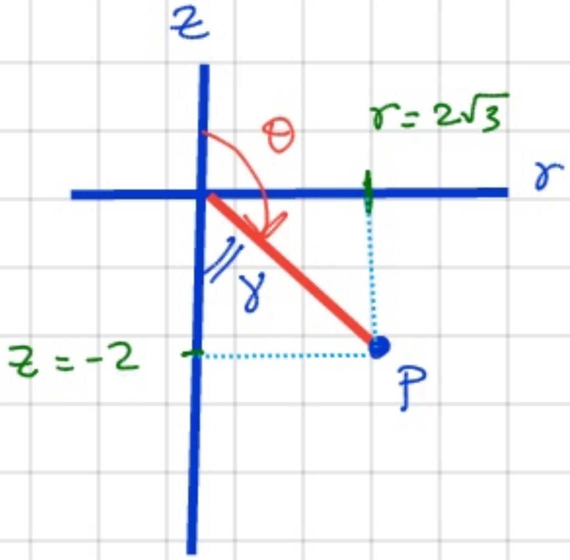
$$R = \sqrt{r^2 + z^2}$$
$$= \sqrt{(2\sqrt{3})^2 + (-2)^2}$$
$$= \sqrt{4(3) + 4}$$
$$= \sqrt{16}$$
$$\boxed{R = 4}$$

$$\phi = \phi \Rightarrow \boxed{\phi = \frac{\pi}{3}}$$

Then P in spherical coord. is:

$$\boxed{P = (R, \theta, \phi) = \left(4, \frac{2\pi}{3}, \frac{\pi}{3}\right)}$$

$$\theta = \tan^{-1}\left(\frac{r}{z}\right)$$
$$= \pi - \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$
$$= \pi - \frac{\pi}{3}$$
$$\boxed{\theta = \frac{2\pi}{3}}$$



$$\gamma = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$
$$\gamma = \frac{\pi}{3}$$

3.3.4 Distance between two points:

In Cartesian coordinates, the distance d between $P_1 = (x_1, y_1, z_1)$ & $P_2 = (x_2, y_2, z_2)$ is:

$$d = |\vec{R}_{12}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

In cylindrical, to convert from Cartesian:

$$x = r\cos\phi \ , \quad y = r\sin\phi \ , \quad z = z$$

Then

$$d = \left[(r_2\cos\phi_2 - r_1\cos\phi_1)^2 + (r_2\sin\phi_2 - r_1\sin\phi_1)^2 + (z_2-z_1)^2 \right]^{1/2}$$

$$= \Big[r_2^2\cos^2\phi_2 + r_1^2\cos^2\phi_1 - 2r_1r_2\cos\phi_2\cos\phi_1$$
$$+ r_2^2\sin^2\phi_2 + r_1^2\sin^2\phi_1 - 2r_1r_2\sin\phi_2\sin\phi_1 + (z_2-z_1)^2 \Big]^{1/2}$$

$$= \Big[r_2^2(\cos^2\phi_2 + \sin^2\phi_2) + r_1^2(\cos^2\phi_1 + \sin^2\phi_1)$$
$$- 2r_1r_2(\cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1) + (z_2-z_1)^2 \Big]^{1/2}$$

$$d = \sqrt{r_2^2 + r_1^2 - 2r_1r_2\cos(\phi_2 - \phi_1) + (z_2-z_1)^2}$$

In spherical, to convert from Cartesian:

$x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, $z = R\cos\theta$

$$d = \Big[\left(R_2\sin\theta_2\cos\phi_2 - R_1\sin\theta_1\cos\phi_1\right)^2 + \left(R_2\sin\theta_2\sin\phi_2 - R_1\sin\theta_1\sin\phi_1\right)^2 + \left(R_2\cos\theta_2 - R_1\cos\theta_1\right)^2 \Big]^{1/2}$$

$$= \Big[R_2^2\sin^2\theta_2\cos^2\phi_2 + R_1^2\sin^2\theta_1\cos^2\phi_1 - 2R_1R_2\sin\theta_2\sin\theta_1\cos\phi_2\cos\phi_1 + R_2^2\sin^2\theta_2\sin^2\phi_2 + R_1^2\sin^2\theta_1\sin^2\phi_1 - 2R_1R_2\sin\theta_2\sin\theta_1\sin\phi_2\sin\phi_1 + R_2^2\cos^2\theta_2 + R_1^2\cos^2\theta_1 - 2R_1R_2\cos\theta_2\cos\theta_1 \Big]^{\frac{1}{2}}$$

$$= \Big[R_2^2\left(\sin^2\theta_2\cos^2\phi_2 + \sin^2\theta_2\sin^2\phi_2 + \cos^2\theta_2\right) + R_1^2\left(\sin^2\theta_1\cos^2\phi_1 + \sin^2\theta_1\sin^2\phi_1 + \cos^2\theta_1\right) - 2R_1R_2\left(\sin\theta_2\sin\theta_1\left[\cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1\right] + \cos\theta_2\cos\theta_1\right) \Big]^{1/2}$$

$$d = \Big[R_2^2 \left(\sin^2\theta_2 \cos^2\phi_2 + \sin^2\theta_2 \sin^2\phi_2 + \cos^2\theta_2 \right)$$
$$+ R_1^2 \left(\sin^2\theta_1 \cos^2\phi_1 + \sin^2\theta_1 \sin^2\phi_1 + \cos^2\theta_1 \right)$$
$$- 2R_1R_2 \left(\sin\theta_2 \sin\theta_1 \left[\cos\phi_2 \cos\phi_1 + \sin\phi_2 \sin\phi_1 \right] + \cos\theta_2 \cos\theta_1 \right) \Big]^{1/2}$$

$$= \Big[R_2^2 \left(\sin^2\theta_2 \underbrace{\left[\cos^2\phi_2 + \sin^2\phi_2 \right]}_{1} + \cos^2\theta_2 \right)$$
$$+ R_1^2 \left(\sin^2\theta_1 \underbrace{\left[\cos^2\phi_1 + \sin^2\phi_1 \right]}_{1} + \cos^2\theta_1 \right)$$
$$- 2R_1R_2 \left(\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1) \right) \Big]^{1/2}$$

$$\boxed{d = \sqrt{R_2^2 + R_1^2 - 2R_1R_2 \left(\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1) \right)}}$$

Example: Find the distance between the following pairs of points:

(a) $P_1 = (1, 2, 3)$ and $P_2 = (-2, -3, -2)$ in Cartesian.

(b) $P_1 = (1, \frac{\pi}{4}, 3)$ and $P_2 = (3, \frac{\pi}{4}, 4)$ in cylindrical.

(c) $P_1 = (4, \frac{\pi}{2}, 0)$ and $P_2 = (3, \pi, 0)$ in spherical.

(a) $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{(-2-1)^2 + (-3-2)^2 + (-2-3)^2}$$

$$= \sqrt{9 + 25 + 25}$$

$$d = 7.68$$

(b) $$d = \sqrt{r_2^2 + r_1^2 - 2 r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2}$$

$$= \sqrt{3^2 + 1^2 - 2(1)(3)\cos\left(\frac{\pi}{4} - \frac{\pi}{4}\right) + (4-3)^2}$$

$$= \sqrt{9 + 1 - 6 + 1}$$

$$d = 2.24$$

(c) (c) $P_1 = (4, \frac{\pi}{2}, 0)$ and $P_2 = (3, \pi\ 0)$ in spherical.

$$d = \sqrt{R_2^2 + R_1^2 - 2R_1R_2\left(\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1\cos(\phi_2 - \phi_1)\right)}$$

$$= \sqrt{3^2 + 4^2 - 2(4)(3)\left[\cos(\pi)\cos(\tfrac{\pi}{2}) + \sin(\pi)\sin(\tfrac{\pi}{2})\cos(0-0)\right]}$$

$$= \sqrt{9 + 16 - 24\ (0 + 0)}$$

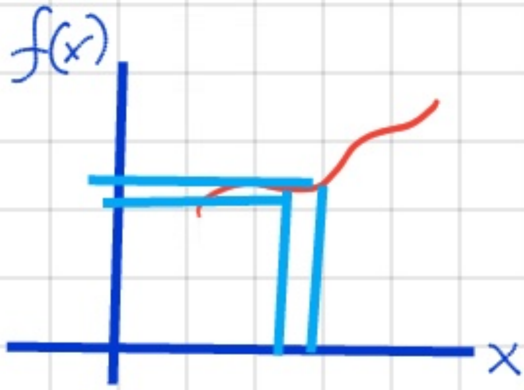
$$d = 5$$

3.4

To describe the differential spatial variations of scalars and vectors, three operators are presented.

1. Gradient
2. Divergence
3. Curl

1 Gradient



Note if a function f depends on only x

then $\frac{df}{dx} = \frac{\partial f}{\partial x}$

as a result: $df = \frac{\partial f}{\partial x} dx$

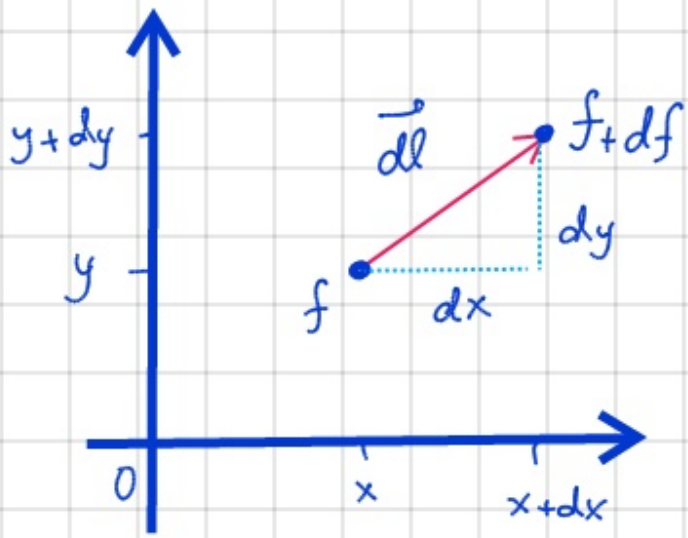
if function f depends on x and y then df is the sum of the partial differentials with respect to all of the independent variables as:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Consider a scalar quantity that is continuous and differentiable function of the coordinates and has the value f at a certain point. To find how f changes over distance $\vec{dl}$,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

This df is a scalar product of $\vec{A}$ & $\vec{dl}$ as $\vec{A} \cdot \vec{dl}$ where



$$\vec{A} = \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y}$$

and

$$\vec{dl} = \hat{x}\, dx + \hat{y}\, dy$$

where the components of $\vec{A}$ are the rates of change of f with distance along the coordinate axes, and is called the gradient of the scalar quantity f.

Gradient is abbreviated as "grad" and is indicated by the symbol ∇, called "del".

So here $\vec{A} = \text{grad } f =: \nabla f$

In three dimensions the operator ∇ is defined as:

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Thus,

$$\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$

and $|\nabla f| = \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right]^{\frac{1}{2}}$

from $df = \vec{A}\cdot\vec{dl} = \nabla f \cdot \vec{dl}$

The magnitude of df is

$$df = \nabla f \cdot \vec{dl} = |\nabla f|\,|\vec{dl}| \cos\theta$$

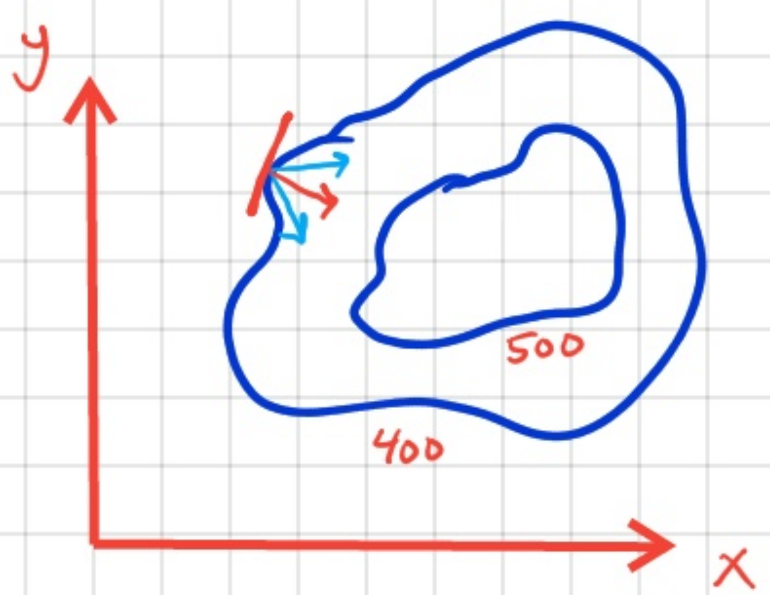
where θ is the angle between the vectors ∇f & $\vec{dl}$.

Since

$$df = \nabla f \cdot \vec{dl} = |\nabla f|\,|\vec{dl}| \cos\theta$$

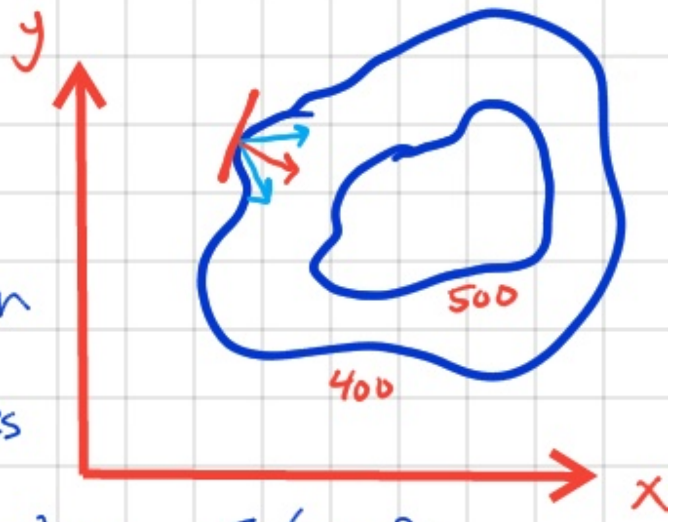
Then the direction we should choose for $\vec{dl}$ in order to have df to be maximum is when $\theta = 0$, which is the same direction as ∇f.

Thus, the gradient of f is a vector whose magnitude and direction are those of the maximum space rate of change of f.



Example:

Consider E the elevation above sea level, it's a function of the x & y coordinates measured on a horizontal plane, $E(x,y)$.



Points of constant elevation are joined together by "contour lines". Then the gradient of the elevation E at a given point has the following properties:

(a) it is perpendicular to the contour line at that point.

(b) its magnitude is equal to the maximum rate of change of elevation with displacement measured in a horizontal plane at that point.

(c) it points toward an increase in elevation.

- In cylindrical coordinates, the gradient operator can be expressed as:

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

remember that: $\vec{dl} = \hat{r}\,dr + \hat{\phi}\,r\,d\phi + \hat{z}\,dz$

- In spherical coordinates, the gradient operator can be expressed as:

$$\nabla = \hat{R}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}$$

remember that: $\vec{dl} = \hat{R}\,dR + \hat{\theta}\,R\,d\theta + \hat{\phi}\,R\sin\theta\,d\phi$

Properties of the gradient operator:

(a) $\nabla(U+V) = \nabla U + \nabla V$

(b) $\nabla(UV) = U\nabla V + V\nabla U$

(c) $\nabla V^n = nV^{n-1}\nabla V$, for any n

Example: Find the gradient of each of the following scalar functions and then evaluate it at the given point.

(a) $v_1 = 24 v_0 \cos\left(\frac{\pi}{3} y\right) \sin\left(\frac{2\pi}{3} z\right)$

at $P_1 = (x, y, z) = (3, 2, 1)$

in Cartesian coordinates

(b) $v_2 = v_0 e^{-2r} \sin 3\phi$ at $P_2 = (r, \phi, z) = (1, \frac{\pi}{2}, 3)$

in cylindrical coordinates

(c) $v_3 = v_0 \frac{a}{R} \cos 2\theta$ at $P_3 = (R, \theta, \phi) = (2a, 0, \pi)$

in spherical coordinates

(a) $v_1 = 24 v_0 \cos\left(\frac{\pi}{3} y\right) \sin\left(\frac{2\pi}{3} z\right)$

at $P_1 = (x, y, z) = (3, 2, 1)$

Solution: In Cartesian coordinates

$$\nabla v_1 = \hat{x} \frac{\partial v_1}{\partial x} + \hat{y} \frac{\partial v_1}{\partial y} + \hat{z} \frac{\partial v_1}{\partial z}$$

$$= \hat{x}(0) + \hat{y}\left[-24 v_0 \left(\frac{\pi}{3}\right) \sin\left(\frac{\pi}{3} y\right) \sin\left(\frac{2\pi}{3} z\right)\right]$$
$$+ \hat{z}\left[24 v_0 \cos\left(\frac{\pi}{3} y\right) \left(\frac{2\pi}{3}\right) \cos\left(\frac{2\pi}{3} z\right)\right]$$

$$\nabla v_1 = 8\pi v_0 \left[-\hat{y} \sin\left(\frac{\pi}{3} y\right) \sin\left(\frac{2\pi}{3} z\right) + \hat{z}\, 2 \cos\left(\frac{\pi}{3} y\right) \cos\left(\frac{2\pi}{3} z\right)\right]$$

$$\nabla v_1 \Big|_{P_1} = 8\pi v_0 \left[-\hat{y} \sin\left(\frac{\pi}{3}(2)\right) \sin\left(\frac{2\pi}{3}(1)\right) + \hat{z}\, 2 \cos\left(\frac{\pi}{3}(2)\right) \cos\left(\frac{2\pi}{3}(1)\right)\right]$$

$$= 8\pi v_0 \left[-\hat{y} \sin^2\left(\frac{2\pi}{3}\right) + \hat{z}\, 2 \cos^2\left(\frac{2\pi}{3}\right)\right]$$

$$= 8\pi v_0 \left[-\hat{y} \left(\frac{\sqrt{3}}{2}\right)^2 + \hat{z}\, 2 \left(\frac{-1}{2}\right)^2\right]$$

$$\nabla v_1 \Big|_{P_1} = 2\pi v_0 \left[-\hat{y}\, 3 + \hat{z}\, 2\right]$$

(b) $v_2 = v_0 e^{-2r} \sin 3\phi$ at $P_2 = (r, \phi, z) = (1, \frac{\pi}{2}, 3)$

Solution: In cylindrical coordinates

$$\nabla v_2 = \hat{r} \frac{\partial v_2}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial v_2}{\partial \phi} + \hat{z} \frac{\partial v_2}{\partial z}$$

$$= \hat{r} \left[v_0 (-2) e^{-2r} \sin 3\phi \right] + \hat{\phi} \frac{1}{r} \left[v_0 e^{-2r} (3) \cos 3\phi \right] + 0$$

$$\nabla v_2 = v_0 e^{-2r} \left[-\hat{r}\, 2 \sin 3\phi + \hat{\phi} \frac{3}{r} \cos 3\phi \right]$$

$$\nabla v_2 \Big|_{P_2} = v_0 e^{-2(1)} \left[-\hat{r}\, 2 \sin 3\frac{\pi}{2} + \hat{\phi} \frac{3}{1} \cos 3\frac{\pi}{2} \right]$$

$$= v_0 e^{-2} \left[-\hat{r}\, 2(-1) + \hat{\phi}\, 3\, (0) \right]$$

$$\nabla v_2 \Big|_{P_2} = \hat{r}\, 2 v_0 e^{-2}$$

(c) $v_3 = v_0 \dfrac{a}{R} \cos 2\theta$ at $P_3 = (R, \theta, \phi) = (2a, 0, \pi)$

Solution: In spherical coordinates

$$\nabla v_3 = \hat{R} \frac{\partial v_3}{\partial R} + \hat{\theta} \frac{1}{R} \frac{\partial v_3}{\partial \theta} + \hat{\phi} \frac{1}{R \sin\theta} \frac{\partial v_3}{\partial \phi}$$

$$= \hat{R} \left[v_0 a \left(\frac{-1}{R^2} \right) \cos 2\theta \right]$$

$$+ \hat{\theta} \frac{1}{R} \left[v_0 \frac{a}{R} (-2) \sin 2\theta \right] + 0$$

$$\nabla v_3 = \frac{v_0 a}{R^2} \left[-\hat{R} \cos 2\theta - \hat{\theta}\, 2 \sin 2\theta \right]$$

$$\nabla v_3 \Big|_{P_3} = \frac{v_0 a}{(2a)^2} \left[-\hat{R} \cos(2(0)) - \hat{\theta}\, 2 \sin(2(0)) \right]$$

$$= \frac{v_0}{4a} \left[-\hat{R}\,(1) - 0 \right]$$

$$\nabla v_3 \Big|_{P_3} = -\hat{R} \frac{v_0}{4a}$$

2 Divergence :

For a vector field $\vec{E}$, the flux density is defined as the amount of flux crossing a unit surface, as:

$$\text{"Flux density of } \vec{E}\text{"} = \frac{\vec{E} \cdot d\vec{s}}{|d\vec{s}|}$$

$$= \frac{\vec{E} \cdot \hat{n}\, |d\vec{s}|}{|d\vec{s}|}$$

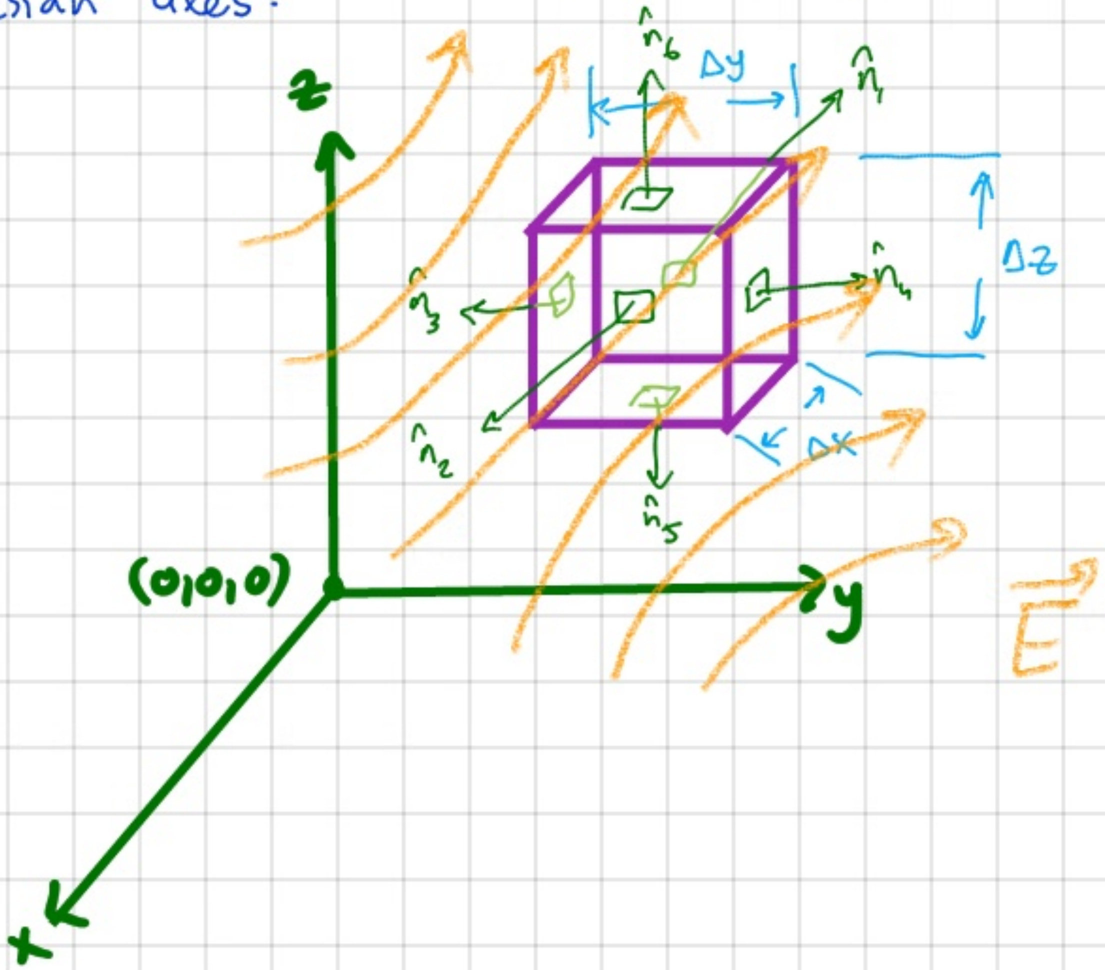
$$= \vec{E} \cdot \hat{n}$$

where $\hat{n}$ is the normal of differential area $d\vec{s}$.

The total flux crossing a closed surface S is

$$\text{Total flux} = \oint_S \vec{E} \cdot d\vec{s} := F \qquad \text{(equ 1)}$$

Consider a differential volume in the Cartesian coordinates, with its edges alignning with the Cartesian axes.

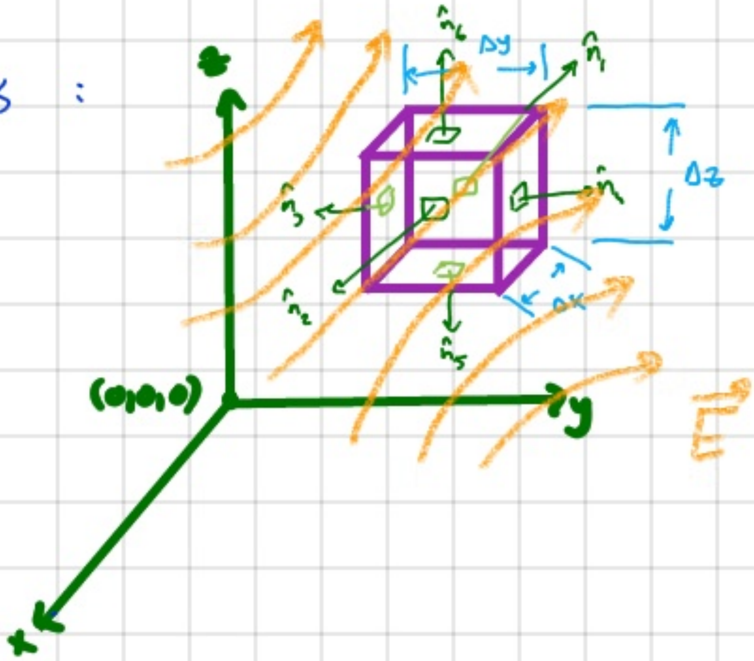


Let vector field $\vec{E}$ be defined as:

$$\vec{E} = \hat{x} E_x + \hat{y} E_y + \hat{z} E_z$$

The total flux of $\vec{E}$ through all the faces of the cube of volume $\Delta v = \Delta x \Delta y \Delta z$ is calculated by adding the fluxes through all of its six faces.

Then total flux F is :



$$F = \sum_{i \in \{1,2,\dots,6\}} F_i$$

For face 1 :

$$F_1 = \int_{\text{face } 1} \vec{E} \cdot d\vec{s}$$

$$= \int (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \cdot (-\hat{x}\, dy\, dz)$$

$$\simeq -E_{x,C1} \Delta y \Delta z$$

where $E_{x,C1}$ is the value of E_x at the center of face 1.

For face 2 :

$$F_2 = \int_{\text{face } 2} \vec{E} \cdot d\vec{s}$$

$$= \int (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) \cdot (\hat{x}\, dy\, dz)$$

$$\simeq E_{x,C2} \Delta y \Delta z$$

where $E_{x,C2}$ is the value of E_x at the center of face 2.

Since the center of face 1 is Δx apart from the center of face 2; then $E_{x,c2}$ in terms of $E_{x,c1}$ is :

$$E_{x,c2} = E_{x,c1} + \frac{\partial E_x}{\partial x}\Delta x \quad \left(\begin{array}{l}\text{the 1st order approx.}\\ \text{from Taylor's series}\end{array}\right)$$

Then
$$F_1 + F_2 = -E_{x,c1}\Delta y \Delta z + E_{x,c2}\Delta y \Delta z$$
$$= \left[-E_{x,c1} + \left(E_{x,c1} + \frac{\partial E_x}{\partial x}\Delta x\right)\right]\Delta y \Delta z$$
$$= \frac{\partial E_x}{\partial x}\Delta x \Delta y \Delta z$$

$$F_1 + F_2 = \frac{\partial E_x}{\partial x}\Delta v$$

Similarly,

$$F_3 + F_4 = \frac{\partial E_y}{\partial y}\Delta v$$

and

$$F_5 + F_6 = \frac{\partial E_z}{\partial z}\Delta v$$

Then $F = \sum_{i \in \{1,2,\dots,6\}} f_i$

$$= \frac{\partial E_x}{\partial x}\Delta v + \frac{\partial E_y}{\partial y}\Delta v + \frac{\partial E_z}{\partial z}\Delta v$$

$$F = \underbrace{\left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right]}\Delta v$$

$$F = \left[\operatorname{div}\vec{E}\right]\Delta v \quad \sim \text{equ } 2$$

from equ 1, $F = \oint_S \vec{E}\cdot d\vec{s}$

Then from equ 1 & equ 2:

$$F = \operatorname{div}\vec{E}\,\Delta v = \oint_S \vec{E}\cdot d\vec{s} \quad \sim \text{equ } 3$$

where $\operatorname{div}\vec{E}$ is a scalar function called divergence of $\vec{E}$.

The divergence of $\vec{E}$ at a point is defined as the total flux per unit volume over a closed surface as the volume goes to zero.

Since $\operatorname{div}\vec{E}\,\Delta v = \oint_S \vec{E}\cdot d\vec{s}$

divide by Δv, then take the limit as $\Delta v \to 0$,

then

$$\operatorname{div}\vec{E} := \lim_{\Delta v \to 0} \frac{1}{\Delta v} \oint_S \vec{E}\cdot d\vec{s} \qquad \text{(equ 4)}$$

where S surface encloses the element volume Δv.

Since $\operatorname{div}\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

$\operatorname{div}\vec{E}$ can be denoted as $\nabla\cdot\vec{E}$

since $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

and $\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$

then

$$\operatorname{div}\vec{E} = \nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

From the result equ. 3,

$$\operatorname{div} \vec{E}\, \Delta v = \oint_S \vec{E} \cdot d\vec{s}$$

the left-hand side can be expressed in terms of dv to give $\int_v \operatorname{div} \vec{E}\, dv$, then

$$\int_v \operatorname{div} \vec{E}\, dv = \oint_S \vec{E} \cdot d\vec{s} \quad \text{(Divergence Theorem)}$$

From the definition of $\operatorname{div} \vec{E}$ in equ 4,

$$\operatorname{div} \vec{E} := \lim_{\Delta v \to 0} \frac{1}{\Delta v} \oint_S \vec{E} \cdot d\vec{s}$$

(a) $\operatorname{div} \vec{E}$ is positive if the total flux on surface S is outward, which is also the case when a source of field lines is contained in volume Δv

(b) $\operatorname{div} \vec{E}$ is negative if the total flux on surface S is inward, which is also the case when a sink of field lines is contained in volume Δv

(c) $\operatorname{div} \vec{E}$ is zero if the vector field $\vec{E}$ is a uniform field, which is also the case when the same amount of flux enters Δv and leaves it. Such vector field is called divergenceless.

So for a vector field $\vec{E}$, the divergence of $\vec{E}$ is:

In Cartesian coord.:

$$\nabla.\vec{E} = \operatorname{div}\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

In cylindrical coord.:

$$\nabla.\vec{E} = \operatorname{div}\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$$

$$= \frac{1}{r}\left[E_r + r\frac{\partial E_r}{\partial r}\right] + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$$

$$\nabla.\vec{E} = \operatorname{div}\vec{E} = \frac{E_r}{r} + \frac{\partial E_r}{\partial r} + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$$

In spherical coord.:

$$\nabla.\vec{E} = \frac{1}{R^2\sin\theta}\left[\frac{\partial}{\partial R}(R^2\sin\theta E_R) + \frac{\partial}{\partial \theta}(R\sin\theta E_\theta) + \frac{\partial}{\partial \phi}(RE_\phi)\right]$$

$$\nabla.\vec{E} = \frac{2E_R}{R} + \frac{\partial E_R}{\partial R} + \frac{E_\theta \cot\theta}{R} + \frac{1}{R}\frac{\partial E_\theta}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial \phi}$$

Properties of the divergence :

For any field vectors $\vec{E_1}$ & $\vec{E_2}$,

$$\nabla . (\vec{E_1} + \vec{E_2}) = \nabla . \vec{E_1} + \nabla . \vec{E_2}$$

Example: Determine the divergence of the following vector field and then evaluate it at the indicated point

(a) $\vec{E} = \hat{x}3x^2 + \hat{y}2z + \hat{z}x^2z$ at $(2,-2,0)$

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2z)$$

$$= 6x + 0 + x^2$$

$$\boxed{\nabla \cdot \vec{E} = x^2 + 6x}$$

$$\nabla \cdot \vec{E}\Big|_{(2,-2,0)} = (2)^2 + 6(2)$$

$$= 4 + 12$$

$$\boxed{\nabla \vec{E}\Big|_{(2,-2,0)} = 16}$$

(b) $\vec{E} = \hat{r}\, r\cos\phi + \hat{\phi}\, r\sin\phi + \hat{z}\, 3z$ at $(2,0,3)$

$$\nabla\cdot\vec{E} = \frac{E_r}{r} + \frac{\partial E_r}{\partial r} + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$$

$$= \frac{1}{r}(r\cos\phi) + \frac{\partial}{\partial r}(r\cos\phi) + \frac{1}{r}\frac{\partial}{\partial \phi}(r\sin\phi)$$

$$+ \frac{\partial}{\partial z}(3z)$$

$$= \cos\phi + \cos\phi + \cos\phi + 3$$

$$\boxed{\nabla\cdot\vec{E} = 3\cos\phi + 3}$$

$$\nabla\cdot\vec{E}\Big|_{(2,0,3)} = 3\,\overset{1}{\cancel{\cos(0)}} + 3$$

$$\boxed{\nabla\cdot\vec{E}\Big|_{(2,0,3)} = 6}$$

(c) $\vec{E} = \hat{R}\left(\frac{a^3\cos\theta}{R^2}\right) - \hat{\theta}\left(\frac{a^3\sin\theta}{R^2}\right)$ at $\left(\frac{a}{2}, 0, \pi\right)$

$$\nabla\cdot\vec{E} = \frac{2E_R}{R} + \frac{\partial E_R}{\partial R} + \frac{E_\theta \cot\theta}{R} + \frac{1}{R}\frac{\partial E_\theta}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

$$= \frac{2}{R}\left(\frac{a^3\cos\theta}{R^2}\right) + \frac{\partial}{\partial R}\left(\frac{a^3\cos\theta}{R^2}\right) + \left(\frac{-a^3\sin\theta}{R^2}\right)\frac{\cot\theta}{R}$$

$$+ \frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{-a^3\sin\theta}{R^2}\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}(0)$$

$$= \frac{2a^3\cos\theta}{R^3} + a^3\cos\theta\left(\frac{-2R}{R^4}\right) - \frac{a^3\sin\cancel{\theta}}{R^3}\frac{\cos\theta}{\cancel{\sin\theta}}$$

$$- \frac{a^3}{R^3}\cos\theta + 0$$

$$= \frac{a^3}{R^3}\cos\theta\left(2 - 2 - 1 - 1\right)$$

$$\boxed{\nabla\cdot\vec{E} = -2\frac{a^3}{R^3}\cos\theta}$$

$$\nabla\cdot\vec{E}\Big|_{\left(\frac{a}{2},0,\pi\right)} = -2\frac{a^3}{\left(\frac{a}{2}\right)^3}\cos(0) = -2\frac{\cancel{a^3}}{\cancel{a^3}}2^3(1)$$

$$\boxed{\nabla\cdot\vec{E}\Big|_{(a/2,0,\pi)} = -16}$$

Example: If $\vec{E} = \hat{R} A R$ in spherical coordinates,

(a) calculate the flux of $\vec{E}$ through a spherical surface of radius a, centered at the origin

(b) verify the divergence theorem by calculating the volume integral of the divergence of the field $\vec{E}$ over the volume bounded by the surface of radius a.

(a) $F = \oint_S \vec{E} \cdot d\vec{s}_R$, $d\vec{s}_R = R^2 \sin\theta \, d\theta \, d\phi \, \hat{R}$

$$= \oint_S (\hat{R} A R) \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{R}) \text{ at } R = a$$

$$= A R^3 \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= A(a^3) \left[-\cos\theta \Big|_0^{\pi} \right] \left[\phi \Big|_0^{2\pi} \right]$$

$$= A a^3 (-\cos\pi + \cos 0)(2\pi - 0)$$

$$\boxed{F = 4\pi A a^3}$$

(b) The divergence theorem is:

$$\int_{v} \operatorname{div} \vec{E}\, dv = \oint_{s} \vec{E} \cdot d\vec{s}$$

From (a), $\oint_{s} \vec{E} \cdot d\vec{s} = 4\pi A a^3$

For the left side $\int_{v} \operatorname{div} \vec{E}\, dv$, since $\vec{E} = \hat{R} A R$,

$$\nabla \cdot \vec{E} = \frac{2E_R}{R} + \frac{\partial E_R}{\partial R} + \frac{E_\theta \cot\theta}{R} + \frac{1}{R}\frac{\partial E_\theta}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial \phi}$$

$$= \frac{2(AR)}{R} + \frac{\partial}{\partial R}(AR) + 0 + 0 + 0$$

$$= 2A + A$$

$$\nabla \cdot \vec{E} = 3A$$

and $dv = R^2 \sin\theta\, dR\, d\theta\, d\phi$

Then $\int_v \operatorname{div}\vec{E}\, dv = \int_v 3A\, R^2 \sin\theta\, dR\, d\theta\, d\phi$

$$= 3A \int_{R=0}^{a} R^2 dR \int_{\theta=0}^{\pi} \sin\theta\, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 3A \left[\frac{R^3}{3} \Big|_0^a \right] \left[-\cos\theta \Big|_0^{\pi} \right] \left[\phi \Big|_0^{2\pi} \right]$$

$$= \frac{\cancel{3}A}{\cancel{3}} \left(a^3 - 0 \right) \left(-\cos\pi + \cos 0 \right) \left(2\pi - 0 \right)$$

$$\int_v \operatorname{div}\vec{E}\, dv = 4\pi A a^3$$

As a result: $\int_v \operatorname{div}\vec{E}\, dv = \oint_S \vec{E} \cdot d\vec{s}$

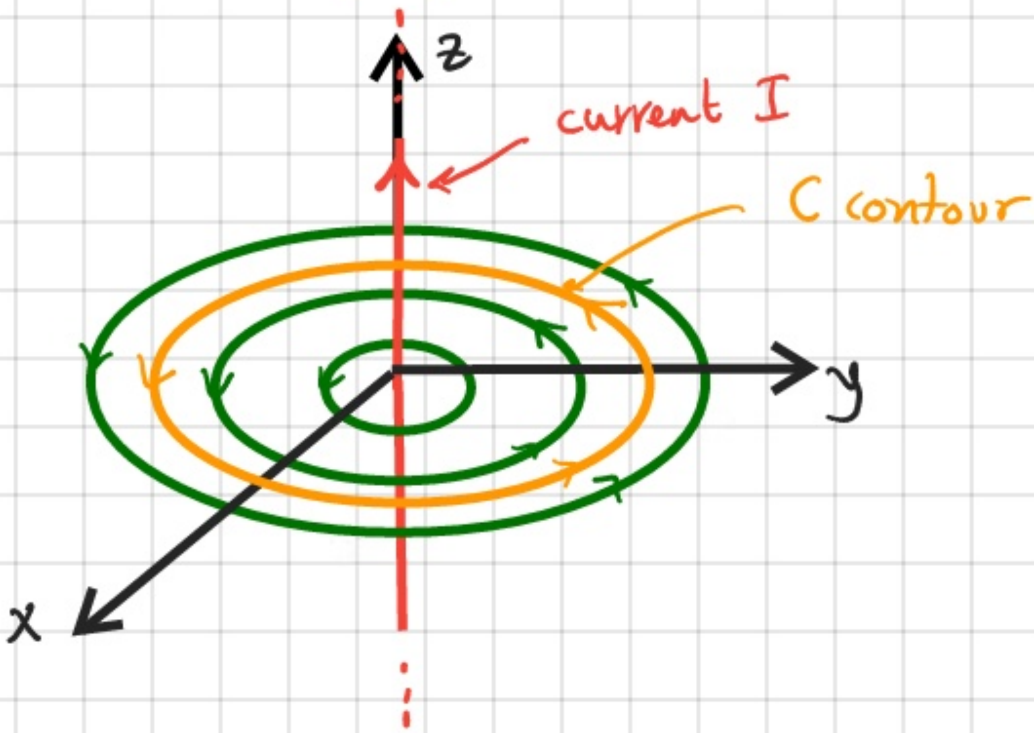
and the divergence theorem is verified.

3 Curl :

For a vector field $\vec{B}$, the circulation of $\vec{B}$ is defined as the line integral of $\vec{B}$ around a closed contour C as :

$$\text{circulation} = \oint_C \vec{B} \cdot d\vec{l}$$

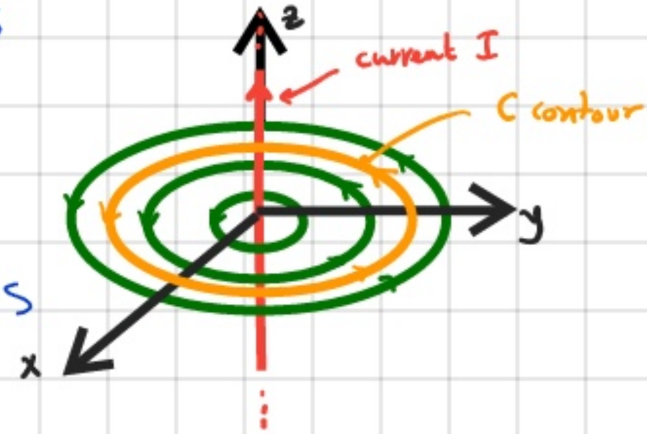
To understand the physical meaning, consider the magnetic flux density $\vec{B}$ caused by an infinite wire carrying a dc current I.



If the current is in free space, then

$$\vec{B} = \hat{\phi}\,\frac{\mu_0 I}{2\pi r}$$

For a circular contour C of radius r, the differential length vector is



$$\vec{dl} = \hat{\phi}\, r\, d\phi$$

Then the circulation of $\vec{B}$ is

$$\text{circulation} = \oint_C \vec{B}\cdot\vec{dl}$$

$$= \int_{\phi=0}^{2\pi} \hat{\phi}\,\frac{\mu_0 I}{2\pi \cancel{r}}\cdot\hat{\phi}\,\cancel{r}\,d\phi$$

$$= \frac{\mu_0 I}{2\pi}\,\phi\Big|_0^{2\pi}$$

$$= \frac{\mu_0 I}{2\pi}\,(2\pi - 0)$$

$$\text{circulation} = \mu_0 I$$

- In this case, the circulation is not zero, however, if contour C was in the xz-plane or the yz-plane, then the circulation would be zero, since the direction of $\vec{dl}$ wouldn't be in the $\hat{\phi}$ direction causing the dot product to be zero. Thus, the circulation depends on the choice of contour C.
- Curl $\vec{B}$ is the circulation of $\vec{B}$ per unit area, with area Δs of the contour C being oriented such that the circulation is maximum, as the area goes to zero

That is as follows:

$$\text{curl } \vec{B} =: \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{n} \oint_C \vec{B} \cdot \vec{dl} \right]_{max}$$

$\hat{n}$: the unit normal of Δs

- $\nabla \times \vec{B}$ denotes curl $\vec{B}$.

- For a vector field $\vec{B}$ in Cartesian coordinates:

$$\vec{B} = \hat{x} B_x + \hat{y} B_y + \hat{z} B_z$$

$$\nabla \times \vec{B} = \hat{x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) - \hat{y}\left(\frac{\partial B_z}{\partial x} - \frac{\partial B_x}{\partial z}\right) + \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

or simply:

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

- For a vector field $\vec{B}$ in cylindrical coord.:

$$\vec{B} = \hat{r} B_r + \hat{\phi} B_\phi + \hat{z} B_z$$

$$\nabla \times \vec{B} = \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ B_r & rB_\phi & B_z \end{vmatrix}$$

- for a vector field $\vec{B}$ in spherical coord.

$$\vec{B} = \hat{R} B_R + \hat{\theta} B_\theta + \hat{\phi} B_\phi$$

$$\nabla \times \vec{B} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \hat{R} & R\hat{\theta} & R\sin\theta\, \hat{\phi} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ B_R & R B_\theta & R\sin\theta\, B_\phi \end{vmatrix}$$

- Note: The curl of a vector field $\vec{B}$ is another vector field

- Recall: $\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

- For any two vector fields $\vec{A}$ & $\vec{B}$ and scalar V:

(a) $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

(b) $\nabla \cdot (\nabla \times \vec{A}) = 0 \Leftrightarrow \text{div}(\text{curl } \vec{A}) = 0$

$\left(\text{since } \vec{C} \cdot (\underbrace{\vec{C} \times \vec{D}}_{\perp \vec{C}}) = 0 \text{ , with } \vec{C} := \nabla, \vec{D} := A\right)$

(c) $\nabla \times (\nabla V) = 0 \Leftrightarrow \text{curl}(\text{grad } V) = 0$

$\left(\text{since } \vec{C} \times \vec{C} = 0 \text{ , with } \vec{C} := \nabla\right)$

- **Stoke's Theorem**: The integral of the tangentical component of a vector field $\vec{B}$ around contour C is equal to the integral of the normal component of curl $\vec{B}$ over surface S.

That is:

$$\oint_C \vec{B} \cdot \vec{dl} = \int_S (\nabla \times \vec{B}) \cdot \vec{ds}$$

It follows from the definition of curl $\vec{B}$.

- If curl $\vec{B} = 0$, the vector field is called conservative or irrotational, because its circulation is zero for any contour chosen.

Example: Determine the curl of the following vector field and then evaluate it at the indicated point

(a) $\vec{B} = \hat{x}\, y\cos ax + \hat{z}\,(y+e^x)$ at the origin

$$\nabla\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos ax & 0 & y+e^x \end{vmatrix}$$

$$= \hat{x}\left[\frac{\partial}{\partial y}(y+e^x) - \frac{\partial}{\partial z}(0)\right] - \hat{y}\left[\frac{\partial}{\partial x}(y+e^x) - \frac{\partial}{\partial z}(y\cos ax)\right]$$

$$+\hat{z}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(y\cos ax)\right]$$

$$= \hat{x}\left[(1+0)-0\right] - \hat{y}\left[(0+1.e^x) - (0)\right] + \hat{z}\left[0 - (\cos ax)\right]$$

$$\nabla\times\vec{B} = \hat{x} - \hat{y}\,e^x - \hat{z}\cos ax$$

$$\nabla\times\vec{B}\Big|_{(0,0,0)} = \hat{x} - \hat{y}\,e^{(0)} - \hat{z}\cos(0)$$

$$\nabla\times\vec{B}\Big|_{(0,0,0)} = \hat{x} - \hat{y} - \hat{z}$$

(b) $\vec{B} = \hat{r}\, 10\, e^{-2r} \cos\phi + \hat{z}\, 10 \sin\phi$ at $(2, 0, 3)$

$$\nabla \times \vec{B} = \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ B_r & rB_\phi & B_z \end{vmatrix}$$

$$= \frac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 10e^{-2r}\cos\phi & r(0) & 10\sin\phi \end{vmatrix}$$

$$= \frac{1}{r} \Big\{ \hat{r} \Big[\frac{\partial}{\partial \phi}(10\sin\phi) - \underbrace{\cancel{\frac{\partial}{\partial z}(r(0))}}_{0} \Big]$$

$$- r\hat{\phi} \Big[\frac{\partial}{\partial r}(10\sin\phi) - \frac{\partial}{\partial z}(10e^{-2r}\cos\phi) \Big]$$

$$+ \hat{z} \Big[\underbrace{\cancel{\frac{\partial}{\partial r}(r(0))}}_{0} - \frac{\partial}{\partial \phi}(10e^{-2r}\cos\phi) \Big] \Big\}$$

$$= \hat{r}\left(\frac{1}{r} 10 \cos\phi\right) - \hat{\phi}\,\frac{1}{r}\, r\,(0 - 0) + \hat{z}\,\frac{1}{r}\left(-10e^{-2r}(-\sin\phi)\right)$$

$$\nabla \times \vec{B} = \hat{r}\,\frac{10}{r}\cos\phi + \hat{z}\,\frac{10e^{-2r}}{r}\sin\phi$$

$$\nabla \times \vec{B}\Big|_{(2,0,3)} = \hat{r}\,\frac{\cancelto{5}{10}}{\cancel{2}}\,\underbrace{\cancel{\cos(0)}}_{1} + \hat{z}\,\frac{10e^{-2(2)}}{2}\,\underbrace{\cancel{\sin(0)}}_{0}$$

$$\nabla \times \vec{B}\Big|_{(2,0,3)} = \hat{r}\, 5$$

(c) $\vec{B} = \hat{\theta}\, 12\sin\theta$ at $\left(3, \frac{\pi}{6}, 0\right)$

$$\nabla \times \vec{B} = \frac{1}{R^2 \sin\theta} \begin{vmatrix} \hat{R} & R\hat{\theta} & R\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ B_R & RB_\theta & R\sin\theta\, B_\phi \end{vmatrix}$$

$$= \frac{1}{R^2 \sin\theta} \begin{vmatrix} \hat{R} & R\hat{\theta} & R\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 12\sin\theta R & 0 \end{vmatrix}$$

$$= \frac{1}{R^2 \sin\theta} \Big\{ \hat{R}\left(\cancel{\frac{\partial}{\partial\theta}(0)}^{0} - \frac{\partial}{\partial\phi}(12\sin\theta R)\right) - R\hat{\theta}\left(\cancel{\frac{\partial}{\partial R}(0)}^{0} - \cancel{\frac{\partial}{\partial\phi}(0)}^{0}\right) + R\sin\theta\,\hat{\phi}\left(\frac{\partial}{\partial R}(12\sin\theta R) - \cancel{\frac{\partial}{\partial\theta}(0)}^{0}\right)\Big\}$$

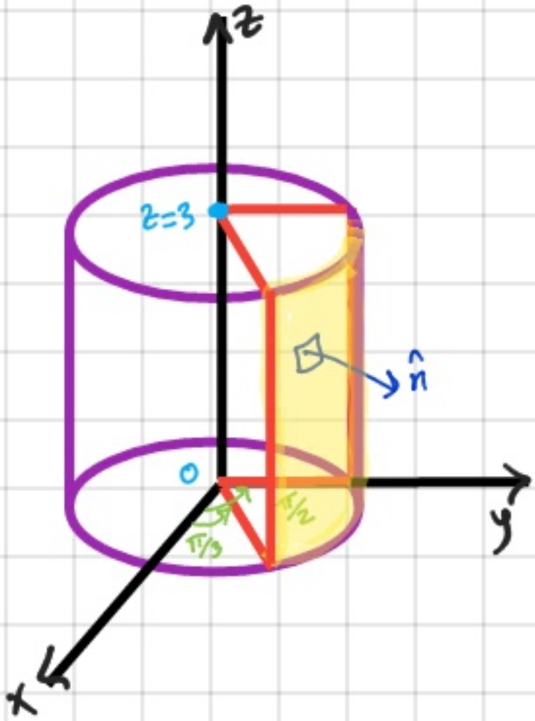
$$= \frac{\hat{R}}{R^2\sin\theta}(0) + \frac{R\sin\theta\,\hat{\phi}}{R^2\sin\theta}(12\sin\theta)$$

$$\nabla \times \vec{B} = \hat{\phi}\,\frac{12\sin\theta}{R}$$

$$\nabla \times \vec{B}\Big|_{(3,\pi/6,0)} = \hat{\phi}\,\frac{\cancel{12}^{4}}{\cancel{3}}\,\cancel{\sin\left(\frac{\pi}{6}\right)}^{\frac{1}{2}} = \hat{\phi}\,2$$

Example: Verify Stoke's Theorem,

For a vector field $\vec{B} = \hat{z}\frac{\cos\phi}{r}$ for a segment of a cylindrical surface defined by $r = 2$, $\frac{\pi}{3} \leq \phi \leq \frac{\pi}{2}$, and $0 \leq z \leq 3$.



Stoke's Theorem is:

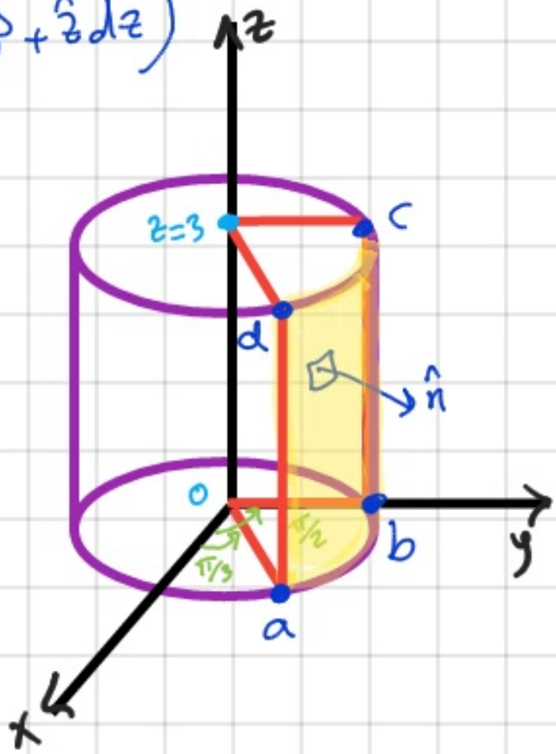
$$\int_S (\nabla \times \vec{B}) \cdot d\vec{S} = \oint_C \vec{B} \cdot d\vec{l}$$

The surface S (the yellow) is bounded by contour C (the edges of the yellow surface).

- For the right hand side, choose the direction of C such that the surface normal $\hat{n}$ is $+\hat{r}$ by the right hand rule

For the right hand side, with $\vec{B} = \hat{z}\dfrac{\cos\phi}{r}$:

$$\oint_C \vec{B}\cdot d\vec{l} = \oint_C \hat{z}\,\frac{\cos\phi}{r}\cdot\left(\hat{r}\,dr + \hat{\phi}\,r\,d\phi + \hat{z}\,dz\right)$$



$$= \oint_C \frac{\cos\phi}{r}\,dz$$

$$= \int_b^c \frac{\cos\phi}{r}\,dz\Bigg|_{r=2,\ \phi=\frac{\pi}{2}}$$

$$+ \int_d^a \frac{\cos\phi}{r}\,dz\Bigg|_{r=2,\ \phi=\frac{\pi}{3}}$$

$$= \frac{\cancelto{0}{\cos\frac{\pi}{2}}}{2}\int_{z=0}^{3} dz + \frac{\cos\frac{\pi}{3}}{2}\int_{z=3}^{0} dz$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)\left(z\Big|_3^0\right)$$

$$= \frac{1}{4}\left(0-3\right)$$

$$\boxed{\oint_C \vec{B}\cdot\vec{dl} = -\frac{3}{4}}$$

For the left-hand side, with $\vec{B} = \hat{z}\dfrac{\cos\phi}{r}$:

$\int_S (\nabla\times\vec{B})\cdot d\vec{s}$, first we find $\nabla\times\vec{B}$

$$\nabla\times\vec{B} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ B_r & rB_\phi & B_z \end{vmatrix}$$

$$= \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\cos\phi}{r} \end{vmatrix}$$

$$= \frac{1}{r}\left\{\hat{r}\left(\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{r}\right) - \cancel{\frac{\partial}{\partial z}(0)}\right) - r\hat{\phi}\left(\frac{\partial}{\partial r}\left(\frac{\cos\phi}{r}\right) - \cancel{\frac{\partial}{\partial z}(0)}\right) + \hat{z}\left(\cancel{\frac{\partial}{\partial r}(0)} - \cancel{\frac{\partial}{\partial\phi}(0)}\right)\right\}$$

$$= \frac{\hat{r}}{r}\frac{1}{r}(-\sin\phi) - \frac{\cancel{r}\hat{\phi}}{\cancel{r}}\cos\phi\left(\frac{-1}{r^2}\right)$$

$$\nabla\times\vec{B} = -\frac{\hat{r}}{r^2}\sin\phi + \frac{\hat{\phi}}{r^2}\cos\phi$$

Then:

$$\int_S (\nabla\times\vec{B})\cdot d\vec{S} = \int_S \left(-\frac{\hat{r}}{r^2}\sin\phi + \frac{\hat{\phi}}{r^2}\cos\phi\right)\cdot (\hat{r}\, r\,d\phi\,dz)$$

$$= -\int_S \frac{\sin\phi}{r^2} r\,d\phi\,dz$$

$$= \frac{-1}{r}\int_{\phi=\frac{\pi}{3}}^{\pi/2} \sin\phi\,d\phi \int_{z=0}^{3} dz \quad \left(\text{at the surface } r=2\right)$$

$$= \frac{-1}{2}\left(-\cos\phi\Big|_{\frac{\pi}{3}}^{\pi/2}\right)\left(z\Big|_0^3\right)$$

$$= \frac{1}{2}\left(\underbrace{\cos\frac{\pi}{2}}_{0} - \underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}}\right)(3-0)$$

$$\boxed{\int_S (\nabla\times\vec{B})\cdot d\vec{S} = -\frac{3}{4}}$$

As a result: $\int_S (\nabla\times\vec{B})\cdot d\vec{S} = \oint_C \vec{B}\cdot d\vec{l}$, then Stoke's Theorem is verified.

Scalar Laplacian Operator :

- The divergence of the gradient of a scalar V is called Laplacian of V and is denoted by $\nabla^2 V$ (it's pronounced del square of V).
- For a scalar function V defined in Cartesian coord., it's gradient is:

$$\nabla V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}$$

$$:= \hat{x}E_x + \hat{y}E_y + \hat{z}E_z = \vec{E}$$

Then the divergence of ∇V (or $\vec{E}$) is :

$$\nabla\cdot(\nabla V) = \nabla\cdot\vec{E}$$

$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla\cdot(\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

That is ; $\nabla^2 V = \nabla\cdot(\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

- for a scalar function V defined in cylindrical coord. :

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

- for a scalar function V defined in spherical coord. :

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right)$$

$$+ \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

The Laplacian Operator and Vector fields:

- If the scalar Laplacian operator is applied to a vector field, it acts on each component in turn and generates a vector field.
- For a vector field $\vec{E}$ in Cartesian coord.:

$$\vec{E} = \hat{x} E_x + \hat{y} E_y + \hat{z} E_z$$

the vector Laplacian of $\vec{E}$ is

$$\nabla^2 \vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{E}$$

$$\boxed{\nabla^2 \vec{E} = \hat{x} \nabla^2 E_x + \hat{y} \nabla^2 E_y + \hat{z} \nabla^2 E_z}$$

Also

$$\nabla^2 \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla \times (\nabla \times \vec{E})$$

Chapters 4 & 5 :

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ϵ and σ	μ
Governing equations		
• Differential form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• Integral form	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\epsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

4.1 Maxwell's Equations:

$$\nabla \cdot \vec{D} = \rho_v \quad \sim (1)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \sim (2)$$

$$\nabla \cdot \vec{B} = 0 \quad \sim (3)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \sim (4)$$

$\vec{E}$: electric field intensity

$\vec{D}$: electric flux density

ϵ: electric permitivity

$$\vec{D} = \epsilon \vec{E}$$

$\vec{H}$: magnetic field intensity

$\vec{B}$: magnetic flux density

μ: magnetic permeability

$$\vec{B} = \mu \vec{H}$$

Under static conditions, none of the quantities appearing in Maxwell's equations are functions of time. i.e. $\frac{\partial}{\partial t} = 0$.

This is when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that ρ_v and $\vec{J}$ are constant in time.

Equ.s from (1) to (4) decouple and become:

Electrostatics:

$$\nabla \cdot \vec{D} = \rho_v , \quad \nabla \times \vec{E} = 0$$

Magnetostatics:

$$\nabla \cdot \vec{B} = 0 , \quad \nabla \times \vec{H} = \vec{J}$$

4.2 Charge and Current Distributions:

In electromagnetics, charges and currents may be distributed over a volume of space, across a surface, or along a line.

When these charge distributions are in motion, they form current distributions.

4.2.1 Charge Densities:

In atomic scale, the charge distribution is discrete, in electromagnetics, we study the distribution at a larger scale where the distribution is continuous.

(A) Volume charge distribution

(B) Surface charge distribution

(C) Line charge distribution

(A) Volume charge distribution

ρ_v : volume charge density

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv} \quad (C/m^3)$$

Δq is the charge contained in Δv.

The total volume charge contained in volume v is

$$Q = \int_v \rho_v \, dv \quad (C)$$

(B) Surface charge distribution

ρ_s : surface charge density

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad (C/m^2)$$

Δq is the charge present across an element surface area ΔS

The total surface charge present on surface S is

$$Q = \int_S \rho_s \, ds \quad (C)$$

(C) Line charge distribution

ρ_l : line charge density

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad (C/m)$$

Δq is the charge held to a line Δl,

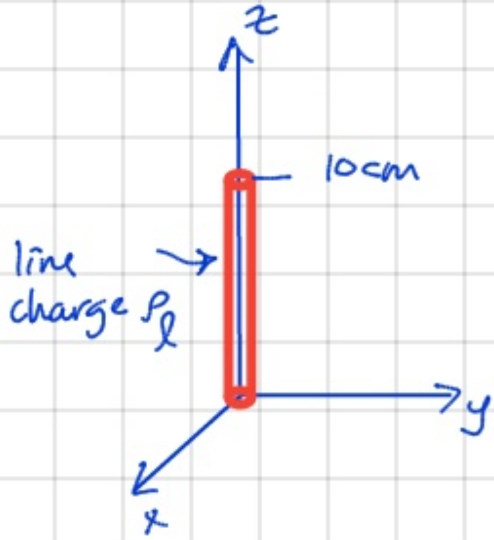
it doesn't need to be straight.

The total line charge contained in length l is

$$Q = \int_l \rho_l \, dl \quad (C)$$

Example:

Calculate the total charge Q contained in a cylindrical tube oriented along the z-axis as shown. The line density is $\rho_\ell = 2z$, where z is the distance in meters from the bottom end of the tube. The tube length is 10 cm.



Solution:

Tube length is 0.1 m,

$$Q = \int_\ell \rho_\ell \, d\ell$$

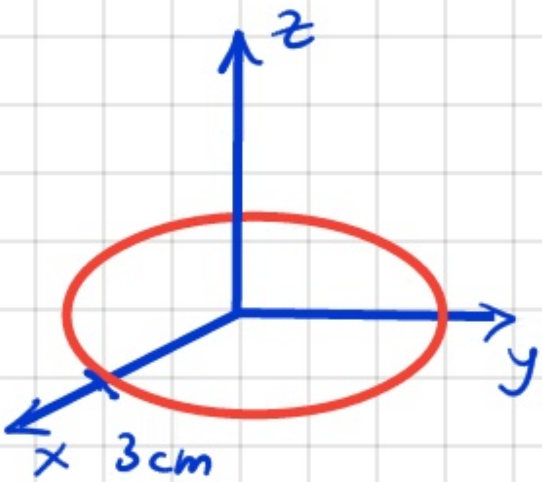
$$= \int_{\ell=0}^{0.1} 2z \, dz$$

$$= 2\frac{z^2}{2} \Big|_{\ell=0}^{0.1}$$

$$= (0.1)^2 - 0$$

$$Q = 0.01 \text{ (C)}$$

Example: The circular disk of electric charge shown is characterized by an azimuthally symmetric surface charge density that increases linearly with r from zero at the center to 6 C/m^2 at $r = 3$ cm. Find the total charge present on the disk surface.

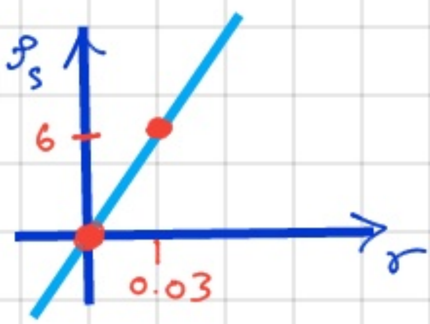


Solution:

$\rho_s = 0$ at $r = 0$ m

$\rho_s = 6$ at $r = 0.03$m

increases linearly, then



$$\rho_s(r) = \left(\frac{6-0}{0.03-0}\right) r$$

$$\rho_s(r) = 200\, r \quad (C/m^2)$$

$$Q = \int_S \rho_s \, ds = \int_S (200 r)(r \, d\phi \, dr)$$

$$Q = \int_S \rho_s \, ds = \int_S (200r)(r\,d\phi\,dr)$$

$$= 200 \int_{r=0}^{0.03} r^2 dr \int_{\phi=0}^{2\pi} d\phi$$

$$= 200 \left(\frac{r^3}{3} \Big|_{r=0}^{0.03} \right) \left(\phi \Big|_{\phi=0}^{2\pi} \right)$$

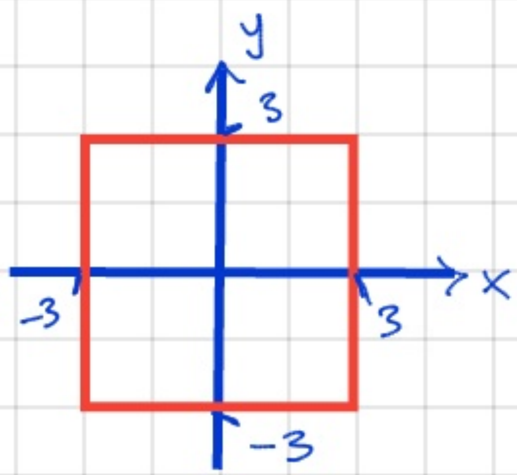
$$= \frac{200}{3} \left((0.03)^3 - 0 \right) \left(2\pi - 0 \right)$$

$$= \frac{400\pi}{\cancel{3}_{1}} \left(\cancel{27}^{9} \times 10^{-6} \right)$$

$$Q = 11.31 \ (mC)$$

Example: A square plate residing in the x-y plane is situated in the space defined by $-3m \le x \le 3m$ and $-3m \le y \le 3m$. Find the total charge on the plane if the surface charge density is

$\rho_s = 4y^2 \; (\mu C/m^2)$.



Solution:

$$Q = \int_S \rho_s \, ds = \int_S 4y^2 \times 10^{-6} \, dx \, dy$$

$$= 4\times10^{-6} \int_{x=-3}^{3} dx \int_{y=-3}^{3} y^2 \, dy$$

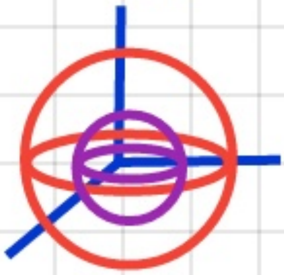
$$= 4\times10^{-6} \left(x \Big|_{x=-3}^{3} \right) \left(\frac{y^3}{3} \Big|_{y=-3}^{3} \right)$$

$$= \frac{4\times10^{-6}}{3} \Big(3-(-3) \Big) \Big((3)^3 - (-3)^3 \Big)$$

$$= \frac{4\times10^{-6}}{3} (6)(27+27)$$

$$Q = 432 \; (\mu C)$$

Example: A thick spherical shell centered at the origin extends between $R = 2$ cm and $R = 3$ cm. If the volume charge density is $\rho_v = 3R \times 10^{-4}\ (C/m^2)$, find the total charge contained in the shell.



Solution:

$$Q = \int_v \rho_v \, dv = \int_v 3R \times 10^{-4} \, R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= 3 \times 10^{-4} \int_{R=0.02}^{0.03} R^3 dR \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

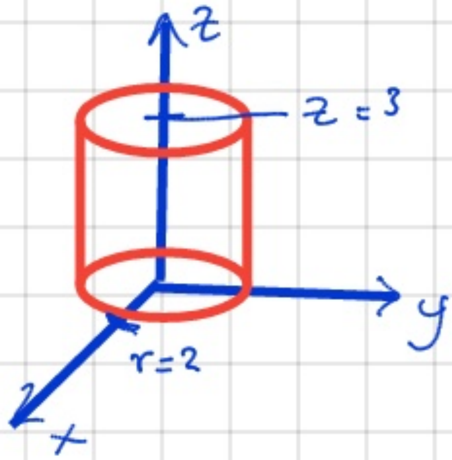
$$= 3 \times 10^{-4} \left(\frac{R^4}{4} \Big|_{R=0.02}^{0.03} \right) \left(-\cos\theta \Big|_{\theta=0}^{\pi} \right) \left(\phi \Big|_{\phi=0}^{2\pi} \right)$$

$$= \frac{3}{4} \times 10^{-4} \left[(0.03)^4 - (0.02)^4 \right] \left[-\cos\pi + \cos 0 \right] \left[2\pi - 0 \right]$$

$$Q = 0.61 \times 10^{-9} = 0.61\ (nC)$$

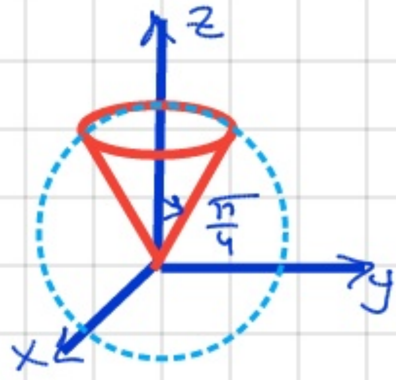
Problem 4.2. Find the total charge contained in a cylindrical volume defined by $r \leq 2\,m$ and $0 \leqslant z \leq 3\,m$ if $\rho_v = 20rz$ (mC/m^2).

Solution:



Problem 4.3. Find the total charge contained in a round-top cone defined by $R \leq 2\,m$ and $0 \leq \theta \leq \frac{\pi}{4}$, given that $\rho_v = 10 R^2 \cos^2\theta \ (mC/m^3)$.

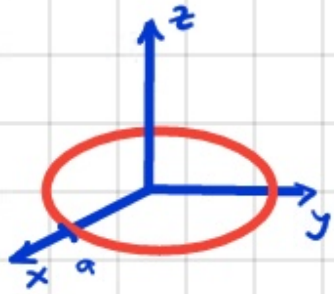
Solution:



Problem 4.5. Find the total charge on a circular disk defined by $r \leq a$ and $z = 0$ if :

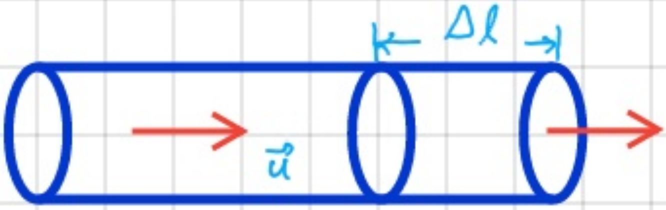
(d) $\rho_s = \rho_{s0} \, e^{-r} \sin^2\phi \; (C/m^2)$

where ρ_{s0} is a constant.



Solution :

4.2.2. Current Density :



$\Delta q = \rho_v \, u \, \Delta s_c \, \Delta t$

ρ_v : volume charge density, Δs_c : cross section area

$\vec{u}$: charge velocity along the tube axis

Over a period Δt, the charges move a distance

$$\Delta l = |\vec{u}| \Delta t = u \Delta t$$

The amount of charge that crosses the tube's cross sectional area Δs_c in time Δt is as follows.

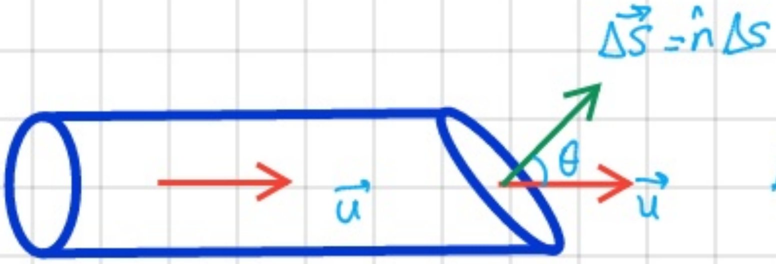
Since $\rho_v = \frac{\Delta q}{\Delta v}$ then

$$\Delta q = \rho_v \, \Delta v$$

$$= \rho_v \, \Delta s_c \, \Delta l$$

$$\Delta q = \rho_v \, \Delta s_c \, u \, \Delta t$$

In a more general case:



$$\Delta q = \rho_v \, \vec{u} \cdot \Delta\vec{S} \, \Delta t$$
$$= \rho_v \, u \, \Delta S \, \Delta t \cos\theta$$

ρ_v : volume charge density , ΔS : surface area

$\vec{u}$: charge velocity along the tube axis

The amount of charge that flows through a surface ΔS with normal $\hat{n}$, where $\hat{n}$ is not necessarily parallel to $\vec{u}$ (the charge velocity along the tube axis), is as follows.

$$\Delta q = \rho_v \, \Delta v$$
$$= \rho_v \, \Delta l \, \Delta S$$
$$= \rho_v \, (\vec{u}\,\Delta t) \cdot (\Delta\vec{S})$$

$$\Delta q = \rho_v \, \Delta t \, \vec{u} \cdot \Delta\vec{S} \qquad \sim \text{equ (1)}$$

where $\Delta\vec{S} = \hat{n}\,\Delta S$

Since

$$\Delta q = \rho_v \, \Delta t \, \vec{u} \cdot \Delta \vec{S} \quad \sim \text{equ (1)}$$

Then $\dfrac{\Delta q}{\Delta t} = \rho_v \, \vec{u} \cdot \Delta \vec{S} =: \Delta I$

where ΔI is the total current flowing in the tube.

So $\Delta I = \rho_v \, \vec{u} \cdot \Delta \vec{S}$

$$\Delta I = \vec{J} \cdot \Delta \vec{S}$$

with $\vec{J} = \rho_v \, \vec{u} \quad (A/m^2)$

where $\vec{J}$ is the current density in ampere per square meter.

In a general form, for an arbitrary surface S, the total current flow through the tube is

$$I = \int_S \vec{J} \cdot d\vec{S} \quad (A)$$

(1) Convection current and convection current density:

Convection current is when the current is due to the actual movement of electrically charged matter. For example, the electron beam of a scanning electron microscope. Here, $\vec{J}$ is called convection current density.

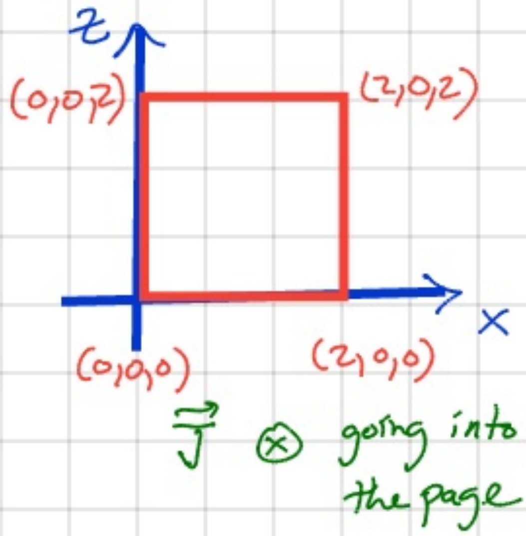
(2) Conduction current and conduction current density.

Conduction current is when the current is due to the movement of charged particles relative to their host material. For example, in a metal wire, if a voltage is applied across the ends of the wire, then the electrons in the outmost electron shells of the atoms can move from one atom to the next, where the electrons that emerge from the wire are not necessarily the same electrons that entered the wire at the other end. Here, $\vec{J}$ is called conduction current density.

- Conduction current obeys Ohm's law, however, convection current does not.

Problem 4.6: If $\vec{J} = \hat{y}\,4xz$ (A/m^2), find the current I flowing through a square with corners $(0,0,0)$, $(2,0,0)$, $(2,0,2)$, and $(0,0,2)$.

Solution:



$$I = \int_S \vec{J} \cdot d\vec{s}$$

$$= \int_S \hat{y}\,4xz \cdot \hat{y}\,dx\,dz$$

$$= 4 \int_{x=0}^{2} x\,dx \int_{z=0}^{2} z\,dz$$

$$= 4 \left(\left. \frac{x^2}{2} \right|_{x=0}^{2} \right) \left(\left. \frac{z^2}{2} \right|_{z=0}^{2} \right)$$

$$= \frac{\cancel{4}}{\cancel{4}} (2^2 - 0)(2^2 - 0)$$

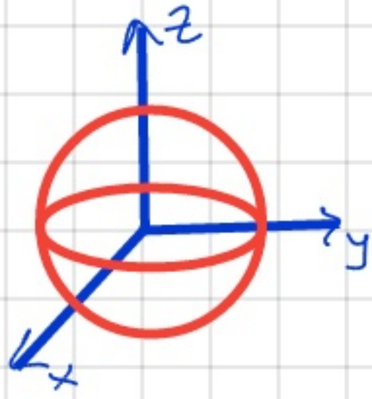
$$\boxed{I = 16\ A}$$

Problem 4.7: If $\vec{J} = \hat{R}\, 5/R$ (A/m^2), find I through the surface $R = 5\,m$.

Solution:

Problem 4.7: If $\vec{J} = \hat{R}\, 5/R$ (A/m^2), find I through the surface $R = 5$ m.

Solution:



$$I = \int_S \vec{J} \cdot d\vec{s}$$

$$= \int_S \hat{R}\frac{5}{R} \cdot \hat{R}\, R^2 \sin\theta \, d\theta \, d\phi$$

$$= 5R \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi \qquad (\text{at } R = 5\text{m})$$

$$= 25 \left(-\cos\theta \Big|_{\theta=0}^{\pi} \right) \left(\phi \Big|_{\phi=0}^{2\pi} \right)$$

$$= 25 \left(-\cos\pi + \cos 0 \right) \left(2\pi - 0 \right)$$

$$\boxed{I = 100\pi \text{ A}}$$

Problem 4.8: An electron beam shaped like a circular cylinder of radius r_0 carries a charge density given by

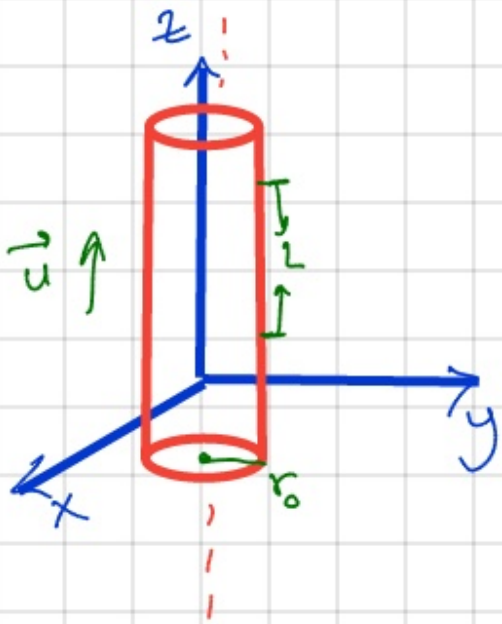
$$\rho_v = \frac{-\rho_0}{1+r^2} \qquad (C/m^3)$$

where ρ_0 is a positive constant and the beam's axis is coincident with the z-axis.

(a) Determine the total charge contained in length L of the beam.

(b) If the electrons are moving in the $+z$ direction with uniform speed u, determine the magnitude and direction of the current crossing the z-plane

(a) Determine the total charge contained in length L of the beam.



Solution:

$$Q = \int_v \rho_v \, dv$$

$$= \int_v \frac{-\rho_o}{1+r^2} \, r \, dr \, d\phi \, dz$$

$$= -\rho_o \int_{r=0}^{r_o} \frac{r}{1+r^2} dr \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{L} dz$$

let $x = 1+r^2 \Rightarrow dx = 2r\,dr \Rightarrow \frac{dx}{2} = r\,dr$

when $r=0$, $x=1$; $r=r_o$, $x = 1+r_o^2$

$$Q = -\rho_o \int_{x=1}^{1+r_o^2} \frac{1}{x} \frac{dx}{2} \left(\phi \Big|_{\phi=0}^{2\pi} \right) \left(z \Big|_{z=0}^{L} \right)$$

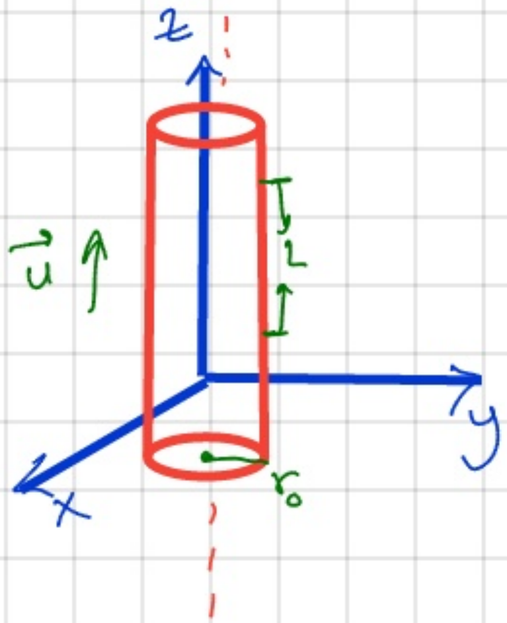
$$= \frac{-\rho_o}{2} \left(\ln x \Big|_{x=1}^{1+r_o^2} \right) (2\pi - 0)(L - 0)$$

$$= -\rho_o \frac{2\pi}{2} L \left(\ln(1+r_o^2) - \cancel{\ln 1}^{\,0} \right)$$

$$Q = -\rho_o \pi L \ln(1+r_o^2) \quad \text{(C)}$$

(b) If the electrons are moving in the $+z$ direction with uniform speed u, determine the magnitude and direction of the current crossing the z-plane ($z = z_{\text{something}}$ plane or simply the xy-plane)

Solution:



$$\vec{J} = \rho_v \vec{u}$$

$$= \frac{-\rho_0}{1+r^2} \hat{z} |\vec{u}|$$

$$\boxed{\vec{J} = \frac{-\rho_0}{1+r^2} \hat{z}\, u}$$

$$\vec{I} = \int_S \vec{J} \cdot d\vec{s}$$

$$= \int_S \frac{-\rho_0}{1+r^2} \hat{z} u \cdot \hat{z}\, r\,dr\,d\phi$$

$$= -\rho_0 u \int_{r=0}^{r_0} \frac{r\,dr}{1+r^2} \int_{\phi=0}^{2\pi} d\phi$$

$$I = -\rho_0 u \int_{r=0}^{r_0} \frac{r\,dr}{1+r^2} \int_{\phi=0}^{2\pi} d\phi$$

let $x = 1+r^2 \Rightarrow dx = 2r\,dr \Rightarrow \frac{dx}{2} = r\,dr$

when $r=0$, $x=1$; $r=r_0$, $x = 1+r_0^2$

$$I = -\rho_0 u \int_{x=1}^{1+r_0^2} \frac{1}{x} \frac{dx}{2} \left(\phi \Big|_{\phi=0}^{2\pi} \right)$$

$$= -\rho_0 \frac{u}{2} \left(\ln x \Big|_{x=1}^{1+r_0^2} \right) \left(2\pi - 0 \right)$$

$$= -\rho_0 u \frac{2\pi}{2} \left(\ln(1+r_0^2) - \cancel{\ln 1}^{\,0} \right)$$

$$I = -\rho_0 u \pi \ln(1+r_0^2) \quad (A)$$

Since ρ_0 is a positive constant, then $\vec{J}$ is in the $(-\hat{z})$ direction as well as I.

Problem 4.9. A circular beam of charge of radius a consists of electrons moving with a constant speed u along the $+z$ direction. The beam's axis is coincident with the z-axis and the electron charge density is given by

$$\rho_v = -cr^2 \qquad (C/m^3)$$

where c is a constant and r is the radial distance from the axis of the beam.

(a) Determine the charge density per unit length.

(b) Determine the current crossing the z-plane

(a) Determine the charge density per unit length.

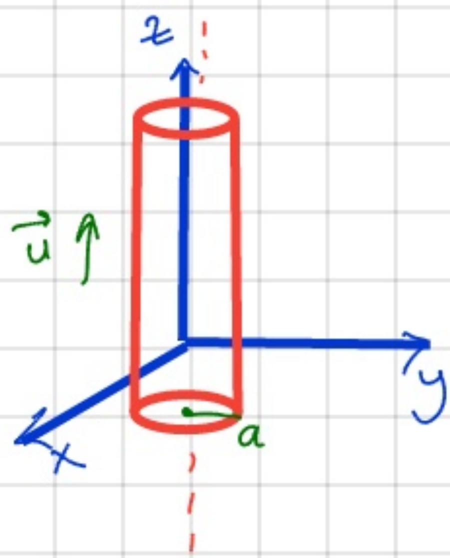
Solution:

Since $\rho_v = \dfrac{\Delta q}{\Delta v}$ and $\Delta v = \Delta s \Delta l$

then $\rho_v = \dfrac{\Delta q}{\Delta s \Delta l} \Rightarrow \dfrac{\Delta q}{\Delta l} = \rho_v \Delta s$

since $\dfrac{\Delta q}{\Delta l} = \rho_l$ then $\boxed{\rho_l = \rho_v \Delta s}$

For a differential area element ds,



$$\rho_l = \int_S \rho_v \, ds$$

$$= \int_S -cr^2 \,(r\,dr\,d\phi)$$

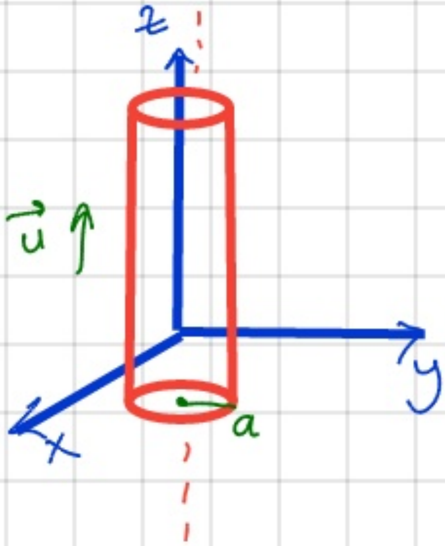
$$= -c \int_{r=0}^{a} r^3 \, dr \int_{\phi=0}^{2\pi} d\phi$$

$$= -c \left(\frac{r^4}{4} \Big|_{r=0}^{a} \right) \left(\phi \Big|_{\phi=0}^{2\pi} \right) = \frac{-c}{4}\left(a^4 - 0\right)\left(2\pi - 0\right)$$

$$\boxed{\rho_l = \frac{-c}{2} \pi a^4}$$

(b) Determine the current crossing the z-plane ($z = z_{\text{something}}$ plane or simply the xy-plane)

Solution:



$$\vec{J} = \rho_v \, \vec{u}$$

$$= -cr^2 \, \hat{z} \, |\vec{u}|$$

$$\boxed{\vec{J} = -\hat{z} \, c r^2 u \quad (A/m^2)}$$

$$I = \int_S \vec{J} \cdot d\vec{s} = \int_S -\hat{z} \, c r^2 u \cdot \hat{z} \, r dr d\phi$$

$$= -cu \int_{r=0}^{a} r^3 dr \int_{\phi=0}^{2\pi} d\phi$$

$$= -cu \left(\frac{r^4}{4} \Big|_{r=0}^{a} \right) \left(\phi \Big|_{\phi=0}^{2\pi} \right)$$

$$= \frac{-cu}{4} \left(a^4 - 0 \right) \left(2\pi - 0 \right)$$

$$\boxed{I = -\frac{cu}{2} \pi a^4 \quad (A)}$$

also $\boxed{I = \rho_\ell \, u \; (A)}$ since $\rho_\ell = -\frac{c\pi a^4}{2}$

4.3. Coulomb's law :

. The fundamental quantity of charge is a single electron, the unit an electron is measured by is the coulomb (C). The magnitude of e is $e = 1.6 \times 10^{-19}$ (C), where the charge of a single electron is $q_e = -e$, and the charge of a single proton is $q_p = +e$.

- Coulomb's experiments demonstrated that:

(1) Two like charges repel one another, whereas two charges of opposite polarity attract.

(2) The force acts along the line joining the charges.

(3) The force strength is proportional to the product of the magnitudes of the two charges and inversely proportional to the square of the distance between them.

- Coulomb's law when expressed mathematically is:

$$\vec{F}_{e21} = \hat{R}_{12} \frac{q_1 q_2}{4\pi\epsilon_0 R_{12}^2} \quad (N) \quad \text{(in free space)}$$

where $\vec{F}_{e21}$ is the electrical force acting on charge q_2 due to charge q_1 when both are in free space,

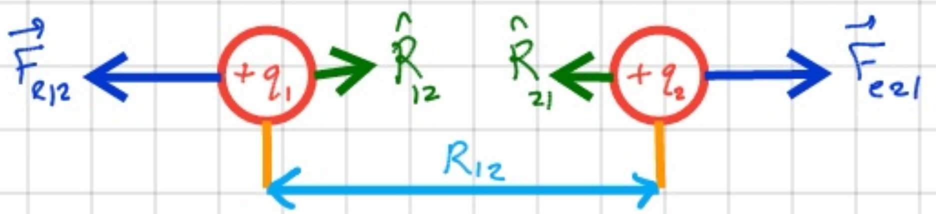
R_{12} is the distance between the two charges,

$\hat{R}_{12}$ is the unit vector pointing from charge q_1 to q_2,

ϵ_0 is a constant called the electrical permittivity of free space, where $\epsilon_0 = 8.854 \times 10^{-12}$ (F/m)

⇒ farad per meter

permittivity is the ability of a material to store electric energy in an electric field



- The force $\vec{F}_{e12}$ acting on charge q_1 due to charge q_2 is equal to force $\vec{F}_{e21}$ in magnitude but opposite in direction:

$$\vec{F}_{e12} = -\vec{F}_{e21}$$

- If any point charge q' is present in an electric field $\vec{E}$ due to other charges, the point charge will experience a force acting on it equal to

$$\vec{F} = q'\vec{E}$$

- Properties for an electric charge:

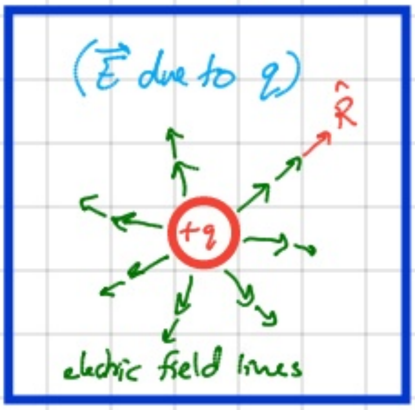
(1) Law of conservation of electric charge:

The net electric charge can neither be created nor destroyed.

(2) Principle of linear superposition:

The total vector electric field at a point in space due to a system of point charges is equal to the vector sum of the electric fields at that point due to the individual charges

- The electric field intensity $\vec{E}$ in free space due to any charge q is :



$$\vec{E} = \hat{R}\,\frac{q}{4\pi\epsilon_0 R^2} \quad (V/m)$$

where R is the distance between the charge and the observation point,

$\hat{R}$ is the radial unit vector pointing away from the charge

- The electric field intensity $\vec{E}$ in any medium due to any charge q is :

$$\vec{E} = \hat{R}\,\frac{q}{4\pi\epsilon R^2} \quad (V/m)$$

where ϵ is the permittivity of the material in which the electric field is measured, with $\epsilon = \epsilon_r \epsilon_0$

where ϵ_r is the relative permittivity (or dielectric constant) of the material and it is dimensionless.

for free-space (a vacuum) $\epsilon_r = 1$

- The electric flux density $\vec{D}$ is given by

$$\vec{D} = \epsilon \vec{E} \quad (C/m^2)$$

- For most materials and under a wide range of conditions, ϵ is independent of both the magnitude and direction of $\vec{E}$.
- If ϵ is independent of the magnitude of $\vec{E}$, then the material is said to be linear, because $\vec{D}$ and $\vec{E}$ are related linearly
- If ϵ is independent of the direction of $\vec{E}$, then the material is said to be isotropic.

(isotropic (meaning): not varying in magnitude according to the direction of measurement)

4.3.1 Electric Field Due to Multiple Point Charges:

- Since the electric field intensity $\vec{E}$ in any medium due to any charge q is:

$$\vec{E} = \hat{R}\frac{q}{4\pi\epsilon R^2} =: \frac{q}{4\pi\epsilon}\frac{\vec{R}}{R}\frac{1}{R^2} = \frac{q}{4\pi\epsilon}\frac{\vec{R}}{R^3}$$

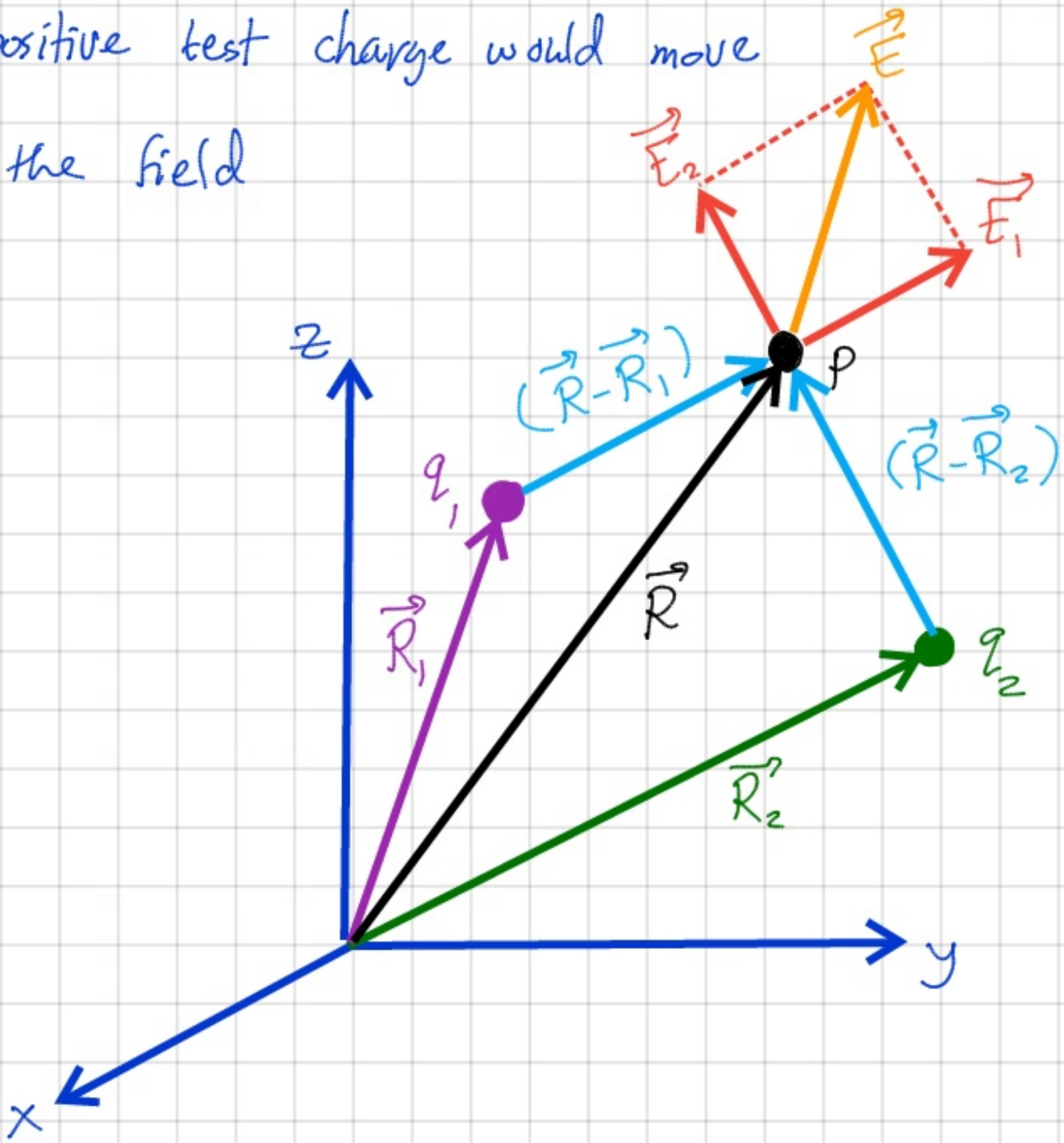
and from the principle of linear superposition:

> The total vector electric field at a point in space due to a system of point charges is equal to the vector sum of the electric fields at that point due to the individual charges.

Then the electric field $\vec{E}$ at point P with position vector $\vec{R}$ due to charges $q_1, q_2, \dots, q_N$ located at points with position vectors $\vec{R}_1, \vec{R}_2, \dots, \vec{R}_N$, equals the vector sum of the electric fields induced by all the individual charges as:

$$\vec{E} = \frac{1}{4\pi\epsilon}\sum_{i=1}^{N} q_i \frac{(\vec{R}-\vec{R}_i)}{|\vec{R}-\vec{R}_i|^3} \qquad (V/m)$$

- The electric field direction is the direction a positive test charge would move in the field



The electric field $\vec{E}$ at point P due to two charges q_1 and q_2 is equal to the vector sum of $\vec{E}_1$ and $\vec{E}_2$.

Example : Two point charges with $q_1 = 2 \times 10^{-5}$ C and $q_2 = -4 \times 10^{-5}$ C are located in free space at points with Cartesian coordinates $(1, 3, -1)$ and $(-3, 1, -2)$, respectively. Find :

(a) The electric field $\vec{E}$ at $(3, 1, -2)$

(b) The force on a 8×10^{-5} C charge located at that point.

All distances are in meters.

Solution: (a) Since $\vec{E} = \dfrac{1}{4\pi\epsilon_0} \displaystyle\sum_{i=1}^{2} q_i \dfrac{\vec{R} - R_i}{|\vec{R} - \vec{R}_i|^3}$

$\vec{R}_1$ is from $(0,0,0)$ to $(1, 3, -1)$ then

$\vec{R}_1 = \hat{x}(1-0) + \hat{y}(3-0) + \hat{z}(-1-0)$

$\vec{R}_1 = \hat{x} + 3\hat{y} - \hat{z}$

Similarly, from $P_2 = (-3, 1, -2)$, $\vec{R}_2 = -3\hat{x} + \hat{y} - 2\hat{z}$

and from $P = (3, 1, -2)$, $\vec{R} = 3\hat{x} + \hat{y} - 2\hat{z}$

$\vec{R}_1 = \hat{x} + 3\hat{y} - \hat{z}$, $\vec{R}_2 = -3\hat{x} + \hat{y} - 2\hat{z}$, $\vec{R} = 3\hat{x} + \hat{y} - 2\hat{z}$

$$\vec{R} - \vec{R}_1 = (3\hat{x} + \hat{y} - 2\hat{z}) - (\hat{x} + 3\hat{y} - \hat{z})$$

$$\vec{R} - \vec{R}_1 = 2\hat{x} - 2\hat{y} - \hat{z}$$

and $|\vec{R} - \vec{R}_1| = \sqrt{2^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3$

$$\vec{R} - \vec{R}_2 = (3\hat{x} + \hat{y} - 2\hat{z}) - (-3\hat{x} + \hat{y} - 2\hat{z}) = 6\hat{x}$$

$$|\vec{R} - \vec{R}_2| = \sqrt{6^2} = 6$$

Then $\vec{E} = \dfrac{1}{4\pi\epsilon_0} \displaystyle\sum_{i=1}^{2} q_i \dfrac{\vec{R} - R_i}{|\vec{R} - \vec{R}_i|^3}$

$$= \frac{1}{4\pi\epsilon_0}\left[q_1 \frac{\vec{R} - \vec{R}_1}{|\vec{R} - \vec{R}_1|^3} + q_2 \frac{\vec{R} - \vec{R}_2}{|\vec{R} - \vec{R}_2|^3} \right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[2\times10^{-5} \frac{(2\hat{x} - 2\hat{y} - \hat{z})}{(3)^3} - 4\times10^{-5} \frac{(6\hat{x})}{(6)^3} \right]$$

$$= \frac{10^{-5}}{4\pi\epsilon_0 (216)} \left[(\cancel{32 - 24})\hat{x} - 32\hat{y} - 16\hat{z} \right]$$

8

$$\vec{E} = \frac{10^{-5}}{108\pi\epsilon_0} (\hat{x} - 4\hat{y} - 2\hat{z}) \quad (V/m)$$

(b) The force on a 8×10^{-5} C charge located at that point.

Solution: Since at that point

$$\vec{E} = \frac{10^{-5}}{108\pi\epsilon_0}\left(\hat{x} - 4\hat{y} - 2\hat{z}\right) \quad (V/m)$$

Then $\vec{F} = q\vec{E}$

$$= (8 \times 10^{-5}) \frac{10^{-5}}{108\pi\epsilon_0}\left(\hat{x} - 4\hat{y} - 2\hat{z}\right)$$

$$\vec{F} = \frac{2 \times 10^{-10}}{27\pi\epsilon_0}\left(\hat{x} - 4\hat{y} - 2\hat{z}\right) \quad (N)$$

4.3.2 Electric Field Due to a Charge Distribution:

Since the electric field for discrete point charges is

$$\vec{E} = \frac{1}{4\pi\epsilon} \sum_{i=1}^{N} q_i \frac{(\vec{R} - \vec{R}_i)}{|\vec{R} - \vec{R}_i|^3} \qquad (V/m)$$

We can extend it to continuous charge distributions as follows. The differential electric field at a point P due to a differential amount of charge dq is

$$d\vec{E} = \frac{dq}{4\pi\epsilon} \frac{\hat{R}}{R^2}$$

where $\hat{R}$ is the unit vector pointing from dq to P.

Then by integrating the electric fields due to all differential charges we get $\vec{E}$ depending on the charge distribution:

(A) Volume charge distribution
(B) Surface charge distribution
(c) Line charge distribution

Since
$$\boxed{d\vec{E} = \frac{dq}{4\pi\epsilon}\frac{\hat{R}}{R^2}}$$

where $\hat{R}$ is the unit vector pointing from dq to P.

(A) Volume charge distribution $\left(\rho_v = \frac{dq}{dv}\right)$:

$$\vec{E} = \int_v d\vec{E} = \frac{1}{4\pi\epsilon}\int_v \rho_v \frac{\hat{R}}{R^2}\, dv$$

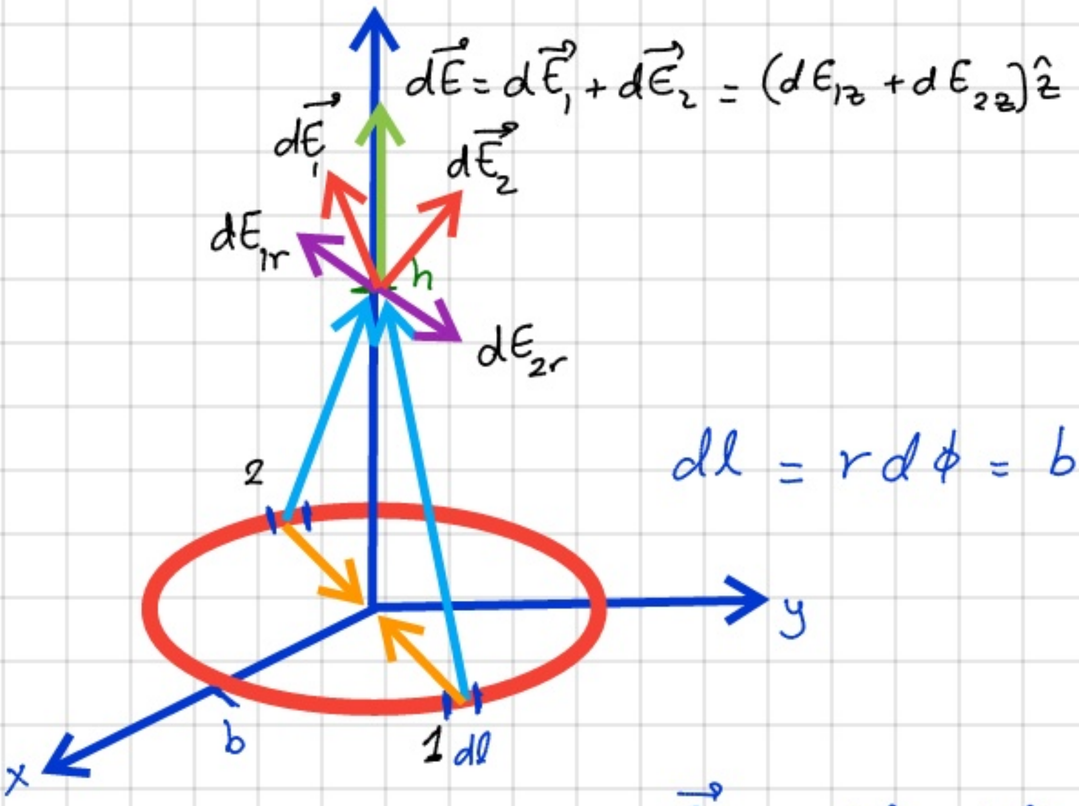
(B) Surface charge distribution $\left(\rho_s = \frac{dq}{ds}\right)$:

$$\vec{E} = \int_S d\vec{E} = \frac{1}{4\pi\epsilon}\int_S \rho_s \frac{\hat{R}}{R^2}\, ds$$

(C) Line charge distribution $\left(\rho_l = \frac{dq}{dl}\right)$:

$$\vec{E} = \int_l d\vec{E} = \frac{1}{4\pi\epsilon}\int_l \rho_l \frac{\hat{R}}{R^2}\, dl$$

Example: A ring of charge of radius b is characterized by a uniform line charge density of positive polarity ρ_ℓ. The ring resides in free space and is positioned in the xy-plane as shown. Determine the electric field intensity $\vec{E}$ at a point $P = (0,0,h)$ along the axis of the ring at a distance h from its center.



$$d\ell = r\,d\phi = b\,d\phi$$

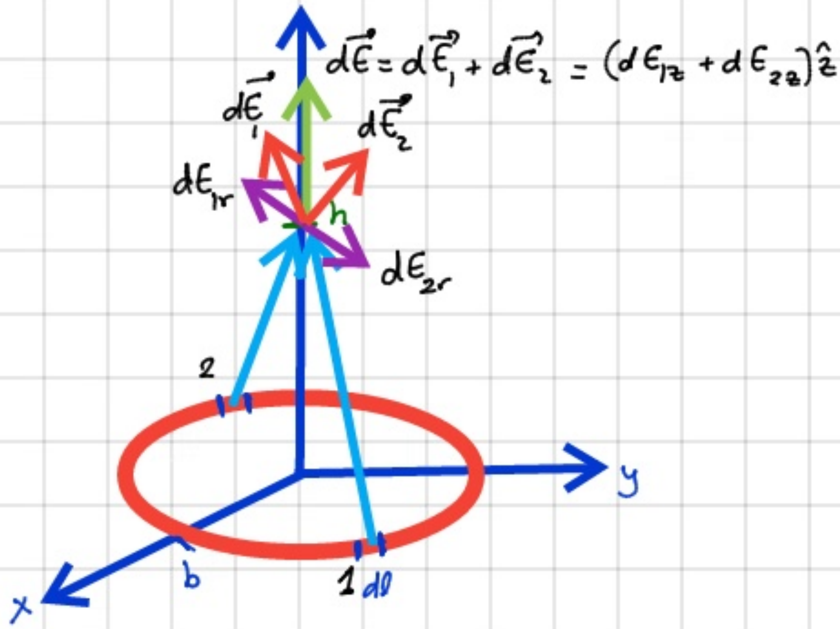
$$\vec{R}_1 = -b\hat{r} + h\hat{z}$$

$$\vec{R}_2 = b\hat{r} + h\hat{z}$$

$$\vec{R}_1 = -b\hat{r} + h\hat{z}$$

$$\vec{R}_2 = b\hat{r} + h\hat{z}$$

$$dl = b\, d\phi$$



$$\vec{E} = \int_l d\vec{E} = \frac{1}{4\pi\epsilon} \int_l \rho_l \frac{\hat{R}}{R^2}\, dl$$

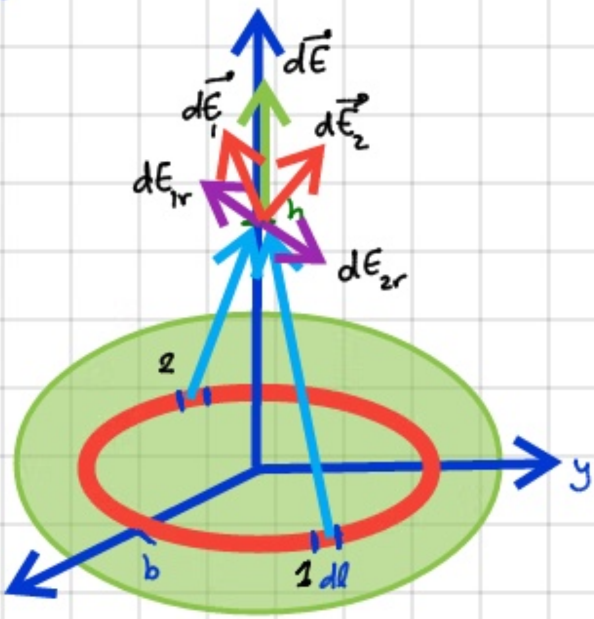
$$= \frac{1}{4\pi\epsilon_0}\left[\int_{\phi=0}^{\pi} \rho_l \frac{(-b\hat{r} + h\hat{z})}{(\sqrt{b^2+h^2})^3}\, b\,d\phi + \int_{\phi=\pi}^{2\pi} \rho_l \frac{(b\hat{r} + h\hat{z})}{(\sqrt{b^2+h^2})^3}\, b\,d\phi\right]$$

$$= \frac{\rho_l b}{4\pi\epsilon_0} \frac{1}{(b^2+h^2)^{3/2}}\left[(-b\hat{r} + h\hat{z})\int_{\phi=0}^{\pi} d\phi + (b\hat{r} + h\hat{z})\int_{\phi=\pi}^{2\pi} d\phi\right]$$

$$= \frac{\rho_l b}{4\pi\epsilon_0 (b^2+h^2)^{3/2}}\left[(\cancel{-b\hat{r}} + h\hat{z})\pi + (\cancel{b\hat{r}} + h\hat{z})\pi\right]$$

$$\vec{E} = \frac{\rho_l b h \hat{z}}{2\epsilon_0 (b^2+h^2)^{3/2}} = \frac{Q h \hat{z}}{4\pi\epsilon_0 (b^2+h^2)^{3/2}} \quad (Q = 2\pi b \rho_l)$$

Example: Find the electric field at point P with Cartesian coordinates (0 0, h) due to a circular disk of radius a and uniform charge density ρ_s residing in the xy-plane, using the results from the previous problem.



For the ring, $(Q = 2\pi b\,\rho_\ell)$

$$\vec{E} = \frac{\rho_\ell\, b\, h\, \hat{z}}{2\epsilon_0 (b^2+h^2)^{3/2}}$$

$$= \frac{Q h \hat{z}}{4\pi\epsilon_0 (b^2+h^2)^{3/2}}$$

For the whole disk, $ds = \overbrace{(r d\phi)}^{\text{ring}}\, dr$

for the ring, $\rho_\ell = \frac{dq}{dl} \Rightarrow$ for any r, $Q = 2\pi r \rho_\ell$

since $Q = \int \rho_\ell\, dl = \int_{\phi=0}^{2\pi} \rho_\ell\, r\, d\phi = 2\pi r \rho_\ell$

for the disk, $\rho_s = \frac{dq}{ds} \Rightarrow$ for any r

$$Q = \int_S \rho_s\, ds = \int_S \rho_s\, r\, dr\, d\phi = \rho_s \int_{\phi=0}^{2\pi} d\phi \int r\, dr$$

$$Q = 2\pi \rho_s \int r\, dr$$

Then for the disk $d\vec{E} = \frac{Q h \hat{z}}{4\pi\epsilon_0 (r^2+h^2)^{3/2}}$, for any r,

$$d\vec{E} = \frac{2\pi \rho_s\, r\, dr}{4\pi\epsilon_0 (r^2+h^2)^{3/2}}\, h\, \hat{z}$$

$$d\vec{E} = \frac{\cancel{2\pi}\rho_s r\,dr}{{}_2\cancel{4\pi}\epsilon_0 (r^2+h^2)^{3/2}}\, h\,\hat{z}$$

$$\vec{E} = \frac{\rho_s h}{2\epsilon_0}\,\hat{z}\int_{r=0}^{a} \frac{r\,dr}{(r^2+h^2)^{3/2}}$$

let $r^2+h^2 = u$ | when $r=0$, $u=h^2$

$2r\,dr = du$ | when $r=a$, $u=a^2+h^2$

$r\,dr = \frac{du}{2}$

Then

$$\vec{E} = \frac{\rho_s h}{2\epsilon_0}\,\hat{z}\int_{u=h^2}^{a^2+h^2} \frac{1}{2}\,u^{-3/2}\,du$$

$$= \frac{\rho_s h}{4\epsilon_0}\,\hat{z}\left[\left.\frac{u^{-1/2}}{-1/2}\right|_{u=h^2}^{a^2+h^2}\right]$$

$$\vec{E} = -\hat{z}\,\frac{\rho_s h}{2\epsilon_0}\left[\frac{1}{\sqrt{a^2+h^2}} - \frac{1}{\sqrt{h^2}}\right]$$

$$= \frac{-\hat{z}\,\rho_s h}{2\epsilon_0}\left[\frac{|h| - \sqrt{a^2+h^2}}{|h|\sqrt{a^2+h^2}}\right]$$

($\vec{E} > 0$ when $h > 0$, $\vec{E} < 0$ when $h < 0$)

Example: Evaluate $\vec{E}$ due to an infinite sheet of charge density ρ_s by letting $a \to \infty$ using the results from the previous problem.

From the last example:

$$\vec{E} = \frac{-\hat{z}\rho_s h}{2\epsilon_0}\left[\frac{|h| - \sqrt{a^2+h^2}}{|h|\sqrt{a^2+h^2}}\right]$$

$$\left(\vec{E} > 0 \text{ when } h > 0 \ , \ \vec{E} < 0 \text{ when } h < 0\right)$$

if $a \to \infty$ then $\vec{E} \to \frac{-\hat{z}\rho_s}{2\epsilon_0}$

since $\lim\limits_{a\to\infty}\left[\frac{|h| - \sqrt{a^2+h^2}}{|h|\sqrt{a^2+h^2}}\right]$

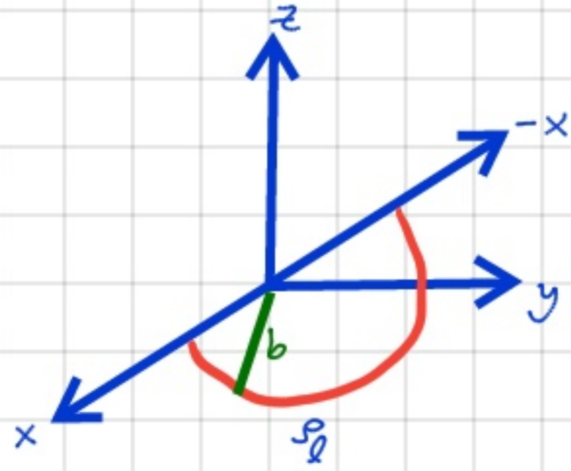
$$= \lim_{a\to\infty} \frac{-\frac{1}{2}(2a)(a^2+h^2)^{-\frac{1}{2}}}{|h|\left(\frac{1}{2}\right)(2a)(a^2+h^2)^{-\frac{1}{2}}} = \frac{-1}{|h|}$$

then $$\vec{E} = -\frac{\hat{z}\rho_s}{2\epsilon_0}\left(\frac{h}{-|h|}\right)$$

$$\Rightarrow \vec{E} = \frac{\hat{z}\rho_s}{2\epsilon_0}\left(\frac{h}{|h|}\right)$$

$$\left(\vec{E} > 0 \text{ when } h > 0 \ , \ \vec{E} < 0 \text{ when } h < 0\right)$$

Problem 4.10 A line of charge of uniform density ρ_ℓ occupies a semicircle of radius b as shown. Use the material presented in Example 4.4 to determine the electric field at the origin.



$$\vec{E} = \int_\ell d\vec{E} = \frac{1}{4\pi\epsilon} \int_\ell \rho_\ell \frac{\hat{R}}{R^2} \, d\ell$$

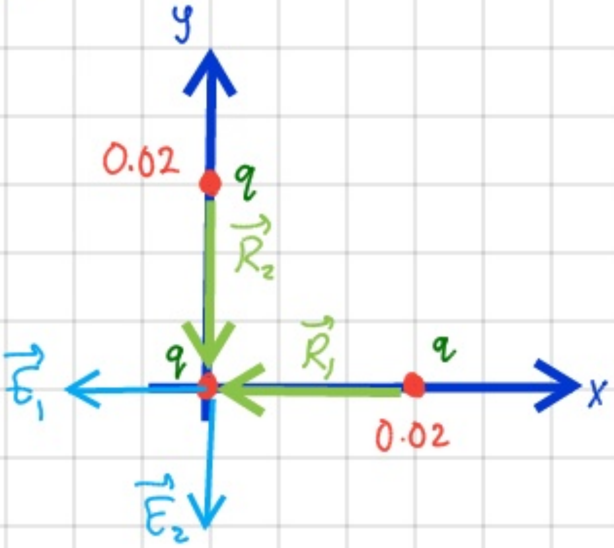
$$= \frac{1}{4\pi\epsilon_0} \int_{\phi=0}^{\pi} \rho_\ell \frac{(-b\hat{r} + h\hat{z})}{\left(\sqrt{b^2+h^2}\right)^3} \, b\,d\phi \qquad (\text{at } h=0)$$

$$= \frac{\rho_\ell b}{4\pi\epsilon_0} \frac{-b\hat{r}}{(b^2)^{3/2}} \int_{\phi=0}^{\pi} d\phi$$

$$= \frac{-\rho_\ell b^2}{4\pi\epsilon_0} \frac{\hat{r}}{b^3} \left(\phi \Big|_{\phi=0}^{\pi} \right) = \frac{-\rho_\ell}{4\pi\epsilon_0} \frac{\hat{r}}{b} (\pi - 0)$$

$$\boxed{\vec{E} = \frac{-\rho_\ell}{4\epsilon_0 b} \hat{r}}$$

Problem 4.12 Three point charges, each with $q = 3\,nC$, are located at the corners of a triangle in the xy-plane, with one corner at the origin, another at (2 cm, 0, 0), and the third at (0, 2 cm, 0). Find the force acting on the charge located at the origin.



$$\vec{F} = q\vec{E}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

$$= \frac{1}{4\pi\epsilon}\left[q\frac{\vec{R}_1}{R_1^3} + q\frac{\vec{R}_2}{R_2^3}\right]$$

$$= \frac{q}{4\pi\epsilon}\left[\frac{-0.02\,\hat{x}}{(0.02)^3} + \frac{-0.02\,\hat{y}}{(0.02)^3}\right]$$

$$= \frac{-q}{4\pi\epsilon}\,\frac{1}{4\times10^{-4}}\,(\hat{x}+\hat{y})$$

$$\vec{E} = \frac{-q\times10^4}{8\pi\epsilon}\,(\hat{x}+\hat{y})$$

$$\vec{F} = q\vec{E} = -\frac{q^2\times10^4}{8\pi\epsilon}\,(\hat{x}+\hat{y})$$

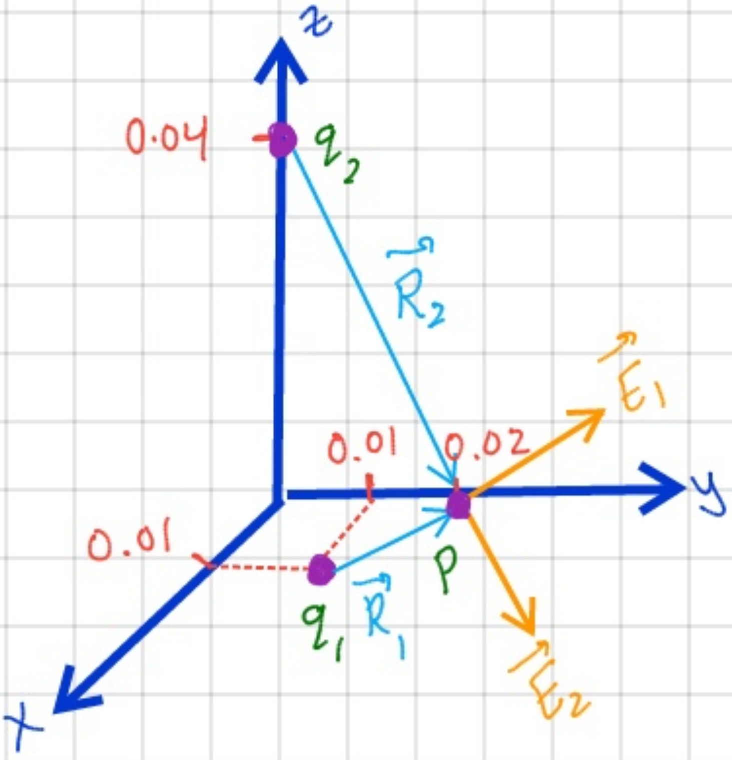
Problem 4.13 Charge $q_1 = 6\,\mu C$ is located at (1 cm, 1 cm, 0) and charge q_2 is located at (0, 0, 4 cm). What should q_2 be so that $\vec{E}$ at (0, 2 cm, 0) has no y component.

$\vec{R}_1$: (0.01, 0.01, 0) to (0, 0.02, 0)

$\vec{R}_1 = -0.01\hat{x} + 0.01\hat{y}$

$\vec{R}_2$: (0, 0, 0.04) to (0, 0.02, 0)

$\vec{R}_2 = 0.02\hat{y} - 0.04\hat{z}$



$\vec{E} = \vec{E}_1 + \vec{E}_2$

$$= \frac{1}{4\pi\epsilon}\left[q_1\frac{\vec{R}_1}{R_1^3} + q_2\frac{\vec{R}_2}{R_2^3}\right]$$

$$= \frac{1}{4\pi\epsilon}\left[\frac{6\times10^{-6}(-\hat{x}+\hat{y})\times10^{-2}}{\left(\sqrt{(.01)^2+(.01)^2}\right)^3} + q_2\frac{(2\hat{y}-4\hat{z})\times10^{-2}}{\left(\sqrt{(.02)^2+(.04)^2}\right)^3}\right]$$

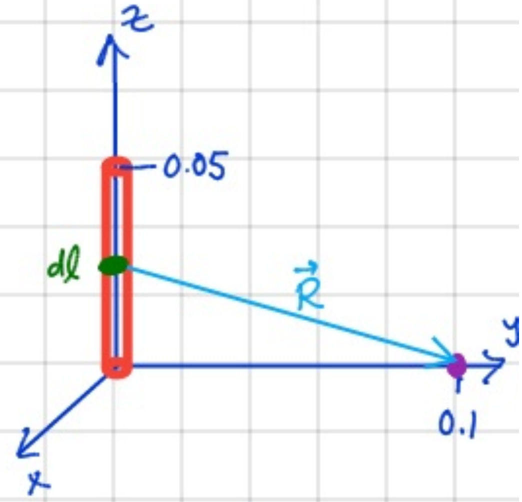
$$\vec{E} = \frac{10^{-2}}{4\pi\epsilon\times10^{-2}}\left[-2.12\times10^{-6}\hat{x} + 2.12\times10^{-6}\hat{y} + q_2 0.0224\hat{y} - q_2 0.0447\hat{z}\right]$$

$\vec{E}$ would have no y-component when

$$2.12\times10^{-6}\hat{y} + q_2 0.0224\hat{y} = 0$$

$$q_2 = \frac{-2.12\times10^{-6}\hat{y}}{0.0224\hat{y}} = -94.64\,\mu C$$

Problem 4.14 A line of charge with uniform density $\rho_\ell = 8\,\mu\, C/m$ exists in air along the z axis between $z=0$ and $z=5$ cm. Find $\vec{E}$ at (0, 10 cm, 0).



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_\ell \rho_\ell \frac{\vec{R}}{R^3}\, dl$$

$\vec{R} = 0.1\hat{y} - z\hat{z}$

$dl = dz$

$$= \frac{1}{4\pi\epsilon_0}\int_{z=0}^{0.05} 8\times10^{-6}\frac{(0.1\hat{y} - z\hat{z})}{[(0.1)^2 + z^2]^{3/2}}\, dz$$

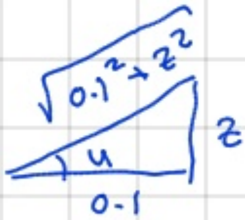
$$= \frac{8\times10^{-6}}{4\pi\epsilon_0}\left[0.1\hat{y}\int_z \frac{dz}{[0.1^2+z^2]^{3/2}} - \hat{z}\int_z \frac{z\,dz}{[0.1^2+z^2]^{3/2}}\right]$$

$\longrightarrow$ find the int.

$$\vec{E} = \frac{2\times10^{-6}}{\pi\epsilon_0}\left[\hat{y}\, 4.47 - \hat{z}\, 1.056\right]$$

1st int.

$$\int_{z} \frac{0.1\,dz}{(0.01+z^2)^{3/2}} \Rightarrow \text{let}$$



$\tan u = \frac{z}{0.1}$

$$z = 0.1 \tan u$$
$$dz = 0.1 \sec^2 u \, du$$
$$\Rightarrow u = \tan^{-1} \frac{z}{0.1}$$

when $z=0 \Rightarrow u=0$

when $z=0.05 \Rightarrow$
$$u = \tan^{-1}\left(\frac{0.05}{0.1}\right)$$
$$u = 26.565 \text{ deg.}$$

also
$$0.1^2 + z^2 = 0.1^2 + 0.1^2 \tan^2 u$$
$$= 0.1^2(1+\tan^2 u)$$
$$= 0.1^2 \sec^2 u$$

Then the integration becomes:

$$\int_{u=0}^{26.565^\circ} \frac{0.1^2 \sec^2 u \, du}{(0.1^2 \sec^2 u)^{3/2}}$$

$$\int_{u=0}^{26.565^\circ} \frac{0.1^2 \sec^2 u \, du}{(0.1^2 \sec^2 u)^{3/2}} = \int_{u} \frac{0.1^2 \sec^2 u \, du}{(0.1^2)^{3/2} (\sec^2 u)^{3/2}}$$

$$= \int \frac{0.1^2}{(0.1)^3} \frac{\sec^2 u}{\sec^3 u} du = \frac{1}{0.1} \int_{u} \frac{1}{\sec u} du$$

$$= 10 \int_{u=0}^{26.565^\circ} \cos u \, du = 10 \left\{ \sin u \Big|_{u=0}^{26.565} \right\} = 4.47$$

(OR) Since $\tan u = z/0.1$, then $\sin u = \frac{z}{\sqrt{0.1^2+z^2}}$

$$\int_{z} \frac{0.1\,dz}{(0.01+z^2)^{3/2}} = \frac{10\,z}{\sqrt{0.1^2+z^2}} \Bigg|_{z=0}^{0.05} = 4.47$$

2nd int. :

$$\int_z \frac{z\,dz}{[0.1^2+z^2]^{3/2}}$$

let $u = 0.1^2 + z^2$ when $z=0 \to u = 0.1^2$

$du = 2z\,dz$ when $z=0.05 \to$

$\frac{du}{2} = z\,dz$ $u = 0.0125$

$$\int_u \frac{1}{2}\frac{du}{u^{3/2}} = \frac{1}{2}\int_u u^{-3/2}\,du = \cancel{\frac{1}{2}}\left[\frac{u^{-1/2}}{-\cancel{1/2}}\Bigg|_{u=0.01}^{0.0125}\right]$$

$$= \left[\frac{-1}{\sqrt{0.0125}} + \frac{1}{\sqrt{0.01}}\right] = 1.056$$

(or) simply

$$\int_z \frac{z\,dz}{[0.1^2+z^2]^{3/2}} = \frac{-1}{\sqrt{0.1^2+z^2}}\Bigg|_{z=0}^{0.05} = \left[-8.944 + 10\right] = 1.056$$

4.4 Gauss's Law

- Gauss's Law can be used to find $\vec{E}$, however, it is limited to symmetrical charge distribution. While Coulomb's law can be used to find $\vec{E}$ for any specified charge distribution.
- The differential form of Gauss's law is:

$$\nabla \cdot \vec{D} = \rho_v \quad \sim (1)$$

where $\vec{D}$ is the flux density and $\vec{D} = \epsilon \vec{E}$.

- To convert the differential form of Gauss's law to an integral form, multiply both sides of (1) by dv and integrate over an arbitrary volume v:

$$\nabla \cdot \vec{D}\, dv = \rho_v\, dv$$

$$\int_v \nabla \cdot \vec{D}\, dv = \int_v \rho_v\, dv \quad \sim (2)$$

$$= Q$$

Then from the divergence theorem, equ (2) can be written as:

$$\int_v \nabla \cdot \vec{D} \, dv = \int_v \rho_v \, dv = Q \quad \sim (2)$$

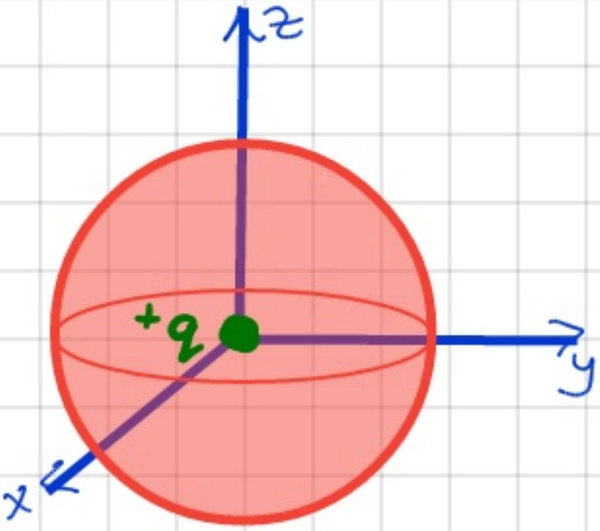
$$\int_v \nabla \cdot \vec{D} \, dv = \oint_s \vec{D} \cdot d\vec{S} = Q \sim (3)$$

Then the integral form of Gauss's law is:

$$\oint_s \vec{D} \cdot d\vec{S} = Q \quad \sim (4)$$

where the surface S inclosing the volume v is called Gassian surface that should be chosen cleverly.

Example: Use the integral form of Gauss's law to find the electric field $\vec{E}$ due to a single isolated positive point charge q by enclosing it with a closed spherical Gaussian surface S of an arbitrary radius R centered at q.



Solution: Since $\vec{D} = \epsilon \vec{E}$, $\vec{E}$ is radially outward along the unit vector $\hat{R}$ with an equal magnitude at all points of S.

While $\vec{D} = \hat{R} D_R$, since $\oint_S \vec{D} \cdot d\vec{S} = Q$

$$\oint_S \vec{D} \cdot d\vec{S} = \oint_S R D_R \cdot \hat{R} R^2 \sin\theta \, d\theta \, d\phi$$

$$= D_R R^2 \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= D_R R^2 \left[-\cos\theta \Big|_{\theta=0}^{\pi} \right] \left[\phi \Big|_{\phi=0}^{2\pi} \right]$$

$$= D_R R^2 \left[-\cos\pi + \cos 0 \right] \left[2\pi - 0 \right]$$

$$Q = 4\pi R^2 D_R$$

Since $Q = 4\pi R^2 D_R$, $D_R = \dfrac{Q}{4\pi R^2}$

Then $\vec{D} = \hat{R} D_R$

$$\vec{D} = \frac{\hat{R}\,Q}{4\pi R^2}$$

Since $\vec{D} = \epsilon \vec{E}$, then

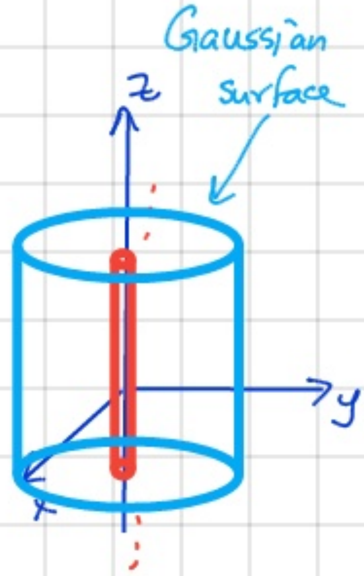
$$\vec{E} = \frac{1}{\epsilon}\vec{D}$$

$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon}\frac{\hat{R}}{R^2}}$$

- Since at every point on the surface S the direction of $d\vec{S}$ is along its outward normal, then only the normal component of $\vec{D}$ at the surface contributes to the integral for the integral form of Gauss's law

$$\oint_S \vec{D} \cdot d\vec{S} = Q$$

Example: Use Gauss's law to find an expression for $\vec{E}$ due to an infinitely long line with uniform charge density ρ_ℓ that resides along the z-axis in free space.



Solution: Since the line charge density is along an infinite length, then $\vec{D}$ has one component in the $+\hat{r}$ direction, so the Gaussian surface will be a cylinder with radius r and length h around the line charge as shown. So

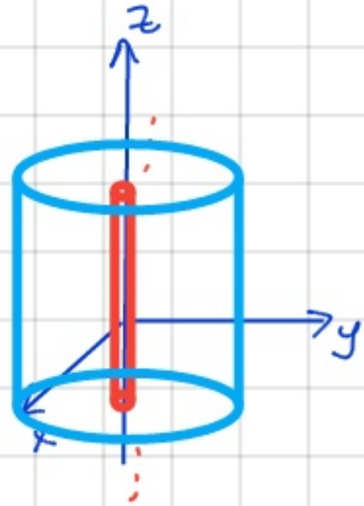
$$\vec{D} = \hat{r} D_r$$

and $$Q = \int_\ell \rho_\ell \, d\ell = \int_{z=u}^{u+h} \rho_\ell \, dz$$ (where u is arbitrary)

$$Q = \rho_\ell \left(z \Big|_{z=u}^{u+h} \right) = \rho_\ell \left[u+h-u \right] = \rho_\ell h$$

$$Q = \rho_\ell h$$

Since $\oint_S \vec{D} \cdot d\vec{s} = Q$



$$\oint_S \hat{r} D_r \cdot \hat{r}\, r\, d\phi\, dz = \rho_l h$$

$$r D_r \int_{\phi=0}^{2\pi} d\phi \int_{z=u}^{u+h} dz = \rho_l h$$

$$r D_r \left[\phi \Big|_{\phi=0}^{2\pi} \right] \left[z \Big|_{z=u}^{u+h} \right] = \rho_l h$$

$$r D_r (2\pi - 0)(u + h - u) = \rho_l h$$

$$2\pi r h D_r = \rho_l h$$

$$D_r = \frac{\rho_l}{2\pi r}$$

Then $\vec{D} = \hat{r} D_r$

$$\vec{D} = \hat{r} \frac{\rho_l}{2\pi r}$$

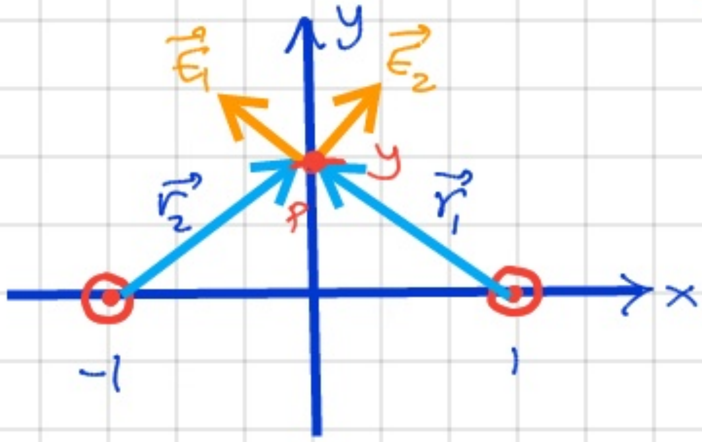
Since $\vec{D} = \epsilon_0 \vec{E}$, then

$$\boxed{\vec{E} = \hat{r} \frac{\rho_l}{2\pi \epsilon_0 r}}$$

- This formula $\vec{E} = \hat{r}\frac{\rho_\ell}{2\pi\epsilon_0 r}$

is applicable for any infinite line of charge, regardless of its location and direction as long as $\hat{r}$ is properly defined as the radial distance vector from the line charge to the observation point. That is, $\hat{r}$ is perpendicular to the line of charge.

Example: Two infinite lines, each carrying a uniform charge density ρ_ℓ, reside in free space parallel to the z-axis at $x=1$ and $x=-1$. Determine $\vec{E}$ at an arbitrary point along the y-axis



Since $\vec{E} = \hat{r}\dfrac{\rho_\ell}{2\pi\epsilon_0 r}$

$\vec{r}_1 = -\hat{x} + y\hat{y}$ | $\vec{r}_2 = \hat{x} + y\hat{y}$

$|\vec{r}_1| = \sqrt{1+y^2}$ | $|\vec{r}_2| = \sqrt{1+y^2}$

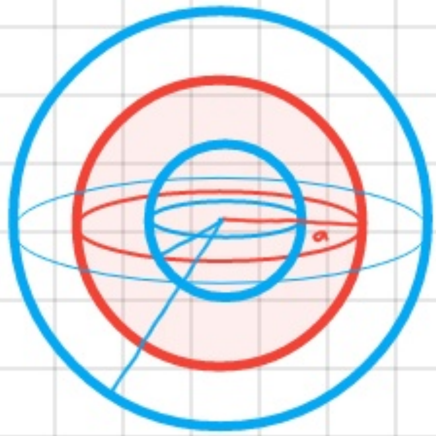
Then $\vec{E}\Big|_{\text{point } P} = \vec{E}_1\Big|_{\text{point } P} + \vec{E}_2\Big|_{\text{point } P}$

$$= \frac{-\hat{x}+y\hat{y}}{\left(\sqrt{1+y^2}\right)^2}\frac{\rho_\ell}{2\pi\epsilon_0} + \frac{\hat{x}+y\hat{y}}{\left(\sqrt{1+y^2}\right)^2}\frac{\rho_\ell}{2\pi\epsilon_0}$$

$$= \frac{2y\hat{y}\rho_\ell}{2\pi\epsilon_0(1+y^2)}$$

$$\vec{E} = \frac{\hat{y}\,y\,\rho_\ell}{\pi\epsilon_0(1+y^2)}$$

Example: A thin spherical shell of radius a carries a uniform surface charge density ρ_s. Use Gauss's law to determine $\vec{E}$ everywhere in free space.



Solution: $\vec{D} = \hat{R} D_R$,

$$\oint_S \vec{D} \cdot d\vec{s} = \oint_S \hat{R} D_R \cdot \hat{R} R^2 \sin\theta \, d\theta \, d\phi$$

$$= D_R R^2 \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= D_R R^2 \left(-\cos\theta \Big|_{\theta=0}^{\pi}\right)\left(\phi \Big|_{\phi=0}^{2\pi}\right)$$

$$= D_R R^2 4\pi = Q$$

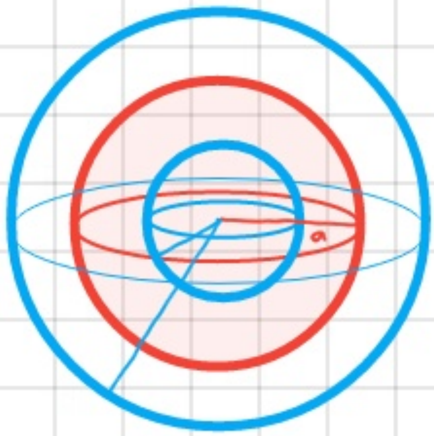
Then $D_R = \dfrac{Q}{4\pi R^2}$ where

$$Q = \int_S \rho_s \, ds = \int_S \rho_s R^2 \sin\theta \, d\theta \, d\phi$$

$$= \rho_s R^2 \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \rho_s R^2 \left[-\cos\theta \Big|_0^{\pi}\right]\left[\phi \Big|_0^{2\pi}\right]$$

$$Q = 4\pi \rho_s R^2 \Big|_{R=a} = 4\pi \rho_s a^2$$



Since $D_R = \dfrac{Q}{4\pi R^2}$ with $\vec{D} = \hat{R} D_R$

and the total charge on the surface

with radius a is $Q = 4\pi \rho_s a^2$

For $R < a$, then $Q = 0$ (no charge enclosed),

then $D_R = \dfrac{0}{4\pi R^2} = 0$

For $R > a$, $Q = 4\pi \rho_s a^2$ (all charge enclosed),

then $D_R = \dfrac{4\pi \rho_s a^2}{4\pi R^2} = \dfrac{\rho_s a^2}{R^2}$

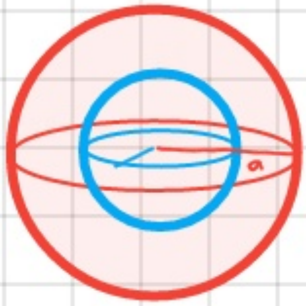
Then $D_R = \begin{cases} 0 & , R < a \\ \dfrac{\rho_s a^2}{R^2} & , R \geq a \end{cases}$

Then since $\vec{E} = \dfrac{1}{\epsilon_0} \vec{D} = \dfrac{1}{\epsilon_0} \hat{R} D_R$

$$\vec{E} = \begin{cases} 0 & , R < a \\ \dfrac{\rho_s a^2}{\epsilon_0 R^2} \hat{R} & , R \geq a \end{cases}$$

Example: A spherical volume of radius a contains a uniform volume charge density ρ_v. Use Gauss's law to determine $\vec{D}$ for: (a) $R \leq a$, (b) $R \geq a$

(a) For $R \leq a$: $\vec{D} = \hat{R} D_R$



$$\oint_S \vec{D} \cdot d\vec{s} = \oint_S \hat{R} D_R \cdot \hat{R} R^2 \sin\theta \, d\theta \, d\phi$$

$$= R^2 D_R \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= R^2 D_R \left(-\cos\theta \Big|_0^{\pi} \right) \left(\phi \Big|_0^{2\pi} \right)$$

$$= 4\pi R^2 D_R = Q$$

where $Q = \int_v \rho_v \, dv = \int_v \rho_v \, R^2 \sin\theta \, dR \, d\theta \, d\phi$

$$= \rho_v \int_{R=0}^{R} R'^2 dR' \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \rho_v \left(\frac{R'^3}{3} \Big|_0^{R} \right) \left(-\cos\theta \Big|_0^{\pi} \right) \left(\phi \Big|_0^{2\pi} \right)$$

$$= \rho_v \frac{1}{3} (R^3 - 0)(2)(2\pi)$$

$$Q = \frac{4\pi}{3} R^3 \rho_v$$

Since $\oint_S \vec{D} \cdot d\vec{s} = 4\pi R^2 D_R = Q$ and

$Q = \frac{4\pi}{3} R^3 \rho_v$, then

$$4\pi R^2 D_R = \frac{4\pi}{3} R^3 \rho_v$$

$$D_R = \frac{\rho_v}{3} R$$

Then $\vec{D} = \hat{R} D_r$

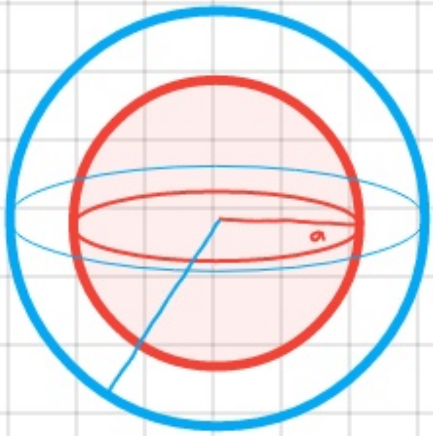
$$\vec{D} = \frac{\rho_v}{3} R\hat{R} \quad (\text{for } R \leq a)$$

(b) For $R \geq a$

from (a) for a general R:

$$\oint_S \vec{D} \cdot d\vec{S} = 4\pi R^2 D_R = Q$$

and Q for $R \geq a$ is



$$Q\Big|_{R\geq a} = \frac{4\pi}{3} R^3 \rho_v = \frac{4\pi}{3} (a)^3 \rho_v$$

$$Q = \frac{4\pi}{3} a^3 \rho_v$$

Then $\quad 4\pi R^2 D_R = \frac{4\pi a^3}{3} \rho_v$

$$D_R = \frac{\rho_v}{3} \frac{a^3}{R^2}$$

Then since $\vec{D} = D_r \hat{R}$

$$\vec{D} = \frac{\rho_v a^3}{3} \frac{\hat{R}}{R^2} \quad (\text{for } R \geq a)$$

4.5 Electric Scalar Potential:

- The voltage difference V between two points in a circuit represents the amount of work (or potential energy) required to move a unit charge from one point to the other.
- Voltage is short for voltage potential and synonymous with electric potential.
- The electric field present in the circuit gives rise to voltage differences across circuit elements.
- The relationship between the electric field $\vec{E}$ and the electric potential V is presented in this section.

4.5.1 Electric Potential as a Function of Electric Field:

- The differential electric potential energy (dW) per unit charge is called the differential electric potential also called differential voltage (dV), that is:

$$\frac{dW}{q} = dV \sim (1) \qquad \left(\text{Units: } \frac{J}{C} := V\right)$$

- The work done, or energy expended, in moving any object a differential distance vector $\vec{dl}$ while exerting a force $\vec{F}_{exert}$ is:

$$dW = \vec{F}_{exert} \cdot \vec{dl} \sim (2)$$

for a point charge q in the presence of an electric field $\vec{E}$, the force acting on the charge q ($\vec{F}_E$) is:

$$\vec{F}_E = q\vec{E} \sim (3)$$

to move q at a constant speed with zero acceleration, the net force acting on q should be:

$$\vec{F}_{exert} + \vec{F}_E = 0 \sim (4)$$

Since : $\frac{dW}{q} = dV$ ~ (1)

$dW = \vec{F}_{exert} \cdot d\vec{l}$ ~ (2)

$\vec{F}_E = q\vec{E}$ ~ (3)

$\vec{F}_{exert} + \vec{F}_E = 0$ ~ (4)

Then
$$dV = \frac{1}{q}\, dW$$
$$= \frac{1}{q}\, \vec{F}_{exert} \cdot d\vec{l}$$
$$= \frac{1}{q}\,(-\vec{F}_E) \cdot d\vec{l}$$
$$= \frac{1}{q}\,(-q\vec{E}) \cdot d\vec{l}$$

$$\boxed{dV = -\vec{E} \cdot d\vec{l}} \sim (5)$$

- Then the potential difference corresponding to moving a point charge from point P_1 to P_2 is found by integrating (5) along any path between the points, as
$$\int_{P_1}^{P_2} dV = \int_{P_1}^{P_2} -\vec{E} \cdot d\vec{l}$$

$$\boxed{V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l}} \sim (6)$$

where V_1 & V_2 are the electric potential at points P_1 & P_2, respectively. The unit of V is volt (V) where $V =: \frac{J}{C}$

- Kirchhoff's voltage law states that the net voltage drop around a closed loop is zero. Then (6) becomes:

$$V_{11} = V_1 - V_1 = -\int_{P_1}^{P_1} \vec{E} \cdot d\vec{l} = 0$$

which is a line integral of the electrostatic field (acceleration = zero) around any closed contour as follows:

$$\oint_C \vec{E} \cdot d\vec{l} = 0 \quad \text{(integral form)}$$

by applying Stoke's theorem for a closed contour C surrounding surface S, then

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{S} = \oint_C \vec{E} \cdot d\vec{l} = 0$$

Then $\nabla \times \vec{E} = 0$ is the differential form.

recall: when $\nabla \times \vec{E} = 0$ the vector field is called irrotational. It is also called conservative since the electrostatic field $\vec{E}$ is conservative since it is not a time-varying function.

- Recall: In nodal analysis, a node in a circuit can't be assigned an absolute voltage, its voltage is measured relative to a reference node with a voltage equal to zero, which is the ground. Here a point in space can't have an absolute electric potential V, its potential is usually measured relative to a reference point chosen to be at infinity. We assume that $V = 0$ when the point is at infinity, as a result, the electric potential V at any point P is:

$$V = -\int_{\infty}^{P} \vec{E} \cdot d\vec{l} \qquad (\text{unit}: V)$$

4.5.2. Electric Potential Due to Point Charges :

- Since the electric field due to a point charge q located at the origin is

$$\vec{E} = \frac{q}{4\pi\epsilon} \frac{\hat{R}}{R^2}$$

where $\vec{R}$ is the position vector from the charge to the observation point.

- To find the electric potential at a point P with position vector $\vec{R}$ (q located at the origin):

$$V = -\int_{\infty}^{R} \vec{E} \cdot d\vec{l} = -\int_{\infty}^{R} \left(\frac{q}{4\pi\epsilon} \frac{\hat{R}}{R^2} \right) \cdot \hat{R}\, dR$$

$$= \frac{-q}{4\pi\epsilon} \int_{\infty}^{R} \frac{dR}{R^2} = \frac{-q}{4\pi\epsilon} \left[\frac{-1}{R} \Big|_{\infty}^{R} \right]$$

$$V = \frac{q}{4\pi\epsilon} \frac{1}{R}$$

- To find the electric potential at a point P with position vector $\vec{R}$ and q located at position vector $\vec{R}_1$

$$V = \frac{q}{4\pi\epsilon} \frac{1}{|\vec{R} - \vec{R}_1|} \quad (V)$$

where $|\vec{R} - \vec{R}_1|$ is the length of the displacement vector from the charge q to point P $(\vec{R} - \vec{R}_1)$.

- For N discrete point charges $q_1, q_2, \ldots, q_N$ residing at position vectors $R_1, R_2, \ldots, R_N$, the electric potential is:

$$V = \frac{1}{4\pi\epsilon} \sum_{i=1}^{N} \frac{q_i}{|\vec{R} - \vec{R}_i|} \quad (V)$$

4.5.3 Electric Potential Due to Continuous Distributions:

Since $$dV = \frac{dq}{4\pi\epsilon}\frac{1}{R}$$

where $\hat{R}$ is the unit vector pointing from dq to the observation point P.

(A) Volume charge distribution $\left(\rho_v = \frac{dq}{dv}\right)$:

$$V = \int_v dV = \frac{1}{4\pi\epsilon}\int_v \rho_v \frac{1}{R}\, dv$$

(B) Surface charge distribution $\left(\rho_s = \frac{dq}{ds}\right)$:

$$V = \int_S dV = \frac{1}{4\pi\epsilon}\int_S \rho_s \frac{1}{R}\, ds$$

(C) Line charge distribution $\left(\rho_l = \frac{dq}{dl}\right)$:

$$V = \int_l dV = \frac{1}{4\pi\epsilon}\int_l \rho_l \frac{1}{R}\, dl$$

4.5.4. Electric Field as a Function of Electric Potential:

Since $dV = -\vec{E} \cdot d\vec{l}$

and for a scalar function V (from the grad.):

$$dV = \nabla V \cdot d\vec{l}$$

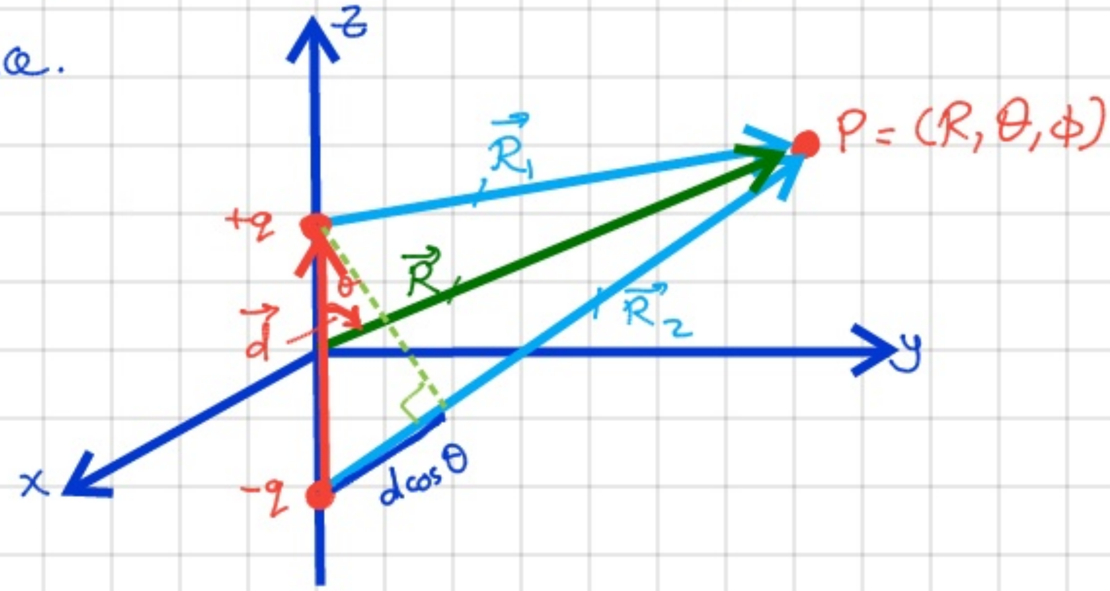
By comparing the above two equ.s:

$$\boxed{\vec{E} = -\nabla V}$$

- Usually it's easier to find $\vec{E}$ through the negative of the grad. of V than to use Coulomb's law conceptually and computationally.

Example: Electric field of an Electric Dipole:

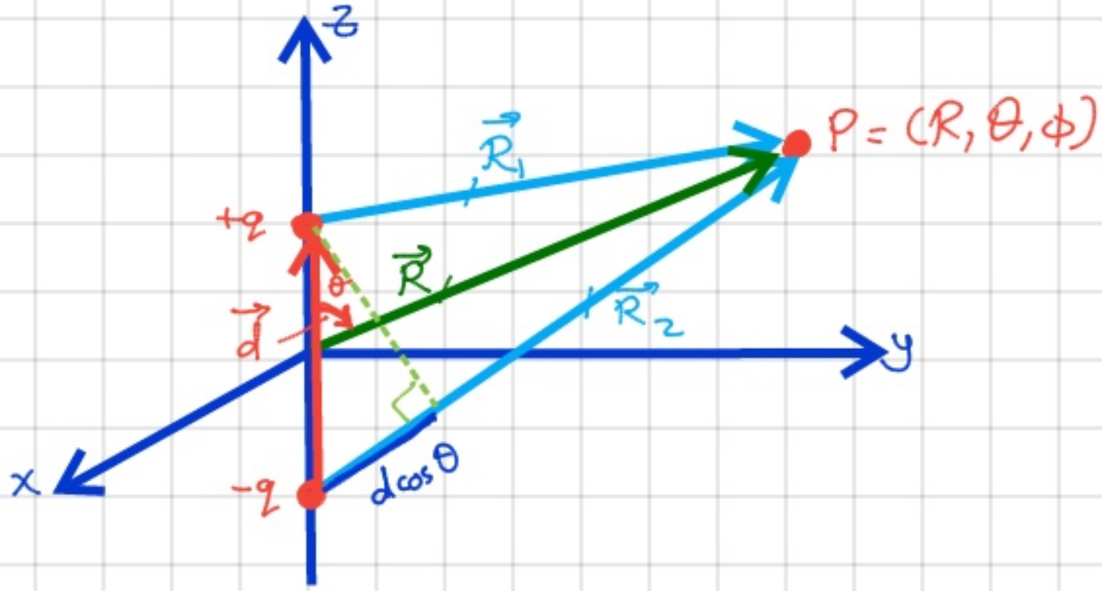
An electric dipole consists of two point charges of equal magnitude but opposite polarity, separated by a distance d. Determine V and $\vec{E}$ at any point P, given that P is at a distance $R \gg d$ from the dipole center, and the dipole resides in free space.



$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{R_1} + \frac{-q}{R_2}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{R_2 - R_1}{R_1 R_2}\right]$$

for $R \gg d$: $R_1 R_2 \simeq R^2$, R_1 & R_2 are approx. parallel, so $R_2 - R_1 \simeq d\cos\theta$.

Then $V = \frac{q}{4\pi\epsilon_0}\frac{d\cos\theta}{R^2}$



- For dipole in this orientation:

$$V = \frac{q}{4\pi\epsilon_0} \frac{d\cos\theta}{R^2}$$

- To find V for any dipole orientation:

for a unit vector $\hat{R}$ pointing from the center of the dipole to the observation point P,

$$d\cos\theta = \vec{d} \cdot \hat{R}$$

Then $$V = \frac{q}{4\pi\epsilon_0} \frac{\vec{d} \cdot \hat{R}}{R^2} = \frac{\vec{P} \cdot \hat{R}}{4\pi\epsilon_0 R^2}$$

where $\vec{P} = q\vec{d}$ is called the dipole moment.

- For dipole in this orientation:

$$V = \frac{q}{4\pi\epsilon_0}\frac{d\cos\theta}{R^2}$$

- For the electric field $\vec{E}$:

$$\vec{E} = -\nabla V$$

$$= -\left(\hat{R}\frac{\partial V}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\right)$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\hat{R}\,\partial\left(\frac{\cos\theta}{R^2}\right) + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{R^2}\right) + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\underbrace{\left(\frac{\cos\theta}{R^2}\right)}_{0}\right]$$

$$= \frac{-qd}{4\pi\epsilon_0}\left[\hat{R}\cos\theta\left(\frac{-2R}{R^4}\right) + \hat{\theta}\frac{1}{R}\left(\frac{-\sin\theta}{R^2}\right)\right]$$

$$\vec{E} = \frac{qd}{4\pi\epsilon_0}\frac{1}{R^3}\left[\hat{R}\,2\cos\theta + \hat{\theta}\sin\theta\right]$$

(All therse results are when $R \gg d$)

4.5.5. Poisson's Equation :

Since the differential form of Gauss's law is

$$\nabla \cdot \vec{D} = \rho_v$$

and $\vec{D} = \epsilon \vec{E}$, then

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon} .$$

Also since $\vec{E} = -\nabla V$, then

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon}$$

$$\nabla \cdot (-\nabla V) = \frac{\rho_v}{\epsilon}$$

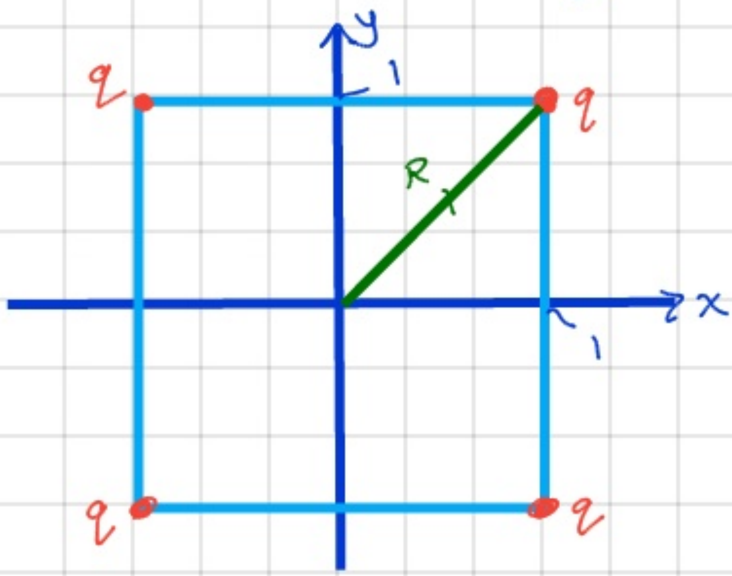
$$\nabla \cdot (\nabla V) = \frac{-\rho_v}{\epsilon}$$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}}$$ (Poisson's equation)

- If the medium considered has no charge ($\rho_v = 0$), then Poisson's equation becomes :

$$\boxed{\nabla^2 V = 0}$$ (Laplace's equation)

Example: Determine the electric potential at the origin due to four 20 μC charges residing in free space at the corners of a 2m × 2m square centered about the origin in the xy-plane.



$$V = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{4} \frac{q_i}{R_i} = 4\left(\frac{q}{4\pi\epsilon_0 R}\right)$$

$$= \frac{4 \times 20 \times 10^{-6}}{4\pi\epsilon_0 \sqrt{1^2+1^2}}$$

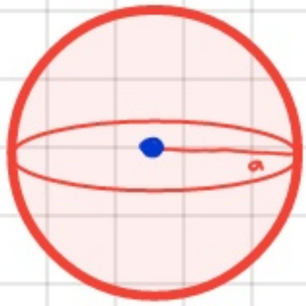
$$V = \frac{\sqrt{2} \times 10^{-5}}{\pi\epsilon_0} \ (V)$$

Example: A spherical shell of radius a has a uniform surface charge density ρ_s. Determine:

(a) the electric potential at the center of the shell

(b) the electric field at the center of the shell.

Solution:



(a) $$V = \frac{1}{4\pi\epsilon} \int_S \rho_s \frac{1}{R} \, dS \Big|_{R=a}$$

$$= \frac{1}{4\pi\epsilon} \int_S \rho_s \frac{1}{R} \left(R^2 \sin\theta \, d\theta \, d\phi \right) \Big|_{R=a}$$

$$= \frac{\rho_s R}{4\pi\epsilon} \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi \Big|_{R=a}$$

$$= \frac{\rho_s a}{4\pi\epsilon} \left[-\cos\theta \Big|_{\theta=0}^{\pi} \right] \left[\phi \Big|_{\phi=0}^{2\pi} \right]$$

$$= \frac{\rho_s a}{4\pi\epsilon} \left[-\cos\pi + \cos 0 \right] \left[2\pi - 0 \right]$$

$$V = \frac{\rho_s a}{\epsilon}$$

(b) $\vec{E} = -\nabla V$

$$= -\nabla\left(\frac{\rho_s a}{\epsilon}\right)$$

$$= -\left(\hat{R}\frac{\partial V}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\right)$$

$$= -\hat{R}\frac{\partial}{\partial R}\cancel{\left(\frac{\rho_s a}{\epsilon}\right)} - \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta}\cancel{\left(\frac{\rho_s a}{\epsilon}\right)} - \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}\cancel{\left(\frac{\rho_s a}{\epsilon}\right)}$$

$\vec{E} = 0$

4.6 Conductors:

- The electromagnetic constitutive parameters of a material medium are
  (1) its electrical permittivity ϵ
  (2) its magnetic permeability μ
  (3) its conductivity σ
- A material is homogeneous if its constitutive parameters do not vary from point to point.
- A material is isotropic if its constitutive parameters are independent of direction.
- Here, all materials are assumed to be homogeneous and isotropic.

- Conductivity (σ) of a material is a measure of how easily electrons can travel through the material under the influence of an externally applied electric field.
- Materials are classified into:

(a) Conductors (metals)

(b) Dielectrics (isulators)

(c) Semiconductors

- In conductors, loosely attached electrons are present in the outermost shells of its atoms.

(1) In the absence of an external electric field: these free electrons move in random directions with varying speeds, their random motion has a zero average current through the conductor

(2) In the presence of an external electric field: these free electrons move from one atom to the next in the direction opposite to the external electric field (electrons have a negative charge compared to the positive test charge). Their movement gives a conduction current:

$\vec{J} = \sigma \vec{E}$ (A/m^2) : amp/meter^2 (Ohm's law)

σ : its unit is in Siemens/meter : (S/m)

- For a perfect conductor $\sigma = \infty$ (& superconductors)

- In dielectrics, the electrons are tightly bound to the atoms, so it is difficult to detach them under the influence of an externally applied electric field. So, no significant conduction current can flow.
- For a perfect dielectric $\sigma = 0$
- In semiconductors, the materials allow for conduction currents even though their conductivities are much smaller than those of metals

4.6.1 Drift Velocity

- In conducting materials, the drift velocity $\vec{u}_e$ of electrons is related to the externally applied electric field $\vec{E}$ as

$$\vec{u}_e = -\mu_e \vec{E} \quad (m/s)$$

where μ_e is the electron mobility (unit: $\frac{m^2}{V \cdot s}$)

- In semiconductors, the current flow is due to the movement of both electrons and holes.

Hole drift velocity $\vec{u}_h = \mu_h \vec{E}$ (m/s)

Electron drift velocity $\vec{u}_e = -\mu_e \vec{E}$ (m/s)

Since the conduction current density in a medium containing a volume density ρ_v of charges moving with veclocity $\vec{u}$ is

$$\vec{J} = \rho_v \vec{u}$$

Then the conduction current density in Semiconductors is :

$$\vec{J} = \vec{J}_e + \vec{J}_h$$
$$= \rho_{ve} \vec{u}_e + \rho_{vh} \vec{u}_h \qquad (A/m^2)$$
$$= -N_e e \vec{u}_e + N_h e \vec{u}_h$$
$$= -N_e e (-\mu_e \vec{E}) + N_h e \mu_h \vec{E}$$
$$= (N_e \mu_e + N_h \mu_h) e \vec{E}$$
$$= (-\rho_{ve} \mu_e + \rho_{vh} \mu_h) \vec{E}$$
$$= \sigma \vec{E}$$

where N_e & N_h : are the number of free electrons and the number of free holes per unit volume.

e : is the absolute charge of a single hole or electron with $e = 1.6 \times 10^{-19}$ C.

$\rho_{ve} = -N_e e$ and $\rho_{vh} = N_h e$

and $\sigma = -\rho_{ve} \mu_e + \rho_{vh} \mu_h = (N_e \mu_e + N_h \mu_h) e$

- When comparing a good conductor to a semiconductor : $N_h \mu_h \ll N_e \mu_e$

then the conductivity for a conductor is

$$\sigma = -\rho_{ve}\mu_e = N_e \mu_e e \quad (S/m)$$

compared to the conductivity for a semiconductor:

$$\sigma = -\rho_{ve}\mu_e + \rho_{vh}\mu_h = (N_e\mu_e + N_h\mu_h)e \quad (S/m)$$

- Since $\vec{J} = \sigma \vec{E}$:

(1) for a perfect dielectric $\sigma = 0$, then

$\vec{J} = 0$ regardless of $\vec{E}$

(2) for a perfect conductor $\sigma = \infty$, then

$\vec{E} = 0$ inside it regardless of $\vec{J}$, no charges inside a perfect conductor.

As a result, it is an equipotential medium

(equal potential at every point in the perfect conductor)

since $V_{21} = -\int \vec{E} \cdot d\vec{l} = 0$ since $\vec{E} = 0$

- The conductivity σ in most metals is on the order of 10^6 S/m. It is common to treat them as perfect conductors and to set $\vec{E} = 0$ inside them.
- Each conductor is an equipotential medium, however, the potential difference between a conductor and another conductor is not necessarily zero, each conductor is an equipotential medium, but the presence of different distributions of charges on their two surfaces can generate a potential difference between them.

Example: Conduction Current in a Copper Wire:

A 2mm diameter copper wire with conductivity of 5.8×10^7 S/m and electron mobility of $0.0032 \ \frac{m^2}{V.s}$ is subjected to an electric field of 20 (mV/m). Find:

(a) the volume charge density of the free electrons

(b) the current density

(c) the current flowing in the wire

(d) the electron drift velocity

(e) the volume density of the free electrons

(a) the volume charge density of the free electrons

$$\rho_{ve} = -\frac{\sigma}{\mu_e} = -\frac{5.8 \times 10^{7}}{0.0032} = -1.81 \times 10^{10} \ (C/m^3)$$

(b) the current density

$$J = \sigma E = 5.8 \times 10^{7} \times 20 \times 10^{-3} = 1.16 \times 10^{6} \ (A/m^2)$$

(c) the current flowing in the wire

$$I = \int_S \vec{J} \cdot d\vec{s} = J \text{ (area)}$$

$$= 1.16 \times 10^{6} \left(\frac{2 \times 10^{-3}}{2}\right)^2 \pi = 3.64 \ (A)$$

(d) the electron drift velocity

$$u_e = -\mu_e E = -0.032 \times 20 \times 10^{-3} = -6.4 \times 10^{-5} \ m/s$$

(e) the volume density of the free electrons

$$N_e = -\frac{\rho_{ve}}{e} = \frac{1.81 \times 10^{10}}{1.6 \times 10^{-19}} = 1.13 \times 10^{29} \text{ electron}/m^3$$

Example: Determine the density of free electrons in aluminum, given that its conductivity is 3.5×10^7 S/m and its electron mobility is $0.0015 \ \frac{m^2}{V.s}$

Solution: Since $\sigma = N_e \mu_e e$, then

$$N_e = \frac{\sigma}{\mu_e e} = \frac{3.5 \times 10^7}{0.0015 \times 1.6 \times 10^{-19}}$$

$$N_e = 1.46 \times 10^{29} \text{ electrons}/m^3$$

Example: The current flowing through a 100 m long conducting wire of uniform cross section has a density of 3×10^5 (A/m^2). Find the voltage drop along the length of the wire if the wire material has a conductivity of 2×10^7 S/m.

Solution: Since $J = \sigma E$ and

$V = E\ell$ (positive since the direction of $\vec{E}$ is from the point with higher potential to the point with lower potential, which is the same direction of the current density $\vec{J} = \sigma\vec{E}$). Then

$$V = E\ell = \frac{J}{\sigma}\ell = \frac{3\times10^5 \times 100}{2\times10^7} = 1.5 \text{ V}$$

4.6.2 Resistance

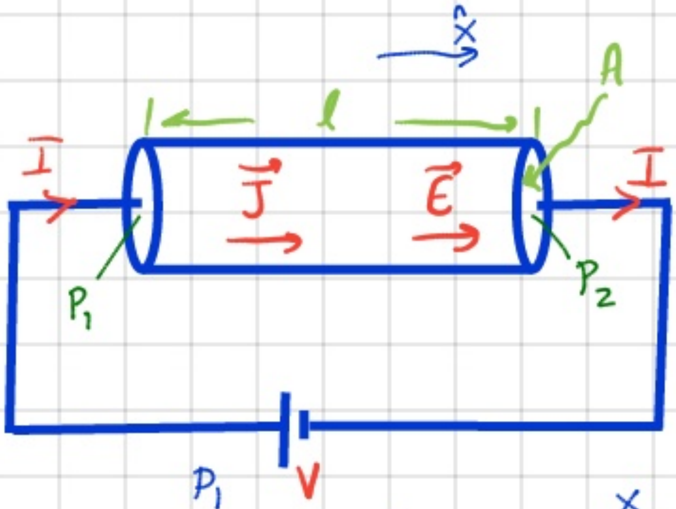
- In a general form, the resistance R for any resistor with arbitrary shape is found from the voltage V across the resistor and the current I by:

$$R = \frac{V}{I} = \frac{-\int_l \vec{E} \cdot d\vec{l}}{\int_S \vec{J} \cdot d\vec{s}} = \frac{-\int_l \vec{E} \cdot d\vec{l}}{\int_S \sigma \vec{E} \cdot d\vec{s}} \quad (\Omega)$$

where l is the path between two specified points and S is the surface of the resistor where $\vec{J}$ crosses.

- The conductance G is the reciprocal of R with the unit Siemens (S) or (Ω^{-1})

Example: For a linear resistor of cross section A and length ℓ connected to a dc voltage source V. The resistor R is as follows.



$$V = V_1 - V_2 = -\int_{P_2}^{P_1} \vec{E} \cdot d\vec{\ell} = -\int_{x_2}^{x_1} \hat{x} E_x \cdot \hat{x}\, dx$$

$$= -E_x(x_1 - x_2) = E_x(x_2 - x_1) = E_x \ell \quad (V)$$

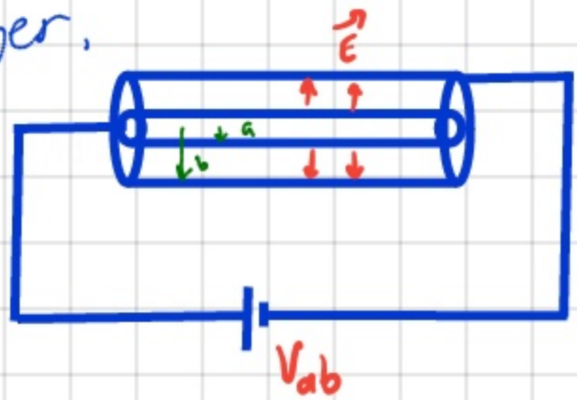
$$I = \int_S \vec{J} \cdot d\vec{S} = \int_S \sigma \vec{E} \cdot d\vec{S} = \sigma \int_S E_x \hat{x} \cdot \hat{x}\, dy\, dz$$
$$= \sigma E_x A$$

Then $R = \dfrac{V}{I} = \dfrac{E_x \ell}{\sigma E_x A} = \dfrac{\ell}{\sigma A} \; (\Omega)$

and $G = \dfrac{1}{R} = \dfrac{\sigma A}{\ell} \; (\Omega^{-1})$

Example: Conduction of Coaxial Cable :

The radii of the inner and outer conductors of a coaxial cable of length ℓ are a & b, respectively. The insulation material has conductivity σ. Obtain an expression for G', the conductance per unit length of the insulation layer.



$$G = \frac{I}{V} = \frac{\int_s \vec{J} \cdot d\vec{s}}{-\int_\ell \vec{E} \cdot d\vec{\ell}}$$

$\vec{J}$ from the inner conductor to the outer conductor is $\vec{J} = \hat{r} J_r = \hat{r} \frac{I}{\text{area}} = \hat{r} \frac{I}{2\pi r \ell}$ at a distance r from the axis of the center of the coaxial cable.

Since $\vec{J} = \sigma \vec{E}$, then $\vec{E} = \frac{\vec{J}}{\sigma} = \hat{r} \frac{I}{2\pi r \ell \sigma}$

Then $V_{ab} = V_a - V_b = -\int_b^a \vec{E} \cdot d\vec{l}$

$$= -\int_b^a \hat{r} \frac{I}{2\pi r l \sigma} \cdot \hat{r}\, dr$$

$$V_{ab} = \frac{I}{2\pi l \sigma} \ln \frac{b}{a}$$

Then $G' = \frac{G}{l} = \frac{1}{l} \frac{I}{V}$

$$= \frac{1}{l} \frac{I}{I \ln \frac{b}{a}} 2\pi l \sigma$$

$$G' = \frac{2\pi \sigma}{\ln \frac{b}{a}} \quad (S/m)$$

4.6.3 Joule's Law:

To find the power dissipated in a conducting medium under the influence of an externally applied electric field $\vec{E}$:

- A volume Δv contains electron and hole charge of : $q_e = \rho_{ve} \Delta v$ and $q_h = \rho_{vh} \Delta v$
- The electric forces acting on q_e & q_h from $\vec{E}$:

$$\vec{F}_e = q_e \vec{E} = \rho_{ve} \Delta v \vec{E}$$

$$\vec{F}_h = q_h \vec{E} = \rho_{vh} \Delta v \vec{E}$$

- The work (energy) expended by $\vec{E}$ to move q_e & q_h a distance Δl_e & Δl_h, respectively:

$$\Delta W = \vec{F}_e \cdot \vec{\Delta l_e} + \vec{F}_h \cdot \vec{\Delta l_h}$$

- The power is the time rate of energy change as:

$$\Delta P = \frac{\Delta W}{\Delta t} = \vec{F}_e \cdot \frac{\vec{\Delta l_e}}{\Delta t} + \vec{F}_h \cdot \frac{\vec{\Delta l_h}}{\Delta t}$$

$$\Delta P = \frac{\Delta W}{\Delta t} = \vec{F}_e \cdot \frac{\Delta \vec{l}_e}{\Delta t} + \vec{F}_h \cdot \frac{\Delta \vec{l}_h}{\Delta t}$$

$$= \vec{F}_e \cdot \vec{u}_e + \vec{F}_h \cdot \vec{u}_h$$

$$= q_e \vec{E} \cdot \vec{u}_e + q_h \vec{E} \cdot \vec{u}_h$$

$$= \rho_{ve} \Delta v \, \vec{E} \cdot \vec{u}_e + \rho_{vh} \Delta v \, \vec{E} \cdot \vec{u}_h$$

$$= \left(\rho_{ve} \vec{E} \cdot \vec{u}_e + \rho_{vh} \vec{E} \cdot \vec{u}_h \right) \Delta v$$

$$= \vec{E} \cdot \left(\rho_{ve} \vec{u}_e + \rho_{vh} \vec{u}_h \right) \Delta v$$

$$\Delta P = \vec{E} \cdot \vec{J} \, \Delta v$$

For a volume v, the total dissipated power is

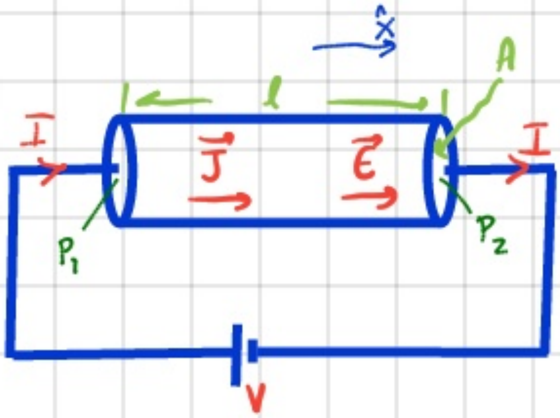
$$P = \int_v \vec{E} \cdot \vec{J} \, dv \quad (W) \qquad \text{(Joule's Law)}$$

From Ohm's law, $\vec{J} = \sigma \vec{E}$, then

$$P = \int_v \vec{E} \cdot \sigma \vec{E} \, dv$$

$$P = \int_v \sigma |\vec{E}|^2 \, dv \quad (W)$$

Example: For a linear resistor of cross section A and length l connected to a dc voltage source V.



The electric field $\vec{E}$ was found where $|\vec{E}| = E_x$. The volume $v = l A$.

Then $$P = \int_v \sigma |\vec{E}|^2 \, dv$$

$$= \int_S \sigma E_x \, ds \int_l E_x \, dl$$

(by separating the volume integral into a product of a surface integral and a line integral)

Then $P = (\sigma E_x A)(E_x l)$

$P = IV$.

Since $R = V/I$, then $P = I^2 R$ and $P = \dfrac{V^2}{R}$

Example: A 50 m long copper wire has a circular cross section with radius $r = 2$ cm. Given that the conductivity of copper is 5.8×10^7 S/m, determine:

(a) the resistance R of the wire

(b) the power dissipated in the wire if the voltage across its length is 1.5 mV.

(a) We found that for a linear resistor:

$$R = \frac{l}{\sigma A} = \frac{50}{5.8 \times 10^7 \times (0.02)^2 \pi} = 6.9 \times 10^{-4}\ \Omega$$

(b) $$P = \frac{V^2}{R} = \frac{(1.5 \times 10^{-3})^2}{6.9 \times 10^{-4}} = 3.3 \text{ mW}$$

Example: Repeat part (b) by applying

$$P = \int_v \sigma |\vec{E}|^2 \, dv .$$

Since $V = 1.5 \times 10^{-3}$ and $l = 50$

Then $|\vec{E}| = \dfrac{1.5 \times 10^{-3}}{50} = 3 \times 10^{-5}$

Then $P = \int_v \sigma |\vec{E}|^2 \, dv$

$$= 5.8 \times 10^{7} \times (3 \times 10^{-5})^2 \int_0^{0.02} r\,dr \int_0^{2\pi} d\phi \int_0^{50} dz$$

$$= 5.8 \times 10^{7} \times (3 \times 10^{-5})^2 \frac{(0.02)^2}{2} (2\pi)(50)$$

$$P = 3.3 \text{ mW}$$

4.7 Dielectrics

- In dielectrics, the electrons in the outmost atomic shells are strongly bound to the atom.
- In dielectric nonpolar molecules,

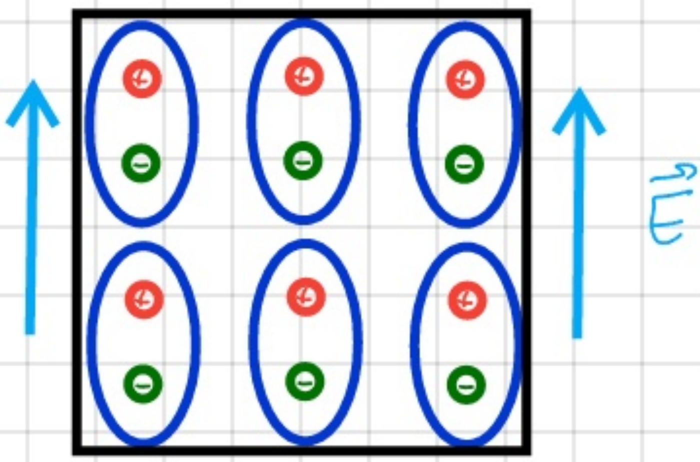
  (1) in the absence of an electric field $\vec{E}$, the positively charged nucleus generates an electric field that attracts and holds the electron cloud around it.

(2) in the presence of an electric field $\vec{E}$, the electrons still can not move, instead, the electric field will polarize the atoms or molecules by moving the center of the electron cloud away from positively charged nucleus.

The polarized atom or molecule can be represented by an electric dipole causing a polarization field.

- The polarization field is weaker than the electric field $\vec{E}$ and in opposite direction to $\vec{E}$. As a result, the net electric field present in the dielectric material caused by $\vec{E}$ is smaller than $\vec{E}$.

- A dielectric medium polarized by an external electric field $\vec{E}$ will have its dipoles aligned linearly along the electric field $\vec{E}$ as:

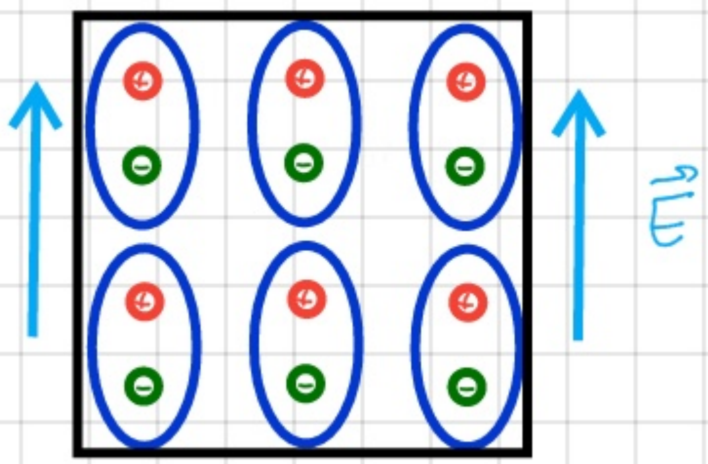


As a result, the upper and lower edges of the dielectric material will display a positive and negative surface charge densities, respectively.

- In dielectric polar materials,

(1) in the absence of an electric field $\vec{E}$, the permanent dipoles are randomly oriented causing no net dipole electric field.

(2) in the presence of an electric field $\vec{E}$, the permanent dipoles align along the direction of the electric field $\vec{E}$ (similar to the nonpolar materials).



4.7.1 Polarization Field

- In free space: $\vec{D} = \epsilon_0 \vec{E}$.

However, in dielectric material:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

where $\vec{P}$ is the electric polarization field, it is produced by the electric field $\vec{E}$ and depends on the material properties.

- Dielectric materials are:

(1) linear: if $\vec{P}$ is directly proportional to the magnitude of $\vec{E}$

(2) isotropic: if $\vec{P}$ and $\vec{E}$ are in the same direction

(3) anisotropic: if $\vec{P}$ and $\vec{E}$ may have different directions

(4) homogeneous: if ϵ, μ, and σ are constant throughout the material.

- We will study here only linear, isotropic, and homogeneous material, where

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

where χ_e (Chi) /kī/ : is the electric susceptibility of the material, where

$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E}$$

$$\vec{D} = \epsilon_0 (1 + \chi_e) \vec{E}$$

$$\vec{D} = \epsilon \vec{E}$$

where the permittivity of the material

$$\epsilon = \epsilon_0 (1 + \chi_e)$$

also it could be defined by the relative permittivity ϵ_r as $\epsilon_r = \epsilon / \epsilon_0$

4.7.2 Dielectric Breakdown

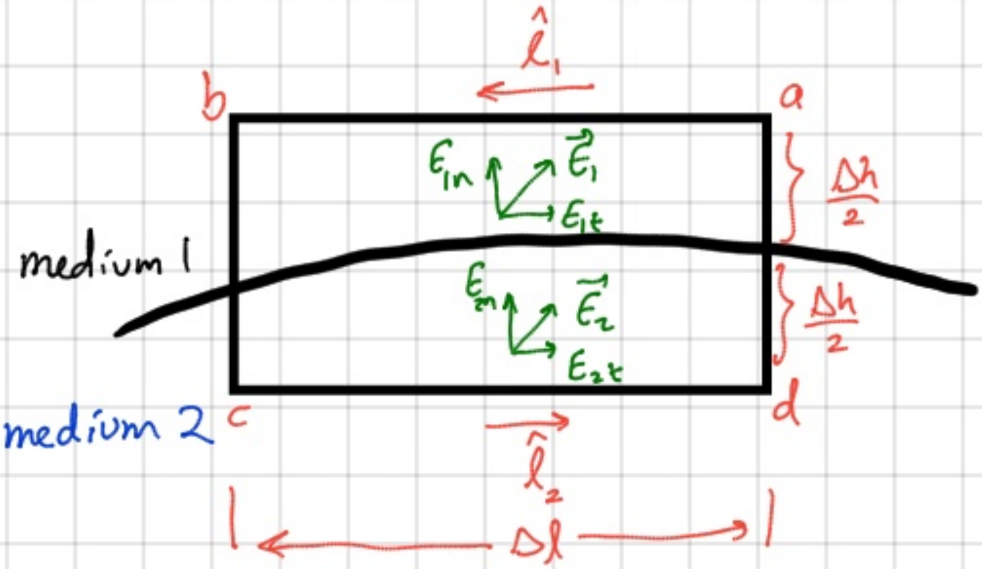
- Here we assume that the magnitude of $\vec{E}$ does not exceed a value called dielectric strength E_{ds} of the material. Since beyond E_{ds} the electrons will detach from the molecules and accelerate through the material in the form of a conduction current causing the dielectric material to suffer a permanent damage due to electron collisions with the molecular structure. This change is called dielectric breakdown. So the dielectric strength E_{ds} is the largest magnitude of $\vec{E}$ that the material can sustain without breaking down.

- A charged thundercloud at electric potential V relative to the ground causes an electric field $E = \frac{V}{d}$, where d is the height of the cloud base above the ground. If V is large so that E exceeds the dielectric strength of air, ionization occurs and a lightning discharge follows, and V is the breakdown voltage V_{br}

4.8 Electric Boundary Conditions

- Boundary conditions specify how the components of vector fields tangential and normal to an interface between two media relate across the interface.
- Here we will study the set of boundary conditions for $\vec{E}$, $\vec{D}$, and $\vec{J}$, applicable at the interface between any two dissimilar media.
- The boundary conditions are derived assuming electrostatic conditions and they are valid for time-varying electric fields as well.

(a) For the tangential components :



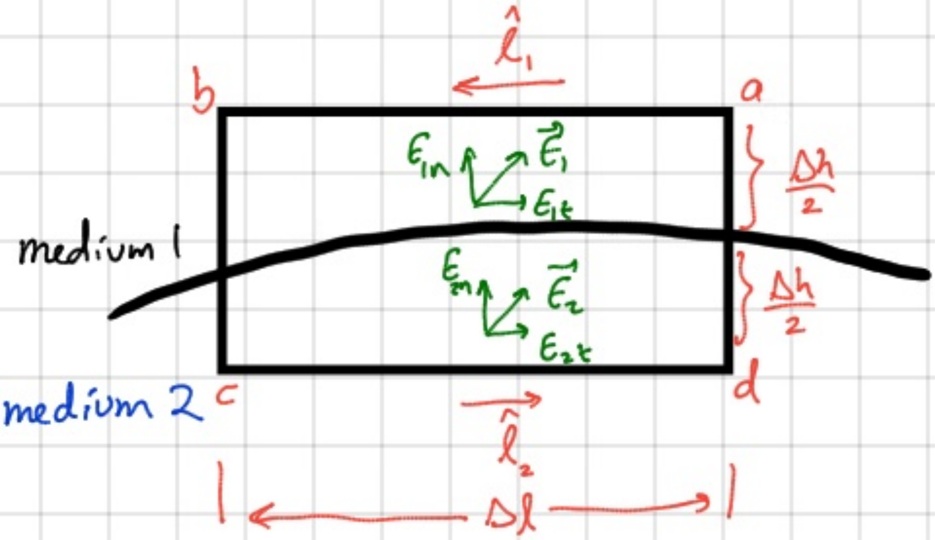
We apply the conservative property of the electric field, by finding the line integral of the electrostatic field around a closed path is zero, and letting $\Delta h \to 0$, then :

$$\oint_C \vec{E} \cdot d\vec{l} = \int_a^b \vec{E}_1 \cdot \hat{l}_1 \, dl + \int_c^d \vec{E}_2 \cdot \hat{l}_2 \, dl = 0$$

where $\hat{l}_1$ & $\hat{l}_2$ are unit vectors along ab & cd

$\vec{E}_1$ & $\vec{E}_2$ are the electric fields in medium 1 & 2

and $\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$, $\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$



$$\oint_C \vec{E}\cdot d\vec{l} = \int_a^b \vec{E}_1 \cdot \hat{l}_1\, dl + \int_c^d \vec{E}_2 \cdot \hat{l}_2\, dl = 0$$

Since $\hat{l}_1 = -\hat{l}_2$, then $(\vec{E}_1 - \vec{E}_2)\cdot \hat{l}_1 = 0$

which are the tangential components of $\vec{E}_1$ & $\vec{E}_2$

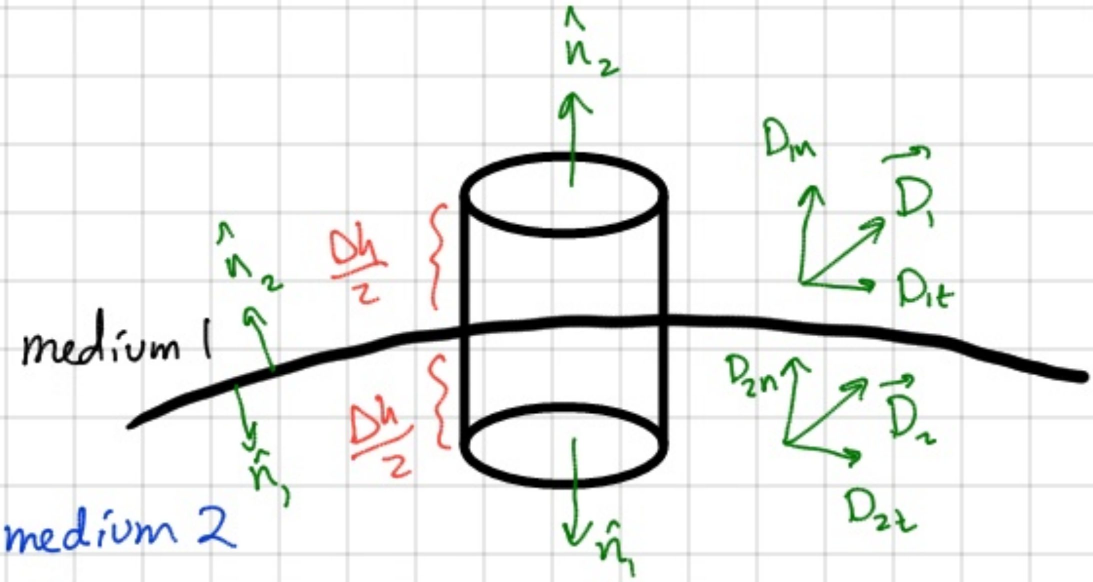
as $\vec{E}_{1t} = \vec{E}_{2t}$ ~ (1)

That is, the tangential component of the electric field is continuous across the boundary between any two media.

Since $\vec{D} = \epsilon \vec{E}$, then using (1) :

$$\frac{\vec{D}_{1t}}{\epsilon_1} = \frac{\vec{D}_{2t}}{\epsilon_2}$$

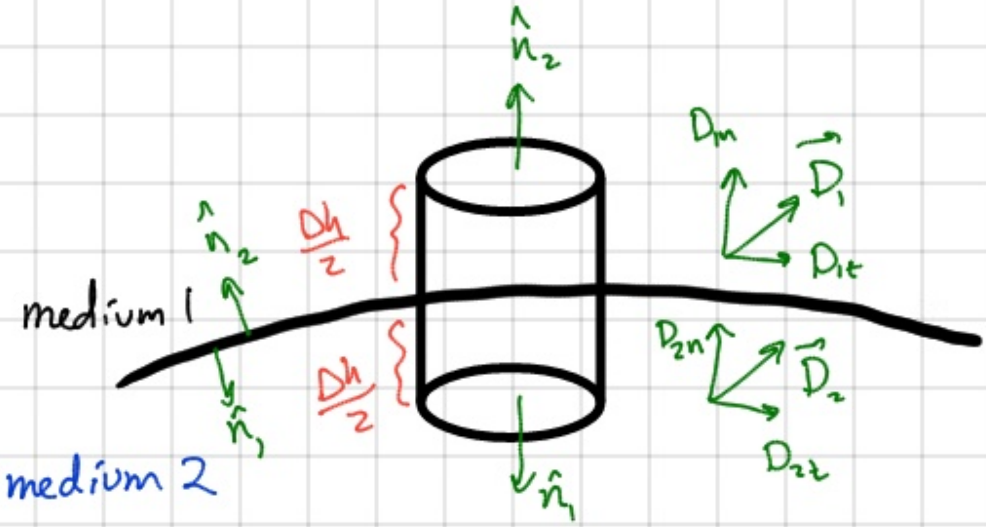
(b) For the normal components:



We apply Gauss's law, the total outward flux of $\vec{D}$ through the cylindrical surfaces must equal the total charge enclosed in the cylinder. Letting $\Delta h \to 0$, then $Q = \rho_s \Delta s$ and:

$$\oint_S \vec{D} \cdot d\vec{s} = \int_{top} \vec{D}_1 \cdot \hat{n}_2 \, ds + \int_{bottom} \vec{D}_2 \cdot \hat{n}_1 \, ds = \rho_s \Delta s$$

where $\hat{n}_1$ & $\hat{n}_2$ are the outward normal unit vectors of the bottom & top surfaces.



$$\oint_S \vec{D}\cdot d\vec{S} = \int_{top} \vec{D}_1 \cdot \hat{n}_2\, ds + \int_{bottom} \vec{D}_2 \cdot \hat{n}_1\, ds = \rho_s \Delta S$$

Since $\hat{n}_1 = -\hat{n}_2$, then $(\vec{D}_1 - \vec{D}_2)\cdot \hat{n}_2 = \rho_s$

Also, $\vec{D}_1 = \vec{D}_{1t} + \vec{D}_{1n}$, $\vec{D}_2 = \vec{D}_{2t} + \vec{D}_{2n}$, then

$$\boxed{D_{1n} - D_{2n} = \rho_s} \quad (C/m^2) \quad \sim (2)$$

That is, the normal component of $\vec{D}$ changes suddenly at a charged boundary between two different media in an amount equal to the surface charge density.

Since $\vec{D} = \epsilon \vec{E}$, using (2):

$$\boxed{\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s}$$

Summary:

(a) From Stokes's Theorem;

$$\nabla \times \vec{E} = 0 \leftrightarrow \oint_C \vec{E} \cdot \vec{dl} = 0$$

$\vec{E}$ has a continuous tangential component across a boundary.

(b) From divergence theorem:

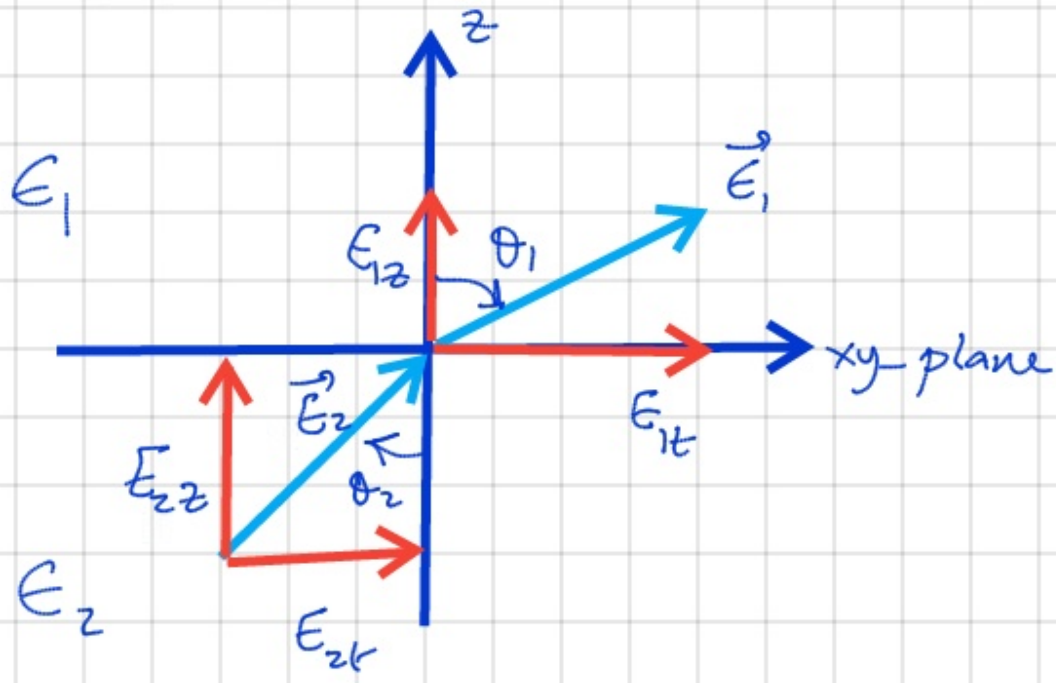
$$\nabla \cdot \vec{D} = \rho_v \leftrightarrow \oint_S \vec{D} \cdot d\vec{s} = Q$$

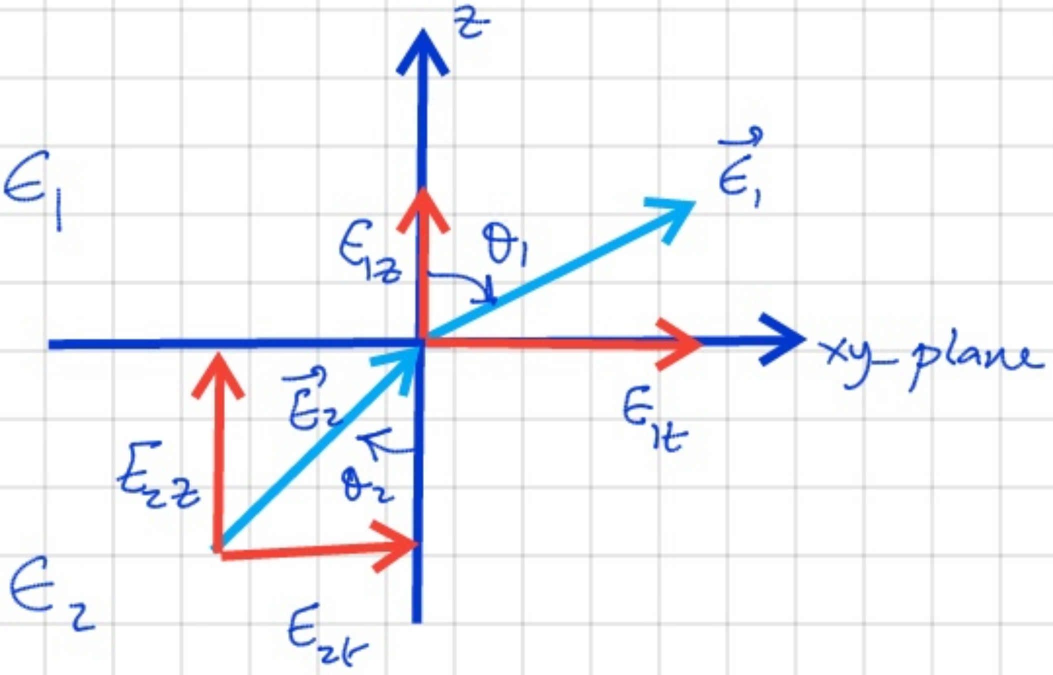
the normal component of $\vec{D}$ changes by ρ_s across a charged boundary.

Example: If the xy-plane is a charge-free boundary separating two dielectric media with permittivities ϵ_1 and ϵ_2 as shown. If the electric field in medium 1 is $\vec{E}_1 = \hat{x}E_{1x} + \hat{y}E_{1y} + \hat{z}E_{1z}$. Find

(a) the electric field $\vec{E}_2$ in medium 2

(b) the angles θ_1 & θ_2





Let $\vec{E}_2 = \hat{x} E_{2x} + \hat{y} E_{2y} + \hat{z} E_{2z}$

For the tangential components of $\vec{E}$:

the x & y components are the tangential,

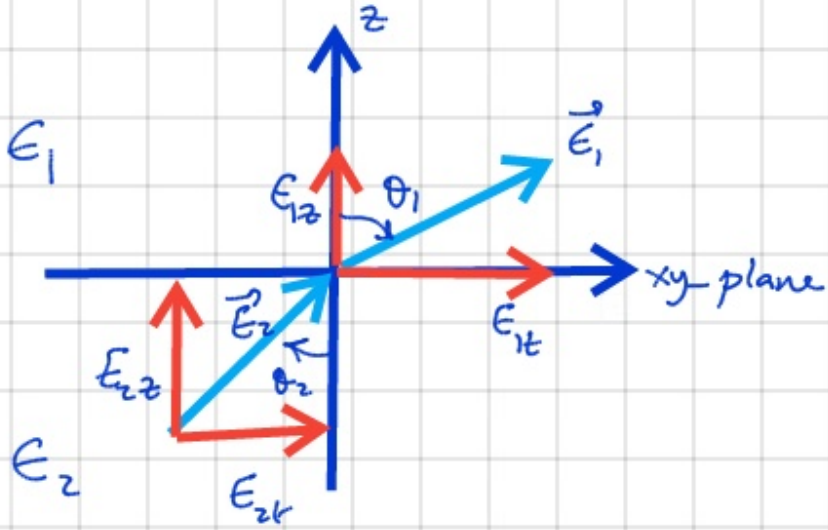
then $E_{2x} = E_{1x}$ and $E_{2y} = E_{1y}$

For the normal components of $\vec{E}$:

$$\epsilon_1 E_{1z} - \epsilon_2 E_{2z} = \rho_s = 0$$

then, $E_{2z} = \dfrac{\epsilon_1}{\epsilon_2} E_{1z}$

Then $\vec{E}_2 = \hat{x} E_{1x} + \hat{y} E_{1y} + \hat{z} \dfrac{\epsilon_1}{\epsilon_2} E_{1z}$



(b) $\vec{E}_{1t} = \hat{x}E_{1x} + \hat{y}E_{1y}$, $\vec{E}_{2t} = \hat{x}E_{2x} + \hat{y}E_{2y}$

then
$$E_{1t} = \sqrt{E_{1x}^2 + E_{1y}^2}, \quad E_{2t} = \sqrt{E_{2x}^2 + E_{2y}^2}$$

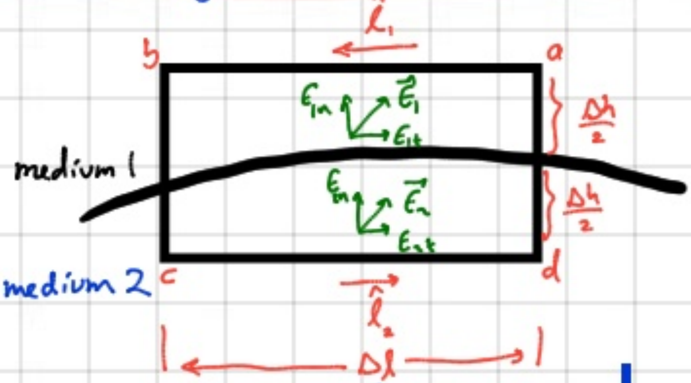
Then $\theta_1 = \tan^{-1}\left(\dfrac{E_{1t}}{E_{1z}}\right) = \tan^{-1}\left(\dfrac{\sqrt{E_{1x}^2 + E_{1y}^2}}{E_{1z}}\right)$

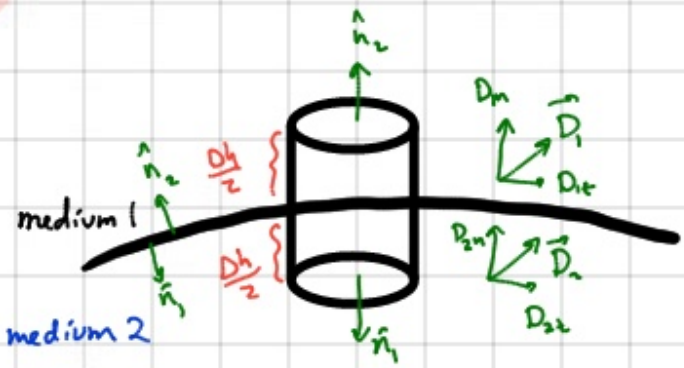
$$\theta_2 = \tan^{-1}\left(\frac{E_{2t}}{E_{2z}}\right) = \tan^{-1}\left(\frac{\sqrt{E_{2x}^2 + E_{2y}^2}}{E_{2z}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{\frac{\epsilon_1}{\epsilon_2}E_{1z}}\right)$$

We can write: $\dfrac{\tan\theta_2}{\tan\theta_1} = \dfrac{\epsilon_2}{\epsilon_1}$

For any two mediums:





$$\vec{E}_{1t} = \vec{E}_{2t}$$

$$\frac{\vec{D}_{1t}}{\epsilon_1} = \frac{\vec{D}_{2t}}{\epsilon_2}$$

$$D_{1n} - D_{2n} = \rho_s$$

$$\epsilon E_{1n} - \epsilon_2 E_{2n} = \rho_s$$

The normal components of $\vec{E}_1, \vec{E}_2, \vec{D}_1, \vec{D}_2$ are along $\hat{n}_2$

4.8.1 Dielectric-Conductor Boundary:

Let medium 1: dielectric & medium 2: perfect conductor.

Since for a perfect conductor $\sigma = \infty$, then $\vec{E} = \frac{\vec{J}}{\sigma} = 0$. Thus,

$\vec{E}_2 = 0$ & $\vec{D}_2 = \epsilon \vec{E}_2 = 0$, then E_{2t}, E_{2n}, D_{2t}, and D_{2n} are 0.

Then the above boundary conditions become:

$$\vec{E}_{1t} = \vec{E}_{2t} = 0$$

$$\vec{D}_{1t} = \vec{D}_{2t} = 0$$

$$D_{1n} = \rho_s, \quad (D_{2n} = 0)$$

$$\epsilon_1 E_{1n} = \rho_s, \quad (E_{2n} = 0)$$

$$D_{1n} = \epsilon_1 E_{1n} = \rho_s$$

Since $\boxed{D_{1n} = \epsilon_1 E_{1n} = \rho_s}$ with D_{1n} & E_{1n} normal to the interface and outward from the conductor, and $\vec{D}_{1t}$ & $\vec{E}_{1t}$ are zero, then

$$\vec{D}_1 = \epsilon_1 \vec{E} = \hat{n} \rho_s$$

where $\hat{n}$ is normally outward from the conductor

Then if ρ_s is negative, the directions of $\vec{D}$ & $\vec{E}$ are toward the conductor.

For an infinite conducting slab placed in a uniform electric field $\vec{E}_1$. The media above and below the slab have permitivity ϵ_1

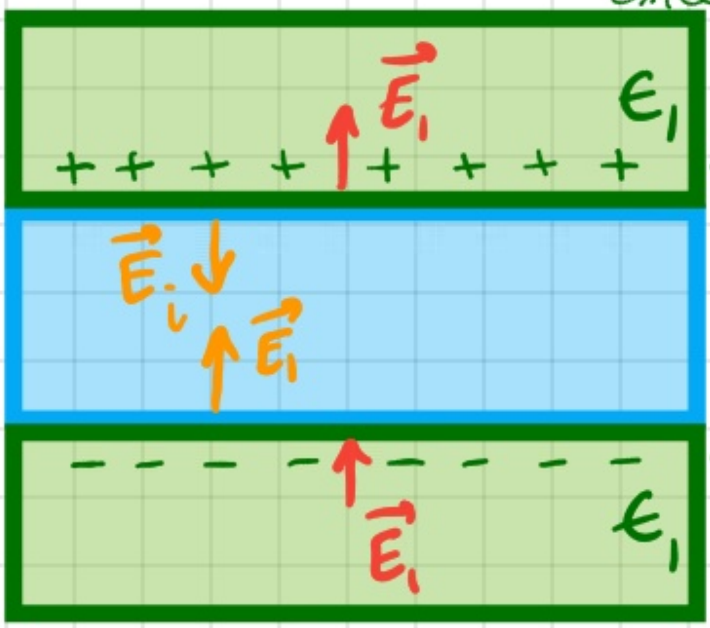
Since $\vec{D}_1 = \epsilon_1 \vec{E}_1 = \hat{n} \rho_s$

Inside the conductor:

$\vec{E} = \vec{E}_1 + \vec{E}_i$

$\vec{E} = 0$

$\Rightarrow \vec{E}_i = -\vec{E}_1$



$\rho_s = \epsilon_1 E_1$

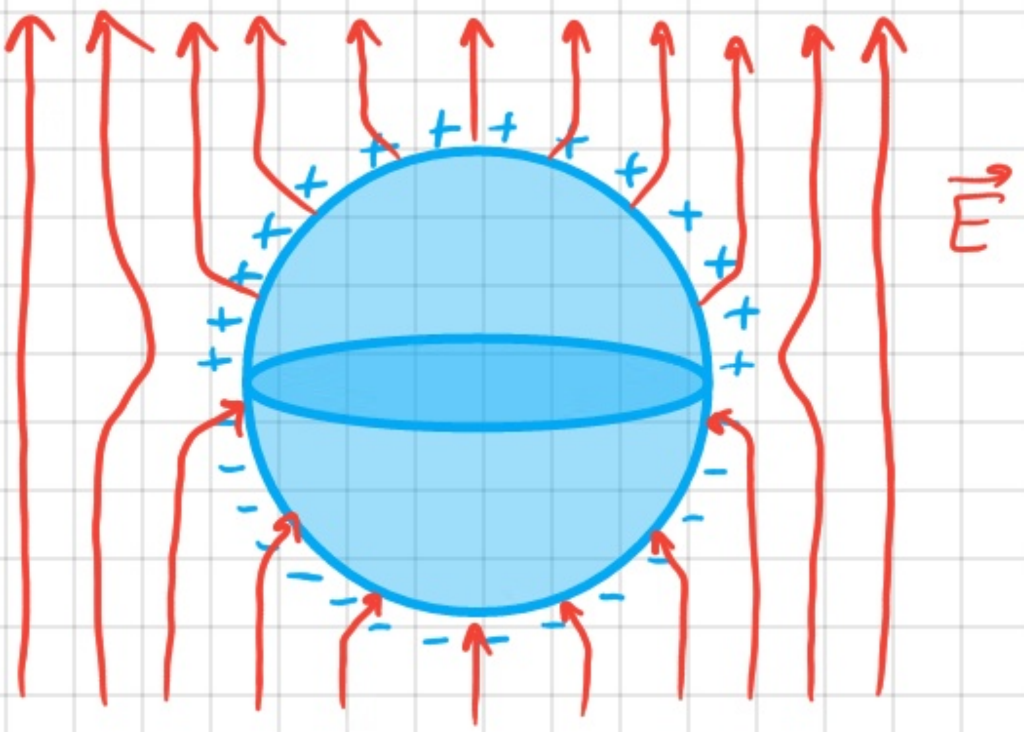
$-\rho_s = \epsilon_1 E_1$

The electric field $\vec{E}_1$ induces the surface charges ρ_s and $-\rho_s$. Then the presence of the surface charges ρ_s & $-\rho_s$ induces $\vec{E}_i$ in the conductor, then inside the conductor $\vec{E} = \vec{E}_1 + \vec{E}_i$,

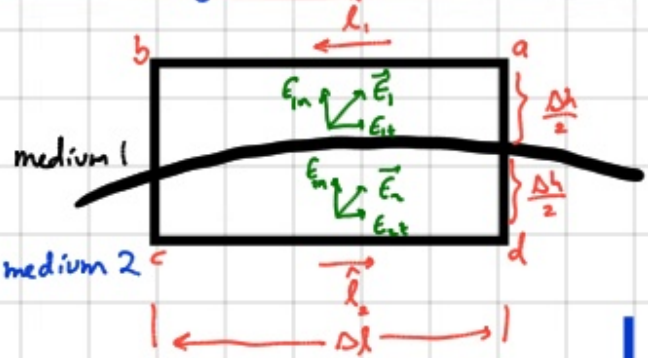
since $\vec{E} = 0$ in the conductor, then $\vec{E}_i = -\vec{E}_1$.

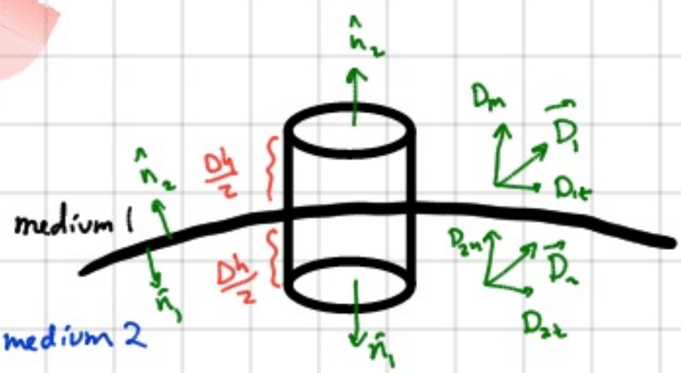
For a metal sphere placed in an electrostatic field, the positive charges will accumulate on the upper hemisphere while the negative charges will accumulate on the lower hemisphere.

Also, since $\vec{E}$ is always normal to a conductor boundary, then the electric field lines will bend to satisfy being normal to the conductor boundary



For any two mediums :





$$\vec{E}_{1t} = \vec{E}_{2t}$$

$$\frac{\vec{D}_{1t}}{\epsilon_1} = \frac{\vec{D}_{2t}}{\epsilon_2}$$

$$D_{1n} - D_{2n} = \rho_s$$

$$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$$

The normal components of $\vec{E}_1, \vec{E}_2, \vec{D}_1, \vec{D}_2$ are along $\hat{n}_2$

4.8.2 Conductor-Conductor Boundary :

Both mediums are not perfect conductors. Their permitivity is ϵ & ϵ_2 for mediums 1 & 2, respectively.

Then boundary conditions are the above equations.

In addition, since $\vec{J} = \sigma \vec{E}$ for conductors, then

$$\frac{\vec{J}_{1t}}{\sigma_1} = \frac{\vec{J}_{2t}}{\sigma_2}$$

$$\frac{\epsilon_1}{\sigma_1} J_{1n} - \frac{\epsilon_2}{\sigma_2} J_{2n} = \rho_s$$

For electrostatic, ρ_s should remain constant in time, that is by having $\vec{J}$ continuous across the interface.

For $\vec{J}_{1t}$ & $\vec{J}_{2t}$: they are tangential to the interface, thus, no charge transfer. However, for J_{1n} & J_{2n}, to guarantee continuity we should have

$J_{1n} = J_{2n}$, then for electrostatic, the equation

$\frac{\epsilon_1}{\sigma_1} J_{1n} - \frac{\epsilon_2}{\sigma_2} J_{2n} = \rho_s$, becomes :

$$J_{1n} \left[\frac{\epsilon_1}{\sigma_1} - \frac{\epsilon_2}{\sigma_2} \right] = \rho_s$$

4.9 Capacitance:

- A capacitor is any two conducting bodies separated by an insulating (dielectric) medium, regardless of their sizes and shapes.
- If a dc voltage source is connected across the two conducting bodies, then charges accumulate on the surfaces of the conductors, a $+Q$ on the surface of the conductor connected to the positive source terminal and a $-Q$ on the surface of the conductor connected to the negative source terminal.
- When a conductor has excess charge, it distributes the charge on its surface in such a manner as to maintain a zero electric field everywhere within the conductor, thereby ensuring that the electric potential is the same at every point in the conductor.

- The capacitance of a two-conductor configuration is defined as $C = \frac{Q}{V}$ (C/V or F)

where V is the potential (voltage) difference between the conductors.

- At either interface, the dielectric-conductor boundary at the conductor surface is

$\vec{D} = \epsilon \vec{E} = \hat{n} \rho_s$, with $\hat{n}$ being normally outward from the conducting surface. Then the normal component of $\vec{E}$ at any point on the surface of either conductors is

$$E_n = \hat{n} \cdot \vec{E} = \frac{\rho_s}{\epsilon}$$

where ϵ is the permittivity of the dielectric medium

- The charge Q is the integration of ρ_s over surface S of the positive surface: $Q = \int_S \rho_s \, dS$

- The voltage V between any arbitrary points, P_1 on the positive conductor and point P_2 on the negative conductor is :

$$V = V_{12} = -\int_{P_2}^{P_1} \vec{E} \cdot d\vec{l}$$

- For the charge Q on the positive conductor:

$$Q = \int_S \rho_s \, ds$$

from $Q = \int_s \rho_s \, ds = \oint_s \vec{D} \cdot d\vec{s}$:

$$Q = \int_S \rho_s \, ds = \oint_S \vec{D} \cdot d\vec{s} = \oint_S \epsilon \vec{E} \cdot d\vec{s} = \int_S \epsilon E_n \, ds$$

(OR) from $E_n = \hat{n} \cdot \vec{E} = \frac{\rho_s}{\epsilon}$:

$$Q = \int_S \rho_s \, ds = \int_S \epsilon E_n \, ds = \int_S \epsilon \vec{E} \cdot d\vec{s}$$

- Then the capacitance C :

$$C = \frac{Q}{V} = \frac{\int_s \epsilon \vec{E} \cdot d\vec{s}}{-\int_l \vec{E} \cdot d\vec{l}} \quad (F)$$

- Since $C = \frac{Q}{V} = \frac{\int_s \epsilon \vec{E} \cdot d\vec{s}}{-\int_l \vec{E} \cdot d\vec{l}}$

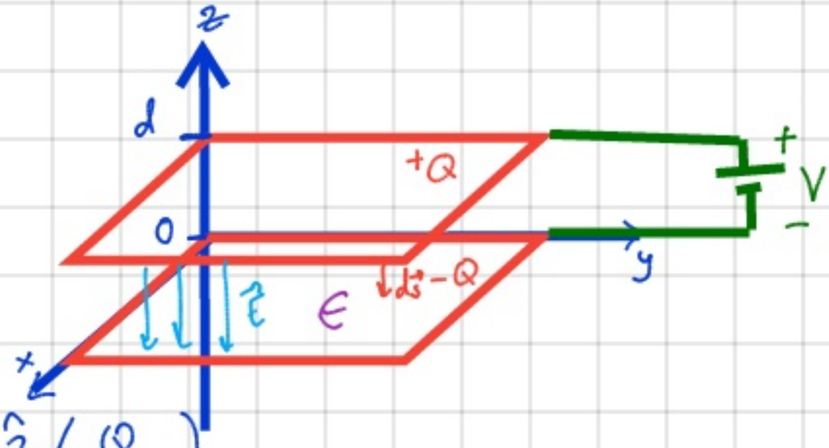
the value of C is independent of $|\vec{E}|$ since it is in both the numerator and denominator. C depends on the geometry of the conductors, the relative positions of the conductors, and the permittivity of the dielectric insulating material.

- If the dielectric material between the two conductors is not a perfect dielectric, then $\sigma \neq 0$, thus, current can flow through the non-perfect dielectric material, this dielectric material will then have a resistance R:

$$R = \frac{V}{I} = \frac{-\int_l \vec{E} \cdot dl}{\int_s \vec{J} \cdot d\vec{s}} = \frac{-\int_l \vec{E} \cdot dl}{\int_s \sigma \vec{E} \cdot d\vec{s}}$$

- Then $RC = \frac{\epsilon}{\sigma}$, if either R or C is known, we can find the other

Example: Obtain an expression for the capacitance C of a parallel-plate capacitor consisting of two parallel plates each of surface area A and separated by a distance d. The capacitor is filled with a dielectric material with permittivity ϵ. Also, determine the breakdown voltage if $d = 1$ cm and the dielectric material is quartz with a dielectric strength $E_{ds} = 30$ MV/m. Ignore the fringing field (the electric field lines near the capacitor edges).



$\rho_s = \frac{Q}{A}$ and $\vec{E} = -\hat{z}E$

since $E_n = \hat{n}\cdot\vec{E} = \frac{\rho_s}{\epsilon}$,

then $\vec{E} = -\hat{z}\left(\frac{\rho_s}{\epsilon}\right) = -\hat{z}\left(\frac{Q}{A\epsilon}\right)$

$$V = -\int_0^d \vec{E}\cdot d\vec{l} = -\int_0^d \left(-\hat{z}\frac{Q}{A\epsilon}\right)\cdot(\hat{z}\,dz) = \frac{Qd}{A\epsilon} = Ed$$

$$C = Q/V = \frac{A\epsilon}{d}$$

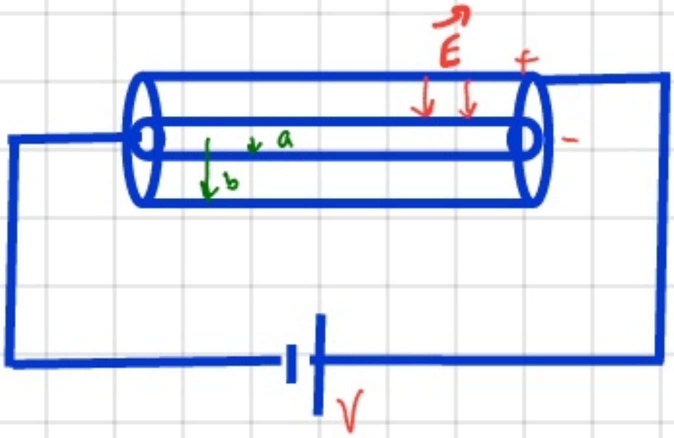
Since $E_{ds} = 30\ MV/m$ and $V = Ed$, then

$$V_{br} = E_{ds}\, d$$

$$= 30 \times 10^{6} \left(1 \times 10^{-2} \right)$$

$$V_{br} = 3 \times 10^{5}\ V$$

Example: Obtain an expression for the capacitance of the coaxial line shown.



$\vec{E} = -\hat{r} E$

and $E_n = \hat{n} \cdot \vec{E} = \frac{\rho_s}{\epsilon}$

at conductor surface.

Then $\vec{E} = -\hat{r} \frac{\rho_s}{\epsilon}$

$$= -\frac{\hat{r}}{\epsilon}\left(\frac{Q}{2\pi r l}\right)$$

$$\vec{E} = \frac{-\hat{r} Q}{2\pi r \epsilon l}$$

$$V = -\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b \frac{-\hat{r} Q}{2\pi r \epsilon l} \cdot \hat{r}\, dr$$

$$V = \frac{Q}{2\pi \epsilon l} \ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{V} = \frac{2\pi \epsilon l}{\ln(b/a)}$$

4.10 Electrostatic Potential Energy طاقة الوضع الكهربائي

A source connected to a capacitor expends energy in charging the capacitor. If the dielectric material separating the two plates is a perfect dielectric, then $\sigma = 0$ and no current flow through the dielectric material, which means no resistance. Thus, no energy loss happens in the capacitor, the energy is stored in the dielectric medium in the form of electrostatic potential energy (W_e).

The amount of stored energy W_e is related to Q, C, and V.

- If a capacitor is charged by changing the voltage across it from $v=0$ to $v=V$, where one conductor has a $+q$ accumulated charge and the other conductor has a $-q$, then a charge q has been transferred from one conductor to the other. In such case $v = \frac{1}{C} q$ since $\left(i = \frac{dq}{dt} = C\frac{dv}{dt} \right)$

- Since the voltage (v) across the capacitor is the amount of differential work (dW_e) or the differential electrostatic potential energy required to transfer an differential charge (dq) from one conductor to the other as : $dW_e = v\, dq$

$$dW_e = \frac{q}{C}\, dq$$

then for a total charge Q, the total amount of work (W_e) is : $W_e = \int_0^Q \frac{1}{C} q\, dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2$ (J)

- For a parallel-plate capacitor, we found that

$$C = \frac{\epsilon A}{d} \quad \text{and} \quad V = Ed$$

Then $W_e = \frac{1}{2} C V^2$

$$= \frac{1}{2} \frac{\epsilon A}{d} (Ed)^2$$

$$= \frac{1}{2} \epsilon E^2 (Ad)$$

$$W_e = \frac{1}{2} \epsilon E^2 v \quad (J) \sim (1)$$

where $v = Ad$ is the volume of the capacitor.

The expression of W_e in equ (1) shows that the energy expended in charging the capacitor is stored in the electric field present in the dielectric material between the two conductors. This expression is valid for any dielectric material with any volume as:

$$W_e = \frac{1}{2} \int_v \epsilon E^2 \, dv \quad (J)$$

- The electrostatic energy density (w_e) is the electrostatic potential energy W_e per unit volume:

$$w_e = \frac{W_e}{v} = \frac{1}{2}\epsilon E^2 \quad (J/m^3)$$

- For a parallel-plate capacitor, to move the plates by a differential distance $\vec{dl}$, the mechanical work dW while exerting a force $\vec{F}_{exert}$ is:

$$dW = \vec{F}_{exert} \cdot \vec{dl} \quad \sim (2)$$

The oppositely charged plates are attracted to each other by an electrical force $\vec{F}_E$. To move the plates in a constant speed with zero acceleration, the net force acting on the moving plates is

$$\vec{F}_{exert} + \vec{F}_E = 0$$

The mechanical work dW spent in moving the plates is provided by spending the electrostatic potential energy dW_e as:

$$dW = -dW_e$$

- dW_e in terms of the gradient of W_e is

$$dW_e = \nabla W_e \cdot \vec{dl}$$

- Since $dW = \vec{F}_{exert} \cdot \vec{dl}$, $dW = -dW_e$, $\vec{F}_{exert} + \vec{F}_E = 0$,

then
$$dW_e = \nabla W_e \cdot \vec{dl}$$
$$-dW = \nabla W_e \cdot \vec{dl}$$
$$-\vec{F}_{exert} \cdot \vec{dl} = \nabla W_e \cdot \vec{dl}$$
$$\Rightarrow \boxed{\vec{F}_{exert} = -\nabla W_e} \sim (3)$$

If we let the vertical spacing between the parallel-plates denoted by d as z, then

$$W_e = \frac{1}{2}\frac{Q^2}{C} = \frac{Q^2}{2}\frac{z}{\epsilon A}$$

Then using equ (3)

$$\vec{F}_{exert} = -\nabla W_e = -\hat{z}\frac{\partial}{\partial z}\left(\frac{Q^2 z}{2\epsilon A}\right) = \frac{-\hat{z}\,Q^2}{2\epsilon A}$$

Since $E = \frac{\rho_s}{\epsilon} = \frac{Q}{A\epsilon}$, then $\boxed{\vec{F}_{exert} = \frac{-\hat{z}}{2}\epsilon A E^2}$

Example: The radii of the inner and outer conductors of a coaxial cable are 2 cm and 5 cm, respectively, and the insulating material between them has a relative permittivity of 4. The charge density on the outer conductor is $\rho_\ell = 10^{-4}$ C/m. Use the expression for $\vec{E}$ found in the previous example to calculate the total energy stored in a 20 cm length of the cable.

From previous example $\vec{E} = \dfrac{-\hat{r}\,Q}{2\pi r \epsilon \ell} = -\hat{r}\dfrac{\rho_\ell}{2\pi r \epsilon}$

$$W_e = \frac{1}{2}\int_v \epsilon E^2 dv = \frac{1}{2}\int_v \epsilon \left(\frac{\rho_\ell}{2\pi r \epsilon}\right)^2 dv$$

$$= \frac{1}{2}\frac{\rho_\ell^2}{4\pi^2\epsilon}\int \frac{r\,dr\,d\phi\,dz}{r^2} = \frac{\rho_\ell^2}{8\pi^2\epsilon}\int_{0.02}^{0.05}\frac{dr}{r}\int_0^{2\pi}d\phi\int_0^{0.20}dz$$

$$= \frac{\rho_\ell^2}{8\pi^2\epsilon_0\epsilon_r}\left(\ln 0.05 - \ln 0.02\right)(2\pi)(0.20)$$

$$= \frac{10^{-8}}{4\pi\epsilon_0(4)}\ln\frac{5}{2}\,(.2) = 4.12 \text{ (J)}$$

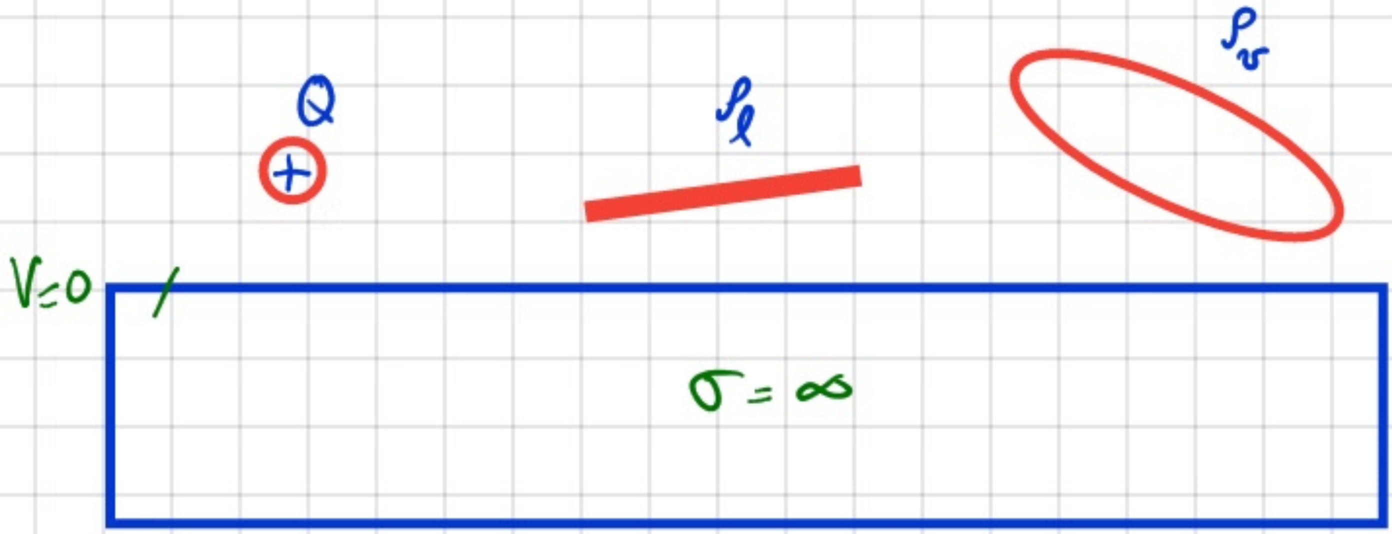
4.11 Image Method

- Any given charge configuration above an infinite, perfectly conducting plane is electrically equivalent to the combination of the given charge configuration and its image configuration, with the conducting plane removed. This applies to any charge distribution.

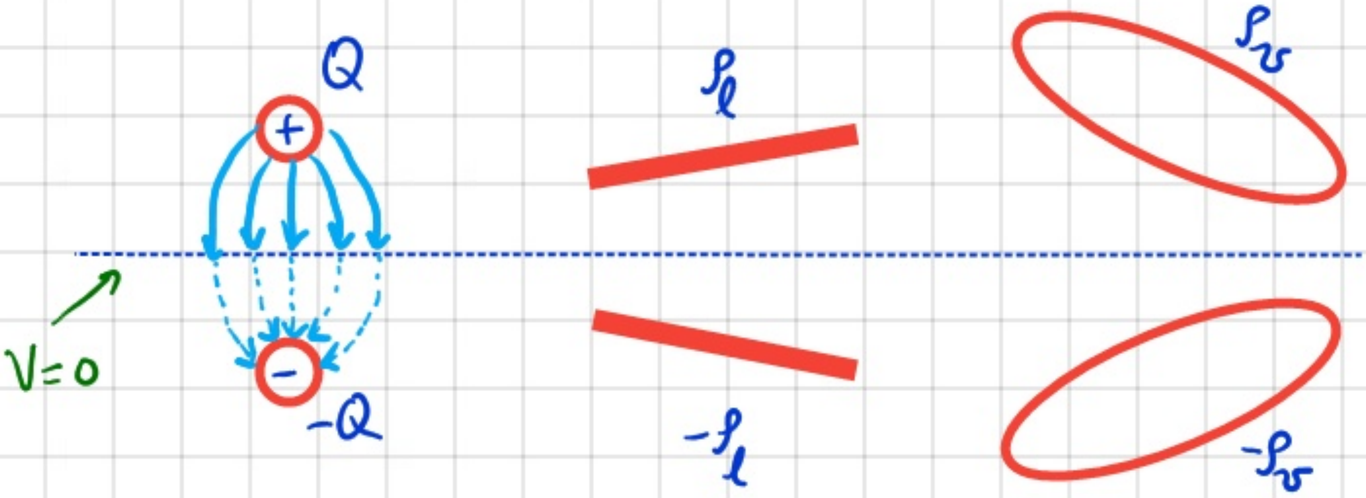
- To find the charge induced on the plane from the found electric field $\vec{E}$:

$$\rho_s = E_n \epsilon = (\hat{n} \cdot \vec{E}) \epsilon$$

where $\hat{n}$ is the outward normal unit vector to the infinite, perfectly conducting plane



Q
ρ_l
ρ_v
$V=0$
$\sigma=\infty$



Q
ρ_l
ρ_v
$V=0$
$-Q$
$-\rho_l$
$-\rho_v$

The image method is used to find V or $\vec{E}$ at any point in the space above the plane as well as the surface charge distribution on the plate, since the three methods we used previously can not be used, they are :

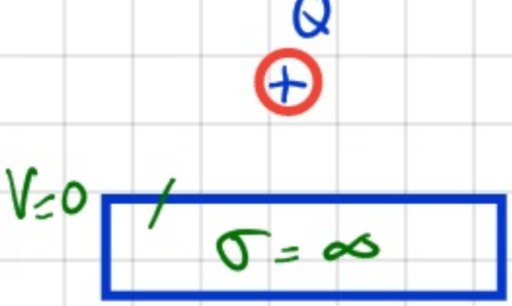
(a) Coulomb's law : it requires knowledge of the magnitudes and locations of all charges, however this case induces a charge Q on the plate with an unknown and nonuniform distribution

(b) Gauss's law : it is difficult to choose a Gaussian surface, since $\vec{E}$ has to be either tangential or normal to the Gaussian surface.

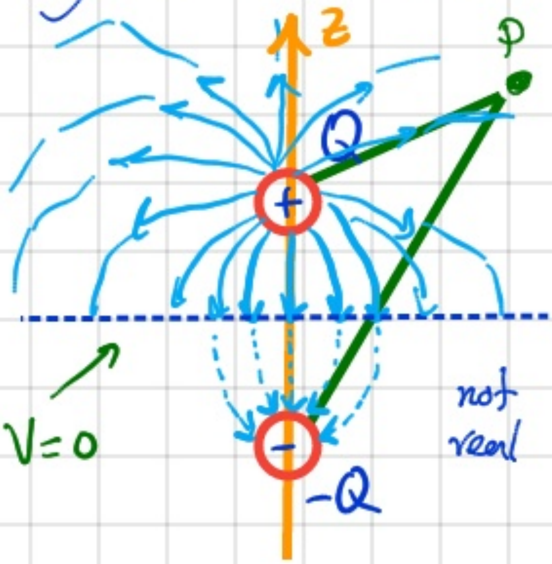
(c) Using $\vec{E} = -\nabla V$: it is mathematically difficult

Example: Use image theory to determine $\vec{E}$ at an arbitrary point $P=(x,y,z)$ in the region $z>0$ due to a charge Q in free-space at a distance d above a grounded conducting plate residing in the $z=0$ plane.

Solution:



$\Rightarrow$



$$\vec{R}_1 = x\hat{x} + y\hat{y} + (z-d)\hat{z}$$

$$\vec{R}_2 = x\hat{x} + y\hat{y} + (z+d)\hat{z}$$

For $z \geqslant 0$:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{Q\vec{R}_1}{R_1^3} + \frac{-Q\vec{R}_2}{R_2^3}\right]$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\left[\frac{x\hat{x}+y\hat{y}+(z-d)\hat{z}}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} - \frac{x\hat{x}+y\hat{y}+(z+d)\hat{z}}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right]$$

Example: Using the result from the previous example to find the surface charge density ρ_s on the surface of the conducting plane.

Solution: For a perfect conductor, at its surface

$$\rho_s = E_n \epsilon = (\hat{n} \cdot \vec{E}) \epsilon$$

Here, $$\vec{E} = \frac{Q}{4\pi\epsilon_0}\left[\frac{x\hat{x} + y\hat{y} + (z-d)\hat{z}}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{x\hat{x} + y\hat{y} + (z+d)\hat{z}}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

At the surface, $z=0$:

$$\rho_s = (\hat{n} \cdot \vec{E}) \epsilon_0 \Big|_{z=0}$$

$$= \frac{Q}{4\pi\epsilon_0} \epsilon_0 \left[\frac{(z-d)\hat{z}}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(z+d)\hat{z}}{[x^2+y^2+(z+d)^2]^{3/2}}\right]\Bigg|_{z=0}$$

$$= \frac{Q}{4\pi}\left[\frac{-d\hat{z} - d\hat{z}}{[x^2+y^2+d^2]^{3/2}}\right]$$

$$\rho_s = \frac{-Q}{2\pi} \frac{d}{[x^2+y^2+d^2]^{3/2}}$$

4.41 A cylindrical bar of silicon has a radius of 4mm and length of 8 cm. If a voltage of 5V is applied between the ends of the bar and $\mu_e = 0.13\ (m^2/V.s)$, $\mu_h = 0.05\ (m^2/V.s)$, $N_e = 1.5 \times 10^{16}$ electrons/m^2, and $N_h = N_e$, find the following:

(a) The conductivity of silicon

(b) The current I flowing in the bar

(c) The drift velocities $\vec{u}_e$ and $\vec{u}_h$

(d) The resistance of the bar

(e) The power dissipated in the bar

(a) The conductivity of silicon :

Since silicon is a semiconductor, $N_e = N_h = 1.5 \times 10^{16}$

$$\sigma = (N_e \mu_e + N_h \mu_h)\, e$$

$$= 1.5 \times 10^{16} (0.13 + 0.05)\, 1.6 \times 10^{-19}$$

$$\sigma = 4.32 \times 10^{-4} \ (S/m)$$

(b) The current I flowing in the bar:

$I = JA = \sigma E A$ and $V = E\ell$

Then $I = \sigma \dfrac{V}{\ell}(r^2 \pi)$

$$= 4.32 \times 10^{-3} \left(\frac{5}{0.08}\right)\left(4 \times 10^{-3}\right)^2 \pi$$

$$I = 1.36 \times 10^{-6} \ A$$

(c) The drift velocities $\vec{u}_e$ and $\vec{u}_h$:

$$|\vec{u}_e| = -\mu_e |\vec{E}| = -0.13 \frac{V}{\ell} = -0.13\left(\frac{5}{0.08}\right) = -8.125$$

in the opposite direction of $\vec{E}$

$$|\vec{u}_h| = \mu_h |\vec{E}| = 0.05 \frac{V}{\ell} = 0.05\left(\frac{5}{0.08}\right) = 3.125$$

in the direction of $\vec{E}$

(d) The resistance of the bar

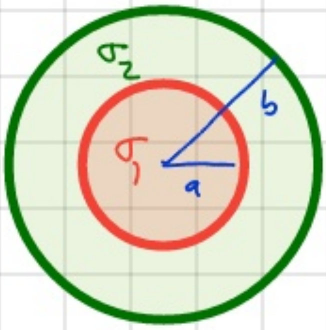
$$R = \frac{V}{I} = \frac{5}{1.36 \times 10^{-6}} = 3.676 \times 10^{6}\ \Omega$$

(e) The power dissipated in the bar

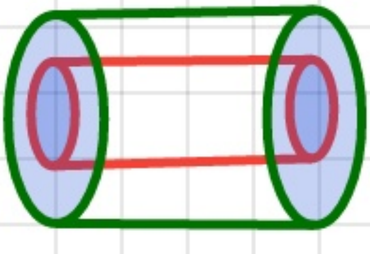
$$P = VI = 5\,(1.36 \times 10^{-6}) = 6.8 \times 10^{-6}\ W$$

4.44 A coaxial resistor of length l consists of two concentric cylinders. The inner cylinder has radius a and is made of a material with conductivity σ_1, and the outer cylinder, extending between $r=a$ and $r=b$, is made of a material with conductivity σ_2. If the two ends of the resistor are capped with conducting plates, show that the resistance between the two ends is

$$R = l/\left[\pi\left(\sigma_1 a^2 + \sigma_2 (b^2-a^2)\right)\right]$$



Each material is a resistor.
The two conducting plates let the voltage across the resistors the same. The electric field in the resistors will be parallel to the axis of the resistors. They have same voltage with different current, Thus, the two resistors are parallel. Then



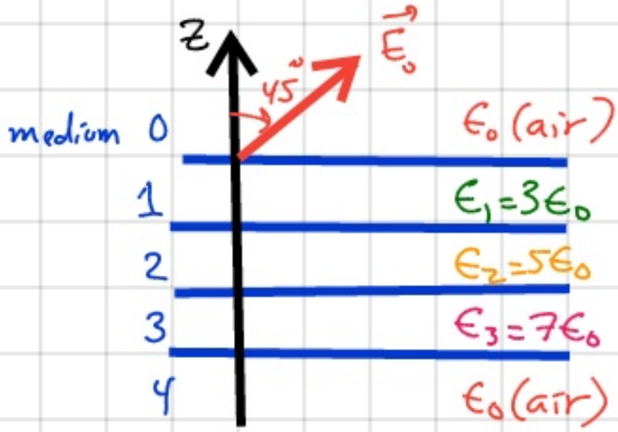
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{\sigma_1 A_1}{l_1} + \frac{\sigma_2 A_2}{l_2}$$

$$\frac{1}{R} = \frac{\sigma_1 a^2 \pi}{l} + \frac{\sigma_2 (b^2\pi - a^2\pi)}{l}$$

$$R = \frac{l}{\pi\left(\sigma_1 a^2 + \sigma_2 (b^2-a^2)\right)}$$

4.51 In the figure shown, three planar dielectric slabs of equal thickness but different dielectric constants. If $\vec{E}_0$ in air makes an angle $45°$ with respect to the z-axis, find the angle of $\vec{E}$ in each of the layers.

Since $\dfrac{\tan\theta_a}{\tan\theta_b} = \dfrac{\epsilon_a}{\epsilon_b}$,



$$\theta_1 = \tan^{-1}\left[\frac{\epsilon_1}{\epsilon_0}\tan\theta_0\right]$$
$$= \tan^{-1}\left[\frac{3\epsilon_0}{\epsilon_0}\tan 45\right]$$
$$\theta_1 = 71.57°$$

$$\theta_2 = \tan^{-1}\left[\frac{\epsilon_2}{\epsilon_1}\tan\theta_1\right] = \tan^{-1}\left[\frac{5\epsilon_0}{3\epsilon_0}\tan 71.57\right] = 78.69°$$

$$\theta_3 = \tan^{-1}\left[\frac{\epsilon_3}{\epsilon_2}\tan\theta_2\right] = \tan^{-1}\left[\frac{7\epsilon_0}{5\epsilon_0}\tan 78.69\right] = 81.87°$$

$$\theta_4 = \tan^{-1}\left[\frac{\epsilon_4}{\epsilon_3}\tan\theta_3\right] = \tan^{-1}\left[\frac{\epsilon_0}{7\epsilon_0}\tan 81.87\right] = 45°$$

4.56 Figure P4.56(a) depicts a capacitor consisting of two parallel, conducting plates separated by a distance d. The space between the plates contains two adjacent dielectrics, one with permittivity ϵ_1 and surface area A_1 and another with ϵ_2 and A_2.

The objective of this problem is to show that the capacitance C of the configuration shown in Fig. P4.56(a) is equivalent to two capacitances in parallel, as illustrated in Fig. P4.56(b), with

$$C = C_1 + C_2 \tag{4.133}$$

where

$$C_1 = \frac{\epsilon_1 A_1}{d} \tag{4.134}$$

$$C_2 = \frac{\epsilon_2 A_2}{d} \tag{4.135}$$

To this end, proceed as follows:

(a) Find the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ in the two dielectric layers.

(b) Calculate the energy stored in each section and use the result to calculate C_1 and C_2.

(c) Use the total energy stored in the capacitor to obtain an expression for C. Show that (4.133) is indeed a valid result.

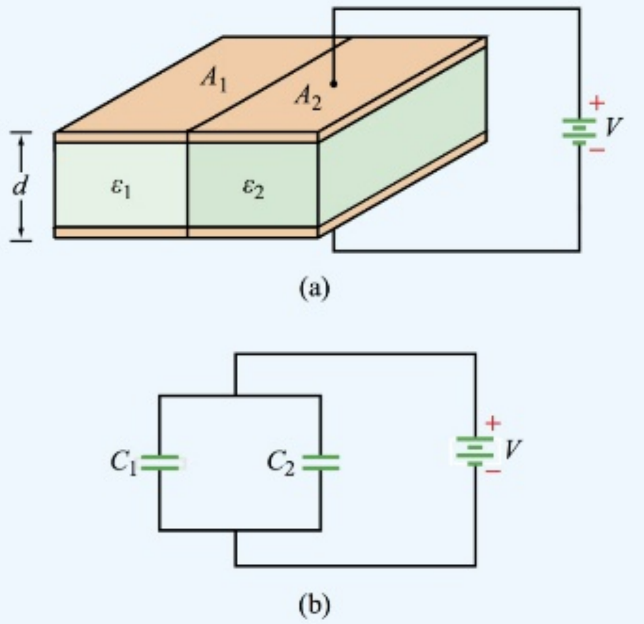
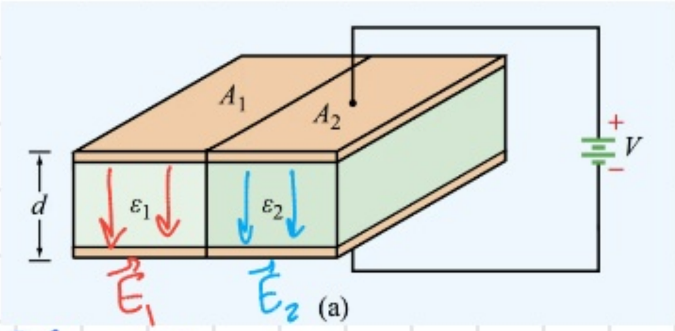


Figure P4.56 (a) Capacitor with parallel dielectric section, and (b) equivalent circuit.

(a) The electric fields $\vec{E}_1$ and $\vec{E}_2$ in the two dielectric layers



Since $V = E\ell$, then

$$E_1 = E_2 = \frac{V}{d}$$ in the direction shown.

(b) Calculate the energy stored in each section and use the results to calculate C_1 and C_2

Since $W_e = \frac{1}{2}\int_v \epsilon E^2 \, dv$, then

$$W_{e_1} = \frac{1}{2}\int \epsilon_1 \frac{V^2}{d^2}\, dv = \frac{1}{2}\epsilon_1 \frac{V^2}{d^2}(A_1 d) = \frac{1}{2}\epsilon_1 \frac{V^2}{d} A_1$$

similarly $W_{e2} = \frac{1}{2}\epsilon_2 \frac{V^2}{d} A_2$

Since $W_e = \frac{1}{2} C V^2$, by comparing :

$$C_1 = \frac{\epsilon_1 A_1}{d} \quad \text{and} \quad C_2 = \frac{\epsilon_2 A_2}{d}$$

(c) Use the total energy stored in the capacitor to obtain an expression for C. Show that
$C = C_1 + C_2$ is indeed a valid result.

$$W_{e_{Total}} = W_{e_1} + W_{e_2}$$

$$= \frac{1}{2}\epsilon_1 \frac{V^2}{d} A_1 + \frac{1}{2}\epsilon_2 \frac{V^2}{d} A_2$$

$$W_{e_{Total}} = \frac{1}{2}\frac{V^2}{d}\left(\epsilon_1 A_1 + \epsilon_2 A_2\right)$$

and $$W_{e_{Total}} = \frac{1}{2} C_{eq} V^2$$

Then $$\frac{1}{2} C_{eq} V^2 = \frac{1}{2}\frac{V^2}{d}\left(\epsilon_1 A_1 + \epsilon_2 A_2\right)$$

$$C_{eq} = \frac{\epsilon_1 A_1}{d} + \frac{\epsilon_2 A_2}{d}$$

$$C_{eq} = C_1 + C_2$$

4.58 The capacitor shown in Fig. P4.58 consists of two parallel dielectric layers. Use energy considerations to show that the equivalent capacitance of the overall capacitor, C, is equal to the series combination of the capacitances of the individual layers, C_1 and C_2, namely

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{4.136}$$

where

$$C_1 = \epsilon_1 \frac{A}{d_1}, \qquad C_2 = \epsilon_2 \frac{A}{d_2}.$$

(a) Let V_1 and V_2 be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields E_1 and E_2? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for E_1 and E_2 in terms of ϵ_1, ϵ_2, V, and the indicated dimensions of the capacitor.

(b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for C.

(c) Show that C is given by Eq. (4.136).

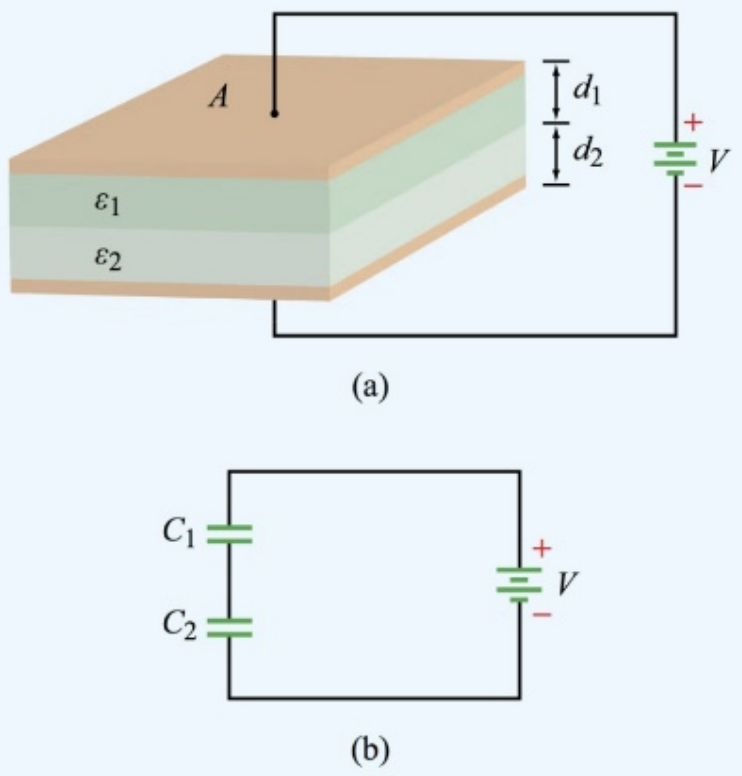
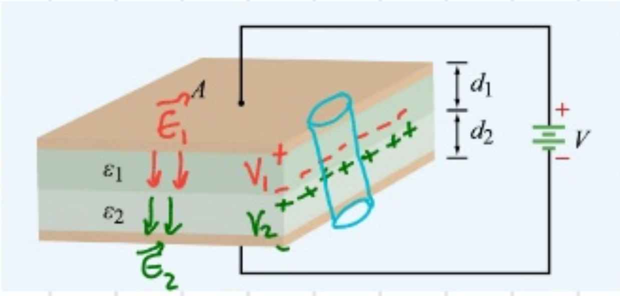


Figure P4.58 (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.58).

(a) Let V_1 and V_2 be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields E_1 and E_2? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for E_1 and E_2 in terms of ϵ_1, ϵ_2, V, and the indicated dimensions of the capacitor.



$V = V_1 + V_2$, since $V = E\ell$

$V = E_1 d_1 + E_2 d_2 \sim (1)$

Since $D_{1n} - D_{2n} = \rho_s$ for the two dielectric materials here, the total surface charge at the interface surface is zero since both have equal surface densities $\frac{Q}{A}$ and $\frac{-Q}{A}$

$D_{1n} - D_{2n} = 0 \Rightarrow D_{1n} = D_{2n}$, since $\vec{D} = \epsilon\vec{E}$ and $\vec{E}_1$ & $\vec{E}_2$ have only normal components, then

$\epsilon_1 E_1 = \epsilon_2 E_2 \sim (2)$

From (1) & (2):

$V = E_1 d_1 + \frac{\epsilon_1}{\epsilon_2} E_1 d_2$

$V = E_1 \left[d_1 + \frac{\epsilon_1}{\epsilon_2} d_2 \right]$

$E_1 = \dfrac{V}{d_1 + \frac{\epsilon_1}{\epsilon_2} d_2}$

Also, from (1) & (2):

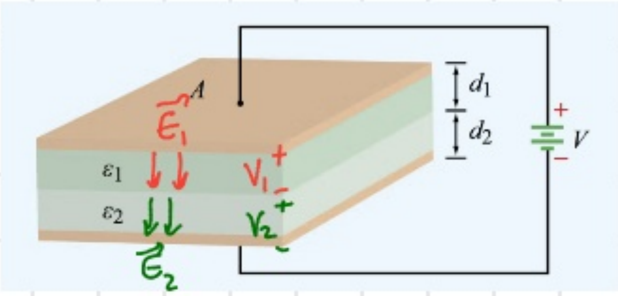
$V = \frac{\epsilon_2}{\epsilon_1} E_2 d_1 + E_2 d_2$

$V = E_2 \left[\frac{\epsilon_2}{\epsilon_1} d_1 + d_2 \right]$

$E_2 = \dfrac{V}{\frac{\epsilon_2}{\epsilon_1} d_1 + d_2}$

(b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for C.

(c) Show that C is given by Eq. (4.136).



Since $W_e = \frac{1}{2}\int_v \epsilon E^2 \, dv$, then

$$W_{e_1} = \frac{1}{2}\int_v \epsilon_1 E_1^2 \, dv = \frac{1}{2}\epsilon_1 E_1^2 A d_1$$

$$= \frac{1}{2}\epsilon_1 A d_1 \left(\frac{V}{d_1 + \frac{\epsilon_1}{\epsilon_2} d_2} \right)^2$$

$$= \frac{1}{2} \frac{\epsilon_1 A d_1 V^2 \epsilon_2^2}{(\epsilon_2 d_1 + \epsilon_1 d_2)^2}$$

$$W_{e_1} = \frac{1}{2} V^2 A d_1 \frac{\epsilon_1 \epsilon_2^2}{(\epsilon_2 d_1 + \epsilon_1 d_2)^2}$$

Similarly, $W_{e_2} = \frac{1}{2} V^2 A d_2 \frac{\epsilon_2 \epsilon_1^2}{(\epsilon_1 d_2 + \epsilon_2 d_1)^2}$

Then $W_{e_{Total}} = W_{e_1} + W_{e2}$, with $W_{e_{Total}} = \frac{1}{2} C_{eq} V^2$,

$$\frac{1}{2} C_{eq} V^2 = \frac{1}{2} V^2 A \frac{\epsilon_1 \epsilon_2}{(\epsilon_1 d_2 + \epsilon_2 d_1)^2} \left(d_1 \epsilon_2 + d_2 \epsilon_1 \right)$$

$$C_{eq} = \frac{A \epsilon_1 \epsilon_2}{\epsilon_1 d_2 + \epsilon_2 d_1}$$

(c) Since $C_1 = \frac{\epsilon_1 A}{d_1}$ & $C_2 = \frac{\epsilon_2 A}{d_2}$

and from (b)

$$C_{eq} = \frac{A \epsilon_1 \epsilon_2}{\epsilon_1 d_2 + \epsilon_2 d_1}$$

$$= \frac{A \epsilon_1 \epsilon_2}{\epsilon_1 d_2 + \epsilon_2 d_1} \cdot \frac{\frac{A}{d_1 d_2}}{\frac{A}{d_1 d_2}}$$

$$= \frac{\frac{A \epsilon_1}{d_1} \cdot \frac{A \epsilon_2}{d_2}}{\frac{A \epsilon_1}{d_1} + \frac{A \epsilon_2}{d_2}}$$

$$C_{eq} = \frac{C_1 \cdot C_2}{C_1 + C_2}$$

4.61 With reference to **Fig. P4.61**, charge Q is located at a distance d above a grounded half-plane located in the x–y plane and at a distance d from another grounded half-plane in the x–z plane. Use the image method to

(a) Establish the magnitudes, polarities, and locations of the images of charge Q with respect to each of the two ground planes (as if each is infinite in extent).

(b) Find the electric potential and electric field at an arbitrary point $P = (0, y, z)$.

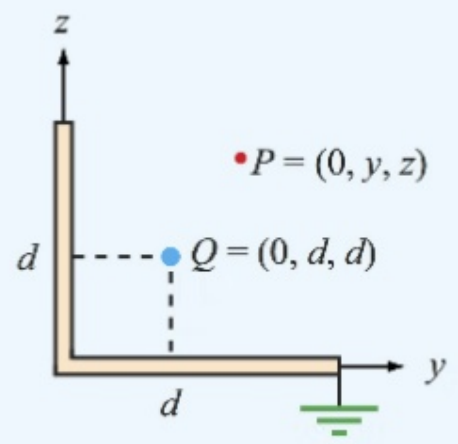
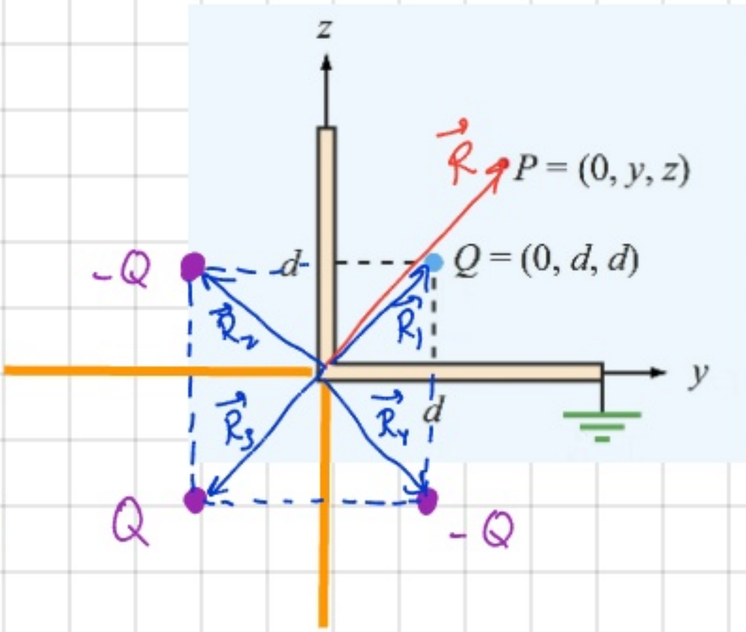


Figure P4.61 Charge Q next to two perpendicular, grounded, conducting half-planes.

For 4 point charges :



$$V = \frac{1}{4\pi\epsilon} \sum_{i=1}^{4} \frac{q_i}{|\vec{R} - \vec{R}_i|}$$

where $\vec{R} = y\hat{y} + z\hat{z}$

$\vec{R}_1 = d\hat{y} + d\hat{z}$, $\vec{R}_2 = -d\hat{y} + d\hat{z}$,

$\vec{R}_3 = -d\hat{y} - d\hat{z}$, $\vec{R}_4 = d\hat{y} - d\hat{z}$

$$|\vec{R} - \vec{R}_1| = \sqrt{(y-d)^2 + (z-d)^2}$$

$$|\vec{R} - \vec{R}_2| = \sqrt{(y+d)^2 + (z-d)^2}$$

$$|\vec{R} - \vec{R}_3| = \sqrt{(y+d)^2 + (z+d)^2}$$

$$|\vec{R} - \vec{R}_4| = \sqrt{(y-d)^2 + (z+d)^2}$$

$$V = \frac{Q}{4\pi\epsilon}\left[\frac{1}{\sqrt{(y-d)^2 + (z-d)^2}} - \frac{1}{\sqrt{(y+d)^2 + (z-d)^2}} + \frac{1}{\sqrt{(y+d)^2 + (z+d)^2}} - \frac{1}{\sqrt{(y-d)^2 + (z+d)^2}}\right]$$

while $E = -\nabla V$

$$E = -\nabla V$$

$$= -\nabla\left(\frac{Q}{4\pi\epsilon}\left[\frac{1}{\sqrt{(y-d)^2+(z-d)^2}} - \frac{1}{\sqrt{(y+d)^2+(z-d)^2}} + \frac{1}{\sqrt{(y+d)^2+(z+d)^2}} - \frac{1}{\sqrt{(y-d)^2+(z+d)^2}}\right]\right)$$

$$= \frac{-Q}{4\pi\epsilon}\left[\frac{\hat{x}(0) - \hat{y}(y-d) - \hat{z}(z-d)}{((y-d)^2+(z-d)^2)^{3/2}} - \frac{\hat{x}(0) - \hat{y}(y+d) - \hat{z}(z-d)}{((y+d)^2+(z-d)^2)^{3/2}} + \frac{\hat{x}(0) - \hat{y}(y+d) - \hat{z}(z+d)}{((y+d)^2+(z+d)^2)^{3/2}} - \frac{\hat{x}(0) - \hat{y}(y-d) - \hat{z}(z+d)}{((y-d)^2+(z+d)^2)^{3/2}}\right]$$

Chapters 4 & 5

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ϵ and σ	μ
Governing equations		
• Differential form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• Integral form	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\epsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

- In electrostatic, stationary charges produce static electric fields, here in magnetostatics, steady non-time-varying currents produce static magnetic fields.
- In electrostatic, the electric field intensity $\vec{E}$ and the electric flux density $\vec{D}$ in linear and isotropic mediums have the relation:

$$\vec{D} = \epsilon \vec{E}$$

- In magnetostatic, the magnetic field intensity $\vec{H}$ and the magnetic flux density $\vec{B}$ in dielectric and metals except in ferromagnetic materials:

$$\vec{B} = \mu \vec{H}$$ with $\mu = \mu_0$ (for most dielectric and metals)

5.1 Magnetic Forces and Torques:

- Electric force $\vec{F_e}$:

If any point charge q is present in an electric field $\vec{E}$ due to other charges, the point charge will experience a force acting on it equal to

$$\vec{F_e} = q\,\vec{E}$$

- Magnetic force $\vec{F_m}$:

Experiments show that a charge q moving with a velocity $\vec{u}$ in a magnetic flux density $\vec{B}$, $\vec{B} = \mu \vec{H}$ where $\vec{H}$ is the magnetic field, experiences a magnetic force $\vec{F_m}$ acting on it equal to

$$\vec{F_m} = q\,\vec{u} \times \vec{B} \qquad (N)$$

$\vec{B}$ is measured in tesla (T) $:= \left(\frac{N}{C \cdot m/s}\right)$

if q is positive, $\vec{F_m}$ is in the direction of $\vec{u} \times \vec{B}$

if q is negative, $\vec{F_m}$ is in the direction of $-\vec{u} \times \vec{B}$

Since $\vec{F}_m = q\,\vec{u} \times \vec{B}$

$$F_m = q u B \sin\theta$$

where θ is the angle between $\vec{u}$ & $\vec{B}$

F_m is maximum if $\theta = 90^\circ$

$F_m = 0$ $\vec{u}$ & $\vec{B}$ are parallel or in opposite

- Electric and magnetic forces :

If a charged partical is in the present of both an electric field $\vec{E}$ and a mgnetic flux density $\vec{B}$, then the total electromagnetic force $\vec{F}$ acting on it is equal to

$$\vec{F} = \vec{F}_e + \vec{F}_m$$

$$\vec{F} = q\vec{E} + q\,\vec{u} \times \vec{B}$$

$$\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

- The difference between electric and magnetic forces:

1. Their direction: the electric force is in the direction of the electric field (positive q), while the magnetic force is perpendicular to the magnetic field $\vec{H}$ (and $\vec{B}$, $\vec{B} = \mu \vec{H}$)

2. The electric force acts on a charged partical whether or not it is moving, the magnetic force acts on a charged partical only when it is moving, $\vec{F}_m = q\,\vec{u} \times \vec{B}$ where $\vec{u} \neq 0$

3. The electric force expends energy in moving a charged partical, while the magnetic force does no work when a charged partical is moved

- To find the work performed when a charged partical is moved a differential distance $\vec{dl}$ in a magnetic flux density $\vec{B}$:

since $\vec{F_m}$ & $\vec{u}$ are perpendicular,

$$dW = \vec{F_m} \cdot \vec{dl} = \vec{F_m} \cdot (\vec{u}\, dt) = 0$$

Since $dW = 0$, no work is done on the moved charged partical, this means the kinetic energy of the charged partical doesn't change, this means that the magnetic flux density can change the direction of the charged partical not its speed $|\vec{u}|$.

Example: An electron moving in the positive x-direction perpendicular to a magnetic field is deflected in the negative z direction. What is the direction of the magnetic field.

$$\vec{F}_m = q\,\vec{u} \times \vec{B}$$

here q is an electron, then

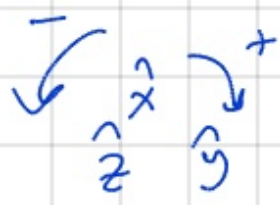
$$\vec{F}_m = -e\,\vec{u} \times \vec{B}$$

$$\vec{F}_m = -e(\hat{x}\,u) \times \vec{B}$$

since $\vec{F}_m = -\hat{z}\,F_m$

then $-\hat{z}\,F_m = -eu\,\hat{x} \times \vec{B}$

$$\hat{z}\,F_m = eu\,\hat{x} \times \vec{B}$$



for $\hat{x} \times \vec{B}$ to be in $\hat{z}$ direction

then $\vec{B}$ has to be in the $\hat{y}$ direction

which is also the direction of the magnetic field $\vec{H}$.

Example: A proton moving with a speed 2×10^{6} m/s through a magnetic field with magnetic flux density of 2.5 T experiences a magnetic force of magnitude 4×10^{-13} N. What is the angle between the magnetic field and the proton's velocity.

$$\vec{F}_m = q\,\vec{u}\times\vec{B}$$

$$F_m = q\,u\,B\sin\theta$$

$$4\times10^{-13} = 1.6\times10^{-19}\left(2\times10^{6}\right)(2.5)\sin\theta$$

$$\sin\theta = 0.5$$

$$\theta = \sin^{-1}0.5$$

$$\theta = 30^\circ \text{ or } 150^\circ$$

Since $\sin 30 = \sin 150 = 0.5$

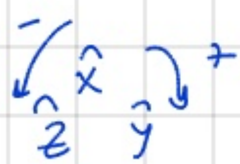
Example: A charged partical with velocity $\vec{u}$ is moving in a medium containing a uniform electric field $\vec{E} = \hat{x} E$ and a uniform magnetic flux density $\vec{B} = \hat{y} B$. What should $\vec{u}$ be so that the partical experiences no net force on it.

From $\vec{E}$: $\vec{F}_e = q\vec{E} = \hat{x} q E$

From $\vec{B}$: $\vec{F}_m = q\vec{u} \times \vec{B} = qB\,\vec{u} \times \hat{y}$

For the net force on the charged partical to be zero, $\vec{u} = \hat{x}u_x + \hat{y}u_y + \hat{z}u_z$ should satisfy:

$$\vec{F}_e + \vec{F}_m = 0 \Rightarrow -\vec{F}_e = \vec{F}_m$$



$$-\hat{x} q E = qB\left(\hat{x}u_x + \hat{y}u_y + \hat{z}u_z\right) \times \hat{y}$$

$$-\hat{x} q E = qB\left(\hat{z}u_x + (0)u_y - \hat{x}u_z\right)$$

$\Rightarrow$ we have 3 conditions :

1. $\left[\hat{z}(0) = \hat{z}u_x qB\right] \Rightarrow u_x = 0$

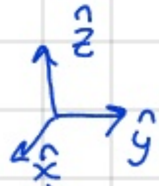
2. $\left[\hat{y}(0) = (0)u_y qB\right] \Rightarrow u_y$ can equal any constant

3. $\left[-\hat{x} q E = -\hat{x}u_z qB\right] \Rightarrow u_z = \frac{E}{B}$

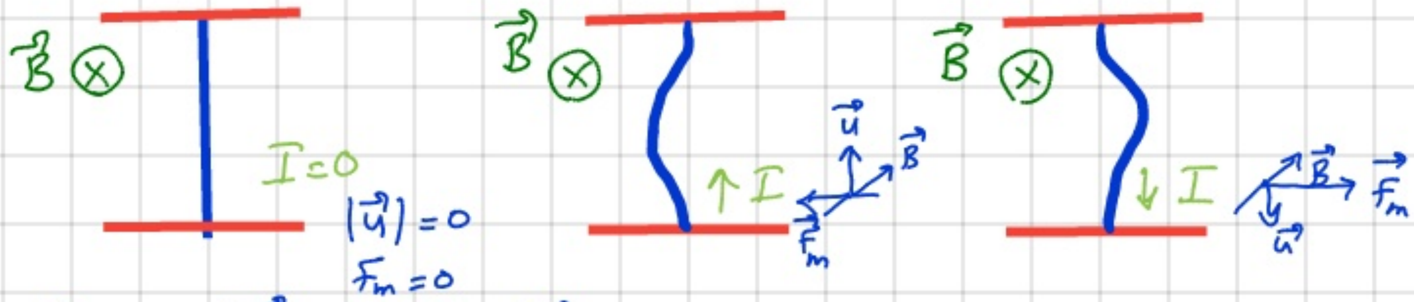
As a result, $\vec{u} = \hat{y}u_y + \hat{z}\frac{E}{B}$

5.1.1 Magnetic Force on a Current-Carrying Conductor

- When a current-carrying wire is placed in a magnetic field, it experiences a force equal to the sum of the magnetic forces acting on the charged particals moving within the wire.
- For example, let a slightly flexible wire be placed vertically along the z-axis in a magnetic flux density $\vec{B}$ along the $-\hat{x}$ direction.



$(I=0)$ (I upward) (I downward)



Since $\vec{F}_m = q\,\vec{u}\times\vec{B}$, then when

(1) $I=0$, $\vec{F}_m = 0$, (2) I: upward, $\vec{F}_m = -\hat{y}\,F_m$

(3) I: downward, $\vec{F}_m = \hat{y}\,F_m$

- To find the relation between $\vec{F}_m$ and I,

in a conducting wire, the current is electrons, then the free electron charge density causing the current is $\rho_{ve} = -N_e e$ where N_e is the number of moving electrons per volume, then to find the total differential charge dQ contained in a differential volume dv of the wire:

$$dQ = \rho_{ve}\, dv$$

$$= \rho_{ve}\, A\, dl$$

$$dQ = -N_e\, e\, A\, dl$$

Then the differential magnetic force on dQ is

$$d\vec{F}_m = dQ\, \vec{u}_e \times \vec{B}$$

$$d\vec{F}_m = -N_e\, e\, A\, dl\, \vec{u}_e \times \vec{B}$$

where $\vec{u}_e$ is the drift velocity of the electrons.

$$d\vec{F}_m = -N_e e A\, dl\, \vec{u}_e \times \vec{B}$$

For the conventional current (i.e. positive charges), it moves a differential length $\vec{dl}$ in the positive direction along $\vec{dl}$, so the electrons move a drift velocity $\vec{u}_e$ in the negative direction along $\vec{dl}$,

Thus, $dl\,\vec{u}_e = -\vec{dl}\, u_e$. Then

$$d\vec{F}_m = N_e e A u_e\, \vec{dl} \times \vec{B}$$

Since $I = \int_S \vec{J} \cdot d\vec{s}$

$= \int_S \rho_v \vec{u} \cdot d\vec{s}$, $\vec{u}$: for positive charges

$= \int_S \rho_{ve} (-\vec{u}_e) \cdot d\vec{s}$

$= \rho_{ve} (-u_e) A$

$= (-N_e e)(-u_e) A$

$$I = N_e e u_e A$$

Then $d\vec{F}_m = I\, \vec{dl} \times \vec{B}$

- for a closed contour C carrying a current I, the total magnetic force is:

$$d\vec{F}_m = I\, d\vec{l} \times \vec{B}$$

$$\vec{F}_m = I \oint_C d\vec{l} \times \vec{B} \qquad (N)$$

- If $\vec{B}$ is uniform, then

$$\vec{F}_m = I \oint_C d\vec{l} \times \vec{B}$$

$$= I \left(\oint_C d\vec{l} \right) \times \vec{B}$$

$$= I\,(0) \times \vec{B}$$

$$\vec{F}_m = 0$$

As a result, the total magnetic force on any closed current loop in a uniform magnetic flux density $\vec{B}$ (& magnetic field $\vec{H}$) is zero.

- In the study of magnetostatics, all currents should flow through closed paths. Since for charges to be non time-varying (static), they should flow in a closed loop. If they did not, then the positive charges would accumulate at one end and the negative charges would accumulate at the other end, thus, they will violate the static assumption for charge distribution.

- Since $\vec{F}_m = I \oint_C d\vec{l} \times \vec{B}$ for a closed contour, then for a length l segment on closed loop wire residing in a uniform magnetic flux density $\vec{B}$, $\vec{F}_m = I \int_l d\vec{l} \times \vec{B} = I \vec{l} \times \vec{B}$

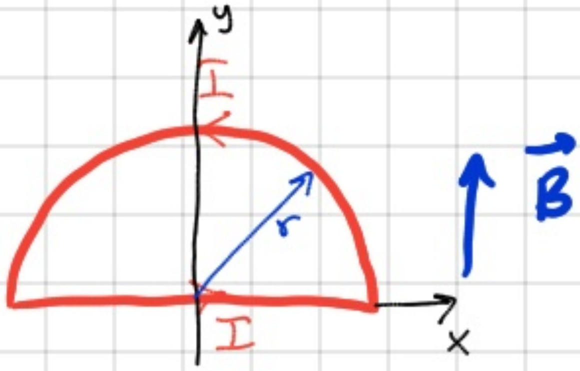
where $\vec{l}$ is a vector direction from point a to b indifferent to the path taken.

Example: For a semicircular conductor that lies in the xy-plane and carries a current I as shown. If it is exposed to a uniform magnetic flux density $\vec{B} = \hat{y} B_0$. Determine:

(a) the magnetic force $\vec{F}_1$ on the straight section

(b) the magnetic force $\vec{F}_2$ on the curved section

(a) $\vec{l} = \hat{x}\, 2r$



$$\vec{F}_1 = I \vec{l} \times \vec{B}$$

$$= I (\hat{x}\, 2r) \times \hat{y}\, B_0$$

$$\vec{F}_1 = \hat{z}\, 2rIB_0$$

(b) $\vec{l} = \hat{\phi}\, r d\phi$, $\vec{B} = \hat{y} B_0 = (\hat{r}\sin\phi + \hat{\phi}\cos\phi) B_0$

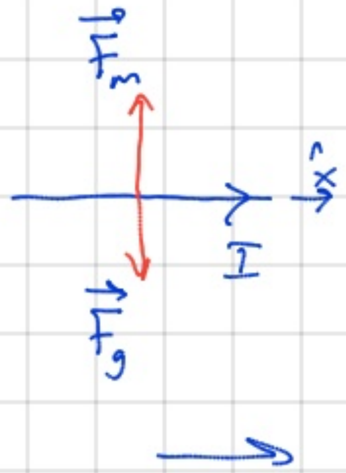
$$\vec{F}_2 = I \int_l d\vec{l} \times \vec{B} = I \int_{\phi=0}^{\pi} (\hat{\phi}\, r d\phi) \times (\hat{r}\sin\phi + \hat{\phi}\cos\phi) B_0$$

$$= I \int_{\phi=0}^{\pi} (-\hat{z}\, r \sin\phi + 0) B_0\, d\phi = -\hat{z} I B_0 r \left(-\cos\phi \Big|_0^{\pi}\right)$$

$$\vec{F}_2 = -\hat{z} I_0 B_0 r (-\cos\pi + \cos 0) = -\hat{z}\, 2rIB_0$$

Example: A horizontal wire with a mass per unit length of 0.2 kg/m carries a current of 4A in the x-direction. If the wire is placed in a uniform magnetic flux density $\vec{B}$, what should be the direction and minimum magnitude of $\vec{B}$ in order to balance the wire horizontally.
Hint: the acceleration due to gravity is $\vec{g} = -\hat{z}\, 9.8\,\text{m/s}^2$

for a length ℓ,



$$\frac{\vec{F}_g}{\ell} = \frac{m}{\ell}\vec{g} = 0.2\,(-\hat{z}\, 9.8)$$

$$\frac{\vec{F}_m}{\ell} = \frac{I\vec{\ell}\times\vec{B}}{\ell} = I\,\frac{(\ell\,\hat{x})}{\ell}\times\vec{B}$$

Then $\quad \dfrac{\vec{F}_m}{\ell} + \dfrac{\vec{F}_g}{\ell} = 0 \quad \Rightarrow \quad \dfrac{\vec{F}_m}{\ell} = -\dfrac{\vec{F}_g}{\ell}$

That is, $\vec{F}_m$ should be in the $+\hat{z}$ direction

$$-\hat{z}\,(.2)(9.8) = -I\,\hat{x}\times\vec{B}$$

$$+\hat{z}\,1.96 = +I\,\hat{x}\times(\hat{x}B_x + \hat{y}B_y + \hat{z}B_z)$$

$$+\hat{z}\,1.96 = I\left((0)B_x + \hat{z}B_y - \hat{y}B_z\right)$$

$$+\hat{z}\,1.96 \quad = I\left((0)B_x + \hat{z}\,B_y - \hat{y}\,B_z\right)$$

(1) $0\,\hat{x} = 0\,B_x I$, B_x can be anything

(2) $0\,\hat{y} = -\hat{y}\,B_z I$, $B_z = 0$

(3) $\hat{z}\,1.96 = \hat{z}\,B_y I$, $B_y = \dfrac{1.96}{I} = \dfrac{1.96}{4} = 0.49$

Then

$$\vec{B} = \hat{x}\,B_x + \hat{y}\,0.49 \quad (T)$$

For a minimum magnitude of $\vec{B}$

$$\vec{B} = \hat{y}\,0.49 \quad (T)$$

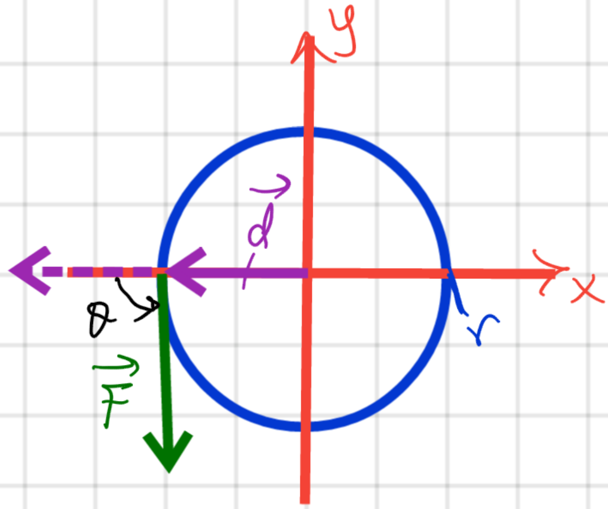
5.1.2 Magnetic Torque on a Current-Carrying Loop:

- In general, for a rigid body that can rotate about a fixed axis, the torque $\vec{T}$ is

$$\vec{T} = \vec{d} \times \vec{F} \quad (\text{N.m})$$

where $\vec{d}$ is called the moment arm and is the perpendicular distance between a point on the rotation axis and the point at which the force $\vec{F}$ is applied.

Example: For a disk in the xy-plane, if a force $\vec{F}$ is applied on the disk and makes an angle θ with $\vec{d}$.

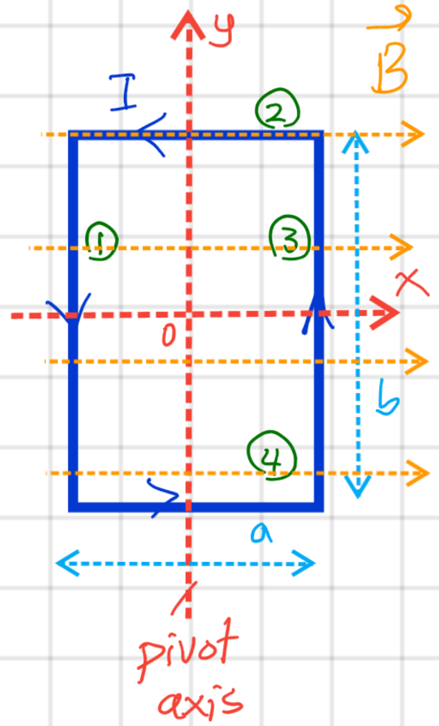


Then the torque $\vec{T} = \vec{d} \times \vec{F}$

$$\vec{T} = \hat{z}\, r F \sin\theta$$

- For the magnetic torque on a conducting loop under the influence of magnetic forces:

Example: Magnetic Field in the Plane of the Loop:



Consider the rectangular conducting loop shown. It is constructed from rigid wire and carries a current I. The loop lies in the xy-plane and is allowed to pivot about the y-axis. Under the influence of an externally generated uniform magnetic flux density $\vec{B} = \hat{x} B_0$. Then the magnetic forces on the four arms are as follows:

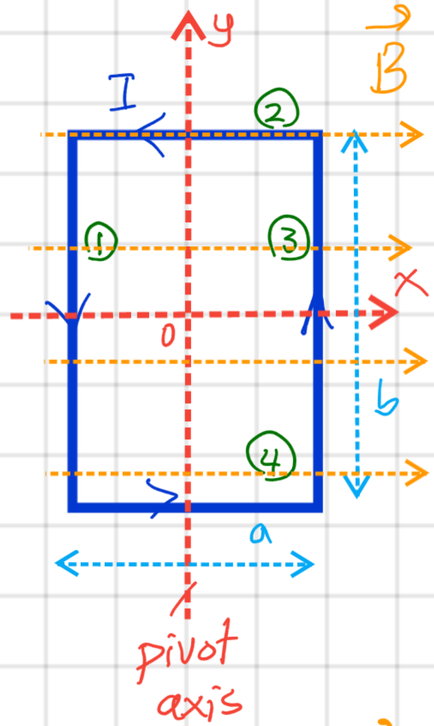
1st arm : $\vec{F}_1 = I \vec{l}_1 \times \vec{B} = I(-\hat{y} b) \times \hat{x} B_0 = \hat{z} I b B_0$

3rd arm : $\vec{F}_3 = I \vec{l}_3 \times \vec{B} = I(\hat{y} b) \times \hat{x} B_0 = -\hat{z} I b B_0$

2nd arm : $\vec{F}_2 = I\vec{l}_2 \times \vec{B} = I(-\hat{x}a) \times \hat{x}B_o = 0$

4th arm : $\vec{F}_4 = I\vec{l}_4 \times \vec{B} = I(\hat{x}a) \times \hat{x}B_o = 0$

Then the non-zero forces $\vec{F}_1$ & $\vec{F}_3$ produce a total torque of:

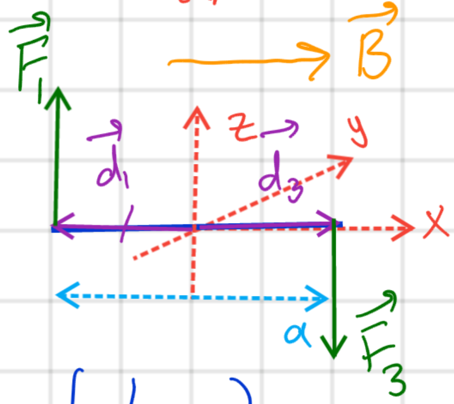


$$\vec{T} = \vec{T}_1 + \vec{T}_3$$

$$= \vec{d}_1 \times \vec{F}_1 + \vec{d}_3 \times \vec{F}_3$$

$$= \left(-\hat{x}\frac{a}{2}\right) \times \left(\hat{z} I b B_o\right) + \left(\hat{x}\frac{a}{2}\right) \times \left(-\hat{z} I b B_o\right)$$

$$= \frac{ab}{2} I B_o \left[\hat{y} + \hat{y}\right]$$



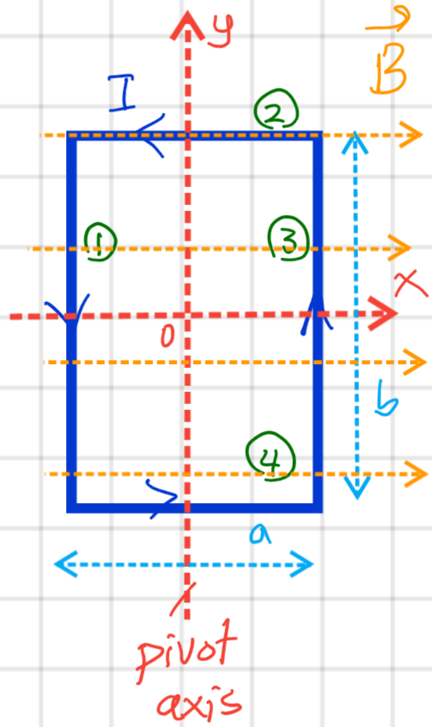
$$\vec{T} = \hat{y}(ab) I B_o$$

$$\vec{T} = \hat{y} A I B_o \quad , \quad A = ab \text{ (area of loop)}$$

Here the angles θ_1 & θ_2 are 90°, as soon as the loop starts to rotate, the angles change causing a decrease in $\vec{T}$ till $\vec{T} = 0$ after 90° of rotation making θ_1 & θ_2 equal 0°.

Example: Magnetic Field Perpendicular to the Axis of a Rectangular Loop and Makes an Angle θ with the Loop's Surface Normal $\hat{n}$:

This case differs from the previous example only by the angle between the loop and the magnetic flux density $\vec{B}$. Here the angle between $\vec{B}$ and the loop's surface normal $\hat{n}$ is θ.



Then $\vec{T} = \vec{T}_1 + \vec{T}_3$

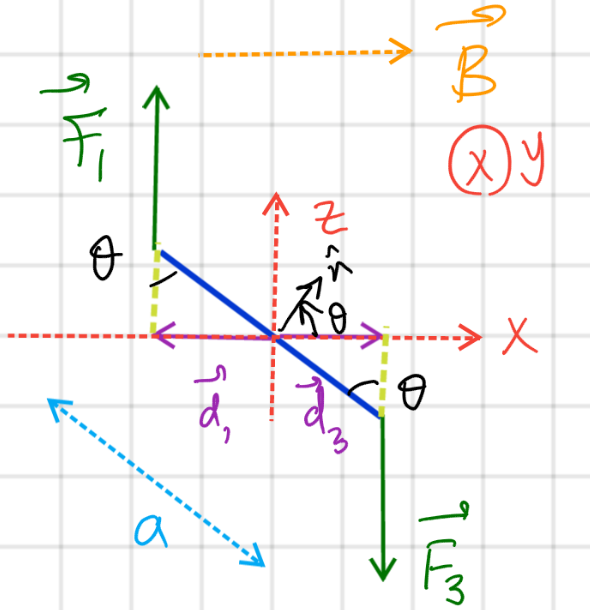
$$= \vec{d}_1 \times \vec{F}_1 + \vec{d}_3 \times \vec{F}_3$$

$$= \left(-\hat{x}\,\frac{a}{2}\sin\theta\right)\times\left(\hat{z}\,I b B_0\right)$$

$$+ \left(\hat{x}\,\frac{a}{2}\sin\theta\right)\times\left(-\hat{z}\,I b B_0\right)$$

$$= \frac{ab}{2}\,I B_0 \sin\theta\,(\hat{y}+\hat{y})$$

$$\vec{T} = \hat{y}\,A I B_0 \sin\theta \qquad \sim (1)$$



- If the loop consists of N turns, each contributing a torque of magnitude from (1)

$$|\vec{T}| = AIB_0 \sin\theta$$

Then the total magnitude of the torque is:

$$|\vec{T}| = NAIB_0 \sin\theta$$

For $\hat{n}$ the surface normal of the loop, which is from the right-hand-rule the direction of the thumb when the four fingers are in the direction of the current I, let $m = NAI$, it is called the magnetic moment, then

$$\vec{m} = \hat{n}\, m \qquad (A.m^2)$$

Then the total torque $\vec{T}$ in terms of $\vec{m}$ is

$$\boxed{\vec{T} = \vec{m} \times \vec{B}}$$ (this is valid for any $\vec{B}$ and for any loop shape)

$$= (NAI)\, \hat{n} \times \vec{B}$$

$$|\vec{T}| = NAIB \sin\theta$$

where θ : angle between $\hat{n}$ & $\vec{B}$

Example: A square coil of 100 turns and 0.5 m long sides is in a region with a uniform magnetic flux density of 0.2 T. If the maximum magnetic torque exerted on the coil is 4×10^{-2} (N.m), what is the current flowing in the coil.

Solution: The maximum torque is when $\sin\theta = 1$.

Then $\vec{T} = \vec{m} \times \vec{B}$

$$T_{max} = NIAB\,(\cancelto{1}{\sin\theta})$$

$$4\times10^{-2} = (100)\, I\, (0.5)^2\, (0.2)$$

$$I = \frac{4\times10^{-2}}{(100)(25\times10^{-2})(0.2)}$$

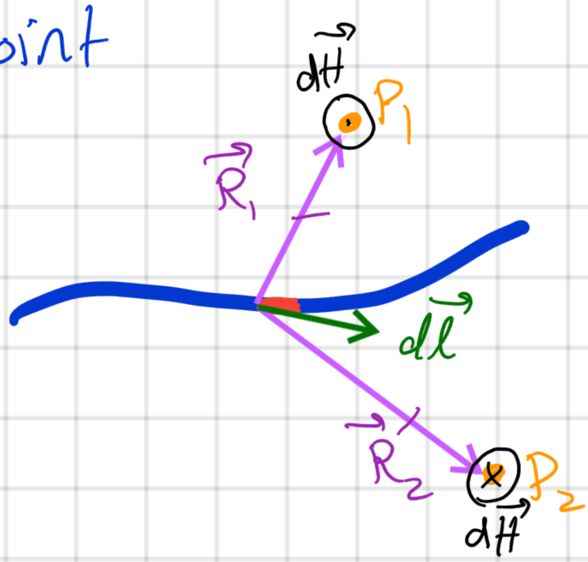
$$I = 8\times10^{-3}\ A$$

5.2 The Biot-Savart Law :

Their Law states that the differential magnetic field $d\vec{H}$ generated by a steady current I $\left(\frac{\partial I}{\partial t} = 0\right)$ flowing through a differential length vector $d\vec{l}$ is

$$d\vec{H} = \frac{1}{4\pi} \frac{I \, d\vec{l} \times \hat{R}}{R^2} \qquad (A/m)$$

where $\vec{R} = \hat{R} R$ is the distance vector between $d\vec{l}$ and the observation point, where $d\vec{l}$ is along the direction of the current I, and $\vec{R}$ points from the current element to the observation point



- To determine the total magnetic field $\vec{H}$ due to a conductor of finite size with a current density, the total $\vec{H}$ will be the integration over $d\vec{H}$. Then Biot-Savart law becomes:

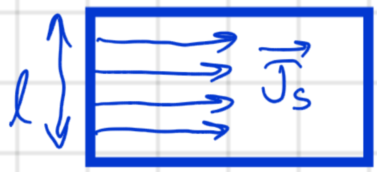
(1) For current I along a line path l:

$$\vec{H} = \frac{I}{4\pi} \int_l \frac{d\vec{l} \times \hat{R}}{R^2} \quad (A/m)$$

(2) For surface current density $\vec{J}_s$ over a surface S:

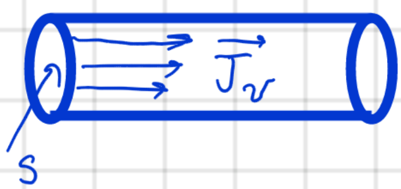
$$\vec{H} = \frac{1}{4\pi} \int_S \frac{\vec{J}_s \times \hat{R}}{R^2} \, ds$$

where $I = \int_l J_s \, dl$



(3) For volume current density $\vec{J}_v$ over a volume v:

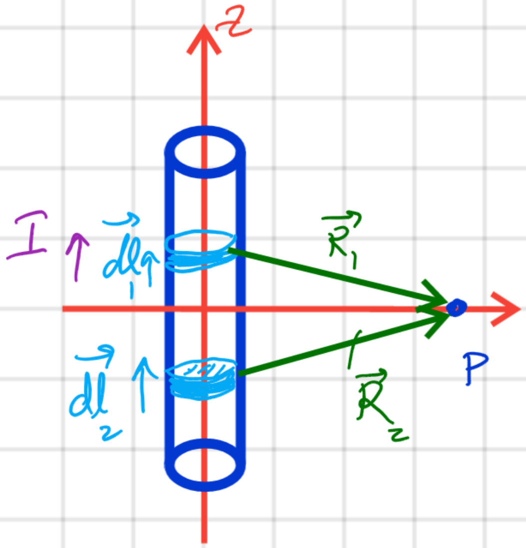
$$\vec{H} = \frac{1}{4\pi} \int_v \frac{\vec{J}_v \times \hat{R}}{R^2} \, dv$$



where $I = \int_S \vec{J}_v \cdot d\vec{s}$

Example: Magnetic Field of a Linear Conductor:

A free-standing linear conductor of length l carries a current I along the z-axis as shown. Determine the magnetic flux density $\vec{B}$ at a point P located at a distance r in the xy-plane.

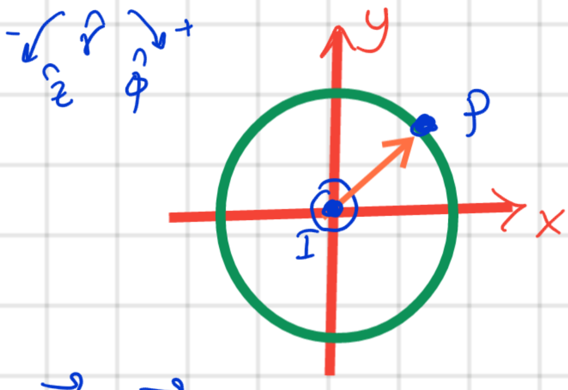


Solution: $\vec{dl}_1 = \vec{dl}_2 = \hat{z}\,dz$

$\vec{R}_1 = \hat{r}r - \hat{z}z$, $\vec{R}_2 = \hat{r}r + \hat{z}z$

$\vec{dl}_1 \times \vec{R}_1 = (\hat{z}\,dz) \times (\hat{r}r - \hat{z}z) = \hat{\phi}\, r\,dz$

$\vec{dl}_2 \times \vec{R}_2 = (\hat{z}\,dz) \times (\hat{r}r + \hat{z}z) = \hat{\phi}\, r\,dz$



so $\vec{dl}_1 \times \vec{R}_1 = \vec{dl}_2 \times \vec{R}_2 = \hat{\phi}\, r\,dz$

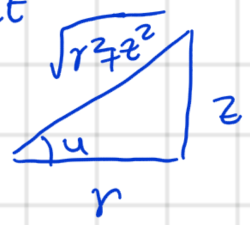
$$\vec{H} = \frac{I}{4\pi}\int_l \frac{\vec{dl} \times \hat{R}}{R^2} = \frac{I}{4\pi}\int_l \frac{\vec{dl} \times \vec{R}}{R^3}$$

$$= \frac{I}{4\pi}\int_l \frac{\hat{\phi}\, r\,dz}{(r^2+z^2)^{3/2}}$$

$$\vec{H} = \frac{I}{4\pi} \int \frac{\hat{\phi}\, r\, dz}{(r^2+z^2)^{3/2}}$$

$$= \frac{\hat{\phi}\, I\, r}{4\pi} \int \frac{dz}{(r^2+z^2)^{3/2}}$$

let



$\tan u = \frac{z}{r}$ | $z = r\tan u$

$dz = r \frac{du}{\cos^2 u}$

for $$\int \frac{dz}{(r^2+z^2)^{3/2}} = \int \frac{1}{(r^2 + r^2\tan^2 u)^{3/2}} \frac{r}{\cos^2 u}\, du$$

$$= \int \frac{r}{(r^2)^{3/2}} \frac{1}{(1+\tan^2 u)^{3/2}} \frac{1}{\cos^2 u}\, du$$

$$= \int \frac{r}{r^3} \frac{1}{\left(\frac{1}{\cos^2 u}\right)^{3/2}} \frac{1}{\cos^2 u}\, du$$

$$= \frac{1}{r^2} \int \cos u\, du = \frac{1}{r^2} \sin u$$

from above $\sin u = \frac{z}{\sqrt{r^2+z^2}}$, then

$$\int \frac{dz}{(r^2+z^2)^{3/2}} = \frac{z}{r^2} \frac{1}{(r^2+z^2)^{\frac{1}{2}}}$$

$$\vec{H} = \frac{\hat{\phi}\, I r}{4\pi} \int \frac{dz}{(r^2+z^2)^{3/2}}$$

$$= \frac{\hat{\phi}\, I r}{4\pi} \left[\frac{z}{r^2} \frac{1}{(r^2+z^2)^{\frac{1}{2}}} \Bigg|_{z=-\frac{l}{2}}^{\frac{l}{2}} \right]$$

$$= \frac{\hat{\phi}\, I}{4\pi r} \frac{1}{\sqrt{r^2+\frac{l^2}{4}}} \left[\frac{l}{2} + \frac{l}{2} \right]$$

$$= \frac{\hat{\phi}\, I l}{4\pi r \, \frac{1}{2}\sqrt{4r^2+l^2}}$$

$$\vec{H} = \frac{\hat{\phi}\, I l}{2\pi r \sqrt{4r^2+l^2}}$$

$$\vec{B} = \mu_0 \vec{H} = \frac{\hat{\phi}\, \mu_0 I l}{2\pi r \sqrt{4r^2+l^2}}$$

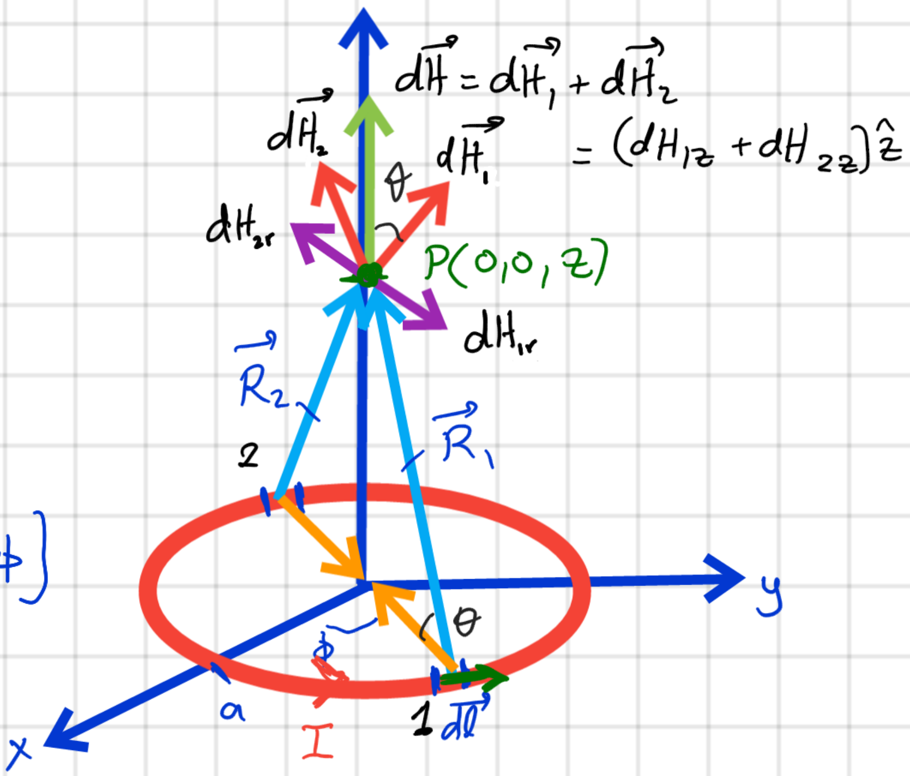
For $l \gg r$:

$$\vec{B} \simeq \frac{\hat{\phi}\, \mu_0 I l}{2\pi r \sqrt{l^2}} = \frac{\hat{\phi}\, \mu_0 I}{2\pi r}$$ (for infinitely long wire)

Example: Magnetic Field of a Circular Loop:

A circular loop of radius a carries a steady current I.

Determine the magnetic field $\vec{H}$ at a point on the axis of the loop.



$$d\vec{H} = \frac{I}{4\pi} \frac{d\vec{l} \times \hat{R}}{R^2}$$

$$d\vec{l} = a\,d\phi\,\hat{\phi}$$

$$= a\,d\phi\left[-\hat{x}\sin\phi + \hat{y}\cos\phi\right]$$

$$\vec{R} = -a\hat{r} + z\hat{z}$$

$$= -a\cos\phi\,\hat{x} - a\sin\phi\,\hat{y} + z\hat{z}$$

$$d\vec{l} \times \vec{R} = a\,d\phi\left[\hat{z}a\sin^2\phi + \hat{z}a\cos^2\phi + \hat{y}z\sin\phi + \hat{x}z\cos\phi\right]$$

$$= a\,d\phi\left[z(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + a\hat{z}\right]$$

$$|\vec{R}| = \sqrt{a^2 + z^2}$$

$$\vec{H} = \frac{I}{4\pi}\int_{\phi=0}^{2\pi} \frac{a\,d\phi\left[z(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + a\hat{z}\right]}{(a^2+z^2)^{3/2}}$$

$$\vec{H} = \frac{I}{4\pi} \frac{a}{(a^2+z^2)^{3/2}} \left[z\left(\sin\phi \Big|_0^{2\pi}\right)\hat{x} + z\left(-\cos\phi \Big|_0^{2\pi}\right)\hat{y} + a\left(\phi \Big|_0^{2\pi}\right)\hat{z} \right]$$

$$= \frac{I}{4\pi} \frac{a}{(a^2+z^2)^{3/2}} \left[z(0)\hat{x} + z(-1+1)\hat{y} + a(2\pi)\hat{z} \right]$$

$$= \frac{I}{4\pi} \frac{a^2\, 2\pi\, \hat{z}}{(a^2+z^2)^{3/2}}$$

$$\vec{H} = \frac{I\, a^2\, \hat{z}}{2(a^2+z^2)^{3/2}} \quad \sim (1)$$

when $z=0$ at the center of the loop, from (1):

$$\vec{H} = \hat{z}\frac{I a^2}{2 a^3} = \hat{z}\frac{I}{2a}$$

when $z \gg a$, from (1):

$$\vec{H} = \hat{z}\frac{I a^2}{2|z|^3}$$

5.2.2 Magnetic Field of a Magnetic Dipole :

A current loop with dimensions much smaller than the distance between the loop and the observation point is called magnetic dipole. It has magnetic field lines similar to a permanent magnet, and to the pattern of the electric field lines of an electric dipole.

Since from the previous example, when $z \gg a$

$$\vec{H} = \hat{z}\,\frac{I a^2}{2|z|^3}$$

for a magnetic moment $\vec{m} = \hat{z} m = \hat{z} I A N = \hat{z} I a^2 \pi$

then $\vec{H} = \hat{z}\,\dfrac{m}{2\pi|z|^3}$ for a point on the loop's axis.

For any point $P = (R, \theta, \phi)$ in spherical coordinates

for $R \gg a$: $\vec{H} = \dfrac{m}{4\pi R^3}\left(\hat{R}\, 2\cos\theta + \hat{\theta}\sin\theta\right)$

Example: A semi-infinite conductor extends between $z=0$ and $z=\infty$ along the z-axis. If the current I in the conductor flows along the positive z direction, find $\vec{H}$ at a point in the xy-plane at a radial distance r from the conductor.

For $l \gg r$ and an infinitly long wire from $-\infty < z < \infty$:

$$\vec{B} \simeq \hat{\phi} \frac{\mu_0 I}{2\pi r}$$

then for a semi-infinitly long wire from $0 < z < \infty$:

$$\vec{B} \simeq \frac{1}{2} \hat{\phi} \frac{\mu_0 I}{2\pi r}$$

, since each half of the infinite wire contributes half the $\vec{B}$ and both in the same direction, then:

$$\vec{B} = \frac{\hat{\phi} \mu_0 I}{4\pi r}$$

Then $\vec{H} = \dfrac{\vec{B}}{\mu} = \dfrac{\hat{\phi} I}{4\pi r}$

Example: A wire carrying a current of 4 A is formed into a circular loop. If the magnetic field at the center of the loop is 20 A/m, what is the radius of the loop if the loop has

(a) only one turn, (b) 10 turns.

(a) Since $\vec{H}$ at the center of a loop in the xy-plane, z=0, with

$$\vec{H} = \hat{z}\frac{I}{2a}$$

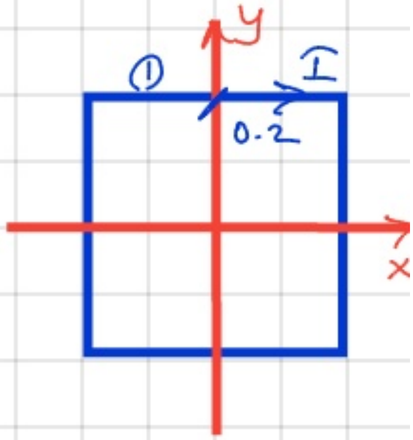
$$a = \frac{1}{2}\frac{I}{|\vec{H}|} = \frac{1}{2}\frac{4}{20} = 0.1 \text{ m}$$

(b) for 10 turns:

$$\vec{H} = \hat{z}\frac{IN}{2a}$$

$$a = \frac{1}{2}\frac{IN}{a} = \frac{1}{2}\frac{4(10)}{20} = 1 \text{ m}$$

Example: A wire is formed into a square loop and placed in the xy-plane with its center at the origin and each of its sides parallel to either the x or y axes. Each side is 40 cm in length, and the wire carries a current of 5A whose direction is clockwise when the loop is viewed from above. Calculate the magnetic field at the center of the loop.

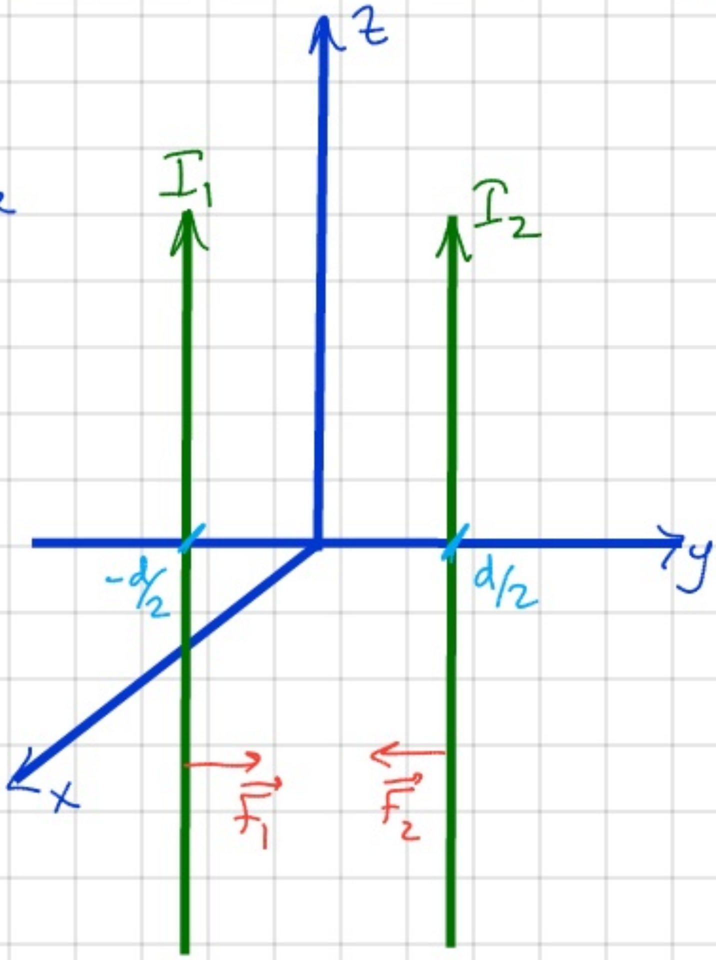


Each segment will contribute the same $\vec{H}$ at the center with the direction into the page ($-\hat{z}$-axis).

$$\vec{H} = 4\vec{H}_1 = -4\hat{z}\frac{I\ell}{2\pi d\sqrt{4d^2+\ell^2}} = \frac{-2\hat{z}I\ell}{\pi\frac{\ell}{2}\sqrt{\ell^2+\ell^2}}$$

$$= \frac{-4\hat{z}I}{\pi\sqrt{2}\,\ell} = \frac{-4\hat{z}(5)}{\pi\sqrt{2}(0.4)} = -\hat{z}\,11.25$$

5.2.3 Magnetic Force Between Two Parallel Conductors:



For two current carrying conductors placed a distance d apart, they will exert a magnetic force on the other. Assume their length is infinitely long. Their currents are I_1 & I_2 in the z-direction. Then for conductor 1, the magnetic flux density $\vec{B}_1$ due to I_1 at the location of conductor 2 is :

$\vec{B}_1 = -\hat{x}\dfrac{\mu_0 I_1}{2\pi d}$, then $\vec{F}_2$ the force exerted on conductor 2 due to $\vec{B}_1$ is $\vec{F}_2 = I_2 \, l \, \hat{z} \times \vec{B}_1$

$$\vec{F_2} = I_2 l \hat{z} \times \vec{B_1} = I_2 l \hat{z} \times \left(\frac{-\hat{x} \mu_0 I_1}{2\pi d} \right)$$

$$= -\hat{y} \mu_0 \frac{I_1 I_2 l}{2\pi d}$$

Similarly, $\vec{B_2} = \hat{x} \frac{\mu_0 I_2}{2\pi d}$

then $\vec{F_1} = I_1 l \hat{z} \times \vec{B_2} = I_1 l \hat{z} \times \left(\frac{\hat{x} \mu_0 I_2}{2\pi d} \right)$

$$= \hat{y} \mu_0 \frac{I_1 I_2 l}{2\pi d}$$

forces per unit lengths are:

$$\vec{F_2}' = \frac{\vec{F_2}}{l} = -\hat{y} \mu_0 \frac{I_1 I_2}{2\pi d}$$

$$\vec{F_1}' = \frac{\vec{F_1}}{l} = \hat{y} \mu_0 \frac{I_1 I_2}{2\pi d}$$

Since I_1 & I_2 are in the same direction, the force between the two wires is attraction with equal force.

If I_1 & I_2 are in opposite directions, the wires repel one another with equal force

5.3 Maxwell's Magnetostatic Equations:

5.3.1 Gauss's Law for Magnetism:

Gauss's law for electricity defined the net outward flux of the electric flux density $\vec{D}$ through a closed Gauss surface equals the enclosed net charge, mathematically:

$$\nabla \cdot \vec{D} = \rho_v \longleftrightarrow \oint_S \vec{D} \cdot d\vec{s} = Q$$

where $Q = \int_v \rho_v \, dv$, and v is enclosed by surface S.

Gauss's law for magnetism is

$$\nabla \cdot \vec{B} = 0 \longleftrightarrow \oint_S \vec{B} \cdot d\vec{s} = 0$$

This means that there is no magnetic equivalence to an electric charge Q or charge density ρ_v in magnetism, since no matter how small a permanent magnet is divided, each new piece always has two poles, isolated monopoles do not exist in nature. Thus, magnetic field lines always form continuous closed loops.

5.3.2 Ampere's Law :

For the electrostatic field $\vec{E}$, the line integral of $\vec{E}$ around any closed contour C is zero, thus, $\vec{E}$ is conservative. Mathematically :

$$\nabla \times \vec{E} = 0 \quad \longleftrightarrow \quad \oint_C \vec{E} \cdot d\vec{l} = 0$$

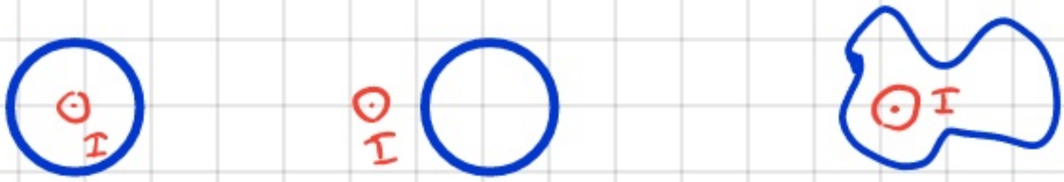
applied to a surface S with contour C.

For magnetostatic,

$$\nabla \times \vec{H} = \vec{J} \quad \longleftrightarrow \quad \oint_C \vec{H} \cdot d\vec{l} = I$$

this is Ampere's law. where I is the total current passing through S, and $I = \int_S \vec{J} \cdot d\vec{s}$, and the direction of the contour path C follows the right-hand-rule in the direction of the four fingers while the thumb is aligned with the direction of the current I.

- In words, Ampere's law states that the line integral of $\vec{H}$ around a closed path is equal to the current traveling through the surface bounded by that path.



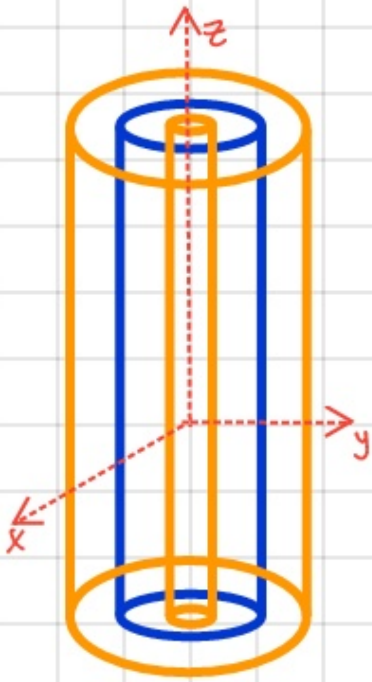
$\int \vec{H} \cdot d\vec{l} = I$ $\qquad$ $\int \vec{H} \cdot d\vec{l} = 0$ $\qquad$ $\int \vec{H} \cdot d\vec{l} = I$

- Ampere's law is limited to symmetric current distributions allowing a convenient choice for an Amperian contour

Example: Magnetic field of a long wire:

A long infinite straight wire of radius a carries a steady current I that is uniformly distributed over its cross section. Determine the magnetic field $\vec{H}$ a distance r from the wire axis for:

(a) $r \leq a$ (inside the wire) (b) $r \geq a$ (outside the wire)



Solution: Let I be along $+z$-axis, then choose the Amperian countor to be circular around the z-axis since I is in the z-direction (right-hand-rule).

Thus, $\vec{H} = H\hat{\phi}$ and $\vec{dl} = \hat{\phi}\, r\, d\phi$, For (a) $r \leq a$ case :

$\oint_c \vec{H}_1 \cdot \vec{dl}_1 = I_1$, here $\vec{H}_1 = H_1 \hat{\phi}$ and $\vec{dl}_1 = \hat{\phi} r_1 \, d\phi$,

Since I is uniformly distributed over the cross section then

$$\frac{I}{\pi a^2} = \frac{I_1}{\pi r_1^2} \Rightarrow I_1 = I \left(\frac{r_1}{a}\right)^2$$

Then $\oint_c \vec{H}_1 \cdot \vec{dl}_1 = I_1$

$$\int_{\phi=0}^{2\pi} (H_1 \hat{\phi}) \cdot (\hat{\phi} r_1 \, d\phi) = I \left(\frac{r_1}{a}\right)^2$$

$$H_1 r_1 \int_{\phi=0}^{2\pi} d\phi = I \frac{r_1^2}{a^2}$$

$$H_1 r_1 (2\pi) = I \frac{r_1^2}{a^2}$$

$$H_1 = I \frac{r_1}{2\pi a^2}$$

So, $\vec{H}_1 = H_1 \hat{\phi} = \hat{\phi} \dfrac{I r_1}{2\pi a^2}$ for $r_1 \leq a$

For (b) $r \geq a$ case:

$\oint_C \vec{H}_2 \cdot d\vec{l}_2 = I_2$, here $\vec{H}_2 = H_2 \hat{\phi}$ and $d\vec{l}_2 = \hat{\phi} r_2 \, d\phi$,

Then $\oint_C \vec{H}_2 \cdot d\vec{l}_2 = I$

$$\int_{\phi=0}^{2\pi} (H_2 \hat{\phi}) \cdot (\hat{\phi} r_2 \, d\phi) = I$$

$$H_2 r_2 \int_{\phi=0}^{2\pi} d\phi = I$$

$$H_2 r_2 (2\pi) = I$$

$$H_2 = \frac{I}{2\pi r_2}$$

So, for a general $r \geq a$:

$$\vec{H}_2 = H_2 \hat{\phi} = \hat{\phi} \frac{I}{2\pi r}$$

this expression was found from $\vec{B} = \hat{\phi} \frac{\mu_0 I}{2\pi r}$ for an infinite wire using Biot-Savart law.

Example: The metal niobium becomes a superconductor with zero electrical resistance when it is cooled to below 9 K, but its superconductive behavior ceases when the magnetic flux density at its surface exceeds 0.12 T. Determine the maximum current that a 0.1 mm diameter niobium wire can carry and remain superconductive.

From the previous example, at the surface

$$\vec{H} = \hat{\phi}\frac{I}{2\pi a}, \quad \text{then} \quad \vec{B} = \hat{\phi}\frac{\mu_0 I}{2\pi a}$$

$$I = \frac{2\pi a}{\mu_0} B$$

$$= \frac{2\pi\left(\frac{0.1}{2}\times 10^{-3}\right) 0.12}{4\pi \times 10^{-7}}$$

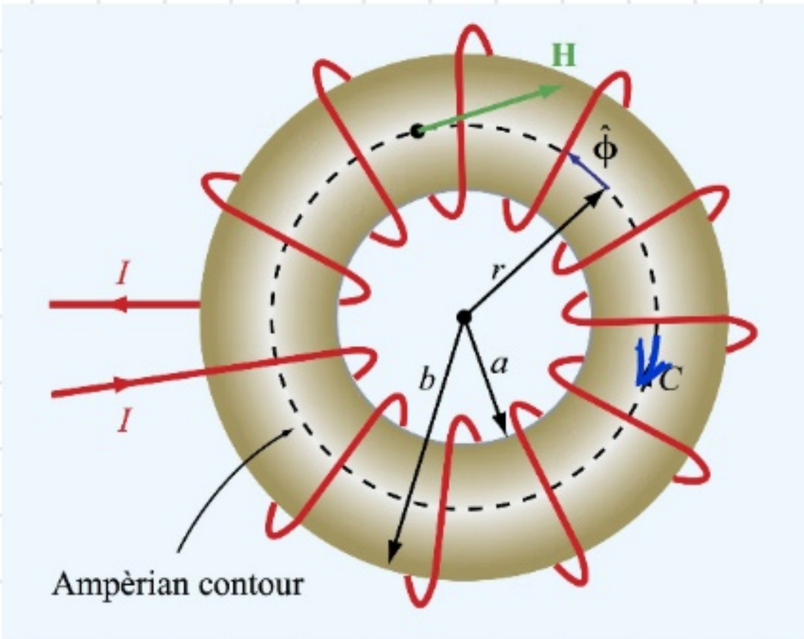
$$I = 30\text{ A}$$

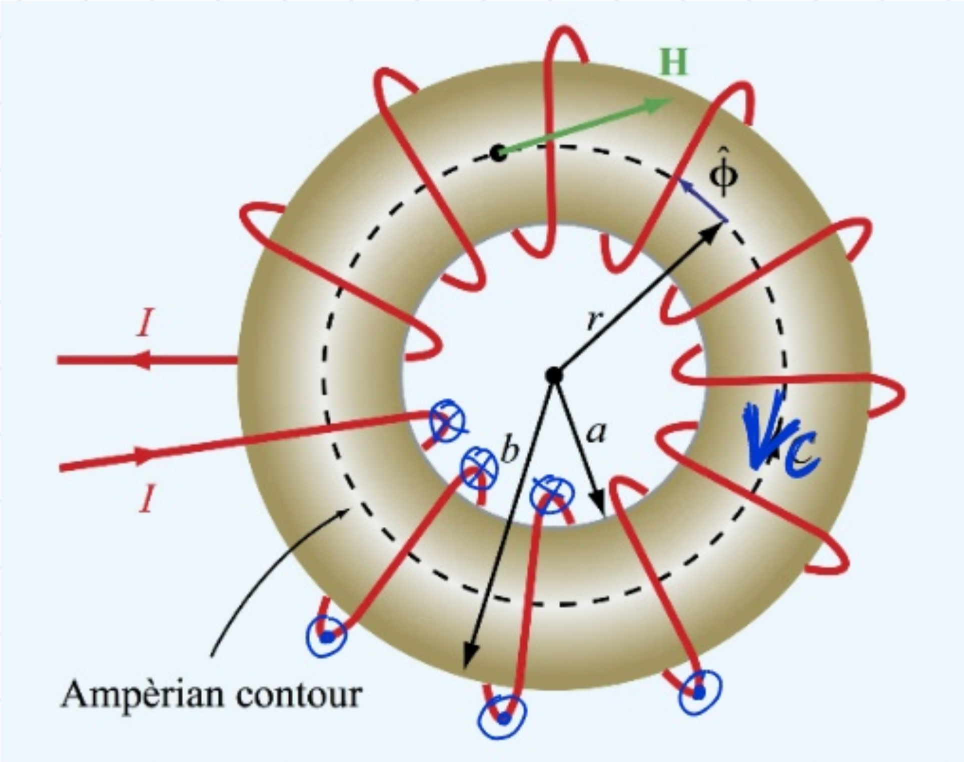
Example: Magnetic field inside a toroidal coil:

A toroidal coil is a doughnut shaped structure wrapped in closely spaced turns of wire. For clarity, the figure shown has the turns spaced far apart, in practice they are wound in a closely spaced arrangement to form approximately circular loops. For a toroid with N turns carrying a current I, determine the magnetic field $\vec{H}$ in each of the following three regions:

(a) $r<a$ (b) $a<r<b$ (c) $r>b$

all in the azimuthal symmetry plane of the toroid.





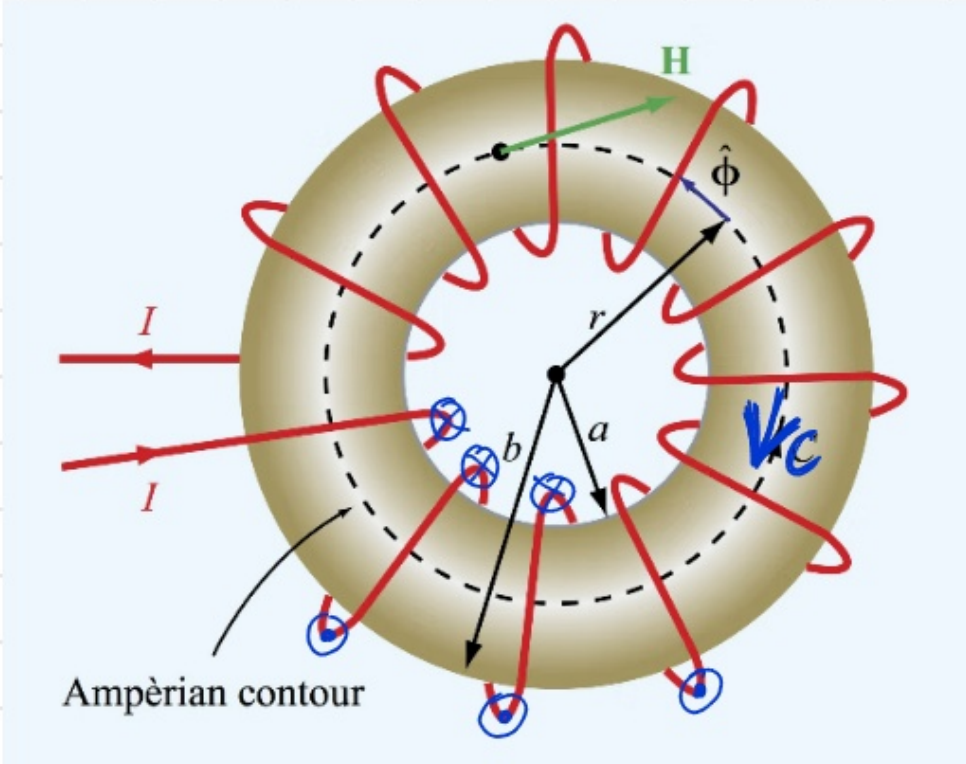
For each loop, using the right-hand-rule, the thumb points in the direction of the current, then the four fingers point in the $-\hat{\phi}$ direction which is the direction of $\vec{H}$.

Then for $(a<r<b)$:

$$\oint_C \vec{H}\cdot d\vec{l} = \int_{\phi=0}^{2\pi} (-\hat{\phi} H)\cdot(-\hat{\phi}\, r\, d\phi) = NI$$

$$Hr\int_{\phi=0}^{2\pi} d\phi = NI \Rightarrow H = \frac{NI}{2\pi r} \Rightarrow \vec{H} = -\hat{\phi}\frac{NI}{2\pi r}$$

$(a<r<b)$



For $r<a$, no current flows through the surface of the contour with radius $r<a$, then

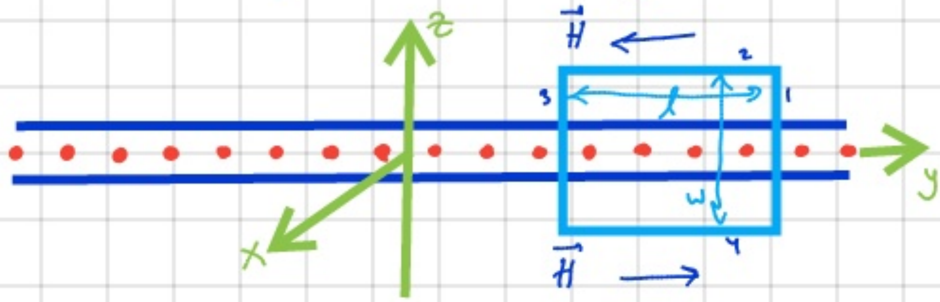
$$\oint_C \vec{H}\cdot d\vec{l} = I = 0 \Rightarrow \vec{H}=0 \quad (r<a)$$

For $r>b$, the net current flowing through the surface of the contour with radius $r>b$ is zero since NI flows into the page and NI out of the page, hence a net of zero. Then

$$\oint_C \vec{H}\cdot d\vec{l} = NI - NI = 0 \Rightarrow \vec{H}=0 \quad (r>b)$$

Example: Magnetic field of an infinite current sheet:

The xy-plane contains an infinite current sheet with surface current density $\vec{J_s} = \hat{x} J_s$. Find the magnetic field $\vec{H}$ everywhere in space.



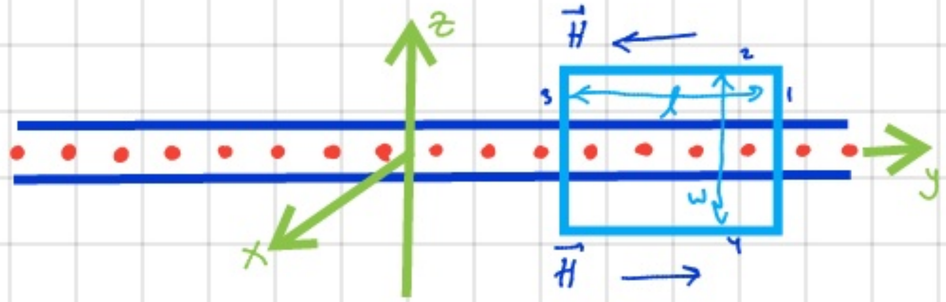
Using right-hand-rule, contour direction is CCW since $\vec{J_s}$ is in the $\hat{x}$-direction. Also, using the right-hand-rule, when the thumb is in the direction of the current, then the four fingers point in the direction of $\vec{H}$,

so if $z > 0$, $\vec{H} = -\hat{y} H$ and if $z < 0$, $\vec{H} = \hat{y} H$.

To find the value of $|\vec{H}|$, $\vec{H} \cdot \vec{dl}$:

for segments 1 & 3, $\vec{H} \cdot \vec{dl}$ equals zero since $\cos\theta = 0$,

while for segments 2 & 4, $\vec{H} \cdot \vec{dl}$ is not zero



Then $$\oint_C \vec{H}\cdot d\vec{l} = I$$

$$\int_2 (-\hat{y}H)\cdot(-\hat{y}\,dl) + \int_4 (\hat{y}H)\cdot(\hat{y}\,dl) = \int_l J_s\,dl$$

$$Hl + Hl = J_s l$$

$$\Rightarrow \quad H = \frac{J_s}{2}$$

Then $$\vec{H} = \begin{cases} -\hat{y}\,\dfrac{J_s}{2}, & z>0 \\ \hat{y}\,\dfrac{J_s}{2}, & z<0 \end{cases}$$

5.4 Vector Magnetic Potential :

From Gauss's law of magnetism : $\nabla \cdot \vec{B} = 0$,

this means that $\nabla \cdot (\nabla \times \vec{A}) = 0$, for any

vector field $\vec{A}$. Here $\vec{A}$ is the magnetic potential.

Thus, $\vec{B} = \nabla \times \vec{A}$.

From Ampere's law : $\nabla \times \vec{H} = \vec{J}$, and since

$\vec{B} = \mu \vec{H}$, then $\nabla \times \vec{B} = \mu \vec{J}$ and

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \mu \vec{J}$$

Since the Laplacian of $\vec{A}$ is

$$\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})$$

then $\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J}$.

$$\Rightarrow \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} \quad \sim (1)$$

Since for a vector $\vec{A}$:

$$\nabla^2 \vec{A} = \hat{x} \nabla^2 A_x + \hat{y} \nabla^2 A_y + \hat{z} \nabla^2 A_z \quad \sim (2)$$

If we choose $\nabla \cdot \vec{A}$ to be zero, then (1) becomes:

$$\nabla^2 \vec{A} = -\mu \vec{J} \quad \sim (3)$$

for $\vec{J} = \hat{x} J_x + \hat{y} J_y + \hat{z} J_z$, from (2) & (3):

$$\Rightarrow \quad \nabla^2 A_x = -\mu J_x$$

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

So, to find A_x for a current density J_x distributed over a volume v:

$$\Rightarrow \quad A_x = \frac{\mu}{4\pi} \int_v \frac{J_x}{R} \, dv$$

Similarly, to find A_y for a current density J_y distributed over a volume v:

$$\Rightarrow \quad A_y = \frac{\mu}{4\pi} \int_v \frac{J_y}{R} \, dv$$

and to find A_z for a current density J_z distributed over a volume v:

$$\Rightarrow A_z = \frac{\mu}{4\pi} \int_v \frac{J_z}{R} \, dv$$

Then the three solutions can be combined into:

$$\vec{A} = \frac{\mu}{4\pi} \int_v \frac{\vec{J}}{R} \, dv \quad \sim (*)$$

. If the current density is distributed over a surface, $\vec{J_s}$, then $(*)$ becomes:

$$\vec{A} = \frac{\mu}{4\pi} \int_S \frac{\vec{J_s}}{R} \, ds$$

. If the current density is distributed over a line, then $(*)$ becomes:

$$\vec{A} = \frac{\mu}{4\pi} \int_l \frac{I \, \vec{dl}}{R}$$

The magnetic flux ϕ on a surface S is the total magnetic flux density passing through it:

$$\phi = \int_S \vec{B} \cdot d\vec{s}$$

Since $\vec{B} = \nabla \times \vec{A}$, and using Stokes's theorem:

$$\phi = \int_S (\nabla \times \vec{A}) \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

where the contour C bounds the surface S.

Problem 5.27: In a given region of space, the vector magnetic potential is given by

$\vec{A} = \hat{x}\, 5\cos \pi y + \hat{z}(2+\sin \pi x)$

(a) Determine $\vec{B}$

(b) Use $\phi = \int_S \vec{B} \cdot d\vec{s}$ to calculate the magnetic flux passing through a square loop with 0.25 m long edges if the loop is in the xy-plane, its center is at the origin, and its edges are parallel to the x- and y- axes.

(c) Calculate ϕ again using

$$\phi = \oint_C \vec{A} \cdot \vec{dl}$$

(a) $\vec{B} = \nabla \times \vec{A}$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5\cos \pi y & 0 & 2+\sin \pi x \end{vmatrix}$$

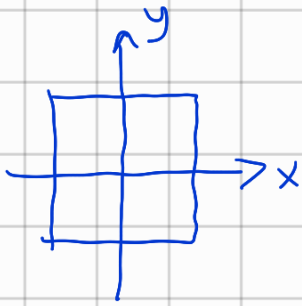
$$= \hat{x}\left[\frac{\partial}{\partial y}(2+\sin\pi x) - 0\right]$$

$$- \hat{y}\left[\frac{\partial}{\partial x}(2+\sin\pi x) - \frac{\partial}{\partial z}(5\cos\pi y)\right]$$

$$+ \hat{z}\left[0 - \frac{\partial}{\partial y}(5\cos\pi y)\right]$$

$$= \hat{x}(0) - \hat{y}\left[\pi\cos\pi x - 0\right] + \hat{z}\left[+5\pi\sin\pi y\right]$$

$$\vec{B} = -\hat{y}\,\pi\cos\pi x + \hat{z}\,5\pi\sin\pi y$$



(b) $$\Phi = \int_S \vec{B}\cdot d\vec{s}$$

$$= \int_S \left[-\hat{y}\,\pi\cos\pi x + \hat{z}\,5\pi\sin\pi y\right]\cdot\hat{z}\,dx\,dy$$

$$= 5\pi \iint_{x\,y} \sin\pi y\,dx\,dy$$

$$= 5\pi \int_x dx \int_y \sin\pi y\,dy$$

$$= 5\pi\left(x\Big|_{-0.125}^{0.125}\right)\left(-\frac{1}{\pi}\cos\pi y\Big|_{-0.125}^{0.125}\right) = 0$$

(c) $\Phi = \oint_C \vec{A} \cdot d\vec{l}$

$$= \int_{x=-0.125}^{0.125} \vec{A} \cdot (-\hat{x}\, dx) \Big|_{y=0.125} + \int_{x=-0.125}^{0.125} \vec{A} \cdot \hat{x}\, dx \Big|_{y=-0.125}$$

$$+ \int_{y=-0.125}^{0.125} \vec{A} \cdot (-\hat{y}\, dy) \Big|_{x=-0.125} + \int_{y=-0.125}^{0.125} \vec{A} \cdot \hat{y}\, dy \Big|_{x=0.125}$$

$$= \int_{x=-0.125}^{0.125} \left[\hat{x}\, 5\cos\pi y + \hat{z}(2+\sin\pi x)\right] \cdot (-\hat{x}\, dx) \Big|_{y=0.125}$$

$$+ \int_{x=-0.125}^{0.125} \left[\hat{x}\, 5\cos\pi y + \hat{z}(2+\sin\pi x)\right] \cdot \hat{x}\, dx \Big|_{y=-0.125}$$

$$+ \int_{y=-0.125}^{0.125} \left[\hat{x}\, 5\cos\pi y + \hat{z}(2+\sin\pi x)\right] \cdot (-\hat{y}\, dy) \Big|_{x=-0.125}$$

$$+\int_{y=-0.125}^{0.125}\left[\hat{x}\,5\cos\pi y+\hat{z}(2+\sin\pi x)\right]\cdot\hat{y}\,dy\,\Bigg|_{x=0.125}$$

$$=-5\cos\pi y\int_{x=-0.125}^{0.125}dx\,\Bigg|_{y=0.125}$$

$$+5\cos\pi y\int_{x=-0.125}^{0.125}dx\,\Bigg|_{y=-0.125}\qquad+0+0$$

$$=-5\cos(\pi(0.125))\left(x\Big|_{-0.125}^{0.125}\right)$$

$$+5\cos(\pi(-0.125))\left(x\Big|_{-0.125}^{0.125}\right)$$

$$=-5\cos(\pi(0.125))(0.125(2))$$

$$+5\cos(\pi(0.125))(0.125(2))$$

$$=0$$

5.5. Magnetic Properties of Materials:

The magnetic moment $\vec{m}$ of a loop of area A and current I has a magnitude $m = AI$ and direction normal to the plane of the loop (using the right-hand rule).

- Magnetization in a material is due to atomic scale current loops associated with:

(1) orbital motions of electrons around the nucleus

(2) motion of the protons inside the nucleus.

(3) the electron spin.

- The magnetic moment due to proton motion is three orders of magnitude smaller than that of an electron, so the total magnetic moment of an atom is the sum of the magnetic moments of its electrons.

- The magnetic behavior of a material depends on the interaction of the magnetic dipole moments of its atoms with an external magnetic field.

5.5.1. Electron orbital and spin magnetic moments

- An electron of charge $-e$ moving at a constant speed u in a circular orbit of radius r completes one revolution in time $T = \frac{2\pi r}{u}$.

This circular motion makes up a loop with current $I = \frac{-e}{T} = \frac{-eu}{2\pi r}$.

Then, the magnitude of the orbital magnetic moment

$$m_o = IA = \frac{-eu}{2\pi r}(r^2\pi) = \frac{-eur}{2}$$

The angular momentum of the electron with a mass m_e is : $L_e = m_e u r$

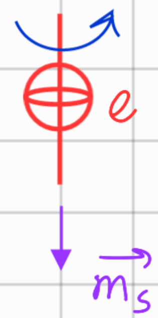
Then, $m_o = \dfrac{-eur}{2}$

$$m_o = \frac{-eL_e}{2m_e}$$

The orbital angular momentum for the electron is always some integer multiple of $\hbar = \frac{h}{2\pi}$ where h is Planck's constant. That is, $L_e = 0, \hbar, 2\hbar, 3\hbar, \ldots$

The smallest nonzero magnitude of the orbital magnetic moment of an electron is $m_o = \dfrac{-e\hbar}{2m_e}$.

- The spin magnetic moment $\vec{m}_s$ due to its spinning motion about its own axis is



$$\vec{m}_s = \frac{-e\hbar}{2m_e}$$

which is equal to the minimum orbital magnetic moment m_o.

- The electrons of atoms with an even number of electrons usually exist in pairs, each pair has opposite spinning directions, thus canceling each others' spin magnetic moments. If the number of electrons is odd, then the atom has a nonzero net spin magnetic moment due to its unpaired electron.

5.5.2 Magnetic Permeability

The magnetization vector $\vec{M}$ is defined as the vector sum of the magnetic dipole moments of the atoms contained in a unit volume of the material ($1\,m^3$). In most magnetic material:

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is the magnetic susceptibility of the material.

For nonferromagnetic materials:

$$\vec{B} = \mu_0 (\vec{H} + \chi_m \vec{H}) = \mu_0 (1 + \chi_m) \vec{H} = \mu \vec{H}$$

where μ is the magnetic permeability of the material.

The relative permeability μ_r :

$$\mu_r = \frac{\mu}{\mu_0} = \frac{\mu_0(1+\chi_m)}{\mu_0} = 1+\chi_m$$

Materials are usually classified as :

(1) diamagnetic : $\chi_m < 0$

(2) paramagnetic : $\chi_m > 0$

For both diamagnetic & paramagnetic

$|\chi_m|$ is on the order of 10^{-5}

$$\Rightarrow \chi_m \ll 1$$

$$\Rightarrow \mu_r \simeq 1$$

$$\Rightarrow \mu \simeq \mu_0$$

$$\Rightarrow \vec{B} \simeq \mu_0 \vec{H}$$

(3) ferromagnetic : $|\chi_m|$: very large.

e.g. for purified iron :

$$|\mu_r| = 2\times 10^5$$

Example: If a certain type of iron with 8.5×10^{28} atoms/m^2 contributes one electron per atom to align its spin magnetic moment along the direction of the applied field, find

(a) the spin magnetic moment of a single electron, given that $m_e = 9.1 \times 10^{-31}$ kg and $\hbar = 1.06 \times 10^{-34}$ J.s

(b) the magnitude of the magnetic vector $\vec{M}$.

(a) $$m_s = \frac{-e\hbar}{2m_e} = \frac{-1.6 \times 10^{-19} \times 1.06 \times 10^{-34}}{2 \times 9.1 \times 10^{-31}}$$

$$= -9.3 \times 10^{-24} \quad A.m^2$$

(b) $$M = n\, m_s = 8.5 \times 10^{28} \, (9.3 \times 10^{-24})$$

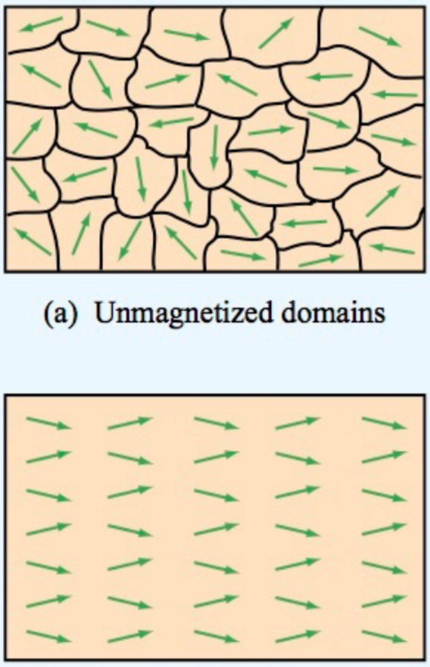
$$= 79.05 \times 10^{4} \quad A/m$$

5.5.3 Magnetic Hysteresis of ferromagnetic materials

For ferromagnetic materials, their magnetic moments align along the direction of an external magnetic field and remain partially magnetized even after the removal of the external field.

These materials are used in permanent magnets.

They have magnetized domains, on the order of $10^{-10} m^3$, within which the magnetic moments of all atoms, on the order of 10^{19} atom, are aligned. With no external magnetic field, the domains take random orientations resulting in a zero net magnetization.

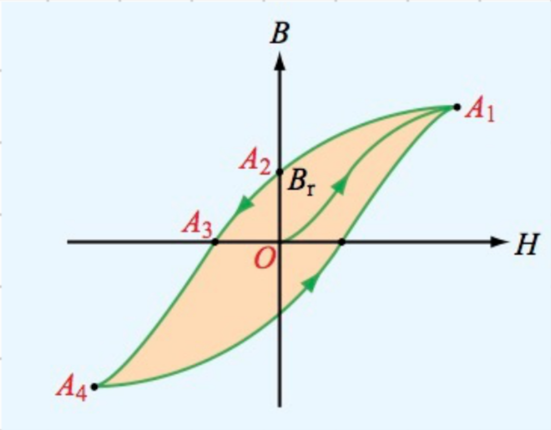


(a) Unmagnetized domains

(b) Magnetized domains

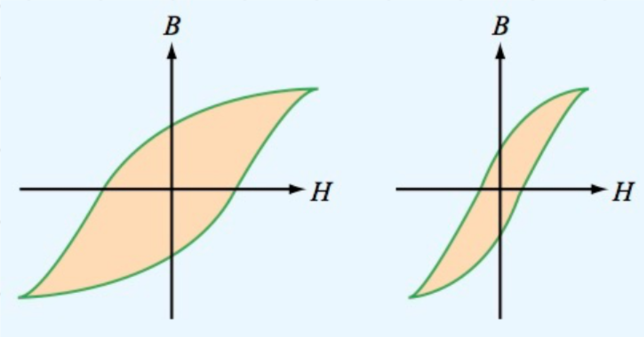
When an unmagnetized sample of a ferromagnetic material is placed in an external magnetic field, the domains partially align with the external field. The magnetization behavior is described in terms of its B-H magnetization curve.

Suppose we start with an unmagnitized sample of iron, denoted by O in the figure shown. Let the sample have



a wire wound around it. To increase H on the sample, increase the current I passing through the wire. B will increase as H increases reaching point A_1 (a saturation condition). If we decrease H back to zero (by decreasing the current I), then the curve will follow the path from A_1 to A_2. At A_2, $H=0$ while $B = B_r$, this is called the residual flux density. This iron material is now magnetized and ready to be used as a permanent magnet, that is due to the fact that a large fraction of its magnetized domains have remained aligned.

If we reverse the direction of H and increase its value, then B will decrease from B_r to zero at point A_3. If H continues to increase, the magnetization reaches point A_4 (a saturation condition). If we decrease H back to zero then increased agin in the positive direction, the curve will follow the path from A_4 to A_1. This process is called magnetic hysteresis. It implies that the magnetization process in ferromagnetic materials depends on the magnetic field and on the magnetic history of the material.



(a) Hard material (b) Soft material

- Hard ferromagnetic materials have wide hysteresis loops, they can't be easily demagnetized because they have a large residual magnetization Br. However, soft ferromagnetic materials have narrow hysteresis loops, they can be easily magnetized and demagnetized.
- To demagnetize a ferromagnetic material, it is subjected to several hysteresis cycles while gradually decreasing the range of the applied magnetic field.

5.6 Magnetic Boundary Conditions

The boundary conditions using Gauss's law:

- For the electric flux density:

$$\oint_S \vec{D} \cdot d\vec{s} = Q \Rightarrow D_{1n} - D_{2n} = \rho_s$$

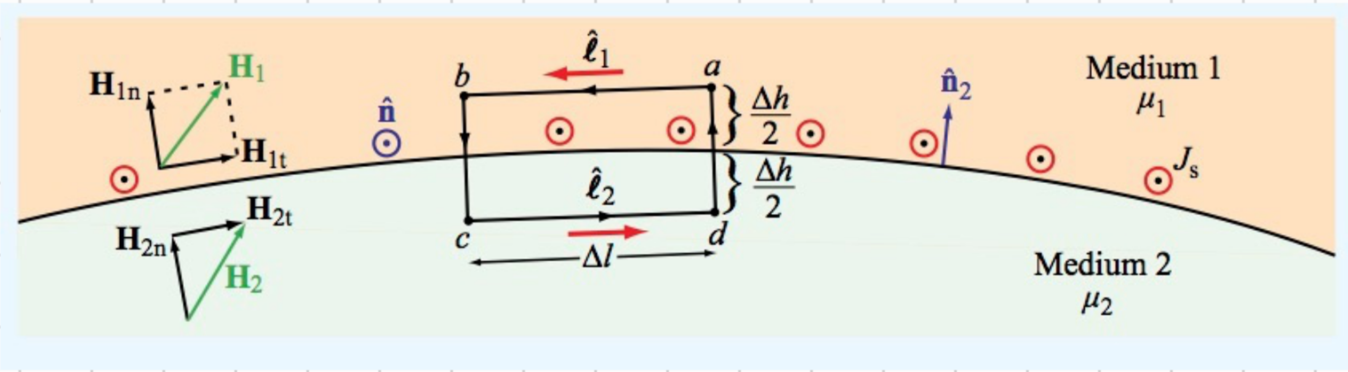
- For the magnetic flux density:

$$\oint_S \vec{B} \cdot d\vec{s} = 0 \Rightarrow B_{1n} = B_{2n}$$

and since $\vec{B} = \mu \vec{H}$, then

$$\mu_1 H_{1n} = \mu_2 H_{2n}$$

For the tangential components, using Ampere's law for a closed rectangular contour of sides Δl & Δh, then let $\Delta h \to 0$:



$$\oint_C \vec{H}\cdot\vec{dl} = \int_a^b \vec{H}_1\cdot\hat{l}_1\,dl + \int_c^d \vec{H}_2\cdot\hat{l}_2\,dl = I$$

where $I = J_s\,\Delta l$, where J_s is the component of $\vec{J}_s$ normal to the contour.

If the normal to the contour is $\hat{n}$, then

$J_s = \vec{J}_s\cdot\hat{n}$, then

$$\oint_C \vec{H}\cdot\vec{dl} = \vec{H}_1\cdot\hat{l}_1\,\Delta l + \vec{H}_2\cdot\hat{l}_2\,\Delta l = \vec{J}_s\cdot\hat{n}\,\Delta l$$

also $\hat{l}_1 = \hat{n}\times\hat{n}_2$ and $\hat{l}_1 = -\hat{l}_2$

$$\Rightarrow (\vec{H}_1 - \vec{H}_2)\cdot(\hat{n}\times\hat{n}_2) = \vec{J}_s\cdot\hat{n}$$

since $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A})$

then

$$\hat{n}\cdot(\hat{n}_2\times[\vec{H}_1-\vec{H}_2]) = \vec{J}_s\cdot\hat{n}$$

If we assume that the normal to medium 2, $\hat{n}_2 = \hat{z}$, then the tangential components could be in the $\hat{x}$ & $\hat{y}$ directions. Then these tangential components are either parallel to $\vec{J}_s$ or orthogonal, all in the xy-plane:

* The component of $(\vec{H}_1 - \vec{H}_2)$ parallel to $\vec{J}_s$:

$\hat{n}_2 \times$ [parallel to $\vec{J}_s$] = { a vector orthogonal to both $\hat{n}_2$ & $\vec{J}_s$

then for this component f $(\vec{H}_1 - \vec{H}_2)$ that is orthogonal to $\hat{n}_2$ and parallel to $\vec{J}_s$:

$$H_{1t} - H_{2t} = 0 \Rightarrow H_{1t} = H_{2t}$$

* For the component of $(\vec{H}_1 - \vec{H}_2)$ orthogonal to $\vec{J}_s$:

$\hat{n}_2 \times$ [orthogonal to $\vec{J}_s$] = { a vector orthogonal to $\hat{n}_2$ and parallel to $\vec{J}_s$

then for this component of $(\vec{H}_1 - \vec{H}_2)$ that is orthogonal to $\hat{n}_2$ and orthogonal to $\vec{J}_s$:

$$H_{1t} - H_{2t} = J_s$$

- Surface currents only exist on the surface of perfect conductors and superconductors

$\left[\sigma = \infty \Rightarrow J = \sigma E \Rightarrow E = 0 \text{ (inside)}\right]$

- For materials with a finite conductivity, $\sigma \neq \infty$, $J_s = 0$ and always

$$H_{1t} = H_{2t}$$

Example:
Determine the angle between $\vec{H_1}$ and $\hat{n}_2 = \hat{z}$ if $\vec{H_2} = 3\hat{x} + 2\hat{z}$, $\mu_{r_1} = 2$, $\mu_{r_2} = 8$, and $\vec{J_s} = 0$.

$\Rightarrow H_{1t} = H_{2t} \Rightarrow H_{1x} = H_{2x} = 3$

also $B_{1n} = B_{2n} \Rightarrow B_{1z} = B_{2z}$

$$\Rightarrow \mu_1 H_{1z} = \mu_2 H_{2z}$$

$$H_{1z} = \frac{\mu_2}{\mu_1} H_{2z}$$

$$= \frac{\mu_0 \mu_{r_2}}{\mu_0 \mu_{r_1}} (2)$$

$$= \frac{8}{2} (2) = 8$$

$$\Rightarrow \vec{H}_1 = \vec{H}_{1t} + \vec{H}_{1n}$$

$$\vec{H}_1 = 3\hat{x} + 8\hat{z}$$

Since $\vec{H}_1 \cdot \hat{z} = H_1 \cos\theta$ (θ : angle between $\vec{H}_1$ & $\hat{z}$)

$$(3\hat{x} + 8\hat{z}) \cdot \hat{z} = \sqrt{3^2 + 8^2} \cos\theta$$

$$8 = \sqrt{73} \cos\theta$$

$$\cos\theta = \frac{8}{\sqrt{73}}$$

$$\theta = \cos^{-1}\left(\frac{8}{\sqrt{73}}\right)$$

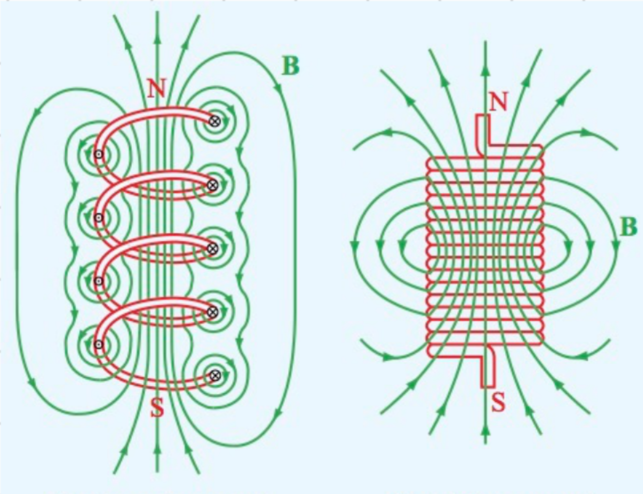
$$\theta = 20.56^\circ$$

5.7 Inductance

An inductor can store energy in the magnetic field near its current carrying conductors. To find the inductance, we should find the magnetic flux density in the region.

5.7.1. Magnetic field in a solenoid

The magnetic flux density in the inner region of a tightly wound solenoid of length l and radius a, and N turns carrying current I, with



(a) Loosely wound solenoid (b) Tightly wound solenoid

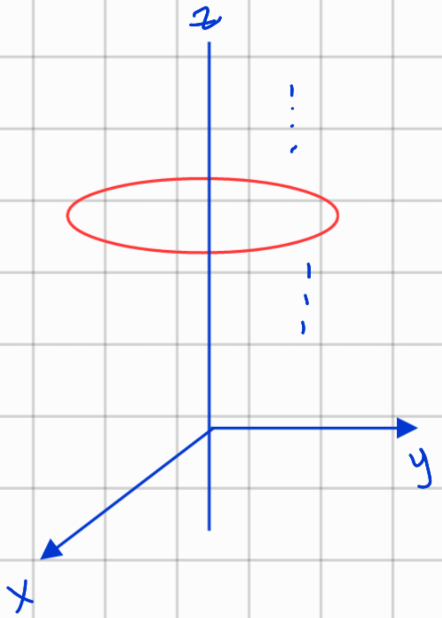
n turns per unit length, $n = N/l$. Assume each winding to be approximatly a circular loop. We found that:

$$\vec{B} = \frac{\mu_0 I a^2 \hat{z}}{2\left[a^2 + (z_p - z_0)^2\right]^{3/2}}$$

where z_p is the location of the point at which we are finding $\vec{B}$ along the axis of a loop with the center of such loop at z_0.

$$dB = \frac{\mu_0 I a^2}{2\left[a^2 + (z_0 - z_p)^2\right]^{3/2}}\, n\, dz_0$$

$$B = \int_{z_0=0}^{l} \frac{\mu_0 I a^2 n \, dz_0}{2\left[a^2 + (z_0 - z_p)^2\right]^{3/2}}$$



let $z = z_0 - z_p \Rightarrow dz = dz_0$

for $z_0 = 0 \Rightarrow z = -z_p$

$z_0 = l \Rightarrow z = l - z_p$

$$B = \int_{z=-z_p}^{l-z_p} \frac{\mu_0 I n a}{2} \frac{a\, dz}{(a^2+z^2)^{3/2}}$$

$$= \frac{\mu_0 I n a}{2} \left(\frac{1}{a} \frac{z}{\sqrt{a^2+z^2}} \Bigg|_{z=-z_p}^{l-z_p} \right)$$

$$B = \frac{\mu_0 I n}{2} \left[\frac{l - z_p}{\left[a^2 + (l - z_p)^2\right]^{\frac{1}{2}}} + \frac{z_p}{\left[a^2 + z_p^2\right]^{\frac{1}{2}}} \right]$$

$$\vec{B} = \frac{\mu_0 I n}{2}\left[\frac{l - z_p}{\left[a^2 + (l - z_p)^2\right]^{\frac{1}{2}}} + \frac{z_p}{\left[a^2 + z_p^2\right]^{\frac{1}{2}}}\right]\hat{z} \quad (**)$$

For an infinitely long solenoid $(l \gg a)$:

$$\vec{B} = \frac{\mu_0 I n}{2}\left[\frac{l - z_p}{|l - z_p|} + \frac{z_p}{|z_p|}\right]\hat{z}$$

$$= \frac{\mu_0}{2} I n \frac{1}{|z_p||l - z_p|}\left[(l - z_p)|z_p| + z_p|l - z_p|\right]\hat{z}$$

$$= \frac{\mu_0}{2} I n \frac{1}{|z_p||l - z_p|}(l - z_p)(z_p + z_p)\,\hat{z}$$

$$\vec{B} = \mu_0 I n \,\hat{z}$$

$\vec{B}$ is independent of the location of the point at which we are finding $\vec{B}$

Example : Use (**) to obtain an expression for $\vec{B}$ at a point on the axis of a very long solenoid but situated at its end points. How does $\vec{B}$ at the end points compare to $\vec{B}$ at the mindpoint of the solenoid.

$$\vec{B} = \frac{\mu_0 I n}{2}\left[\frac{l - z_p}{\left[a^2+(l-z_p)^2\right]^{\frac{1}{2}}} + \frac{z_p}{\left[a^2+z_p^2\right]^{\frac{1}{2}}}\right]\hat{z} \quad (**)$$

at the ends of the solenoid, $z_p = 0$ or $z_p = l$

for $z_p = 0 \Rightarrow \vec{B} = \frac{\mu_0 I n}{2}\frac{l}{(a^2+l^2)^{\frac{1}{2}}}\hat{z}$

however, for $l \gg a$

$$\vec{B} = \mu_0 \frac{In}{2}\hat{z}$$

when $z_p = l$, $\vec{B} = \frac{\mu_0 I n}{2}\frac{l}{(a^2+l^2)^{\frac{1}{2}}}\hat{z}$, for $l \gg a$

$$\vec{B} = \frac{\mu_0 I n}{2}\hat{z}$$

So, $\vec{B} = \frac{\mu_0 I n}{2} \hat{z}$ for both ends,

which is half the value of $\vec{B}$ for midpoint as found above where

$$\vec{B} = \mu_0 I n \hat{z}.$$

5.7.2 Self-inductance

The self-inductance of any conducting structure is the ratio of the magnetic flux on a surface to the current flowing through the structure.

$$L = \frac{\Phi}{I}$$

For the solenoid, for a single loop:

$$\phi = \int_S \vec{B} \cdot d\vec{s} = \int_S \mu I n \hat{z} \cdot \hat{z} \, ds$$

$$\phi = \mu I n A$$ (A: is the cross sectional area of the loop)

for N loops $\Rightarrow$ $\phi_{Total} = N \mu I n A$

$$= N \mu I \frac{N}{l} A$$

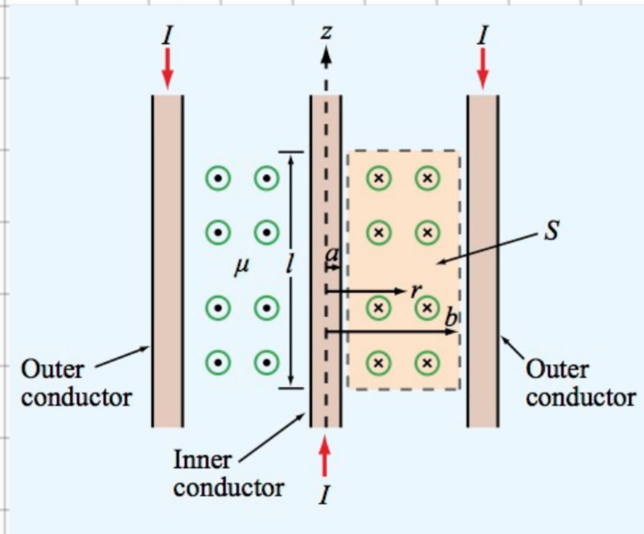
$$= N^2 \frac{\mu I}{l} A$$

So for N loops, $L = \frac{\phi_{total}}{I}$

$$L = N^2 \frac{\mu}{l} A$$

Example:

Find the expression for the inductance per unit length of a coaxial transmission line with inner and outer conductors of radii a and b and an insulating material of permeability μ.



We found that $\vec{B} = \hat{\phi} \frac{\mu I}{2\pi r}$

The flux ϕ through surface S is

$$\phi = \int_S \vec{B} \cdot d\vec{s} = \int_S -\hat{x} \frac{\mu I}{2\pi y} \cdot (-\hat{x}\, dy\, dz)$$

$$= \frac{\mu I}{2\pi} \int_{y=a}^{b} \frac{dy}{y} \int_{z=0}^{l} dz$$

$$= \frac{\mu I}{2\pi} \left(\ln y \Big|_a^b \right) \left(z \Big|_0^l \right)$$

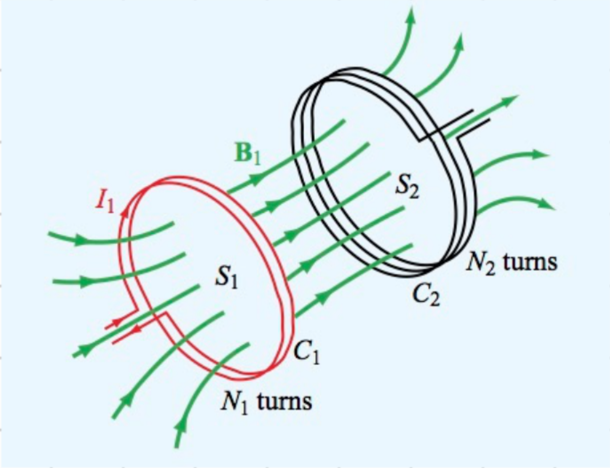
$$\phi = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

L' : L per unit length

$$L' = \frac{L}{l} = \frac{\phi}{Il} = \frac{\mu I l}{2\pi} \frac{1}{Il} \ln \frac{b}{a}$$

$$L' = \frac{\mu}{2\pi} \ln \frac{b}{a}$$

5.7.3. Mutual inductance



Magnetic coupling between two different conducting structures is described in terms of the mutual inductance between them.

- The magnetic flux $\vec{B}_1$ generated by I_1 results in a flux ϕ_{12} through loop 2 as

$$\phi_{12} = \int_{S_2} \vec{B}_1 \cdot d\vec{s}$$

and if loop 2 has N_2 turns, then

$$L_{12} = \frac{N_2}{I_1} \int_{S_2} \vec{B}_1 \cdot d\vec{s}$$

5-8 Magnetic energy

To study the energy stored in an inductor, consider an inductor with inductance L connected to a current source that increases from $i=0$ to $i=I$, we know that

$$v = L\frac{di}{dt}, \quad \text{and} \quad P = iv$$

Since

$$W_m = \int P\,dt = \int iv\,dt = \int i L\,di$$

$$= L\int_{i=0}^{I} i\,di = L\left.\frac{i^2}{2}\right|_{i=0}^{I}$$

$$W_m = \frac{1}{2}LI^2$$: magnetic energy stored in the inductor

Consider the solenoid inductor,

$$L = \frac{\mu N^2 A}{l} \quad \text{and} \quad B = \frac{\mu N I}{l} \Rightarrow I = \frac{Bl}{\mu N}$$

Then, $W_m = \frac{1}{2}LI^2$

$$= \frac{1}{2}\left(\frac{\mu N^2 A}{\ell}\right)\left(\frac{B^2 \ell^2}{\mu^2 N^2}\right)$$

$$= \frac{\mu}{2} A \ell \, \frac{B^2}{\mu^2}$$

$$= \frac{1}{2} \mu A \ell H^2$$

$$W_m = \frac{1}{2} \mu H^2 v \quad \sim (1)$$

where $v = A\ell$: the volume of the interior of the solenoid

and the magnetic energy density w_m :

$$w_m = \frac{W_m}{v} = \frac{1}{2} \mu H^2 \quad \sim (2)$$

The expressions (1) & (2) can be generalized for any medium with magnetic field $\vec{H}$. So the total magnetic energy stored in a magnetic field $\vec{H}$ for any volume v containing a material with permeability μ:

$$W_m = \frac{1}{2}\int_v \mu H^2 \, dv$$

Example:

Find the magnetic energy stored in a coaxial cable of length ℓ and inner and outer radii a and b. The current flowing through the cable is I and its insulation material has permeability μ.

$$B = \frac{\mu I}{2\pi r} \Rightarrow H = \frac{I}{2\pi r}$$

Then $$W_m = \frac{1}{2}\int_v \mu \frac{I^2}{4\pi^2 r^2} \, dv$$

$$= \frac{\mu I^2}{8\pi^2}\int_v \frac{1}{r^2} \, r\,dr\,d\phi\,dz$$

$$= \frac{\mu}{8\pi^2} I^2 \int_{r=a}^{b} \frac{dr}{r} \int_{\phi=0}^{2\pi} d\phi \int_{z=0}^{l} dz$$

$$= \frac{\mu}{8\pi^2} I^2 \left(\ln r \Big|_a^b \right) \left(\phi \Big|_0^{2\pi} \right) \left(z \Big|_0^{l} \right)$$

$$= \frac{\mu}{8\pi^2} I^2 \ln \frac{b}{a} (2\pi)(l)$$

$$W_m = \frac{\mu}{4\pi} I^2 l \ln \frac{b}{a}$$

since we found that $L = \frac{\mu l}{2\pi} \ln \frac{b}{a}$

$$\Rightarrow W_m = \left(\frac{\mu l}{2\pi} \ln \frac{b}{a} \right) \frac{I^2}{2}$$

$$W_m = \frac{L}{2} I^2$$