EICoM
ELECTRICAL COMPUTER MECHATRONICS

تقدم لجنة

تلخيص لمادة:

معالجة اشارات رقمية

من شرح:

د.عبدالكريم البياتي

جزيل الشكر للطالب:

حسين صافو

The Hashemite University
Faculty of Engineering
Course Syllabus

Course Title:	Digital Signal Processing	**Course Number:**	110409422
Department:	Electrical Engineering	**Designation:**	compulsary
Prerequisite(s):	Signals and systems		
Instructor:	Dr. Abdul Karim Al-Bayati	**Instructor's Office:**	*E3053*
Instructor's e-mail:	Karim_bayati@yahoo.com		
Office Hours:			
Time:		**Class Room:**	
Course description:	This course aims to introduce the student to the principles of digital signal processing. Students will learn the basic types of discrete-time signals and systems, specifically LTI systems. Students are introduced to Z-transform and frequency domain representation of discrete-time signals and system, and the discrete Fourier transform and its properties and applications and the efficient way of DFT computation which is the FFT.		
Textbook(s):	J. G. Proakis, D. G. Manolakis, Digital signal processing, principles algorithms and applications (4th edition), Prentice Hall. Chapters: 2,3,4,5,7,8,10		
Other required material:	1- A.V. Oppenheim, R.W. Schafer, Discrete-time signal processing, Prentice Hall. 2- S.K. Mitra, Digital Signal processing, a computer based approach (2nd ed. 2001), McGraw Hill. 3- T.J. Terell, L.K. Shark, Digital signal; processing, A student guide (2000), Macmillan press ltd. 4- V.K. Ingle, J.G.Proakis, Digital signal processing using MATLAB (2000), Brooks/Cole.		
Course objectives:	*The student should be able to* 1. Understand the basic types of discrete-time signals and systems. 2. Analyze discrete-time systems in the time and frequency domains. 3. Use the Z-transform in analysis of discrete-time systems, and utilize its properties. 4. Understand the DFT and its properties and applications. 5. Use the DFT for spectral analysis and for signal filtering. 6. Implement FFT algorithm for computing the DFT using radix-2,4.		
Topics covered:	1. Basic elements of DSP systems, Advantages of Digital over analog signal processing. 2. Discrete-time signals and systems, classification of signals, analysis of LTI discrete-time systems, convolution, solution of difference equations, impulse response of LTI systems. 3. The Z-transform, the inverse transform, properties of the z-transform, analysis of LTI systems in the z-domain 4. Frequency analysis of discrete-time signals. 5. Frequency analysis of discrete-time systems. 6. Discrete Fourier transform and its properties and applications in spectral analysis and filtering. 7. The Fast Fourier transform method for computing the DFT, radix-2, and radix-4.		
Grading Plan:	First Exam (25 Points) Second Exam (25 Points) Final Exam (50 Points)		
General Notes:	Attendance is mandatory. No more than 15% no excuse absent is permitted.		

Chapter 1
Preface introduction

Wednesday 2/6/13 8:46 AM

Comparison between Discrete & Analog Devices:

DSP:

for High order filter, it is simpler, more accuracy, no aging

more apps like adaptive filters(Varactors) due to wireless ch. (motion)

high flexibility (Software Defined Radio): can switch from FM to CDMA by switch; because it's a program.

Limitation:

Complexity: high sampling rate

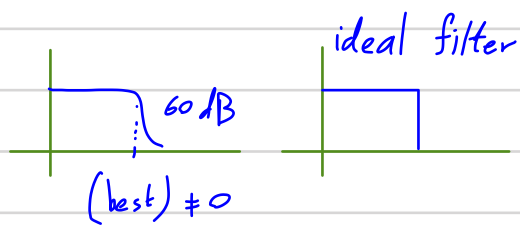
Cost, size.

Analog:

good for simple filters.

But:

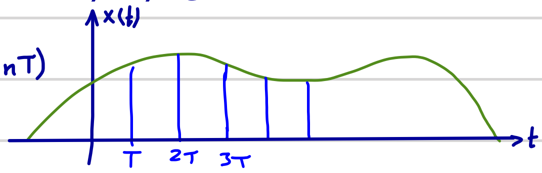
limited by 60 dB at cut off freq.

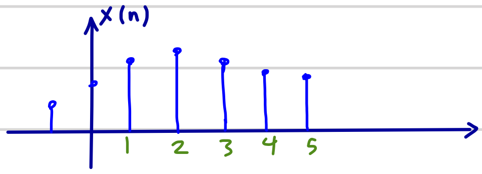


Chapter 2
Discrete-time Signals and Systems

Monday 2/11/13 8:12 AM

$x(n) \triangleq x(t=nT)$



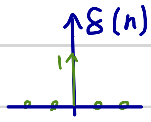


n: integer $n = (-\infty, \infty)$

Some basic signals:-

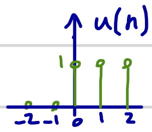
1) unit sample sequence

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



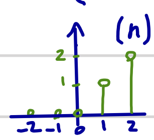
2) unit step signal

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



3) unit ramp signal

$$u_r(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Classification of signals:-

Energy signal:

$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$. $x(n)$ is an energy signal $\overset{\text{iff}}{\Longleftrightarrow} 0 < E < \infty$

Power signal:

$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$. $x(n)$ is a power signal $\overset{\text{iff}}{\Longleftrightarrow} 0 < P < \infty$

EXA: let $x(n) = u(n)$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1$$

$$= \lim_{N \to \infty} \frac{N+1}{2N+1} = 1/2 \qquad 0 < P < \infty$$

$\therefore$ $x(n) = u(n)$ is a power signal.

Periodic / Aperiodic signals:

$x(n)$ is periodic with period = N if $x(n+N) = x(n)$

N: smallest integer satisfying condition.

If there is no N-value satisfying condition, x(n) is Aperiodic.

EXA: $x(n) = \sin(2\pi f_0 n)$

Solution

$$\sin[2\pi f_0(n+N)] = \sin(2\pi f_0 n)$$

$$= \sin(2\pi f_0 n + 2\pi f_0 N)$$

$f_0 N = k$ (integer)

$\boxed{f_0 = \frac{k}{N}}$ ← integer, ← ″

rational

EXA: determine periodicity

a) $x(n) = \cos(0.01\pi n)$

$$= \cos \frac{2\pi}{200} n$$

$$= \cos 2\pi \left(\frac{1}{200}\right) n$$

$\therefore$ $x(n)$ is periodic with period $= 200$ $(N = 200)$.

EXA: $x(n) = \cos \frac{30\pi}{105} n$

$$= \cos 2\pi \frac{15}{105} n$$

$f_0 = \frac{15}{105}$ rational $\longrightarrow$ $x(n)$ is periodic.

$f_0 = \frac{1}{7} \longrightarrow \boxed{N = 7}$

EXA: $x(n) = \sin 3n$

$$= \sin 2\pi \left(\frac{3}{2\pi}\right) n$$

$f_0 = \frac{3}{2\pi} \longrightarrow$ NOT rational.

$\therefore$ $x(n)$ is NOT periodic

Find periodicity:

$$x(n) = \cos\frac{\pi}{2}n - \sin\frac{\pi n}{8} + 3\cos\left(\frac{\pi n}{4} + \frac{\pi}{3}\right)$$

x_1 $\quad$ x_2 $\quad$ x_3

$x_1(n) = \cos 2\pi \frac{1}{4} n \longrightarrow f_0 = \frac{1}{4}$ (periodic) $\quad N_1 = 4$

$x_2(n) = \sin\frac{\pi n}{8} = \sin 2\pi \frac{1}{16} n \longrightarrow f_0 = \frac{1}{16}$ (per.) $\quad N_2 = 16$

$x_3(n) = \cos\left(\frac{\pi n}{4} + \frac{\pi}{3}\right) = \cos\left(2\pi\frac{1}{8}n + \frac{\pi}{3}\right) \longrightarrow f_0 = \frac{1}{8}$ (per.) $\quad N_3 = 8$

$\therefore$ $x(n)$ is periodic $\quad N = LCM[N_1, N_2, N_3] = LCM[4, 16, 8] = 16$

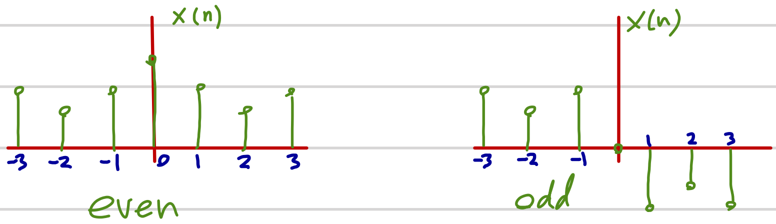
Least Common Multiple

$$\boxed{N = 16}$$

Symmetric (even) and Antisymmetric (odd):

x(n) is even (symmetric) if x(-n) = x(n) $\qquad$ even

x(n) is odd (Antisymmetric) if x(-n) = -x(n) $\qquad$ odd



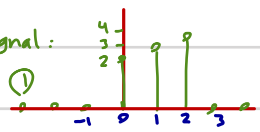
$x(n) = x_e(n) + x_o(n) \quad — ①$

$x(-n) = x_e(n) + x_o(-n)$

$\quad = x_e(n) - x_o(n) \quad — ②$

$$\boxed{\begin{aligned} x_e(n) &= \tfrac{1}{2}[x(n) + x(-n)] \\ x_o(n) &= \tfrac{1}{2}[x(n) - x(-n)] \end{aligned}}$$

Signal: ①



(method of writing discrete-time signal)

② $x(n) = [0\ 0\ 0\ \dots\ 0\ \underset{\uparrow}{2}\ 3\ 4\ 0\ 0\dots]$

$n=0$

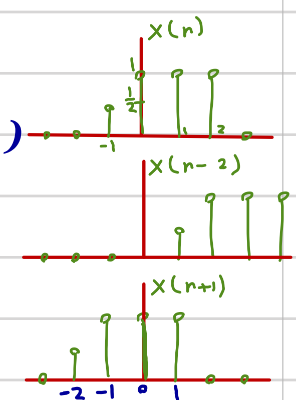
Manipulation in time:

Shift in time:-

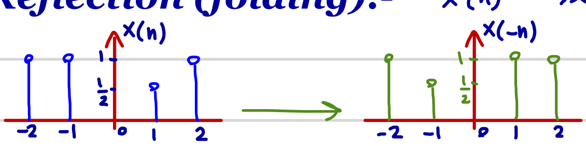
x(n - k) is a delay of x(n) by k-units (shift to the right by k-units)

$\hookrightarrow$ +ve integer

x(n+k) is a shift left by k-units.



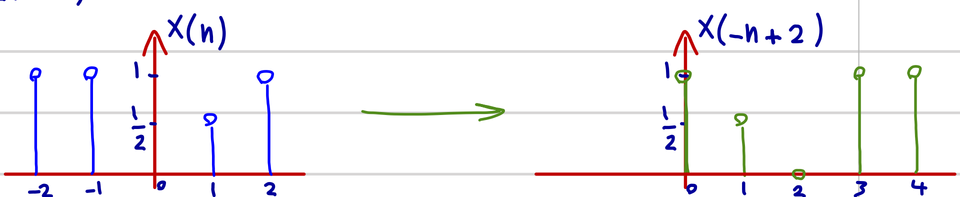
Reflection (folding):- $\quad x(n) \longrightarrow x(-n)$



Reflection + Shift $\quad x(-n+2) = x(-[n-2])$

1- fold x(n).

2- shift right by 2.



Discrete-time System:
Is a device or algorithm that operates on a discrete-time input signal to produce an output discrete-time signal according to mathematical expression or any rate.

EXA:

$y(n) = x(n-1)$ delay sys.

$x(n) = [0\ 0\ 2\ \underset{\uparrow}{1}\ -3\ 5]$

$y(n) = [0\ 0\ \underset{\underset{n=0}{\uparrow}}{2}\ 1\ -3\ 5]$

$n=0 \quad y(0) = x(0-1) = x(-1)$

$n=1 \quad y(1) = x(1-1) = x(0)$

$n=2 \quad y(2) = x(1)$

EXA: $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$ {moving avg fn.}

$x(n) = [0\ 0 \ldots 0\ 0\ 3,\ 2, 1, \underset{\underset{n=0}{\uparrow}}{0}, 1, 2, 3, 0\ 0 \ldots]$

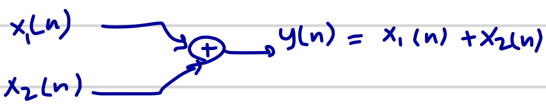
at $n=0 \quad y(0) = \frac{1}{3}[x(1) + x(0) + x(-1)]$
$= \frac{1}{3}[1 + 0 + 1] = 2/3$

$n=1 \quad y(1) = \frac{1}{3}[x(2) + x(1) + x(0)]$
$= \frac{1}{3}[2 + 1 + 0] = 1$

$\therefore y(n) = [0\ 0 \ldots 0, 1, \frac{5}{3}, 2, 1, \underset{\uparrow}{\frac{2}{3}}, 1, 2, \frac{5}{3}, 1, 0 \ldots]$

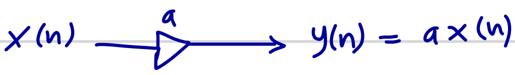
Elements of discrete-time system:

Adder



$\Rightarrow$ Digital Adder.

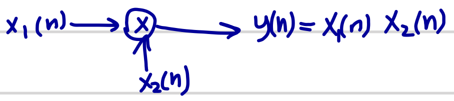
Constant Multiplier



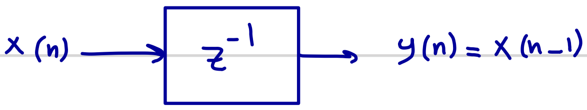
$\Rightarrow$ Digital Multiplier.

Wednesday 2/13/13 8:03 AM

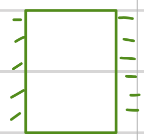
Signal Multiplier



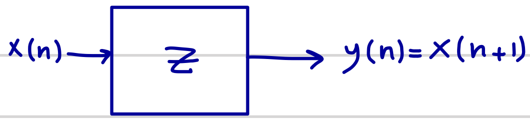
Unit delay element



causal $\Rightarrow$ Shift register.



Unit advance element

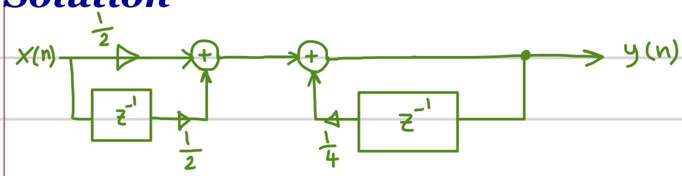


like voice recognition. $\Rightarrow$ // ↗

EXA: $y(n) = \frac{1}{4} y(n-1) + \frac{1}{2} x(n) + \frac{1}{2} x(n-1)$

constant coefficients diff. eqn. Rep. y in block diagram.

Solution



Classification of discrete-time systems:

Static / dynamic system

A system is static or memoryless if y(n) depends on input samples at time (n) only. Like:-

$$\left.\begin{aligned} y(n) &= a\, x(n) \\ y(n) &= n\, x(n) + b x^3(n) \\ y(n) &= (n+2)\, x^2(n) \end{aligned}\right\} \text{Static}$$

A system is dynamic (with memory) if y(n) depends on previous or future x(n). Like:-

$$\left.\begin{aligned} y(n) &= x(n) + 3x(n-1) \\ y(n) &= 2x(n+1) + x(n) \end{aligned}\right\} \text{Dynamic.}$$

Time-invariant, time-varying:

A system is time-invariant if

given $x(n) \longrightarrow y(n)$

implies $x(n-k) \longrightarrow y(n-k)$ for any $x(n)$ & any k

Test of invariance of a system:

1- $y(n) = T[x(n)]$

2- Find $y(n-k)$

3- Delay i/p by k-units and det. o/p $y(n,k) \triangleq \underbrace{T[x(n-k)]}_{}$ ← o/p

4- If $y(n-k) = y(n,k) \Rightarrow$ System is time-invariant.

EXA: determine invariance of $y(n) = \boxed{n\,x(n)} = T[x(n)]$

Solution

$$y(n-k) = (n-k)\,x(n-k)$$

$$y(n,k) = T[x(n-k)] = n\,x(n-k)$$

$$y(n-k) \neq y(n,k) \Rightarrow \text{System is time-varying.}$$

EXA: $y(n) = x(-n)$ folding

Solution

$$y(n-k) = x(-(n-k)) = x(-n+k)$$

$$y(n,k) = T[x(n-k)] = x(-n-k) \neq y(n-k)$$

System is time-varying.

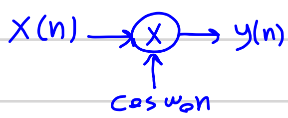
EXA: $y(n) = x(n)\cos \omega_0 n$

solution

$$y(n-k) = x(n-k)\cos \omega_0 (n-k)$$

$$y(n,k) \triangleq T[x(n-k)] = x(n-k)\cos \omega_0 n$$

$$y(n-k) \neq y(n,k) \Rightarrow \text{Time-varying.}$$



EXA: $y(n) = x(n) - x(n-1)$

Solution

$$y(n-k) = x(n-k) - x(n-k-1)$$

$$y(n,k) = x(n-k) - x(n-1-k) = y(n-k)$$

Time-invarient.

Linear / Non-linear systems:

a system is linear if

$$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$

Test:-

1- let $x(n) = a_1 x_1(n) + a_2 x_2(n)$

2- $y(n) = T[x(n)]$ $\qquad y_1(n) = T[x_1(n)]$

3- $a_1 y_1(n) + a_2 y_2(n) \neq y(n)$ $\qquad y_2(n) = T[x_2(n)]$

Equality $\Rightarrow$ Linear system.

EXA: investigate linearity $y(n) = x(n^2)$

Solution

let $x(n) = a_1 x_1(n) + a_2 x_2(n)$

$$y(n) = T[a_1 x_1(n) + a_2 x_2(n)] = a_1 x_1(n^2) + a_2 x_2(n^2)$$

$$y_1(n) = T[x_1(n)] = x_1(n^2)$$

$$y_2(n) = T[x_2(n)] = x_2(n^2)$$

$$a_1 y_1(n) + a_2 y_2(n) = a_1 x_1(n^2) + a_2 x_2(n^2) = y(n)$$

System is Linear.

EXA: $y(n) = x^2(n)$

Solution

let $x(n) = \underbrace{a_1 x_1(n) + a_2 x_2(n)}$

$y(n) \triangleq T[\;\;] = T[a_1 x(n) + a\; x_2(n)]$

$= a_1^2 x^2(n) + a_2^2 x_2^2(n) + 2 a_1 a_2 x_1(n) x_2(n)$

$y(n) = T[x_1(n)] = x_1^2(n)$ $\quad$ $a_1 y_1(n) + a_2 y_2(n) \neq y(n)$

$y_2(n) = \quad = x_2^2(n)$ $\quad$ $= a_1 x_1^2(n) + a_2 x_2^2(n)$

System is Non-linear.

EXA: $y(n) = Ax(n) + B$

solution

let $x(n) = a_1 x_1(n) + a_2 x_2(n)$

$y(n) = A[a_1 x_1(n) + a_2 x_2(n)] + B = A a_1 x_1(n) + A a_2 x_2(n) + B$

$y_1 = T[x_1(n)] = A x_1(n) + B$

$y_2 = \quad = A x_2(n) + B$

$a_1 y_1(n) + a_2 y_2(n) = a_1 A x_1(n) + a_1 B + a_2 A x_2(n) + a_2 B = a_1 A x_1(n) + a_2 A x_2(n) + (a_1 + a_2) B \neq y(n)$

$\Rightarrow$ NON- linear.

Causal / Anticausal systems:

A system is causal if output at time = n depends on input at present and previous time (x(n), x(n-1) ---) and not on future samples (x(n+1), x(n+2)---)

EXA: $y(n) = x(n) - x(n-1)$ $\quad$ causal

$y(n) = x(n) + 3x(n+4)$ $\quad$ Anticausal

$y(n) = (n+2)\,x(n)$ $\quad$ causal

$y(n) = x(-n)$

$n = -1 \Rightarrow y(-1) = \overset{\text{future}}{x(1)}$

$n = 1 \Rightarrow y(1) = x(-1)$ $\quad$ Anticausal

$y(n) = x(2n)$

$n = 2 \Rightarrow y(2) = x(4)$ $\quad$ Anticausal.

Monday 2/18/13 8:11 AM

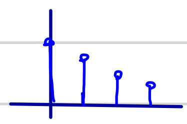
EXA: $y(n) = x(n^2)$

$y(2) = x(4)$ Anti causal.

Response of LTI systems to arbitrary inputs (convolution sum):

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

LTI



$x(n) = \cdots x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + \cdots$

$= \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$

Define:-

$h(n,k) \triangleq T[\delta(n-k)]$

$y(n) = T[x(n)]$

$= T\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} T[x(k)\,\delta(n-k)]$ Linearity ✓

$= \sum_{k=-\infty}^{\infty} x(k)\,T[\delta(n-k)] = \sum_{k=-\infty}^{\infty} x(k)\,h(n,k)$

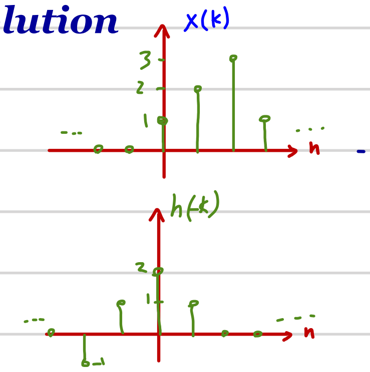
$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)} = \sum x(k)\,h(-(k-n))$$

Convolution Sum

EXA: find the output of the LTI system whose $h(n) = \{1, \underset{\uparrow n=0}{2}, 1, -1\}$

For the input $x(n) = \{\underset{\uparrow n=0}{1}, 2, 3, 1\}$

Solution



$y(0) = 2\times1 + 1\times2 = 4$

$y(1) = 1\times1 + 2\times2 + 1\times3 = 8$

$y(-1) = 1\times1 = 1$

$\therefore$ output : $n = (-1, 5)$

$\Rightarrow y(n) = [\cdots 0, 0, 1, \underset{\uparrow n=0}{4}, 8, 8, 3, -2, -1, 0, 0 \cdots]$

Properties of convolution:

1) commutative law:-

$x(n) * h(n) = h(n) * x(n)$

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) \quad = \quad h(n) \rightarrow \boxed{x(n)} \rightarrow y(n)$

$x(n) \rightarrow \boxed{h_1(n)} \rightarrow \boxed{h_2(n)} \rightarrow y(n) \quad = \quad x(n) \rightarrow \boxed{h_2(n)} \rightarrow \boxed{h_1(n)} \rightarrow y(n)$

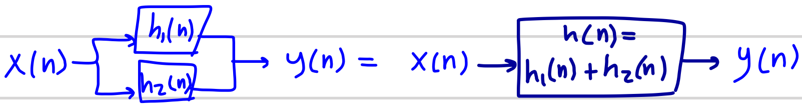
2) associative law:-

$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$

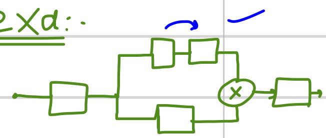
$x(n) \rightarrow \boxed{h_1(n)} \rightarrow \boxed{h_2(n)} \rightarrow y(n) \quad = \quad x(n) \rightarrow \boxed{h(n) = h_1(n) * h_2(n)} \rightarrow y(n)$

3) Distributive law:-

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$



exa:



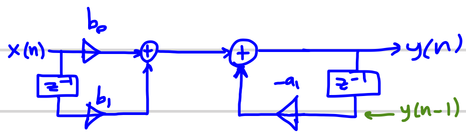
LTI systems described by different equations:
Structures (block diagrams):-

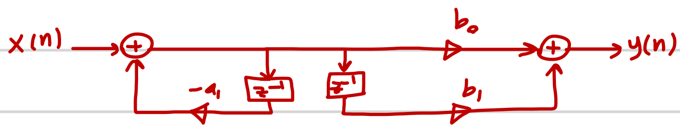
$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

indicate the order

1st order system:-

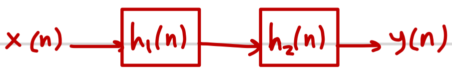
$$y(n) = -a_1\, y(n-1) + b_0 x(n) + b_1 x(n-1)$$

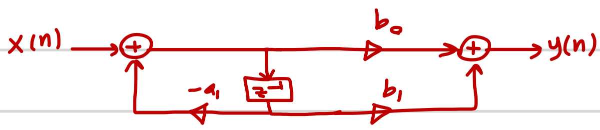




Direct form-I simulation Diagram

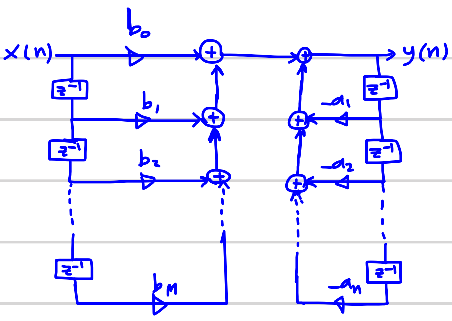
$h_2(n)$

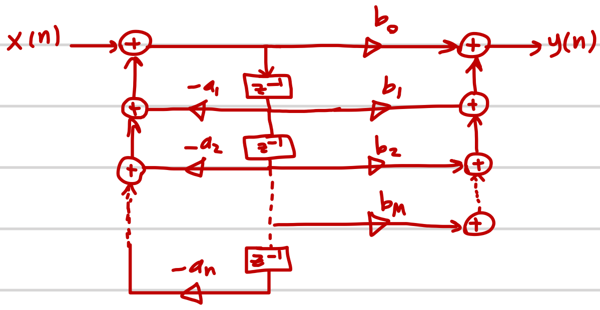




$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

Direct form-II simulation Diagram → reduce Shift registers.

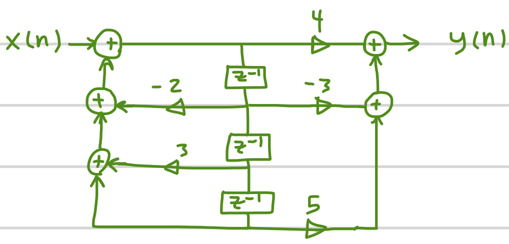




EXA:

$$y(n) = -2y(n-1) + 3y(n-2) + y(n-3) + 4x(n) - 3x(n-1) + 5x(n-3)$$

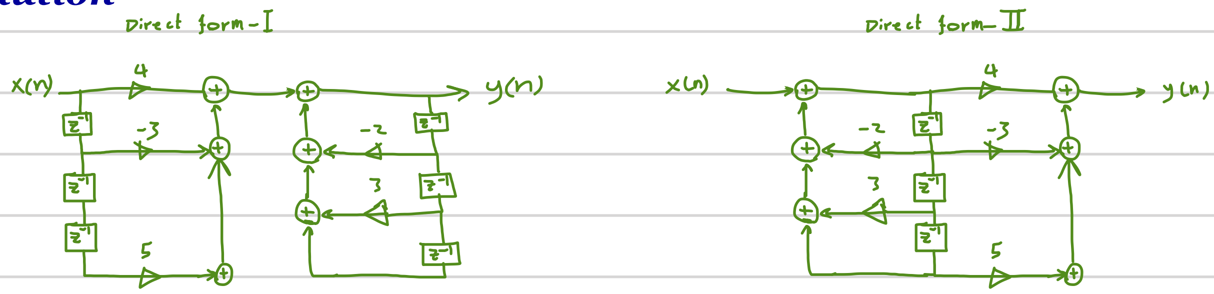
Solution



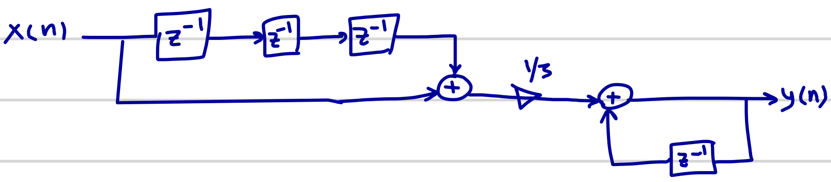
Wednesday 2/20/13 8:10 AM

EXA: $y(n) = -2y(n-1) + 3y(n-2) + y(n-3) + 4x(n) - 3x(n-1) + 5x(n-3)$.

Solution



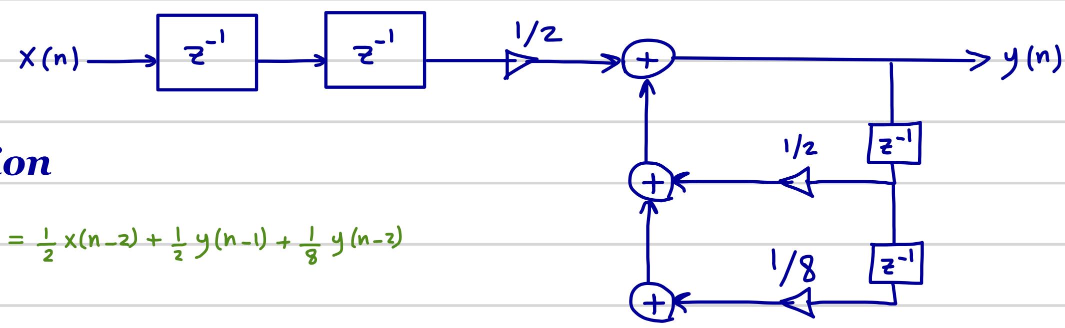
EXA:



Solution

$$y(n) = \frac{1}{3}\left[x(n) + x(n-3)\right] + y(n-1)$$

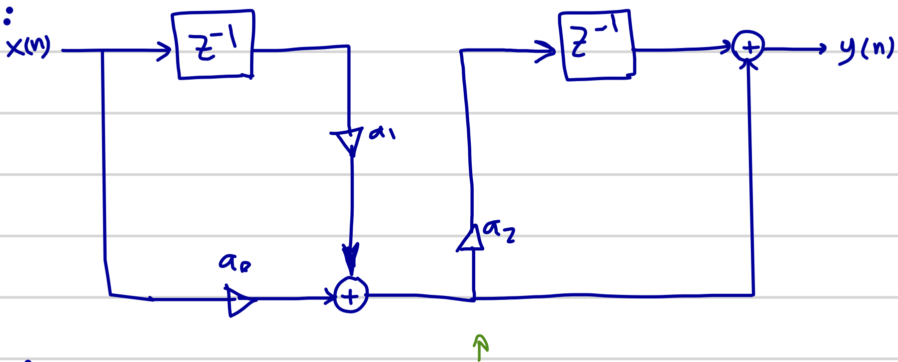
EXA:



solution

$$y(n) = \frac{1}{2}x(n-2) + \frac{1}{2}y(n-1) + \frac{1}{8}y(n-2)$$

EXA:



Solution

$$a_1 x(n-1) + a_0 x(n) = w(n)$$

$$y(n) = a_2 w(n-1) + w(n)$$

$$= a_2\left[a_1 x(n-2) + a_0 x(n-1)\right] + a_1 x(n-1) + a_0 x(n)$$

$$= a_0 x(n) + (a_1 + a_0 a_2) x(n-1) + a_1 a_2 x(n-2).$$

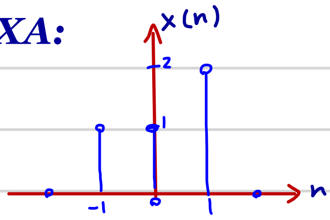
Correlation of discrete - time signals:

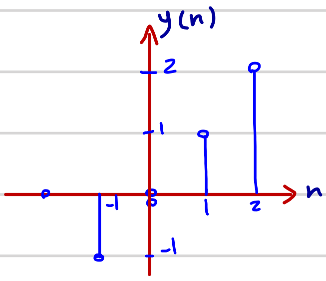
$$r_{xy}(\ell) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-\ell) = \sum_{n=-\infty}^{\infty} x(n+\ell)\, y(n)$$

$$r_{yx}(\ell) = \sum y(n)\, x(n-\ell) = \sum y(n+\ell)\, x(n)$$

$$\boxed{r_{xy}(\ell) = r_{yx}(-\ell)}$$

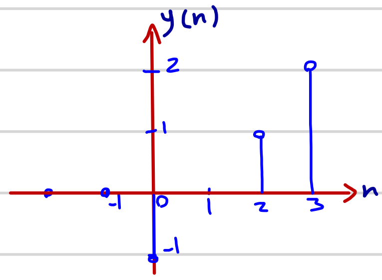
EXA:





Solution

$r_{xy}(0) = (-1 * 1) + 0 + 1 * 2 = 1$



$r_{xy}(1) = (-1 * 1) + 0 = -1$

$\vdots$

$\therefore\ r_{xy}(\ell) = [\,0, 0, 2, 3, 5, \underset{\uparrow\ \ell=0}{1}, -1, -2, 0, 0\,]$

Relation to convolution:

$$\boxed{r_{xy}(\ell) = x(\ell) * y(-\ell)}$$

If $y(n) = x(n)$

$$r_{xx}(\ell) = x(\ell) * x(-\ell)$$

$$\boxed{r_{xx}(\ell) = r_{xx}(-\ell)}$$

Input - output correlation:

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$r_{yx}(\ell) = y(\ell) * x(-\ell) = [x(\ell) * h(\ell)] * x(-\ell) = h(\ell) * [x(\ell) * x(-\ell)]$$

$$\boxed{r_{yx}(\ell) = r_{xx}(\ell) * h(\ell)}$$

usually $\delta(\ell)$ ✓

$$r_{yy}(\ell) = y(\ell) * y(-\ell) = [x(\ell) * h(\ell)] * [x(-\ell) * h(-\ell)]$$

$$\boxed{r_{yy}(\ell) = r_{xx}(\ell) * r_{hh}(\ell)}$$

Radar detect:-

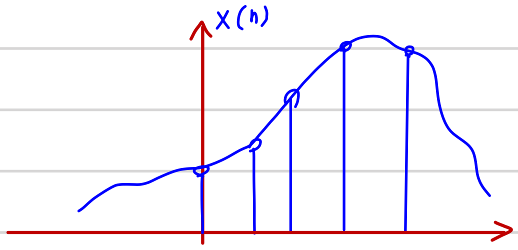
$$y(n) = \alpha x(n-D) + w(n)$$

$$r_{xy}(\ell) = \sum_{n=-\infty}^{\infty} x(n)\, y(n-\ell) = \sum_{n=-\infty}^{\infty} x(n)[\alpha x(n-\ell-D) + w(n-\ell)]$$

$$= r_{xw}(\ell) + \alpha r_{xx}(\ell + D)$$

$$r_{xy}(-D) = \alpha r_{xx}(0) + r_{xw}(-D) \qquad \{\, |r_{xx}(0)| \geq |r_{xx}(\ell)| \,\}$$

Chapter
The
Z - Transform



$$x(t) = \cdots + x(-2)\,\delta(t+2) + x(-1)\,\delta(t+1) + x(0)\,\delta(t) + x(1)\,\delta(t-1) + \cdots$$

$$\left\{ \begin{array}{l} \delta(t) \xleftrightarrow{\mathcal{L}} 1 \\ \delta(t-t_0) \xleftrightarrow{\mathcal{L}} 1 \cdot e^{-st_0} \end{array} \right\}$$

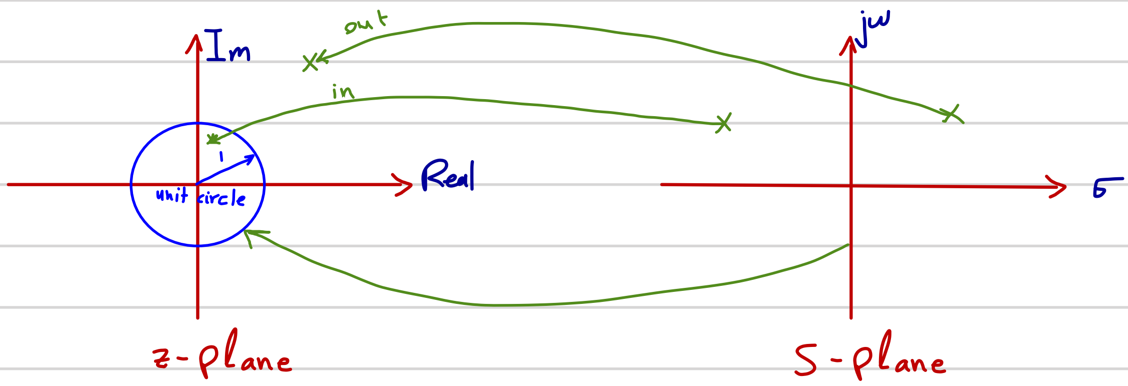
$$\mathcal{L}[x(t)] = x(-2)e^{2s} + x(-1)e^{s} + x(0) + x(1)e^{-s} + x(2)e^{-2s} + \cdots$$

let $\boxed{z = e^{s}}$ $\quad X(z) = X(s)\big|_{z=e^{s}}$ $\Rightarrow$ $X(z) = x(-2)z^{2} + x(-1)z + x(0) + x(1)z^{-1} + x(-2)z^{-2}$

$\therefore$ $\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}}$ $\quad$ like : $x(n) = [\underset{\substack{\uparrow \\ n=0}}{1}, \frac{1}{2}] \longrightarrow X(z) = 1z^{0} + \frac{1}{2}z^{-1} = 1 + \frac{1}{2}z^{-1}$

Z vs S - plane:-

$$z = e^{s} = e^{\sigma + j\omega} = (e^{\sigma})\,e^{j\omega}$$



Region of convergence:-

Is the set of z-values for which x(z) has a finite value.

Monday 2/25/13 8:06 AM

EXA: find Z-transform & ROC (region of convergence)

$$x_1(n) = \{1, 2, \underset{\uparrow n=0}{5}, 7, 0, 1\}$$

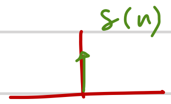
Solution

$$X_1(z) = 1 \cdot z^{+2} + 2z^{+1} + 5z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} = z^2 + 2z + 5 + 7z^{-1} + 0 + z^{-3}$$

ROC : all z-plane except $z=0$, $z=\infty$

$$x_2(n) = \delta(n)$$

$X_2(z) = 1$, ROC: All z-plane. $\left\{ x_2(n) = [\underset{\uparrow n=0}{1}] \right.$



$\}$

$$x_3(z) = \delta(n-4)$$

$X_3(z) = 1z^{-4} = z^{-4}$, ROC : all z- except $z=0$.

$$x_4(n) = \delta(n+3)$$

$X_4(z) = z^3 = 1 \cdot z^{-(-3)} = z^3$, ROC: All z-plane except $z=\infty$.

EXA:

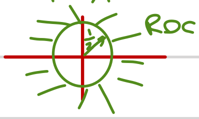
$$x(n) = \left(\tfrac{1}{2}\right)^n u(n)$$

Solution

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^n$$

$$\left\{ 1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A} \text{ ; if } |A| < 1 \right\} \Rightarrow X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \quad ; \quad \left|\tfrac{1}{2} z^{-1}\right| < 1$$

ROC : $|z| > \frac{1}{2}$



Region of convergence:

$$z = re^{j\theta}$$

$$\Rightarrow X(z) = \sum x(n) r^{-n} e^{-jn\theta}$$

$$|X(z)| = \left|\sum x(n) r^{-n} e^{-jn\theta}\right| \le \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} e^{-jn\theta} \right|$$

$$\le \sum_{n=-\infty}^{\infty} |x(n) r^{-n}|$$

$$\le \sum_{n=-\infty}^{\infty} |x(n) r^{-n}|$$

$$\le \sum_{n=-\infty}^{-1} |x(n) r^{-n}| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$

$$\le \underbrace{\sum_{n=1}^{\infty} |x(-n) r^{n}|}_{\text{Anticausal part}} + \underbrace{\sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|}_{\text{causal part of } x(n)}$$

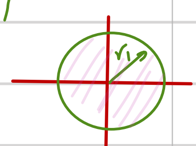
Anticausal part $\rightarrow$ ROC : $r < r_1$

causal part of $x(n)$ $\rightarrow$ ROC : $r > r_2$

r → $|z|$; r_2 → like $\frac{1}{2}$ (example)

Anticausal signal

ROC : $|z| < r_1$



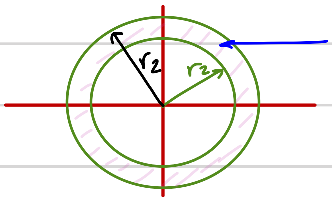
Causal signal

ROC : $r > r_2$
$\{|z|\}$

Causal + Anticausal:-

Case-1 : $r_2 < |z| < r_1$

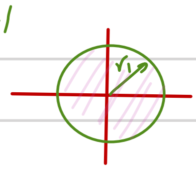
if $r_1 > r_2$



ROC

Anticausal signal

ROC : $|z| < r_1$

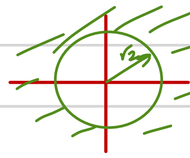


case-2 : $r_2 > r_1$ No intersection

$X(z)$ does not exist.

Causal signal

ROC : $r > r_2$
$[|z|]$

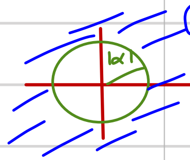


ROC

EXA: $x(n) = \alpha^n u(n)$

Solution

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1-\alpha z^{-1}}, \quad \text{ROC: } |\alpha z^{-1}| < 1 \Rightarrow |z| > |\alpha|$$



ROC

$$\boxed{\alpha^n u(n) \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}}}$$

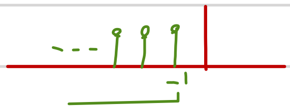
ROC

$|z| > |\alpha|$

EXA: $x(n) = -\alpha^n u(-n-1)$

Solution

$x(n) = -\alpha^n u(-(n+1))$



$$X(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} \quad ; \text{ let } \ell = -n$$

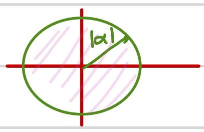
$$X(z) = -\sum_{\ell=1}^{\infty} (\alpha^{-1} z)^\ell \qquad \left\{ A + A^2 + A^3 + \cdots = A[1 + A + A^2 + \cdots] = \frac{A}{1-A} \; ; |A| < 1 \right\}$$

$$X(z) = \frac{-\alpha^{-1} z}{1-\alpha^{-1} z} \qquad \div \alpha^{-1} z \Rightarrow X(z) = \frac{-1}{\alpha z^{-1} - 1} = \frac{1}{1-\alpha z^{-1}}$$

$$\boxed{\alpha^n u(-n-1) \xleftrightarrow{z} \frac{-1}{1-\alpha z^{-1}}}$$

ROC

$|z| < |\alpha|$



$\{ |\alpha^{-1} z| < 1 \Rightarrow |z| < |\alpha| \}$

Properties of Z - transform:

1) Linearity:-

if $x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$

then $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{z} a_1 X_1(z) + a_2 X_2(z)$

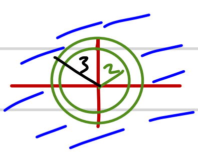
ROC: Intersection of ROCs of $X_1(z)$ & $X_2(z)$

EXA: Find z-tr. & ROC of: $x(n) = 3 . \underbrace{(2)^n u(n)}_{} - 4 . \underbrace{(3)^n u(n)}_{x_2(n)}$

Solution

$\frac{1}{1-2z^{-1}}$, ROC $|z| > 2$ $\qquad$ $\frac{1}{1-3z^{-1}}$, ROC $|z| > 3$

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}, \quad \text{ROC : } |z| > 3$$



ROC.

EXA: $x(n)=(\frac{1}{3})^n u(n) - 3(2)^n u(-n-1)$

$$X(z)=\frac{1}{1-\frac{1}{3}z^{-1}}+\frac{3}{1-2z^{-1}} \quad ; \text{ROC: } \frac{1}{3}<|z|<2$$

EXA: $x(n)=(\frac{1}{2})^n u(n) + (\frac{1}{4})^n u(-n-1)$

$$X(z)=\frac{1}{1-\frac{1}{2}z^{-1}}-\frac{1}{1-\frac{1}{4}z^{-1}} \quad , \left.\begin{matrix}|z|>\frac{1}{2}\\ |z|<\frac{1}{4}\end{matrix}\right\} \text{doesn't exist.}$$

EXA: $x(n)=\cos(\omega_0 n)\,u(n)$

$$= \frac{1}{2}e^{j\omega_0 n}u(n)+\frac{1}{2}e^{-j\omega_0 n}u(n) = \frac{1}{2}(e^{j\omega_0})^n u(n)+\frac{1}{2}(e^{-j\omega_0})^n u(n)$$

$$\Rightarrow X(z) = \frac{1}{2}\frac{1}{1-e^{j\omega_0}z^{-1}}+\frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}} = \frac{1}{2}\frac{1-e^{-j\omega_0}z^{-1}+1-e^{j\omega_0}z^{-1}}{(1+z^{-2}-e^{j\omega_0}z^{-1}-e^{-j\omega_0}z^{-1})} = \frac{1}{2}\frac{2-2(\cos\omega_0)z^{-1}}{1+z^{-2}-2(\cos\omega_0)z^{-1}}$$

$$\boxed{\cos(\omega_0 n)u(n) \xleftrightarrow{z} \frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}} \qquad \text{ROC: } |z|>1 \qquad \{|e^{j\omega_0}|\}$$

$$\boxed{\sin(\omega_0 n)u(n) \xleftrightarrow{z} \frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}} \qquad \text{ROC: } |z|>1$$

if $u(-n-1) \longrightarrow$ z-side $\times(-1)$ ***

Wednesday 2/27/13 8:23 AM

EXA: $X(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{elsewhere} \end{cases}$

Solution

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=0}^{N-1} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{N-1} (z^{-1})^n \qquad \left\{ \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \; ; \; \sum_{n=0}^{4} a^n = [1+a+a^2+\dots] - [a^5+a^6+a^7+\dots] = 1+a+a^2+a^3+a^4 = \frac{1}{1-a} - a^5\left(\frac{1}{1-a}\right) \right\}$$

$$\Rightarrow X(z) = \begin{cases} \dfrac{1-z^{-N}}{1-z^{-1}} & z \ne 1 \\ N & z = 1 \end{cases}$$

If the linear combination of several signals gives a finite-length sequence, then ROC is determined by the finite duration nature of the resulting sequence.

* $\sin(\omega_0 n)\, u(n) \xleftrightarrow{z} \dfrac{1 - z^{-1}\sin\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}$

Time - Shift property:

$$x(n) \xleftrightarrow{z} X(z)$$

$$x(n-k) \xleftrightarrow{z} z^{-k} X(z) \;;$$

ROC is the same as $X(z)$ [except in special cases]

* proof:-

$\sum_{n=-\infty}^{\infty} x(n-k)\, z^{-n}$ let $m = n-k$

$$\Rightarrow \sum_{n=-\infty}^{\infty} x(m)\, z^{-(m+k)} = \sum_{n=-\infty}^{\infty} x(m)\, z^{-m}\, z^{-k} = X(z)\, z^{-k}$$

EXA: $x_1(n) = [\underset{\uparrow n=0}{1}, 2, 5, 7, 0, 0]$

$\rightarrow X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3}$

$x_1(n-2) = [\underset{\uparrow n=0}{0}, 0, 1, 2, 5, 7, 0, 0]$

$\rightarrow X_2(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5}$

$= z^{-2}(1 + 2z^{-1} + 5z^{-2} + 7z^{-3})$

Scaling in the Z-domain:

$x(n) \xleftrightarrow{z} X(z)$; ROC: $r_1 < |z| < r_2$

$a^n x(n) \xleftrightarrow{z} X(a^{-1}z)$ ROC: $|a|r_1 < |z| < |a|r_2$

$a^n x(n) \Rightarrow \sum a^n x(n) z^{-n} = \sum x(n) (a^{-1}z)^{-n}$; $r_1 < |a^{-1}z| < r_2 \Rightarrow |a| r_1 < |z| < |a| r_2$

EXA: $x(n) = a^n \cos(\omega_o n) u(n)$

Solution

$\cos(\omega_o n) u(n) \xleftrightarrow{z} \frac{1 - z^{-1}\cos\omega_o}{1 - 2z^{-1}\cos\omega_o + z^{-2}}$ $\infty > |z| > 1$

$a^n \cos(\omega_o n) u(n) \xleftrightarrow{z} \frac{1 - az^{-1}\cos\omega_o}{1 - 2az^{-1}\cos\omega_o + a^2 z^{-2}}$ $|z| > |a|$

EXA: $x(n) = (\frac{1}{3})^n \Big[\underbrace{2(\frac{1}{4})^n u(n) + 3(\frac{1}{2})^n u(-n-1)}_{x_1(n)} \Big]$

Solution

$X_1(z) = \frac{2}{1-\frac{1}{4}z^{-1}} - \frac{3}{1-\frac{1}{2}z^{-1}}$ $\frac{1}{4} < |z| < \frac{1}{2}$

$X(z) = X_1(a^{-1}z) = X_1(3z)$

$= \frac{2}{1-\frac{1}{4}(3z)^{-1}} - \frac{3}{1-\frac{1}{2}(3z)^{-1}}$ $\frac{1}{12} < |z| < \frac{1}{6}$

Time reversal property:

$\sum_{n=-\infty}^{\infty} x(-n) z^{-n} \Rightarrow \sum_{m=-\infty}^{\infty} x(m) z^{m} = \sum x(m) (z^{-1})^{-m}$; $r_1 < |z^{-1}| < r_2 \Rightarrow \frac{1}{r_2} < |z| < \frac{1}{r_1}$

$$\boxed{\begin{array}{llll} \text{If} & x(n) \xleftrightarrow{z} X(z) & & r_1 < |z| < r_2 \\ \text{then} & x(-n) \xleftrightarrow{z} X(z^{-1}) & & \frac{1}{r_2} < |z| < \frac{1}{r_1} \end{array}}$$

EXA: Find z-tr. for $x(n) = u(n)$

Solution

$u(n) = 1^n u(n) = \alpha^n u(n)$

$u(n) \xleftrightarrow{z} \frac{1}{1-z^{-1}}$ $\infty > |z| > 1$

EXA: $x(n) = u(-n)$

Solution

$\rightarrow x(-(n+1))$

$X(z) = \frac{1}{1-z}$ $0 < |z| < 1$

$\left\{ \begin{array}{l} u(-n-1) \xleftrightarrow{z} \frac{-1}{1-z^{-1}} \quad |z| < 1 \\ \frac{z}{1-z} = \frac{1}{z^{-1}-1} = \frac{-1}{1-z^{-1}} \end{array} \right\}$

Differentiation in the z-domain:

$$X(z) = \sum x(n) z^{-n}$$

$$\frac{dX(z)}{dz} = \sum -n\, x(n)\, z^{-n-1} \Rightarrow -z\frac{dX(z)}{dz} = \sum \boxed{n\, x(n)}\, z^{-n}$$

If $x(n) \xleftrightarrow{z} X(z)$

then $n x(n) \xleftrightarrow{z} -z\dfrac{dX(z)}{dz}$ } same ROC

EXA: Let $x(n) = n a^n u(n)$

Solution

$$a^n u(n) \xleftrightarrow{z} \frac{1}{1 - a z^{-1}} \qquad |z|>|a|$$

$$n a^n u(n) \xleftrightarrow{z} -z\frac{d}{dz}(1-az^{-1})^{-1} = -z\left(-(1-az^{-1})^{-2}(az^{-2})\right)$$

$$\boxed{n a^n u(n) \xleftrightarrow{z} \frac{az^{-1}}{(1-az^{-1})^2}} \qquad |z|>|a|$$

$$\boxed{n u(n) \xleftrightarrow{z} \frac{z^{-1}}{(1-z^{-1})^2}} \qquad |z|>|1|$$

EXA: find inverse transform of

$$X(z) = \log(1+az^{-1}) \quad ; \quad ROC: |z|>|a|$$

Solution

$$-z\frac{dX(z)}{dz} = \frac{(-z)(-az^{-2})}{1+az^{-1}} = \frac{az^{-1}}{1+az^{-1}} = \frac{az^{-1}}{1-(-a)z^{-1}}$$

$$a(-a)^n u(n) \xleftrightarrow{z} \frac{a}{1-(-a)z^{-1}}$$

$$\underbrace{a(-a)^{n-1} u(n-1)}_{n x(n)} \xleftrightarrow{z} \frac{az^{-1}}{1+az^{-1}}$$

$$\therefore\ x(n) = \frac{a(-a)^{n-1}u(n-1)}{n}$$

Convolution property:

$$x_1(n) \xleftrightarrow{z} X_1(z)$$

$$x_2(n) \xleftrightarrow{z} X_2(z)$$

then

$$x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z)\, X_2(z)$$; ROC: at least the intersection of ROCs of $X_1(z)$ & $X_2(z)$.

EXA: Find the convolution of:

$$x_1(n) = [\underset{\uparrow n=0}{1}, -2, 1]$$

$$x_2(n) = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

Solution

$$y(n) = x_1(n) * x_2(n)$$

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$\Rightarrow Y(z) = X_1(z)\,X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

$$\therefore\ y(n) = [\underset{\uparrow n=0}{1}\ \ -1\ \ 0\ \ 0\ \ 0\ \ 0\ \ -1\ \ 1]$$

The system fn. for LTI systems:

Let $x(n)$: i/p

$y(n)$: o/p

$h(n)$: impulse response.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

like: $y(n) = \frac{1}{2} y(n-1) + x(n)$

$$y(n) = x(n) * h(n)$$

$$Y(z) = X(z)\,H(z) \Rightarrow \boxed{H(z) = \frac{Y(z)}{X(z)}}$$ System fn.

$$y(n) = Z^{-1}[Y(z)]$$

Monday 3/4/13 8:02 AM

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Apply z-transform:

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z) \Rightarrow Y(z)\left[1+\sum_{k=1}^{N} a_k z^{-k}\right] = X(z)\sum_{k=0}^{M} b_k z^{-k}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

Poles and zeros:
Zeros of H(z) are values of z for which x(z) = 0
Poles of H(z) are values of z for which x(z) = ∞

Stability:
-An LTI system is BIBO stable iff the ROC of the system fn. includes the unit circle.
-A causal LTI system is stable if all poles of H(z) are inside the unit circle.

$$H(z) = \frac{- - -}{(z-p_1)(z-p_2)}$$

EXA: find the convolution of

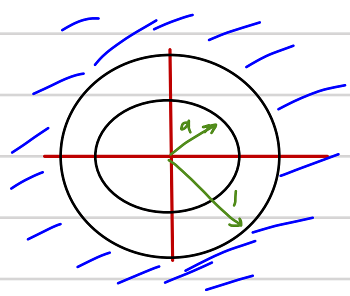
$$x_1(n) = a^n u(n), \quad x_2(n) = u(n) \quad |a| < 1$$

Solution

$$y(n) = x_1(n) * x_2(n)$$

$$Y(z) = X_1(z) X_2(z) \Rightarrow X_1(z) = \frac{1}{1-az^{-1}} \quad |z|>|a|, \quad X_2(z) = \frac{1}{1-z^{-1}} \quad |z|>1$$

$$\Rightarrow Y(z) = \frac{1}{(1-az^{-1})(1-z^{-1})} = \left(\frac{1}{1-a}\right)\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right) \qquad ROC: |z|>1$$



$$y(n) = \frac{1}{1-a}\left(u(n) - a(a)^n u(n)\right)$$

EXA: det. The system fn. H(z) and the impulse response h(n) of the system described by:-

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

Solution

Take z-transform

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z) \Rightarrow Y(z)\left[1-\frac{1}{2}z^{-1}\right] = 2X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-\frac{1}{2}z^{-1}}, \quad h(n) = Z^{-1}[H(z)] = 2\left(\frac{1}{2}\right)^n u(n).$$

EXA: find the z-transform of:-

$$x(n) = (n-2)\,(0.5)^{n-2} \cos\left(\frac{\pi}{2}(n-2)\right) u(n-2)$$

Solution

$$n\ \underbrace{0.5^{n} \cos\left(\tfrac{\pi}{3}n\right)u(n)} \Rightarrow 0.5^{n}\left(\cos\tfrac{\pi}{3}n\right)u(n) \xleftrightarrow{z} \frac{1-\left(0.5\cos\frac{\pi}{3}\right)z^{-1}}{1-2\times0.5\times\left(\cos\frac{\pi}{3}\right)z^{-1}+0.5^{2}z^{-2}} = \frac{1-0.25z^{-1}}{1-0.5z^{-1}+0.25z^{-2}} \qquad |z|>\frac{1}{2}$$

$$n\ 0.5^{n}\cos\left(\tfrac{\pi}{3}n\right)u(n) \xleftrightarrow{z} -z\frac{d}{dz}\left(\quad\right) = \frac{0.25z^{-3}-0.5z^{-4}+0.0625z^{-5}}{1-z^{-1}+0.75z^{-2}-0.25z^{-3}+0.0625z^{-4}}$$

$$X(z) = z^{-2}\left[-z\frac{d}{dz}\left(\frac{1-0.25z^{-1}}{1-0.5z^{-1}+0.25z^{-2}}\right)\right]$$

EXA: find z-transform & ROC of:-

$$x(n) = \left(a^{n}+a^{-n}\right)u(n)$$

Solution

$$x(n) = a^{n}u(n) + \left(\frac{1}{a}\right)^{n}u(n)$$

$$X(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-\frac{1}{a}z^{-1}}, \qquad ROC:\ |z| > \max\left\{|a|, \left|\frac{1}{a}\right|\right\}$$

EXA: $\frac{1}{2}\left(n^{2}+n\right)\left(\frac{1}{3}\right)^{n-1}u(n-1)$

Solution

$$\left(\frac{1}{3}\right)^{n}u(n) \xleftrightarrow{z} \frac{1}{1-\frac{1}{3}z^{-1}} \qquad |z|>\frac{1}{3}$$

$$\left(\frac{1}{3}\right)^{n-1}u(n-1) \xleftrightarrow{z} \frac{z^{-1}}{1-\frac{1}{3}z^{-1}} \qquad |z|>\frac{1}{3}$$

$$n\left(\frac{1}{3}\right)^{n-1}u(n-1) \xleftrightarrow{z} -z\frac{d}{dz}\left[\frac{z^{-1}}{1-\frac{1}{3}z^{-1}}\right] = -z\,\frac{\left(1-\frac{1}{3}z^{-1}\right)\left(-z^{-2}\right)-z^{-1}\left(\frac{1}{3}z^{-2}\right)}{\left(1-\frac{1}{3}z^{-1}\right)^{2}} = \frac{z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)^{2}}$$

$$n^{2}\left(\frac{1}{3}\right)^{n-1}u(n-1) \xleftrightarrow{z} -z\frac{d}{dz}\left[\frac{z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)^{2}}\right] = \frac{z^{-1}+\frac{1}{3}z^{-2}}{\left(1-\frac{1}{3}z^{-1}\right)^{3}}$$

$$\Rightarrow X(z) = \frac{1}{2}\left[\frac{z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)^{2}} + \frac{z^{-1}+\frac{1}{3}z^{-2}}{\left(1-\frac{1}{3}z^{-1}\right)^{3}}\right], \qquad ROC:\ |z| > \frac{1}{3}.$$

EXA:

$$x(n) = \begin{cases} \left(\frac{1}{3}\right)^{n} & n \geq 0 \qquad u(n) \\ \left(\frac{1}{2}\right)^{-n} & n < 0 \qquad u(-n-1) \end{cases}$$

Solution

$$x(n) = \left(\frac{1}{3}\right)^{n}u(n) + 2^{n}u(-n-1)$$

$$X(z) = \underbrace{\frac{1}{1-\frac{1}{3}z^{-1}}}_{\hookrightarrow |z|>\frac{1}{3}} - \underbrace{\frac{1}{1-2z^{-1}}}_{\hookrightarrow |z|<2}, \qquad ROC:\ \frac{1}{3} < |z| < 2$$

EXA: find z-transform of:-

$$x(n) = (n+1)\left(\tfrac{1}{3}\right)^{n-2} u(n+3)$$

Solution

$$x(n) = (n+1)\left(\tfrac{1}{3}\right)^{-5}\left(\tfrac{1}{3}\right)^{n+3} u(n+3)$$

$$\left(\tfrac{1}{3}\right)^{n} u(n) \xleftrightarrow{z} \frac{1}{1-\frac{1}{3}z^{-1}} \qquad |z| > \tfrac{1}{3} \quad \Rightarrow \quad \left(\tfrac{1}{3}\right)^{n+3} u(n+3) \xleftrightarrow{z} \frac{z^{3}}{1-\frac{1}{3}z^{-1}} \qquad |z| > \tfrac{1}{3}$$

$$X(z) = 3^{5}\left[\frac{z^{3}}{1-\frac{1}{3}z^{-1}} + (-z)\frac{d}{dz}\left(\frac{z^{3}}{1-\frac{1}{3}z^{-1}}\right)\right], \quad ROC: \ |z| > \tfrac{1}{3}.$$

$$= \;---\; = 3^{5}\left[\frac{z^{3}}{1-\frac{1}{3}z^{-1}} + \frac{-3z^{3}+\frac{4}{3}z^{2}}{1-\frac{1}{3}z^{-1}}\right]$$

Inversion of z-transform:
1) contour integration:-

$$\boxed{x(n) = \frac{1}{2\pi j}\oint_{C} X(z)\, z^{n-1}\, dz}$$

; $\oint$: ***Integration over a closed path that encloses the origin and lies within ROC of X(z).***

2) long division.
3) Partial fraction and table look-up.

Long division:

$$X(z) = \frac{P_1(z)}{P_2(z)} = \sum C_n z^{-n}$$

EXA: find the inverse transform for:-

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

a) ROC: $|z| > 1$, b) ROC: $|z| < 0.5$

Solution

a) causal:- (normal order)

$$\begin{array}{r|l} & 1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2} \\ \hline 1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} & 1 \\ & -\left(1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right) \\ \hline & \frac{3}{2}z^{-1}-\frac{1}{2}z^{-2} \\ & -\left(\frac{3}{2}z^{-1}-\frac{9}{4}z^{-2}+\frac{3}{4}z^{-3}\right) \\ \hline & \frac{7}{4}z^{-2}-\frac{3}{4}z^{-3} \\ & -\left(\frac{7}{4}z^{-2}-\frac{21}{8}z^{-3}+\frac{7}{8}z^{-4}\right) \\ \hline & \frac{15}{8}z^{-3}+--- \end{array}$$

$$\therefore X(z) = \underset{\uparrow\, x(0)}{1} + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + --- = \sum x(n)\, z^{-n}$$

$$\Rightarrow x(n) = [\underset{\uparrow\, n=0}{1}, \tfrac{3}{2}, \tfrac{7}{4}, \dots]$$

b) $|z| < 0.5$ (Long division in reverse order).

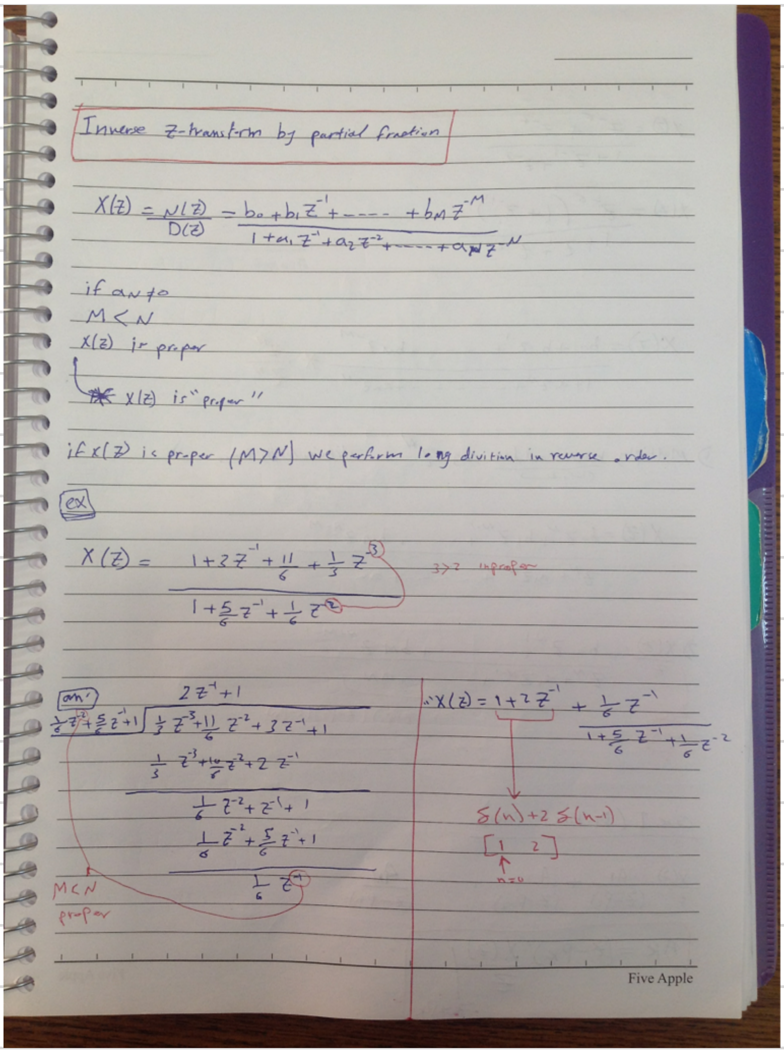
$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + \cdots \\
\hline
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \\
 & \underline{1 - 3z + 2z^2} \\
 & 3z - 2z^2 \\
 & \underline{3z - 9z^2 + 6z^3} \\
 & 7z^2 - 6z^3
\end{array}
$$

$\therefore \; x(n) = \{ \cdots, 14, 6, 2, 0, \underset{\uparrow_{n=0}}{0} \}$

$$\alpha^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - \alpha z^{-1}}$$

$$n\,\alpha^n u(n) \overset{z}{\longleftrightarrow} \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$$

Wednesday 3/6/13 8:25 AM



Inverse z-transform by partial fraction

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

if $a_N \neq 0$

$M < N$

$X(z)$ is proper

$X(z)$ is "proper"

if $X(z)$ is proper $(M > N)$ we perform long division in reverse order.

ex

$$X(z) = \frac{1 + 3z^{-1} + \frac{11}{6} + \frac{1}{3} z^{-3}}{1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

3 > 2 improper

an:

$$\frac{1}{6} z^{-2} + \frac{5}{6} z^{-1} + 1 \;\big)\; \frac{1}{3} z^{-3} + \frac{11}{6} z^{-2} + 3 z^{-1} + 1 \quad = 2z^{-1} + 1$$

$\frac{1}{3} z^{-3} + \frac{10}{6} z^{-2} + 2 z^{-1}$

$\frac{1}{6} z^{-2} + z^{-1} + 1$

$\frac{1}{6} z^{-2} + \frac{5}{6} z^{-1} + 1$

$\frac{1}{6} z^{-1}$

M < N proper

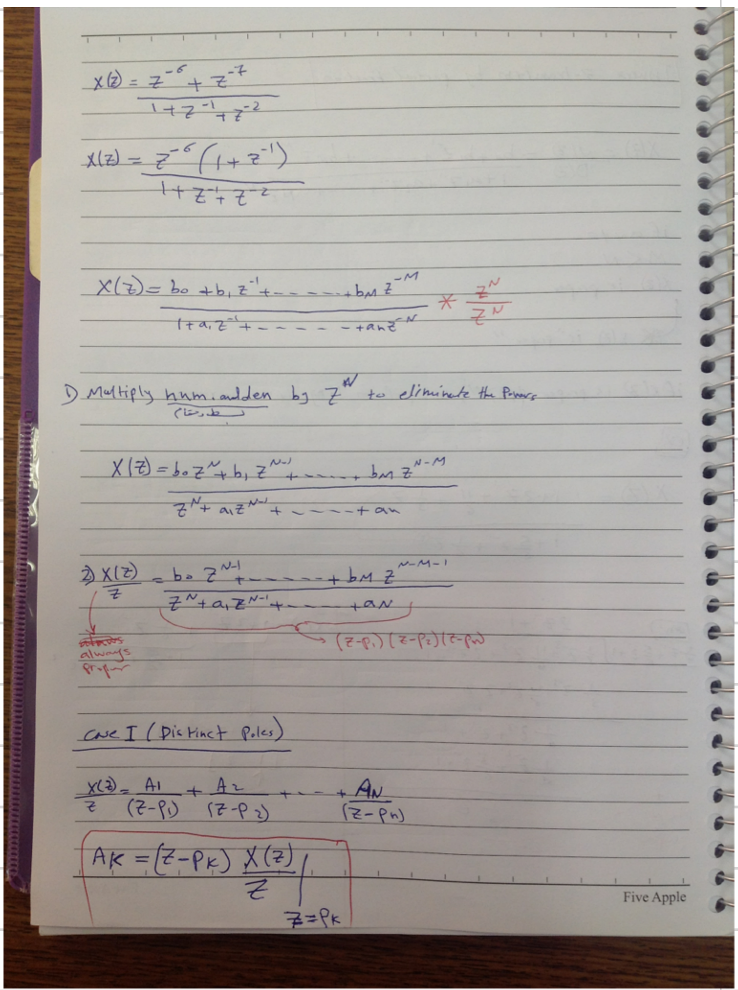
$$\therefore X(z) = 1 + 2z^{-1} + \frac{\frac{1}{6} z^{-1}}{1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

$\delta(n) + 2\delta(n-1)$

$[1 \quad 2]$

$n = 0$

Five Apple



$$X(z) = \frac{z^{-6} + z^{-7}}{1 + z^{-1} + z^{-2}}$$

$$X(z) = \frac{z^{-6}(1 + z^{-1})}{1 + z^{-1} + z^{-2}}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} * \frac{z^N}{z^N}$$

1) Multiply num. and den by z^N to eliminate the powers

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

2) $$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

always proper

$(z-p_1)(z-p_2)(z-p_N)$

Case I (Distinct poles)

$$\frac{X(z)}{z} = \frac{A_1}{(z-p_1)} + \frac{A_2}{(z-p_2)} + \cdots + \frac{A_N}{(z-p_N)}$$

$$\boxed{A_K = (z-p_K)\frac{X(z)}{z}\Big|_{z=p_K}}$$

Five Apple

EXA: find PF expansion of:-

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Solution

1- $$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

2- $$\frac{X(z)}{z} = \frac{z}{z^2 - 1.5z + 0.5} = \frac{z}{(z-1)(z-0.5)}$$

$$\frac{X(z)}{z} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

$$A_1 = (z-1)\frac{X(z)}{z}\Big|_{z=1} = \frac{z}{z-0.5}\Big|_{z=1} = \frac{1}{1-0.5} = 2$$

$$A_2 = (z-0.5)\frac{X(z)}{z}\Big|_{z=0.5} = \frac{z}{z-1}\Big|_{z=0.5} = \frac{0.5}{0.5-1} = -1$$

$$\therefore \frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

EXA: find PFE of:-

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Solution

$$X(z) = \frac{z^2+z}{z^2-z+0.5} \Rightarrow \frac{X(z)}{z} = \frac{z+1}{z^2-z+0.5} = \frac{z+1}{(z-P_1)(z-P_2)} = \frac{A_1}{z-P_1} + \frac{A_2}{z-P_2}$$

$$P_1 = \tfrac{1}{2} + j\tfrac{1}{2}, \quad P_2 = \tfrac{1}{2} - j\tfrac{1}{2}$$

$$A_1 = (z-P_1)\frac{X(z)}{z}\Big|_{z=P_1} = \frac{z+1}{z-P_2}\Big|_{z=P_1} = \frac{\frac{1}{2}+j\frac{1}{2}+1}{\frac{1}{2}+j\frac{1}{2}-\frac{1}{2}-j\frac{1}{2}} = \boxed{\tfrac{1}{2}-j\tfrac{3}{2}}$$

$$A_2 = (z-P_2)\frac{X(z)}{z}\Big|_{z=P_2} = \frac{z+1}{z-P_1}\Big|_{z=P_2} = \tfrac{1}{2}+j\tfrac{3}{2}$$

Case-II: Repeated poles

EXA: $X(z) = \dfrac{1}{(1+z^{-1})(1-z^{-1})^2}$

Solution

$$X(z) = \frac{z^3}{(z+1)(z-1)^2} \Rightarrow \frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2}$$

$$* \quad \frac{1}{(z+1)(z-1)^L} = \frac{A_1}{z+1} + \frac{A_2}{(z-1)^1} + \frac{A_3}{(z-1)^2} + \cdots + \frac{A_{L+1}}{(z-1)^L}$$

$$A_1 = (z+1)\frac{X(z)}{z}\Big|_{z=-1} = \frac{z^2}{(z-1)^2}\Big|_{z=-1} = \frac{1}{(-1-1)^2} = \frac{1}{4}$$

$$A_3 = (z-1)^2\frac{X(z)}{z}\Big|_{z=1} = \frac{z^2}{z+1}\Big|_{z=1} = \frac{1}{2}$$

$$A_2 \Rightarrow \frac{z^2}{z+1} = \frac{A_1(z-1)^2}{z+1} + A_2(z-1) + A_3$$

$$\frac{d}{dz}\left(\frac{z^2}{z+1}\right)\Big|_{z=1} = \left(\frac{2A_1(z-1)(z+3)}{z+1} + A_2 + 0\right)\Big|_{z=1}$$

$$\therefore \boxed{A_2 = \frac{d}{dz}\left[(z-1)^2\frac{X(z)}{z}\right]\Big|_{z=1}} \qquad \left\{\frac{1}{n!}\ ;\ n: \text{no. of derivatives}\right\}$$

$$A_2 = \frac{d}{dz}\left(\frac{z^2}{z+1}\right)\Big|_{z=1} = \frac{(z+1)(2z)-z^2}{(z+1)^2}\Big|_{z=1} = \frac{3}{4} \checkmark$$

$$\therefore \frac{X(z)}{z} = \frac{1/4}{z+1} + \frac{3/4}{z-1} + \frac{1/2}{(z-1)^2}$$

Inversion step:

- Distinct poles:-

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \cdots + \frac{A_N}{z-p_N}$$

$z^{-1} \longrightarrow$

$$X(z) = \frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}} + \cdots + \frac{A_N}{1-p_N z^{-1}}$$

$$Z^{-1}\left[\frac{1}{1-p_k z^{-1}}\right] = \begin{cases} (p_k)^n u(n) & \text{If ROC}: |z| > |p_k| \\ -(p_k)^n u(-n-1) & \text{"} \quad \text{"} : |z| < |p_k| \end{cases}$$

- Multiple poles:-

$$Z^{-1}\left[\frac{p z^{-1}}{(1-p z^{-1})^2}\right] = \begin{cases} n p^n u(n) & \text{If ROC}: |z| > p_k \\ -n p^n u(-n-1) & \text{"} \quad \text{"} : |z| < p_k \end{cases}$$

EXA: find inverse tr. of:-

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}} \longrightarrow \text{a) } |z|>1, \quad \text{b) } |z|<0.5, \quad \text{c) } 0.5<|z|<1$$

Solution

$$\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5} \Rightarrow X(z) = \frac{1}{1-0.5z^{-1}}$$

a) $|z| > 1 \Longrightarrow x(n) = 2u(n) - (0.5)^n u(n)$

b) $|z| < 0.5 \Longrightarrow x(n) = -2u(-n-1) + (0.5)^n u(-n-1)$

c) $0.5<|z|<1 \Longrightarrow x(n) = -2u(-n-1) - (0.5)^n u(n)$

EXA: find the causal signal inversal tr. of:-

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

Solution

$$\frac{X(z)}{z} = \frac{1/4}{z+1} + \frac{3/4}{z-1} + \frac{1/2}{(z-1)^2} \Rightarrow X(z) = \frac{1/4}{1+z^{-1}} + \frac{3/4}{1-z^{-1}} + \frac{0.5z^{-1}}{(1-z^{-1})^2}$$

$$\therefore x(n) = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}u(n) + 0.5n\,u(n)$$

- Complex conjugate poles:-

$$X(z) = \frac{A_k}{1-p_k z^{-1}} + \frac{A_k^*}{1-p_k^* z^{-1}} + \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \cdots$$

$$\underbrace{A_k}_{|A_k|e^{j\alpha_k}} (\underbrace{p_k}_{r_k e^{jB_k}})^n u(n) + \underbrace{A_k^*}_{|A_k|e^{-j\alpha_k}}(\underbrace{p_k^*}_{r_k e^{-jB_k}})^n u(n) \Rightarrow \boxed{x(n) = 2|A_k|\, r_k^n \cos(B_k n + \alpha_k)\, u(n)}$$

Suggested problems:-

Ch. 2
7-16-17-22-23-27-30-31-33-34-46-47-48-49-51-53-57-58

Ch. 3
1-2-4-8-11-14-15-16-19-25-26-38-40-41-43-51-55-56

Monday 3/11/13 8:09 AM

$$X(z)=\frac{A_k}{1-P_k z^{-1}}+\frac{A_k^*}{1-P_k^* z^{-1}}$$

$$\left.\begin{aligned}P_k &= r_k e^{j\beta_k}\\ A_k &= |A_k| e^{j\alpha_k}\end{aligned}\right\} \quad 2|A_k| r_k^n \cos(\beta_k n+\alpha_k)\, u(n) \longleftrightarrow \frac{A_k}{1-P_k z^{-1}}+\frac{A_k^*}{1-P_k^* z^{-1}}$$

Solution of difference equations with zero-initial conditions:

1) take z-tr. of difference eqn. :- $H(z)=\frac{Y(z)}{X(z)}$

2) take z-tr. of $x(n)\longrightarrow X(z)$

3) obtain $Y(z)=H(z)\,X(z)$

4) $y(n)=\mathcal{Z}^{-1}[Y(z)]$.

EXA: find the unit step response of the system described by:-

$$y(n)=0.9\,y(n-1)-0.81\,y(n-2)+x(n)$$

Solution

$$x(n)=u(n)$$

$$Y(z)=0.9z^{-1}Y(z)-0.81z^{-2}Y(z)+X(z)$$

$$\Rightarrow Y(z)\left[1-0.9z^{-1}+0.81z^{-2}\right]=X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1-0.9z^{-1}+0.81z^{-2}}, \qquad X(z)=\frac{1}{1-z^{-1}}$$

$$Y(z)=X(z)H(z)=\frac{1}{1-0.9z^{-1}+0.81z^{-2}}\times\frac{1}{1-z^{-1}}=\frac{1}{(1-0.9e^{j\frac{\pi}{3}}z^{-1})(1-0.9e^{j\frac{\pi}{3}}z^{-1})(1-z^{-1})}$$

$$=\frac{0.542-j0.049}{1-0.9e^{j\pi/3}z^{-1}}+\frac{0.542+j0.049}{1-0.9e^{-j\pi/3}z^{-1}}+\frac{1.099}{1-z^{-1}}$$

$$\therefore\ y(n)=1.088(0.9)^n\cos\left(\frac{\pi}{3}n-5.2^\circ\right)u(n)+1.09u(n).$$

EXA: find the convolution of :-

$$x_1(n)=\left(\tfrac{1}{3}\right)^n u(n)+2^n u(-n-1)$$

$$x_2(n)=\left(\tfrac{1}{2}\right)^n u(n)$$

Solution

$$y(n)=x_1(n)*x_2(n)$$

$$X_1(z)=\frac{1}{1-\frac{1}{3}z^{-1}}-\frac{1}{1-2z^{-1}}=\frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} \qquad \tfrac{1}{3}<|z|<2$$

$$X_2(z)=\frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z|>\tfrac{1}{2}$$

$$Y(z) = \frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})(1-\frac{1}{2}z^{-1})} \qquad \frac{1}{2} < |z| < 2$$

$$= \frac{2}{1-\frac{1}{3}z^{-1}} - \frac{4/3}{1-2z^{-1}} + \frac{10/3}{1-\frac{1}{2}z^{-1}} \qquad \frac{1}{2} < |z| < 2$$

$$\therefore\ y(n) = 2\left(\tfrac{1}{3}\right)^n u(n) + \tfrac{4}{3}(2)^n u(-n-1) + \tfrac{10}{3}\left(\tfrac{1}{2}\right)^n u(n).$$

EXA: a causal LTI system with I/O

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n) - \tfrac{1}{4}\left(\tfrac{1}{2}\right)^{n-1} u(n-1)$$

$$y(n) = \left(\tfrac{1}{3}\right)^n u(n).$$

Find: a) h(n) b) the difference eqn. c) direct-form II realization
d) Is the system stable?

Solution

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{4}\,\frac{z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$$

$$H(z) = \frac{1-\frac{1}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}} = \frac{-2}{1-\frac{1}{3}z^{-1}} + \frac{3}{1-\frac{1}{4}z^{-1}}$$

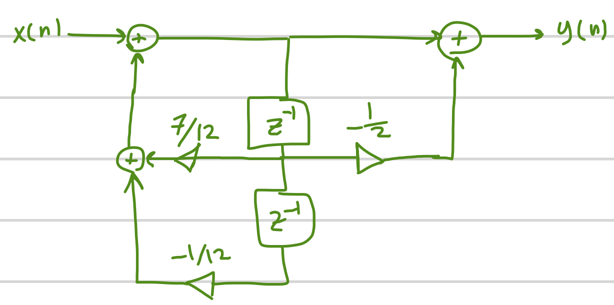
$$h(n) = Z^{-1}[H(z)] = -2\left(\tfrac{1}{3}\right)^n u(n) + 3\left(\tfrac{1}{4}\right)^n u(n)$$

b) $Y(z) - \frac{7}{12}z^{-1}Y(z) + \frac{1}{12}z^{-2}Y(z) = X(z) - \frac{1}{2}z^{-1}X(z)$

Take inverse tr.

$$y(n) = \tfrac{7}{12}y(n-1) - \tfrac{1}{12}y(n-2) + x(n) - \tfrac{1}{2}x(n-1)$$

c)



d) Stable system ; (Poles are inside unit circle).

EXA: find the causal signal of x(z):-

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + 4z^{-1} + 4z^{-2}}$$

Solution

$$X(z) = \frac{1}{4} + \underbrace{\frac{z^{-1} + \frac{3}{4}}{1 + 4z^{-1} + 4z^{-2}}}_{X_1(z)}$$

$$\frac{X_1(z)}{z} = \frac{\frac{3}{4}z + 1}{(z+2)^2} = \frac{A_1}{z+2} + \frac{A_2}{(z+2)^2}$$

$$A_2 = (z+2)^2 \frac{X_1(z)}{z}\Big|_{z=-2} = \left(\frac{3}{4}z + 1\right)\Big|_{z=-2} = -\frac{1}{2} \quad , \quad A_1 = \frac{d}{dz}\left[(z+2)^2 \frac{X_1(z)}{z}\right]_{z=-2} = \frac{d}{dz}\left(\frac{3}{4}z + 1\right)\Big|_{z=-2} = \frac{3}{4}$$

$$\frac{X_1(z)}{z} = \frac{-0.5}{(z+2)^2} + \frac{3/4}{z+2} \Rightarrow X(z) = \frac{3/4}{1 + 2z^{-1}} - \frac{0.5z^{-1}}{(1 + 2z^{-1})^2} = \frac{1}{4} + \frac{3/4}{1 - (-2)z^{-1}} - \underbrace{\frac{0.5z^{-1}}{1 - (-2)z^{-1}}}$$

$$\left(\frac{1}{4} \frac{(-2)z^{-1}}{(1 - (-2)z^{-1})^2}\right)$$

$$x(n) = \frac{1}{4}\delta(n) + \frac{3}{4}(-2)^n u(n) + \frac{1}{4} n (-2)^n u(n)$$

Inverse transform using contour integration:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

$$= \sum_{\substack{\text{All poles of } X(z)z^{n-1} \\ \text{inside } C}} \text{residues of } X(z) z^{n-1} \quad ;$$

C: closed path (circle) enclosing the origin and within ROC of X(z).

Cauchy residues Theorem:

1) sample poles

$$\text{Res}\left[\frac{f(z)}{z - z_0} \text{ at } z_0\right] = \begin{cases} f(z_0) & \text{if } z_0 \text{ is inside } C \\ 0 & z_0 \text{ outside } C. \end{cases}$$

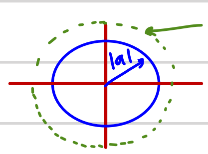
2) multiple poles

$$\text{Res}\left[\frac{f(z)}{(z - z_0)^k} \text{ at } z_0\right] = \frac{1}{(k-1)!} \frac{d^{k-1} f(z)}{dz^{k-1}}\Big|_{z=z_0}$$

EXA: find inverse tr. of:-

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

Solution



$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz = \frac{1}{2\pi j} \oint_C \frac{z^{n-1}}{1 - az^{-1}} dz = \frac{1}{2\pi j} \oint_C \frac{z^n}{(z - a)} dz$$

$$x(n) = \text{Res}[z = a] \quad , \quad z_0 = a \, , \quad f(z) = z^n \quad \Rightarrow x(n) = z^n\Big|_{z=a} = a^n$$

Let $n = -1 \Rightarrow x(n) = \frac{1}{2\pi j} \oint_C \frac{z^{-1}}{(z - a)} dz = \frac{1}{2\pi j} \oint_C \frac{1}{z(z - a)} dz = \text{Res}(z = a) + \text{Res}(z = 0)$

$$\Rightarrow x(-1) = \frac{1}{z}\Big|_{z=a} + \frac{1}{z - a}\Big|_{z=0} = \frac{1}{a} + \frac{1}{-a} = 0.$$

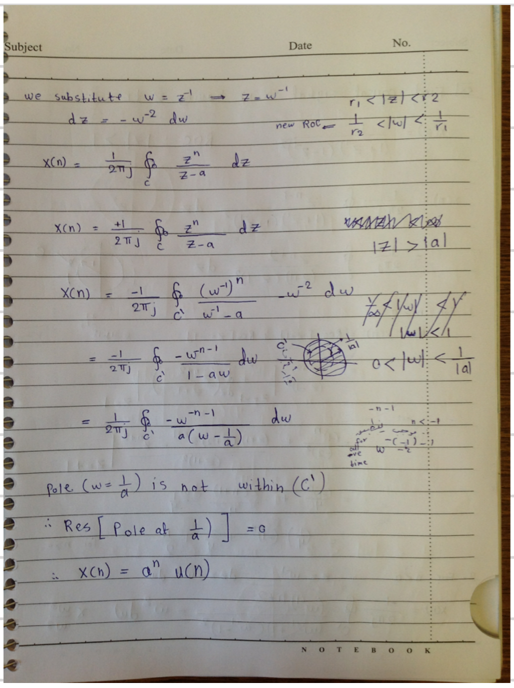
$$x(-2) = \frac{1}{2\pi j} \oint_C \frac{z^{-2}}{(z - a)} dz = \frac{1}{2\pi j} \oint_C \frac{1}{z^2(z - a)} dz = \underbrace{\text{Res}(z = 0)} + \underbrace{\text{Res}(z = a)} = \frac{1}{a^2} - \frac{1}{a^2} = 0$$

$$\frac{d}{dz} f(z)\Big|_{z=0} = \frac{d}{dz}(z-a)^{-1}\Big|_{z=0} = -(z-a)^{-2}\Big|_{z=0} = -\frac{1}{a^2} \qquad \frac{1}{z^2}\Big|_{z=a} = \frac{1}{a^2}$$

Wednesday 3/13/13 8:22 AM

$\frac{1}{1-az^{-1}}$ $|a|>1$ $\Rightarrow$ 1- For $n \geq 0 \Rightarrow a^n$

2- for $n<0 \Rightarrow \left.\begin{matrix} n=-1 \\ n=-2 \end{matrix}\right\} \Rightarrow 0$


Subject Date No.

we substitute $w = z^{-1} \Rightarrow z = w^{-1}$ $\quad r_1 < |z| < r_2$

$dz = -w^{-2}\, dw$ $\quad$ new RoC $\Leftarrow \frac{1}{r_2} < |w| < \frac{1}{r_1}$

$x(n) = \frac{1}{2\pi j} \oint_c \frac{z^n}{z-a}\, dz$

$x(n) = \frac{+1}{2\pi j} \oint_c \frac{z^n}{z-a}\, dz$ $\quad |z| > |a|$

$x(n) = \frac{-1}{2\pi j} \oint_{c'} \frac{(w^{-1})^n}{w^{-1}-a} -w^{-2}\, dw$ $\quad$ ~~$\frac{1}{\infty} < |w| < 1$~~ $|w|<1$

$= \frac{-1}{2\pi j} \oint_{c'} \frac{-w^{n-1}}{1-aw}\, dw$ $\quad 0 < |w| < \frac{1}{|a|}$

$= \frac{1}{2\pi j} \oint_{c'} \frac{-w^{-n-1}}{a(w-\frac{1}{a})}\, dw$

Pole $(w=\frac{1}{a})$ is not within (c')

$\therefore$ Res [Pole at $\frac{1}{a}$)] $= 0$

$\therefore x(n) = a^n u(n)$

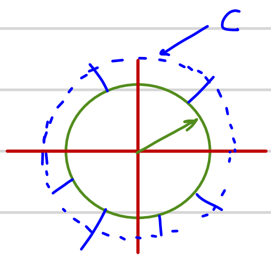
NOTEBOOK

EXA: find causal signal x(n) having Z-tr.

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} \qquad |z|>1$$

Solution

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{(1+z^{-1})(1-z^{-1})}\,dz = \frac{1}{2\pi j}\oint_C \frac{z^{n+2}}{(z+1)(z-1)^2}\,dz$$



1- For $n \geq 0$

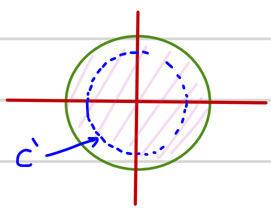
$$x(n) = \text{Res}[z=1] + \text{Res}[z=-1] = \left.\frac{z^{n+2}}{(z-1)^2}\right|_{z=-1} + \left.\frac{d}{dz}\left(\frac{z^{n+2}}{z+1}\right)\right|_{z=1}$$

$$= \frac{(-1)^n}{(-2)^2} + \left.\frac{(z+1)(n+2)z^{n+1} - z^{n+2}}{(z+1)^2}\right|_{z=1} = \frac{1}{4}(-1)^n \ \frac{2(n+2)1}{4} - \frac{1}{4} = \frac{1}{4}(-1)^n + \frac{3}{4} + \frac{n}{2}.$$

2- For $n<0$

$z = w^{-1} \longrightarrow dz = -w^{-2}\,dw$

$$\Rightarrow x(n) = \frac{-1}{2\pi j}\oint_{C'} \frac{(w^{-1})^{n+2}}{(w^{-1}+1)(w^{-1}-1)^2}(-w^{-2})\,dw = \frac{1}{2\pi j}\oint_{C'} \frac{w^{-n-1}}{(1+w)(1-w)^2} \qquad |w|<1$$



* Poles are outside (C') $\Rightarrow$ Res $= 0$.

$\therefore x(n) = \left[\frac{1}{4}(-1)^n + \frac{3}{4} + \frac{n}{2}\right] u(n)$

EXA: find causal signal x(n) having Z-tr.

$$X(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

a) ROC: $0.5 < |z| < 3$

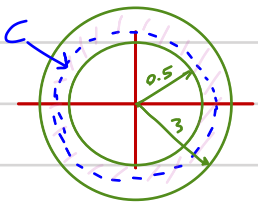
b) ROC: $|z| > 3$

c) ROC: $|z| < 0.5$

Solution

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{z^{n-1}(3-4z^{-1})}{1-3.5z^{-1}+1.5z^{-2}}\,dz = \frac{1}{2\pi j}\oint_C \frac{z^n(3z-4)}{z^2-3.5z+1.5}\,dz = \frac{1}{2\pi j}\oint_C \frac{z^n(3z-4)}{(z-3)(z-0.5)}\,dz$$

a) $0.5 < |z| < 3$



1- For $n \geq 0$

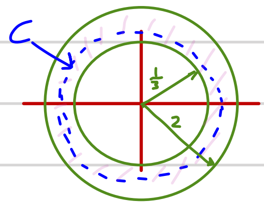
$x(n) = \text{Res}[\text{Pole at } z=0.5]$

$$= \frac{z^n(3z-4)}{z-3}\Big|_{z=0.5} = \left(\tfrac{1}{2}\right)^n.$$

2. For $n < 0$

$z = w^{-1} \longrightarrow dz = -w^{-2}\,dw$

$$X(n) = \frac{1}{2\pi j}\oint_{C'} \frac{w^{-n}(3w^{-1}-4)(-w^{-2})}{(w^{-1}-3)(w^{-1}-\frac{1}{2})}\,dw = \frac{1}{2\pi j}\oint_{C'} \frac{w^{-n}(3w^{-1}-4)}{\frac{3}{2}(w-\frac{1}{3})(w-2)}\,dw \qquad \tfrac{1}{3} < |w| < 2$$

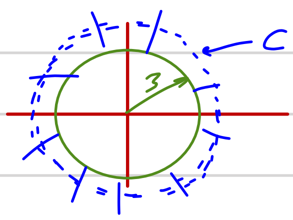


$$\Rightarrow x(n) = \text{Res}[w = \tfrac{1}{3}] = \frac{w^{-n}(3w^{-1}-4)}{\frac{3}{2}(w-2)}\Big|_{w=\frac{1}{3}} = \frac{(\frac{1}{3})^{-n}(3\times 3-4)}{\frac{3}{2}(\frac{1}{3}-2)} = (-2)(3)^n$$

$\therefore\ X(n) = (\tfrac{1}{2})^n u(n) - 2(3)^n u(-n-1)$

b) ROC: $|z| > 3$

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{z^n(3z-4)}{(z-3)(z-\frac{1}{2})}\,dz$$

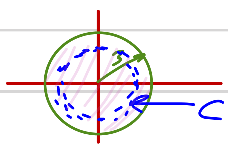


$= \text{Res}[z=3] + \text{Res}[z=\tfrac{1}{2}]$

$$= \frac{z^n(3z-4)}{z-\frac{1}{2}}\Big|_{z=3} + \frac{z^n(3z-4)}{z-3}\Big|_{z=\frac{1}{2}} = 2(3)^n + (\tfrac{1}{2})^n.$$

2- For $n < 0$

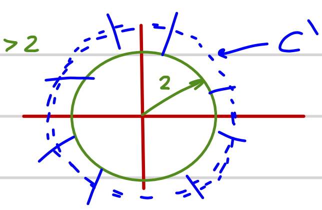
$|z| > 3 \longrightarrow 0 < |w| < \frac{1}{3}$



$x(n) = 0$

$\Big\}\ X(n) = [2(3)^n + (\tfrac{1}{2})^n]\,u(n)$

c) $x(n) = 0$ for $n \geq 0$ $\qquad |z| < 0.5 \Rightarrow |w| > 2$



$X(n) = \text{Res}[w = \tfrac{1}{3}] + \text{Res}[w=2]$

$= -[(\tfrac{1}{2})^n + 2(3)^n]\,u(-n-1)$

Correlation of 2 sequences:

If $x_1(n) \xleftrightarrow{z} X(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$

$$r_{x_1x_2}(\ell) = x_1(n) * x_2(n) \xleftrightarrow{z} R_{x_1x_2}(z) = X_1(z)\,X_2(z^{-1}).$$

ROC: intersection of ROCs of $X_1(z)$ & $X_2(z^{-1})$

EXA: find the autocorrelation of:-

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n)$$

Solution

$|z| > \frac{1}{2}$ $\quad |z| < 2 \quad$ ROC

$$R_{xx}(z) = X(z)\,X(z^{-1}) = \frac{1}{1-\frac{1}{2}z^{-1}} \times \frac{1}{1-\frac{1}{2}z} \qquad \tfrac{1}{2} < |z| < 2$$

$$\frac{R_{xx}(z)}{z} = \frac{4/3}{z-0.5} - \frac{4/3}{z-2} \Rightarrow R_{xx}(z) = \frac{4/3}{1-0.5z^{-1}} - \frac{4/3}{1-2z^{-1}} \qquad \tfrac{1}{2} < |z| < 2$$

$$\therefore\ r_{xx}(\ell) = \tfrac{4}{3}(0.5)^{\ell}u(\ell) + \tfrac{4}{3}(2)^{\ell}u(-\ell-1).$$

Wednesday 3/20/13 8:06 AM

One-sided z-transform:

ex: $y(n) = \frac{1}{2} y(n-1) + x(n)$

$n=0 \longrightarrow y(0) = \frac{1}{2} y(-1) + \overset{1}{x(0)} = 1$

$$\boxed{X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}}$$

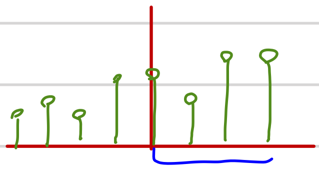
Time - delay property:-

if $x(n) \xleftrightarrow{z^+} X^+(z)$

then $x(n-k) \xleftrightarrow{z^+} z^{-k} X^+(z) + [x(-k) + x(-k+1) z^{-1} + x(-k+2) z^{-2} + \cdots + x(-1) z^{-k+1}]$

$\xleftrightarrow{(k=3)} z^{-3} X^+(z) + x(-3) + x(-2) z^{-1} + x(-1) z^{-2}$

Because : $X^+(z) = X(z)\big|_{u(n)x(n)}$



Time advance property:-

if $x(n) \xleftrightarrow{z^+} X^+(z)$

then $x(n+k) \xleftrightarrow{z^+} z^k X^+(z) - x(0) - x(1) z - x(2) z^2 - \cdots - x(k-1) z^{k-1}$

EXA: find the unit step response of the system

$$y(n) = 0.9 y(n-1) - 0.81 y(n-2) + x(n)$$

With initial conditions $y(-1) = y(-2) = 1$

Solution

Take one-sided z-tr.

$$Y^+(z) = 0.9 [z^{-1} Y(z) + y(-1)] - 0.81 [z^{-2} Y^+(z) + y(-2) + y(-1) z^{-1}]$$

$$\Rightarrow Y^+(z) [1 - 0.9 z^{-1} + 0.81 z^{-2}] = X(z) + 0.9 \times 1 - 0.81 \times 1 - 0.81 \times 1 \times z^{-1}$$

$$= 0.09 - 0.81 z^{-1} + \frac{1}{1 - z^{-1}}$$

$$\therefore Y^+(z) = \frac{1.099}{1 - z^{-1}} + \frac{0.568 + j0.445}{1 - 0.9 e^{j\pi/3} z^{-1}} + \frac{0.568 - j0.445}{1 - 0.9 e^{-j\pi/3} z^{-1}}$$

$$\Rightarrow y(n) = 1.099 u(n) + 1.44 (0.9)^n \cos\left(\frac{\pi}{3} n + 38^\circ\right) u(n)$$

EXA: solve $y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$

for $x(n) = 4^n u(n)$

With initial conditions $y(-1) = 1$, $y(-2) = 2$

Solution

$\{ x^+(z) : \text{causal} \Rightarrow x(n-k) \xleftrightarrow{z^+} z^{-k} x^+(z) + 0 \}$

$$\frac{1}{1-4z^{-1}}$$

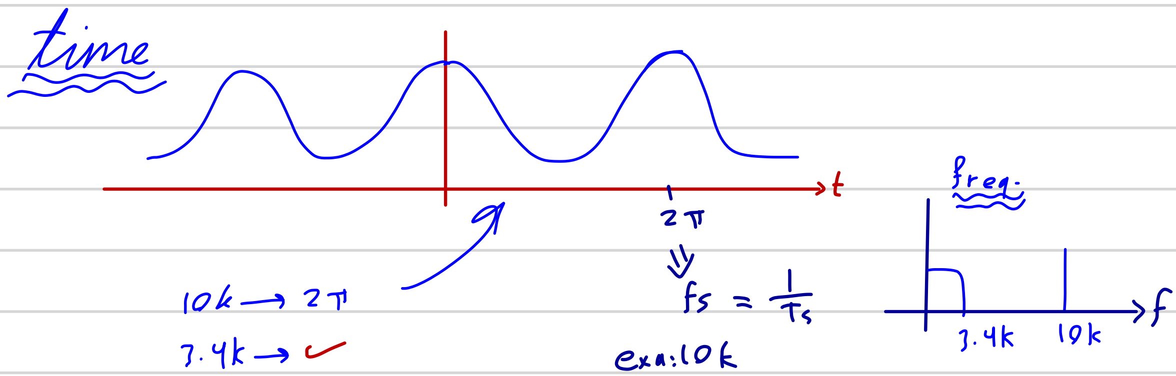
$$y^+(z) - 3\left[z^{-1} y^+(z) + y(-1)\right] - 4\left[z^{-2} y^+(z) + y(-2) + y(-1) z^{-1}\right] = x^+(z) + 2 z^{-1} x^+(z)$$

$$\Rightarrow y^+(z) = \frac{4.8}{4} \frac{4z^{-1}}{(1-4z^{-1})^2} + \frac{10.64}{1-4z^{-1}} + \frac{1.36}{1+z^{-1}}$$

$$\therefore y(n) = 1.2\, n(4)^n u(n) + 10.64(4)^n u(n) + 1.36(-1)^n u(n)$$

Chapter
Frequency Analysis
Of
Discrete - time Signals

For continuous time signals, freq. range ($-\infty \to \infty$)
For discrete time signals, freq. range (0 , 2π) periodic.



Periodic Signals:

$x(n)$ is periodic if $x(n) = x(n+N)$ for all (n).

N: smallest integer satisfying condition.

* $x(n)$ can be rep. by N-harmonics:-

$$\boxed{\begin{aligned} x(n) &= \sum_{k=0}^{N-1} C_k\, e^{\frac{j2\pi kn}{N}} \\ C_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \end{aligned}}$$

Synthesis

Analysis

Derivation of Fourier coefficients:-

$$x(n) = \sum_{k=0}^{N-1} C_k\, e^{j2\pi kn/N}$$

Multiply both sides by $e^{-j2\pi \ell n/N}$ and sum the product from $n = (0 \to N-1)$

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi \ell n}{N}} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} C_k\, e^{j2\pi(k-\ell)n/N}$$

$$= \sum_{k=0}^{N-1} C_k \underbrace{\sum_{n=0}^{N-1} e^{j2\pi(k-\ell)n/N}}_{\frac{1-e^{j2\pi(k-\ell)N/N}}{1-e^{j2\pi(k-\ell)/N}} = \frac{1-1}{(\ )} = \begin{cases} 0 & k \neq \ell \\ N & k = \ell \end{cases}} \qquad \left\{\left(e^{j2\pi(k-\ell)/N}\right)^n\right\}$$

$$RHS = C_0 \times 0 + C_1 \times 0 + \cdots + C_\ell \times N + C_{\ell+1} \times 0 + \cdots + 0$$

$$\therefore \quad \boxed{C_\ell = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi \ell n}{N}}}$$

EXA: find Fourier coefficients for the signals:-

a) $x(n) = \cos\sqrt{2}\,\pi n$

$= \cos 2\pi \frac{\sqrt{2}}{2} n \Rightarrow (2\pi f_o n) \Rightarrow f_o = \frac{1}{\sqrt{2}}$ irrational.

Apperiodic.

b) $x(n) = \cos\frac{\pi}{3} n$

$= \cos 2\pi \frac{1}{6} n \Rightarrow f_o = \frac{1}{6} \Rightarrow \boxed{N = 6}$

$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}} = \frac{1}{6}\sum \cos\frac{\pi}{3} n\, e^{-j\frac{2\pi k n}{6}} \Rightarrow \frac{e^{j\frac{2\pi}{6}n} + e^{-j\frac{2\pi}{6}n}}{2} \longrightarrow \sum C_k e^{j\frac{2\pi k n}{N}}$

$= \frac{1}{2} e^{j2\pi\frac{1}{6}n} + \frac{1}{2} e^{j2\pi(\frac{-1}{6})n}$

$C_1 = \frac{1}{2}$ $\qquad C_{-1} = \frac{1}{2}$

$\therefore \{C_0, C_1, C_2, \dots, C_{N-1}\}$ ⟵ sufficient. (C_k is periodic)

C_1 ✓

$C_{-1} = C_{-1+N} = C_{-1+6} = C_5$

$\therefore \{\; 0 \quad \frac{1}{2} \quad 0 \quad 0 \quad 0 \quad \frac{1}{2} \;\}$

$\quad\;\; C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5$

Wednesday 3/27/13 8:05 AM

EXA: $x(n) = 2 + 2\cos\frac{\pi}{4}n + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi}{4}n$

Solution

$$x(n) = \underset{\substack{\downarrow \\ N_1 = 1}}{2} + 2\cos\frac{\pi}{4}n + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi}{4}n$$

$2\cos\frac{\pi n}{4} = 2\cos 2\pi\frac{1}{8}n \longrightarrow f_0 = \frac{1}{8}$, $\boxed{N_2 = 8}$

$\cos\frac{\pi n}{2} = \cos 2\pi\frac{1}{4}n \longrightarrow f_0 = \frac{1}{4}$, $\boxed{N_3 = 4}$

$\cos\frac{3\pi}{4}n = \cos 2\pi\frac{3}{8}n \longrightarrow f_0 = \frac{3}{8}$, $\boxed{N_4 = 8}$

$$N = LCM(N_1, N_2, N_3, N_4) = LCM(1, 4, 8) = 8$$

$$x(n) = 2 + 2\,\frac{e^{j\frac{\pi n}{4}} + e^{-j\frac{\pi n}{4}}}{2} + \frac{e^{j\frac{\pi n}{8}} + e^{-j\frac{\pi n}{8}}}{2} + \frac{1}{2}\,\frac{e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}}{2} \qquad \left\{\sum e^{j\frac{2\pi k n}{N}}\right\}$$

$$= \underset{C_0}{2} + \underset{C_1}{e^{j2\pi\frac{1}{8}n}} + \underset{C_{-1}}{e^{-j2\pi\frac{1}{8}n}} + \underset{C_2}{\frac{1}{2}e^{j\frac{2\pi n}{4}\times\frac{2}{2}}} + \underset{C_{-2}}{\frac{1}{2}e^{-j\frac{2\pi n}{4}\times\frac{2}{2}}} + \underset{C_3}{\frac{1}{4}e^{j2\pi\frac{3}{8}n}} + \underset{C_{-3}}{\frac{1}{4}e^{-j2\pi\frac{3}{8}n}}$$

$C_0 = 2$

$C_1 = C_{-1} = 1 = C_{-1+8} = C_7$

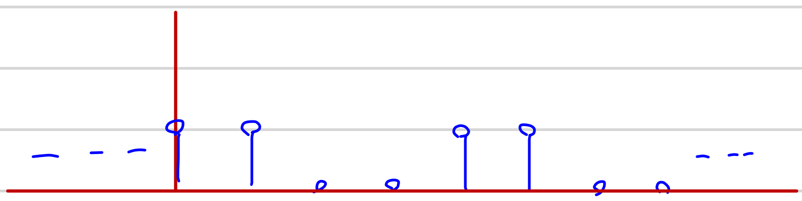
$C_2 = C_{-2} = \frac{1}{2} = C_{-2+8} = C_6$

$C_3 = C_{-3} = \frac{1}{4} = C_{-3+8} = C_5$

$$\vec{C}_k = [\underset{\substack{\uparrow \\ C_0}}{2}, \underset{\substack{\uparrow \\ C_1}}{1}, \tfrac{1}{2}, \tfrac{1}{4}, 0, \tfrac{1}{4}, \tfrac{1}{2}, \underset{\substack{\uparrow \\ C_7}}{1}]$$

EXA: $x(n) = [\underset{\substack{\uparrow \\ n=0}}{1} \quad 1 \quad 0 \quad 0]$

find Fourier coefficients.



Solution

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}} = \frac{1}{4}\left[x(0)\,e^{0} + x(1)\,e^{-j\frac{2\pi k 1}{4}} + 0 + 0\right]$$

$$= \frac{1}{4}\left[1 + 1\,e^{-j\frac{\pi k}{2}}\right] \qquad \boxed{k = 0, 1, 2}$$

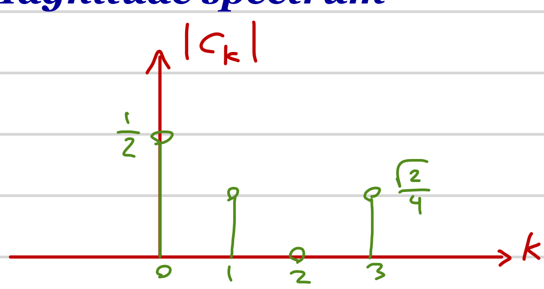
$C_0 = \frac{1}{4}[1+1] = 1/2$

$C_1 = \frac{1}{4}[1 + e^{-j\frac{\pi}{2}\times 1}] = \frac{1}{4}(1-j) = \frac{\sqrt{2}}{4}\angle -\frac{\pi}{4}$

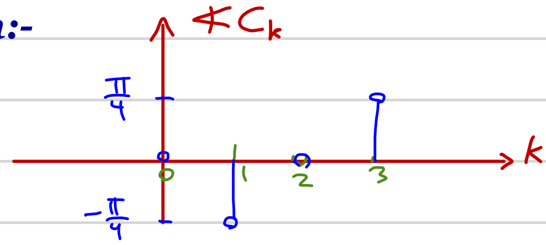
$C_2 = \frac{1}{4}[1 + e^{-j\frac{\pi}{2}\times 2}] = \frac{1}{4}(1-1) = 0$

$C_3 = \frac{1}{4}[1 + e^{-j\frac{\pi}{2}\times 3}] = \frac{1}{4}(1+j) = \frac{\sqrt{2}}{4}\angle\frac{\pi}{4}$

Magnitude spectrum



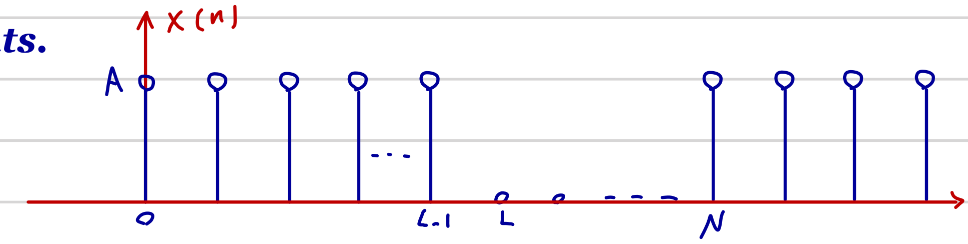
Phase spectrum:-



* If $x(n)$ is real

$\boxed{C_k^* = C_{-k}}$ real $x(n)$ $\Rightarrow$ $\left[\frac{1}{N}\sum x(n)\, e^{-j\frac{2\pi k n}{N}}\right]^* = \frac{1}{N}\sum x(n)\, e^{-j\frac{2\pi(-k)n}{N}} = C_{-k}$

EXA: find Fourier coefficients.



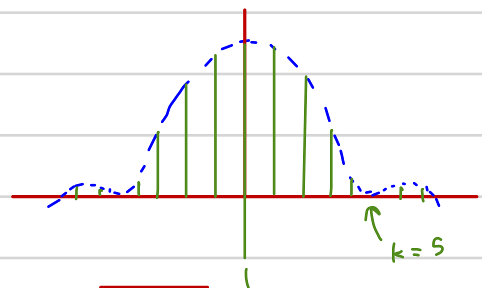
Solution

$$C_k = \frac{1}{N}\sum x(n)\, e^{-j\frac{2\pi k n}{N}} = \frac{1}{N}\sum_{n=0}^{L-1} A\left(e^{-j\frac{2\pi k}{N}}\right)^n$$

$$= \begin{cases} \dfrac{A}{N}\,\dfrac{1 - e^{-j\frac{2\pi}{N}kL}}{1 - e^{-j\frac{2\pi}{N}k}} & k \neq 0 \\[2ex] \dfrac{AL}{N} & k = 0 \end{cases}$$

$$\Rightarrow \frac{e^{-j\pi kL/N}}{e^{-j\pi k/N}} \times \frac{e^{j\pi kL/N} - e^{-j\pi kL/N}}{e^{j\pi k/N} - e^{-j\pi k/N}} \times \frac{2j}{2j} = e^{-j\pi k(L-1)/N}\,\frac{\sin(\pi kL/N)}{\sin(\pi k/N)}$$

$$\therefore\ C_k = \begin{cases} \dfrac{AL}{N} & k = 0 \\[2ex] \dfrac{A}{N}\, e^{-j\pi k(L-1)/N}\,\dfrac{\sin(\pi kL/N)}{\sin(\pi k/N)} & k \neq 0 \end{cases}$$



$\left(\begin{matrix} N = 50 \\ L = 10 \end{matrix}\right)$

$$\frac{\pi k L}{N} = \pi \Rightarrow \boxed{k = \frac{N}{L}}$$

Power spectral density (periodic signals):

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 = \frac{1}{N}\sum x(n)\, x^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\left[\sum_{k=0}^{N-1} C_k^*\, e^{-j\frac{2\pi k n}{N}}\right]$$

$$= \sum_{k=0}^{N-1} C_k^*\left[\frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}}\right] = \sum_{k=0}^{N-1} C_k^*\, C_k$$

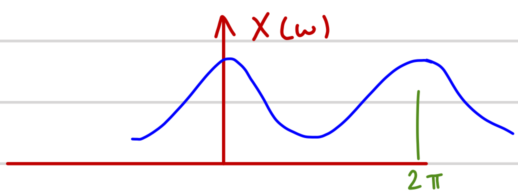
$$= \boxed{\sum_{k=0}^{N-1} |C_k|^2}$$ **Parseval's eqn.**

Fourier Transform for Aperiodic signals:

$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)\, e^{j\omega n}\, d\omega$	$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$

$$X(\omega + 2\pi) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega+2\pi)n} = \sum x(n)\, \cancel{e^{-j2\pi n}}\, e^{-j\omega n} = \sum x(n)\, e^{-j\omega n} = X(\omega)$$

$\therefore$ $X(\omega)$ is periodic with period $= 2\pi$.



Derivation of F.T.:

$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$, multiply by $e^{j\omega m}$ and integrate over $(-\pi, \pi)$

$$\Rightarrow \int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega = \int_{-\pi}^{\pi} \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right] e^{j\omega m} d\omega = \sum_{n=-\infty}^{\infty} x(n) \Big[\underbrace{\int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega}_{I} \Big]$$

$$I = \frac{e^{j\omega(m-n)}}{j(m-n)} \Big|_{-\pi}^{\pi} = 0 \qquad m \neq n$$

$$= \frac{e^{j\pi(m-n)} - e^{-j\pi(m-n)}}{j(m-n)} \times \frac{2j}{2j} = \frac{\sin \pi(m-n)}{j(m-n)} \Rightarrow I = \begin{cases} 2\pi & m = 0 \\ 0 & m \neq 0 \end{cases}$$

$$\therefore \int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega = x(0) \times 0 + x(1) \times 0 + \cdots + x(m) 2\pi - - -$$

$$x(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega$$

Convergence of F.T.:

1) if x(n) is absolutely summable

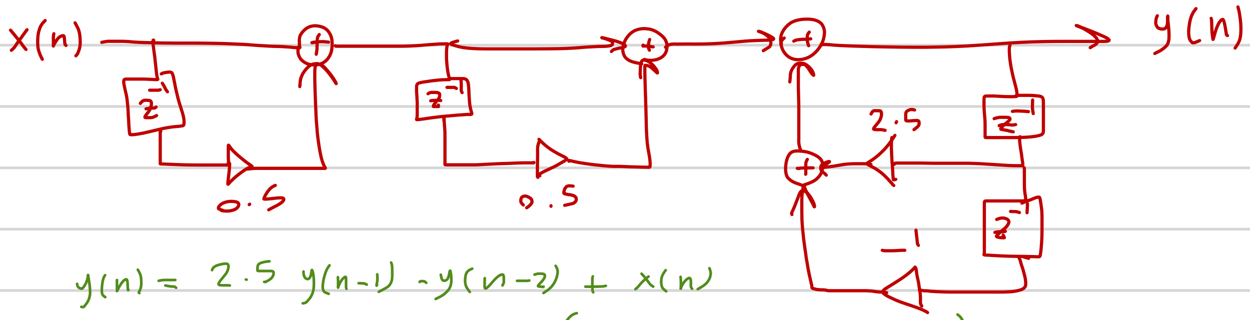
$$\sum_{n=-\infty}^{\infty} |X(\omega)| < \infty$$

then $X(\omega)$ exists

$$\Rightarrow \lim_{N \to \infty} \left[X(\omega) - \sum_{n=-N}^{N} x(n) e^{-j\omega n} \right] = 0$$

Monday 4/1/13 8:25 AM

1st exam-Q3)



$$y(n) = 2.5\,y(n-1) - y(n-2) + x(n) + 0.5x(n-1) + 0.5\left(x(n-1) + 0.5\,x(n-2)\right)$$

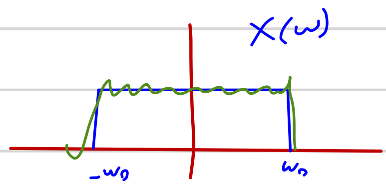
2- if x(n) is square summable:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty \quad \text{(finite energy)}$$

Mean-squared convergence

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |X(\omega) - X_N(\omega)|^2\, d\omega \; ; \quad X_N(\omega) = \sum_{n=-N}^{N} x(n)\, e^{-j\omega n}$$

like : rect.



$$x(n) = \frac{\sin \omega_0 n}{\pi n}$$

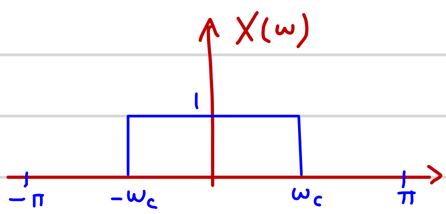
3- if x(n) has a pole on the unit circle.

$$x(n) = u(n)$$

$$X(z) = \frac{1}{1 - z^{-1}}$$

$$X(\omega) = \begin{cases} X(z)\big|_{z=e^{j\omega}} = \frac{1}{1-e^{-j\omega}} & \omega \neq 0, 2\pi k \\ \text{impulse} & \omega = 0 \end{cases}$$

EXA: find x(n) for X(w)



Solution

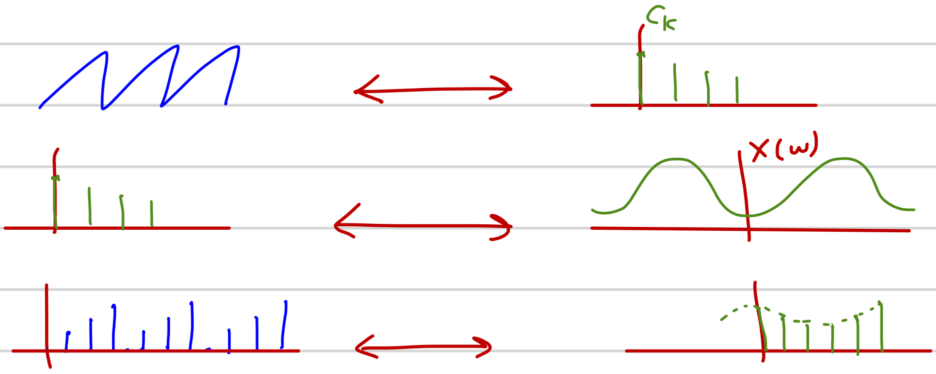
$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1\, e^{j\omega n} d\omega = \frac{1}{2\pi} \frac{e^{j\omega n}}{jn} \Big|_{-\omega_c}^{\omega_c} = \frac{1}{j2\pi n}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right)$$

$$= \frac{\sin \omega_c n}{\pi n} \times \frac{\omega_c}{\omega_c} = \frac{\omega_c}{\pi} \operatorname{Sinc} \omega_c n$$

$$\Rightarrow x(n) = \begin{cases} \frac{\omega_c}{\pi} \operatorname{Sinc} \omega_c n & n \neq 0 \\ \frac{\omega_c}{\pi} & n = 0 \end{cases}$$

Periodicity in one domain <---> discretization in the other domain

Like:-

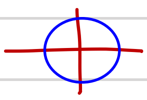


$\Rightarrow$ discr. & periodic.

Relation of Fourier transform to z-transform:

$$X(z) = \sum x(n)\, z^{-n} \qquad z = e^{s} = e^{\sigma + j\omega}$$

$$X(\omega) = X(z)\Big|_{z=e^{j\omega}}$$



If ROC of X(z) includes the unit circle.

Energy density spectrum [Sxx(w)]:

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

ESD

$$S_{xx}(\omega) = |X(\omega)|^2 = X(\omega)\, X^*(\omega)$$

For real x(n):-

$$X^*(\omega) = X(-\omega)$$

$$\Rightarrow \left.\begin{aligned} S_{xx}(\omega) &= X(\omega)\, X(-\omega) \\ S_{xx}(-\omega) &= X(-\omega)\, X(\omega) \end{aligned}\right\} \quad S_{xx}(\omega) = S_{xx}(-\omega)$$

(even fn.)

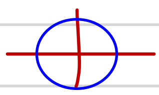
EXA: find F.T. of $x(n) = a^n\, u(n) \qquad |a| < 1$

Solution

$$X(\omega) = \sum x(n)\, e^{-j\omega n}$$

super $\rightarrow$ $X(z) = \dfrac{1}{1 - a z^{-1}} \qquad |z| > |a|$

ROC includes unit circle



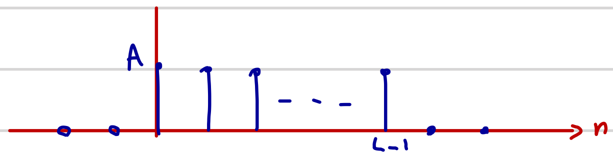
$$\Rightarrow X(\omega) = X(z)\Big|_{z=e^{j\omega}} = \frac{1}{1 - a e^{-j\omega}}$$

Now $S_{xx}(\omega) = X(\omega)\, X^*(\omega)$

$$= \frac{1}{1 - a e^{-j\omega}} \times \frac{1}{1 - a e^{+j\omega}} = \frac{1}{1 + a^2 - a e^{-j\omega} - a e^{j\omega}} = \frac{1}{1 + a^2 - 2a\cos\omega}$$

EXA: Find F.T. of

$$x(n) = \begin{cases} A & 0 \leq n \leq L-1 \\ 0 & \text{———} \end{cases}$$



Solution

$$X(z) = \begin{cases} AL & z = 1 \\ A\,\dfrac{1 - z^{-L}}{1 - z^{-1}} & z \neq 1 \end{cases}$$

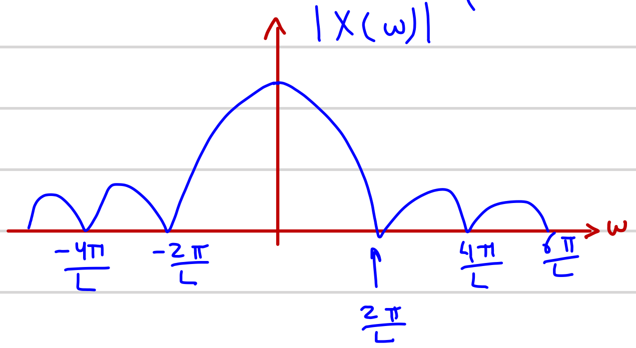
ROC : All z-plane except $z=0$

$\Rightarrow$ includes $|z| = 1$ (unit circle).

$$\Rightarrow \quad X(\omega) = \begin{cases} AL & \omega = 0 \\ A\,\dfrac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} & \omega \neq 0 \end{cases}$$

$$\to A\,\frac{e^{-j\frac{\omega L}{2}}}{e^{-j\frac{\omega}{2}}} \times \frac{e^{j\frac{\omega L}{2}} - e^{-j\frac{\omega L}{2}}}{e^{j\omega/2} - e^{-j\omega/2}} \times \frac{2j}{2j}$$

$$= \begin{cases} Ae^{-j\frac{\omega}{2}(L-1)}\,\dfrac{\sin(\omega L/2)}{\sin(\omega/2)} & \omega \neq 0 \\ AL & \omega = 0 \end{cases}$$

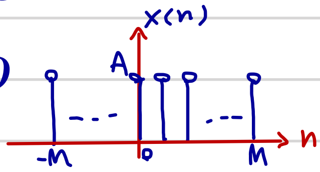


1st null :-

$$\frac{\omega L}{2} = \pi \Rightarrow \omega = \frac{2\pi}{L}.$$

Wednesday 4/3/13 8:06 AM

EXA: find F.T. Of x(n)



Solution

$$X(z) = A\,\frac{1 - z^{-(2M+1)}}{1 - z^{-1}}\, z^{M} \qquad \text{ROC: All z-plane except } z=0,\ z=\infty \text{ (even it's a limiting case)}$$

$$= A\,\frac{\cancel{z^{-M-\frac{1}{2}}}}{\cancel{z^{-1/2}}}\,\frac{z^{M+\frac{1}{2}} - z^{-M-\frac{1}{2}}}{z^{1/2} - z^{-1/2}}\,\cancel{z^{M}} = A\,\frac{z^{M+\frac{1}{2}} - z^{-M-\frac{1}{2}}}{z^{1/2} - z^{-1/2}}$$

$$X(\omega) = X(z)\big|_{z=e^{j\omega}} = A\,\frac{e^{j\omega(M+\frac{1}{2})} - e^{-j(M+\frac{1}{2})}}{e^{j\omega/2} - e^{-j\omega/2}} \times \frac{2j}{2j} = A\,\frac{\sin(M+\frac{1}{2})\omega}{\sin(\omega/2)}$$

EXA: find F.T. of x(n)

$$x(n) = a^{|n|} \qquad |a| < 1$$

Solution

$$x(n) = a^{n}u(n) + a^{-n}u(-n-1)$$

$$X(z) = \frac{1}{1 - az^{-1}} - \frac{1}{1 - \frac{1}{a}z^{-1}} \qquad \text{ROC: } |a| < |z| < \frac{1}{|a|} \implies \text{includes } |z| = 1$$

$$X(\omega) = X(z)\big|_{z=e^{j\omega}} = \frac{1}{1 - ae^{-j\omega}} - \frac{1}{1 - \frac{1}{a}e^{-j\omega}} = \frac{(a - a^{-1})e^{-j\omega}}{1 + e^{-j2\omega} - (a + a^{-1})e^{-j\omega}} \times \frac{ae^{j\omega}}{ae^{j\omega}}$$

$$= \frac{a^{2} - 1}{ae^{j\omega} + ae^{-j\omega} - (a^{2} + 1)} = \frac{1 - a^{2}}{1 + a^{2} - 2a\cos\omega}$$

Symmetry properties of F.T.:

Real & even x(n) <---> Real & even X(w)

$$X(\omega) = \sum x(n)e^{-j\omega n} = \underbrace{\sum_{n=-\infty}^{\infty} \underbrace{x(n)}_{\text{even}}\,\underbrace{\cos\omega n}_{\text{even}}}_{X_R(\omega)} - j\underbrace{\sum_{n=-\infty}^{\infty} \underbrace{x(n)}_{\text{even}}\,\underbrace{\sin\omega n}_{\text{odd}}}_{X_I(\omega)}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\cos\omega n$$

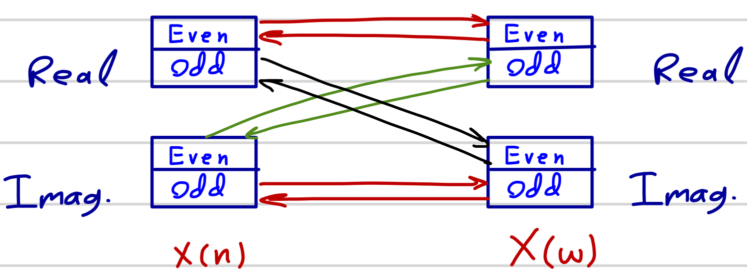
$\longrightarrow$ X(w) is real, X(w) is even in (w).

* X(w): Real & even:-

$$\longrightarrow x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\,d\omega = \frac{1}{2\pi}\Big[\underbrace{\int_{-\pi}^{\pi} X_R(\omega)\cos\omega n\,d\omega}_{X_R(n)} + \underbrace{\int_{-\pi}^{\pi} \underbrace{X_R(\omega)}_{\substack{\text{even}\\ \text{in } \omega}}\,\underbrace{\sin(\omega n)}_{\substack{\text{odd}\\ \text{in } \omega}}\,d\omega}_{X_I(n)}\Big]$$

∴ X(n) is real & even

Summary:-



$$X(\omega) = \sum x(n) e^{-j\omega n}$$
$$X(-\omega) = \sum x(n) e^{j\omega n}$$
$$X^*(-\omega) = \sum x^*(n) e^{-j\omega n}$$

Signal	F.T.
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$j x_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$

Real Signals

real $x(n)$:

$X(\omega) = X^*(-\omega)$

$X_R(\omega) = X_R(-\omega)$

$X_I(\omega) = -X_I(-\omega)$

$|X(\omega)| = |X(-\omega)|$

$\measuredangle X(\omega) = -\measuredangle X(-\omega)$

$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \longleftrightarrow X_R(\omega)$

$x_o(n) = \frac{1}{2}[x(n) - x(-n)] \longleftrightarrow jX_I(\omega)$

Properties of F.T.:

1) Linearity:-

If $x_1(n) \longleftrightarrow X_1(\omega)$, $x_2(n) \longleftrightarrow X_2(\omega)$ then $a_1 x_1(n) + a_2 x_2(n) \longleftrightarrow a_1 X_1(\omega) + a_2 X_2(\omega)$

2) Time shift:-

If $x(n) \longleftrightarrow X(\omega)$ then $x(n-k) \longleftrightarrow e^{-j\omega k} X(\omega)$

3) Time reversal:-

If $x(n) \longleftrightarrow X(\omega)$ then $x(-n) \longleftrightarrow X(-\omega)$

proof:

$$F[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) e^{-j\omega n} \underset{m=-n}{=} \sum_{m=-\infty}^{\infty} x(m) e^{j\omega n} = \sum x(m) e^{-j(-\omega)m} = X(-\omega)$$

4) Convolution Theorem:-

If $x_1(n) \longleftrightarrow X_1(\omega)$, $x_2(n) \longleftrightarrow X_2(\omega)$ then $x(n) = x_1(n) * x_2(n) \longleftrightarrow X(\omega) = X_1(\omega) X_2(\omega)$

EXA: find the convolution of $x_1(n) = x_2(n) = \{1, \underset{\uparrow n=0}{1}, 1\}$

Solution

$$x(n) = x_1(n) * x_2(n) \Rightarrow X_1(\omega) = \sum x(n) e^{-j\omega n} = 1e^{-j\omega(-1)} + 1e^{0} + 1e^{-j\omega(1)} = 1 + 2\cos\omega = X_2(\omega)$$

$$\longrightarrow X(\omega) = X_1(\omega) X_2(\omega) = (1 + 2\cos\omega)^2 = 1 + 4\cos\omega + \underbrace{4\cos^2\omega}_{\frac{1}{2}+\frac{1}{2}\cos 2\omega} = 3 + 4\cos\omega + 2\cos 2\omega$$

$$= 3e^{j0} + 2e^{j\omega} + 2e^{-j\omega} + 1e^{j2\omega} + 1e^{-j2\omega}$$

$$\longrightarrow x(n) = \{1, 2, \underset{\uparrow n=0}{3}, 2, 1\}$$

5) The correlation Theorem:-

If $x_1(n) \longleftrightarrow X_1(\omega)$ then $r_{x_1 x_2}(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(k-n) \longleftrightarrow S_{x_1 x_2}(\omega) = \boxed{X_1(\omega) X_2(-\omega)}$

$x_2(n) \longleftrightarrow X_2(\omega)$

↳ cross-energy density spectrum.

For real x(n)

$$r_{xx}(\ell) \longleftrightarrow X(\omega) X^*(\omega) = |X(\omega)|^2$$

$$r_{xx}(\ell) \longleftrightarrow S_{xx}(\omega)$$

6) Freq. shift:-

If $x(n) \longleftrightarrow X(\omega)$ then $e^{j\omega_0 n} x(n) \longleftrightarrow X(\omega - \omega_0)$

* proof :-

$$\sum x(n) e^{j\omega_0 n} e^{-j\omega n} = \sum x(n)\, e^{-j(\omega-\omega_0)n} \checkmark$$

7) Modulation Theorem:-

If $x(n) \longleftrightarrow X(\omega)$ then $x(n)\cos(\omega_0 n) \longleftrightarrow \frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$

8) Parseval's Theorem:-

If $x_1(n) \longleftrightarrow X_1(\omega)$ then $\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega) X_2^*(\omega)\, d\omega$

$x_2(n) \longleftrightarrow X_2(\omega)$

* proof :-

$$\text{R.H.S.} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega) X_2^*(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} x_1(n)\, e^{-j\omega n}\right] X_2^*(\omega)\, d\omega = \sum x_1(n) \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(\omega)\, e^{-j\omega n}\, d\omega}_{x_2^*(n)}$$

$$= \sum x_1(n)\, x_2^*(n)$$

$$\Rightarrow \quad = \sum x_1(n)\, X_2^*(n)$$

Multiplication of two sequences :-

- If $x_1(n) \longleftrightarrow X_1(\omega)$

  $x_2(n) \longleftrightarrow X_2(\omega)$

then $x_3(n) = x_1(n)\, x_2(n) \longleftrightarrow X_3(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)\, X_2(\omega-\lambda)\, d\lambda$

- Differentiation in freq-domain

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$j \times \left(\frac{dX(\omega)}{d\omega} = \sum x(n)\, e^{-j\omega n}(-jn) \right)$$

$$\Rightarrow j\frac{dX(\omega)}{d\omega} = \sum [n\, x(n)]\, e^{-j\omega n}$$

$$\Rightarrow \boxed{n\, x(n) \longleftrightarrow j\frac{dX(\omega)}{d\omega}}$$

---

ex) Find the spectrum of the periodic signal:-

$x(n) = \{ \ldots\ldots -1, 2, \underset{\uparrow n=0}{1}, 2, -1, 0, -1, 2, \underset{\uparrow}{1}, \ldots\ldots \}$

$N = 6$

$\Rightarrow$ Sol. $c_k = \frac{1}{6}\sum_{n=0}^{5} x(n)\, e^{-j\frac{2\pi}{N}nk}$

$= \frac{1}{6}\left[1 + 2e^{-j\frac{2\pi}{6}k} + -1\, e^{-j\frac{2\pi}{6}2k} + 0 + -1\, e^{-j\frac{2\pi}{6}4k} + 2e^{-j\frac{2\pi}{6}5k} + \cancel{1\, e^{\ldots}} \right]$

$\Rightarrow c_k = \frac{1}{6}\left[1 + 2e^{-j\frac{\pi}{3}k} - 1\, e^{-j\frac{2}{3}\pi k} - e^{-j\frac{4\pi}{3}k} + 2e^{-j\frac{5\pi}{3}k} \right]$

$e^{-j\frac{4\pi}{3}k} \rightarrow e^{j\frac{6\pi}{3}k}$, $\quad e^{-j\frac{5\pi}{3}k} \rightarrow e^{j\frac{\pi}{3}k}$

$c_k = \frac{1}{6}\left[1 + 4\cos(\frac{\pi}{3}k) - \frac{4}{2}\cos(\frac{2\pi}{3}k) \right]$

Note $\rightarrow$ $x(n)$ is real & even $\rightarrow$ spectrum $\rightarrow$ symmetric as in term of cosine

$c_0 = \frac{1}{6}[1 + 4\cos(0) - 2\cos(0)] = \frac{1}{2}$

$c_1 = \frac{1}{6}[1 + 4\cos(\frac{\pi}{3}) - 2\cos(\frac{2\pi}{3})] = 2/3$

$c_2 = 0$

$c_3 = -5/6$

$c_4 = 0$, $\quad c_5 = 2/3$

$\rightarrow c_k = [\frac{1}{2} \quad \frac{2}{3} \quad 0 \quad \frac{-5}{6} \quad 0 \quad \frac{2}{3}]$

$x(n)$ in term of synthesis term

$x(n) = \sum_{k=0}^{N-1} c_k\, e^{j\frac{2\pi}{N}kn}$

$= \frac{1}{2} + \frac{2}{3}e^{j\frac{2\pi}{6}n} - \frac{5}{6}e^{j\pi n} + \frac{2}{3}e^{j\frac{10\pi}{6}n}$

$\rightarrow$ it's useful to find the output of the periodic function..

---

ex):- Find $x(n)$ (with period $= 6$) gives fourier coefficients?

$c_k = \{ \ldots 1, \frac{1}{2}, 0, \frac{1}{2}, 1, \underset{\uparrow n=0}{2}, 1, \frac{1}{2}, 0, \frac{1}{2}, \underset{\uparrow n=5}{1}, \ldots \}$

$\Rightarrow$ ~~even symmetry~~ even function.

$\Rightarrow x(n) = \sum_{n=0}^{5} c_k\, e^{j\frac{2\pi}{N}kn}$

$= 2 + e^{j\frac{\pi}{3}n} + \frac{1}{2}e^{j\frac{2\pi}{3}n} + \frac{1}{2}e^{j\frac{4\pi}{3}n} + e^{j\frac{2\pi}{3}\times 5 n}$

$e^{j\frac{4\pi}{3}n} \rightarrow e^{-j\frac{2\pi}{3}n}$, $\quad e^{j\frac{2\pi}{6}\times 5n} \rightarrow e^{-j\frac{\pi}{3}n}$

$= 2 + 2\cos(\frac{\pi}{3}n) + \cos(\frac{2\pi}{3}n).$

---

ex:- Find F.T of

$$x(n) = 2^n\, u(-n)$$

Sol $\quad x(n) = (\frac{1}{2})^{-n}\, u(-n)$

$(\frac{1}{2})^n u(n) \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}}$, R.O.C $|z| > \frac{1}{2}$ ((includes unit circle))

$w(n) = (\frac{1}{2})^n u(n) \xleftrightarrow{F} \frac{1}{1-\frac{1}{2}e^{-j\omega}} = W(\omega)$

$\Rightarrow X(\omega) = W(-\omega) = \frac{1}{1-\frac{1}{2}e^{j\omega}}$

---

ex:- Find Fourier transform of

$$x(n) = (n+1)\,\alpha^n\, u(n), \quad |\alpha| < 1$$

Sol. $\alpha^n u(n) \xleftrightarrow{z} \frac{1}{1-\alpha z^{-1}}$, $\quad |z| > |\alpha|$ (include a unit circle)

$\rightarrow \alpha^n u(n) \xleftrightarrow{F} \frac{1}{1-\alpha e^{-j\omega}}$

$n\, \alpha^n u(n) \longleftrightarrow j\frac{dX(\omega)}{d\omega}$

$= j\left[\frac{d}{d\omega}\left[(1-\alpha e^{-j\omega})^{-1} \right] \right]$

$$\rightarrow \quad j(-1)\left(1-\alpha e^{-j\omega}\right)^{-2}\alpha\left(j\alpha e^{-j\omega}\right)$$

$$= \frac{\alpha e^{-j\omega}}{\left(1-\alpha e^{-j\omega}\right)^{2}}$$

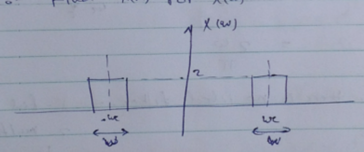
$$\Rightarrow X(\omega) = \frac{1}{1-\alpha e^{-j\omega}} + \frac{\alpha e^{-j\omega}}{\left(1-\alpha e^{-j\omega}\right)^{2}} = \frac{1}{\left(1-\alpha e^{-j\omega}\right)^{2}}$$

ex) Find $x(n)$ for $X(\omega) = \cos^2(\omega)$

Sol $= \quad X(\omega) := \sum_{-\infty}^{\infty} x(n) e^{-j\omega n}$

$$X(\omega) = \frac{1}{2} + \frac{1}{2}\cos(2\omega) = \underbrace{\frac{1}{2}}_{x(0)} + \underbrace{\frac{1}{4}}_{x(-2)} e^{j\omega(2)} + \underbrace{\frac{1}{4}}_{x(2)} e^{-j\omega(2)}$$

ex:- Find $x(n)$ for $X(\omega)$



Sol

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{W}{2}-\omega_c}^{-\omega_c+\frac{W}{2}} 2 e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_c-\frac{W}{2}}^{\omega_c+\frac{W}{2}} 2 e^{j\omega n} d\omega$$

$$= \frac{2}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\frac{W}{2}-\omega_c}^{-\omega_c+\frac{W}{2}} + \frac{2}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_c-\frac{W}{2}}^{\omega_c+\frac{W}{2}}$$

$$= \frac{2}{\pi n}\left[\frac{e^{j(-\frac{W}{2}+\omega_c)n} - e^{j\omega(-\omega_c-\frac{W}{2})}}{j2}\right]$$

$$+ \frac{2}{\pi n}\left[\frac{e^{j(\omega_c+\frac{W}{2})n} - e^{j(\omega_c-\frac{W}{2})n}}{j2}\right]$$

$$x(n) = \frac{2}{\pi n}\sin\left(\left(\omega_c+\frac{W}{2}\right)n\right) - \frac{2}{\pi n}\sin\left(\left(\omega_c-\frac{W}{2}\right)n\right) \qquad n \neq 0$$

at $n=0 \rightarrow \quad x(n) = \frac{1}{2\pi}\int_{-\omega_c-\frac{W}{2}}^{-\omega_c+\frac{W}{2}} 2\,d\omega + \frac{1}{2\pi}\int_{\omega_c-\frac{W}{2}}^{\omega_c+\frac{W}{2}} 2\,d\omega$

$$= \frac{2W}{\pi}$$

→ practicly, there is no ideal filter. → 1- find F^{-1}
2- multiply by area.

ex) Find $Y(\omega)$ in terms of $X(\omega)$

a) $y(n) = x(n) - x(n-1)$

$$Y(\omega) = X(\omega) - e^{-j\omega}X(\omega) = X(\omega)\left[1 - e^{-j\omega}\right]$$

b) $y(n) = x(n) * x(n-1)$

$$Y(\omega) = X(\omega) \cdot X(\omega) e^{-j\omega} = X^2(\omega) e^{-j\omega}$$

ex:- Given

$$x(n) = \{1, 0, -1, \underset{\substack{\uparrow \\ n=0}}{2}, 3\}$$

$X(\omega) = X_R(\omega) + j X_I(\omega)$, Find $y(n)$ whose F.T is

$$Y(\omega) = X_I(\omega) + X_R(\omega) e^{j2\omega}$$

Sol.

$$x_e(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right] \longleftrightarrow X_R(\omega)$$

$$x_e(n+2) = \frac{1}{2}\left[x(n+2) + x^*(-n+2)\right] \longrightarrow X_R(\omega) e^{j2\omega}$$

$$x_o(n) \longleftrightarrow j X_I(\omega)$$

$$-j\, x_o(n) \longleftrightarrow X_I(\omega) \quad \rightarrow \text{by Linearity.}$$

$$y(n) = x_e(n+2) - j x_o(n)$$

where $x(n) = \{1, 0, -1, \underset{\substack{\uparrow \\ n=0}}{2}, 3\}$

$$x^*(-n) = \{3, 2, -1, 0, 1\}$$

$$x_e(n) = \frac{1}{2}\left[x(n) + x^*(-n)\right] = \{\tfrac{1}{2}, 0, 1, \underset{\substack{\uparrow \\ n=0}}{2}, 1, 0, \tfrac{1}{2}\}$$

$$x_o(n) = \frac{1}{2}\{x(n) - x^*(-n)\} = \{\tfrac{1}{2}, 0, -2, \underset{\substack{\uparrow \\ n=0}}{0}, 2, 0, -\tfrac{1}{2}$$

$$x_e(n+2) = \{(\tfrac{1}{2}, \underset{\substack{\uparrow \\ n=0}}{0}, 1, 2, 1, 0, \tfrac{1}{2})\}$$

$$-j x_o(n) = \{(-\tfrac{1}{2}j, 0, 2j, \underset{\substack{\uparrow \\ n=0}}{0}, -2j, 0, \tfrac{1}{2}j)\}$$

~~$y(n) = x_e(n+2) - j x_o(n)$~~

$=$ ~~$\{-\frac{1}{2}, 0, 2.5, 0, -1, 2, 2.5, 0, \frac{1}{2}$~~

$$y(n) = \{\tfrac{1}{2}, 0, 1-\tfrac{j}{2}, 2, 1+j2, \underset{\substack{\uparrow \\ n=0}}{0}, \tfrac{1}{2}-j2, 0, j\tfrac{1}{2}\}$$

ex):- Given $x(n) = [-1, 2, \underset{\substack{\uparrow \\ n=0}}{-3}, 2, -1]$

Find: (without find fourier Transform)

a) $X(0)$

b) $X(\pi)$

c) $\int_{-\pi}^{\pi} X(\omega)\, d\omega$

Sol. $\quad X(\omega) = \sum x(n) e^{-j\omega n}$

a) $X(0) = \sum x(n) = -1 + 2 - 3 + 2 - 1 = -1$

b) $X(\pi) = \sum x(n) e^{-j\pi n}$

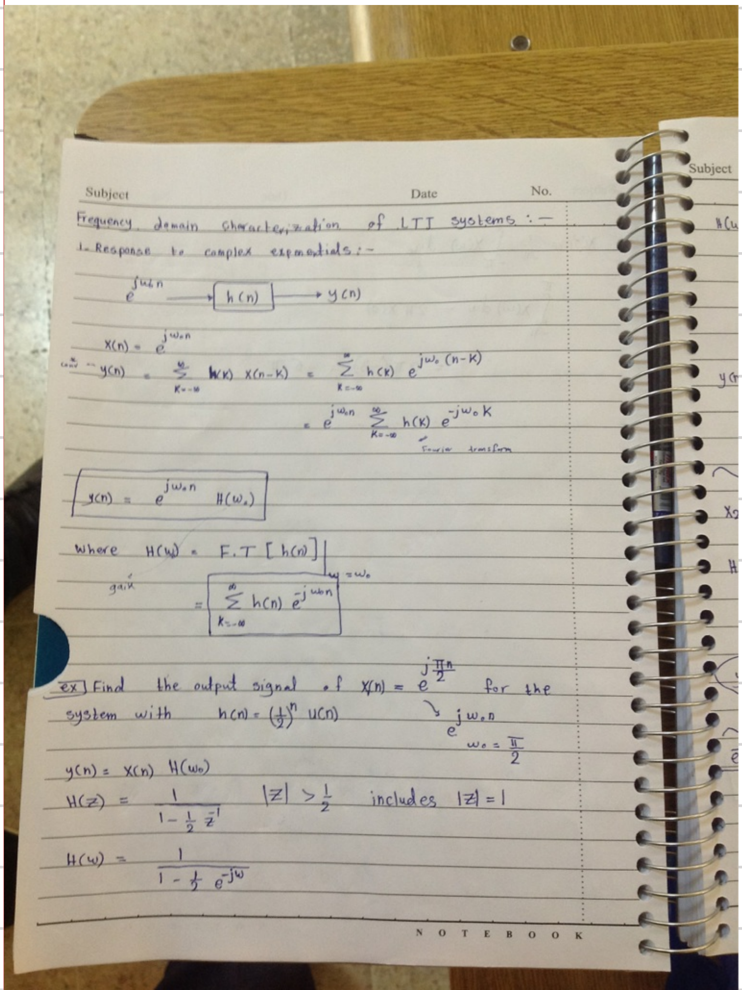
$$= \sum_{n\,\text{even}} x(n) - \sum_{n\,\text{odd}} x(n)$$

$$= [-1, -3, -1] - [2+2] = -9$$

c) $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, d\omega \Rightarrow 2\pi x(0) = -6\pi.$$

Wednesday 4/10/13 8:30 AM



Subject Date No.

Frequency domain characterization of LTI systems :-

1- Response to complex exponentials :-

$$e^{j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$x(n) = e^{j\omega_0 n}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega_0 (n-k)}$$

$$= e^{j\omega_0 n} \underbrace{\sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega_0 k}}_{\text{Fourier transform}}$$

$$\boxed{y(n) = e^{j\omega_0 n}\, H(\omega_0)}$$

where $H(\omega) = F.T\,[h(n)]\Big|_{\omega=\omega_0}$ (gain)

$$= \boxed{\sum_{k=-\infty}^{\infty} h(n)\, e^{-j\omega_0 n}}$$

ex Find the output signal of $x(n) = e^{j\frac{\pi n}{2}}$ for the system with $h(n) = \left(\frac{1}{2}\right)^n u(n)$

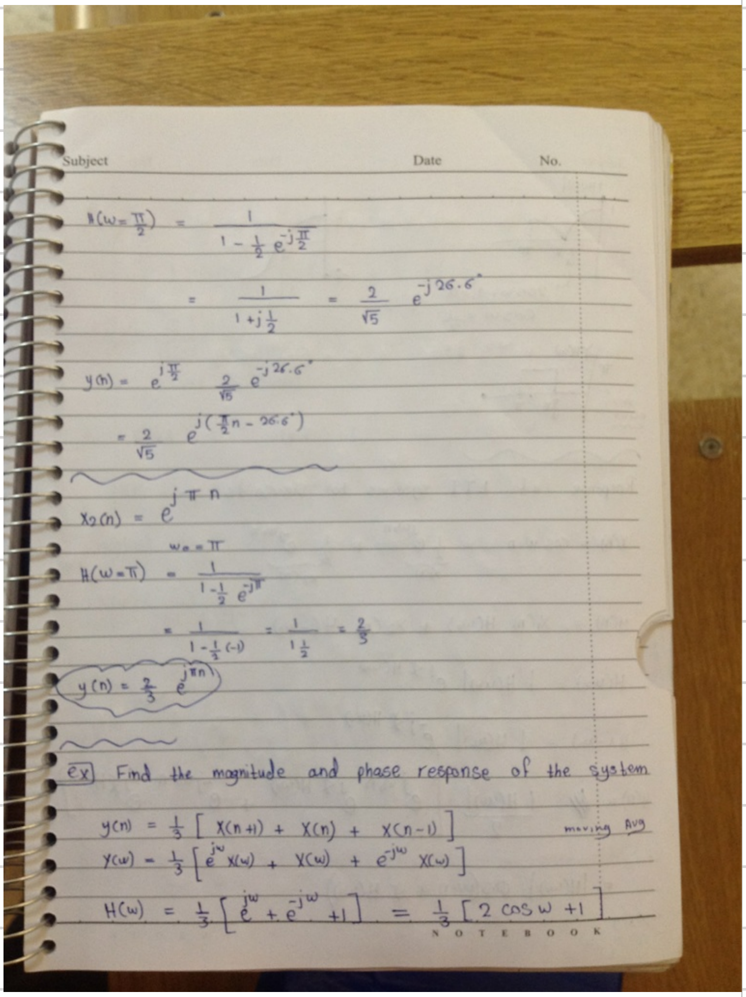
$e^{j\omega_0 n}$, $\omega_0 = \frac{\pi}{2}$

$$y(n) = x(n)\, H(\omega_0)$$

$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \quad |z| > \frac{1}{2} \quad \text{includes } |z| = 1$$

$$H(\omega) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

N O T E B O O K



Subject Date No.

$$H\left(\omega = \frac{\pi}{2}\right) = \frac{1}{1 - \frac{1}{2} e^{-j\frac{\pi}{2}}}$$

$$= \frac{1}{1 + j\frac{1}{2}} = \frac{2}{\sqrt{5}}\, e^{-j26.6^\circ}$$

$$y(n) = e^{j\frac{\pi}{2}}\, \frac{2}{\sqrt{5}}\, e^{-j26.6^\circ}$$

$$= \frac{2}{\sqrt{5}}\, e^{j\left(\frac{\pi}{2}n - 26.6^\circ\right)}$$

$$x_2(n) = e^{j\pi n}$$

$$\omega_0 = \pi$$

$$H(\omega = \pi) = \frac{1}{1 - \frac{1}{2} e^{-j\pi}}$$

$$= \frac{1}{1 - \frac{1}{2}(-1)} = \frac{1}{1\frac{1}{2}} = \frac{2}{3}$$

$$y(n) = \frac{2}{3}\, e^{j\pi n}$$

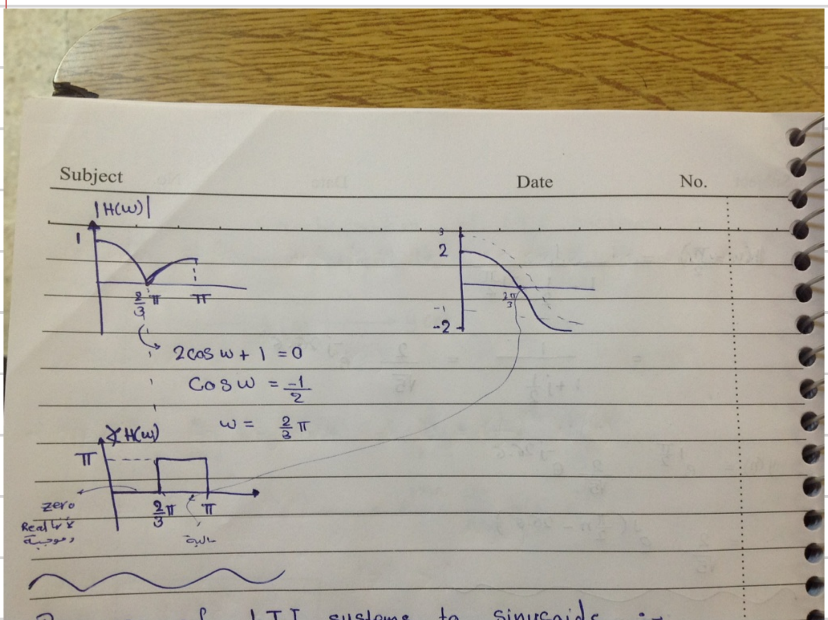
ex Find the magnitude and phase response of the system

$$y(n) = \frac{1}{3}\left[x(n+1) + x(n) + x(n-1)\right] \quad \text{moving Avg}$$

$$Y(\omega) = \frac{1}{3}\left[e^{j\omega} X(\omega) + X(\omega) + e^{-j\omega} X(\omega)\right]$$

$$H(\omega) = \frac{1}{3}\left[e^{j\omega} + e^{-j\omega} + 1\right] = \frac{1}{3}\left[2\cos\omega + 1\right]$$

N O T E B O O K



Subject Date No.

$|H(\omega)|$

1, $\frac{2}{3}\pi$, π

2, -2, $\frac{2\pi}{3}$

$$2\cos\omega + 1 = 0$$

$$\cos\omega = \frac{-1}{2}$$

$$\omega = \frac{2}{3}\pi$$

$\measuredangle H(\omega)$

π, $\frac{2}{3}\pi$, π

zero

Real

Response of LTI systems to sinusoids :-

Response of LTI systems to sinusoids:

$$x(n) = \cos\omega_o n = \underbrace{\tfrac{1}{2} e^{j\omega_o n}}_{x_1(n)} + \underbrace{\tfrac{1}{2} e^{-j\omega_o n}}_{x_2(n)}$$

$$y(n) = x_1(n)\, H(\omega_o) + x_2(n)\, H(-\omega_o)$$

$$\left.\begin{aligned} H(\omega_o) &= |H(\omega_o)|\, e^{j\measuredangle H(\omega_o)} \\ H(-\omega_o) &= |H(\omega_o)|\, e^{-j\measuredangle H(\omega_o)} \end{aligned}\right\} \quad y(n) = \frac{|H(\omega_o)|}{2}\left[e^{j\omega_o n} e^{j\measuredangle H(\omega_o)} + e^{-j\omega_o n} e^{-j\measuredangle H(\omega_o)}\right]$$

$$\longrightarrow \boxed{y(n) = |H(\omega_o)| \cos(\omega_o n + \measuredangle H(\omega_o))}$$

* for $x(n) = \sin\omega_o n$

$$\longrightarrow \boxed{y(n) = |H(\omega_o)| \sin(\omega_o n + \measuredangle H(\omega_o))}$$

Effect of | H(wo) |:-

1- Gain $|H(\omega_o)|$

2- Phase $\measuredangle H(\omega_o)$

EXA: find the output of

$x(n) = 10 - 5\sin\frac{\pi}{2}n + 20\cos\pi n$, applied to LTI system with:-

$h(n) = (\frac{1}{2})^n u(n)$

Solution

$$H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

1^{st} term $\omega_o = 0 \longrightarrow H(\omega=0) = \frac{1}{1-\frac{1}{2}e^{0}} = 2$

2^{nd} term $\omega_o = \frac{\pi}{2} \longrightarrow H(\omega=\frac{\pi}{2}) = \frac{1}{1-\frac{1}{2}e^{-j\pi/2}} = \frac{2}{\sqrt{5}} e^{-j26.6^\circ}$

3^{rd} term $\omega_o = \pi \longrightarrow H(\omega=\pi) = \frac{1}{1-\frac{1}{2}e^{-j\pi}} = \frac{2}{3}$

$$\therefore\ y(n) = 10 \times 2 - 5 \times \frac{2}{\sqrt{5}} \sin\left(\frac{\pi}{n} - 26.6^\circ\right) + 20 \times \frac{2}{3}\cos\pi n$$

Steady state and transient response to Sinusoidal inputs:

Let $x(n) = e^{j\omega_o n}u(n)$, applied to system :

$h(n) = (\frac{1}{2})^n u(n)$

$$X(z) = \frac{1}{1 - e^{j\omega_o}z^{-1}} \ ; \quad |z| > 1 \quad , \quad H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \ ; \quad |z| > \frac{1}{2}$$

$$Y(z) = \frac{1}{1 - e^{j\omega_o}z^{-1}} \times \frac{1}{1 - \frac{1}{2}z^{-1}} \xrightarrow[\text{fraction}]{\text{partial}} y(n) = \underbrace{\left(\frac{1}{1 - 2e^{j\omega_o}}\right)\left(\frac{1}{2}\right)^n u(n)}_{\textbf{Transient response}} + \underbrace{\left(\frac{1}{1 - \frac{1}{2}e^{-j\omega_o}}\right) e^{j\omega_o n} u(n)}_{\textbf{Steady state}}$$

$$x(n) \longrightarrow \boxed{\begin{matrix} h(n) \\ H(\omega) \end{matrix}} \longrightarrow y(n) = x(n) * h(n)$$

$$Y(\omega) = X(\omega)\,H(\omega)$$

$$\hookrightarrow y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega)\,e^{j\omega n}\,d\omega$$

$$* \; S_{yy}(\omega) = |Y(\omega)|^2 = |X(\omega)\,H(\omega)|^2 \longrightarrow \boxed{S_{yy}(\omega) = S_{xx}(\omega)\,|H(\omega)|^2}$$

EXA: an LTI system with $h(n) = \left(\frac{1}{2}\right)^n u(n)$

and input $x(n) = \left(\frac{1}{4}\right)^n u(n)$

Find the output energy spectral density.

Solution

$$X(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}, \qquad H(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$\longrightarrow Y(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}} \times \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$S_{yy}(\omega) = |Y(\omega)|^2 = \frac{1}{|1-\frac{1}{4}e^{-j\omega}|^2\,|1-\frac{1}{2}e^{-j\omega}|^2} = \frac{1}{|1-\frac{1}{4}\cos\omega + j\frac{1}{4}\sin\omega|^2\,|1-\frac{1}{2}\cos\omega + j\frac{1}{2}\sin\omega|^2}$$

$$S_{yy}(\omega) = \frac{1}{\left(\frac{17}{16}-\frac{1}{2}\cos\omega\right)\left(\frac{5}{4}-\cos\omega\right)}$$

System response to periodic signals:

Given $x(n)$ periodic (period $= N$)

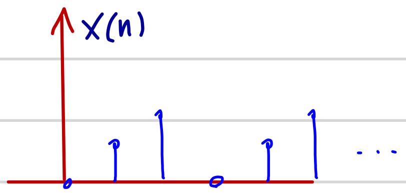
$$x(n) = \sum_{k=0}^{N-1} C_k\, e^{j\frac{2\pi k}{N}n} = \sum_{k=0}^{N-1} x_k(n)$$

$$x_k(n) = C_k\, e^{j\frac{2\pi k}{N}n} \longrightarrow y_k(n) = H\left(\frac{2\pi k}{N}\right) C_k\, e^{j\frac{2\pi k n}{N}}$$

$$\therefore \boxed{y(n) = \sum_{k=0}^{N-1} C_k\, \underbrace{H\left(\frac{2\pi k}{N}\right)}_{F.T.[h(n)]}\, e^{j\frac{2\pi k n}{N}}} \qquad C_{ky} = C_{kx}\, H\left(\frac{2\pi k}{N}\right)$$

An LTI system modifies the shape of an input periodic signal by modifying it's harmonics.

Given periodic x(n)



applied to $y(n) = \frac{1}{2} y(n-1) + 2x(n)$

Solution

1) $H(\omega) = H(z)\big|_{z=e^{j\omega_0}}$

2) Find C_{kx}

3) Find $C_{ky} = C_{kx} H\left(\frac{2\pi k}{N}\right)$

4) $y(n) = \sum_{k=0}^{N-1} C_{ky} e^{j\frac{2\pi k n}{N}}$; $n = 0, 1, \cdots, N-1$

$y(n=0) = [\ldots]$
$=1$
$=2$

Monday 4/15/13 8:07 AM

EXA: a digital filter described by

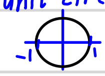
$y(n) = x(n) - x(n-10)$

Find a) Freq. Response $H(\omega)$

b) o/p for $x(n) = \cos\frac{\pi}{10}n + 3\sin\left(\frac{\pi}{3}n + \frac{\pi}{10}\right)$

Solution

(unit circle)



$Y(z) = X(z)\left[1 - z^{-10}\right]$ $\left\{ \text{OR } \; Y(\omega) = X(\omega) - e^{-j10\omega}X(\omega) \right\}$

$H(z) = 1 - z^{-10} \longrightarrow H(\omega) = 1 - e^{-j10\omega}$ → since $y(n)$ is finite ; ROC is all z-plane except $0, \infty$ ⇒ includes

$H(\omega = \frac{\pi}{10}) = 1 - e^{-j10\frac{\pi}{10}} = 1 - e^{-j\pi} = 2\angle 0^\circ$

$H(\omega = \frac{\pi}{3}) = 1 - e^{-j10\frac{\pi}{3} - j2\pi} = 1 - e^{-j4\frac{\pi}{3}} = \sqrt{3}\angle -\pi/6$

$-\pi/15$

$\Rightarrow y(n) = 2\cos\frac{\pi}{10}n + 3\sqrt{3}\sin\left(\frac{\pi}{3}n + \frac{\pi}{10} - \frac{\pi}{6}\right)$

Design of simple digital filters by pole-zero placement:
Freq. response (graphical):-

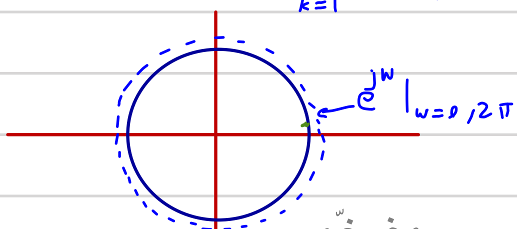
$$H(z) = G\,\frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1})\cdots}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})\cdots}$$

gain

$$H(\omega) = G\,\frac{\prod_{k=1}^{M}(1 - z_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - p_k e^{-j\omega})} \times \frac{e^{j\omega N}}{e^{j\omega N}} = G\, e^{j\omega(N-M)}\,\frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

{Assume $N > M$ → proper}

$$\Rightarrow \boxed{|H(\omega)| = G\,\frac{\prod_{k=1}^{M}|e^{j\omega} - z_k|}{\prod_{k=1}^{N}|e^{j\omega} - p_k|}}$$



مضخّم

Pole-zero placement principle:-
1- place poles near points (frequencies) to be emphasized.
2- place zeros at frequencies to be de-emphasized.
3- all poles are placed inside the unit circle (for the filter to be stable.)
4- Zeros are placed anywhere in z-plane, mostly on the unit circle (for exact zero.)
5- all complex zeros & poles must occur in complex conjugate pairs.

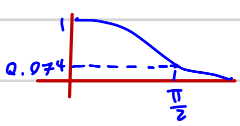
$(z - \rho e^{j\theta})(z - \rho e^{-j\theta})$

$\rightarrow (1)z^2 - (2\rho\cos\theta)z + \rho^2$ → for $y(n)$ (o/p) to be real.

EXA: design a LPF to have one pole to satisfy $|H(\omega)| = 1$ for $\omega = 0$

$|H(\omega)| = 0.074$ for $\omega = \frac{\pi}{2}$

Solution



$H(z) = G\,\frac{1}{1 - pz^{-1}}$

$p = re^{j\theta} = ae^{j0} = a$ → $\omega = 0$ for LPF, but for HPF: $\omega = \pi$ (*To be emphasized*)

$\Rightarrow H(z) = G\,\frac{1}{1 - az^{-1}} = G\,\frac{z}{z - a} \longrightarrow H(\omega) = G\,\frac{e^{j\omega}}{e^{j\omega} - a} \Rightarrow |H(\omega)| = G\,\frac{1}{|e^{j\omega} - a|}$

$$|H(w=0)| = 1 = \frac{G}{|e^{0}-a|} \longrightarrow \boxed{G = 1-a}$$

$$|H(w=\tfrac{\pi}{2})| = 0.074 = \frac{G}{|e^{j\frac{\pi}{2}}-a|} = \frac{G}{|j-a|}$$

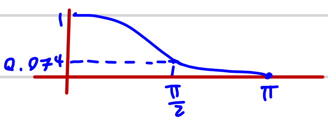
$$\Rightarrow \frac{G^2}{1+a^2} = (0.074)^2 \quad \text{— } (2)$$

$$\Rightarrow \frac{(1-a)^2}{1+a^2} = (0.074)^2 \Rightarrow 0.9945a^2 - 2a + 0.9945 = 0 \longrightarrow a = \underset{\times}{1.11}, \ \underset{\checkmark}{0.9}$$

$$\therefore \ G = 1-a = 1-0.9 = 0.1 \Rightarrow \boxed{H(z) = \frac{0.1}{1-0.9z^{-1}}}$$

EXA: same as the previous one , but with a zero:- $|H(\pi)| = 0$

Solution



$$H(z) = G\,\frac{1-z_1 z^{-1}}{1-p_1 z^{-1}} \ ; \quad z_1 = 1\,e^{j\pi} = -1 \ , \quad p_1 = r e^{j0} = a e^{j0} = a$$

$$\Rightarrow H(z) = G\,\frac{1+z^{-1}}{1-az^{-1}} = G\,\frac{z+1}{z-a} \Rightarrow H(w) = G\,\frac{e^{jw}+1}{e^{jw}-a}$$

$$\longrightarrow |H(w)| = \frac{G|e^{jw}+1|}{|e^{jw}-a|} \Rightarrow |H(w=0)| = 1 = G\,\frac{|e^{0}+1|}{|e^{0}-a|} = \frac{2G}{1-a}$$

$$\Rightarrow |H(\tfrac{\pi}{2})| = G\,\frac{|e^{j\frac{\pi}{2}}+1|}{|e^{j\pi/2}-a|} = G\,\frac{|j+1|}{|j-a|} = 0.074$$

$$\frac{G^2(1+1)}{1+a^2} = (0.074)^2 \Rightarrow \frac{\left(\frac{1-a}{2}\right)^2 \times 2}{1+a^2} = (0.074)^2 \longrightarrow 0.989a^2 - 2a + 0.989 = 0$$

$$\Rightarrow a = 1.16,\ 0.8618 \longrightarrow G = \frac{1-a}{2} = 0.0691$$

$$\therefore \ \boxed{H(z) = 0.0691\,\frac{1+z^{-1}}{1-0.862z^{-1}}}$$

EXA: design a two-pole (identical) LPF to have $|H(0)|=1$, $|H(\frac{\pi}{4})| = \frac{1}{2}$

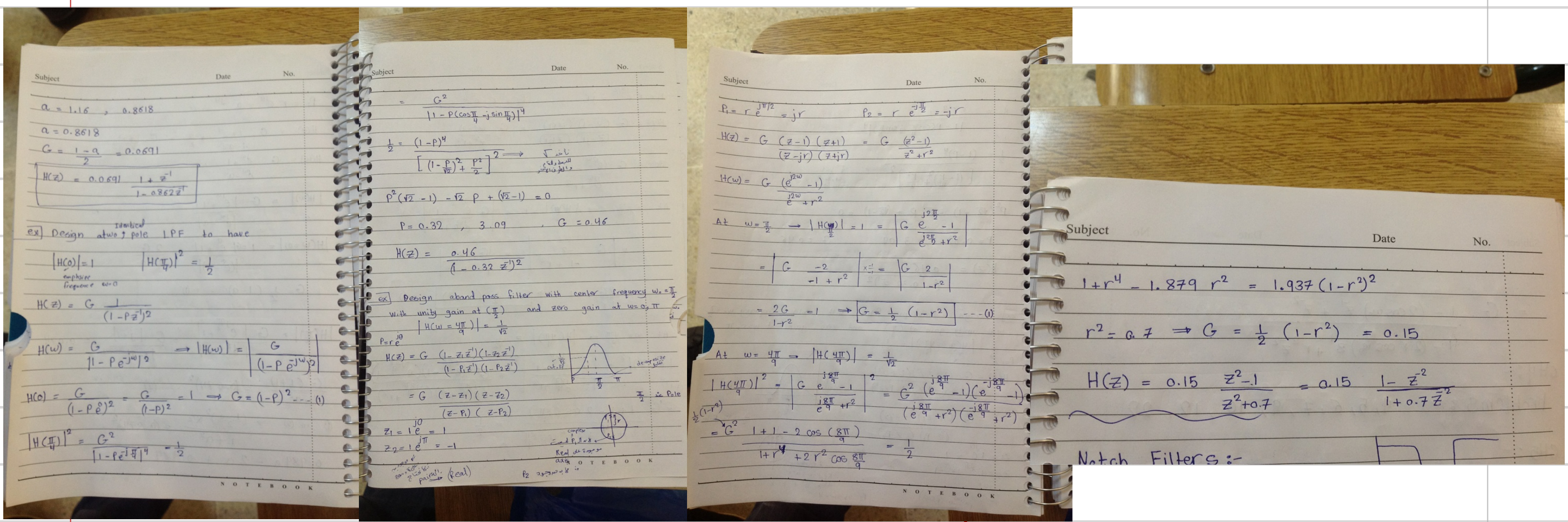
Solution

$$H(z) = G\,\frac{1}{(1-pz^{-1})^2} \Rightarrow |H(w)| = \left|\frac{G}{(1-pe^{-jw})^2}\right| \Rightarrow |H(0)| = \frac{G}{(1-pe^{0})^2} = \frac{G}{(1-p)^2} = 1 \Rightarrow G = (1-p)^2 \quad \text{— } (1)$$

$$|H(\tfrac{\pi}{4})|^2 = \frac{G^2}{|1-pe^{-j\pi/4}|^4} = \frac{G^2}{|1-p(\cos\frac{\pi}{4}-j\sin\frac{\pi}{4})|^4} \Rightarrow \frac{1}{2} = \frac{(1-p)^2}{\left[(1-\frac{p}{\sqrt{2}})^2+\frac{p^2}{2}\right]^2} \Rightarrow p^2(\sqrt{2}-1) - \sqrt{2}\,p + (\sqrt{2}-1) = 0$$

$$p = \boxed{0.32},\ 3.09 . \quad G = 0.46 \Rightarrow \boxed{H(z) = \frac{0.46}{(1-0.32z^{-1})^2}}$$

Wednesday 4/17/13 8:24 AM



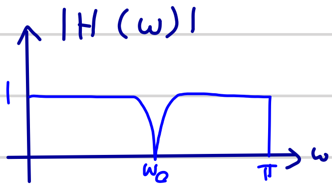
$1 + r^4 - 1.879\, r^2 = 1.937\,(1-r^2)^2$

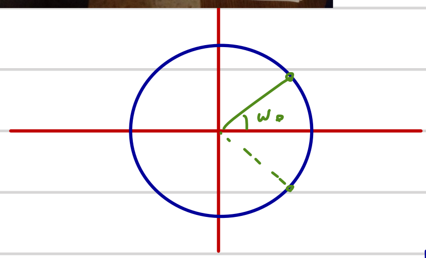
$r^2 = 0.7 \Rightarrow G = \frac{1}{2}(1-r^2) = 0.15$

$H(z) = 0.15 \frac{z^2-1}{z^2+0.7} = 0.15 \frac{1-z^{-2}}{1+0.7z^{-2}}$

Notch Filters :-

Notch Filter:

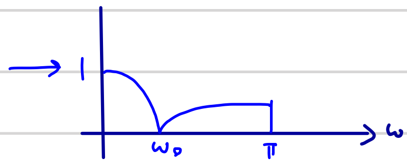




$$H(z) = G(1 - z_1 z^{-1})(1 - z_2 z^{-1})$$

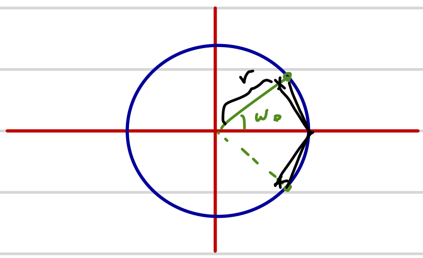
$$z_1 = 1 e^{j\omega_0}, \quad z_2 = 1 e^{-j\omega_0}$$

$$H(z) = G(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})$$



$$\Longrightarrow H(z) = G\,\frac{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}{(1-re^{j\omega_0}z^{-1})(1-re^{-j\omega_0}z^{-1})}$$

$$= G\,\frac{1-(2\cos\omega_0)z^{-1}+z^{-2}}{1-2r\cos\omega_0\, z^{-1}+r^2z^{-2}}$$



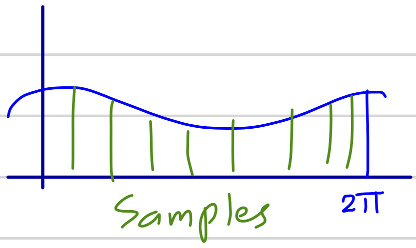
Disadvantage: it has a big delay (impulse response:p^n needs more time until it reaches the steady state.)

Suggested problems:
Chp. 4:-
4,5,6,7,9,10,12,22.
Chp. 5:-
3,4,6,7,8,10,16,17,19,21,22,24,30,32,52,54,62,68

2nd Exam Wednesday 24/4/2013

Chapter 7 The discrete Fourier Transform (DFT)

Applications:
Power spectrum estimation.
Filtering.
Correlation.
Freq. estimation.

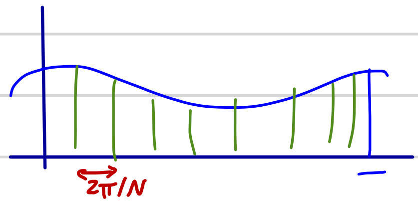


Freq. Domain Sampling and reconstruction of time domain signals:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \Rightarrow X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\left(\frac{2\pi k}{N}\right)n} \quad ; \quad k = 0, 1, \cdots, N-1$$

$$X\left(\frac{2\pi}{N}k\right) = \cdots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi k}{N}n} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N}n} + \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi k}{N}n} + \cdots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi k}{N}n} \quad ; \quad l : \text{integer.}$$



= shift in time by $(-lN)$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN) e^{-j\frac{2\pi k}{N}(n-lN)} = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n-lN) e^{-j\frac{2\pi k}{N}n} = \sum_{n=0}^{N-1} \Bigg[\underbrace{\sum_{l=-\infty}^{\infty} x(n-lN)}_{x_p(n)} \Bigg] e^{-j\frac{2\pi k}{N}n}$$

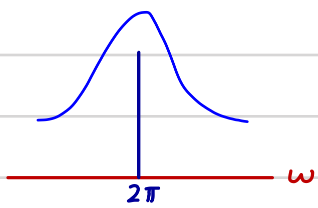
$x_p(n)$: periodic? $\longrightarrow$ $x_p(n+N) = x_p(n)$ ✓

$\Rightarrow$ $x_p(n)$ is periodic $= N$

$$\therefore \quad X\left(\frac{2\pi k}{N}\right) = \sum x_p(n) e^{-j\frac{2\pi k}{N}n}$$

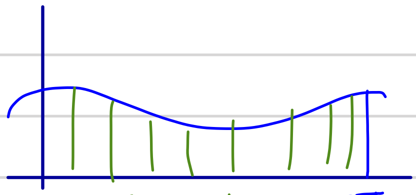
$$= \sum N \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi k}{N}n}$$

$$= N C_k$$



Chapter 7 The discrete Fourier Transform (DFT)

Applications:
Power spectrum estimation.
Filtering.
Correlation.
Freq. estimation.

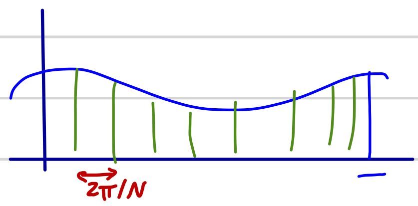


Freq. Domain Sampling and reconstruction of time domain signals:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \Rightarrow X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\left(\frac{2\pi k}{N}\right)n} \quad ; \quad k = 0, 1, \cdots, N-1$$

$$X\left(\frac{2\pi}{N}k\right) = \cdots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi k}{N}n} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N}n} + \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi k}{N}n} + \cdots$$

$$= \sum_{\ell=-\infty}^{\infty} \sum_{n=\ell N}^{\ell N+N-1} x(n) e^{-j\frac{2\pi k}{N}n} \quad ; \quad \ell : \text{integer}.$$



= shift in time by $(-\ell N)$

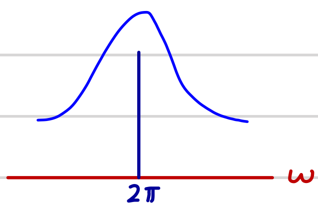
$$= \sum_{\ell=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-\ell N) e^{-j\frac{2\pi k}{N}(n-\ell N)} = \sum_{\ell=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-\ell N) e^{-j\frac{2\pi k}{N}n} = \sum_{n=0}^{N-1} \underbrace{\left[\sum_{\ell=-\infty}^{\infty} x(n-\ell N)\right]}_{x_p(n)} e^{-j\frac{2\pi k}{N}n}$$

$x_p(n)$: periodic? $\longrightarrow$ $x_p(n+N) = x_p(n)$ ✓

$\Rightarrow x_p(n)$ is periodic $= N$

$$\therefore X\left(\frac{2\pi k}{N}\right) = \sum x_p(n) e^{-j\frac{2\pi k}{N}n}$$

$$= \sum N \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi k}{N}n}$$



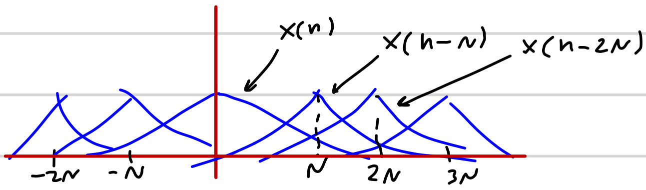
$$= N C_k$$

Monday 4/22/13 8:09 AM

$$x_p(n) = \sum C_k e^{j\frac{2\pi k}{N}n} \qquad C_k = \frac{1}{N} X\left(\frac{2\pi k}{N}\right) \quad \Bigg\} \quad \boxed{x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j\frac{2\pi k n}{N}}}$$

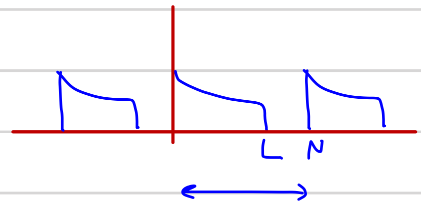
Reconstructed from the samples:

$X\left(\frac{2\pi k}{N}\right)$; $k = 0, 1 \cdots N-1$

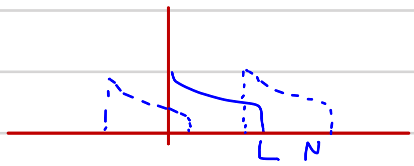


Reconstruction of x(n):
x(n) has a finite duration of 0 < n < L-1
N: no. of samples on X(w).
1- if N >= L

$$x(n) = \begin{cases} x_p(n) & 0 \le n \le N-1 \\ 0 & \text{elsewhere} \end{cases}$$



2- if L >= N

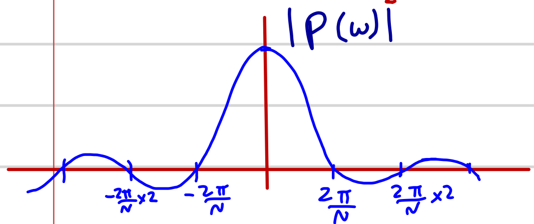


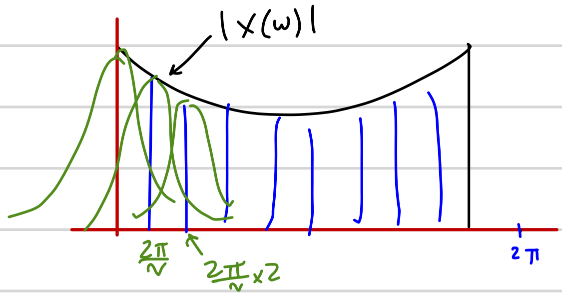
Expressing X(w) in terms of X(2pie*k/N) {k = 0,1,---,N-1}:
Assume N >= L

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \Bigg[\overbrace{\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j\frac{2\pi k n}{N}}}^{x(n) = x_p(n)} \Bigg] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} X\left(\frac{2\pi k}{N}\right)\left[\frac{1}{N}\sum_{k=0}^{N-1} e^{-j\left(\omega - \frac{2\pi k}{N}\right)n}\right]$$

* Define: $$p(\omega) = \frac{1}{N}\sum_{n=0}^{N-1}\left(e^{-j\omega}\right)^n = \frac{1}{N}\,\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \frac{\sin(\omega N/2)}{N \sin(\omega/2)} e^{-j\omega(N-1)/2}$$

1^{st} null at $\frac{\omega N}{2} = \pi \Rightarrow \boxed{\omega = \frac{2\pi}{N}}$



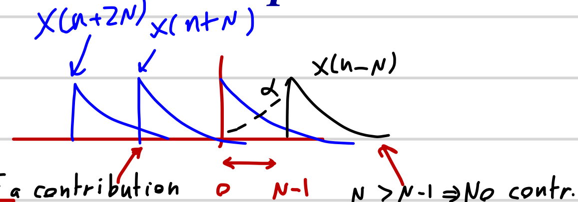


$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) p\left(\omega - \frac{2\pi k}{N}\right)$$

EXA: let $x(n) = a^n u(n) \longleftrightarrow X(\omega)$
The spectrum X(w) is sampled at $\omega = 2\pi k/N$; $k = 0, 1, ---, N-1$
Determine the reconstructed time signal & the reconstructed spectrum.

Solution

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) = \sum_{l=-\infty}^{0} a^{n-lN} = a^n \sum_{l=-\infty}^{0} a^{-lN}$$



$$m = -l \Rightarrow \quad x_p(n) = a^n \sum_{m=0}^{\infty} a^{mN} = a^n \sum_{m=0}^{\infty} (a^N)^m = \boxed{\frac{a^n}{1 - a^N}}$$

$\neq a^n u(n)$ {by scaling}

$$\hat{X}(\omega) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\omega n} = \sum_{n=0}^{N-1} \frac{a^n}{1-a^N} e^{-j\omega n} = \boxed{\frac{1}{1-a^N} \frac{1-(a e^{-j\omega})^N}{1-a e^{-j\omega}}}$$

$$\hat{X}\left(\omega = \frac{2\pi k}{N}\right) = \frac{1}{1-a^N} \frac{1-(a e^{-j\omega})^N}{1-(a e^{-j\omega})} = \frac{1}{1-a^N} \frac{1-a^N e^{-j\frac{2\pi k}{N} N}}{1-a e^{-j\frac{2\pi k}{N}}} = \frac{1}{1-a e^{-j\frac{2\pi k}{N}}}$$

$$X(\omega) = \frac{1}{1-e^{j\omega}}$$ ⟵ samples on $X(\omega)$

The DFT:

DFT: $X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$; $k = 0, 1, \dots, N-1$

IDFT: $x(n) = \frac{1}{N} \sum X(k)\, e^{j\frac{2\pi k n}{N}}$; $n = 0, 1, \dots, N-1$

$\vec{x}\ (N) \longleftrightarrow \vec{X}\ (N)$

Fourier Tr. ⟹ If $L > N$, truncate the signal to N-points.

EXA: let x(n) = $\begin{cases} 1 & 0 \le n \le L-1 \\ 0 & \text{elsewhere} \end{cases}$

Det. N-pt. DFT for $N \geq L$

Solution

$$X(\omega) = \sum_{n=0}^{L-1} x(n)\, e^{-j\omega n} = \sum_{n=0}^{L-1} (e^{-j\omega})^L = \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

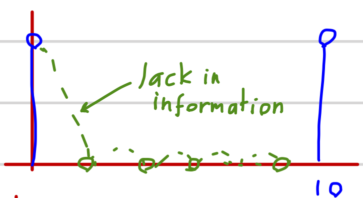
$$\Rightarrow X(k) = \frac{\sin\left(\frac{L\, 2\pi k/N}{2}\right)}{\sin\left(\frac{2\pi k/N}{2}\right)} e^{-j\frac{2\pi k}{N}\left(\frac{L-1}{2}\right)}$$

1. If $N = L$

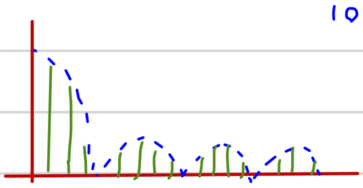
$$X(k) = \frac{\sin(\pi k)}{\sin(\pi k/L)} e^{-j\pi k(L-1)/L} = \begin{cases} L & k = 0 \\ 0 & k = 1, 2, \dots, N-1 \end{cases}$$

$L = N = 10$

(Zero pending)

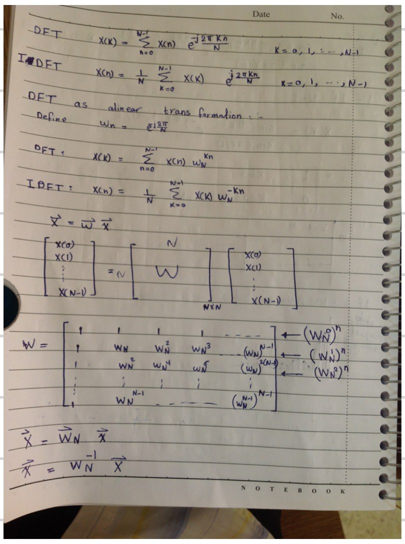


If $L = 10$, $N = 50$



$$X(\omega)\Big|_{\omega = \frac{2\pi k}{N}} = \sum x(n)\, e^{-j\omega n}$$

Monday 4/29/13 8:40 AM



DFT $X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi Kn}{N}}$ $K = 0, 1, \dots, N-1$

IDFT $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi Kn}{N}}$ $K = 0, 1, \dots, N-1$

DFT as alinear trans formation :-

Define $w_n = e^{-j\frac{2\pi}{N}}$

DFT : $X(K) = \sum_{n=0}^{N-1} x(n)\, w_N^{Kn}$

IDFT : $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, w_N^{-Kn}$

$\vec{X} = \vec{w}\, \vec{x}$

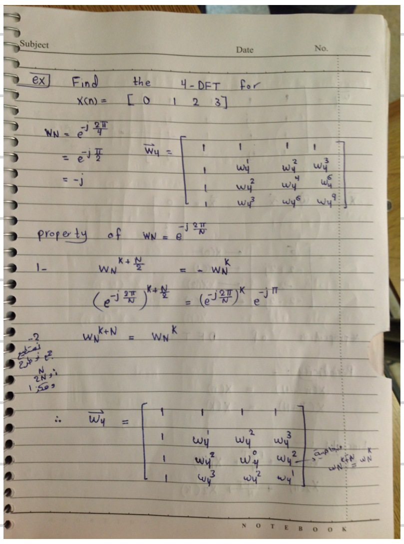
$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \underbrace{\begin{bmatrix} & W & \end{bmatrix}}_{N\times N} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & W_N & W_N^2 & W_N^3 & \cdots (W_N)^{N-1} \\ 1 & W_N^2 & W_N^4 & W_N^6 & (W_N)^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & \cdots & & (W_N^{N-1})^{N-1} \end{bmatrix} \begin{matrix} \leftarrow (W_N^0)^n \\ \leftarrow (W_N^1)^n \\ \leftarrow (W_N^2)^n \\ \\ \\ \end{matrix}$$

$\vec{X} = \vec{W_N}\, \vec{x}$

$\vec{x} = W_N^{-1}\, \vec{X}$

NOTEBOOK



Subject Date No.

ex Find the 4-DFT for

$x(n) = [0 \; 1 \; 2 \; 3]$

$W_N = e^{-j\frac{2\pi}{4}}$

$= e^{-j\frac{\pi}{2}}$

$= -j$

$$\vec{W_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w_4^1 & w_4^2 & w_4^3 \\ 1 & w_4^2 & w_4^4 & w_4^6 \\ 1 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix}$$

property of $W_N = e^{-j\frac{2\pi}{N}}$

1- $W_N^{K+\frac{N}{2}} = -W_N^K$

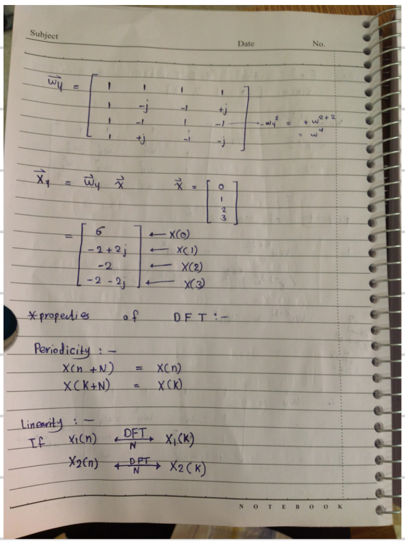
$\left(e^{-j\frac{2\pi}{N}}\right)^{K+\frac{N}{2}} = \left(e^{-j\frac{2\pi}{N}}\right)^K e^{-j\pi}$

2- $W_N^{K+N} = W_N^K$

$$\therefore \vec{W_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w_4^1 & w_4^2 & w_4^3 \\ 1 & w_4^2 & w_4^0 & w_4^2 \\ 1 & w_4^3 & w_4^2 & w_4^1 \end{bmatrix}$$

$W_N^{K+N} = W_N^K$

NOTEBOOK



Subject Date No.

$$\vec{W_4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$$

$-w_4^2 = +w^{2+2}$

$= w^4$

$\vec{X_4} = \vec{W_4}\, \vec{x}$ $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$

$$= \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} \begin{matrix} \leftarrow X(0) \\ \leftarrow X(1) \\ \leftarrow X(2) \\ \leftarrow X(3) \end{matrix}$$

* properties of DFT :-

Periodicity :-

$x(n+N) = x(n)$

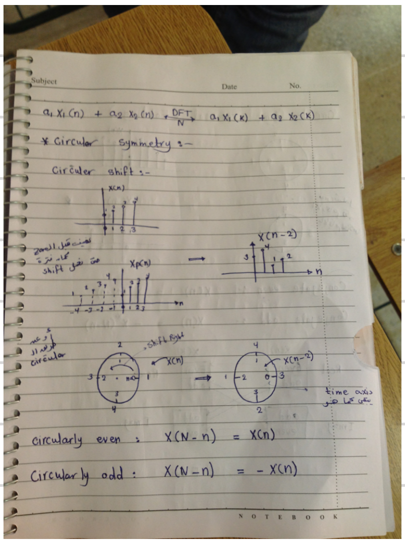
$X(K+N) = X(K)$

Linearity :-

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(K)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(K)$

NOTEBOOK



Subject Date No.

$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(K) + a_2 X_2(K)$

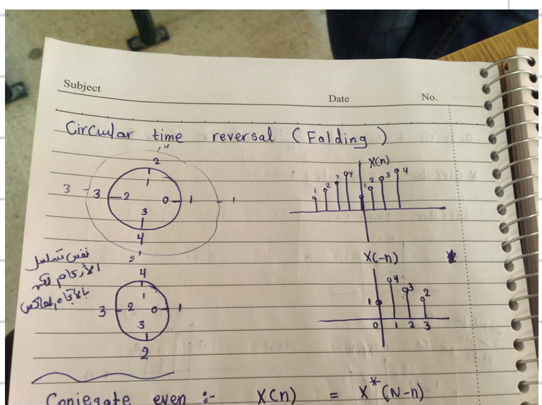
* Circular symmetry :-

Circuler shift :-

circularly even : $x(N-n) = x(n)$

Circularly odd : $x(N-n) = -x(n)$

NOTEBOOK



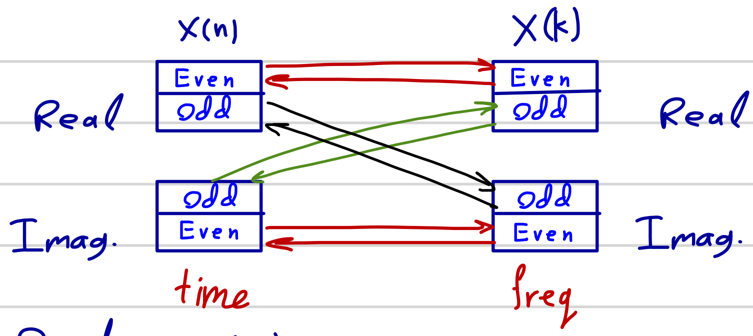
Subject Date No.

Circular time reversal (Folding)

Conjegate even :- $x(n) = x^*(N-n)$

Conjugate even: $x(n) = x^{*}(N-n)$

Conjugate odd: $x(n) = -x^{*}(N-n)$



$x(n)$	$X(k)$
$x^{*}(n)$	$X^{*}(N-k)$
$x^{*}(N-n)$	$X^{*}(k)$
$x_R(n)$	$X_{ce}(k) = \frac{1}{2}[X(k) + X^{*}(N-k)]$
$j\,x_I(n)$	$X_{co}(k) = \frac{1}{2}[X(k) - X^{*}(N-k)]$
$x_{ce}(n) = \frac{1}{2}[x(n) + x^{*}(N-n)]$	$X_R(k)$
$x_{co}(n) = \frac{1}{2}[x(n) - x^{*}(N-n)]$	$j\,X_I(k)$

Real $x(n)$

$X(k) = X^{*}(N-k)$

$X_R(k) = X_R(N-k)$

$X_I(k) = -X_I(N-k)$

$|X(k)| = |X(N-k)|$

$\measuredangle X(k) = -\measuredangle X(N-k)$

Circular convolution:

$$x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$$

$$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$$

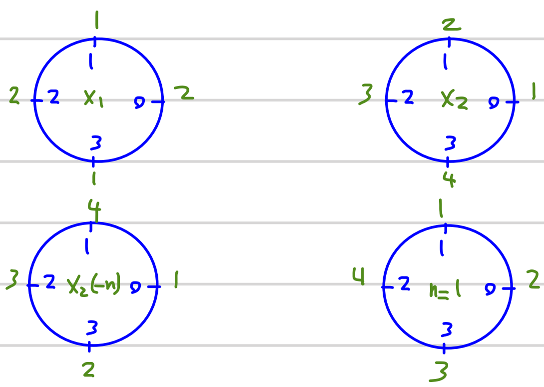
then $\quad x_1(n) \; N \; x_2(n) \xleftrightarrow[N]{DFT} X_1(k)\,X_2(k)$

EXA: Find the Circular convolution of:-

$x_1(n) = [2 \quad 1 \quad 2 \quad 1]$

$x_2(n) = [1 \quad 2 \quad 3 \quad 4]$

Solution



$y(0) = 1\times2 + 4\times1 + 3\times2 + 2\times1 = 14$

$y(1) = 2\times2 + 1\times1 + 4\times2 + 3\times1 = 16$

$y(2) = 14$

$y(3) = 16$

$\Rightarrow y(n) = [14 \quad 16 \quad 14 \quad 16]$

Time reversal

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$$

Circular time shift

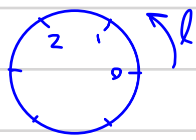
$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$x((n-l))_N \xleftrightarrow[N]{DFT} X(k)\, e^{-j\frac{2\pi k l}{N}}$$

Circular freq. - shift

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$x(n)\, e^{j\frac{2\pi \ell n}{k}} \xleftrightarrow[N]{DFT} X((k-\ell))_N$$



Circular correlation

circular $\tilde{r}_{xy}(\ell) = \sum_{n=0}^{N-1} x(n)\, y^*((n-\ell))_N \longleftrightarrow \tilde{R}_{xy}(k) = \boxed{X(k)\, Y^*(k)}$; $k = 0, 1, \cdots N-1$

$$\tilde{r}_{xx}(\ell) \longleftrightarrow |X(k)|^2 \quad ; \ k = 0, 1, \cdots N-1$$

Multiplication of two sequences

Convolution

$$x_1(n)\, x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} X_1(k) \, (N) \, X_2(k)$$

Parseval's Theorem

$$\boxed{\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)}$$

If $y(n) = x(n)$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Energy

(N in time may be less than in freq. or the opposite ---> so we can calculate the energy with the less symbols.)

Wednesday 5/1/13 8:06 AM

EXA: Given $X(k) = [0.25,\ 0.125 - j0.3,\ 0,\ 0.125 - j0.05,\ 0, \dots]$
is an 8-point DFT of real valued x(n).
Find the remaining points $(X(5), X(6), X(7))$

Solution

$$X(k) = X^*(N-k)$$

$$X(5) = X^*(8-5) = X^*(3) = 0.125 + j0.05$$

$$X(6) = X^*(8-6) = X^*(2) = 0$$

$$X(7) = X^*(8-7) = X^*(1) = 0.125 + j0.3$$

EXA: Complete the energy if the N- point signal

$$x(n) = \cos(2\pi k_0 n / N) \qquad 0 \le n \le N$$

Solution

$$\text{Energy} = \sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi \frac{kn}{N}}$$

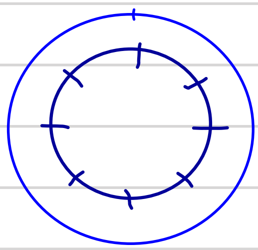
$$\Rightarrow x(n) = \frac{1}{N}\left[\frac{N}{2} e^{\frac{j2\pi k_0 n}{N}} + \frac{N}{2} e^{\frac{-j2\pi k_0 n}{N}}\right]$$

$$\therefore X(k) = [0\ 0 \dots \underset{\uparrow k_0}{\tfrac{N}{2}}\ 0 \dots 0\ \underset{\uparrow N-k_0}{\tfrac{N}{2}}\ 0 \dots 0\ \underset{\uparrow N-1}{0}]$$

$$\Rightarrow E = \frac{1}{N}\left[\left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2\right] = \frac{N}{2}$$

EXA: find the circular auto correlation of

$$x(n) = \cos\left(\frac{2\pi}{N} n\right) \qquad 0 \le n \le N-1$$



Solution

$$\tilde{r}_{xy}(\ell) \longleftrightarrow \tilde{R}_{xx}(k) = X(k)\, Y^*(k)$$

$$\Rightarrow \tilde{r}_{xx}(\ell) \xleftrightarrow[N]{DFT} |X(k)|^2 \ ; \qquad k = 0, 1 \dots N-1$$

$$x(n) = \frac{1}{2} e^{j\frac{2\pi}{N} n} + \frac{1}{2} e^{-j\frac{2\pi}{N} n} \times \frac{N}{N} = \frac{1}{N}\left[\frac{N}{2} e^{j\frac{2\pi}{N} n} + \frac{N}{2} e^{-j\frac{2\pi}{N} n}\right]$$

$$\left.\begin{aligned} \Rightarrow X(-1) &= \frac{N}{2} = X(N-1) \\ X(1) &= \frac{N}{2} \end{aligned}\right\} \quad X(k) = [0\ \underset{\uparrow k=1}{\tfrac{N}{2}}\ 0 \dots 0\ \underset{\uparrow N-1}{\tfrac{N}{2}}]$$

$$e^{j2\pi \frac{N}{N} \ell} \qquad e^{-j\frac{2\pi \ell}{N}}$$

$$\Rightarrow \tilde{R}_{xx}(k) = [0\ \ \frac{N^2}{4}\ \ 0 \dots 0\ \ \frac{N^2}{4}]$$

$$\rightarrow \tilde{r}_{xx}(\ell) = IDFT[\tilde{R}_{xx}(\ell)] = \frac{1}{N}\sum_{k=0}^{N-1} R_{xx}(k)\, e^{j\frac{2\pi k \ell}{N}} = \frac{1}{N}\left[R_{xx}(1)\, e^{j\frac{2\pi \ell}{N}} + R_{xx}(N-1)\, e^{j\frac{2\pi (N-1)\ell}{N}}\right]$$

$$= \frac{1}{N}\left[\frac{N^2}{4} e^{j\frac{2\pi \ell}{N}} + \frac{N^2}{4} e^{-j\frac{2\pi \ell}{N}}\right] = \frac{1}{N}\, 2\, \frac{N^2}{4} \cos\frac{2\pi \ell}{N} = \frac{N}{2} \cos\frac{2\pi \ell}{N}$$

EXA: Given 8- point DFT of

$$x(n) = [1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0]$$

Find the DFT of

$$x_1(n) = [1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1]$$

$$x_2(n) = [0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0]$$

In terms of DFT of x(n)

Solution

$$x_1(n) = x((n-5))_N$$

$$x((n-\ell))_N \longleftrightarrow X(k)\, e^{-j\frac{2\pi k \ell}{N}}$$

$$\Rightarrow X_1(k) = X(k)\, e^{-j\frac{2\pi k 5}{8}}$$

* $x_2(n) = x((n-2))_8$

$$\Rightarrow X_2(k) = X(k)\, e^{-j\frac{2\pi k 2}{8}} = X(k)\, e^{-j\frac{\pi}{2}k}$$

EXA: Given $x(n) = [0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 0]$

a) Find the sequence s(n) whose 6- point DFT is

$$S(k) = W_2^k\, X(k) \qquad \left\{ W_N = e^{-j\frac{2\pi}{N}} \right\}$$

b) Find y(n) whose 6- point DFT is

$$Y(k) = \text{Real}[X(k)]$$

Solution

a) $S(k) = \left(e^{-j\frac{2\pi}{2}}\right)^k X(k) = e^{-j\pi k} X(k)$ $\frac{2\pi k \ell}{N}$

$$= X(k)\, e^{-j\frac{2\pi k\times 3}{2\times 3}} = X(k)\, e^{-j\frac{2\pi k 3}{6}}$$

$$\Rightarrow s(n) = x((n-3))_6$$

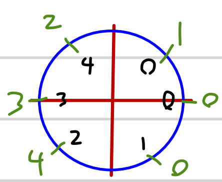
$$\Rightarrow s(n) = [3\ \ 4\ \ 0\ \ 0\ \ 1\ \ 2]$$

b) $Y(k) = \text{real}[X(k)]$

$$x_{ce}(n) \longleftrightarrow X_R$$

$$= \frac{1}{2}[x(n) + x^*(-n)]$$

$$\left.\begin{aligned} x(n) &= [0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 0] \\ x^*(-n) &= [0\ \ 0\ \ 4\ \ 3\ \ 2\ \ 1] \end{aligned}\right\}$$



$$\Rightarrow y(n) = \frac{1}{2}[x(n) + x^*(-n)] = [0\ \ \tfrac{1}{2}\ \ 3\ \ 3\ \ 3\ \ \tfrac{1}{2}]$$

EXA: find the DFT of $\delta(n)$

Solution

$$\delta(n) \longleftrightarrow X(\omega) = 1$$

$$X(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}} \qquad k = 0, 1, \dots, N-1$$

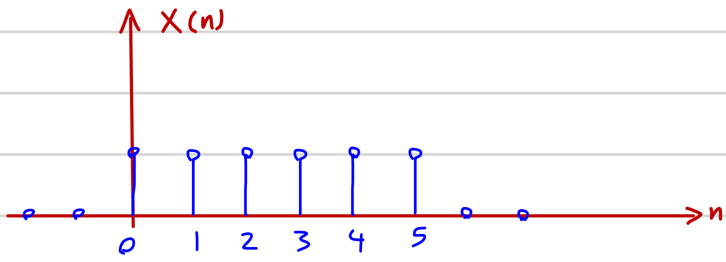
$$X(k) = 1 \qquad k = 0, 1, \dots, N-1$$

$$\left\{ \delta(n - n_0) \longleftrightarrow e^{-j n_0 \omega} \quad X(k) = e^{-j n_0 \frac{2\pi k}{N}} \right\}$$

EXA: Given x(n)

If we sample X(z) at

$$Z = e^{j\frac{2\pi}{4}k}$$



Find the sample the IDFT of

$$X_1(k) = X(z)\Big|_{z = e^{j\frac{2\pi k}{4}}} \qquad k = 0, 1, 2, 3$$

Solution

$x_1(n) = x_p(n)$ $\quad n = 0 \to 3 \quad \left\{ \sum x(n - N) \right\}$

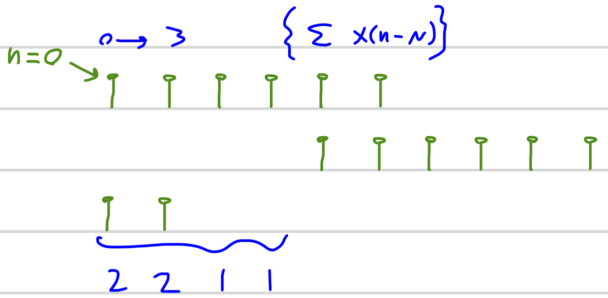
$x(n)$

$x(n-4)$

$x(n-8)$

$x(n+4)$

$x(n+8)$



$$x_1(n) = [\,2 \quad 2 \quad 1 \quad 1\,]$$

$$\int_{-\pi}^{\pi} X(\omega)\, X(\omega-\pi)\, d\omega = \int_{-\pi}^{\pi} \underbrace{X(\omega-\pi)}_{X_1(\omega)} \underbrace{[X^*(\omega)]^*}_{X_2(\omega)} d\omega$$

$X(\omega-\pi) \rightarrow e^{j\pi n} x(n)$, $X_2(\omega) \longrightarrow x^*(-n)$

$$= 2\pi \sum e^{j\pi n} x(n)\, x^*(-n) = -34\pi$$

$y(n) = x(2n)$

$$Y(\omega) = \cdots \sum_{m:\,even} x(m)\, e^{-j\frac{\omega}{2}m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \left(\frac{1+e^{j\pi m}}{2} \right) e^{-j\frac{\omega}{2}m}$$

$$\Rightarrow \frac{1}{2} X\left(\frac{\omega}{2}\right) + \frac{1}{2} X\left(\frac{\omega}{2}-\pi\right)$$ then substitute ✓

Monday 5/6/13 8:30 AM

EXA: Use symmetry properties to calculate DFT of 2-real even & 2-real odd simultaneously using one N-DFT only.

Solution

$$\left.\begin{matrix} x_1^e(n) & x_2^e(n) \\ y_1^o(n) & y_2^o(n) \end{matrix}\right\} \text{real}$$

Generate $x(n) = x_1^e(n) + y_1^o(n) + j\left[x_2^e(n) + y_2^o(n)\right]$

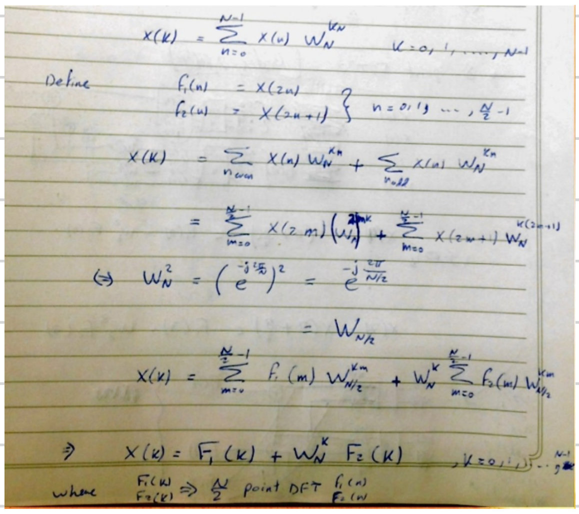
$V_R^e(k) + V_R^o(k) + jV_I^e(k) + jV_I^o(k)$

$$\left.\right\} \; X_1^e(k) = \frac{V_R(k) + V_R(-k)}{2}$$

$$\Rightarrow Y_2^o(k) = \frac{V_R(k) - V_R(-k)}{2} \Rightarrow Y_2^o(k) = -j\left[\frac{V_R(k) - V_R(-k)}{2}\right]$$

$$* \; Y_1^o(k) = \frac{jV_I(k) - jV_I(-k)}{2}, \quad jX_2^e(k) = \frac{jV_I(k) + jV_I(-k)}{2} \Rightarrow X_2^e(k) = -j\left[\frac{jV_I(k) + jV_I(-k)}{2}\right]$$

The Fast Fourier Transform (FFT):



$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \dots, N-1$$

Define

$$\left.\begin{matrix} f_1(n) = x(2n) \\ f_2(n) = x(2n+1) \end{matrix}\right\} n = 0, 1, \dots, \frac{N}{2}-1$$

$$X(k) = \sum_{n\,even} x(n) W_N^{kn} + \sum_{n\,odd} x(n) W_N^{kn}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x(2m)\left(W_N^2\right)^{mk} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_N^{k(2m+1)}$$

$$\Leftrightarrow W_N^2 = \left(e^{-j\frac{2\pi}{N}}\right)^2 = e^{-j\frac{2\pi}{N/2}}$$

$$= W_{N/2}$$

$$X(k) = \sum_{m=0}^{\frac{N}{2}-1} f_1(m) W_{N/2}^{km} + W_N^k \sum_{m=0}^{\frac{N}{2}-1} f_2(m) W_{N/2}^{km}$$

$$\Rightarrow X(k) = F_1(k) + W_N^k F_2(k), \quad k = 0, 1, \dots, N-1$$

where $\left.\begin{matrix} F_1(k) \\ F_2(k) \end{matrix}\right. \Rightarrow \frac{N}{2}$ point DFT $\begin{matrix} f_1(n) \\ f_2(n) \end{matrix}$

$$X(k) = F_1(k) + W_N^k F_2(k) \quad ; \quad k = 0, 1, \dots N-1$$

where: $\left.\begin{matrix} F_1(k) \\ F_1(k) \end{matrix}\right\} \frac{N}{2}$ point DFT $\begin{matrix} f_1(n) \\ f_2(n) \end{matrix}$ $\Rightarrow \begin{matrix} F_1(k+\frac{N}{2}) = F_1(k) \\ F_2(k+\frac{N}{2}) = F_2(k) \end{matrix}$

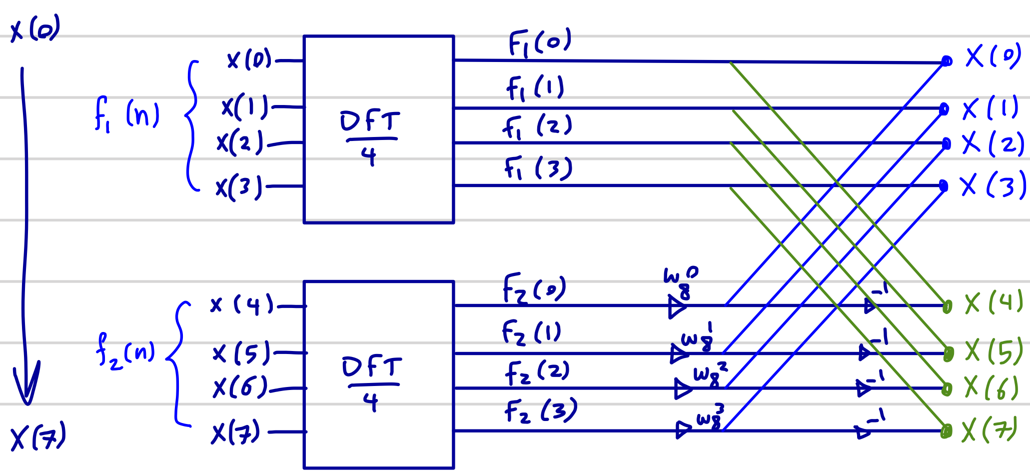
$$\frac{N^2}{4} + \frac{N^2}{4} = \frac{N^2}{2}, \quad W_N^{k+N/2} = -W_N^k$$

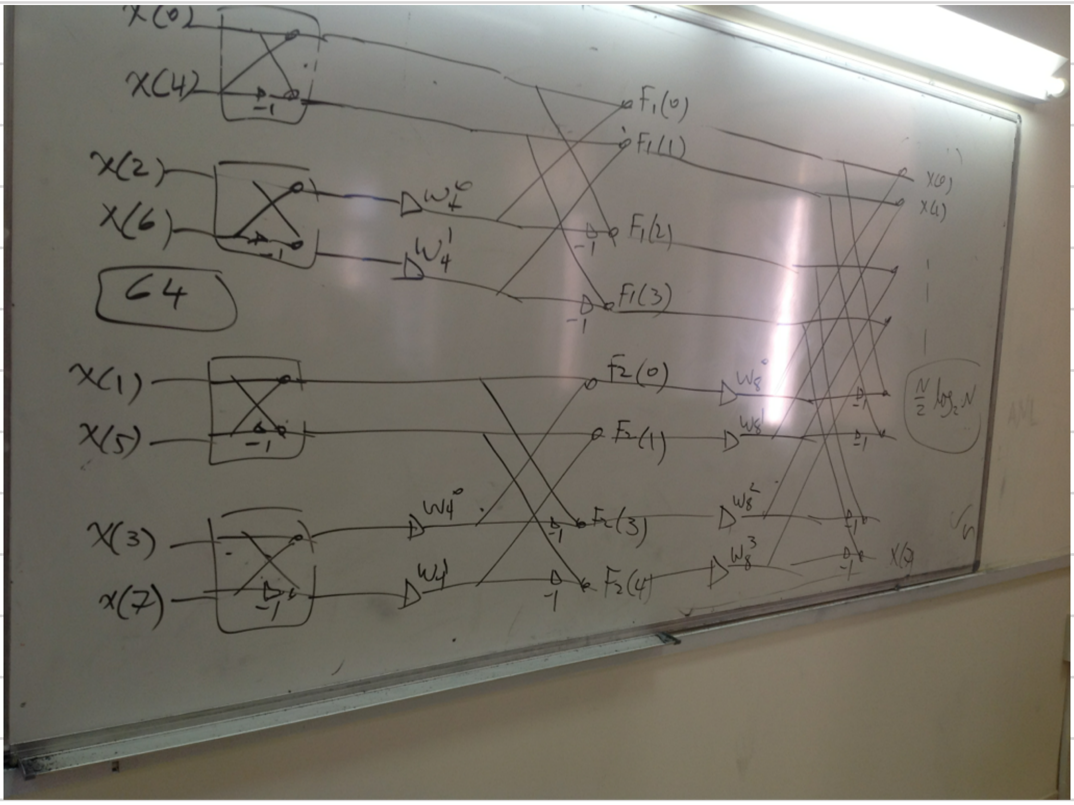
$$* \left.\begin{matrix} X(k) = F_1(k) + W_N^k F_2(k) \\ X(k+\frac{N}{2}) = F_1(k) - W_N^k F_2(k) \end{matrix}\right\} k = 0, 1, \dots \frac{N}{2}-1$$

$$F_1(k) = \underset{\frac{N}{2}}{DFT}\left(f_1(n)\right)$$

※ Radix_2 FFT.

EXA: 8-point FFT:-





	N^2	$\frac{N}{2}\log_2 N$
64	4096	192
1024	10^6	5120

Suggested problems:-
7.1, 4, 7, 8, 9, 11, 14, 19, 20, 25

The End