Design of Digital filters

FIR and IIR Filters

Discrete-time LTI systems can be broadly classified as:

1- systems that have a finite impulse response, i.e their impulse response is zero outside some finite interval, and are denoted by (FIR).

2- systems that have infinite impulse response (IIR).

FIR Filters

In general they are described by the difference equation

$$y(n) = \sum_{k=0}^{M-1} b_K\, x(n-k) = b_0 x(n) + b_1 x(n-1) + \cdots + b_{M-1} x(n-M+1)$$

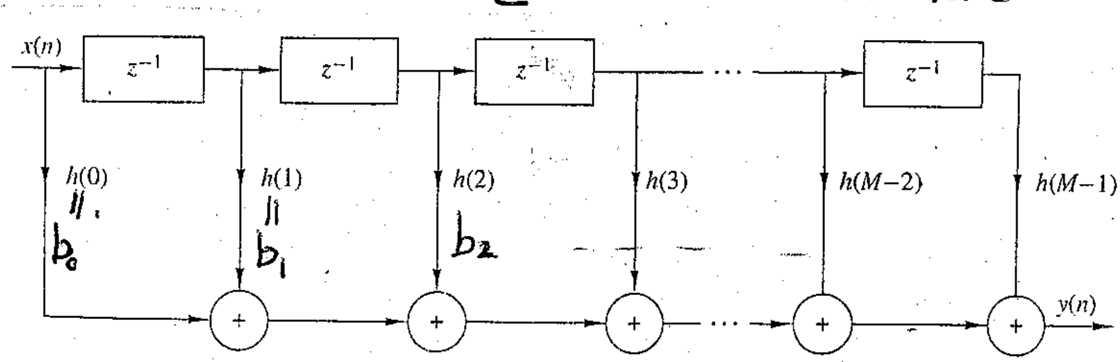
or by the system function

$$\boxed{\delta(n-k) \overset{Z}{\Longleftrightarrow} z^{-k}} \qquad H(z) = \sum_{k=0}^{M-1} b_K\, z^{-k} \quad \left(\text{Since } Y(z) = \sum_{k=0}^{M-1} b_K z^{-k} X(z)\right)$$

$$h(n) = b_0\delta(n) + b_1\delta(n-1) + b_2\delta(n-2) + \cdots$$

Hence the unit sample response is given by $\left[Z^{-1}(H(z))\right]$

$$h(n) = \begin{cases} b_n & 0 \le n \le M-1 \\ 0 & \text{elsewhere} \end{cases}$$



which is finite to M samples.

FIR filters are implemented in practice by implementation of the convolution sum

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k) \quad \left(\text{Filter implementation}\right)$$

IIR Filters

They are generally described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M-1} b_k\, x(n-k)$$

or by the system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

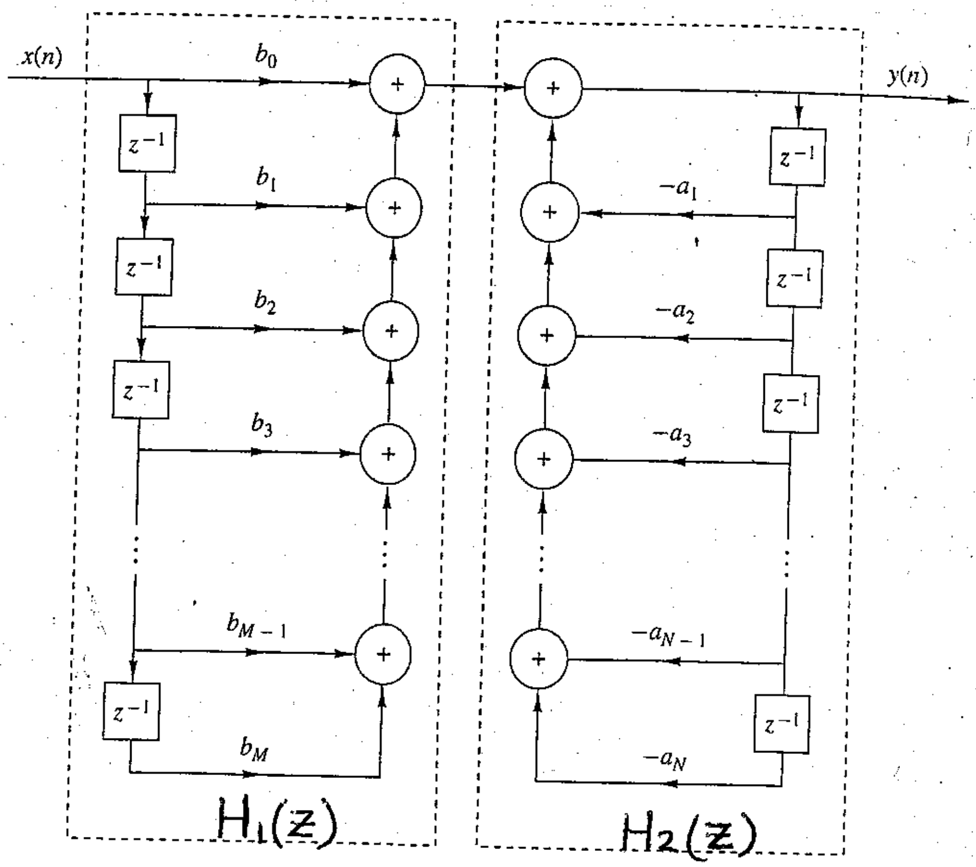
The unit sample response $h(n) = \mathcal{Z}^{-1}[H(z)]$ is generally an <u>infinite</u> sequence.

We can express $H(z)$ as

$$H(z) = H_1(z) H_2(z)$$

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$



Since the impulse response $h(n)$ is infinite, IIR filters are implemented by the itterative process using the difference equation

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \quad (\text{not implementable})$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad (\text{Filter implementation})$$

Digital Filter design process

In this process we determine the coefficients of a causal **FIR** or **IIR** filters that closely approximates the desired frequency response specifications.

Choosing between FIR and IIR filters

1) FIR filters can have exactly Linear-phase response. Such filters cause no distortion to the signal. This is important in several applications. Therefore if Linear-phase is required we select FIR filters.
2) FIR filters are always stable. IIR stability cannot always be guaranteed. (ROC includes unit circle)
3) Round off noise and coefficient quantization errors are much less severe in FIR than in IIR.
4) FIR requires more coefficients for sharp cutoff filters than IIR. Hence for a given amplitude response specification more processing time and storage will be required for FIR implementation.
5) Analog filters can be readily transformed into equivalent IIR digital filters. This is not possible in FIR. However FIR can synthesize arbitrary frequency responses.

In general
- Use IIR when the only important requirements are sharp cutoff filters and high througput.
- Use FIR if the no. of filter coefficients is not too Large and if Little or no phase distortion is desired.

Design of FIR Filters

We focus on the Linear-phase FIR filters.

Symmetric and Asymmetric FIR filters

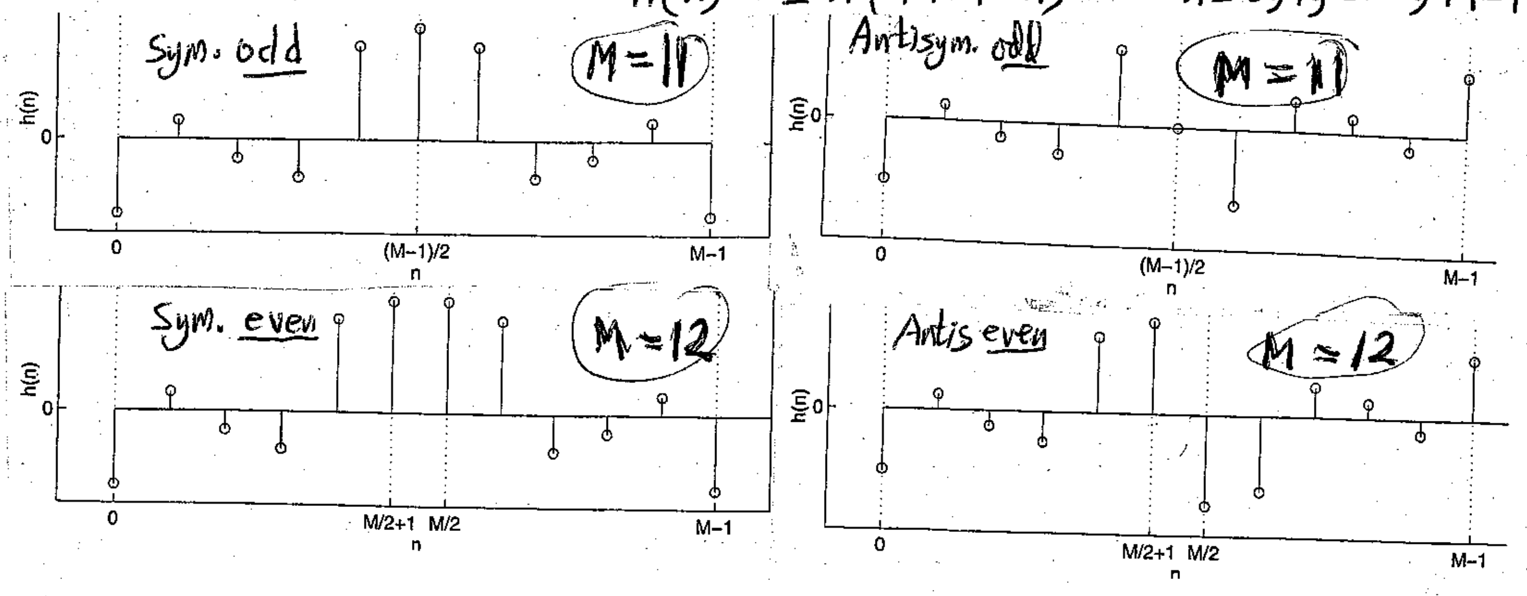
An FIR filter is described by

$$y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_{M-1} x(n-M+1)$$

$$\Rightarrow \quad h(k) = b_k \qquad k = 0, 1, \ldots, M-1$$

An FIR filter has Linear phase characteristic if $h(n)$ satisfies

$$h(n) = \pm h(M-1-n) \qquad n = 0, 1, \ldots, M-1$$



substituting this into $H(z) = \sum_{k=0}^{M-1} h(k) z^{-k}$ —— 1

$$H(z) = h(0) + h(1) z^{-1} + h(2) z^{-2} + \cdots + h(M-2) z^{-(M-2)} + h(M-1) z^{-(M-1)}$$

$$= z^{-(M-1)/2} \left\{ h\left(\frac{M-1}{2}\right) + \sum_{k=0}^{(M-3)/2} h(k) \left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2} \right] \right\}$$

$$H(z) = h\left(\frac{M-1}{2}\right) z^{-\left(\frac{M-1}{2}\right)} + \sum_{k=0}^{(M-3)/2} h(k) z^{-k} \pm h(M-1-k) z^{-(M-1-k)}$$

M: Odd

$$= z^{-(M-1)/2} \sum_{k=0}^{\frac{M}{2}-1} h(k) \left[z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2} \right]$$

M: Even —— 2

To get $H(w)$, we evaluate $H(z)$ in 2 on the unit circl

$$H(w) = H(z)\big|_{z=e^{jw}}$$

Symmetric case:

We have $h(n) = h(M-1-n)$ —— 3

Substituting 3 in 2 we get

$$\boxed{H(w) = H_r(w)\, e^{-jw(M-1)/2}}$$

Here we replace (k) by (n)

$H_r(w)$ is a real function of (w) given as

4a — $H_r(w) = h\left(\frac{M-1}{2}\right) + 2 \sum_{n=0}^{(M-3)/2} h(n) \cos w\left(\frac{M-1}{2} - n\right)$ $\boxed{\text{M: Odd}}$

4b — $H_r(w) = 2 \sum_{n=0}^{(M/2)-1} h(n) \cos w\left(\frac{M-1}{2} - n\right)$ $\boxed{\text{M: Even}}$

Phase characteristic

5 ---- $$\measuredangle H(w) = \begin{cases} -w\left(\frac{M-1}{2}\right) & \text{if } H_r(w) > 0 \\ -w\left(\frac{M-1}{2}\right) + \pi & \text{if } H_r(w) < 0 \end{cases}$$

Antisymmetric case

We have

$$h(n) = -h(M-1-n) \quad \text{——— (6)}$$

For M: odd $\Rightarrow$ center point at $n = (M-1)/2$

$$h\left(\frac{M-1}{2}\right) = 0 \qquad \left(\text{Since } h(0) = -h(-0) = 0\right)$$

M: Even, each terms in $h(n)$ has a matching term of opposite sign.

Using (6) in (2) we get

since $\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$ in eqn. (2)

$$H(w) = H_r(w)\, e^{j[-w(M-1)/2 + \pi/2]}$$

where

$$(7a) \text{——} \quad H_r(w) = 2 \sum_{n=0}^{(M-3)/2} h(n) \sin w\left(\frac{M-1}{2} - n\right) \qquad M: \text{odd}$$

$$(7b) \text{——} \quad H_r(w) = 2 \sum_{n=0}^{(M/2)-1} h(n) \sin w\left(\frac{M-1}{2} - n\right) \qquad M: \text{Even}$$

Phase char.

$$(8) \text{——} \quad \measuredangle H(w) = \begin{cases} \frac{\pi}{2} - w\left(\frac{M-1}{2}\right) & \text{if } H_r(w) > 0 \\ \frac{3\pi}{2} - w\left(\frac{M-1}{2}\right) & \text{if } H_r(w) < 0 \end{cases}$$

Choice of symmetric / Antisymmetric h(n)

I- Antisymmetric $\Longrightarrow$ $h(n) = -h(M-1-n)$

M: odd $\Rightarrow$ From (7a) $H_r(0) = 0$, $H_r(\pi) = 0$

$\Rightarrow$ $\boxed{\text{(Antisymmetric + Odd) is not suitable for LPF, HPF or BSF}}$

M: Even $\Longrightarrow$ From (7b) $H_r(0) = 0$

$\boxed{\text{(Antisymmetric + Even) is not suitable for LPF or BSF}}$

II – Symmetric $\Longrightarrow h(n) = h(M-1-n)$

M: Even $\Longrightarrow$ From (4b) $H_r(\pi) = 0$

$\boxed{\text{(Symmetric + Even) is not suitable for HPF, BSF}}$

Design of Linear-phase FIR filters using Windows

Starting from a desired frequency response $H_d(w)$, we relate it to $h_d(n)$

$$H_d(w) = \sum_{n=0}^{\infty} h_d(n) e^{-jwn}$$

The corresponding unit sample response $h_d(n)$ is given by

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(w) e^{jwn} \, dw$$

But $h_d(n)$ is infinite in duration and must be truncated at some point.

Define
$$w(n) = \begin{cases} 1 & n = 0, 1, \ldots, M-1 \\ 0 & \text{elsewhere} \end{cases}$$

We use this window ($w(n)$) to truncate $h_d(n)$ to get

The FIR practical filter is

$$h(n) = h_d(n)\, w(n)$$

$$h(n) = \begin{cases} h_d(n) & n = 0, 1, \dots, M-1 \\ 0 & \text{elsewhere} \end{cases}$$

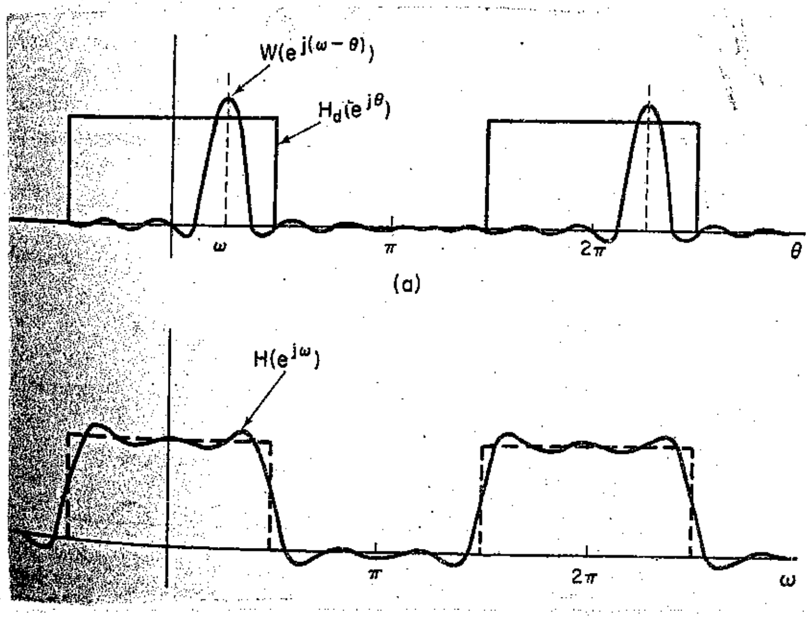
Effect of truncation on $H_d(\omega)$ (Rectangular window)

Multiplication property

$$h(n) = h_d(n)\, w(n) \xleftrightarrow{F} H(\omega) = H_d(\omega) * W(\omega)$$

$$\Rightarrow \quad H(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(v)\, W(\omega - v)\, dv$$

But $$W(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}} = e^{-j\omega(M-1)/2}\, \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$



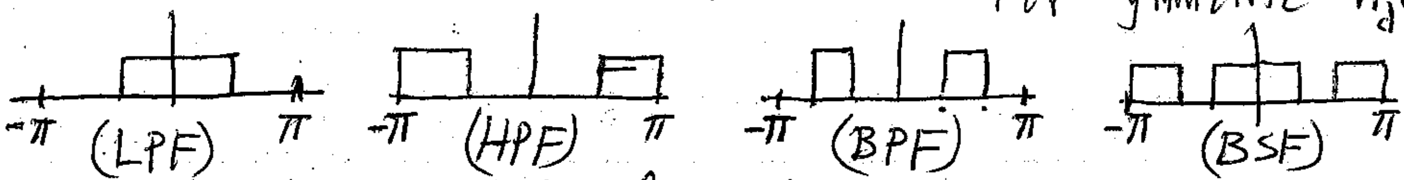
$H(\omega)$ looks like $H_d(\omega)$ on all points except at sharp transitions of $H_d(\omega)$. We need a narrower $W(\omega)$ (ideally an impulse).

85b

We can write $W(w)$ as $W(w) = W_r(w)\, e^{-jw\left(\frac{M-1}{2}\right)}$
where $W_r(w)$ is real and given by
$$W_r(w) = \frac{\sin(wM/2)}{\sin(w/2)} \quad \text{— 1}$$

We also select $H_d(w)$ to have Linear-phase, i.e in the form
$$H_d(w) = H_{dr}(w)\, e^{-jw(M-1)/2} \quad \text{— 2}$$
where $H_{dr}(w)$ is a real function in (w) ~~which~~ can be Low-pass, high-pass, band-pass or band-stop

for symmetric $h_d(n)$



It is also obvious that the form of 2 is necessary since it represents a shift to $h_{dr}(n)$ (i.e $F^{-1}\{H_{dr}(n)\}$) by $n = (M-1)/2$ to the right ($h_{dr}(n)$ is symmetrical about $n=0$) leading to $h_d(n) = h_{dr}(n-(M-1)/2)$ that is symmetric about $n=(M-1)/2$, which when multiplied by $w(n)$ (also symmetrical about $n=(M-1)/2$) gives truncated $h(n)$ that is symmetric about $(M-1)/2$ and hence Linear-phase.
Shifting $h_{dr}(n)$ by $n \neq (M-1)/2$ will result in non-symmetric $h(n)$ and non-Linear phase. If 1 and 2 are satisfied then

$$H(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(v)\, W(w-v)\, dv = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{dr}(v)\, e^{-jv\left(\frac{M-1}{2}\right)}\, W_r(w-v)\, e^{-j(w-v)\left(\frac{M-1}{2}\right)}\, dv$$

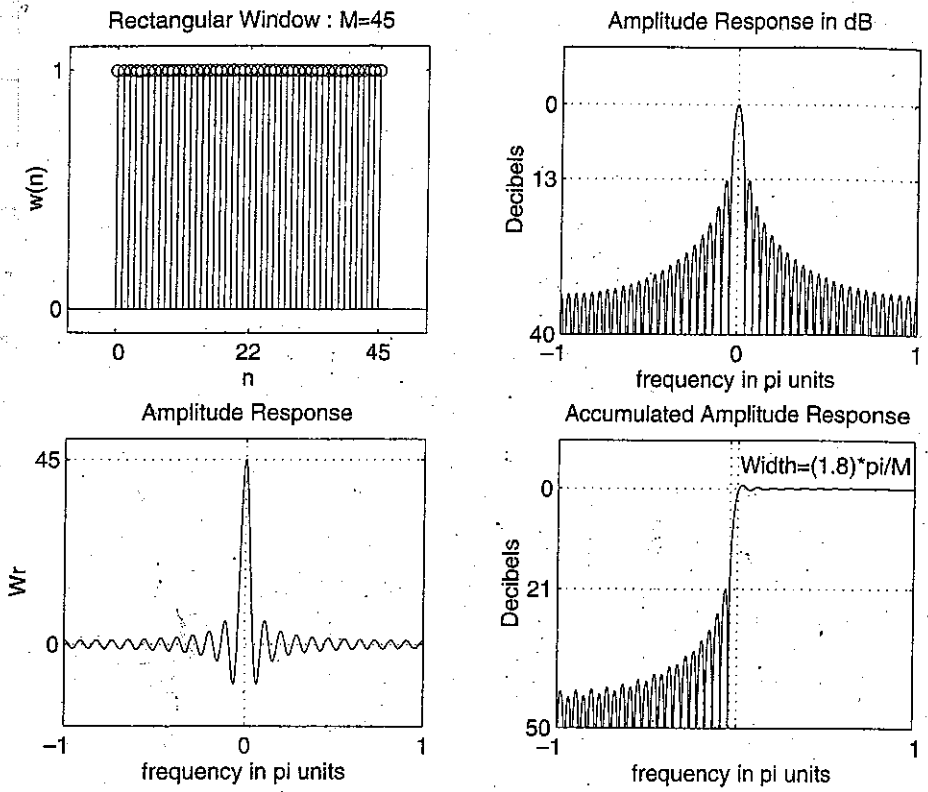
$$\boxed{H(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_r(v)\, W_r(w-v)\, e^{-jw\left(\frac{M-1}{2}\right)}\, dv = H_r(w)\, e^{-jw\left(\frac{M-1}{2}\right)}}$$

where $\boxed{H_r(w) = (1/2\pi)\int_{-\pi}^{\pi} H_r(v)\, W_r(w-v)\, dv}$ Convolution $H_r(w) * W_r(w)$

which has the form of a Linear-phase filter.

We can reduce width of $W(\omega)$ by increasing M.

If we increase M, the width of each side lobe in $W(\omega)$ will decrease,



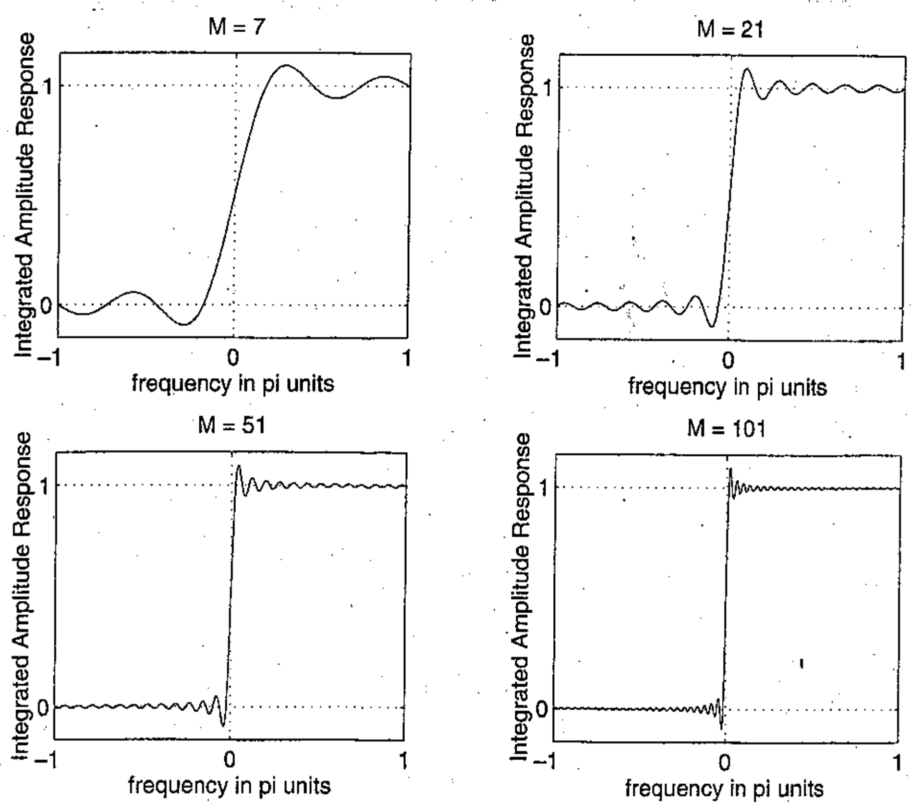
Rectangular window: $M = 45$

but relative amplitudes of sidelobes in $H(\omega)$ will remain constant. (Gibbs phenomenon)

∴ Rectangular window

⇓

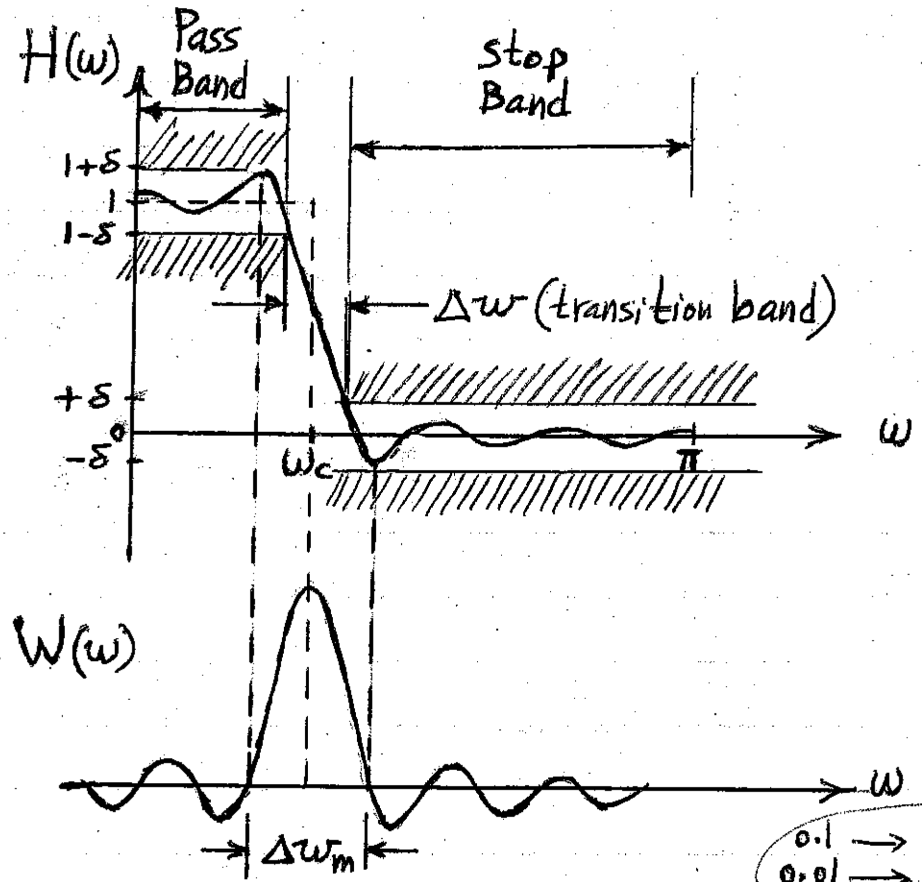
Relatively Large side Lobes in $H(\omega)$ (Gibbs phen.)



Gibbs phenomenon

To reduce sidelobes we use windows that are tapered smoothly to zero at each side so that side Lobes in $H(\omega)$ are reduced at the expense of wider main lobe in $H(\omega)$.

The general shape of the amplitude response of the filter $H(w)$ due to use of windows is as shown



0.1 → 20 dB
0.01 → 40 dB

<u>Definitions:</u>

Stop-band minimum attenuation $= -20 \log_{10}(\delta)$ (dB) $\frac{1}{\delta}$

Pass-band ripple $= -20 \log_{10}(1-\delta)$ (dB) $\frac{1}{1-\delta}$

Transitition width $= \Delta w$

Width of main Lobe $= \Delta w_m$

Cutoff frequency $= \omega_c$

Some commonly used windows

Windows are classified as fixed, and adjustable windows.

A - Fixed Windows

1 - Rectangular window

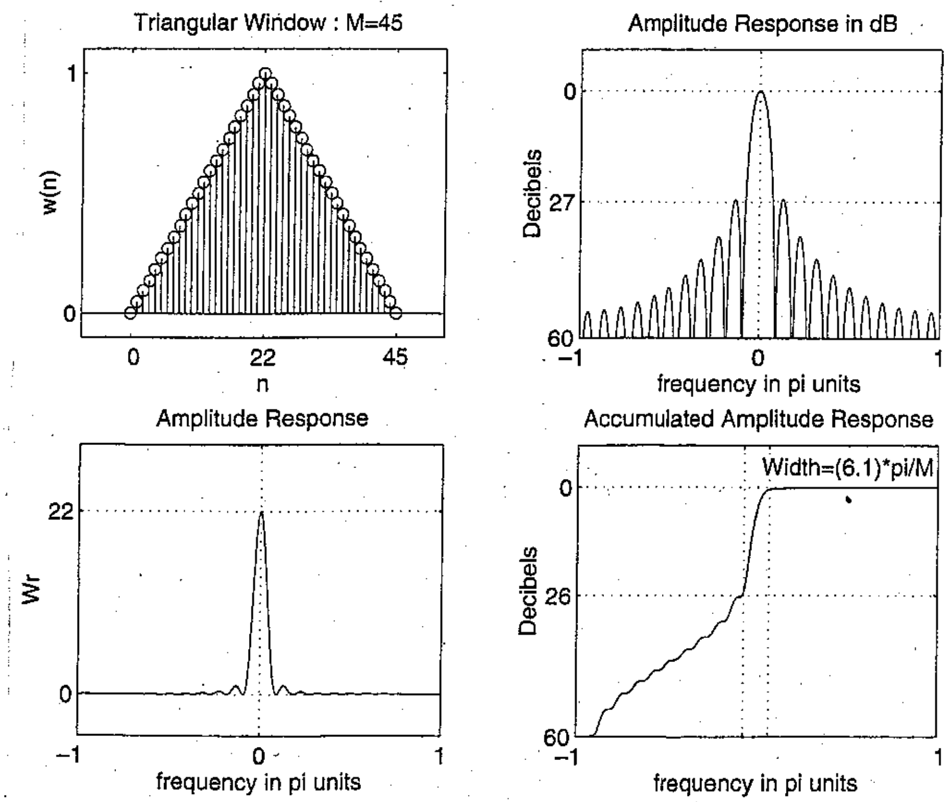
$$w(n) = \begin{cases} 1 & n = 0, 1, ----, M-1 \\ 0 & \text{elsewhere} \end{cases}$$

Figure shown earlier.

2 - Bartlett window

$$w(n) = 1 - \frac{2\left|n - \frac{M-1}{2}\right|}{M-1} \quad \text{for } n = 0, 1, ---, M-1$$

or zero elsewhere



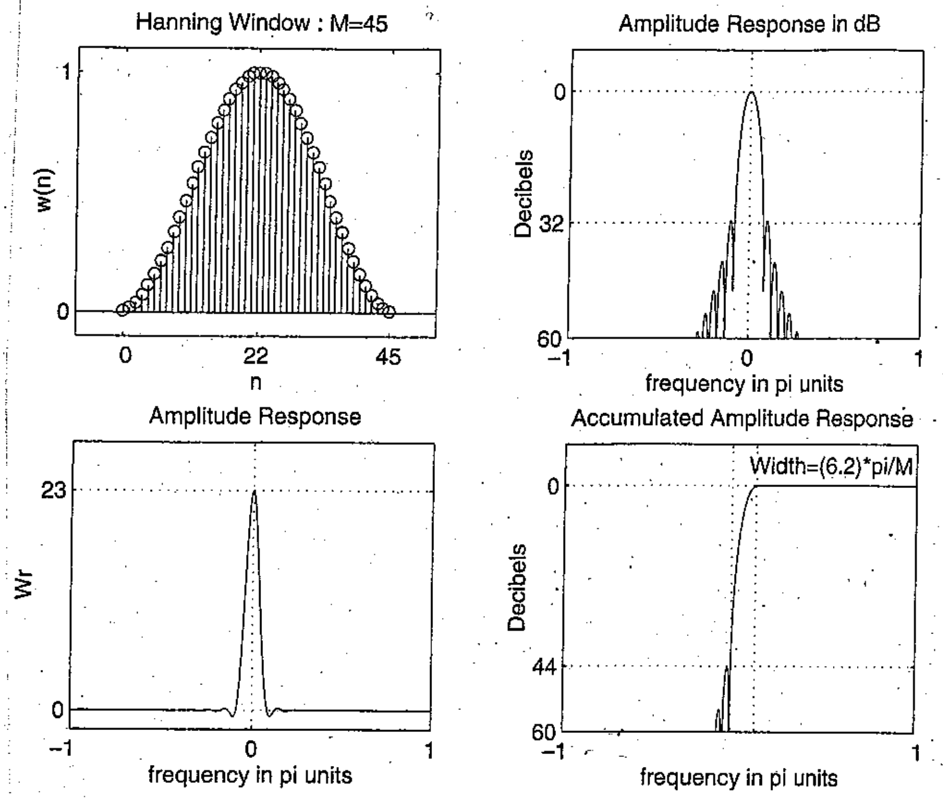
Bartlett (triangular) window: $M = 45$

3- Hanning window

$$w(n) = \frac{1}{2}\left(1 - \cos\frac{2\pi n}{M-1}\right)$$

for $n = 0, 1, ---, M-1$

$= 0$ (elsewhere)



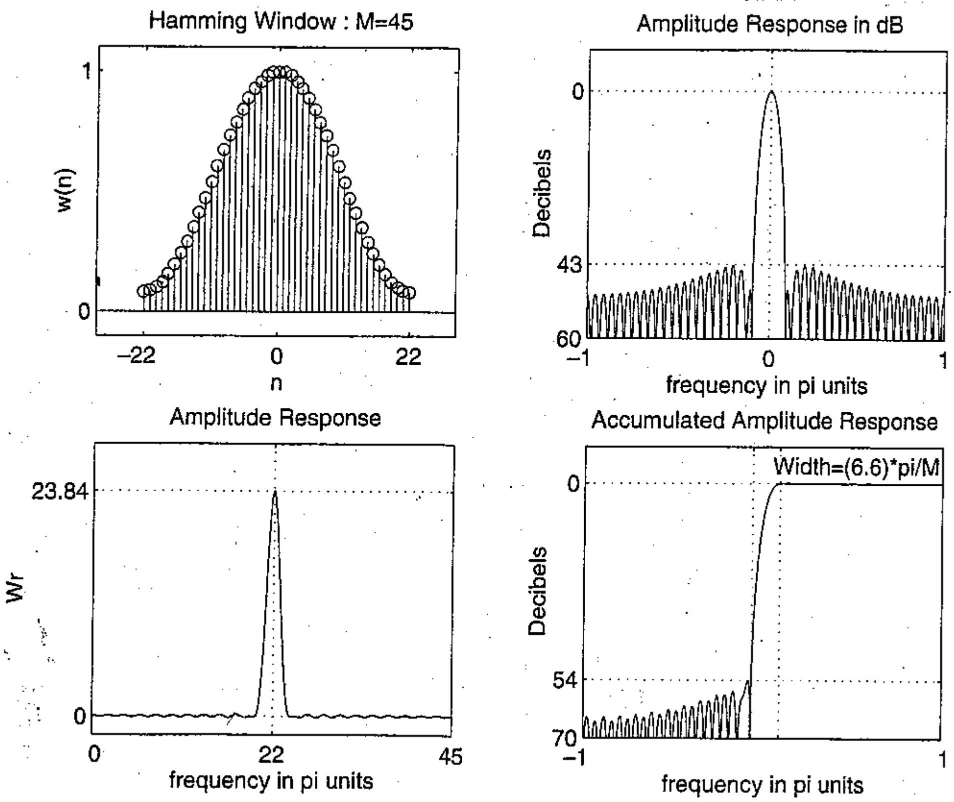
Hanning window: $M = 45$

4- Hamming window

$$w(n) = 0.54 - 0.46\cos\frac{2\pi n}{M-1}$$

for $n = 0, 1, ---, M-1$

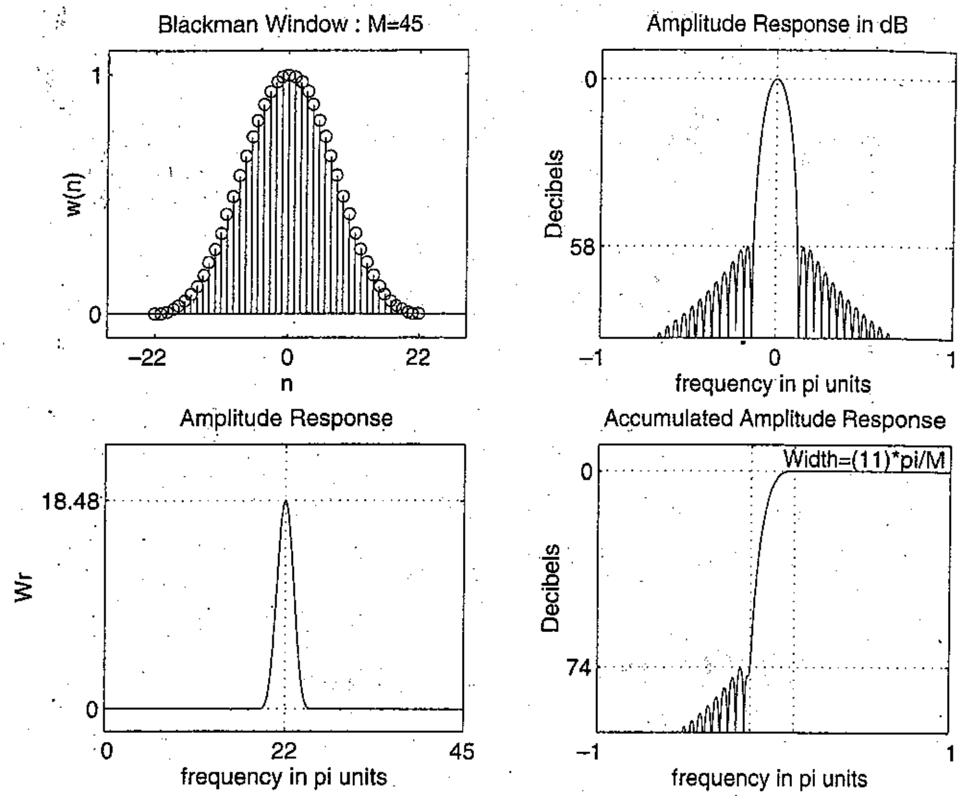
$= 0$ (elsewhere)



Hamming window: $M = 45$

5- Blackman window

$$w(n) = 0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1} \quad \text{for } n=0,1,\dots,M-1$$
$$= 0 \quad (\text{elsewhere})$$



Blackman window: $M = 45$

Window type	Pass-band Ripple (dB)	width of main Lobe	Transition width	Minimum stop-band attenuation (dB)	$w(n)$ for $n=0,1,\dots,M-1$
Rectangular	0.7416	$4\pi/M$	$1.81\pi/M$	21	1
Bartlett		$8\pi/M$	$6.1\pi/M$	25	$1 - \frac{2\left\|n-\frac{M-1}{2}\right\|}{M-1}$
Hanning	0.0546	$8\pi/M$	$6.2\pi/M$	44	$\frac{1}{2}\left(1-\cos\frac{2\pi n}{M-1}\right)$
Hamming	0.0194	$8\pi/M$	$6.6\pi/M$	53	$0.54 - 0.46\cos\frac{2\pi n}{M-1}$
Blackman	0.0017	$12\pi/M$	$11\pi/M$	74	$0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1}$

Design steps using fixed windows

step-1. Specify the "ideal" or desired frequency response $H_D(\omega)$.

step-2. Obtain the impulse response, $h_d(n)$, of the desired filter as

$$h_d(n) = F^{-1}\{H_D(\omega)\}$$

step-3. Select a suitable window that satisfies
i) stop-band attenuation requirement (minimum).
ii) Pass-band ripple (if specified).

step-4. Select filter Length (M) to satisfy the transition-width $(\Delta\omega)$ requirement $(\Delta\omega \propto 1/M)$.

step-5. Obtain $w(n)$ from the chosen window and Length (M) and then determine $h(n)$ as

$$h(n) = h_d(n)\, w(n) \qquad n = 0, 1, \dots, M-1$$

Example on selecting window type and length (M)

Design an FIR Low pass filter (LPF) with specifications
1- Transition width $(TW) \leq 0.1\pi$
2- Stop-band attenuation ≥ 50 dB.
3- Pass-band ripple ≤ 0.25 dB

A- From table of windows

Hamming	53 dB	0.019 dB
Blackman	74 dB	0.0017 dB

Both Hamming and Blackman satisfy attenuation and ripple requirements.

B- Hamming $TW = 6.6\pi/M$ Blackman $TW = 11\pi/M$

$M \geq \dfrac{6.6\pi}{TW}$ (gives a shorter filter) $\Rightarrow$ $M \geq \dfrac{11\pi}{TW}$ (select Hamming)

Using Hamming window
$$M \geq \frac{6.6\pi}{0.1\pi} \geq 66$$

Using Blackman
$$M \geq \frac{11\pi}{0.1\pi} \geq 110$$

$\therefore$ Using Hamming window will satisfy requirements with a shorter filter (67-taps) than Blackman (110).

Selecting $H_d(w)$ with Linear phase

We note that all windows satisfy

$$w(n) = \begin{cases} w(M-n-1) & 0 \leq n \leq M-1 \\ 0 & \text{elsewhere} \end{cases}$$

i.e they are all symmetric about $n = (M-1)/2$, and hence they will have Fourier Transform of the form

$$W(w) = W_r(w)\, e^{-jw(M-1)/2}$$

$W_r(w)$: real and even in (w).

If we select $h_d(n) = h_d(M-n-1)$ which is symmetric about $\left(n = \frac{M-1}{2}\right)$ then the windowed impulse response

$$h(n) = h_d(n)\, w(n)$$

will also have the same symmetry about $n = (M-1)/2$ and will have frequency response that is Linear-phase having the form

$$H(w) = A_r(w)\, e^{-jw(M-1)/2} \qquad A1$$

where $A_r(w)$ is real and even in (w).

If we select $h_d(n)$ to be <u>antisymmetric</u> about $n = (M-1)/2$ (i.e. $h_d(n) = -h_d(M-n-1)$) then the windowed impulse response $h(n)$ will also be antisymmetric about $n = (M-1)/2$ and will have a frequency response that is Linear-phase as

$$H(w) = A_r(w)\, e^{-jw(M-1)/2}\, e^{j\pi/2} \qquad \text{A2}$$

$A_r(w)$ is real and odd in (w).

Using the facts

Real, even $x(n) \Longleftrightarrow$ Real, even $X(w)$

Real, odd $x(n) \Longleftrightarrow$ Imaginary and odd $X(w)$

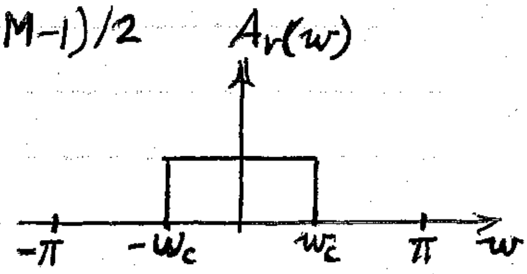
and in order to design a causal and Linear-phase FIR filters we will make the following selections:

Low-Pass Filter (LPF)

select
$$H_d(w) = \begin{cases} 1\, e^{-jw(M-1)/2} & 0 \le |w| \le w_c \\ 0 & \text{elsewhere} \end{cases}$$

to represent the frequency response of an ideal LPF with a cutoff frequency (w_c) rad/sec. We can write it as

$$H_d(w) = A_r(w)\, e^{-jw(M-1)/2}$$



$\underbrace{A_r(w)}_{}$ Even in (w)

$A_r(w) = 1$ for $0 \le |w| \le w_c$

We find $h_d(n)$ as the inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(w)\, e^{jwn}\, dw = \frac{1}{2\pi}\int_{-w_c}^{+w_c} e^{-jw(M-1)/2}\, e^{jwn}\, dw$$

$$\therefore \quad h_d(n) = \frac{1}{2\pi}\int_{-w_c}^{w_c} e^{jw\left(n-\frac{M-1}{2}\right)}\,dw = \frac{1}{2\pi}\left[\frac{e^{jw\left(n-\frac{M-1}{2}\right)}}{j\left(n-\frac{M-1}{2}\right)}\right]_{-w_c}^{w_c}$$

$$= \frac{1}{2\pi j\left(n-\frac{M-1}{2}\right)}\left\{\left[\cancel{\cos w_c\left(n-\tfrac{M-1}{2}\right)} + j\sin w_c\left(n-\tfrac{M-1}{2}\right) - \cancel{\cos\left[(-w_c)\left(n-\tfrac{M-1}{2}\right)\right]}\right.\right.$$

$$\left.\left. - j\sin\left[(-w_c)\left(n-\tfrac{M-1}{2}\right)\right]\right]\right\}$$

$$= \frac{2j\sin\left[(w_c)\left(n-\frac{M-1}{2}\right)\right]}{2\pi j\left(n-\frac{M-1}{2}\right)} = \frac{\sin w_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)}$$

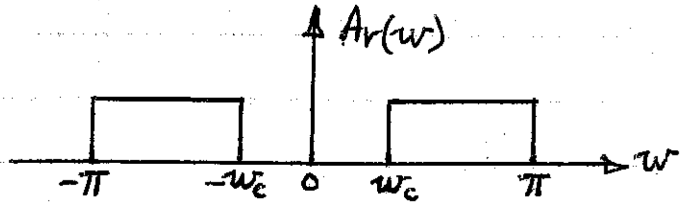
$$\therefore \quad \boxed{h_{LP}(n) = \frac{\sin w_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} \quad \text{for } n = 0, 1, \ldots, M-1 \text{ except if } n = \frac{M-1}{2} : \text{Odd } M \text{ case}}$$

$$\left(\text{for } n = \tfrac{M-1}{2}\right) \; h_{LP}(n) = (1/2\pi)\left[w\right]_{-w_c}^{w_c} = 2w_c/2\pi = \boxed{w_c/\pi}$$

High-Pass Filter (HPF)

Select
$$H_d(w) = \begin{cases} 0 & 0 \le |w| < w_c \\ 1\,e^{-jw(M-1)/2} & w_c \le |w| \le \pi \end{cases}$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(w)\, e^{jwn}\,dw$$



$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-w_c} e^{-jw\frac{M-1}{2}}\, e^{jwn}\,dw + \int_{w_c}^{\pi} e^{-jw\frac{M-1}{2}}\, e^{jwn}\,dw\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-w_c} e^{jw\left(n-\frac{M-1}{2}\right)}\,dw + \int_{w_c}^{\pi} e^{jw\left(n-\frac{M-1}{2}\right)}\,dw\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{e^{jw\left(n-\frac{M-1}{2}\right)}}{j\left(n-\frac{M-1}{2}\right)}\right]_{-\pi}^{-w_c} + \left[\frac{e^{jw\left(n-\frac{M-1}{2}\right)}}{j\left(n-\frac{M-1}{2}\right)}\right]_{w_c}^{\pi}\right\}$$

$$h_d(n) = \frac{1}{2\pi j\left(n-\frac{M-1}{2}\right)}\Bigg[\cancel{\cos\left[-w_c\left(n-\frac{M-1}{2}\right)\right]} + j\sin\left[-w_c\left(n-\frac{M-1}{2}\right)\right] - \cancel{\cos\left[-\pi\left(n-\frac{M-1}{2}\right)\right]}$$

$$- j\sin\left[-\pi\left(n-\frac{M-1}{2}\right)\right] + \cancel{\cos\left[\pi\left(n-\frac{M-1}{2}\right)\right]} + j\sin\left[\pi\left(n-\frac{M-1}{2}\right)\right] - \cancel{\cos\left[w_c\left(n-\frac{M-1}{2}\right)\right]}$$

$$- j\sin\left[w_c\left(n-\frac{M-1}{2}\right)\right]\Bigg]$$

$$= \frac{1}{2\pi j\left(n-\frac{M-1}{2}\right)}\left[2j\sin\pi\left(n-\frac{M-1}{2}\right) - 2j\sin w_c\left(n-\frac{M-1}{2}\right)\right]$$

$$= \frac{1}{\pi\left(n-\frac{M-1}{2}\right)}\left[\sin\pi\left(n-\frac{M-1}{2}\right) - \sin w_c\left(n-\frac{M-1}{2}\right)\right]$$

If $n = \frac{M-1}{2} \Longrightarrow h_d(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-w_c} e^0\,dw + \int_{w_c}^{\pi} e^0\,dw\right]$

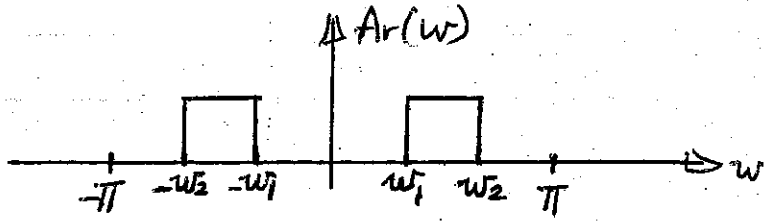
$$= \frac{1}{2\pi}\left\{ w\Big]_{-\pi}^{-w_c} + w\Big]_{w_c}^{\pi} \right\} = \frac{1}{2\pi}\left[-w_c+\pi+\pi-w_c\right]$$

$$= 1 - \frac{w_c}{\pi}$$

$$\therefore \boxed{\begin{aligned} h_{HP}(n) &= \frac{\sin\pi\left(n-\frac{M-1}{2}\right) - \sin w_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} \qquad n = 0, 1, \dots, M-1 \\ &= \left(1-\frac{w_c}{\pi}\right) \qquad \boxed{\text{if } n = \frac{M-1}{2}} \quad \text{case of Odd-}M \end{aligned}}$$

Band-Pass Filter (BPF)

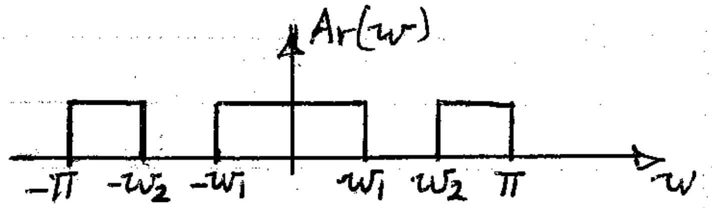
We select
$$H_d(w) = \begin{cases} e^{-jw\left(\frac{M-1}{2}\right)} & w_1 \le |w| \le w_2 \\ 0 & \text{elsewhere} \end{cases}$$



$$\boxed{\begin{aligned} h_{BP}(n) &= \frac{\sin w_2\left(n-\frac{M-1}{2}\right) - \sin w_1\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} \qquad n = 0, 1, \ldots, M-1 \\ &= \left(\frac{w_2 - w_1}{\pi}\right) \quad \text{if } n = \frac{M-1}{2} \quad M: \text{odd} \end{aligned}}$$

Band-stop Filter (BSF)

$$H_d(w) = \begin{cases} e^{-jw\left(\frac{M-1}{2}\right)} & \rightarrow \begin{cases} 0 < |w| \le w_1 \\ w_2 < |w| \le \pi \end{cases} \\ 0 & \text{elsewhere} \end{cases}$$



$$\boxed{\begin{aligned} h_{BS}(n) &= \frac{\sin w_1\left(n-\frac{M-1}{2}\right) - \sin w_2\left(n-\frac{M-1}{2}\right) + \sin \pi\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} \qquad n = 0, 1, \ldots, M. \\ &= \left(1 + \frac{w_1 - w_2}{\pi}\right) \quad \text{if } n = \frac{M-1}{2} \quad \text{Happens if } M: \text{odd} \end{aligned}}$$

Design Examples

Ex) Design an FIR Low-pass filter (LPF) to meet the following specifications:

Cutoff frequency( ) = 1.5 KHz ($w_c = 2\pi f_c$)
Transition width (TW) = 0.5 KHz
Stop-band attenuation (A_s) $\geq$ 50 dB
Sampling frequency (f_s) = 8 KHz

Sampling $\sin 2\pi f_0 t$ ($t = nT$)
$= \sin 2\pi f_0 nT$
$= \sin 2\pi \frac{f_0}{f_s} n$ ($w_0 = \frac{f_0}{f_s} 2\pi$)

1- Convert into $(0, 2\pi)$ radian frequency range. ($f_s \Leftrightarrow 2\pi$)

$$W_c = 2\pi \times (1.5/8) = 0.375\pi$$
$$TW = 2\pi \times (0.5/8) = 0.125\pi$$

2- Select a window that satisfies $A_s > 50$ dB.
Hamming window and Blackman's window both satisfy this requirement. However Hamming window has a smaller TW

$TW(\text{Hamming}) = 6.6\pi / M$ (M: filter Length)
$TW(\text{Blackman}) = 11\pi / M$

We select Hamming to satisfy requirements of (TW) with a smaller length (M) of the filter.

3- Select filter Length to satisfy (TW).
(Hamming) $TW = 6.6\pi / M$
$0.125\pi = 6.6\pi / M$ $M = 6.6/0.125$

we select $M \geq 53$
$M = 53$.

4- Compute window response $w(n)$, desired response $h_d(n)$, and the truncated filter response $h(n)$

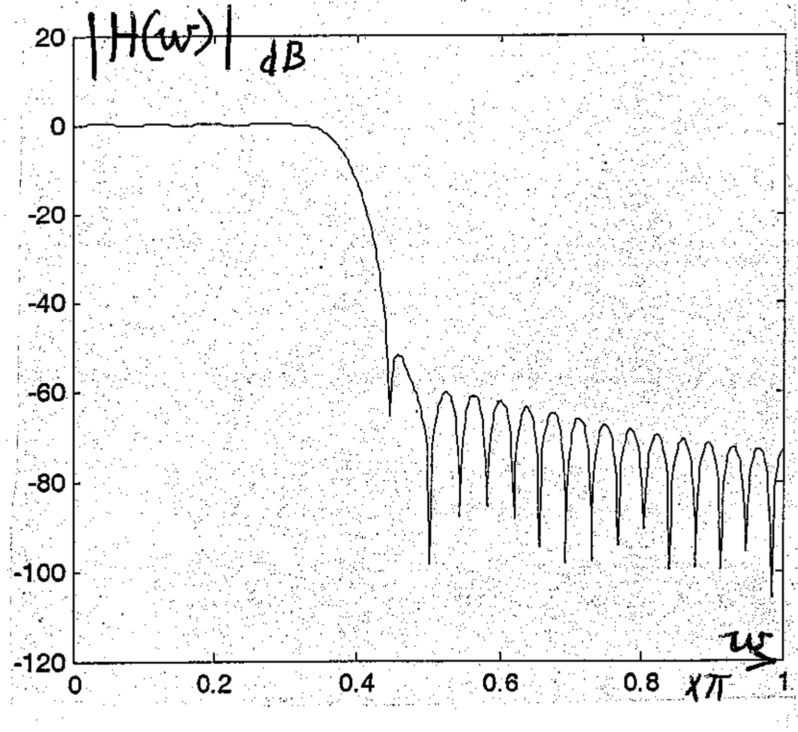
$$h(n) = w(n)\, h_d(n) = w_{Ham}(n) \times h_{LP}(n)$$

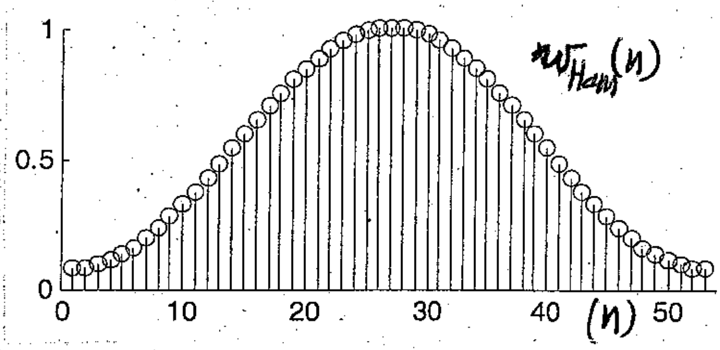
$$w_{Ham}(n) = 0.54 - 0.46 \cos \frac{2\pi n}{M-1}$$

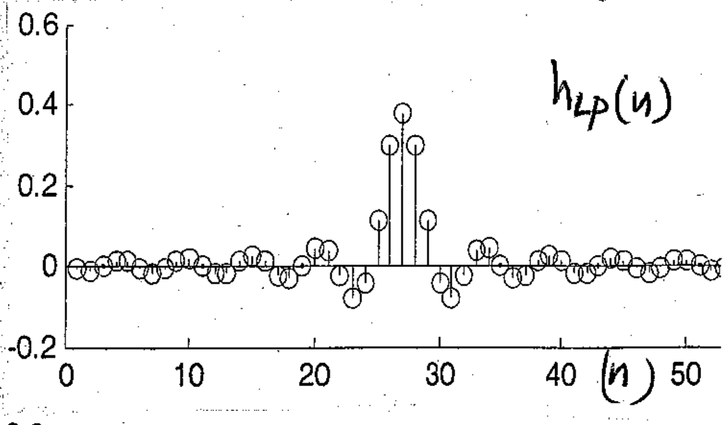
$$h_{LP}(n) = \begin{cases} \dfrac{\sin w_c \left(n - \frac{M-1}{2}\right)}{\pi \left(n - \frac{M-1}{2}\right)} & \text{for } n = 0, 1, \ldots, M-1 \text{ except } n = (M-1)/2 \\ \dfrac{w_c}{\pi} & \text{for } n = \dfrac{M-1}{2} \end{cases}$$

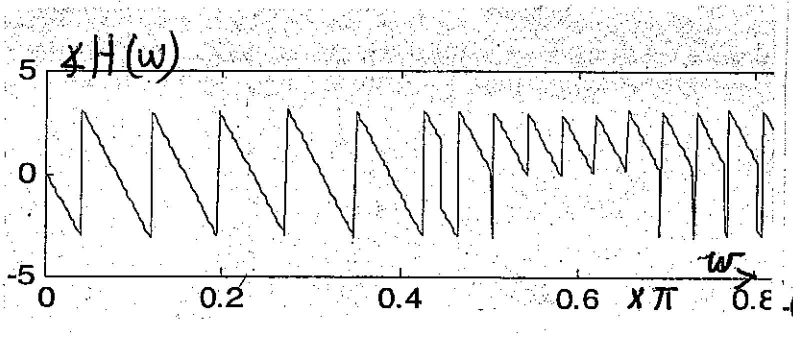
Substituting $w_c = 0.375\pi$, $M = 53$

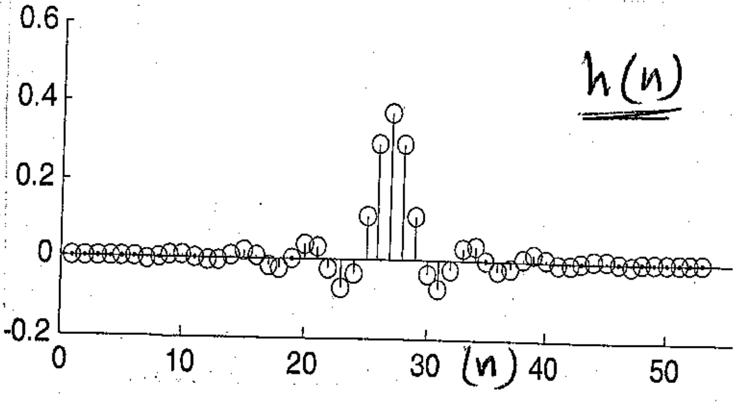
$$h(n) = \begin{cases} \left[0.54 - 0.46 \cos \frac{2\pi n}{52}\right]\left[\dfrac{\sin 0.375\pi \left(n - \frac{52}{2}\right)}{\pi \left(n - \frac{52}{2}\right)}\right] & \text{for } n = 0, 1, \ldots, 52 \text{ except } n = 26 \\ \left[0.54 - 0.46 \cos\left(\frac{2\pi \times 26}{52}\right)\right]\left[\dfrac{0.375\pi}{\pi}\right] & \text{for } n = 26 \end{cases}$$











Ex) Design a Band-Pass Filter (BPF) to meet the following

Pass-Band = 150 — 250 Hz $(\omega_1 - \omega_2)$

TW = 50 Hz

Pass-band ripple (R_p) = 0.1 dB

Stop-band attenuation (A_s) = 40 dB

Sampling frequency = 1 KHz

1- $TW = 2\pi \times (50/1000) = 0.1\pi$

$\omega_1 = 2\pi \times (150/1000) = 0.3\pi$

$\omega_2 = 2\pi \times (250/1000) = 0.5\pi$

2- Select window to satisfy R_p, A_s

Hamming, Hanning, Blackman all satisfy R_p, A_s

TW (Hanning) = $6.2\pi/M$

TW (Hamming) = $6.6\pi/M$

TW (Blackman) = $11\pi/M$

∴ Hanning has the smallest TW for specific (M) and hence it gives the smallest filter size (M).

3- Filter size (M)

(Han) $TW = 6.2\pi/M \Longrightarrow 0.1\cancel{\pi} = 6.2\cancel{\pi}/M$

$M = 62$

4- $h(n) = w_{Han}(n)\, h_{BP}(n)$

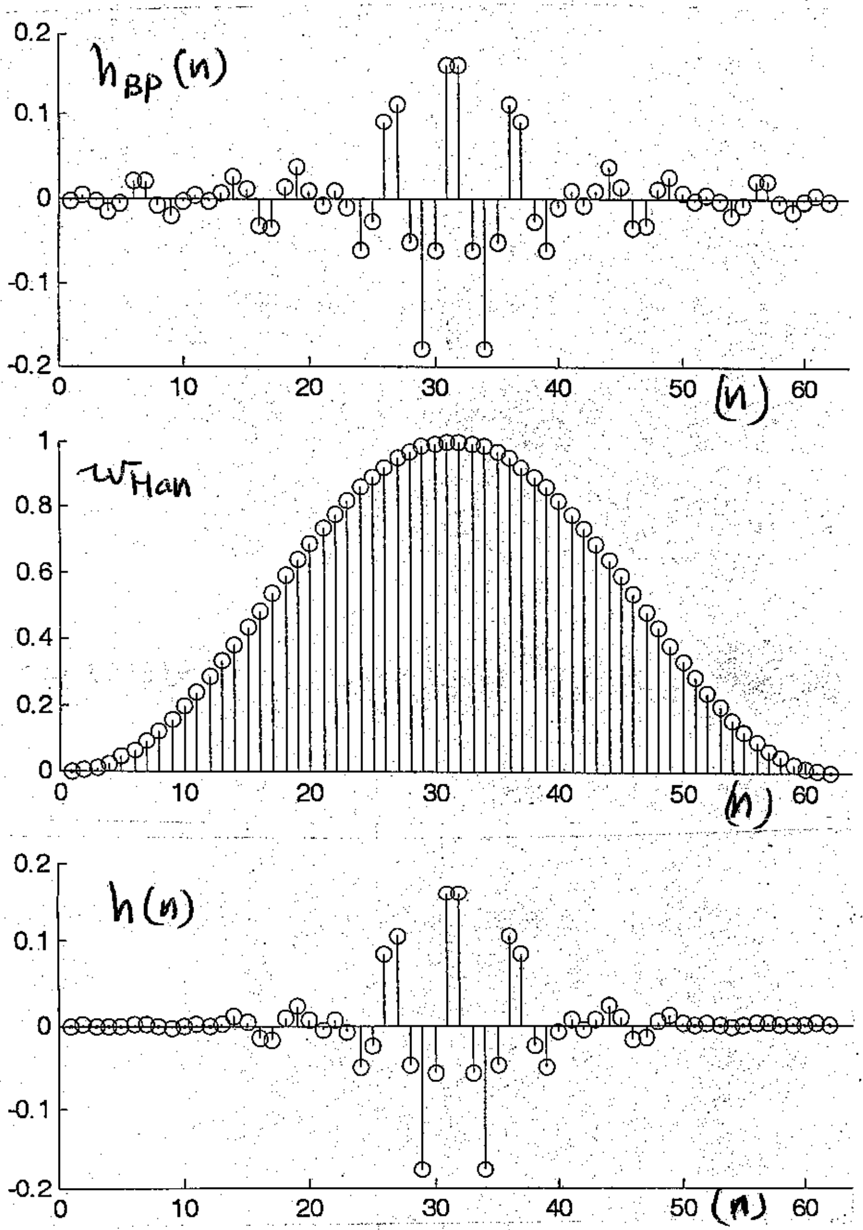
substituting $M = 62$, $\omega_1 = 0.3\pi$, $\omega_2 = 0.5\pi$

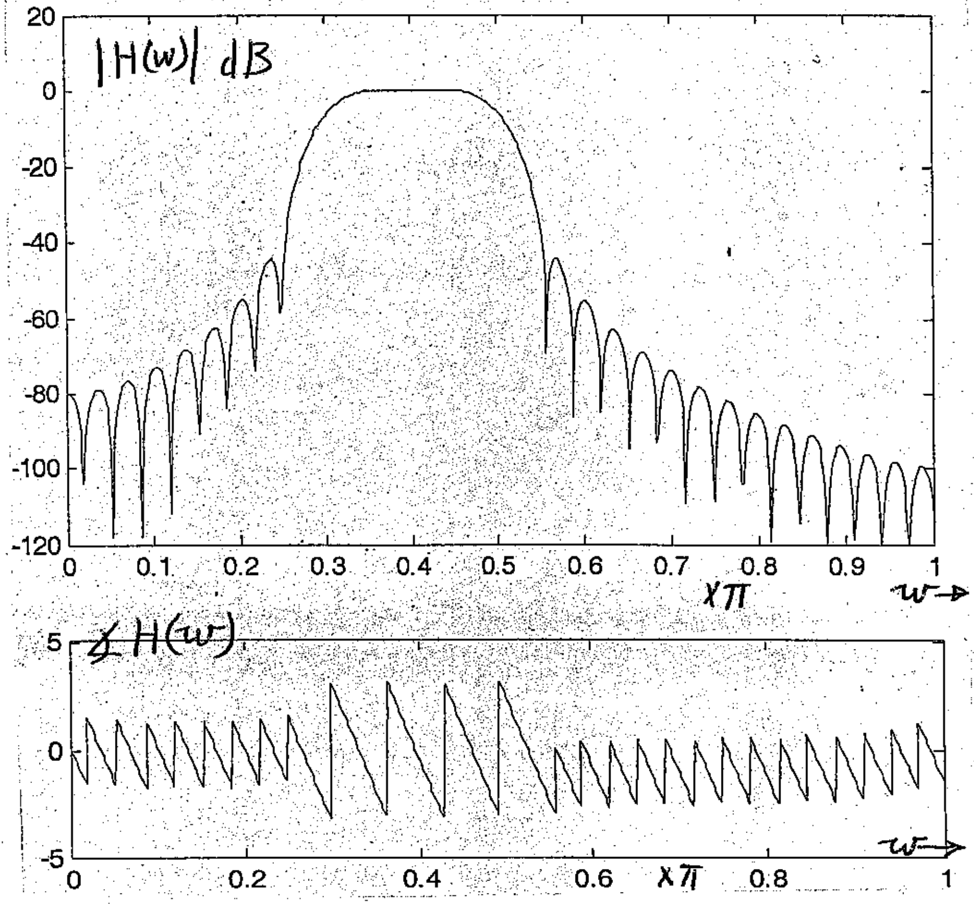
$$h(n) = \left[\frac{1}{2}\left(1 - \cos\frac{2\pi n}{61}\right)\right]\left[\frac{\sin 0.5\pi(n-\frac{61}{2}) - \sin 0.3\pi(n-\frac{61}{2})}{\pi(n-\frac{61}{2})}\right]$$

for $n = 0, 1, \ldots, M-1$

$(n \neq \frac{M-1}{2}$ since M is even$)$

5- Plot $H(\omega)$





M=62, BPF response

Ex) Design a High-Pass filter (HPF) with the following specifications:

Cutoff frequency = 400 Hz
Tran. wid. (TW) = 150 Hz
Stop-band att. $(A_s) \geq 70$ dB
Sampling frequeny = 2000 Hz

1- $W_c = 2\pi \times (400/2000) = 0.4\pi$
$TW = 2\pi \times (150/2000) = 0.15\pi$

2- Select window to satisfy $A_s > 70$ dB
Blackman window is selected.

3- Filter size
(Blackman Window) $\Rightarrow TW = \frac{11\pi}{M}$

$$0.15\cancel{\pi} = \frac{11\cancel{\pi}}{M} \Rightarrow M \geq 74$$

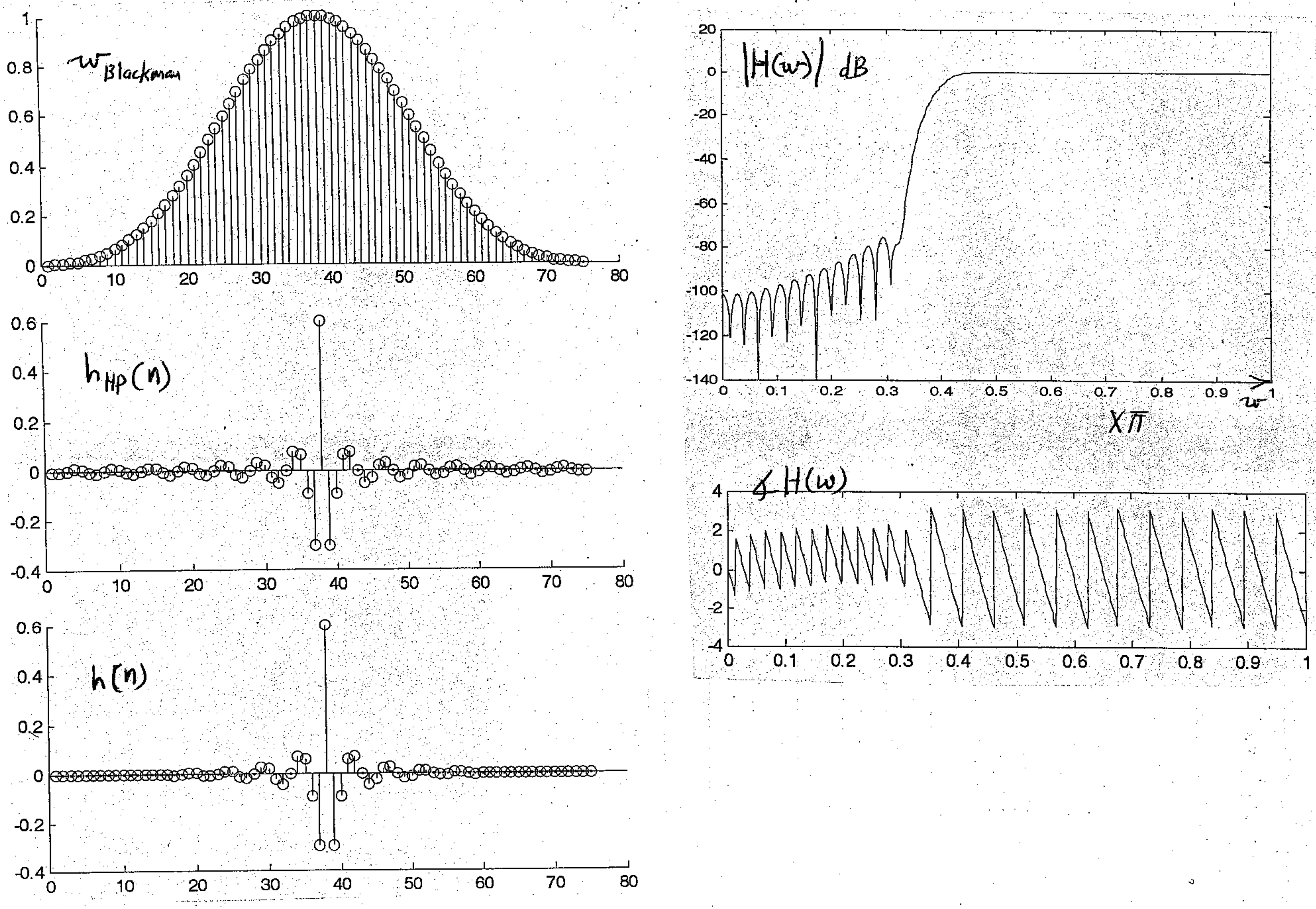
But for (Symmetric, Even) $H_r(\pi) = 0$ ∴ not suitable for HPF
Select $M = 75$

4- Using $M = 75$, $W_c = 0.4\pi$ we write $h(n) = h_{hp}(n)\, w(n)$

$$h(n) = \left[0.42 - 0.5\cos\frac{2\pi n}{74} + 0.08\cos\frac{4\pi n}{74}\right]\left[\frac{\sin\pi(n-\frac{74}{2}) - \sin 0.4\pi(n-\frac{74}{2})}{\pi(n-\frac{74}{2})}\right]$$

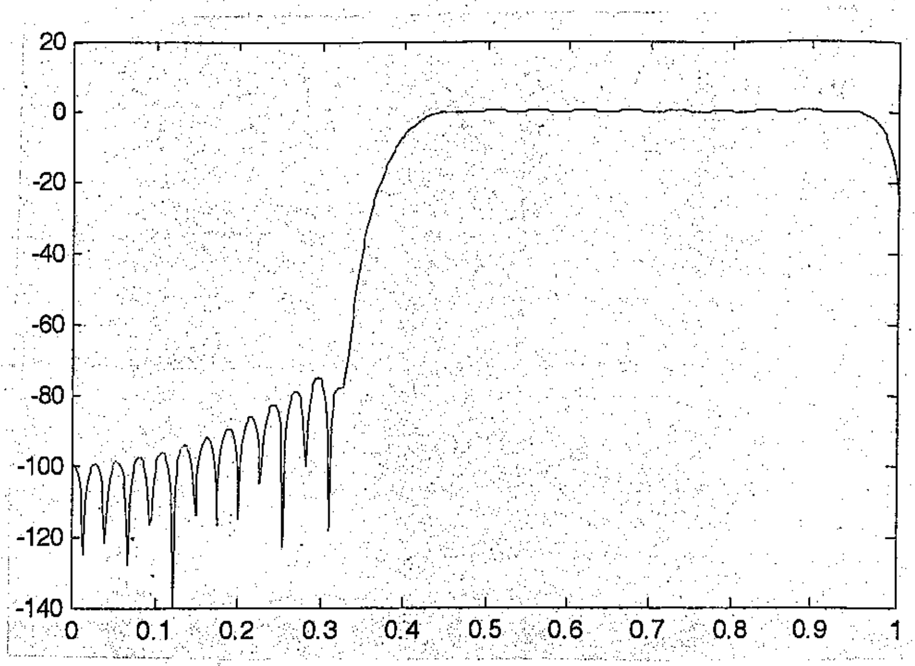
for $n = 0, 1, \ldots, 74$
except for $n = \frac{M-1}{2} = 37$

$$h(n) = \left[0.42 - 0.5\cos\frac{2\pi(37)}{74} + 0.08\cos\frac{4\pi(37)}{74}\right]\left[1 - \frac{0.4\pi}{\pi}\right] \quad \text{for } (n = 37)$$

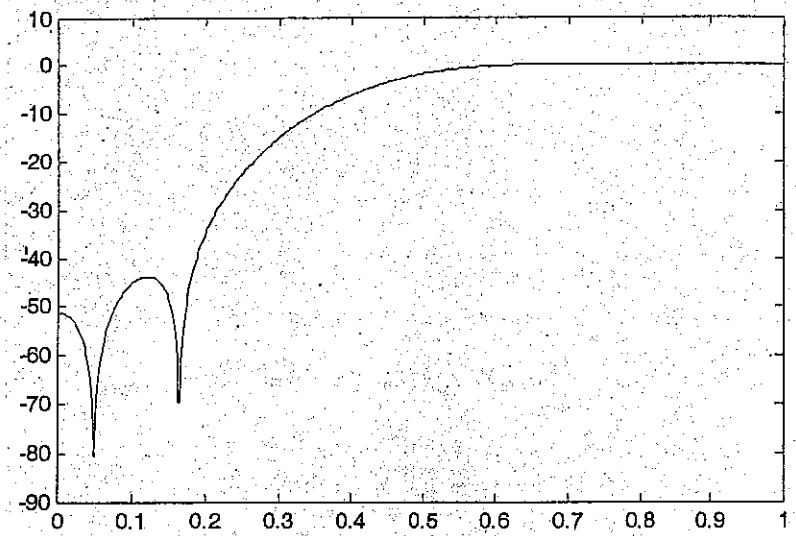


$w_{Blackman}$
$h_{HP}(n)$
$h(n)$
$|H(w)|$ dB
w
$\times\pi$
$\angle H(w)$

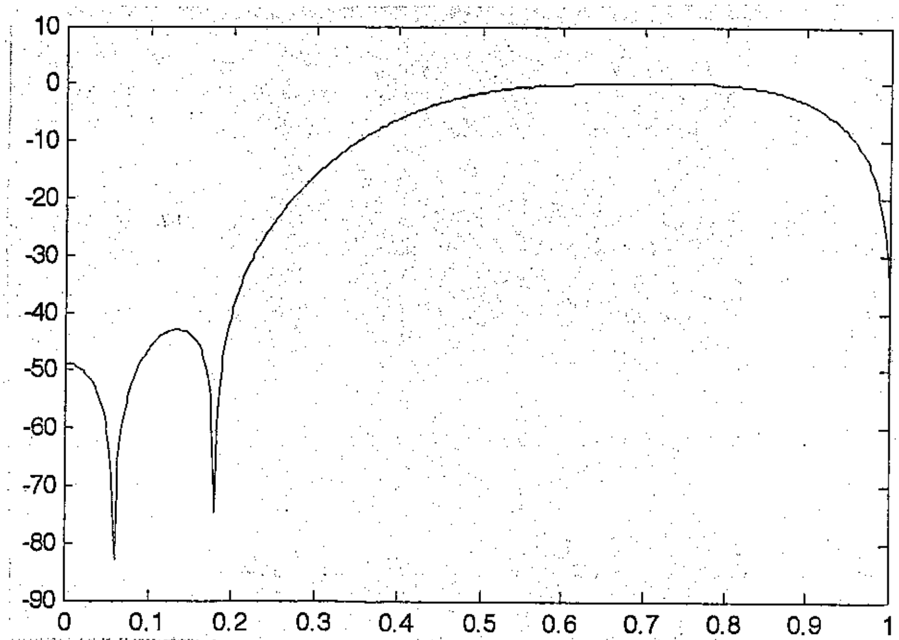
HPF ($M = 76$)
using Blackman window
(note effect of even
length of filter
$H(\pi) = 0$) not
suitable as HPF.
($W_c = 0.4\pi$)

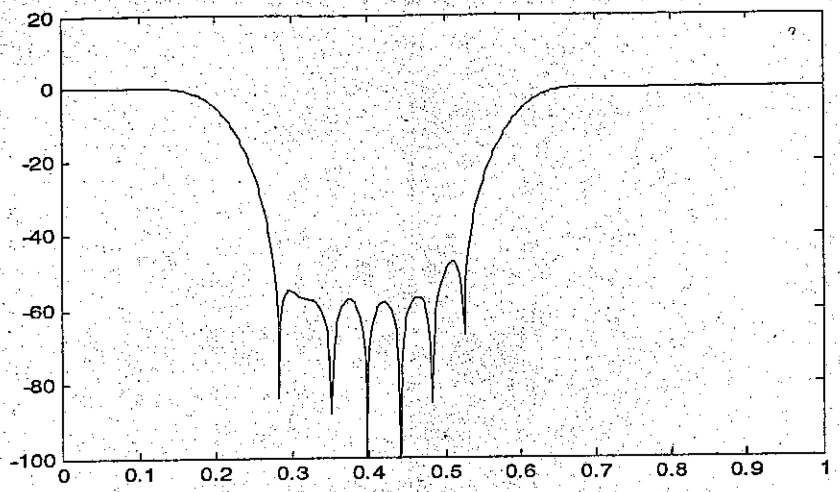


($M = 15$) HPF using
Hanning Window
($W_c = 0.4\pi$)

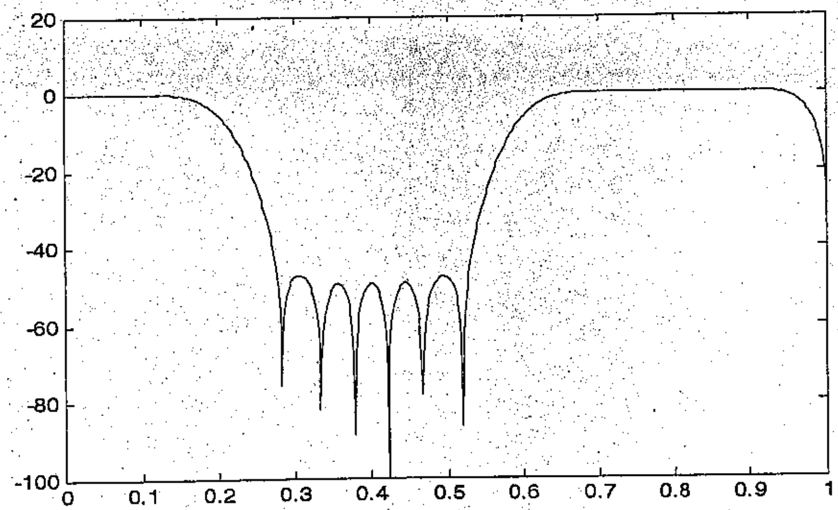


($M = 16$) HPF using
Hanning Window
(also shown that
$H(\pi) = 0$ for
even M)
($W_c = 0.4\pi$)





M= 45



M=46
($H(\pi)=0$)
not suitable for BSF.

(BSF) using Hamming window
($w_1 = 0.2\pi$, $w_2 = 0.6\pi$)

Digital FIR differentiator

For a continuous time signal

If $x(t) \xleftrightarrow{F} X(w)$

$$\frac{\partial x(t)}{\partial t} \xleftrightarrow{F} jw\,X(w)$$

$\therefore$ a differentiator is a filter with frequency response of

$$H(w) = jw$$

For phase-Linearity and causality we will select

$$\boxed{H_d(w) = \underbrace{(jw)}_{} e^{-jw\left(\frac{M-1}{2}\right)}}$$

$H_r(w)$ is imaginary and odd in (w) $\Longleftrightarrow$ $h_d(n)$ real and odd in (n) i.e antisymmetric about $n = \frac{M-1}{2}$

We determine $h_d(n)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(w)\, e^{jwn}\, dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} jw\, e^{-jw\left(\frac{M-1}{2}\right)} e^{jwn}\, dw$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{jw}_{U}\, \underbrace{e^{jw\left(n-\frac{M-1}{2}\right)} dw}_{dV}$$

$U = jw \Longrightarrow dU = j\,dw$

$$V = \int dV = \frac{e^{jw\left(n-\frac{M-1}{2}\right)}}{j\left(n-\frac{M-1}{2}\right)}$$

$$h_d(n) = \frac{1}{2\pi}\Big[UV - \int V\, dU\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{jw\, e^{jw\left(n-\frac{M-1}{2}\right)}}{j\left(n-\frac{M-1}{2}\right)} - \int \frac{e^{jw\left(n-\frac{M-1}{2}\right)}}{\cancel{j}\left(n-\frac{M-1}{2}\right)}\cancel{j}\, dw\right]_{-\pi}^{\pi}$$

$$h_d(n) = \frac{1}{2\pi}\left[\frac{\omega e^{j\omega(n-\frac{M-1}{2})}}{(n-\frac{M-1}{2})} - \frac{e^{j\omega(n-\frac{M-1}{2})}}{j(n-\frac{M-1}{2})^2}\right]_{-\pi}^{\pi}$$

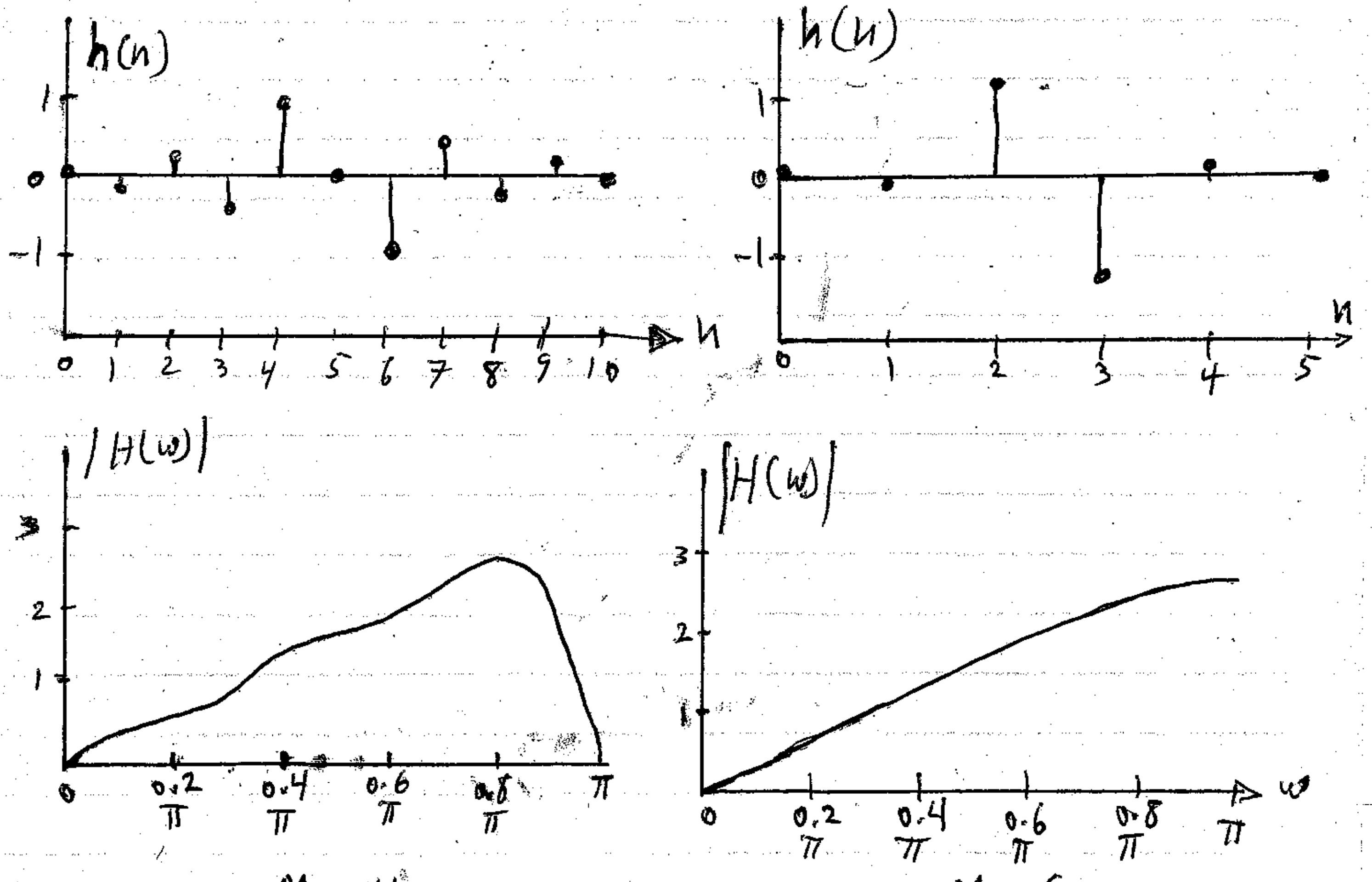
$$\boxed{h_{diff}(n) = \frac{\cos\pi(n-\frac{M-1}{2})}{n-\frac{M-1}{2}} - \frac{\sin\pi(n-\frac{M-1}{2})}{\pi(n-\frac{M-1}{2})^2}}$$

Using a window

$$\boxed{h(n) = h_{diff}(n)\, w(n)}$$

$h_{diff}(n)$ is antisymmetric about $n = \frac{M-1}{2} \Rightarrow \left.\begin{array}{l} H_r(0)=0 \\ H_r(\pi)=0 \end{array}\right\} M \text{ odd}$

$\therefore$ M = odd is not suitable for differentiator.



M = 11 M = 6

Adjustable Windows

Stop-band attenuation (A_s) and transition width (TW) of previous windows were fixed.
The trade off between main-lobe and sidelobe area (i.e TW versus A_s) can be controlled in adjustable windows to allow setting a required (A_s) and then set (TW) by increasing M.

1- Kaiser Window
2- Dolph-Chebyshev Window

We discuss Kaiser Window.

Kaiser Window

$$w(n) = \begin{cases} I_0\left[\beta\left(1-[(n-\alpha)/\alpha]^2\right)^{1/2}\right] & n=0,1,\ldots,M-1 \\ 0 & \text{elsewhere} \end{cases}$$

where

$$A_s = -20\log_{10}\delta$$

$$\alpha = \frac{M-1}{2}$$

$$\beta = \begin{cases} 0.1102\,(A_s-8.7) & \text{If } A_s > 50 \\ 0.5842\,(A_s-21)^{0.4} + 0.07886\,(A_s-21) & \text{If } 21 \le A_s \le 50 \\ 0 & \text{If } A_s < 21 \end{cases}$$

Transition Width $\quad TW = \dfrac{A_s - 8}{2.285(M-1)}$

$$I_0(x) = 1 + \sum_{k=1}^{L}\left[\frac{(x/2)^k}{k!}\right]^2$$ (typically $L \simeq 25$ is sufficient)

Ex) Design a LPF with $\omega_c = 0.5\pi$, $\delta = 0.001$, $TW = 0.2\pi$, using Kaiser window.

1- $A_s = -20\log_{10}\delta = 60\ dB$ (stop-band attenuation)

2- Since $A_s > 50 \Longrightarrow \beta = 0.1102(A_s - 8.7)$

$= 0.1102(60 - 8.7) = \boxed{5.653}$

3- $TW = \dfrac{A_s - 8}{2.285(M-1)}$

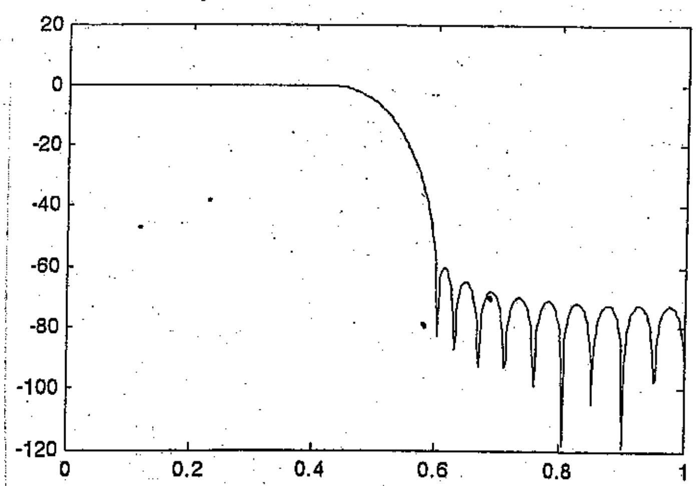
$0.2\pi = \dfrac{A_s - 8}{2.285(M-1)} \Longrightarrow M \simeq 37.2$

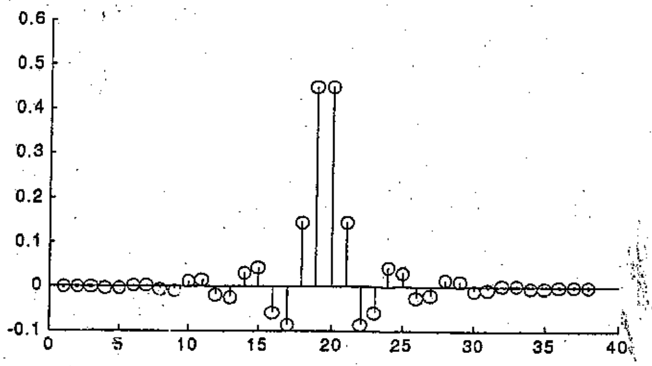
$\boxed{M = 38} \Longrightarrow \alpha = \dfrac{M-1}{2} = \boxed{18.5}$

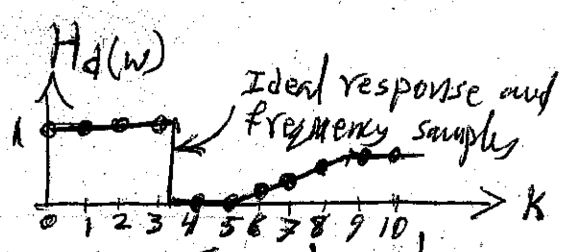
4- $w(n) = I_0\left[\beta\left(1 - [(n-\alpha)/\alpha]^2\right)^{1/2}\right]$

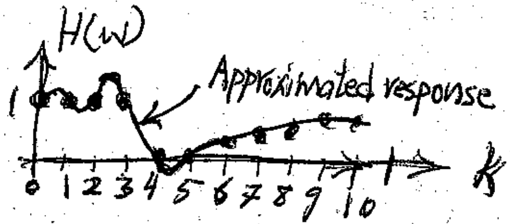
$= I_0\left[5.653\left(1 - [(n-18.5)/18.5]^2\right)^{1/2}\right]$ $(n = 0, 1, \ldots, 37)$

5- $h(n) = h_{Lp}(n)\,w(n)$









Frequency-Sampling design of FIR filters

In this method we specify the desired frequency response $H_d(w)$ at a set of equally spaced frequencies

$$w_k = \frac{2\pi k}{M} \qquad \begin{pmatrix} k = 0, 1, \ldots, M-1 \\ \text{i.e. } 0 \to 2\pi \end{pmatrix}$$

But we have

$$H(w) = \sum_{n=0}^{M-1} h(n)\, e^{-jwn}$$

substituting W_k's

$$H(w_k) = \sum_{n=0}^{M-1} h(n)\, e^{-j2\pi kn/M} \quad (k = 0, 1, \ldots, M-1)$$

Since $H(w_k)$ $(k = 0, 1, \ldots, M-1)$ are known, these set of M-Linear equations in M-unknowns $(h(n),\ n = 0, 1, \ldots, M-1)$ can be solved to obtain $h(n)$

$$\boxed{h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H(w_k)\, e^{j2\pi kn/M} \quad (n = 0, 1, \ldots, M-1) \quad \text{(IDFT)}}$$

We recall that a Linear-phase filter is described by

$$h(n) = \pm\, h(M-1-n) \qquad (n = 0, 1, \ldots, M-1)$$

which corresponds to

$$H(w) = H_r(w)\, e^{-jw(M-1)/2}$$

To ensure obtaining a real $h(n)$, the choice of $H(w_k)$ $(k = 0, 1, \ldots, M-1)$ should satisfy

$$H(w_k) = H^*(w_{-k}) = H^*(w_{M-k})$$

Therefore $H(w_k) \Longrightarrow$ $\boxed{H(w_k) = H_r(w_k)\, e^{j\measuredangle H(w_k)}}$

where

$$\boxed{\begin{array}{l} H_r(w_k) \\ \text{or } H_r\left(\frac{2\pi k}{M}\right) \end{array} = \begin{cases} H_r(0) & k = 0 \\ H_r\left(\frac{2\pi(M-k)}{M}\right) & k = 1, 2, \ldots, M-1 \end{cases}}$$

$$\underset{\text{(Symm.)}}{\measuredangle H(w_k)} = \begin{cases} -\left(\frac{M-1}{2}\right)\underbrace{\left(\frac{2\pi k}{M}\right)}_{w_k} & k=0,\ldots,\left\lfloor\frac{M-1}{2}\right\rfloor \\ +\left(\frac{M-1}{2}\right)\frac{2\pi}{M}(M-k) & k=\left\lfloor\frac{M-1}{2}\right\rfloor+1,\ldots,M-1 \end{cases}$$

[Symmetric $h(n)$]

$$\underset{\text{(Anti-sym)}}{\measuredangle H(w_k)} = \begin{cases} \left(\pm\frac{\pi}{2}\right)-\left(\frac{M-1}{2}\right)\left(\frac{2\pi k}{M}\right) & k=0,\ldots,\left\lfloor\frac{M-1}{2}\right\rfloor \\ -\left(\pm\frac{\pi}{2}\right)+\left(\frac{M-1}{2}\right)\frac{2\pi}{M}(M-k) & k=\left\lfloor\frac{M-1}{2}\right\rfloor+1,\ldots,M-1 \end{cases}$$

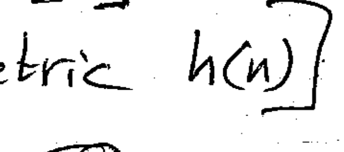
[For Anti-symmetric $h(n)$]

These equations ensure (see equations 41, 42 pages. 92, 93)

1- $H(w_k) = H^*(w_{M-k}) \Longrightarrow$ Real $h(n)$

2- The form $H(w_k) = \underbrace{H_r(w_k)}_{\text{+ve or -ve real}} e^{-jw_k\frac{M-1}{2}} \Longrightarrow$ Linear-phase

Ex) Design a LPF to have $w_c = 0.25\pi$, $A_s = 50\,dB$, $TW = 0.1\pi$.



Let us choose $M = 20$.

$w_p = w_c - TW/2 = 0.25\pi - 0.1\pi/2 = 0.2\pi$

$w_s = w_c + TW/2 = 0.25\pi + 0.1\pi/2 = 0.3\pi$

$k\text{-pass} = \frac{w_p}{2\pi} \times M = \frac{0.2\pi}{2\pi} \times 20 = 2 \Longrightarrow k = \underbrace{0,1,2}_{\text{Pass-band}}$

$k\text{-stop} = \frac{w_s}{2\pi} \times M = \frac{0.3\pi}{2\pi} \times 20 = 3$ (no. of zeros $= 20-3-2 = 15$)

∴ We select

$$H_r(w_k) = [\underset{k=0}{1}\ \underset{k=1}{1}\ \underset{k=2}{1}\ \underbrace{0 \cdots\cdots 0}_{15\text{-zeros}}\ \underset{M-2}{1}\ \underset{k=M-1}{1}]$$

For symmetric even $h(n)$

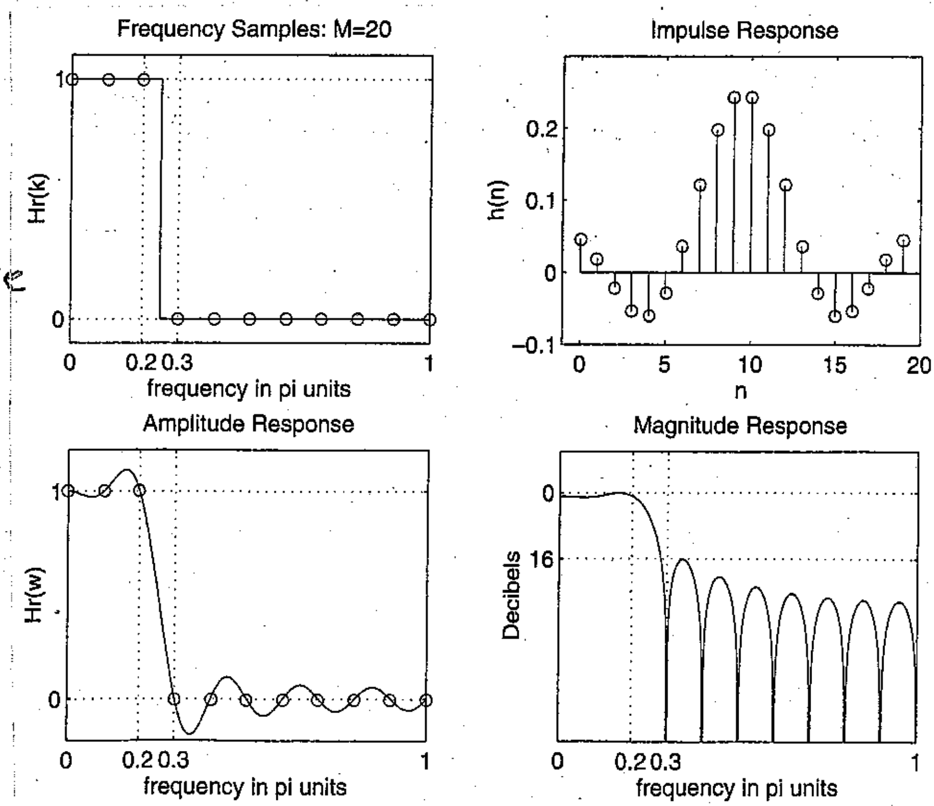
$$\measuredangle H(\omega_k) = \begin{cases} -\left(\frac{M-1}{2}\right)\left(\frac{2\pi k}{M}\right) & k = 0,\dots,\left\lfloor\frac{M-1}{2}\right\rfloor \\ +\left(\frac{M-1}{2}\right)\frac{2\pi}{M}(M-k) & k=\left\lfloor\frac{M-1}{2}\right\rfloor+1,\dots,M-1\end{cases}$$

$\left(\frac{M-1}{2}\right)=9.5$

$$\measuredangle H(\omega_k) = \begin{cases} -9.5\,\frac{2\pi}{20}\,k = -0.95\pi k & (0\le k\le 9) \\ +9.5\,\frac{2\pi}{20}(20-k) = +0.95\pi\,(20-k) & (10\le k\le 19)\end{cases}$$

$$h(n) = \frac{1}{20}\sum_{k=0}^{19} H(\omega_k)\, e^{j2\pi kn/20}$$

- Obviously $A_{s\,min} = 16$ dB is not acceptable.
- Solution is to increase M so that we have points in the transition band.
- Optimum values for transition point amplitudes are obtained from tables.



Approximate Rule

$A_s \simeq (2.5 + 20K_T)$ dB

$TW \simeq \frac{(1+K_T)}{M}\times 2\pi$

$M = \frac{1+K_T}{TW}\times 2\pi$

K_T: no. of freq. (ω_k's) in the transition band.

Let us choose $M = 40$

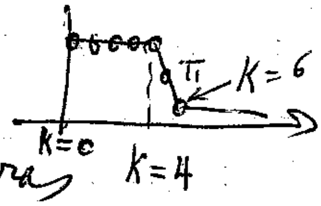
$k=0,1,2,3,4$

$$k\text{-pass} = \frac{0.2\pi}{2\pi} \times 40 = 4 \quad , \quad k\text{-stop} = \frac{0.3\pi}{2\pi} \times 40 = 6$$

Transition width is at $K = 5$

$\therefore$ TW $= 1$ sample.

No. of zeros $= 40 - 5 - 4 - 1 - 1 = 29$ zeros

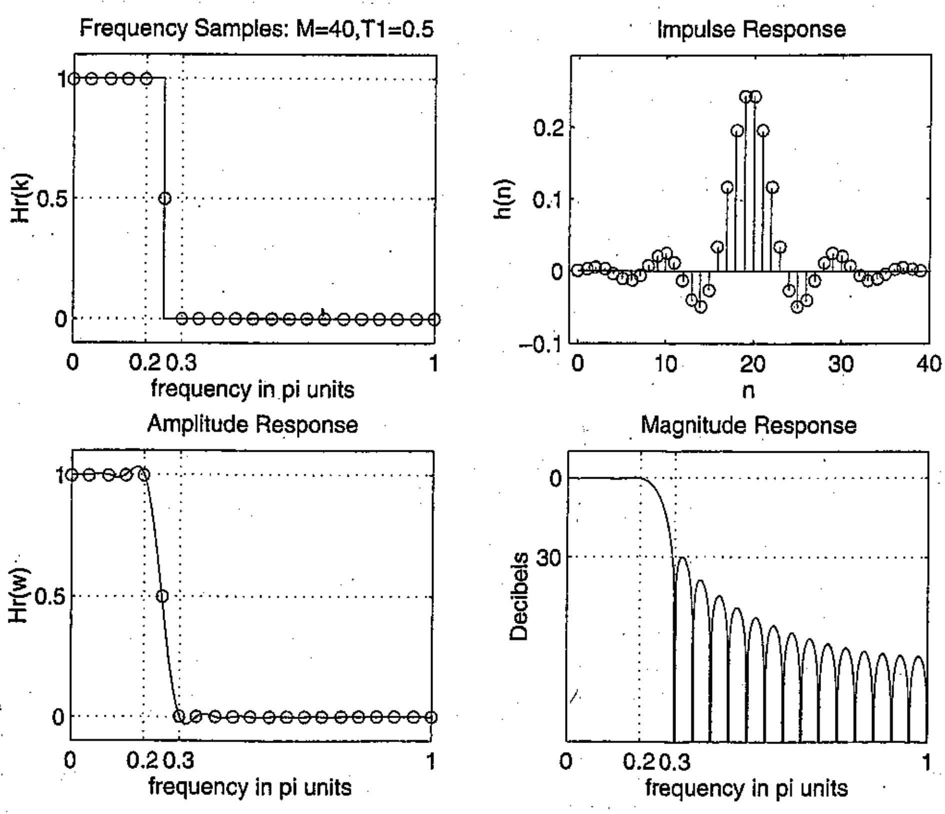


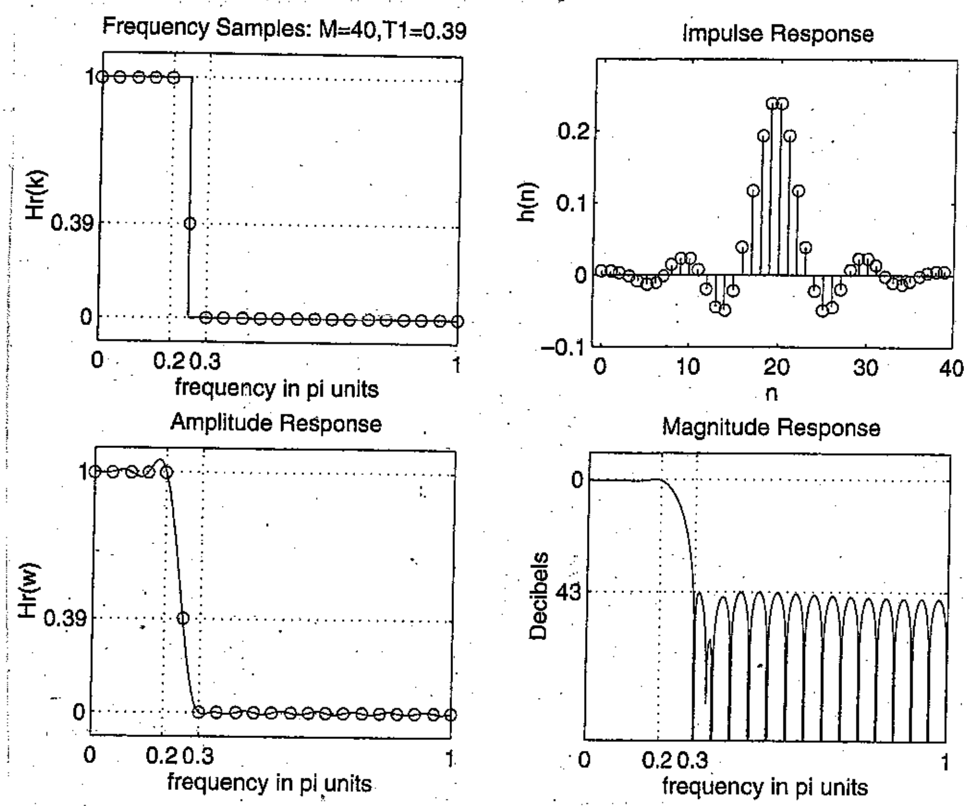
$$\therefore \quad H_r(\omega_k) = [\,1\ 1\ 1\ 1\ 1\ T_1\ \underbrace{0 \cdots 0}_{29\text{-zeros}}\ T_1\ 1\ 1\ 1\ 1\,]$$

$$\frac{M-1}{2} = 19.5$$

$$\measuredangle H(\omega_k) = \begin{cases} -19.5\,\frac{2\pi}{40}\,k = -0.975\,\pi k & 0 \le K \le 19 \\ +19.5\,\frac{2\pi}{40}(40-k) = +0.975\,\pi(40-k) & 20 \le K \le 39 \end{cases}$$

We calculate $\quad h(n) = (1/40) \sum_{k=0}^{39} H_r(\omega_k)\, e^{j2\pi nk/40} \qquad n = 0, 1, \cdots, 39$





Let us choose $M=60$ so that we have two samples in the transition region.

$$k\text{-pass} = \frac{0.2\pi}{2\pi} \times 60 = \underset{\uparrow\, k=0,1,\ldots,6}{6} \quad , \quad k\text{-stop} = \frac{0.3\pi}{2\pi} \times 60 = 9$$

$\Longrightarrow$ two points ($k=7,8$) in the transition region.

No. of zeros $= 60-7-6-2-2 = 43$ zeros

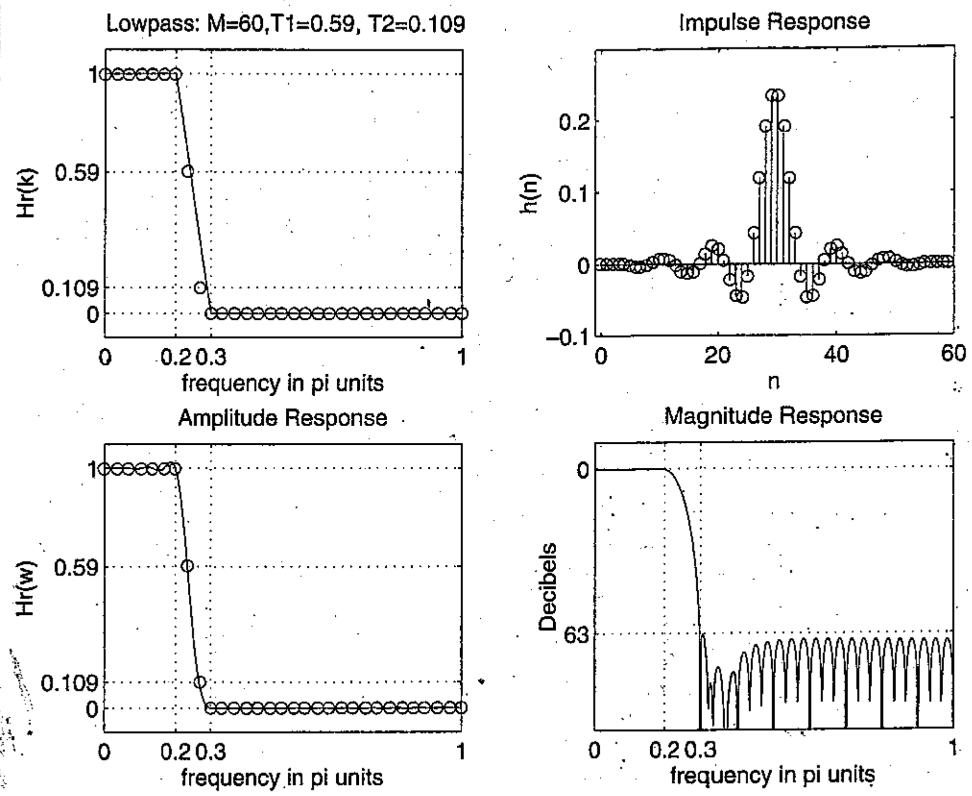
$$\therefore\ H_r(\omega_k) = [\underbrace{1 \cdots 1}_{7\text{-ones}},\ T_1\ T_2\ \underbrace{0 \cdots\cdots 0}_{43\text{-zeros}},\ T_2\ T_1\ \underbrace{1 \cdots 1}_{6\text{-ones}}\ ]$$

Optimum values are (From tables)

$$T_1 = 0.5925, \quad T_2 = 0.1099$$

$$\measuredangle H(\omega_k) = \begin{cases} -29.5\,\dfrac{2\pi}{60}\,k & 0 \le k \le 29 \\[2ex] +29.5\,\dfrac{2\pi}{60}\,(60-k) & 30 \le k \le 59 \end{cases}$$

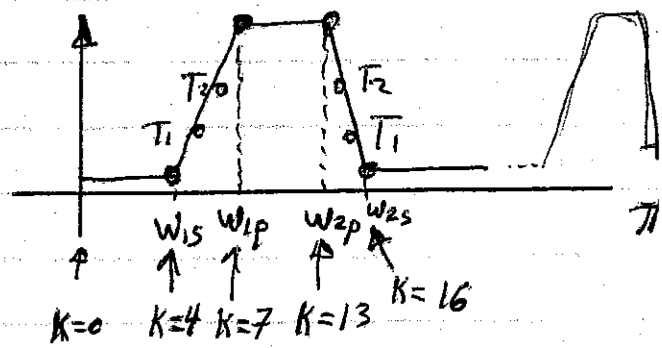
$$h(n) = (1/60)\sum_{k=0}^{59} H(\omega_k)\, e^{j2\pi nk/60}$$



Ex) Design a BPF specified by
$\omega_{1s} = 0.2\pi$, $\omega_{1p} = 0.35\pi$, $\omega_{2p} = 0.65\pi$, $\omega_{2s} = 0.8\pi$
$A_s = 60\,dB$

$$A_s \simeq (25 + 20K_T)\ dB$$
$\Rightarrow K_T = 2$ gives $\simeq 60\,dB$

$$T_W \simeq (1+K_T)(2\pi/M)$$
$$0.15\pi \simeq 3 \times 2\pi/M \Rightarrow \boxed{M = 40}$$
$$k_{1s} = \frac{\omega_{1s}}{2\pi} \times M = \frac{0.2\pi}{2\pi} \times 40 = 4$$



$$k_{1p} = (\omega_{1p}/2\pi)M = (0.35\pi/2\pi)\,40 = 7$$
$$k_{2p} = (\omega_{2p}/2\pi)M = (0.65\pi/2\pi)\,40 = 13$$
$$k_{2s} = (\omega_{2s}/2\pi)M = (0.8\pi/2\pi)\,40 = 16$$

mid ~~zeros~~ ones $= 40-5-2-7-2-4-2-7-2 = 9$

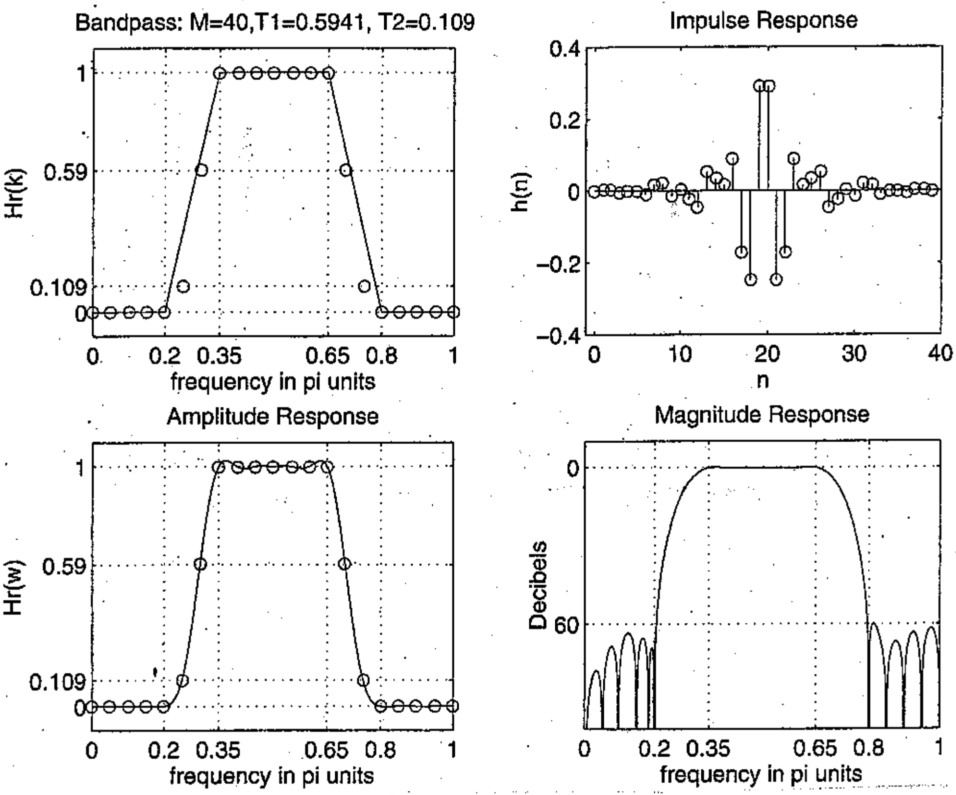
$$H_r(\omega_k) = [\underbrace{0 - 0}_{5}\ T_1\ T_2\ \underbrace{1 - 1}_{7}\ T_2\ T_1\ \underbrace{0 \text{———} 0}_{9\text{-zeros}},\ T_1 T_2 \underbrace{1 - 1}_{7}\ T_2 T_1\ \underbrace{0 - 0}_{4}]$$

(115)

Optimum values for T_1, T_2 ($M=40$) and 7-samples in pass-band

$T_1 = 0.10902$, $T_2 = 0.5941745$

$$\measuredangle H_r(\omega_k) = \begin{cases} -19.5\,\dfrac{2\pi k}{40} & (k = 0, 1, \ldots, 19) \\ +19.5\,\dfrac{2\pi(40-k)}{40} & (k = 20, \ldots, 39) \end{cases}$$



Ex) Design a HPF with $\omega_s = 0.6\pi$, $\omega_p = 0.8\pi$, $A_s = 50$ dB

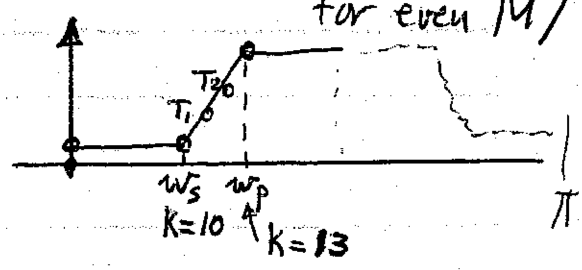
1- Select $K_T = 2$ to satisfy $A_s = 50$ dB

$0.2\pi = TW \simeq (1 + K_T)(2\pi/M) = 3 \times 2\pi/M$

$M = 33$ (slightly larger than required + odd M since $H(\pi) = 0$ for even M)

$k_s = \dfrac{\omega_s}{2\pi} \times M = \dfrac{0.6\pi}{2\pi} \times 33 = 9.9 \simeq 10$

$k_p = \dfrac{\omega_p}{2\pi} \times M = \dfrac{0.8\pi}{2\pi} \times 33 = 13.2 \simeq 13$



Optimum FIR design techique

Window design method and frequency sampling method are easy to implement. However, it does not specify δ_1, δ_2 simultaneously and it is inaccurate in specifying w_c. Also its approximation error is not uniformly distributed over the band intervals.

Optimum design distribute the error uniformly

$\Longrightarrow$ we can obtain lower order filter satisfying given specifications, compared to window or freq. sampling methods.

$h(n)$ is selected to minimize approximation error

$$\min_{\text{over } h(n)} \left[\max \left| W(w)\left[H_d(w) - H(w)\right] \right| \right]$$

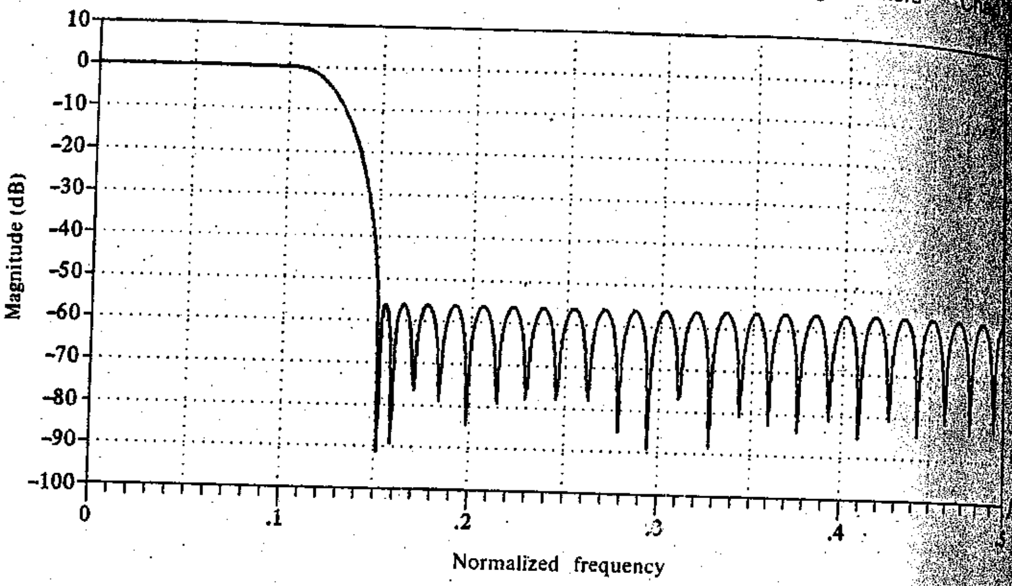


Figure 8.19 Frequency response of $M = 61$ FIR filter in Example 8.2.3.

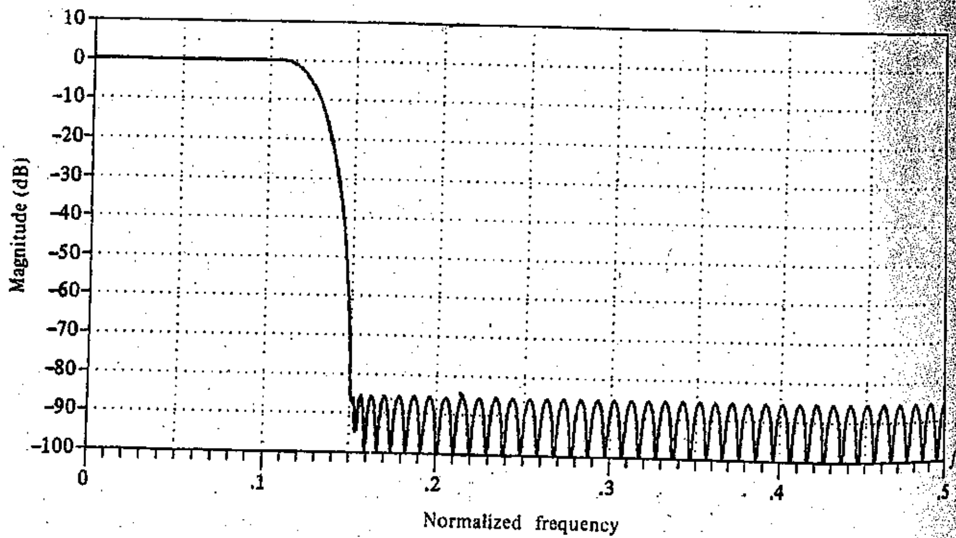


Figure 8.20 Frequency response of $M = 101$ FIR filter in Example 8.2.3.

$(0.1, 1)$, we obtain a filter that has a stopband attenuation of -65 dB and a passband ripple of 0.049 dB.

Example 8.2.4

Design a bandpass filter of length $M = 32$ with passband edge frequencies $f_{p1} = 0.2$ and $f_{p2} = 0.35$ and stopband edge frequencies of $f_{s1} = 0.1$ and $f_{s2} = 0.425$.

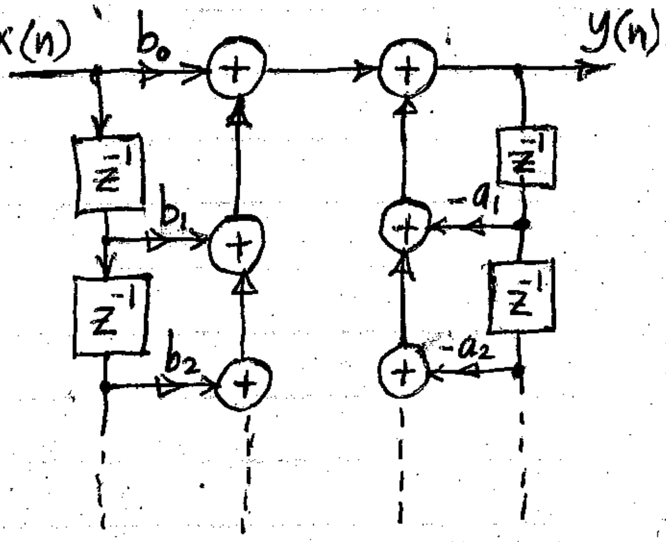
Design of Infinite Impulse Response (IIR) Filters

IIR filters are generally described by the diff. equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M-1} b_k x(n-k)$$

or by the system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$



IIR Digital Filter

Since analog filter design is mature and well developed it becomes a good approach to design IIR filters by conversion from analog filters.

IIR filter design by Impulse Invariance

The objective of this approach is to design an IIR filter that has a unit sample response $h(n)$ that is a sampled version of the impulse of the analog filter i.e.

$$h(n) \equiv h(nT) \qquad n = 0, 1, \cdots$$

T: sampling interval

We start by specifying an analog filter $H_a(s)$ and we

$$\sum_{k=1}^{N} c_k \left(e^{P_k T}\right)^n u(n)$$

assume that it has distinct poles

$$\boxed{H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - P_k}}$$

P_k : Poles of $H_a(s)$

Consequently
$$h_a(t) = \mathcal{L}^{-1}(H_a(s)) = \sum_{k=1}^{N} c_k e^{P_k t} u(t) \quad t \geq 0$$

$$e^{-\alpha t} u(t) \longleftrightarrow \frac{1}{s+\alpha}$$

If we sample $h_a(t)$ at $t = nT$ we get

$$h(n) = h_a(nT) = \sum_{k=1}^{N} c_k \left(e^{P_k T}\right)^n u(n) \quad n \geq 0$$

Now determine a digital filter which has the same $h(n)$ given above

Taking ~~inverse~~ z-transform

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k \left(e^{P_k T}\right)^n \right) z^{-n}$$

$$= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{P_k T} z^{-1}\right)^n$$

$$\alpha^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1 - \alpha z^{-1}}$$
ROC: $|z| > |\alpha|$

If $\mathrm{real}(P_k) < 0$, the inner sum converges

$$\sum_{n=0}^{\infty} \left(e^{P_k T} z^{-1}\right)^n = \frac{1}{1 - e^{P_k T} z^{-1}}$$

ROC: $|z| > |e^{P_k T}|$

$\Rightarrow$ Includes unit circle (stable if ~~H(s) is stable~~)

$$\Rightarrow \boxed{H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{P_k T} z^{-1}}}$$

The digital filter poles are at

$$z_k = e^{P_k T} \qquad k = 1, 2, \cdots, N$$

$$\boxed{H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - P_k}} \Longrightarrow \boxed{H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{P_k T} z^{-1}}}$$

Analog | Digital

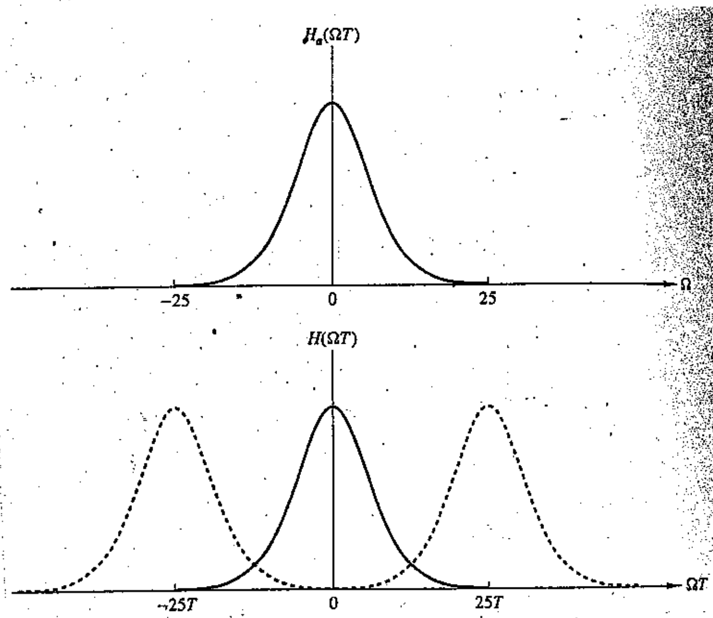
Aliasing Effect

Sampling of $h(t)$ will give $h(n)$ that will have Fourier transform that is a convolution of a frequency-domain impulse train (with $\omega_s = 2\pi f_s = 2\pi / T_s$ separation) with $H_a(\omega)$

$$\underbrace{H(\omega)}_{\text{digital filter}} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} H_a\left[(\omega - k\,\omega_s)\right]$$

Frequency Mapping

$$\boxed{\omega_{dig} = 2\pi \frac{\omega_a}{\omega_s}}$$



Aliasing occurs if $(\omega_s/2) \leq$ max. frequency of $H_a(\omega)$

∴ The resulting filter will have some distortion compared to the analog filter due to aliasing.

Practically if the analog filter has small amplitude at $\omega \geq \omega_s/2$ then the aliasing effect will be small. This effect can be made smaller by increasing the sampling rate. (not suitable for HPF, BSF)

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$$\frac{s+0.1}{s^2+0.2s+9.01}$$

Ex) Convert the analog filter

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

into a digital IIR filter by means of impulse invariance.

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2-(j3)^2} = \frac{s+0.1}{(s+0.1-j3)(s+0.1+j3)}$$

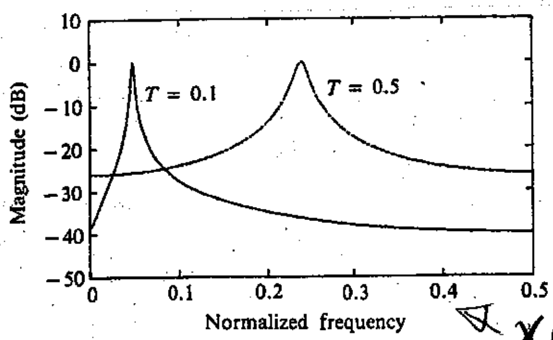
$$= \frac{1/2}{s+0.1-j3} + \frac{1/2}{s+0.1+j3}$$

$P_1 = -0.1+j3$, $P_2 = -0.1-j3$ (Complex conjugates)

$$H(z) = \frac{1/2}{1-e^{(-0.1+j3)T}z^{-1}} + \frac{1/2}{1-e^{(-0.1-j3)T}z^{-1}}$$

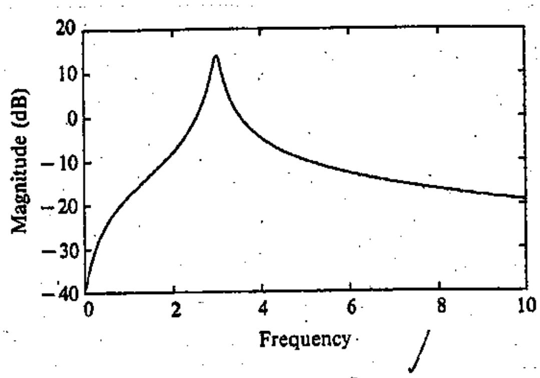
$$= \frac{(1/2)\left[1-e^{-0.1T}z^{-1}e^{j3T}+1-e^{-0.1T}z^{-1}e^{-j3T}\right]}{1-e^{-0.1T}z^{-1}e^{j3T}-e^{-0.1T}z^{-1}e^{-j3T}+e^{-0.2T}z^{-2}}$$

$$= \frac{1-\left(e^{-0.1T}\cos(3T)\right)z^{-1}}{1-\left(2e^{-0.1T}\cos 3T\right)z^{-1}+e^{-0.2T}z^{-2}}$$



× (2π) scale

Digital filter



Analog filter

Frequency Mapping

Conversion from s-plane to z-plane we used the basic substitution used to obtain the z-trans func

$$z = e^{sT} \Rightarrow re^{j\omega_d} = e^{\sigma T}e^{j\omega_a T} \Rightarrow r = e^{\sigma T}$$

$$\underset{\text{digital}}{\omega_d} = \underset{\text{analog}}{\omega_a} T \Rightarrow \boxed{\omega_d = \frac{\omega_a}{f_s} = 2\pi\frac{\omega_a}{\omega_s}}$$

PFE for ex) at ~~P(122)~~ page 122

$$H_a(s) = \frac{A_1}{s+0.1-j3} + \frac{A_2}{s+0.1+j3}$$

$$A_1 = (s+0.1-j3)\,H_a(s)\Big|_{s=-0.1+j3} = \frac{s+0.1}{s+0.1+j3}\Big|_{s=-0.1-j3}$$

$$= \frac{-\cancel{0.1}-j3+\cancel{0.1}}{-\cancel{0.1}-j3+\cancel{0.1}+j3} = \frac{-j3}{-j6} = \boxed{\frac{1}{2}}$$

$$A_2 = \frac{s+0.1}{s+0.1-j3}\Big|_{s=-0.1-j3} = \frac{-\cancel{0.1}-j3+\cancel{0.1}}{-\cancel{0.1}-j3+\cancel{0.1}-j3} = \frac{-j3}{-j6} = \boxed{\frac{1}{2}}$$

$$\boxed{H_a(s) = \frac{0.5}{s-(-0.1+j3)} + \frac{0.5}{s-(-0.1-j3)}}$$

PFE for ex) at page 123

$$H_a(s) = \frac{s+1}{s^2+5s+6} = \frac{s+1}{(s+3)(s+2)} = \frac{A_1}{s+3} + \frac{A_2}{s+2}$$

$$A_1 = (s+3)\,H_a(s)\Big|_{s=-3} = \frac{s+1}{s+2}\Big|_{s=-3} = \frac{-3+1}{-3+2} = \frac{-2}{-1} = \boxed{2}$$

$$A_2 = \frac{s+1}{s+3}\Big|_{s=-2} = \frac{-2+1}{-2+3} = \frac{-1}{1} = \boxed{-1}$$

$$H_a(s) = \frac{2}{s+3} - \frac{1}{s+2} = \boxed{\frac{2}{s-(-3)} - \frac{1}{s-(-2)}}$$

$w_o(\text{analog}) = 3 \text{ rad/sec}$

1- for $f_s = \frac{1}{T} = \frac{1}{0.1} = 10\,Hz \Rightarrow w_s = 20\pi \ \frac{rad}{sec}$

$$w_o(\text{digital}) = 2\pi \frac{w_o(\text{analog})}{w_s} = \frac{3}{20\pi} \times 2\pi = 0.095\pi$$

2- for $T = 0.5 \Rightarrow f_s = \frac{1}{0.5} = 2 \Rightarrow w_s = 4\pi \ \frac{rad}{sec}$

$$w_o(\text{digital}) = \frac{w_o}{w_s} 2\pi = \frac{3}{4\pi} 2\pi = 0.477\pi \ \frac{rad}{sec}$$

Notes

1- It is important to select small value for T to minimize the distortion due to aliasing.

2- Impulse invariance is only suitable for LPF, BPF.

Ex) Convert the analog filters with transfer function

$$H_a(s) = \frac{s+1}{s^2+5s+6}$$

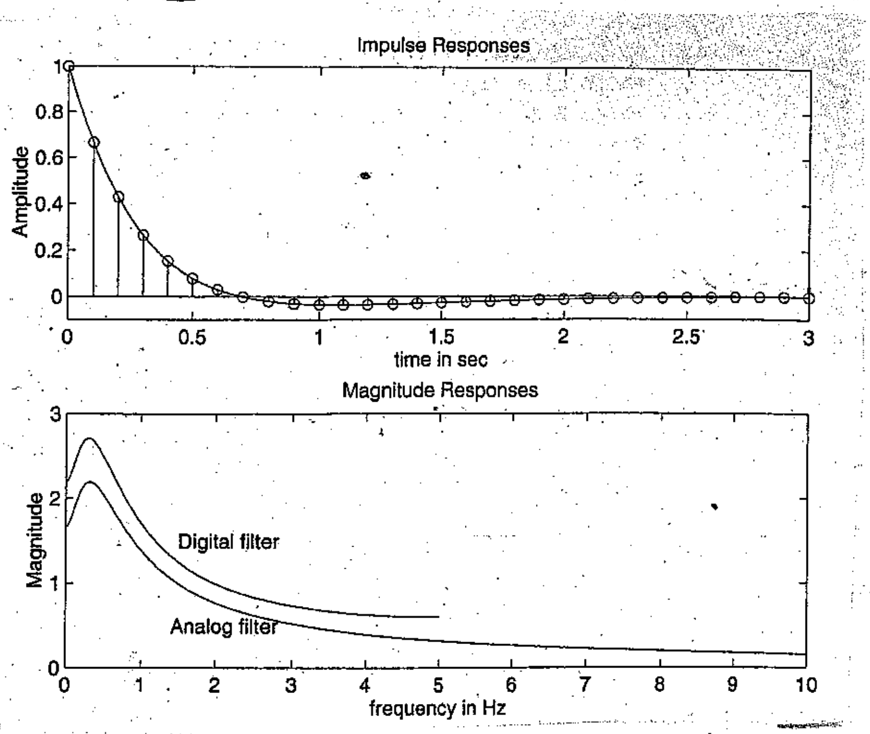
into a digital filter using impulse invariance with sampling period = 0.1 sec.

$$H_a(s) = \frac{2}{s+3} - \frac{1}{s+2} \quad \left(\text{Partial fraction}\right)$$

$$P_1 = -3, \quad P_2 = -2$$

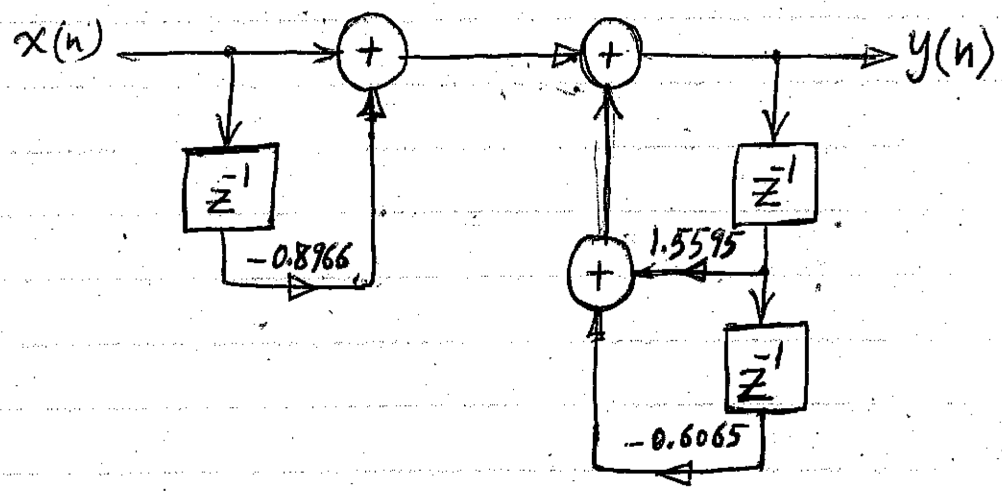
$$\Rightarrow H(z) = \frac{2}{1-e^{-3T}z^{-1}} - \frac{1}{1-e^{-2T}z^{-1}} \quad (\text{substitute } T = 0.1)$$

$$H(z) = \frac{1-0.8966\,z^{-1}}{1-1.5595\,z^{-1}+0.6065\,z^{-2}} = \frac{Y(z)}{X(z)}$$



$$Y(z) - 1.5595\,z^{-1}Y(z) + 0.6065\,z^{-2}Y(z) = X(z) - 0.8966\,z^{-1}X(z)$$
$$y(n) = 1.5595\,y(n-1) - 0.6065\,y(n-2) + x(n) - 0.8966\,x(n-1)$$

<u>The filter implementation</u>



<u>Stability of the designed filter</u>

If the continuous-time filter is stable i.e the real part of P_K is (−ve) then the magnitude of
$$|e^{P_K T}| < 1 \quad \text{(i.e poles inside unit circle)}$$
∴ the corresponding digital filter is also stable.

Design steps using impulse Invariance

Impulse Invariance is suitable only for LPF, BPF provided that their high freq. response is small causing small aliasing effect so that the resulting digital is similar to the analog filter. (for $w > \frac{W_s}{2}$)

Steps:

Given a desired specifications W_{dig} (W_c or W_o) (LPF, BPF) and gain requirement

1- Select a suitable normalized ($W_c = 1 \frac{rad}{sec}$) analog filter.

2- Select T such that

$$W_{dig} = 2\pi \frac{W_{ang}}{W_s} = 2\pi \times \frac{1}{2\pi} T_s$$

$$\boxed{T_s = W_{dig}} \longrightarrow \boxed{W_s = \frac{2\pi}{W_{dig.}}}$$

$$\frac{W_s}{2} = \frac{\pi}{W_{dig.}}$$

3- Apply PFE and Imp. Invar. to get the desired dig. filter.

ex) Design a digital IIR filter to have

$$W_{dig} \text{(Cutoff)} = 0.1\pi$$

that is similar to Butterworth 3rd order LPF

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

(3rd normalized Butterworth)

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$$H(s) = \frac{1}{(s-(-1))[s-(-0.5+j0.866)][s-(-0.5-j0.866)]}$$

$$= \frac{A_1}{s+1} + \frac{A_2}{s-(-0.5+j0.866)} + \frac{A_3}{s-(-0.5-j0.866)}$$

$$A_1 = \left.\frac{1}{s^2+s+1}\right|_{s=-1} = 1$$

$$A_2 = \left.\frac{1}{(s+1)(s+0.5+j0.866)}\right|_{s=-0.5+j0.866} = -0.5-j0.2887$$

$$A_3 = A_2^*$$

$$H(s) = \frac{1}{s-(-1)} + \frac{-0.5-j0.2887}{s-(-0.5+j0.866)} + \frac{-0.5+j0.2887}{s-(-\frac{1}{2}-j0.866)}$$

Using $\boxed{T_s = W_{dig} = \boxed{0.1\pi}}$

$$H(z) = \frac{1}{1-e^{-0.1\pi}} + \frac{-0.5-j0.2887}{1-e^{(-0.5+j0.866)0.1\pi}z^{-1}}$$

$$+ \frac{-0.5+j0.2887}{1-e^{(-0.5-j0.866)0.1\pi}z^{-1}}$$

$$H(z) = \frac{0.0398z^{-1}+0.0323z^{-2}}{1-2.3768z^{-1}+1.9329z^{-2}-0.5335z^{-3}}$$

IIR filter design by the bilinear transformation

This method is based on a mapping from s-plane to z-plane.
Consider the transfer function

$$H_a(s) = \frac{b}{s+a} = \frac{Y(s)}{X(s)}$$

This corresponds to the differential equation

$$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t) \qquad \text{—— (1)}$$

We will integrate the derivative and approximate the integral by the trapezoidal formula

$$y(t) = \int_{t_0}^{t} y'(\tau)\,d\tau + y(t_0)$$

At $\begin{cases} t = nT \\ t_0 = nT - T \end{cases}$

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T) \qquad \text{—— (2)}$$

evaluating (1) at $t = nT$

$$y'(nT) = -a\,y(nT) + b\,x(nT) \qquad \text{—— (3)}$$

We substitute (3) in (2) for the derivatives y'. We denote
$y(n) \equiv y(nT)$, $x(n) \equiv x(nT)$

(2) $\Longrightarrow$
$$\left(1 + \frac{aT}{2}\right)y(n) - \left(1 - \frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$

Taking z-transform
$$\left(1 + \frac{aT}{2}\right)Y(z) - \left(1 - \frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}(1 + z^{-1})X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(bT/2)(1+z^{-1})}{1 + aT/2 - (1 - aT/2)z^{-1}}$$

Or
$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a} \Longrightarrow$$ also applicable to N^{th} order diff. eqn.

∴ Bilinear Transformation (s-plane → z-plane)

$$\boxed{s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)} \qquad \text{—— (4)}$$

Characteristics of the biLinear transform

We start from $z = re^{j\omega}$, $s = \sigma + j\Omega$

we substitute in (4) which may re-written as

$$s = \frac{2}{T}\,\frac{z-1}{z+1} = \frac{2}{T}\,\frac{re^{j\omega}-1}{re^{j\omega}+1} \left(\times \frac{1+re^{-j\omega}}{1+re^{-j\omega}}\right) \text{ conjugate}$$

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$$= \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega} + j\,\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right) = \sigma + j\Omega$$

$$\sigma = \frac{2}{T}\,\frac{r^2-1}{\underbrace{1+r^2+2r\cos\omega}_{(1-r)^2+2r(1+\cos\omega)}}, \quad \Omega = \frac{2}{T}\,\frac{2r\sin\omega}{1+r^2+2r\cos\omega}$$

Conclusions

1- If $r<1 \Rightarrow \sigma < 0$
If $r>1 \Rightarrow \sigma > 0$

s-plane		z-plane
LHP in (s)	$\Rightarrow$	inside unit circle
RHP in (s)	$\Rightarrow$	Outside unit circle

$\therefore$ $\boxed{\text{Causal stable analog filters} \Longrightarrow \Longrightarrow \text{Causal stable digital filters}}$

2- If $r=1 \Rightarrow \sigma = 0$ and

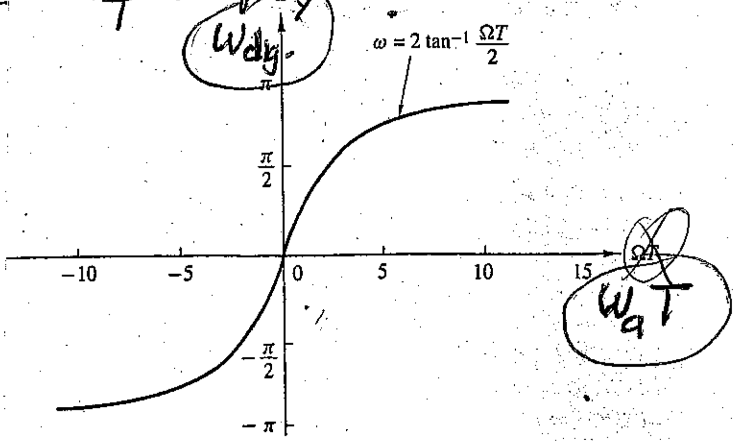
$$\Omega = \frac{2}{T}\,\frac{\sin\omega}{1+\cos\omega} = \frac{2\sin(\omega/2)\cos(\omega/2)}{\cancel{1}+2\cos^2(\omega/2)-\cancel{1}}\,\frac{2}{T}$$

$$\boxed{\Omega = \frac{2}{T}\tan\frac{\omega}{2}} \quad \text{or} \quad \boxed{\omega = 2\tan^{-1}\frac{\Omega T}{2}}$$

(For very small Ω, $\tan\theta = \theta \Rightarrow \Omega = \frac{\omega}{T}$ as before)

Entire range in $(\Omega) \Rightarrow$
$\Longrightarrow -\pi \le \omega \le \pi$

Mapping of frequency is non-Linear
(Frequency Warping)



$w_c = 1$
$w_p = 1$
Normalized

Design Procedure using bilinear transform

1- To satisfy given requirements, we select a suitable analog filter $H(s)$.
2- From requirements, determine cutoff frequency (or edge frequency) ω_p (digital).
3- Obtain an equivalent analog filter cutoff frequency (ω_p') using the relation

for simplicity select $T = 2$

$$\Omega(\text{analog}) = \frac{2}{T}\tan\left(\frac{\omega}{2}\right) \leftarrow \text{digital}$$

(Prewarped freq) $$\omega_p' = \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) \leftarrow \text{digital}$$

Use any T-value

4- **Denormalize** the analog filter by frequency scaling $H(s)$ normalized.
This is achieved by replacing s with s/ω_p'
5- Apply bilinear transformation

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \boxed{\frac{1-z^{-1}}{1+z^{-1}}}$$

$2 \to T$

Ex) Convert the analog filter (with resonant frequency $= 4 \frac{\text{rad}}{\text{sec}}$)

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

into a digital IIR filter by the bilinear transformation so that ω_r (digital resonant freq) $= \pi/2$.

Using $$\Omega(\text{analog}) = \frac{2}{T}\tan\frac{\omega(\text{dig})}{2}$$

$$4 = \frac{2}{T}\tan\frac{\pi/2}{2} = \frac{2}{T} \Rightarrow T = 1/2$$

$$\therefore s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$ substitute in $H_a(s)$

$$H(z) = \frac{0.128 + 0.006 z^{-1} - 0.122 z^{-2}}{1 + 0.0006 z^{-1} + 0.975 z^{-2}}$$

Ex) The normalized LPF ($w_c = 1$) is

$$H(s) = \frac{1}{s^2 + \sqrt{2} s + 1}$$

Use this to design a digital LPF with 3 dB cutoff (w_c) frequency of 150 Hz, and a sampling frequency 1.28 KHz.

$$w_c(\text{digital}) = \frac{w_c(\text{analog})}{w_s(\text{analog})} \times 2\pi = \frac{f_c(\text{analog})}{f_s(\text{analog})} \times 2\pi$$

$$= \frac{150}{1280} \times 2\pi = 0.7363 = 0.234\pi$$

$$\boxed{w_c(\text{prewrapped})} = \frac{2}{T} \tan \frac{w_c(\text{dig.})}{2} = 2 \times 1280 \tan\left(\frac{0.234\pi}{2}\right)$$

$$= 987.5 \text{ rad/sec}$$

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Denormalize filter

$$H_a(\text{denormalized}) = \frac{1}{\left(\frac{s}{987.5}\right)^2 + \sqrt{2}\frac{s}{987.5} + 1}$$

Apply bilinear

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} = 2 f_s \frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = \frac{1}{\left(\frac{2 \times 1280}{987.5}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \sqrt{2}\left(\frac{2 \times 1280}{987.5}\frac{1-z^{-1}}{1+z^{-1}}\right) + 1}$$

$2 \times 1280 \tan(w_d/2)$

$$= \frac{1}{6.721\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 3.666\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1}$$

$$= \frac{(1+z^{-1})^2}{6.721(1 - 2z^{-1} + z^{-2}) + 3.666(1 - z^{-2}) + 1 + 2z^{-1} + z^{-2}}$$

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{11.387 - 11.442 z^{-1} + 4.055 z^{-2}} = \frac{0.0878(1 + 2z^{-1} + z^{-2})}{1 - 1.0048 z^{-1} + 0.3561 z^{-2}}$$