الجامعة الهاشمية
The Hashemite University

ELECTROMAGNETICS II COURSE

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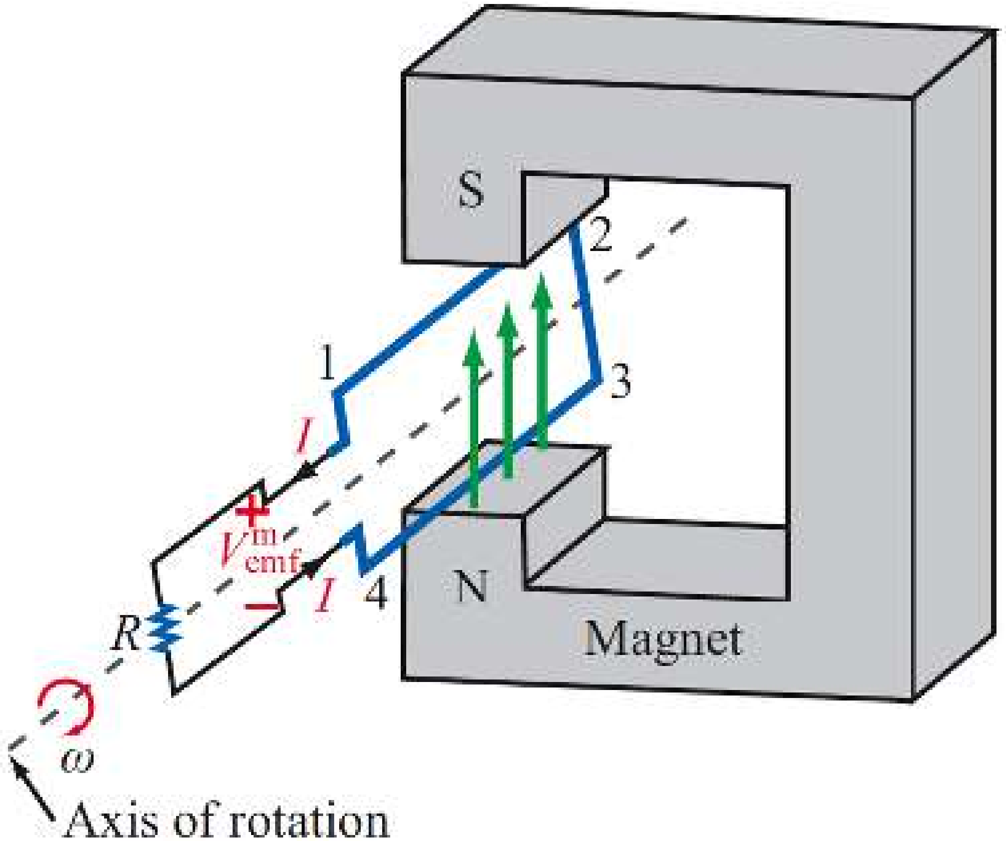


(b) ac generator

6. MAXWELL'S EQUATIONS IN TIME-VARYING FIELDS

7e Applied EM by Ulaby and Ravaioli

Chapter 6 Overview

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Apply Faraday's law to compute the voltage induced by a stationary coil placed in a time-varying magnetic field or moving in a medium containing a magnetic field.
2. Describe the operation of the electromagnetic generator.
3. Calculate the displacement current associated with a time-varying electric field.
4. Calculate the rate at which charge dissipates in a material with known ϵ and σ.

Maxwell's Equations

Table 6-1: Maxwell's equations.

Reference	Differential Form	Integral Form	
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	(6.1)
Faraday's law	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \dfrac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$	(6.2)*
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	(6.3)
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$	(6.4)

*For a stationary surface S.

In this chapter, we will examine Faraday's and Ampère's laws

Faraday's Law

Electromotive force (voltage) induced by time-varying magnetic flux:

$$V_{\text{emf}} = -N\,\frac{d\Phi}{dt} = -N\,\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} \qquad \text{(V)}$$

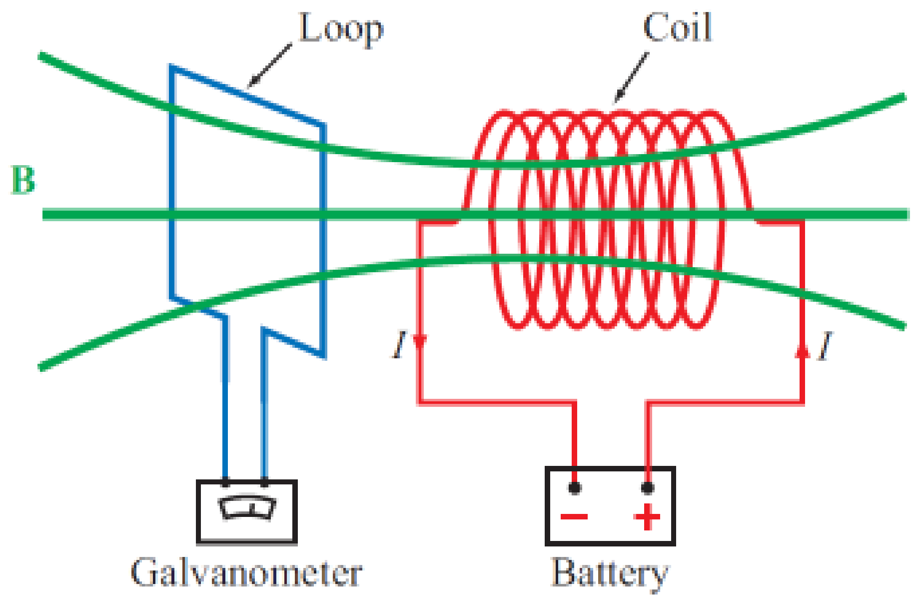


Figure 6-1: The galvanometer (predecessor of the ammeter) shows a deflection whenever the magnetic flux passing through the square loop changes with time.

Magnetic fields can produce an electric current in a closed loop, but only if the magnetic flux linking the surface area of the loop changes with time. The key to the induction process is change.

Three types of EMF

1. A time-varying magnetic field linking a stationary loop; the induced emf is then called the ***transformer emf***, $V_{\text{emf}}^{\text{tr}}$.

2. A moving loop with a time-varying surface area (relative to the normal component of **B**) in a static field **B**; the induced emf is then called the ***motional emf***, $V_{\text{emf}}^{\text{m}}$.

3. A moving loop in a time-varying field **B**.

The total emf is given by

$$V_{\text{emf}} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}}, \tag{6.7}$$

Stationary Loop in Time-Varying B

It is important to remember that $\mathbf{B}_{\text{ind}}$ *serves to oppose the change in* $\mathbf{B}(t)$*, and not necessarily* $\mathbf{B}(t)$ *itself.*

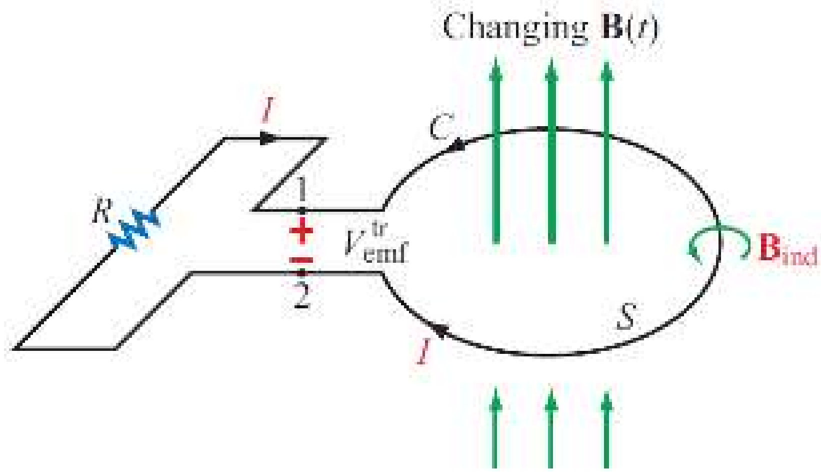
$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad \text{(transformer emf).}$$

The connection between the direction of $d\mathbf{s}$ *and the polarity of* $V_{\text{emf}}^{\text{tr}}$ *is governed by the following right-hand rule: if* $d\mathbf{s}$ *points along the thumb of the right hand, then the direction of the contour C indicated by the four fingers is such that it always passes across the opening from the positive terminal of* $V_{\text{emf}}^{\text{tr}}$ *to the negative terminal.*

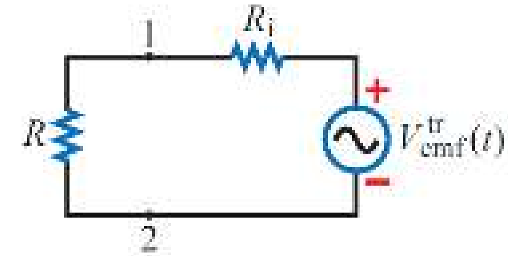
$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R + R_i}. \qquad (6.9)$$

For good conductors, R_i usually is very small, and it may be ignored in comparison with practical values of R.

The polarity of $V_{\text{emf}}^{\text{tr}}$ *and hence the direction of* I *is governed by* ***Lenz's law****, which states that the current in the loop is always in a direction that opposes the change of magnetic flux* $\Phi(t)$ *that produced* I.



(a) Loop in a changing **B** field



(b) Equivalent circuit

Figure 6-2: (a) Stationary circular loop in a changing magnetic field $\mathbf{B}(t)$, and (b) its equivalent circuit.

Example 6-1: Inductor in a Changing Magnetic Field

An inductor is formed by winding N turns of a thin conducting wire into a circular loop of radius a. The inductor loop is in the x–y plane with its center at the origin, and connected to a resistor R, as shown in Fig. 6-3. In the presence of a magnetic field $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$, where ω is the angular frequency, find

(a) the magnetic flux linking a single turn of the inductor,

(b) the transformer emf, given that $N = 10$, $B_0 = 0.2$ T, $a = 10$ cm, and $\omega = 10^3$ rad/s,

(c) the polarity of $V^{\text{tr}}_{\text{emf}}$ at $t = 0$, and

(d) the induced current in the circuit for $R = 1$ kΩ (assume the wire resistance to be much smaller than R).

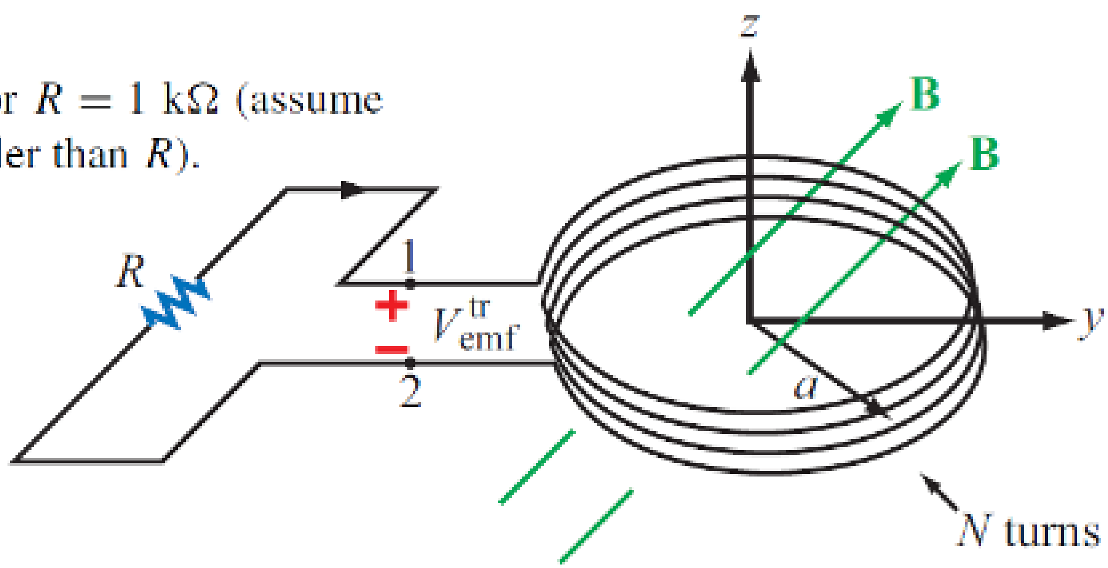


Figure 6-3: Circular loop with N turns in the x–y plane. The magnetic field is $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$ (Example 6-1).

cont.

Example 6-1 Solution

Solution: **(a)** The magnetic flux linking each turn of the inductor is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$
$$= \int_S [B_0(\hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,3)\sin\omega t] \cdot \hat{\mathbf{z}}\, ds$$
$$= 3\pi a^2 B_0 \sin\omega t.$$

(b) To find $V^{\text{tr}}_{\text{emf}}$, we can apply Eq. (6.8) or we can apply the general expression given by Eq. (6.6) directly. The latter approach gives

$$V^{\text{tr}}_{\text{emf}} = -N\frac{d\Phi}{dt}$$
$$= -\frac{d}{dt}(3\pi N a^2 B_0 \sin\omega t)$$
$$= -3\pi N\omega a^2 B_0 \cos\omega t.$$

For $N = 10$, $a = 0.1$ m, $\omega = 10^3$ rad/s, and $B_0 = 0.2$ T,

$$V^{\text{tr}}_{\text{emf}} = -188.5 \cos 10^3 t \qquad \text{(V)}.$$

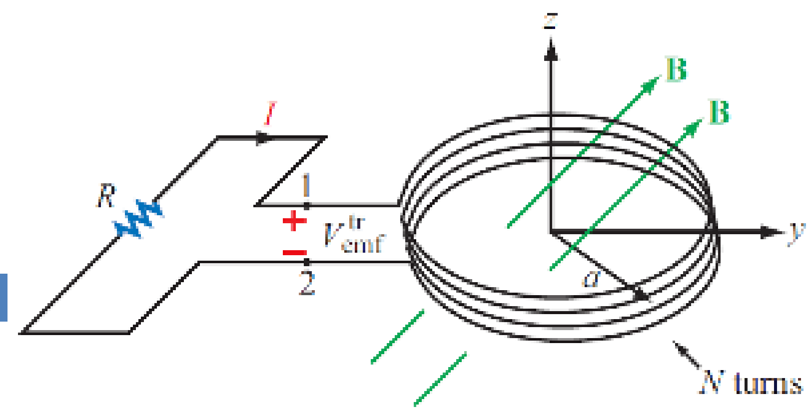


Figure 6-3: Circular loop with N turns in the x–y plane. The magnetic field is $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$ (Example 6-1).

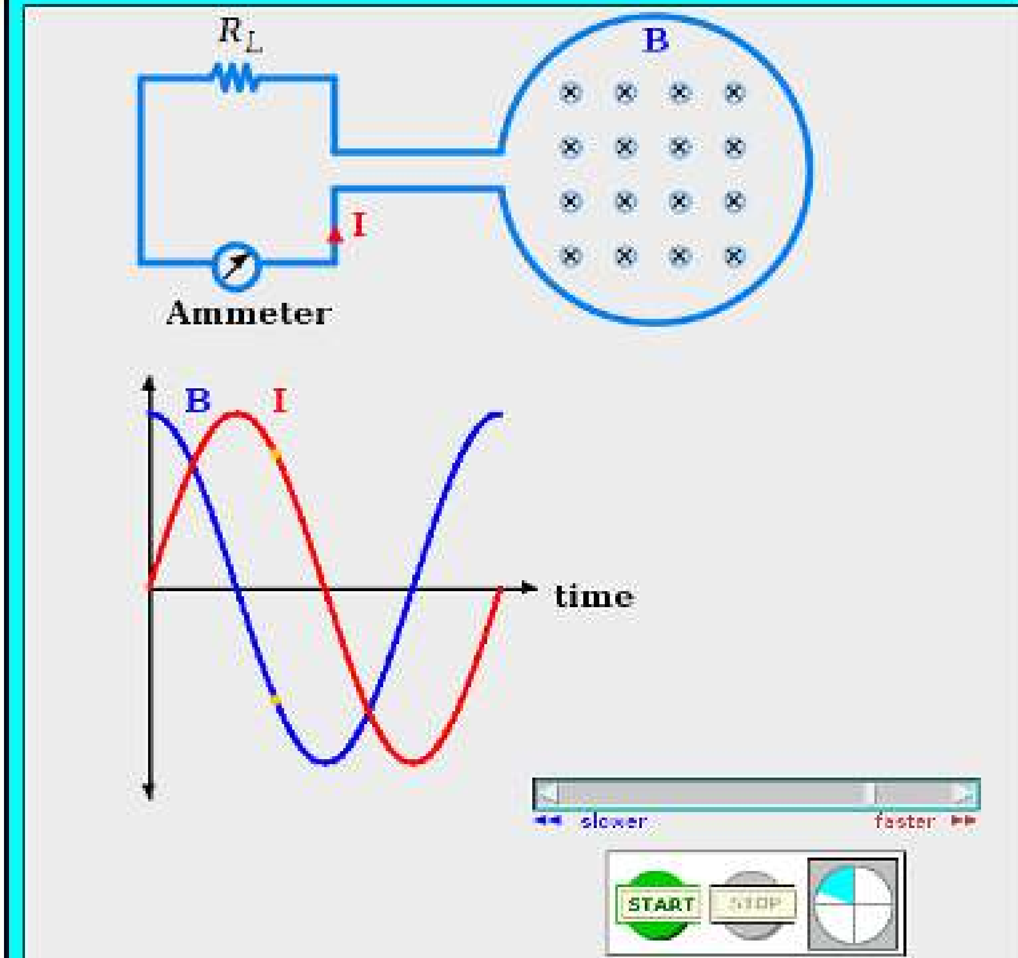
(c) At $t = 0$, $d\Phi/dt > 0$ and $V^{\text{tr}}_{\text{emf}} = -188.5$ V. Since the flux is increasing, the current I must be in the direction shown in Fig. 6-3 in order to satisfy Lenz's law. Consequently, terminal 2 is at a higher potential than terminal 1 and

$$V^{\text{tr}}_{\text{emf}} = V_1 - V_2$$
$$= -188.5 \qquad \text{(V)}.$$

(d) The current I is given by

$$I = \frac{V_2 - V_1}{R}$$
$$= \frac{188.5}{10^3}\cos 10^3 t$$
$$= 0.19 \cos 10^3 t \qquad \text{(A)}.$$

Module 6.1 **Circular Loop in Time-varying Magnetic Field**



Demonstration of Faraday's Law

The circular wire loop shown in the figure is connected to a simple circuit composed of a resistor R_L in series with a current meter. The time-varying magnetic flux linking the surface of the loop induces a V_{emf}, and hence a current through R. The purpose of this demo is to illustrate, in the form of a slow-motion video, how the current I varies with time, in both magnitude and direction, when $B(t) = B_0 \cos \omega t$.

Note that $I(t)$ is a maximum when the slope of $B(t)$ is a maximum, which occurs when B itself is zero. The direction of $I(t)$ is dictated by Lenz's Law.

Applet Design: Janice Richards

Example 6-2: Lenz's Law

Determine voltages V_1 and V_2 across the 2-Ω and 4-Ω resistors shown in Fig. 6-4. The loop is located in the x–y plane, its area is 4 m^2, the magnetic flux density is $\mathbf{B} = -\hat{\mathbf{z}}0.3t$ (T), and the internal resistance of the wire may be ignored.

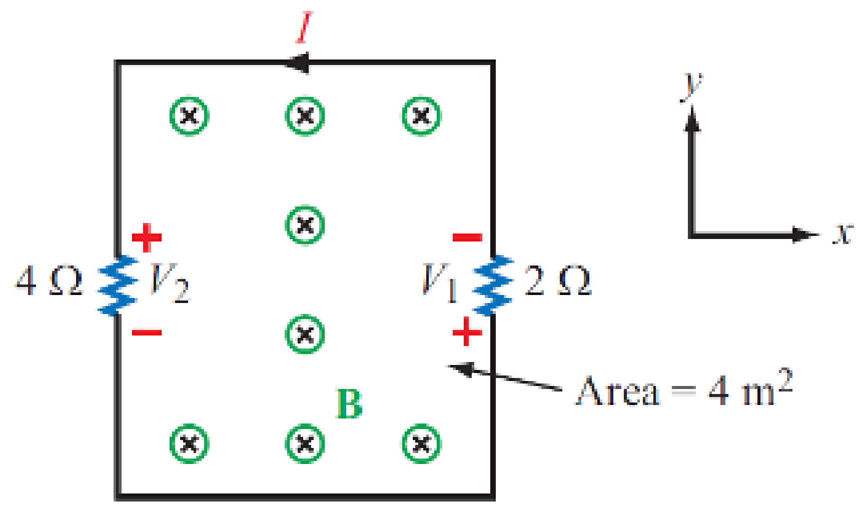


Figure 6-4: Circuit for Example 6-2.

Solution: The flux flowing through the loop is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S (-\hat{\mathbf{z}}0.3t) \cdot \hat{\mathbf{z}}\, ds$$
$$= -0.3t \times 4 = -1.2t \qquad \text{(Wb)},$$

and the corresponding transformer emf is

$$V_{\text{emf}}^{\text{tr}} = -\frac{d\Phi}{dt} = 1.2 \qquad \text{(V)}.$$

The total voltage of 1.2 V is distributed across two resistors in series. Consequently,

$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R_1 + R_2} = \frac{1.2}{2+4} = 0.2 \text{ A},$$

and

$$V_1 = IR_1 = 0.2 \times 2 = 0.4 \text{ V},$$
$$V_2 = IR_2 = 0.2 \times 4 = 0.8 \text{ V}.$$

Ideal Transformer

$$V_1 = -N_1 \frac{d\Phi}{dt} \, .$$

A similar relation holds true on the secondary side:

$$V_2 = -N_2 \frac{d\Phi}{dt} \, .$$

$$\boxed{\frac{V_1}{V_2} = \frac{N_1}{N_2}} \qquad \boxed{\frac{I_1}{I_2} = \frac{N_2}{N_1}}$$

$$R_{\text{in}} = \frac{V_1}{I_1}$$

$$R_{\text{in}} = \frac{V_2}{I_2}\left(\frac{N_1}{N_2}\right)^2 = \left(\frac{N_1}{N_2}\right)^2 R_{\text{L}}. \qquad (6.20)$$

When the load is an impedance Z_L and V_1 is a sinusoidal source, the phasor-domain equivalent of Eq. (6.20) is

$$\boxed{\mathbf{Z}_{\text{in}} = \left(\frac{N_1}{N_2}\right)^2 \mathbf{Z}_{\text{L}}. \qquad (6.21)}$$

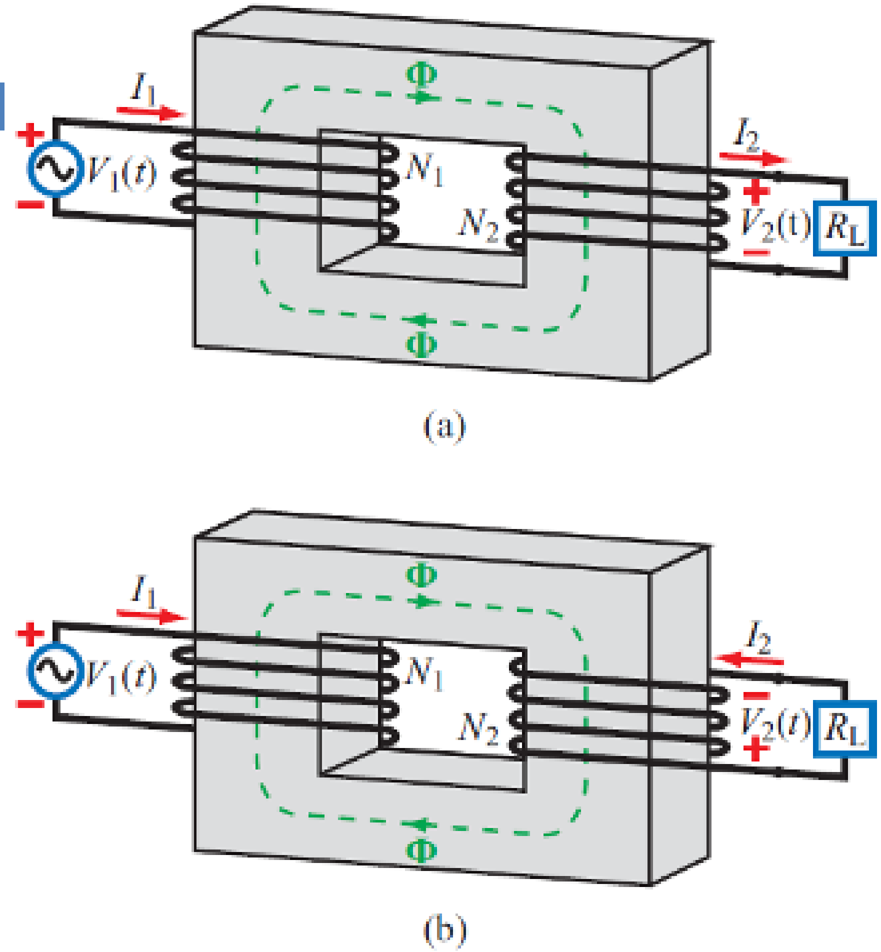


Figure 6-5: In a transformer, the directions of I_1 and I_2 are such that the flux Φ generated by one of them is opposite to that generated by the other. The direction of the secondary winding in (b) is opposite to that in (a), and so are the direction of I_2 and the polarity of V_2.

Motional EMF

Magnetic force on charge *q* moving with velocity **u** in a magnetic field **B**:

$$\mathbf{F}_m = q(\mathbf{u} \times \mathbf{B}).$$

This magnetic force is equivalent to the electrical force that would be exerted on the particle by the electric field Em given by

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{q} = \mathbf{u} \times \mathbf{B}.$$

This, in turn, induces a voltage difference between ends 1 and 2, with end 2 being at the higher potential. The induced voltage is

$$V_{emf}^{m} = V_{12} = \int_{2}^{1} \mathbf{E}_m \cdot d\mathbf{l} = \int_{2}^{1} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

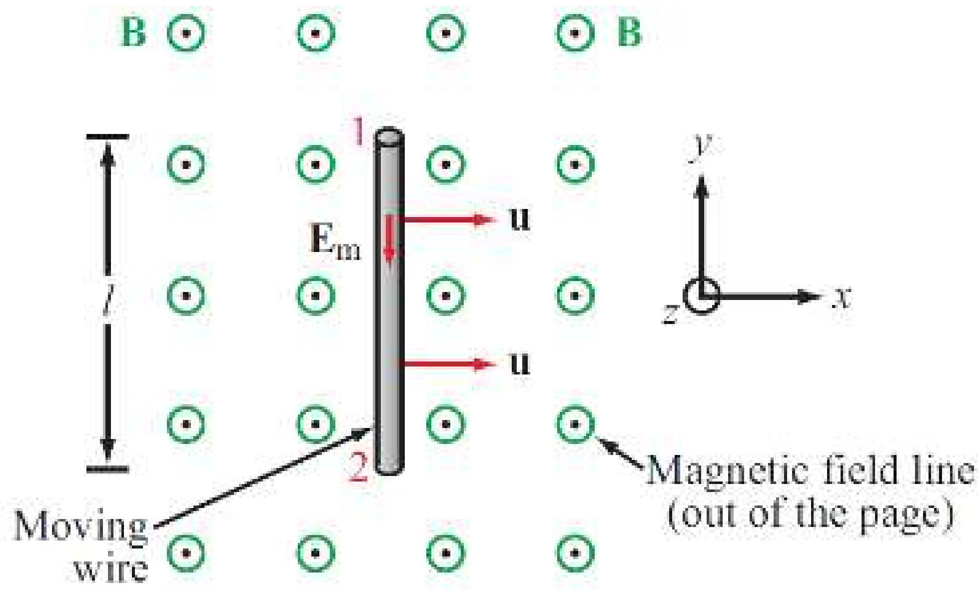


Figure 6-7: Conducting wire moving with velocity **u** in a static magnetic field.

For the conducting wire, $\mathbf{u} \times \mathbf{B} = \hat{\mathbf{x}}u \times \hat{\mathbf{z}}B_0 = -\hat{\mathbf{y}}uB_0$ and $d\mathbf{l} = \hat{\mathbf{y}}\,dl$. Hence,

$$V_{emf}^{m} = V_{12} = -uB_0 l. \tag{6.25}$$

Motional EMF

In general, if any segment of a closed circuit with contour C moves with a velocity $\mathbf{u}$ across a static magnetic field $\mathbf{B}$, then the induced motional emf is given by

$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \qquad \text{(motional emf).} \qquad (6.26)$$

Only those segments of the circuit that cross magnetic field lines contribute to $V_{\text{emf}}^{\text{m}}$.

Example 6-3: Sliding Bar

$$V_{\text{emf}}^{\text{m}} = V_{12} = V_{43} = \int_3^4 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

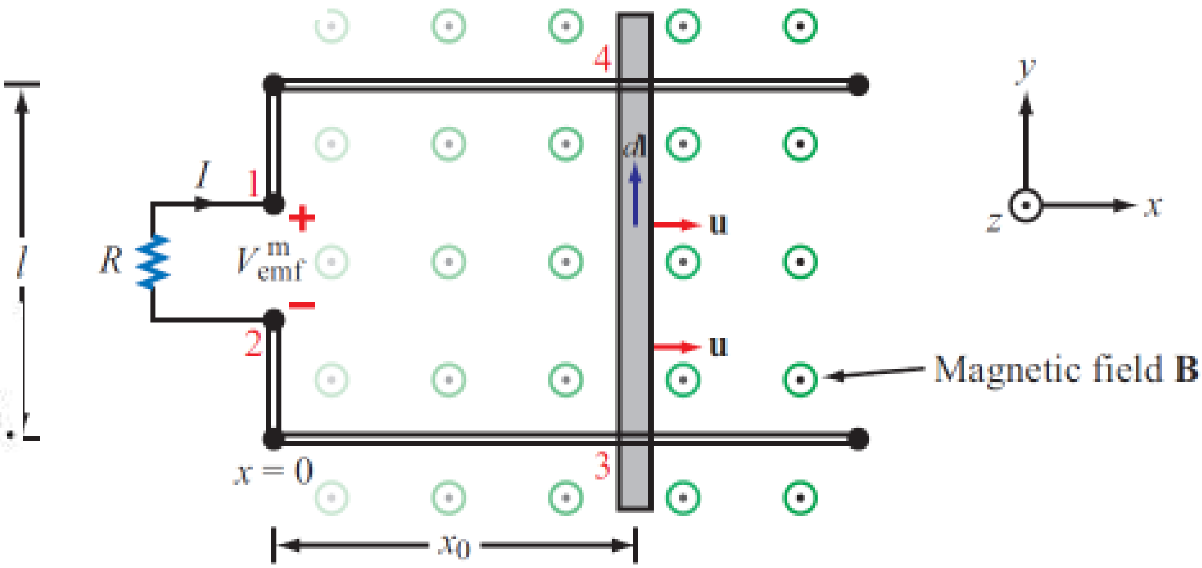
$$= \int_3^4 (\hat{\mathbf{x}}u \times \hat{\mathbf{z}}B_0x_0) \cdot \hat{\mathbf{y}}\, dl = -uB_0x_0l$$

Note that B increases with x

$$\mathbf{B} = \hat{\mathbf{z}}B_0x$$

The length of the loop is related to *u* by *x*0 = *ut*. Hence

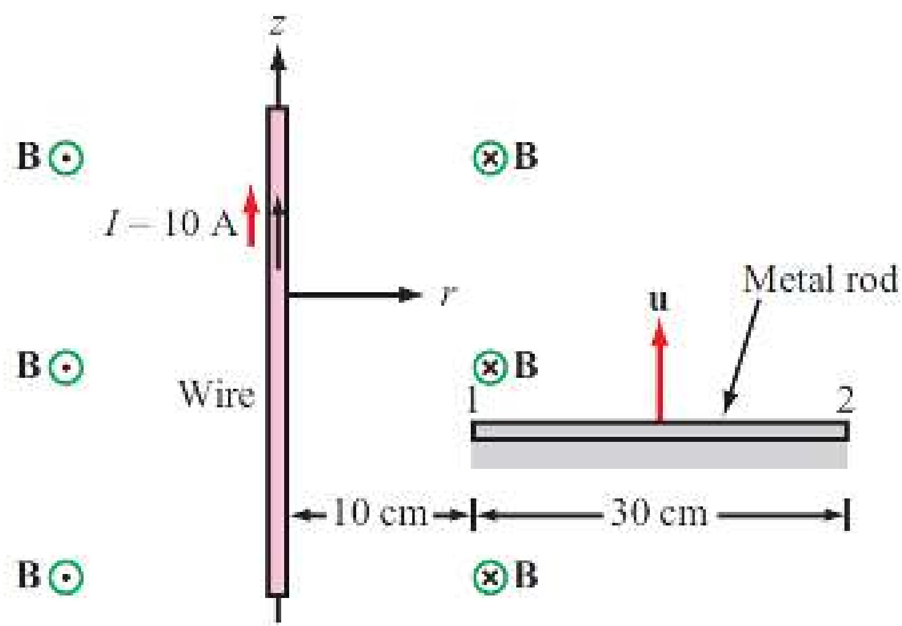
$$V_{\text{emf}}^{\text{m}} = -B_0u^2lt \qquad \text{(V)}.$$



Example 6-5: Moving Rod Next to a Wire

The wire shown in Fig. 6-10 carries a current $I = 10$ A. A 30-cm-long metal rod moves with a constant velocity $\mathbf{u} = \hat{\mathbf{z}}5$ m/s. Find V_{12}.

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$



$$V_{12} = \int_{40\text{ cm}}^{10\text{ cm}} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

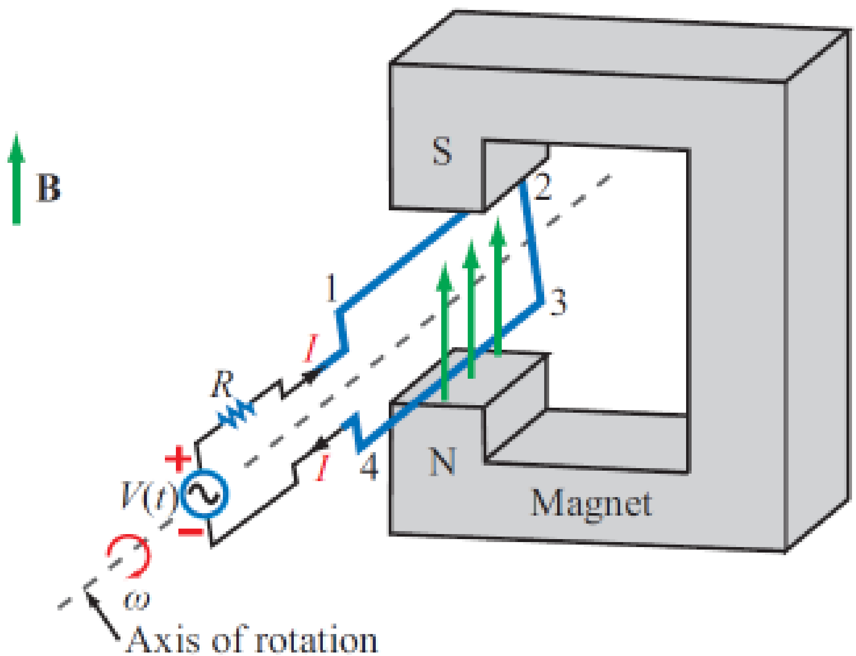
$$= \int_{40\text{ cm}}^{10\text{ cm}} \left(\hat{\mathbf{z}}5 \times \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \right) \cdot \hat{\mathbf{r}}\, dr$$

$$= -\frac{5\mu_0 I}{2\pi} \int_{40\text{ cm}}^{10\text{ cm}} \frac{dr}{r}$$

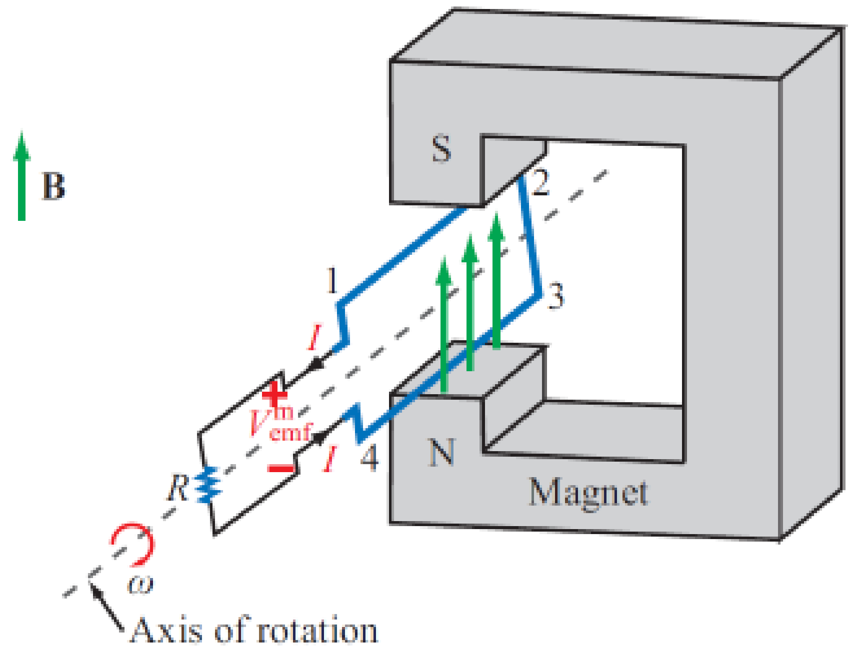
$$= -\frac{5 \times 4\pi \times 10^{-7} \times 10}{2\pi} \times \ln\left(\frac{10}{40}\right)$$

$$= 13.9 \qquad (\mu\text{V}).$$

EM Motor/ Generator Reciprocity



(a) ac motor



(b) ac generator

Motor: Electrical to mechanical energy conversion

Generator: Mechanical to electrical energy conversion

EM Generator EMF

As the loop rotates with an angular velocity *ω about its own* axis, segment 1–2 moves with velocity **u** given by

$$\mathbf{u} = \hat{\mathbf{n}}\omega \frac{w}{2}$$

Also: $\hat{\mathbf{n}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}} \sin\alpha$

Segment 3-4 moves with velocity **–u**. Hence:

$$V_{\text{emf}}^{\text{m}} = V_{14} = \int_{2}^{1} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} + \int_{4}^{3} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$= \int_{-l/2}^{l/2} \left[\left(\hat{\mathbf{n}}\omega \frac{w}{2}\right) \times \hat{\mathbf{z}} B_0\right] \cdot \hat{\mathbf{x}}\, dx$$

$$+ \int_{l/2}^{-l/2} \left[\left(-\hat{\mathbf{n}}\omega \frac{w}{2}\right) \times \hat{\mathbf{z}} B_0\right] \cdot \hat{\mathbf{x}}\, dx.$$

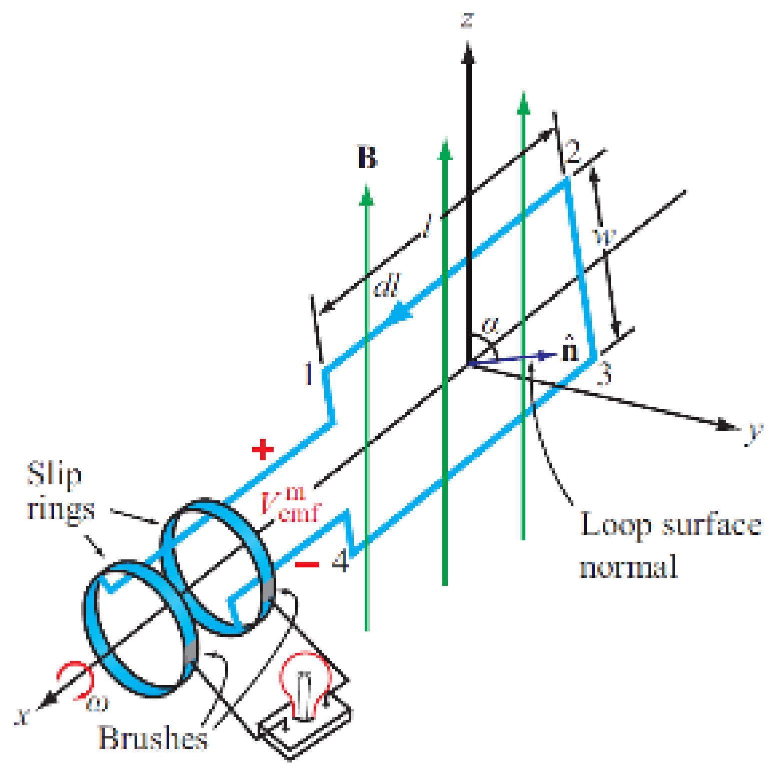


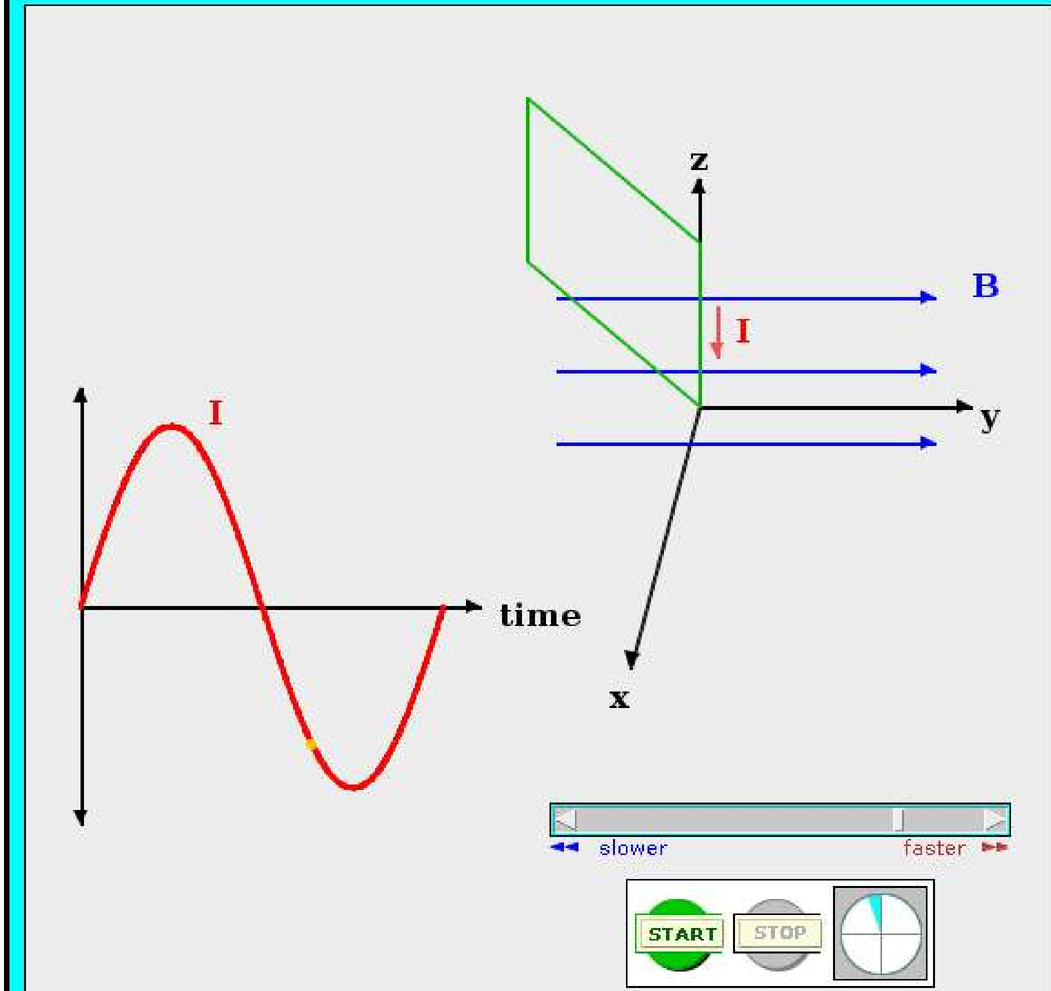
Figure 6-12: A loop rotating in a magnetic field induces an emf.

$$V_{\text{emf}}^{\text{m}} = wl\omega B_0 \sin\alpha = A\omega B_0 \sin\alpha,$$

$$\alpha = \omega t + C_0,$$

$$\boxed{V_{\text{emf}}^{\text{m}} = A\omega B_0 \sin(\omega t + C_0) \qquad \text{(V)}.}$$

Module 6.2 Rotating Wire Loop in Constant Magnetic Field



Demonstration of Motional EMF

A rectangular wire loop of area A rotates at an angular frequency ω in a constant magnetic flux density B_0. The purpose of the demo is to illustrate how the current varies in time relative to the loop's position.

Note the direction of the current and its magnitude, as indicated by its brightness.

$$I_{max} = \omega B_0 A$$

Applet Design: Janice Richards

Tech Brief 12: EMF Sensors

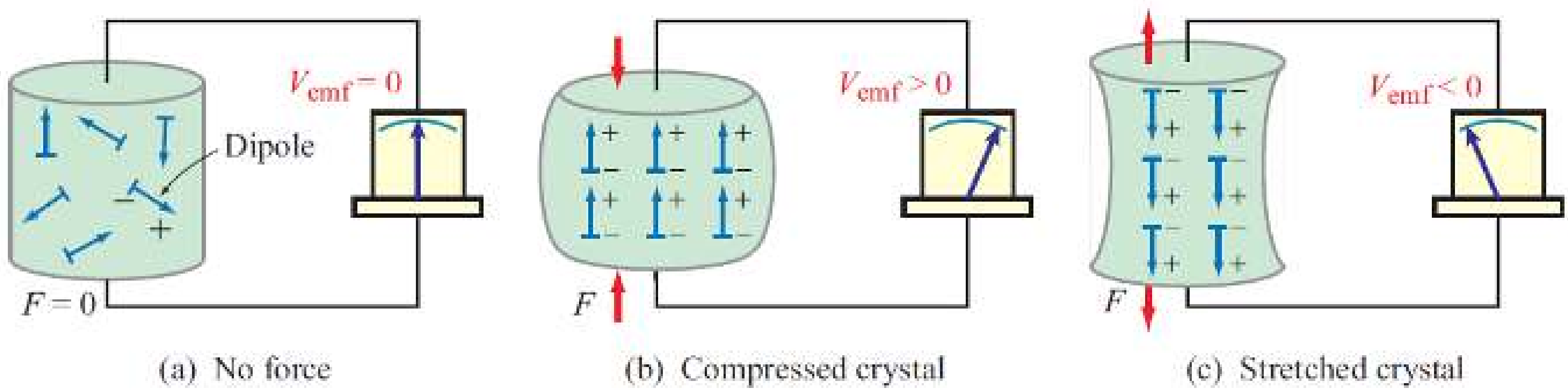


Figure TF12-1: Response of a piezoelectric crystal to an applied force.

- Piezoelectric crystals generate a voltage across them proportional to the compression or tensile (stretching) force applied across them.

- Piezoelectric transducers are used in medical ultrasound, microphones, loudspeakers, accelerometers, etc.

- Piezoelectric crystals are bidirectional: pressure generates emf, and conversely, emf generates pressure (through shape distortion).

Faraday Accelerometer

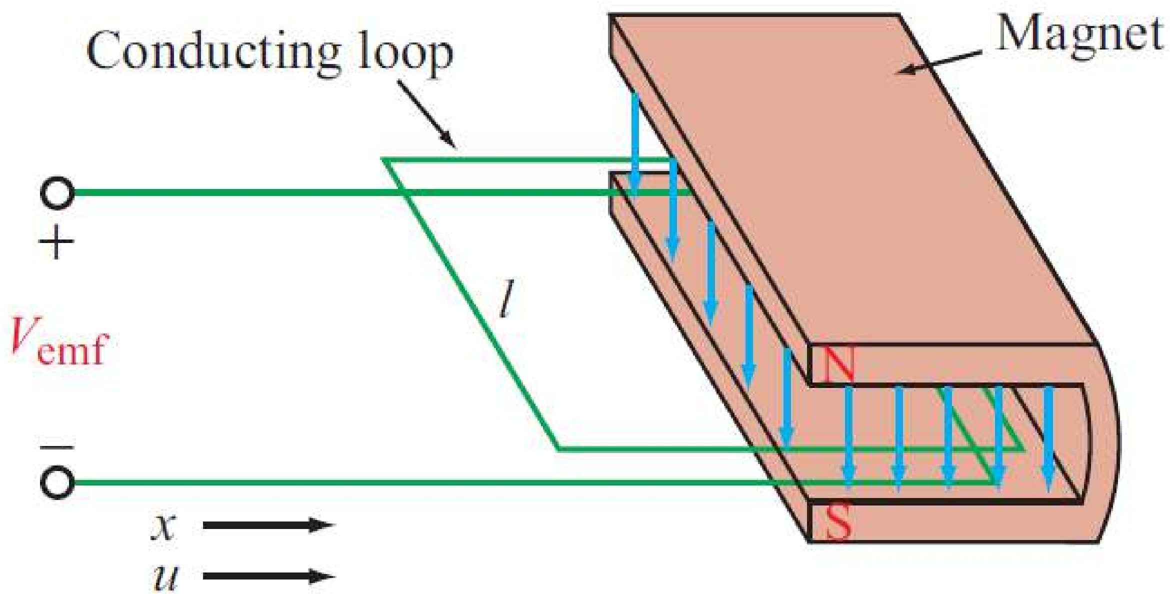


Figure TF12-3: In a Faraday accelerometer, the induced emf is directly proportional to the velocity of the loop (into and out of the magnet's cavity).

The acceleration **a** is determined by differentiating the velocity u with respect to time

The Thermocouple

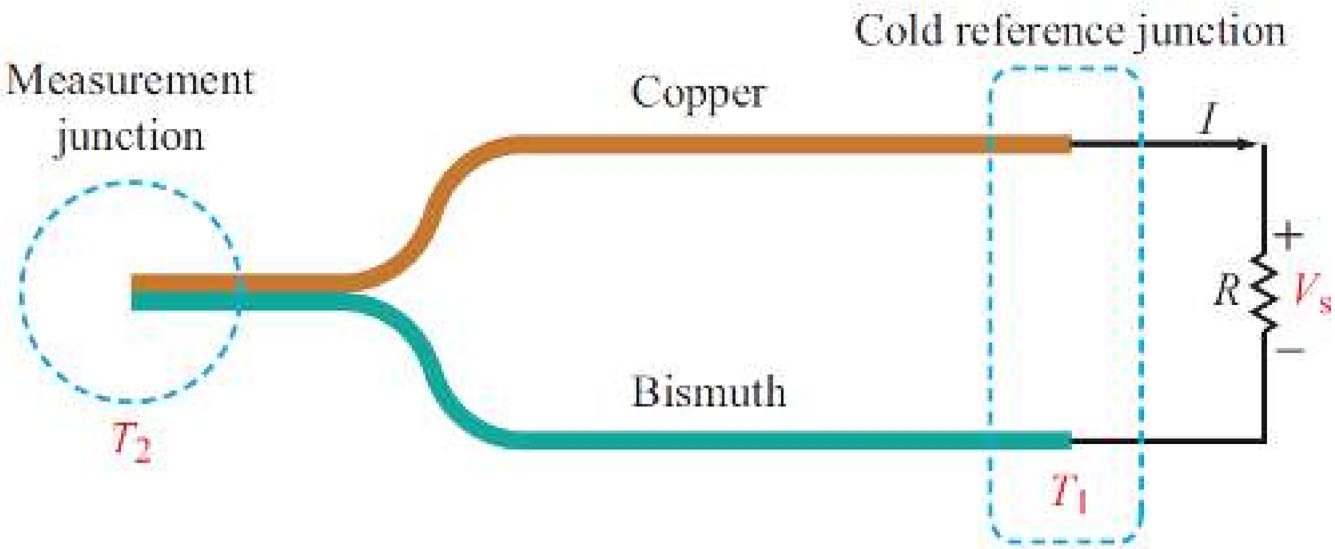


Figure TF12-4: Principle of the thermocouple.

- The thermocouple measures the unknown temperature T_2 at a junction connecting two metals with different thermal conductivities, relative to a reference temperature T_1.
- In today's temperature sensor designs, an artificial cold junction is used instead. The artificial junction is an electric circuit that generates a voltage equal to that expected from a reference junction at temperature T_1.

Displacement Current

Ampère's law in differential form is given by

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{(Ampère's law).} \qquad (6.41)$$

Integrating both sides of Eq. (6.41) over an arbitrary open surface S with contour C, we have

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}. \qquad (6.42)$$

This term is conduction current I_C

This term must represent a current

Application of Stokes's theorem gives:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \qquad \text{(Ampère's law)}$$

Cont.

Displacement Current

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot ds \quad \text{(Ampère's law)}$$

Define the displacement current as:

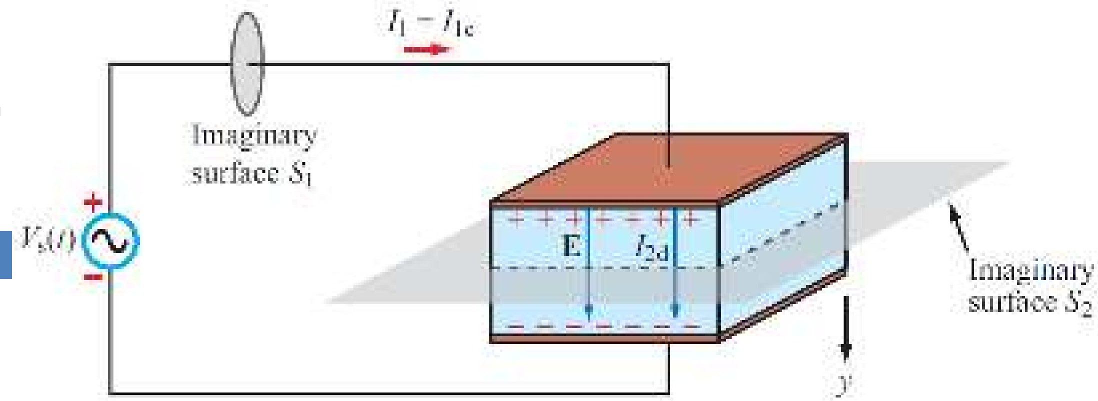
$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}, \qquad (6.44)$$

where $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ represents a *displacement current density*. In view of Eq. (6.44),

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + I_d = I, \qquad (6.45)$$

The displacement current does not involve real charges;
it is an equivalent current that depends on $\partial \mathbf{D}/\partial t$

Capacitor Circuit



Given: Wires are perfect conductors and capacitor insulator material is perfect dielectric.

For Surface S_1:

$$I_1 = I_{1c} + I_{1d}$$

$$I_{1c} = C\frac{dV_C}{dt} = C\frac{d}{dt}(V_0 \cos\omega t) = -CV_0\omega\sin\omega t$$

$I_{1d} = 0$ (**D** = 0 in perfect conductor)

For Surface S_2:

$$I_2 = I_{2c} + I_{2d}$$

$I_{2c} = 0$ (perfect dielectric)

$$\mathbf{E} = \hat{\mathbf{y}}\frac{V_c}{d} = \hat{\mathbf{y}}\frac{V_0}{d}\cos\omega t$$

$$I_{2d} = \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s}$$

$$= \int_A \left[\frac{\partial}{\partial t}\left(\hat{\mathbf{y}}\frac{\varepsilon V_0}{d}\cos\omega t\right)\right]\cdot(\hat{\mathbf{y}}\, ds)$$

$$= -\frac{\varepsilon A}{d}V_0\omega\sin\omega t = -CV_0\omega\sin\omega t$$

Conclusion: $I_1 = I_2$

Example 6-7: Displacement Current Density

The conduction current flowing through a wire with conductivity $\sigma = 2 \times 10^7$ S/m and relative permittivity $\varepsilon_r = 1$ is given by $I_c = 2 \sin \omega t$ (mA). If $\omega = 10^9$ rad/s, find the displacement current.

Solution: The conduction current $I_c = JA = \sigma EA$, where A is the cross section of the wire. Hence,

$$E = \frac{I_c}{\sigma A} = \frac{2 \times 10^{-3} \sin \omega t}{2 \times 10^7 A} = \frac{1 \times 10^{-10}}{A} \sin \omega t \qquad \text{(V/m)}.$$

Application of Eq. (6.44), with $D = \varepsilon E$, leads to

$$\begin{aligned} I_d &= J_d A \\ &= \varepsilon A \frac{\partial E}{\partial t} \\ &= \varepsilon A \frac{\partial}{\partial t}\left(\frac{1 \times 10^{-10}}{A} \sin \omega t\right) \\ &= \varepsilon \omega \times 10^{-10} \cos \omega t = 0.885 \times 10^{-12} \cos \omega t \quad \text{(A)}, \end{aligned}$$

where we used $\omega = 10^9$ rad/s and $\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}$ F/m. Note that I_c and I_d are in phase quadrature (90° phase shift between them). Also, I_d is about nine orders of magnitude smaller than I_c, which is why the displacement current usually is ignored in good conductors.

Boundary Conditions

Table 6-2: Boundary conditions for the electric and magnetic fields.

Field Components	General Form	Medium 1 Dielectric	Medium 2 Dielectric	Medium 1 Dielectric	Medium 2 Conductor
Tangential E	$\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Normal D	$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$
Tangential H	$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$		$H_{1t} = J_s$	$H_{2t} = 0$
Normal B	$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$		$B_{1n} = B_{2n} = 0$	

Notes: (1) ρ_s is the surface charge density at the boundary; (2) $\mathbf{J}_s$ is the surface current density at the boundary; (3) normal components of all fields are along $\hat{\mathbf{n}}_2$, the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of $\mathbf{J}_s$ is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.

Charge Current Continuity Equation

Current I out of a volume is equal to rate of decrease of charge Q contained in that volume:

$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

$$\oint_S \mathbf{J}\cdot d\mathbf{s} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

$$\oint_S \mathbf{J}\cdot d\mathbf{s} = \int_{\mathcal{V}} \nabla\cdot\mathbf{J}\, d\mathcal{V} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

Used Divergence Theorem

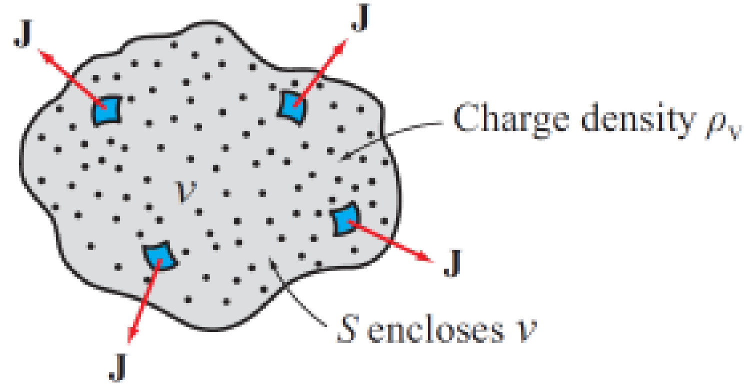


Figure 6-14: The total current flowing out of a volume $\mathcal{V}$ is equal to the flux of the current density $\mathbf{J}$ through the surface S, which in turn is equal to the rate of decrease of the charge enclosed in $\mathcal{V}$.

$$\boxed{\nabla\cdot\mathbf{J} = -\frac{\partial \rho_{\mathrm{v}}}{\partial t}\,, \qquad (6.54)}$$

which is known as the ***charge-current continuity relation***, or simply the ***charge continuity equation***.

Charge Dissipation

Question 1: What happens if you place a certain amount of free charge inside of a material?
Answer: The charge will move to the surface of the material, thereby returning its interior to a neutral state.

Question 2: How fast will this happen?
Answer: It depends on the material; in a good conductor, the charge dissipates in less than a femtosecond, whereas in a good dielectric, the process may take several hours.

Derivation of charge density equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} . \quad (6.58)$$

In a conductor, the point form of Ohm's law, given by Eq. (4.63), states that $\mathbf{J} = \sigma \mathbf{E}$. Hence,

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho_v}{\partial t} . \quad (6.59)$$

Next, we use Eq. (6.1), $\nabla \cdot \mathbf{E} = \rho_v / \varepsilon$, to obtain the partial differential equation

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0. \quad (6.60)$$

Cont.

Solution of Charge Dissipation Equation

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

Given that $\rho_v = \rho_{vo}$ at $t = 0$, the solution of Eq. (6.60) is

$$\rho_v(t) = \rho_{vo} e^{-(\sigma/\varepsilon)t} = \rho_{vo} e^{-t/\tau_r} \qquad (\text{C/m}^3),$$

where $\tau_r = \varepsilon/\sigma$ is called the ***relaxation time constant***.

For copper: $\tau_r = 1.53 \times 10^{-19}$ s

For mica: $\tau_r = 5.31 \times 10^4$ s = 15 hours

EM Potentials

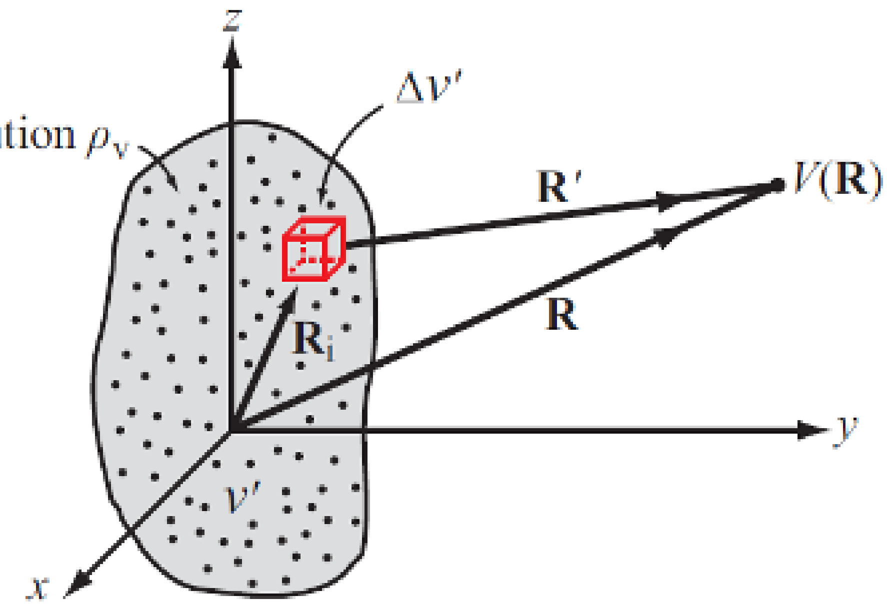


Figure 6-16: Electric potential $V(\mathbf{R})$ due to a charge distribution ρ_v over a volume $\mathcal{V}'$.

Static condition

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v(\mathbf{R}_i)}{R'} \, d\mathcal{V}'$$

Dynamic condition

$$V(\mathbf{R}, t) = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v(\mathbf{R}_i, t)}{R'} \, d\mathcal{V}'$$

Dynamic condition with propagation delay:

$$V(\mathbf{R}, t) = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v(\mathbf{R}_i, t - R'/u_p)}{R'} \, d\mathcal{V}' \quad \text{(V)},$$

Similarly, for the magnetic vector potential:

$$\mathbf{A}(\mathbf{R}, t) = \frac{\mu}{4\pi} \int_{\mathcal{V}'} \frac{\mathbf{J}(\mathbf{R}_i, t - R'/u_p)}{R'} \, d\mathcal{V}' \quad \text{(Wb/m)}.$$

Time Harmonic Potentials

If charges and currents vary sinusoidally with time:

$$\rho_v(\mathbf{R}_i, t) = \rho_v(\mathbf{R}_i)\cos(\omega t + \phi)$$

we can use phasor notation:

$$\rho_v(\mathbf{R}_i, t) = \mathfrak{Re}\left[\tilde{\rho}_v(\mathbf{R}_i)\, e^{j\omega t}\right],$$

with

$$\tilde{\rho}_v(\mathbf{R}_i) = \rho_v(\mathbf{R}_i)\, e^{j\phi}.$$

Expressions for potentials become:

$$\widetilde{V}(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\tilde{\rho}_v(\mathbf{R}_i)\, e^{-jkR'}}{R'}\, d\mathcal{V}' \quad \text{(V)}.$$

$$\widetilde{\mathbf{A}}(\mathbf{R}) = \frac{\mu}{4\pi}\int_{V'} \frac{\widetilde{\mathbf{J}}(\mathbf{R}_i)\, e^{-jkR'}}{R'}\, d\mathcal{V}'.$$

Also:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{(dynamic case).}$$

$$\widetilde{\mathbf{H}} = \frac{1}{\mu}\nabla \times \widetilde{\mathbf{A}}.$$

Maxwell's equations become:

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$$

$$\text{or} \qquad \widetilde{\mathbf{H}} = -\frac{1}{j\omega\mu}\nabla \times \widetilde{\mathbf{E}}.$$

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon\widetilde{\mathbf{E}} \quad \text{or} \quad \widetilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\nabla \times \widetilde{\mathbf{H}}.$$

$$k = \frac{\omega}{u_p}$$

Example 6-8: Relating E to H

In a nonconducting medium with $\varepsilon = 16\varepsilon_0$ and $\mu = \mu_0$, the electric field intensity of an electromagnetic wave is

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 10 \sin(10^{10} t - kz) \qquad \text{(V/m)}. \tag{6.88}$$

Determine the associated magnetic field intensity **H** and find the value of k.

Solution: We begin by finding the phasor $\widetilde{\mathbf{E}}(z)$ of $\mathbf{E}(z,t)$. Since $\mathbf{E}(z,t)$ is given as a sine function and phasors are defined in this book with reference to the cosine function, we rewrite Eq. (6.88) as

$$\begin{aligned}\mathbf{E}(z,t) &= \hat{\mathbf{x}}\, 10 \cos(10^{10} t - kz - \pi/2) \qquad \text{(V/m)} \\ &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right],\end{aligned} \tag{6.89}$$

with $\omega = 10^{10}$ (rad/s) and

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\, 10 e^{-jkz} e^{-j\pi/2} = -\hat{\mathbf{x}}\, j\, 10 e^{-jkz}. \tag{6.90}$$

Cont.

Example 6-8: Relating E to H

To find both $\widetilde{\mathbf{H}}(z)$ and k, we will perform a "circle": we will use the given expression for $\widetilde{\mathbf{E}}(z)$ in Faraday's law to find $\widetilde{\mathbf{H}}(z)$; then we will use $\widetilde{\mathbf{H}}(z)$ in Ampère's law to find $\widetilde{\mathbf{E}}(z)$, which we will then compare with the original expression for $\widetilde{\mathbf{E}}(z)$; and the comparison will yield the value of k. Application of Eq. (6.87) gives

$$\begin{aligned}
\widetilde{\mathbf{H}}(z) &= -\frac{1}{j\omega\mu}\nabla \times \widetilde{\mathbf{E}} \\
&= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -j10e^{-jkz} & 0 & 0 \end{vmatrix} \\
&= -\frac{1}{j\omega\mu}\left[\hat{\mathbf{y}}\,\frac{\partial}{\partial z}(-j10e^{-jkz})\right] \\
&= -\hat{\mathbf{y}}j\,\frac{10k}{\omega\mu}e^{-jkz}. \qquad (6.91)
\end{aligned}$$

Cont.

Example 6-8 cont.

So far, we have used Eq. (6.90) for $\widetilde{\mathbf{E}}(z)$ to find $\widetilde{\mathbf{H}}(z)$, but k remains unknown. To find k, we use $\widetilde{\mathbf{H}}(z)$ in Eq. (6.86) to find $\widetilde{\mathbf{E}}(z)$:

$$\begin{aligned}\widetilde{\mathbf{E}}(z) &= \frac{1}{j\omega\varepsilon}\nabla\times\widetilde{\mathbf{H}} \\ &= \frac{1}{j\omega\varepsilon}\left[-\hat{\mathbf{x}}\frac{\partial}{\partial z}\left(-j\,\frac{10k}{\omega\mu}e^{-jkz}\right)\right] \\ &= -\hat{\mathbf{x}}j\,\frac{10k^2}{\omega^2\mu\varepsilon}e^{-jkz}. \qquad (6.92)\end{aligned}$$

Equating Eqs. (6.90) and (6.92) leads to

$$k^2 = \omega^2\mu\varepsilon,$$

or

$$\begin{aligned}k &= \omega\sqrt{\mu\varepsilon} \\ &= 4\omega\sqrt{\mu_0\varepsilon_0} \\ &= \frac{4\omega}{c} = \frac{4\times10^{10}}{3\times10^8} = 133 \qquad \text{(rad/m)}. \qquad (6.93)\end{aligned}$$

Cont.

Example 6-8 cont.

With k known, the instantaneous magnetic field intensity is then given by

$$\mathbf{H}(z,t) = \mathfrak{Re}\left[\widetilde{\mathbf{H}}(z)\, e^{j\omega t}\right]$$
$$= \mathfrak{Re}\left[-\hat{\mathbf{y}} j\, \frac{10k}{\omega\mu} e^{-jkz} e^{j\omega t}\right]$$
$$= \hat{\mathbf{y}}\, 0.11 \sin(10^{10} t - 133z) \qquad \text{(A/m)}. \qquad (6.94)$$

We note that k has the same expression as the phase constant of a lossless transmission line [Eq. (2.49)].

Summary

Chapter 6 Relationships

Faraday's Law

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}}$$

Transformer

$$V_{\text{emf}}^{\text{tr}} = -N\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} \qquad (N \text{ loops})$$

Motional

$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

Charge-Current Continuity

$$\nabla\cdot\mathbf{J} = -\frac{\partial \rho_{\text{v}}}{\partial t}$$

EM Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

Current Density

Conduction $\quad \mathbf{J}_{\text{c}} = \sigma\,\mathbf{E}$

Displacement $\quad \mathbf{J}_{\text{d}} = \dfrac{\partial \mathbf{D}}{\partial t}$

Conductor Charge Dissipation

$$\rho_{\text{v}}(t) = \rho_{\text{vo}}e^{-(\sigma/\varepsilon)t} = \rho_{\text{vo}}e^{-t/\tau_r}$$

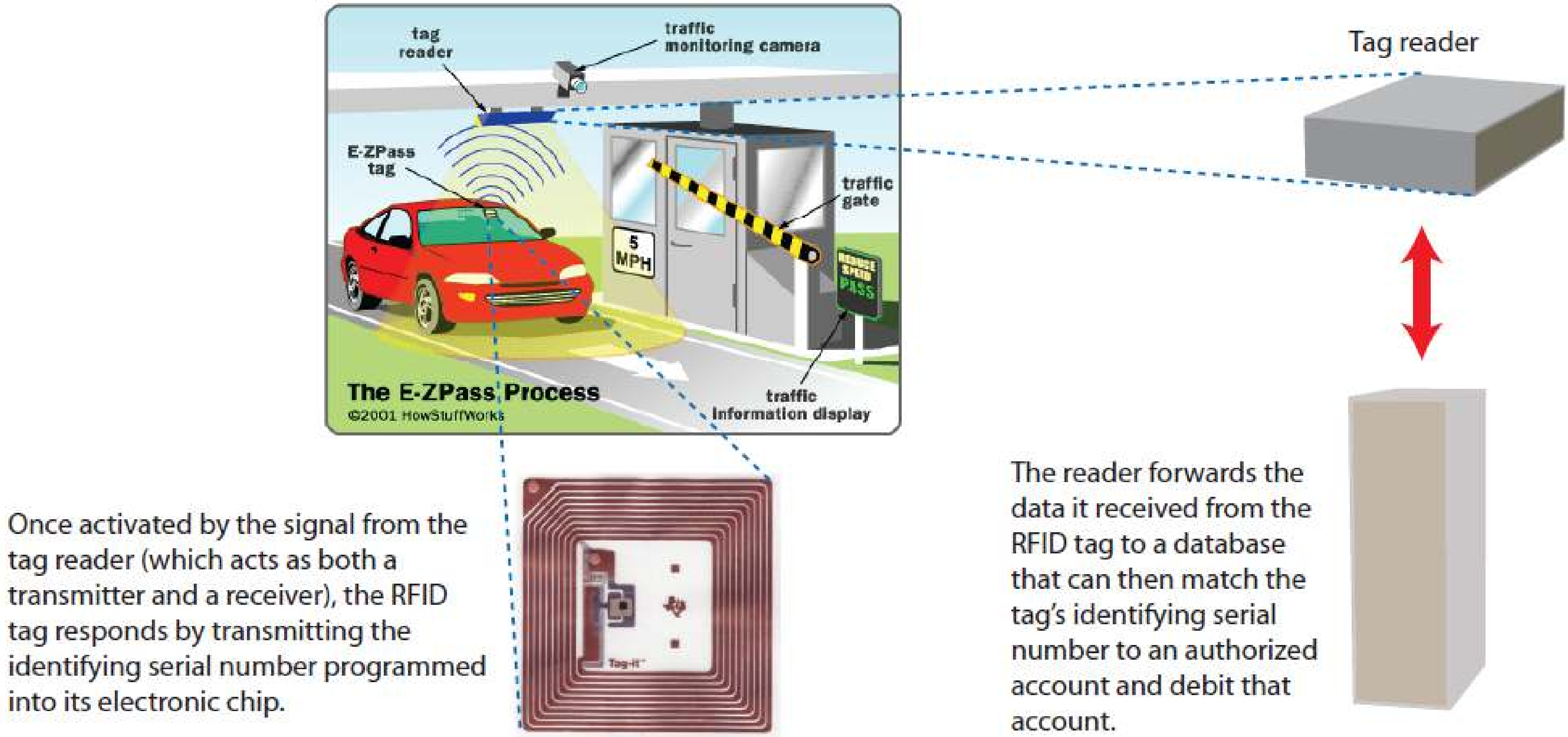


Figure TF13-2: How an RFID system works is illustrated through this EZ-Pass example. (Tag courtesy of Texas Instruments.)

7. PLANE WAVE PROPAGATION

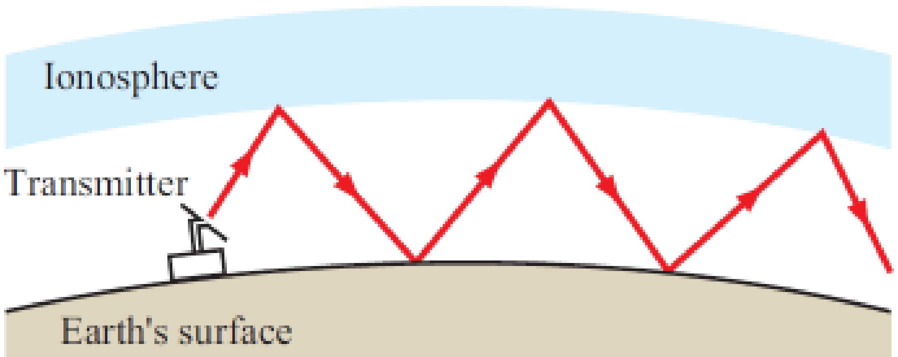
Chapter 7 Overview

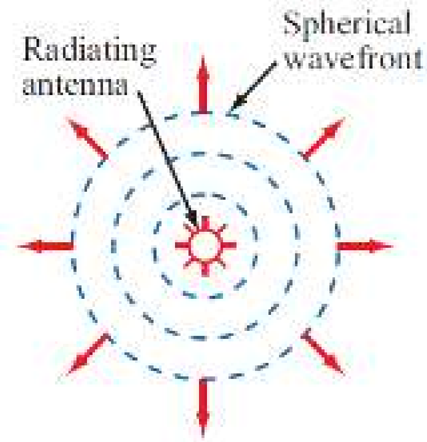
Chapter Contents

Objectives

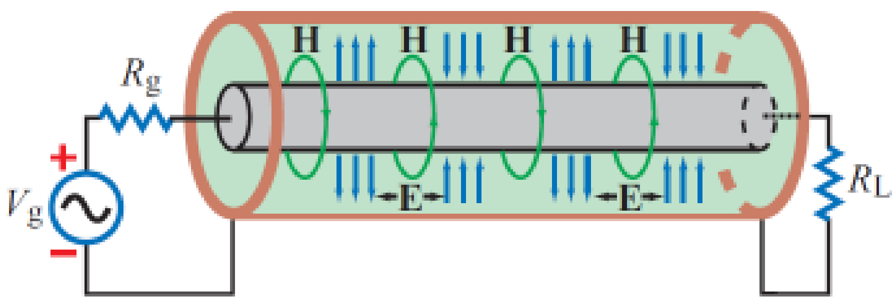
Upon learning the material presented in this chapter, you should be able to:

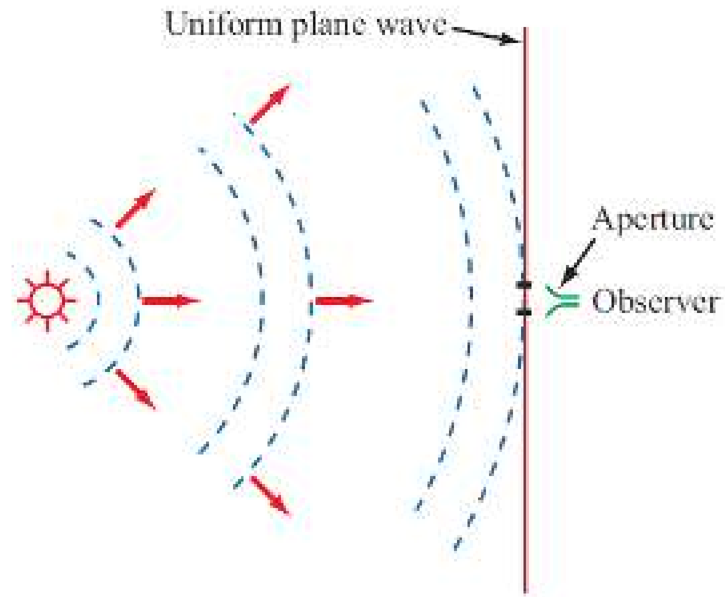
1. Describe mathematically the electric and magnetic fields of TEM waves.
2. Describe the polarization properties of an EM wave.
3. Relate the propagation parameters of a wave to the constitutive parameters of the medium.
4. Characterize the flow of current in conductors and use it to calculate the resistance of a coaxial cable.
5. Calculate the rate of power carried by an EM wave, in both lossless and lossy media.





(a) Spherical wave





(b) Plane-wave approximation

Guided EM Waves

Unbounded EM Waves

Maxwell's Equations

For sinusoidal time variations:

$$\nabla \cdot \widetilde{\mathbf{E}} = \tilde{\rho}_v / \varepsilon,$$
$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}},$$
$$\nabla \cdot \widetilde{\mathbf{H}} = 0,$$
$$\nabla \times \widetilde{\mathbf{H}} = \widetilde{\mathbf{J}} + j\omega\varepsilon\widetilde{\mathbf{E}}.$$

$$\begin{aligned}\nabla \times \widetilde{\mathbf{H}} &= \widetilde{\mathbf{J}} + j\omega\varepsilon\widetilde{\mathbf{E}} \\ &= (\sigma + j\omega\varepsilon)\widetilde{\mathbf{E}} = j\omega\left(\varepsilon - j\frac{\sigma}{\omega}\right)\widetilde{\mathbf{E}}.\end{aligned}$$

By defining the ***complex permittivity*** ε_c as

$$\boxed{\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}, \qquad (7.4)}$$

Eq. (7.3) can be rewritten as

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon_c\widetilde{\mathbf{E}}.$$

For any vector field:

$$\nabla \cdot \nabla \times \widetilde{\mathbf{H}} = 0$$

Hence: $\nabla \cdot \widetilde{\mathbf{E}} = 0$ and $\tilde{\rho}_v = 0$

Consequently, Maxwell's equations become:

$$\nabla \cdot \widetilde{\mathbf{E}} = 0,$$
$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}},$$
$$\nabla \cdot \widetilde{\mathbf{H}} = 0,$$
$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon_c\widetilde{\mathbf{E}}.$$

We will use these to derive the wave equation for EM waves.

Complex Permittivity

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon'', \tag{7.7}$$

with

$$\varepsilon' = \varepsilon, \tag{7.8a}$$

$$\varepsilon'' = \frac{\sigma}{\omega} . \tag{7.8b}$$

For a lossless medium with $\sigma = 0$, it follows that $\varepsilon'' = 0$ and $\varepsilon_c = \varepsilon' = \varepsilon$.

Wave Equations

$$\nabla \cdot \widetilde{\mathbf{E}} = 0, \quad (7.6a)$$

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}, \quad (7.6b)$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0, \quad (7.6c)$$

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon_c\widetilde{\mathbf{E}}. \quad (7.6d)$$

$$\nabla \times (\nabla \times \widetilde{\mathbf{E}}) = -j\omega\mu(\nabla \times \widetilde{\mathbf{H}}). \quad (7.9)$$

Upon substituting Eq. (7.6d) into Eq. (7.9) we obtain

$$\nabla \times (\nabla \times \widetilde{\mathbf{E}}) = -j\omega\mu(j\omega\varepsilon_c\widetilde{\mathbf{E}}) = \omega^2\mu\varepsilon_c\widetilde{\mathbf{E}}. \quad (7.10)$$

From Eq. (3.113), we know that the curl of the curl of $\widetilde{\mathbf{E}}$ is

$$\nabla \times (\nabla \times \widetilde{\mathbf{E}}) = \nabla(\nabla \cdot \widetilde{\mathbf{E}}) - \nabla^2\widetilde{\mathbf{E}}, \quad (7.11)$$

where $\nabla^2\widetilde{\mathbf{E}}$ is the Laplacian of $\widetilde{\mathbf{E}}$, which in Cartesian coordinates is given by

$$\nabla^2\widetilde{\mathbf{E}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\widetilde{\mathbf{E}}. \quad (7.12)$$

In view of Eq. (7.6a), the use of Eq. (7.11) in Eq. (7.10) gives

$$\nabla^2\widetilde{\mathbf{E}} + \omega^2\mu\varepsilon_c\widetilde{\mathbf{E}} = 0, \quad (7.13)$$

which is known as the ***homogeneous wave equation for*** $\widetilde{\mathbf{E}}$. By defining the ***propagation constant*** γ as

$$\gamma^2 = -\omega^2\mu\varepsilon_c, \quad (7.14)$$

Eq. (7.13) can be written as

$$\boxed{\nabla^2\widetilde{\mathbf{E}} - \gamma^2\widetilde{\mathbf{E}} = 0. \quad (7.15)}$$

To derive Eq. (7.15), we took the curl of both sides of Eq. (7.6b) and then we used Eq. (7.6d) to eliminate $\widetilde{\mathbf{H}}$ and obtain an equation in $\widetilde{\mathbf{E}}$ only. If we reverse the process, that is, if we start by taking the curl of both sides of Eq. (7.6d) and then use Eq. (7.6b) to eliminate $\widetilde{\mathbf{E}}$, we obtain a wave equation for $\widetilde{\mathbf{H}}$:

$$\boxed{\nabla^2\widetilde{\mathbf{H}} - \gamma^2\widetilde{\mathbf{H}} = 0. \quad (7.16)}$$

Since the wave equations for $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{H}}$ are of the same form, so are their solutions.

Lossless Media

If the medium is ***nonconducting*** **(σ = 0)**, the wave does not suffer any attenuation as it travels and hence the medium is said to be ***lossless.***

$$\gamma^2 = -\omega^2 \mu\varepsilon. \qquad (7.17)$$

For lossless media, it is customary to define the ***wavenumber*** k as

$$\boxed{k = \omega\sqrt{\mu\varepsilon}\ . \qquad (7.18)}$$

In view of Eq. (7.17), $\gamma^2 = -k^2$ and Eq. (7.15) becomes

$$\nabla^2\widetilde{\mathbf{E}} + k^2\widetilde{\mathbf{E}} = 0. \qquad (7.19)$$

Uniform Plane Wave

For an electric field phasor decomposed in its Cartesian components as

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{y}}\widetilde{E}_y + \hat{\mathbf{z}}\widetilde{E}_z, \qquad (7.20)$$

substitution of Eq. (7.12) into Eq. (7.19) gives

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{y}}\widetilde{E}_y + \hat{\mathbf{z}}\widetilde{E}_z) + k^2(\hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{y}}\widetilde{E}_y + \hat{\mathbf{z}}\widetilde{E}_z) = 0. \qquad (7.21)$$

To satisfy Eq. (7.21), each vector component on the left-hand side of the equation must vanish. Hence,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\widetilde{E}_x = 0, \qquad (7.22)$$

and similar expressions apply to $\widetilde{E}_y$ and $\widetilde{E}_z$.

Cont.

Uniform Plane Wave

A uniform plane wave is characterized by electric and magnetic fields that have uniform properties at all points across an infinite plane.

If this happens to be the x–y plane, then $\mathbf{E}$ and $\mathbf{H}$ do not vary with x and y. Hence, $\partial\widetilde{E}_x/\partial x = 0$ and $\partial\widetilde{E}_x/\partial y = 0$, and Eq. (7.22) reduces to

$$\frac{d^2\widetilde{E}_x}{dz^2} + k^2\widetilde{E}_x = 0. \tag{7.23}$$

General Form of the Solution:

$$\widetilde{E}_x(z) = \widetilde{E}_x^+(z) + \widetilde{E}_x^-(z) = E_{x0}^+ e^{-jkz} + E_{x0}^- e^{jkz}$$

For a wave travelling along +z only:

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x^+(z) = \hat{\mathbf{x}}E_{x0}^+ e^{-jkz}$$

Application of $\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$ yields:

$$\widetilde{H}_y(z) = \frac{k}{\omega\mu}E_{x0}^+ e^{-jkz} = H_{y0}^+ e^{-jkz}$$

Summary: This is a plane wave with

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x^+(z) = \hat{\mathbf{x}}E_{x0}^+ e^{-jkz},$$

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\frac{\widetilde{E}_x^+(z)}{\eta} = \hat{\mathbf{y}}\frac{E_{x0}^+}{\eta}e^{-jkz},$$

with $$\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} \quad (\Omega)$$

Summary from previous slide:

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x^+(z) = \hat{\mathbf{x}}E_{x0}^+ e^{-jkz},$$

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\frac{\widetilde{E}_x^+(z)}{\eta} = \hat{\mathbf{y}}\frac{E_{x0}^+}{\eta}e^{-jkz}.$$

Time-Domain Solution

In the general case, E_{x0}^+ is a complex quantity with magnitude $|E_{x0}^+|$ and phase angle ϕ^+. That is,

$$E_{x0}^+ = |E_{x0}^+|e^{j\phi^+}. \qquad (7.33)$$

The instantaneous electric and magnetic fields therefore are

$$\begin{aligned}\mathbf{E}(z,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\,e^{j\omega t}\right] \\ &= \hat{\mathbf{x}}|E_{x0}^+|\cos(\omega t - kz + \phi^+) \quad \text{(V/m)},\end{aligned} \qquad (7.34a)$$

and

$$\begin{aligned}\mathbf{H}(z,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{H}}(z)\,e^{j\omega t}\right] \\ &= \hat{\mathbf{y}}\frac{|E_{x0}^+|}{\eta}\cos(\omega t - kz + \phi^+) \quad \text{(A/m)}.\end{aligned} \qquad (7.34b)$$

Wave's Phase Velocity

$$\boxed{u_p = \frac{\omega}{k} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{(m/s)}, \qquad (7.35)}$$

and its wavelength is

$$\boxed{\lambda = \frac{2\pi}{k} = \frac{u_p}{f} \qquad \text{(m)}. \qquad (7.36)}$$

In vacuum, $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$, and the phase velocity u_p and the intrinsic impedance η given by Eq. (7.31) are

$$u_p = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3 \times 10^8 \qquad \text{(m/s)}, \qquad (7.37)$$

$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ (\Omega) \approx 120\pi \quad (\Omega), \qquad (7.38)$$

Example 7-1: EM Plane Wave in Air

This example is analogous to the "Sound Wave in Water" problem given by Example 1-1.

The electric field of a 1-MHz plane wave traveling in the $+z$-direction in air points along the x-direction. If this field reaches a peak value of 1.2π (mV/m) at $t = 0$ and $z = 50$ m, obtain expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$, and then plot them as a function of z at $t = 0$.

Solution: At $f = 1$ MHz, the wavelength in air is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{1 \times 10^6} = 300 \text{ m},$$

and the corresponding wavenumber is $k = (2\pi/300)$ (rad/m). The general expression for an x-directed electric field traveling in the $+z$-direction is given by Eq. (7.34a) as

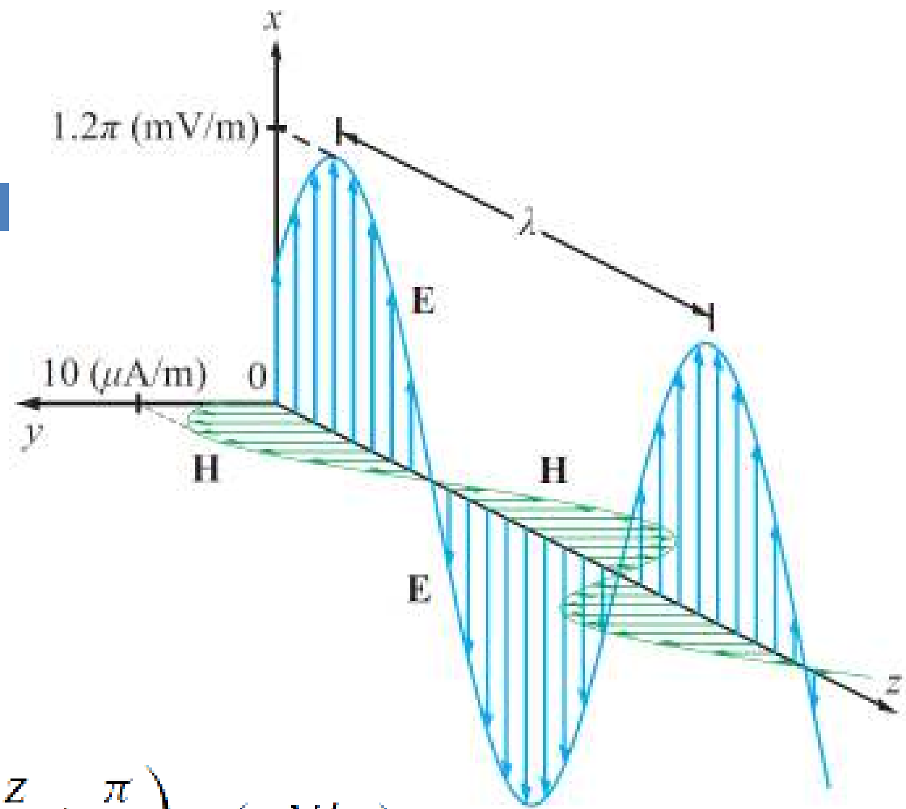
$$\begin{aligned}\mathbf{E}(z, t) &= \hat{\mathbf{x}} |E_{x0}^+| \cos(\omega t - kz + \phi^+) \\ &= \hat{\mathbf{x}}\, 1.2\pi \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \phi^+\right) \quad \text{(mV/m)}.\end{aligned}$$

The field $\mathbf{E}(z, t)$ is maximum when the argument of the cosine function equals zero or a multiple of 2π. At $t = 0$ and $z = 50$ m, this condition yields

$$-\frac{2\pi \times 50}{300} + \phi^+ = 0 \qquad \text{or} \qquad \phi^+ = \frac{\pi}{3}.$$

Cont.

Example 7-1 cont.



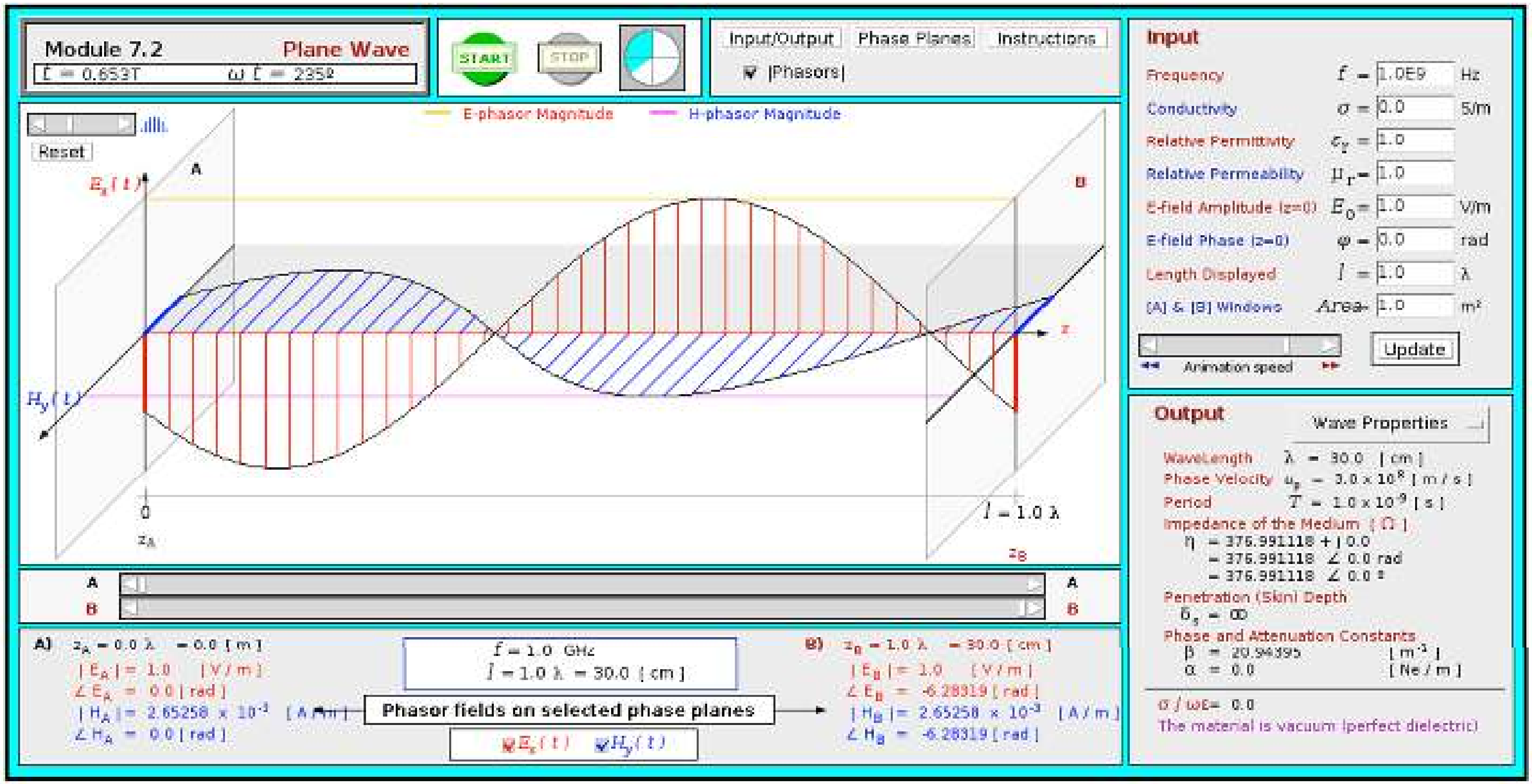
Hence,

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 1.2\pi \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \frac{\pi}{3}\right) \quad \text{(mV/m)},$$

and from Eq. (7.34b) we have

$$\mathbf{H}(z,t) = \hat{\mathbf{y}}\, \frac{E(z,t)}{\eta_0}$$

$$= \hat{\mathbf{y}}\, 10 \cos\left(2\pi \times 10^6 t - \frac{2\pi z}{300} + \frac{\pi}{3}\right) \quad (\mu\text{A/m}),$$



Module 7.2
Plane Wave
START
STOP
Input/Output
Phase Planes
Instructions
|Phasors|
Input
Frequency
Conductivity
Relative Permittivity
Relative Permeability
E-field Amplitude (z=0)
E-field Phase (z=0)
Length Displayed
[A] & [B] Windows
Update
Animation speed
Reset
E-phasor Magnitude
H-phasor Magnitude
Output
Wave Properties
Wavelength
Phase Velocity
Period
Impedance of the Medium
Penetration (Skin) Depth
Phase and Attenuation Constants
The material is vacuum (perfect dielectric)
Phasor fields on selected phase planes

Directional Relation Between **E** and **H**

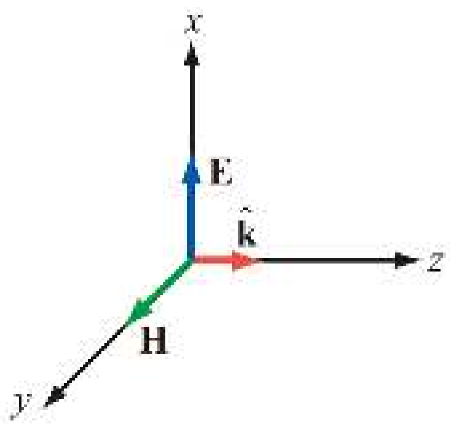


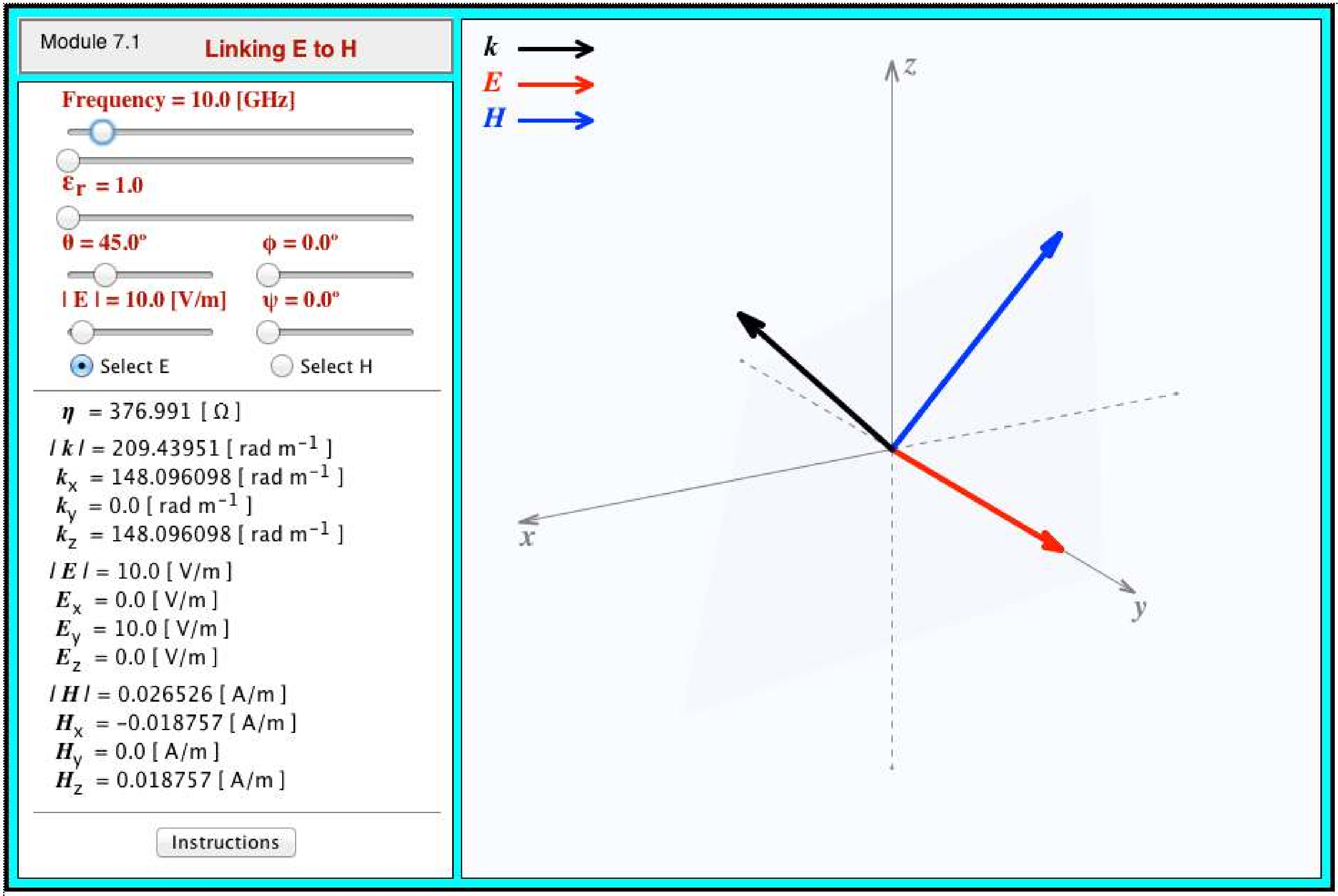
Figure 7-4: A transverse electromagnetic (TEM) wave propagating in the direction $\hat{\mathbf{k}} = \hat{\mathbf{z}}$. For all TEM waves, $\hat{\mathbf{k}}$ is parallel to $\mathbf{E} \times \mathbf{H}$.

For Any TEM Wave

$$\widetilde{\mathbf{H}} = \frac{1}{\eta}\,\hat{\mathbf{k}} \times \widetilde{\mathbf{E}}, \qquad (7.39a)$$

$$\widetilde{\mathbf{E}} = -\eta\,\hat{\mathbf{k}} \times \widetilde{\mathbf{H}}. \qquad (7.39b)$$

The following right-hand rule applies: when we rotate the four fingers of the right hand from the direction of **E** *toward that of* **H***, the thumb points in the direction of wave travel,* $\hat{\mathbf{k}}$.


Module 7.1
Linking E to H
Frequency = 10.0 [GHz]
εr = 1.0
θ = 45.0°
φ = 0.0°
| E | = 10.0 [V/m]
ψ = 0.0°
Select E
Select H
η = 376.991 [Ω]
/ k / = 209.43951 [rad m-1]
kx = 148.096098 [rad m-1]
ky = 0.0 [rad m-1]
kz = 148.096098 [rad m-1]
/ E / = 10.0 [V/m]
Ex = 0.0 [V/m]
Ey = 10.0 [V/m]
Ez = 0.0 [V/m]
/ H / = 0.026526 [A/m]
Hx = -0.018757 [A/m]
Hy = 0.0 [A/m]
Hz = 0.018757 [A/m]
Instructions
k
E
H
z
x
y

Wave decomposition

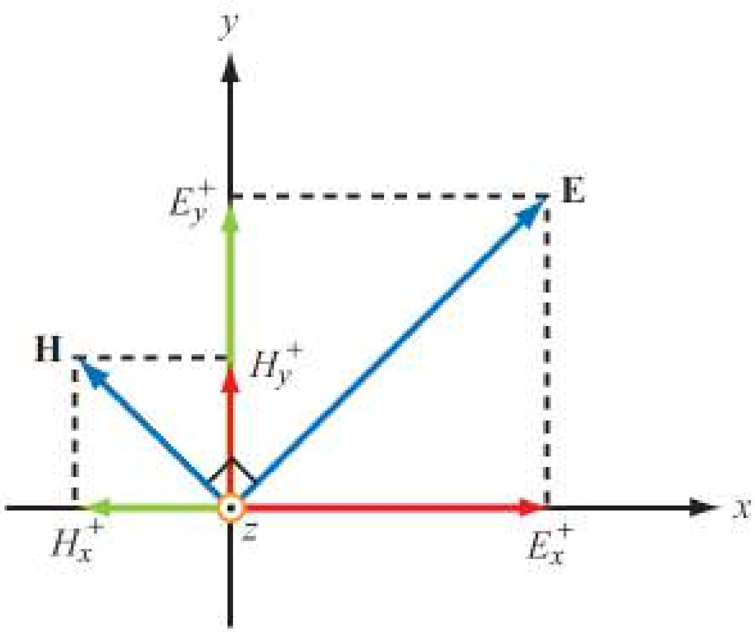


Figure 7-6: The wave (**E**, **H**) is equivalent to the sum of tw waves, one with fields (E_x^+, H_y^+) and another with (E_y^+, H_x^+ with both traveling in the $+z$-direction.

In general, a uniform plane wave traveling in the $+z$-direction may have both x- and y-components, in which case $\widetilde{\mathbf{E}}$ is given by

$$\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\,\widetilde{E}_x^+(z) + \hat{\mathbf{y}}\,\widetilde{E}_y^+(z), \tag{7.43a}$$

and the associated magnetic field is

$$\widetilde{\mathbf{H}} = \hat{\mathbf{x}}\,\widetilde{H}_x^+(z) + \hat{\mathbf{y}}\,\widetilde{H}_y^+(z). \tag{7.43b}$$

Application of Eq. (7.39a) gives

$$\widetilde{\mathbf{H}} = \frac{1}{\eta}\,\hat{\mathbf{z}} \times \widetilde{\mathbf{E}} = -\hat{\mathbf{x}}\,\frac{\widetilde{E}_y^+(z)}{\eta} + \hat{\mathbf{y}}\,\frac{\widetilde{E}_x^+(z)}{\eta}\,. \tag{7.44}$$

By equating Eq. (7.43b) to Eq. (7.44), we have

$$\widetilde{H}_x^+(z) = -\frac{\widetilde{E}_y^+(z)}{\eta}\,, \qquad \widetilde{H}_y^+(z) = \frac{\widetilde{E}_x^+(z)}{\eta}\,. \tag{7.45}$$

Tech Brief 13: RFID Tags

Figure TF13-1: Passive RFID tags were developed in the 1970s for tracking cows.

Overall System View

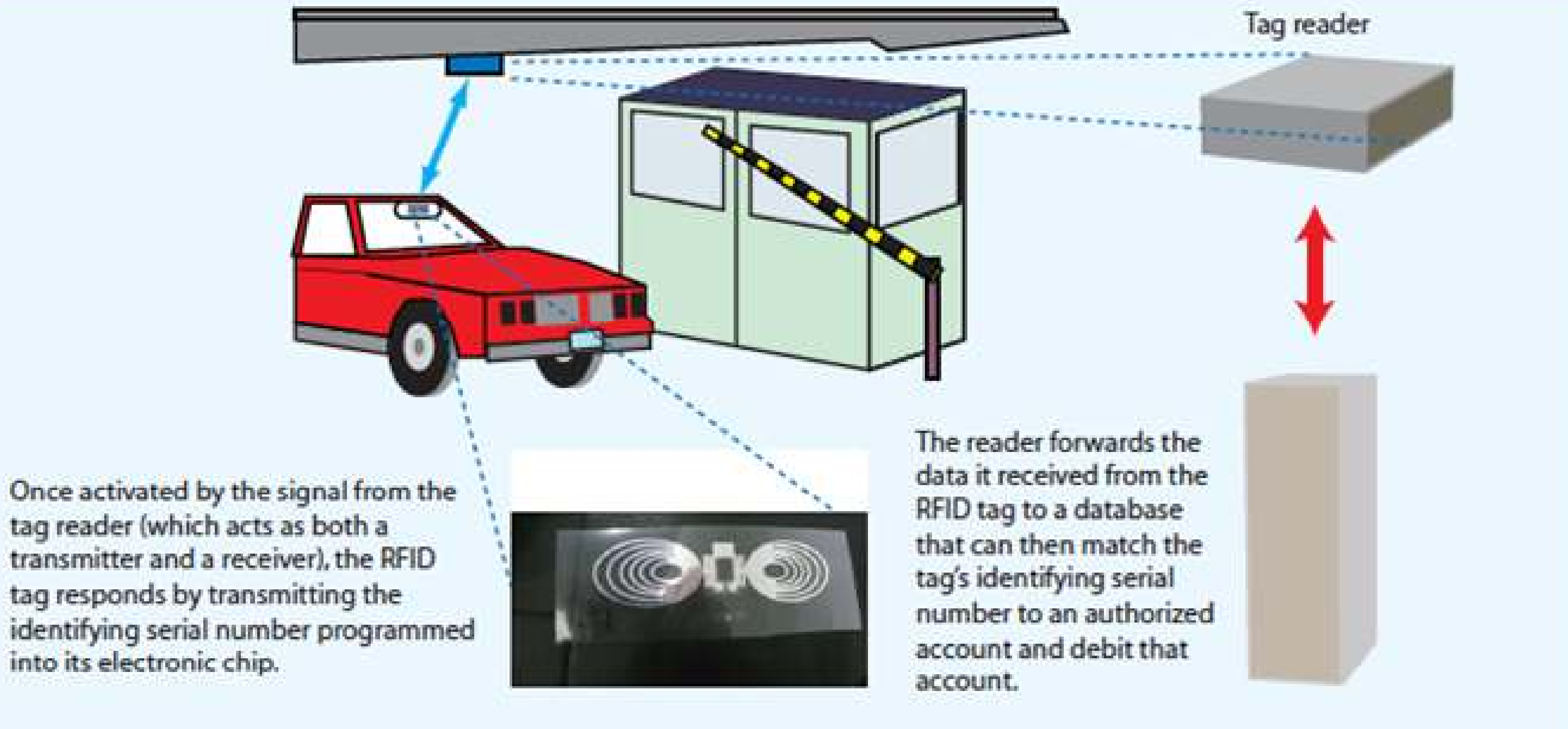


Figure TF13-2 How an RFID system works is illustrated through this EZ-Pass example. The UHF RFID shown is courtesy of Prof. C. F. Huang of Tatung University, Taiwan.

RFID Tag Communication

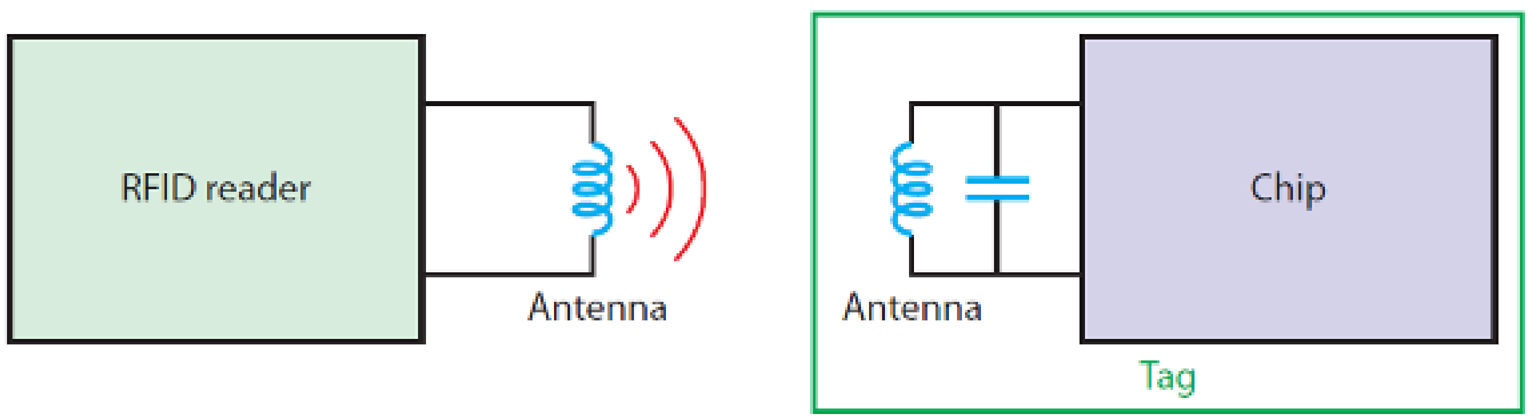


Figure TF13-3: Simplified diagram for how the RFID reader communicates with the tag. At the two lower carrier frequencies commonly used for RFID communication, namely 125 kHz and 13.56 MHz, coil inductors act as magnetic antennas. In systems designed to operate at higher frequencies (900 MHz and 2.54 GHz), dipole antennas are used instead.

RFID Tag Frequencies

Table TT13-1: Comparison of RFID frequency bands.

Band	LF	HF	UHF	Microwave
RFID frequency	125–134 kHz	13.56 MHz	865–956 MHz	2.45 GHz
Read range	≤ 0.5 m	≤ 1.5 m	≤ 5 m	≤ 10 m
Data rate	1 kbit/s	25 kbit/s	30 kbit/s	100 kbit/s
Typical applications	• Animal ID • Automobile key/antitheft • Access control	• Smart cards • Article surveillance • Airline baggage tracking • Library book tracking	• Supply chain management • Logistics	• Vehicle toll collection • Railroad car monitoring

Wave Polarization

*The **polarization** of a uniform plane wave describes the locus traced by the tip of the* **E** *vector (in the plane orthogonal to the direction of propagation) at a given point in space as a function of time.*

Plane wave propagating along +z :

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x(z) + \hat{\mathbf{y}}\widetilde{E}_y(z),$$

with

$$\widetilde{E}_x(z) = E_{x0}e^{-jkz},$$

$$\widetilde{E}_y(z) = E_{y0}e^{-jkz},$$

If: $E_{x0} = a_x,$

$$E_{y0} = a_y e^{j\delta},$$

then

$$\widetilde{\mathbf{E}}(z) = (\hat{\mathbf{x}}a_x + \hat{\mathbf{y}}a_y e^{j\delta})e^{-jkz},$$

and the corresponding instantaneous field is

$$\begin{aligned}\mathbf{E}(z,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\,e^{j\omega t}\right] \\ &= \hat{\mathbf{x}}a_x\cos(\omega t - kz) \\ &\quad + \hat{\mathbf{y}}a_y\cos(\omega t - kz + \delta).\end{aligned}$$

Polarization State

Polarization state describes the trace of **E** as a function of time at a fixed *z*

Magnitude of **E**

$$|\mathbf{E}(z,t)| = [E_x^2(z,t) + E_y^2(z,t)]^{1/2}$$
$$= [a_x^2 \cos^2(\omega t - kz)$$
$$+ a_y^2 \cos^2(\omega t - kz + \delta)]^{1/2}$$

Inclination Angle

$$\psi(z,t) = \tan^{-1}\left(\frac{E_y(z,t)}{E_x(z,t)}\right)$$

Linear Polarization:

$\delta = 0$ or $\delta = \pi$

A wave is said to be linearly polarized if for a fixed z, the tip of $\mathbf{E}(z,t)$ *traces a straight line segment as a function of time. This happens when* $E_x(z,t)$ *and* $E_y(z,t)$ *are* ***in-phase*** *(i.e.,* $\delta = 0$*) or* ***out-of-phase*** *(*$\delta = \pi$*).*

Under these conditions Eq. (7.50) simplifies to

$$\mathbf{E}(0,t) = (\hat{\mathbf{x}}a_x + \hat{\mathbf{y}}a_y)\cos(\omega t - kz) \quad \text{(in-phase)}, \qquad (7.53a)$$

$$\mathbf{E}(0,t) = (\hat{\mathbf{x}}a_x - \hat{\mathbf{y}}a_y)\cos(\omega t - kz) \quad \text{(out-of-phase)}. \qquad (7.53b)$$

Let us examine the out-of-phase case. The field's magnitude is

$$|\mathbf{E}(z,t)| = [a_x^2 + a_y^2]^{1/2}|\cos(\omega t - kz)|, \qquad (7.54a)$$

and the inclination angle is

$$\psi = \tan^{-1}\left(\frac{-a_y}{a_x}\right) \quad \text{(out-of-phase)}. \qquad (7.54b)$$

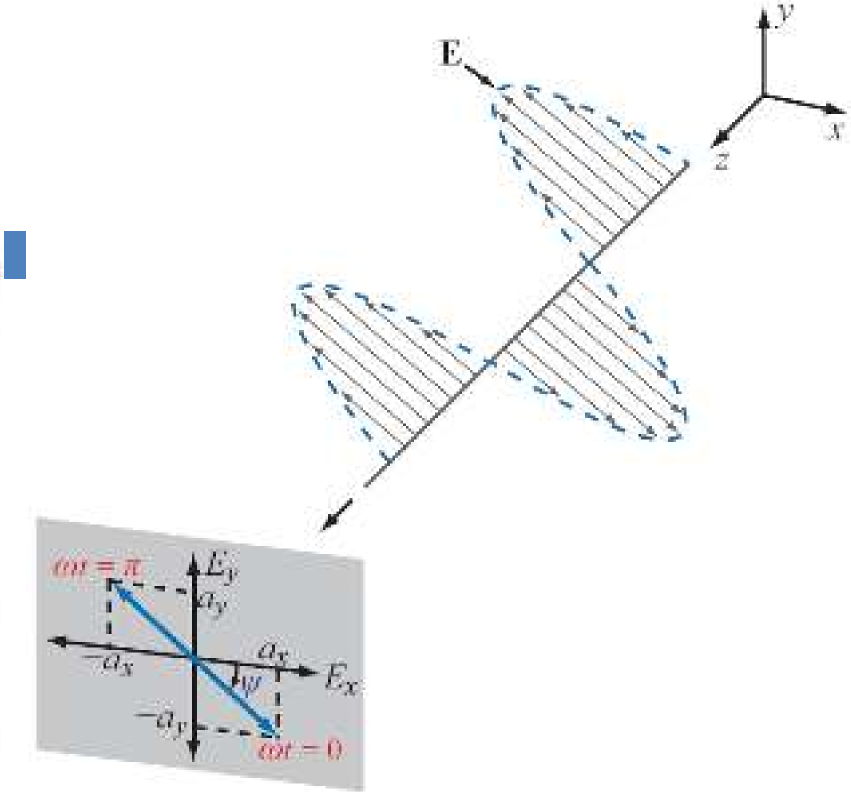


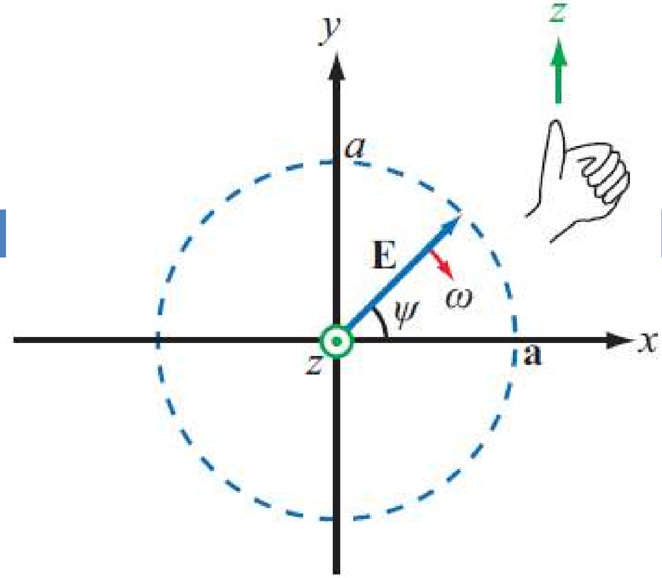
Figure 7-7: Linearly polarized wave traveling in the $+z$-direction (out of the page).

E traces a line(in blue) as the wave traverses a fixed plane

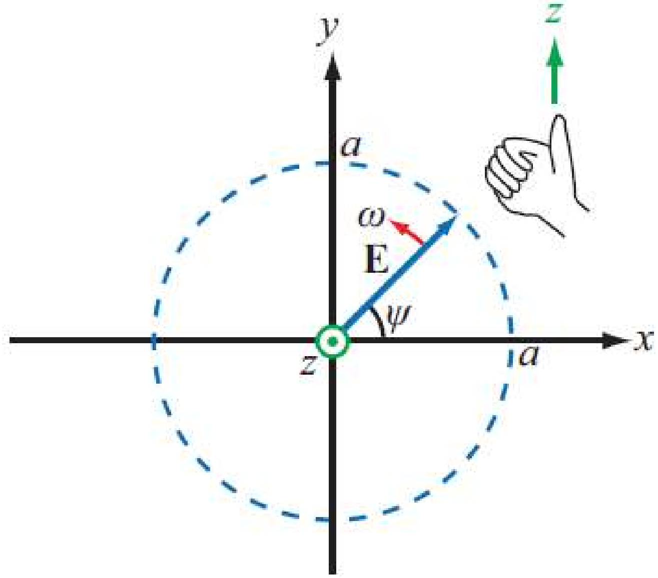
If $a_y = 0$, then $\psi = 0°$ or $180°$, and the wave is x-polarized; conversely, if $a_x = 0$, then $\psi = 90°$ or $-90°$, and the wave is y-polarized.

Polarization Handedness

Polarization handedness is defined in terms of the rotation of **E** *as a function of time in a fixed plane orthogonal to the direction of propagation, which is opposite of the direction of rotation of* **E** *as a function of distance at a fixed point in time.*



(a) LHC polarization



(b) RHC polarization

LH Circular Polarization:

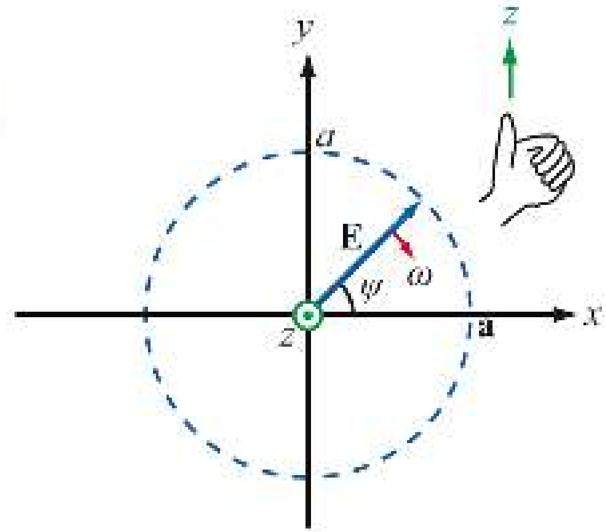
$a_x = a_y = a$ and $\delta = \pi/2$

(a) Left-Hand Circular (LHC) Polarization

For $a_x = a_y = a$ and $\delta = \pi/2$, Eqs. (7.49) and (7.50) become

$$\begin{aligned}\widetilde{\mathbf{E}}(z) &= (\hat{\mathbf{x}}a + \hat{\mathbf{y}}ae^{j\pi/2})e^{-jkz} \\ &= a(\hat{\mathbf{x}} + j\hat{\mathbf{y}})e^{-jkz},\end{aligned}$$

$$\begin{aligned}\mathbf{E}(z,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right] \\ &= \hat{\mathbf{x}}a\cos(\omega t - kz) + \hat{\mathbf{y}}a\cos(\omega t - kz + \pi/2) \\ &= \hat{\mathbf{x}}a\cos(\omega t - kz) - \hat{\mathbf{y}}a\sin(\omega t - kz).\end{aligned}$$



(a) LHC polarization

The corresponding field magnitude and inclination angle are

$$\begin{aligned}|\mathbf{E}(z,t)| &= \left[E_x^2(z,t) + E_y^2(z,t)\right]^{1/2} \\ &= [a^2\cos^2(\omega t - kz) + a^2\sin^2(\omega t - kz)]^{1/2} \\ &= \boxed{a},\end{aligned}$$

$$\begin{aligned}\psi(z,t) &= \tan^{-1}\left[\frac{E_y(z,t)}{E_x(z,t)}\right] \\ &= \tan^{-1}\left[\frac{-a\sin(\omega t - kz)}{a\cos(\omega t - kz)}\right] \\ &= \boxed{-(\omega t - kz)}.\end{aligned}$$

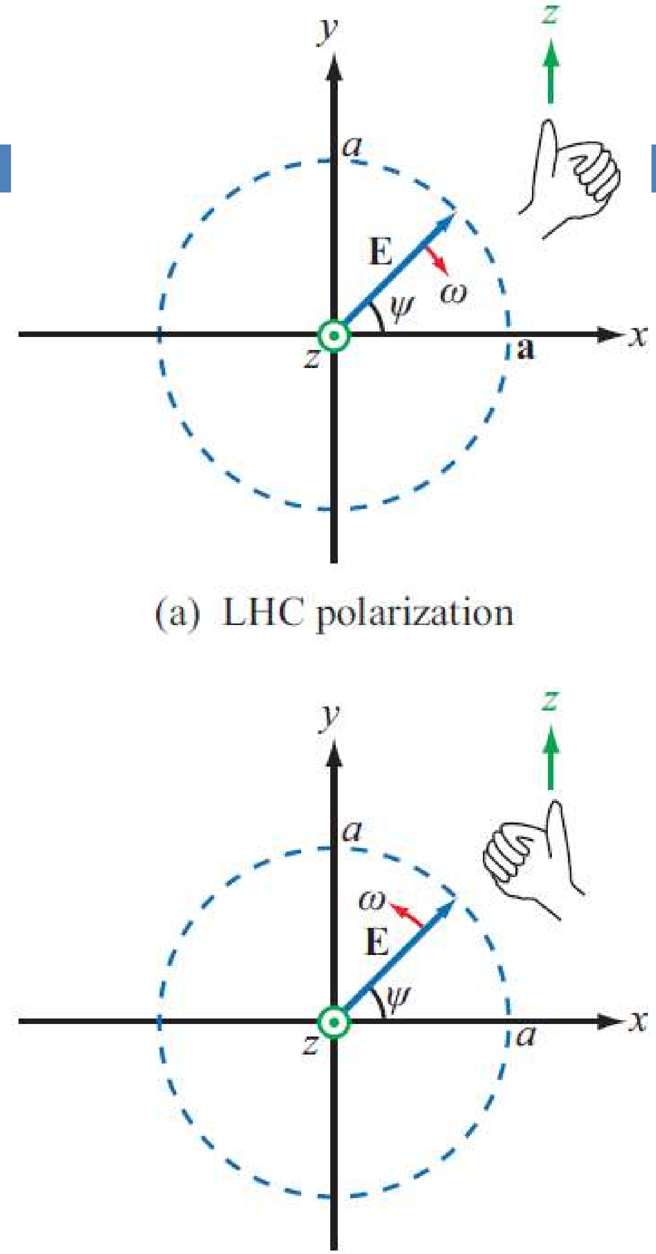
RH Circular Polarization:

$a_x = a_y = a$ and $\delta = -\pi/2$

(b) Right-Hand Circular (RHC) Polarization

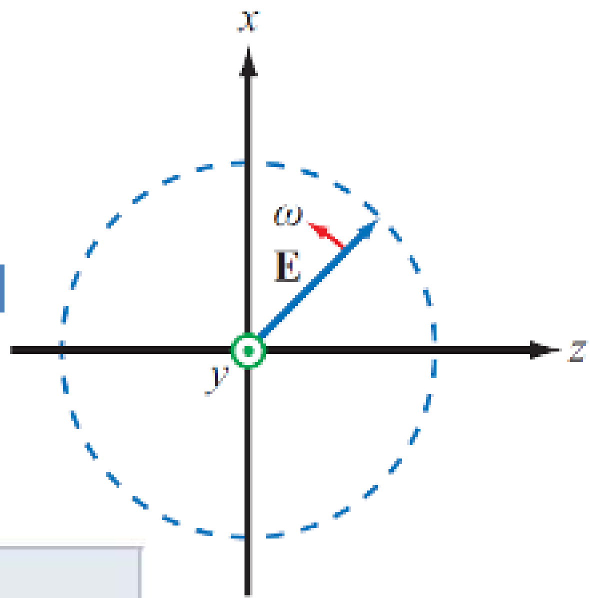
For $a_x = a_y = a$ and $\delta = -\pi/2$, we have

$$\boxed{|\mathbf{E}(z,t)| = a, \qquad \psi = (\omega t - kz).}$$



(a) LHC polarization

(b) RHC polarization



Example 7-2: RHC Polarized Wave

An RHC polarized plane wave with electric field magnitude of 3 (mV/m) is traveling in the $+y$-direction in a dielectric medium with $\varepsilon = 4\varepsilon_0$, $\mu = \mu_0$, and $\sigma = 0$. If the frequency is 100 MHz, obtain expressions for $\mathbf{E}(y, t)$ and $\mathbf{H}(y, t)$.

Solution: Since the wave is traveling in the $+y$-direction, its field must have components along the x- and z-directions. The rotation of $\mathbf{E}(y, t)$ is depicted in Fig. 7-10, where $\hat{\mathbf{y}}$ is out of the page. By comparison with the RHC polarized wave shown in Fig. 7-8(b), we assign the z-component of $\widetilde{\mathbf{E}}(y)$ a phase angle of zero and the x-component a phase shift of $\delta = -\pi/2$.

Cont.

Example 7-2 cont.

Wave with electric field magnitude of 3 (mV/m) traveling in the +y-direction

With $\omega = 2\pi f = 2\pi \times 10^8$ (rad/s), the wavenumber k is

$$k = \frac{\omega\sqrt{\varepsilon_r}}{c} = \frac{2\pi \times 10^8 \sqrt{4}}{3 \times 10^8} = \frac{4}{3}\pi \qquad \text{(rad/m)},$$

$$\begin{aligned}\widetilde{\mathbf{E}}(y) &= \hat{\mathbf{x}}\widetilde{E}_x + \hat{\mathbf{z}}\widetilde{E}_z \\ &= \hat{\mathbf{x}}ae^{-j\pi/2}e^{-jky} + \hat{\mathbf{z}}ae^{-jky} \\ &= (-\hat{\mathbf{x}}j + \hat{\mathbf{z}})3e^{-jky} \qquad \text{(mV/m)},\end{aligned}$$

and application of (7.39a) gives

$$\begin{aligned}\widetilde{\mathbf{H}}(y) &= \frac{1}{\eta}\hat{\mathbf{y}} \times \widetilde{\mathbf{E}}(y) \\ &= \frac{1}{\eta}\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}j + \hat{\mathbf{z}})3e^{-jky} \\ &= \frac{3}{\eta}(\hat{\mathbf{z}}j + \hat{\mathbf{x}})e^{-jky} \qquad \text{(mA/m)}.\end{aligned}$$

Cont.

Example 7-2 cont.

$$\eta = \frac{\eta_0}{\sqrt{\varepsilon_r}} \simeq \frac{120\pi}{\sqrt{4}} = 60\pi \qquad (\Omega).$$

The instantaneous fields $\mathbf{E}(y, t)$ and $\mathbf{H}(y, t)$ are

$$\begin{aligned}\mathbf{E}(y,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(y)\, e^{j\omega t}\right] \\ &= \mathfrak{Re}\left[(-\hat{\mathbf{x}}j + \hat{\mathbf{z}})3e^{-jky}e^{j\omega t}\right] \\ &= 3[\hat{\mathbf{x}}\sin(\omega t - ky) + \hat{\mathbf{z}}\cos(\omega t - ky)] \qquad \text{(mV/m)}\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}(y,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{H}}(y)\, e^{j\omega t}\right] \\ &= \mathfrak{Re}\left[\frac{3}{\eta}(\hat{\mathbf{z}}j + \hat{\mathbf{x}})e^{-jky}e^{j\omega t}\right] \\ &= \frac{1}{20\pi}[\hat{\mathbf{x}}\cos(\omega t - ky) - \hat{\mathbf{z}}\sin(\omega t - ky)] \text{ (mA/m)}.\end{aligned}$$

Elliptical Polarization: General Case

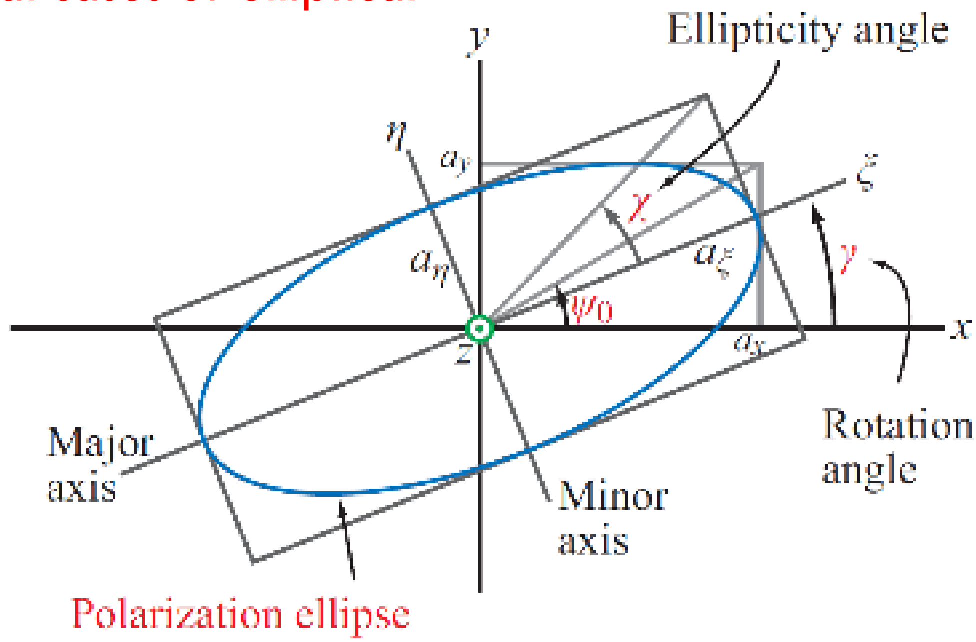
Linear and circular polarizations are special cases of elliptical

$$\tan 2\gamma = (\tan 2\psi_0)\cos\delta \quad (-\pi/2 \le \gamma \le \pi/2),$$
$$\sin 2\chi = (\sin 2\psi_0)\sin\delta \quad (-\pi/4 \le \chi \le \pi/4),$$

where ψ_0 is an ***auxiliary angle*** defined by

$$\tan\psi_0 = \frac{a_y}{a_x} \qquad \left(0 \le \psi_0 \le \frac{\pi}{2}\right).$$

$$\gamma > 0 \text{ if } \cos\delta > 0,$$
$$\gamma < 0 \text{ if } \cos\delta < 0.$$



Positive values of χ, corresponding to $\sin\delta > 0$, are associated with left-handed rotation, and negative values of χ, corresponding to $\sin\delta < 0$, are associated with right-handed rotation.

Example 7-3: Polarization State

Determine the polarization state of a plane wave with electric field

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\,3\cos(\omega t - kz + 30^\circ) - \hat{\mathbf{y}}\,4\sin(\omega t - kz + 45^\circ) \qquad \text{(mV/m)}.$$

Solution: We begin by converting the second term to a cosine reference,

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{x}}\,3\cos(\omega t - kz + 30^\circ) - \hat{\mathbf{y}}\,4\cos(\omega t - kz + 45^\circ - 90^\circ)\\ &= \hat{\mathbf{x}}\,3\cos(\omega t - kz + 30^\circ) - \hat{\mathbf{y}}\,4\cos(\omega t - kz - 45^\circ).\end{aligned}$$

The corresponding field phasor $\widetilde{\mathbf{E}}(z)$ is

$$\begin{aligned}\widetilde{\mathbf{E}}(z) &= \hat{\mathbf{x}}\,3e^{-jkz}e^{j30^\circ} - \hat{\mathbf{y}}\,4e^{-jkz}e^{-j45^\circ}\\ &= \hat{\mathbf{x}}\,3e^{-jkz}e^{j30^\circ} + \hat{\mathbf{y}}\,4e^{-jkz}e^{-j45^\circ}e^{j180^\circ}\\ &= \hat{\mathbf{x}}\,3e^{-jkz}e^{j30^\circ} + \hat{\mathbf{y}}\,4e^{-jkz}e^{j135^\circ},\end{aligned}$$

Cont.

Example 7-3 cont.

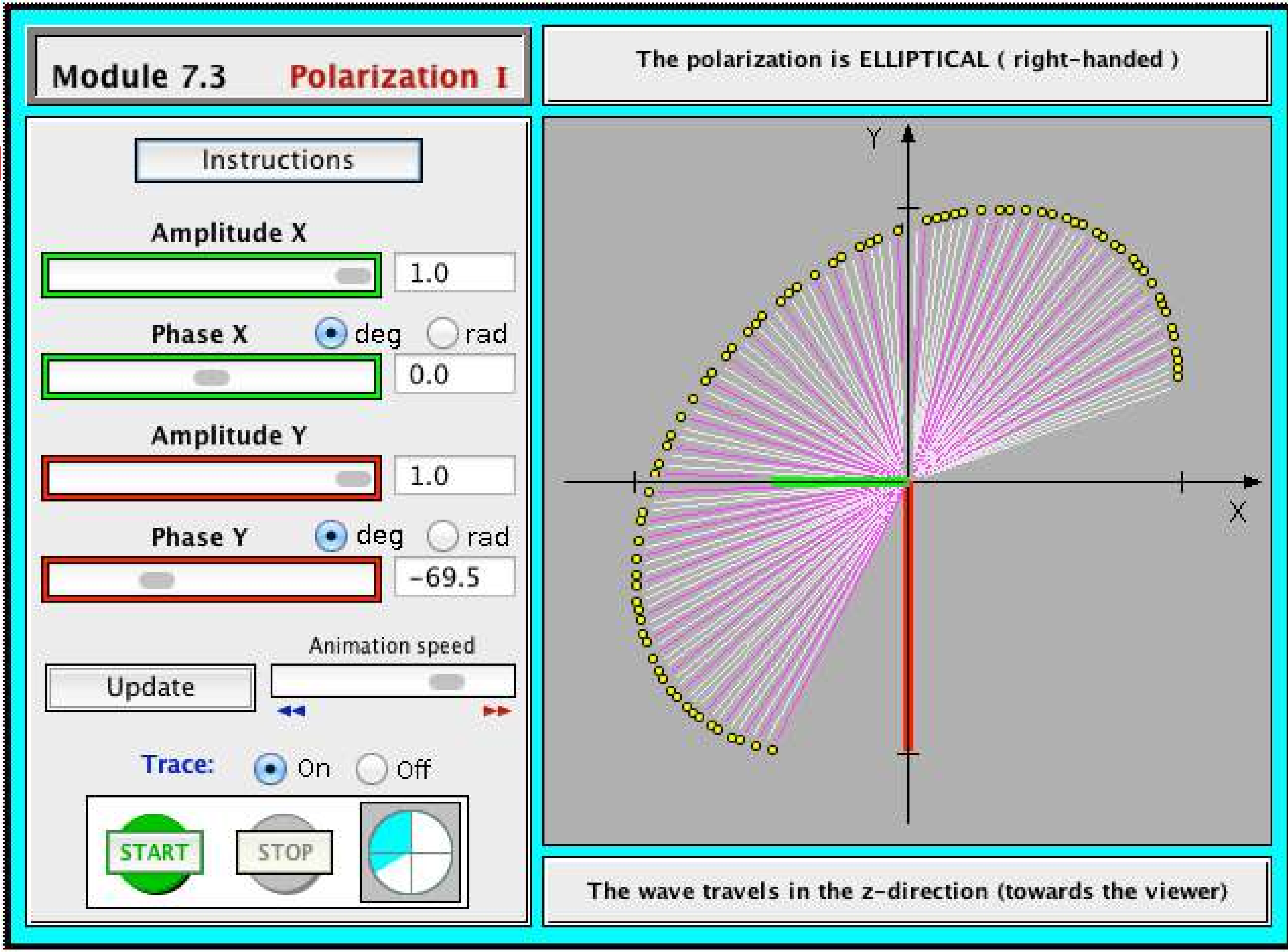
$$\psi_0 = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ.$$

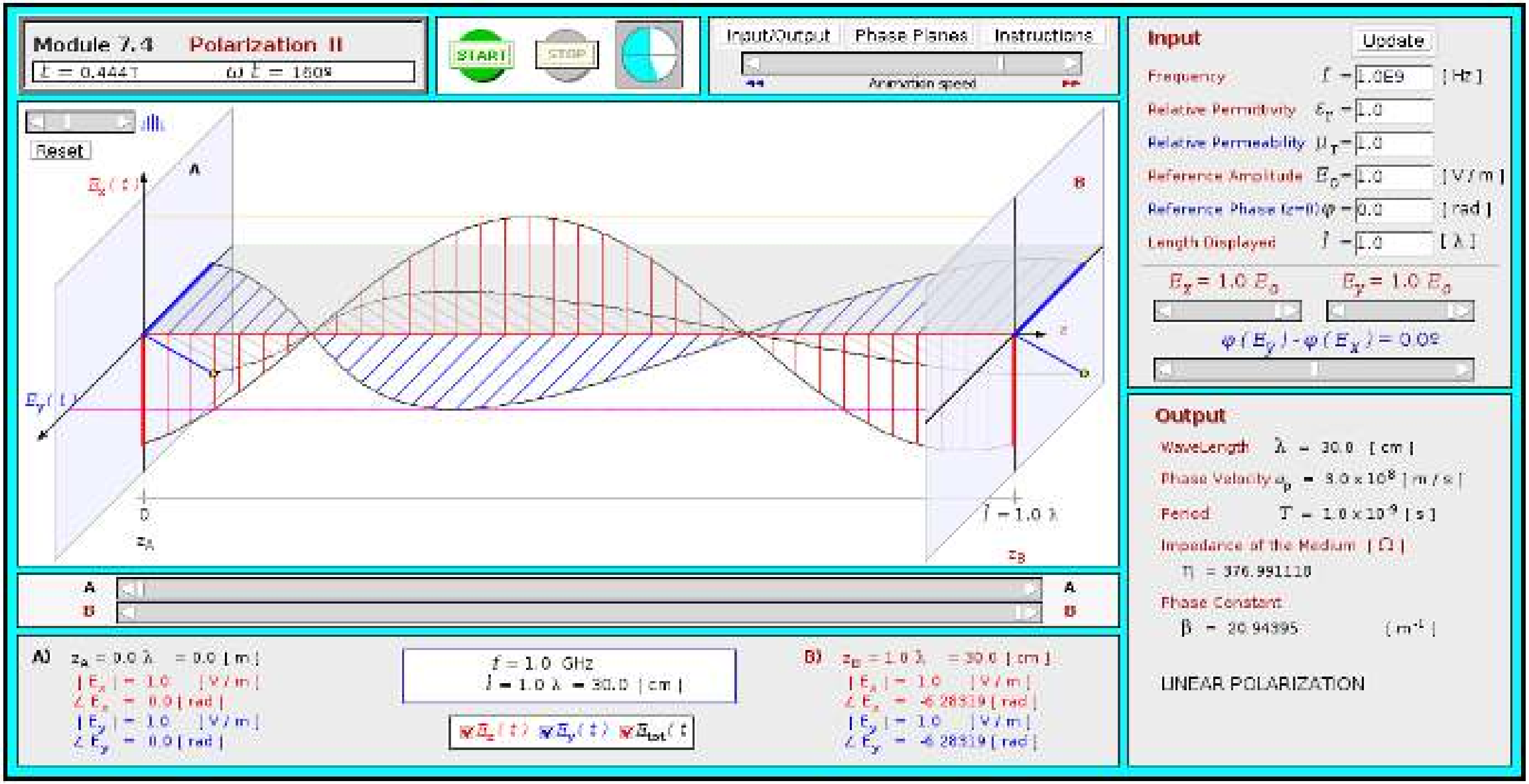
$$\tan 2\gamma = (\tan 2\psi_0)\cos\delta = \tan 106.2^\circ \cos 105^\circ = 0.89,$$

which gives two solutions for γ, namely $\gamma = 20.8^\circ$ and $\gamma = -69.2^\circ$. Since $\cos\delta < 0$, the correct value of γ is -69.2°. From Eq. (7.59b),

$$\sin 2\chi = (\sin 2\psi_0)\sin\delta = \sin 106.2^\circ \sin 105^\circ = 0.93 \quad \text{or} \quad \chi = 34.0^\circ.$$

The magnitude of χ indicates that the wave is elliptically polarized and its positive polarity specifies its rotation as left handed.


Module 7.3 Polarization I
The polarization is ELLIPTICAL (right-handed)
Instructions
Amplitude X
1.0
Phase X
deg
rad
0.0
Amplitude Y
1.0
Phase Y
deg
rad
-69.5
Animation speed
Update
Trace:
On
Off
START
STOP
Y
X
The wave travels in the z-direction (towards the viewer)



Module 7.4 Polarization II
START
STOP
Input/Output
Phase Planes
Instructions
Animation speed
Reset
Input
Update
Frequency
Relative Permittivity
Relative Permeability
Reference Amplitude
Reference Phase (z=0)
Length Displayed
Output
WaveLength λ = 30.0 [cm]
Period
Impedance of the Medium [Ω]
η = 376.991110
Phase Constant
β = 20.94395
f = 1.0 GHz
LINEAR POLARIZATION

Tech Brief 7: LCD

Liquid crystals are neither a pure solid nor a pure liquid, but rather a hybrid of both. One particular variety of interest is the twisted nematic liquid crystal whose rod-shaped molecules have a natural tendency to assume a twisted spiral structure when the material is sandwiched between finely grooved glass substrates with orthogonal orientations.

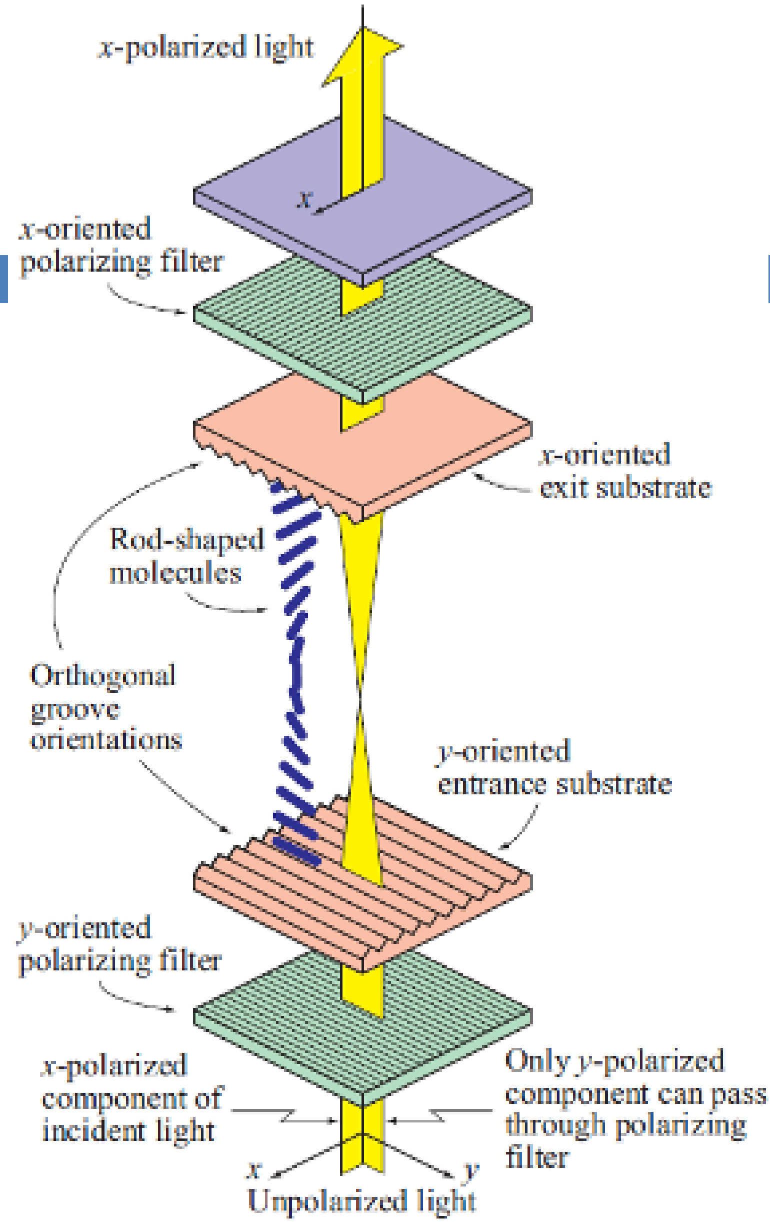


Figure TF14-1: The rod-shaped molecules of a liquid crystal sandwiched between grooved substrates with orthogonal orientations causes the electric field of the light passing through it to rotate by 90°.

Operation of a Single Pixel

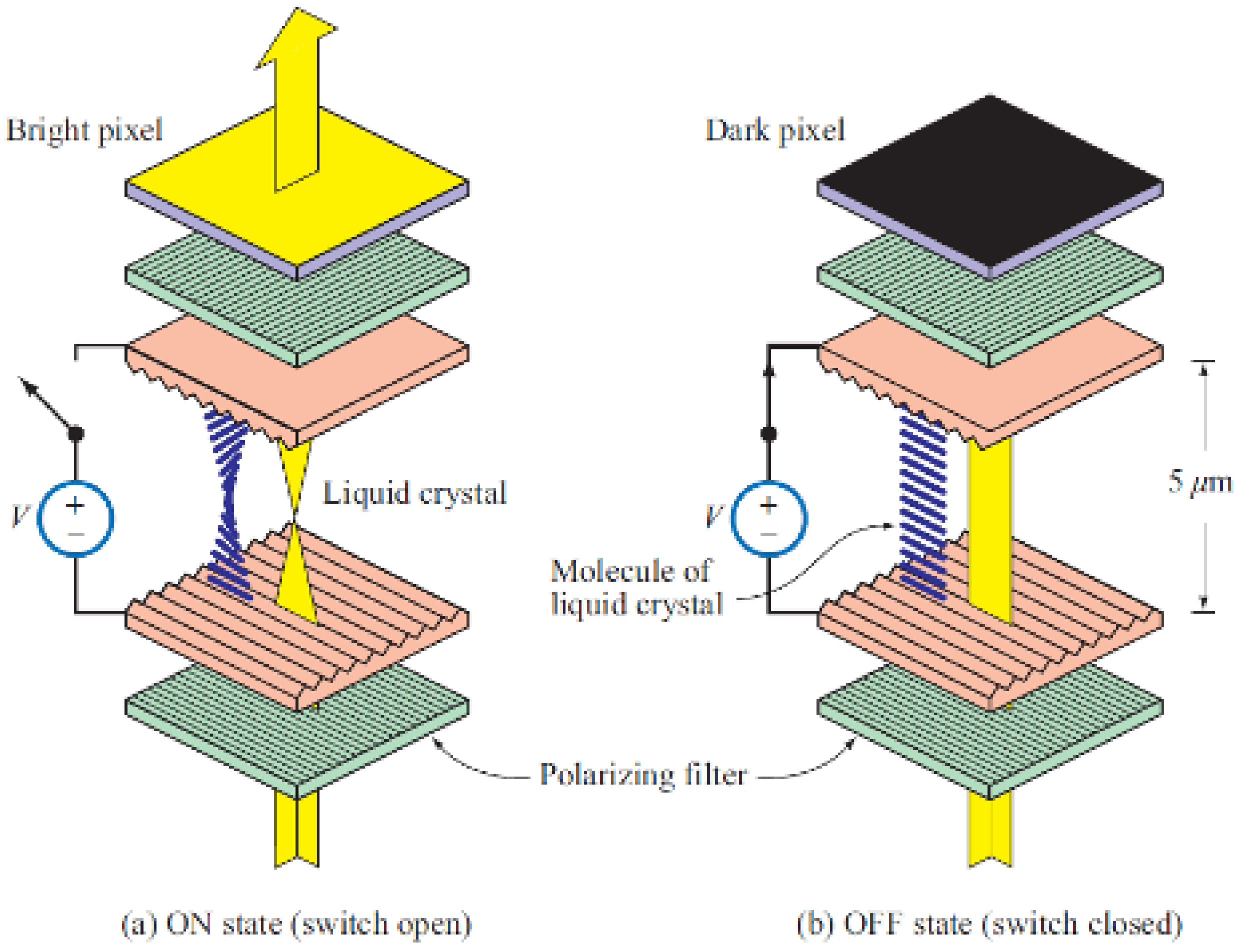


Figure TF14-2: Single-pixel LCD.

LCD 2-D Array

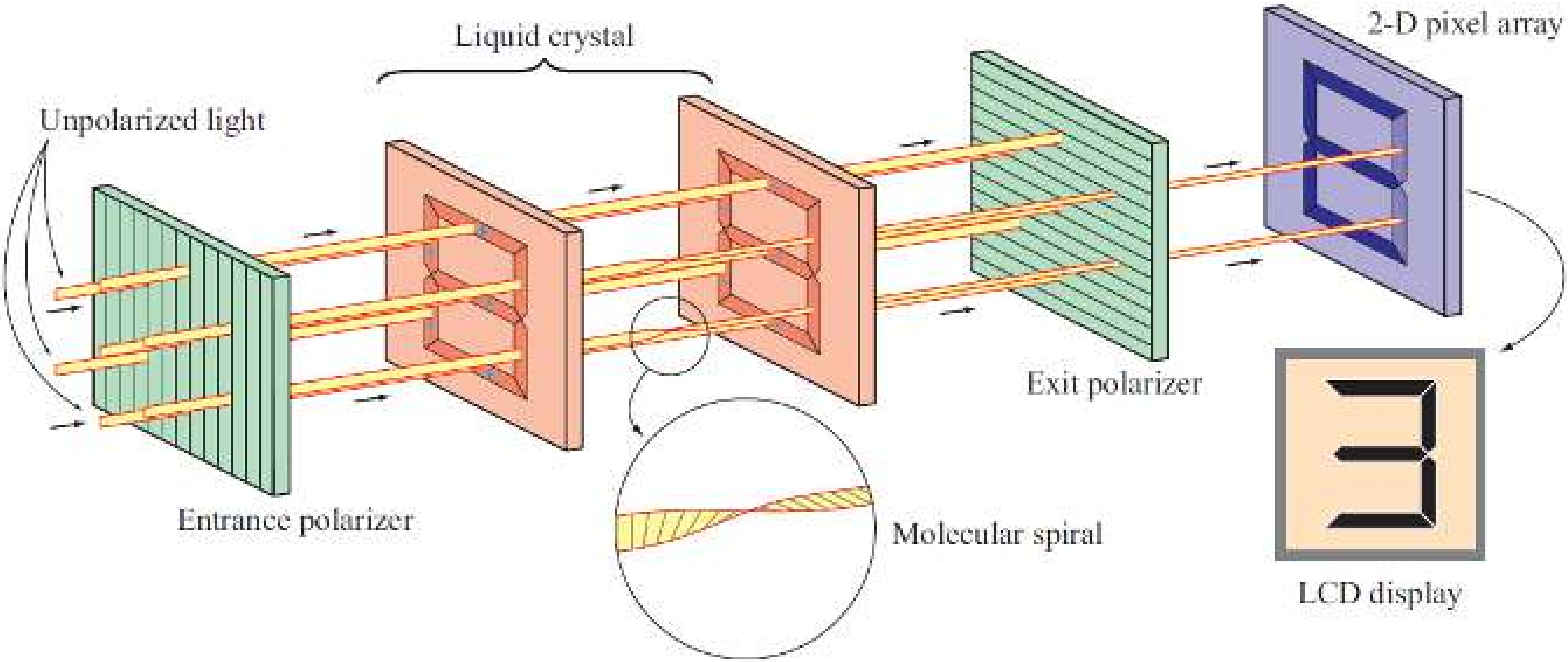


Figure TF14-3: 2-D LCD array.

Lossy Media

For a uniform plane wave with electric field $\widetilde{\mathbf{E}} = \hat{\mathbf{x}}\, \widetilde{E}_x(z)$ traveling along the z-direction, the wave equation given by Eq. (7.61) reduces to

$$\frac{d^2\, \widetilde{E}_x(z)}{dz^2} - \gamma^2\, \acute{E}_x(z) = 0. \qquad (7.67)$$

with

$$\gamma^2 = -\omega^2 \mu \varepsilon_c = -\omega^2 \mu(\varepsilon' - j\varepsilon''), \qquad (7.62)$$

where $\varepsilon' = \varepsilon$ and $\varepsilon'' = \sigma/\omega$. Since γ is complex, we express it as

$$\gamma = \alpha + j\beta, \qquad (7.63)$$

where α is the medium's ***attenuation constant*** and β its ***phase constant***. By replacing γ with $(\alpha + j\beta)$ in Eq. (7.62), we obtain

$$\begin{aligned}(\alpha + j\beta)^2 &= (\alpha^2 - \beta^2) + j2\alpha\beta \\ &= -\omega^2\mu\varepsilon' + j\omega^2\mu\varepsilon''. \qquad (7.64)\end{aligned}$$

Cont.

Lossy Media

The rules of complex algebra require the real and imaginary parts on one side of an equation to equal the real and imaginary parts on the other side. Hence,

$$\alpha^2 - \beta^2 = -\omega^2 \mu \varepsilon', \quad (7.65a)$$

$$2\alpha\beta = \omega^2 \mu \varepsilon''. \quad (7.65b)$$

Solving these two equations for α and β gives

$$\alpha = \omega \left\{ \frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1 \right] \right\}^{1/2} \quad \text{(Np/m)}, \quad (7.66a)$$

$$\beta = \omega \left\{ \frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1 \right] \right\}^{1/2} \quad \text{(rad/m)}. \quad (7.66b)$$

Attenuation

E and **H** fields:

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\widetilde{E}_x(z) = \hat{\mathbf{x}}E_{x0}e^{-\gamma z} = \hat{\mathbf{x}}E_{x0}e^{-\alpha z}e^{-j\beta z}. \quad (7.68)$$

The associated magnetic field $\widetilde{\mathbf{H}}$ can be determined by applying Eq. (7.2b): $\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$, or using Eq. (7.39a): $\widetilde{\mathbf{H}} = (\hat{\mathbf{k}} \times \widetilde{\mathbf{E}})/\eta_c$, where η_c is the ***intrinsic impedance of the lossy medium***. Both approaches give

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}}\,\widetilde{H}_y(z) = \hat{\mathbf{y}}\,\frac{\widetilde{E}_x(z)}{\eta_c} = \hat{\mathbf{y}}\,\frac{E_{x0}}{\eta_c}e^{-\alpha z}e^{-j\beta z}, \quad (7.69)$$

where

$$\boxed{\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \quad (\Omega). \quad (7.70)}$$

Cont.

Attenuation

Magnitude of *E*

$$|\widetilde{E}_x(z)| = |E_{x0}e^{-\alpha z}e^{-j\beta z}| = |E_{x0}|e^{-\alpha z}$$

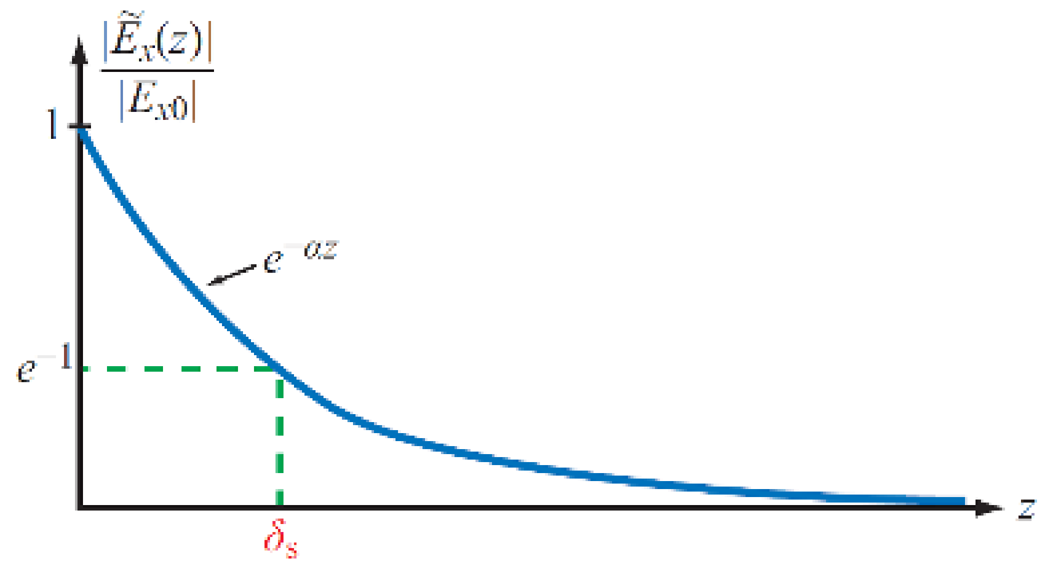


Figure 7-13: Attenuation of the magnitude of $\widetilde{E}_x(z)$ with distance z. The skin depth δ_s is the value of z at which $|\widetilde{E}_x(z)|/|E_{x0}| = e^{-1}$, or $z = \delta_s = 1/\alpha$.

Skin depth

$$\delta_s = \frac{1}{\alpha} \quad \text{(m)}, \qquad (7.72)$$

the wave magnitude decreases by a factor of $e^{-1} \approx 0.37$ (Fig. 7-13). At depth $z = 3\delta_s$, the field magnitude is less than 5% of its initial value, and at $z = 5\delta_s$, it is less than 1%.

This distance δ_s, called the ***skin depth*** *of the medium, characterizes how deep an electromagnetic wave can penetrate into a conducting medium.*

Low and High Frequency Approximations

Table 7-1: Expressions for α, β, η_c, u_p, and λ for various types of media.

	Any Medium	Lossless Medium ($\sigma = 0$)	Low-loss Medium ($\varepsilon''/\varepsilon' \ll 1$)	Good Conductor ($\varepsilon''/\varepsilon' \gg 1$)	Units
$\alpha =$	$\omega \left[\frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1 \right] \right]^{1/2}$	0	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\pi f \mu \sigma}$	(Np/m)
$\beta =$	$\omega \left[\frac{\mu\varepsilon'}{2} \left[\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1 \right] \right]^{1/2}$	$\omega\sqrt{\mu\varepsilon}$	$\omega\sqrt{\mu\varepsilon}$	$\sqrt{\pi f \mu \sigma}$	(rad/m)
$\eta_c =$	$\sqrt{\frac{\mu}{\varepsilon'}} \left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$\sqrt{\frac{\mu}{\varepsilon}}$	$(1 + j)\frac{\alpha}{\sigma}$	(Ω)
$u_p =$	ω/β	$1/\sqrt{\mu\varepsilon}$	$1/\sqrt{\mu\varepsilon}$	$\sqrt{4\pi f/\mu\sigma}$	(m/s)
$\lambda =$	$2\pi/\beta = u_p/f$	u_p/f	u_p/f	u_p/f	(m)

Notes: $\varepsilon' = \varepsilon$; $\varepsilon'' = \sigma/\omega$; in free space, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$; in practice, a material is considered a low-loss medium if $\varepsilon''/\varepsilon' = \sigma/\omega\varepsilon < 0.01$ and a good conducting medium if $\varepsilon''/\varepsilon' > 100$.

Example 7-4: Plane Wave in Seawater

A uniform plane wave is traveling in seawater. Assume that the x–y plane resides just below the sea surface and the wave travels in the $+z$-direction into the water. The constitutive parameters of seawater are $\varepsilon_r = 80$, $\mu_r = 1$, and $\sigma = 4$ S/m. If the magnetic field at $z = 0$ is $\mathbf{H}(0, t) = \hat{\mathbf{y}}\, 100 \cos(2\pi \times 10^3 t + 15°)$ (mA/m),

(a) obtain expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$, and

(b) determine the depth at which the magnitude of $\mathbf{E}$ is 1% of its value at $z = 0$.

Solution: **(a)** Since $\mathbf{H}$ is along $\hat{\mathbf{y}}$ and the propagation direction is $\hat{\mathbf{z}}$, $\mathbf{E}$ must be along $\hat{\mathbf{x}}$. Hence, the general expressions for the phasor fields are

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_{x0} e^{-\alpha z} e^{-j\beta z}, \quad (7.78a)$$

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}} \frac{E_{x0}}{\eta_c} e^{-\alpha z} e^{-j\beta z}. \quad (7.78b)$$

Cont.

Example 7-4: Plane Wave in Seawater

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{\omega\varepsilon_r\varepsilon_0} = \frac{4}{2\pi \times 10^3 \times 80 \times (10^{-9}/36\pi)}$$
$$= 9 \times 10^5.$$

This qualifies seawater as a good conductor at 1 kHz and allows us to use the good-conductor expressions given in Table 7-1:

$$\alpha = \sqrt{\pi f \mu \sigma}$$
$$= \sqrt{\pi \times 10^3 \times 4\pi \times 10^{-7} \times 4}$$
$$= 0.126 \qquad \text{(Np/m)}, \tag{7.79a}$$
$$\beta = \alpha = 0.126 \qquad \text{(rad/m)}, \tag{7.79b}$$
$$\eta_c = (1 + j)\frac{\alpha}{\sigma}$$
$$= (\sqrt{2}\, e^{j\pi/4})\frac{0.126}{4} = 0.044 e^{j\pi/4} \qquad (\Omega). \tag{7.79c}$$

Cont.

Example 7-4: Plane Wave in Seawater

$$\mathbf{E}(z,t) = \Re\mathfrak{e}\left[\hat{\mathbf{x}}|E_{x0}|e^{j\phi_0}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right]$$
$$= \hat{\mathbf{x}}|E_{x0}|e^{-0.126z}\cos(2\pi \times 10^3 t - 0.126z + \phi_0)$$
$$\text{(V/m)}, \qquad (7.80a)$$

$$\mathbf{H}(z,t) = \Re\mathfrak{e}\left[\hat{\mathbf{y}}\,\frac{|E_{x0}|e^{j\phi_0}}{0.044e^{j\pi/4}}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right]$$
$$= \hat{\mathbf{y}}22.5|E_{x0}|e^{-0.126z}\cos(2\pi \times 10^3 t$$
$$- 0.126z + \phi_0 - 45^\circ) \qquad \text{(A/m)}. \qquad (7.80b)$$

Cont.

Example 7-4: Plane Wave in Seawater

At $z = 0$,

$$\mathbf{H}(0, t) = \hat{\mathbf{y}}\, 22.5 |E_{x0}| \cos(2\pi \times 10^3 t + \phi_0 - 45^\circ) \quad \text{(A/m)}. \tag{7.81}$$

By comparing Eq. (7.81) with the expression given in the problem statement,

$$\mathbf{H}(0, t) = \hat{\mathbf{y}}\, 100 \cos(2\pi \times 10^3 t + 15^\circ) \qquad \text{(mA/m)},$$

we deduce that

$$22.5 |E_{x0}| = 100 \times 10^{-3}$$

or

$$|E_{x0}| = 4.44 \qquad \text{(mV/m)},$$

and

$$\phi_0 - 45^\circ = 15^\circ \qquad \text{or} \qquad \phi_0 = 60^\circ.$$

Cont.

Example 7-4: Plane Wave in Seawater

Hence, the final expressions for $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ are

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 4.44e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + 60^\circ) \quad \text{(mV/m)}, \tag{7.82a}$$

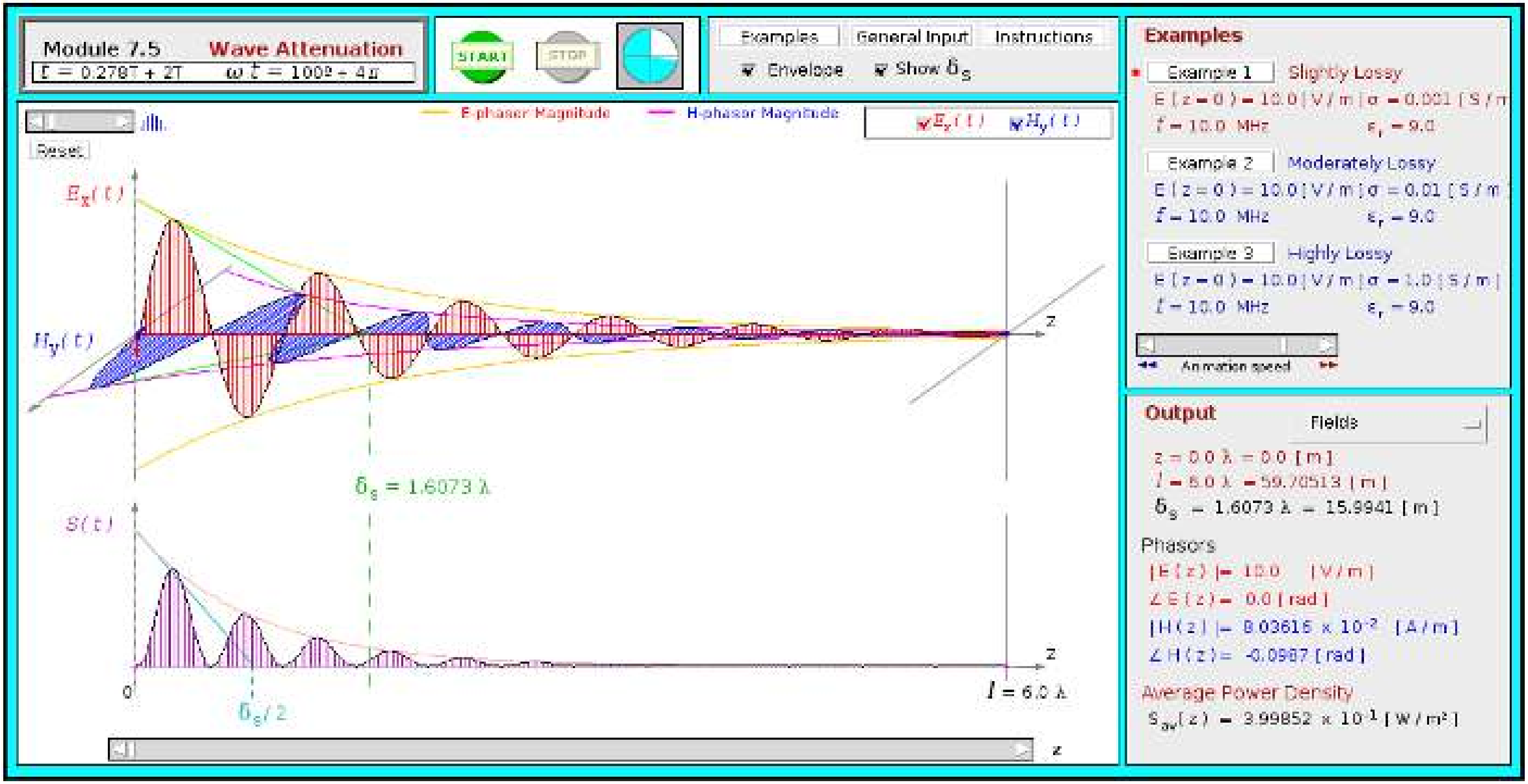
$$\mathbf{H}(z,t) = \hat{\mathbf{y}}\, 100e^{-0.126z} \cos(2\pi \times 10^3 t - 0.126z + 15^\circ) \quad \text{(mA/m)}. \tag{7.82b}$$

(b) The depth at which the amplitude of $\mathbf{E}$ has decreased to 1% of its initial value at $z = 0$ is obtained from

$$0.01 = e^{-0.126z}$$

or

$$\boxed{z = \frac{\ln(0.01)}{-0.126} = 36.55 \text{ m} \approx 37 \text{ m}}$$


Module 7.5
Wave Attenuation
START
STOP
Examples
General Input
Instructions
Envelope
Show δs
Reset
E-phaser Magnitude
H-phasor Magnitude
Ex(t)
Hy(t)
z
δs = 1.6073 λ
S(t)
0
δs/2
l = 6.0 λ
Examples
Example 1
Slightly Lossy
Example 2
Moderately Lossy
Example 3
Highly Lossy
Animation speed
Output
Fields
Phasors
Average Power Density

dc vs ac Current Flow in Conductors

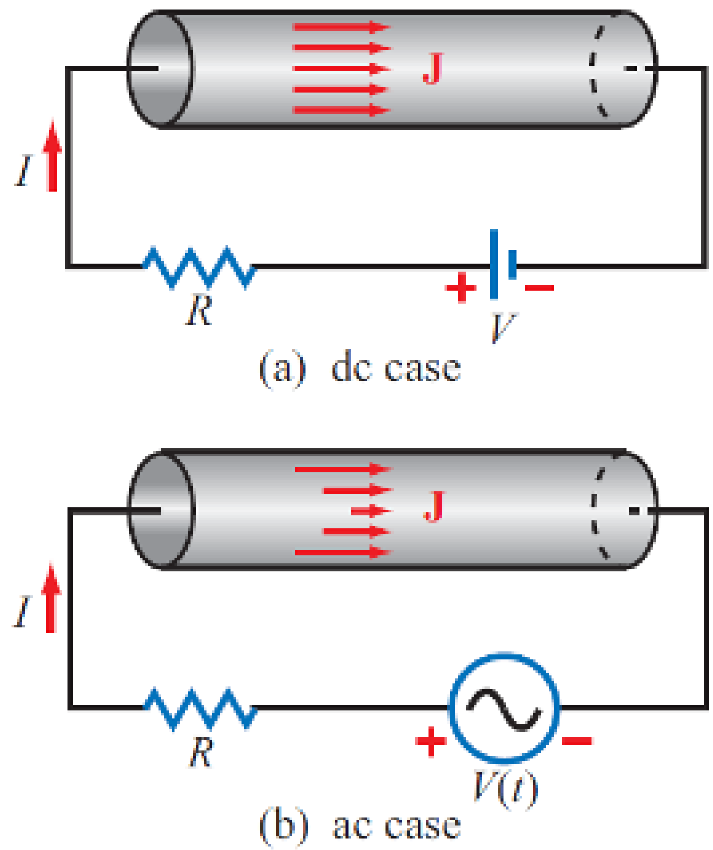


Figure 7-14: Current density **J** in a conducting wire is (a) uniform across its cross section in the dc case, but (b) in the ac case, **J** is highest along the wire's perimeter.

Linear Conductor

For a conductor with E_0 at the surface:

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}} E_0 e^{-\alpha z} e^{-j\beta z}, \qquad (7.83a)$$

$$\widetilde{\mathbf{H}}(z) = \hat{\mathbf{y}} \frac{E_0}{\eta_c} e^{-\alpha z} e^{-j\beta z}. \qquad (7.83b)$$

From $\mathbf{J} = \sigma \mathbf{E}$, the current flows in the x-direction, and its density is

$$\widetilde{\mathbf{J}}(z) = \hat{\mathbf{x}} \widetilde{J}_x(z), \qquad (7.84)$$

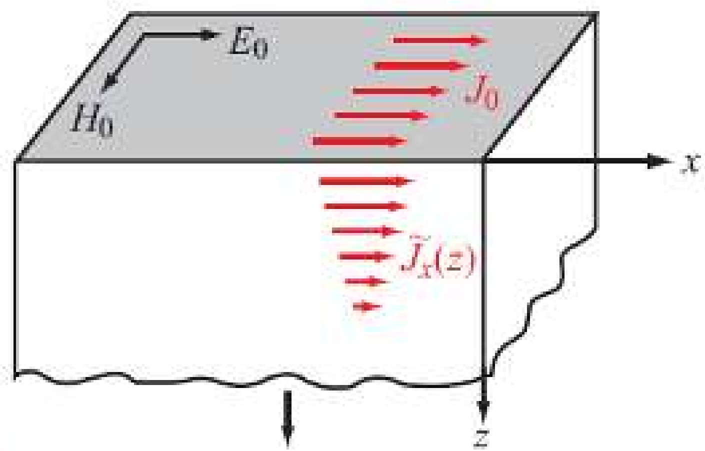
with

$$\widetilde{J}_x(z) = \sigma E_0 e^{-\alpha z} e^{-j\beta z} = J_0 e^{-\alpha z} e^{-j\beta z}, \qquad (7.85)$$

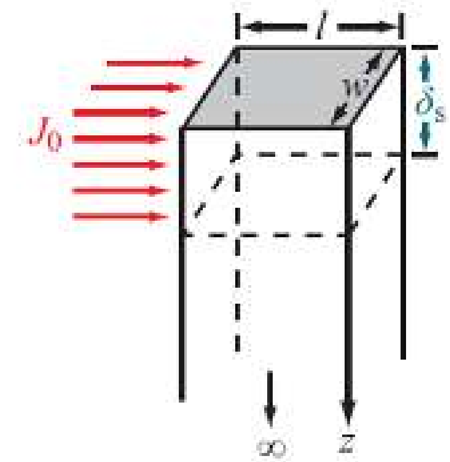
Total current crossing y-z plane:

$$\widetilde{I} = w \int_0^\infty \widetilde{J}_x(z)\, dz$$

$$= w \int_0^\infty J_0 e^{-(1+j)z/\delta_s}\, dz = \frac{J_0 w \delta_s}{(1+j)} \qquad \text{(A)}$$



(a) Exponentially decaying $\widetilde{J}_x(z)$



(b) Equivalent J_0 over skin depth δ_s

Figure 7-15: Exponential decay of current density $\widetilde{J}_x(z)$ with z in a solid conductor. The total current flowing through (a) a section of width w extending between $z = 0$ and $z = \infty$ is equivalent to (b) a constant current density J_0 flowing through a section of depth δ_s.

Surface Impedance

The voltage across a length l at the surface [Fig. 7-15(b)] is given by

$$\widetilde{V} = E_0 l = \frac{J_0}{\sigma} l. \qquad (7.88)$$

Hence, the impedance of a slab of width w, length l, and depth $d = \infty$ (or, in practice, $d > 5\delta_s$) is

$$Z = \frac{\widetilde{V}}{\widetilde{I}} = \frac{1+j}{\sigma \delta_s} \frac{l}{w} \quad (\Omega). \qquad (7.89)$$

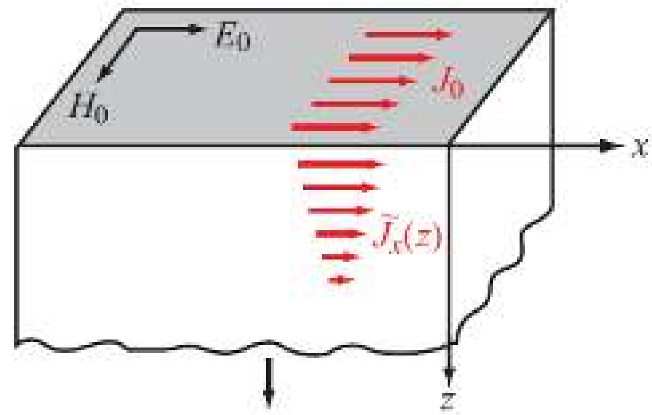
It is customary to represent Z as

$$Z = Z_s \frac{l}{w}, \qquad (7.90)$$

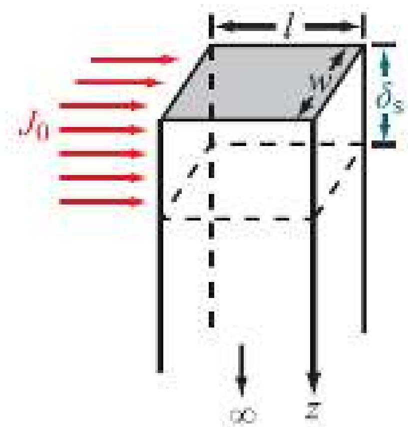
where Z_s, the ***internal*** or ***surface impedance*** of the conductor, is defined as the impedance Z for a length $l = 1$ m and a width $w = 1$ m. Thus,

$$\boxed{Z_s = \frac{1+j}{\sigma \delta_s} \quad (\Omega). \qquad (7.91)}$$

Thus, conductor is equivalent to a resistor in series with an inductor.



(a) Exponentially decaying $\widetilde{J}_x(z)$



(b) Equivalent J_0 over skin depth δ_s

Figure 7-15: Exponential decay of current density $\widetilde{J}_x(z)$ with z in a solid conductor. The total current flowing through (a) a section of width w extending between $z = 0$ and $z = \infty$ is equivalent to (b) a constant current density J_0 flowing through a section of depth δ_s.

ac Resistance of Coaxial Cable

Since in the ac case, most of the current flows through a very thin skin along the outside of the inner conductor and along the inside of the outer conductor, we can use the results of the planar conductor to figure out the resistance of the coax. The procedure leads to the following expression for the resistance per unit length:

$$R' = R'_1 + R'_2 = \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right) \quad (\Omega/\mathrm{m})$$

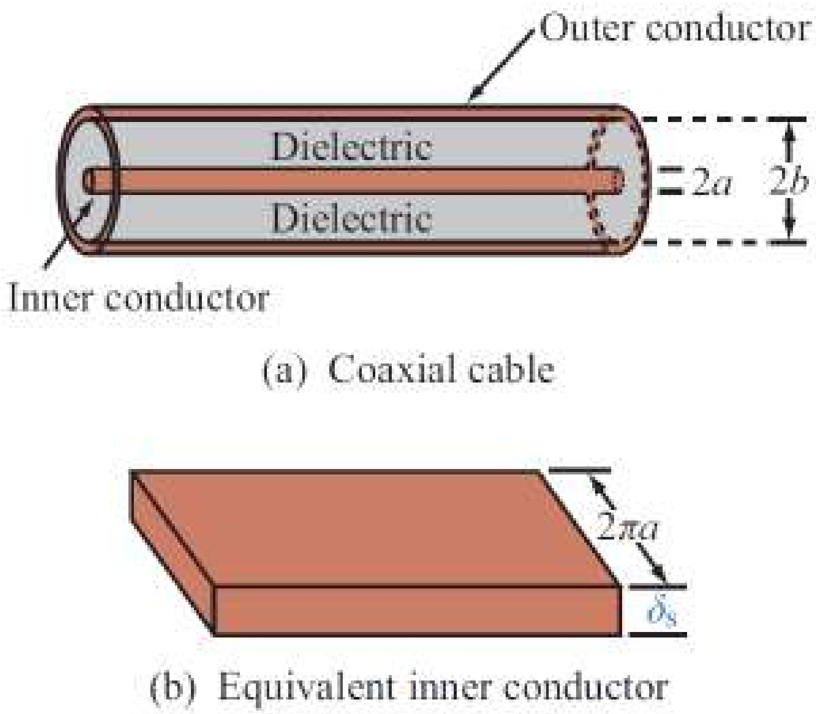
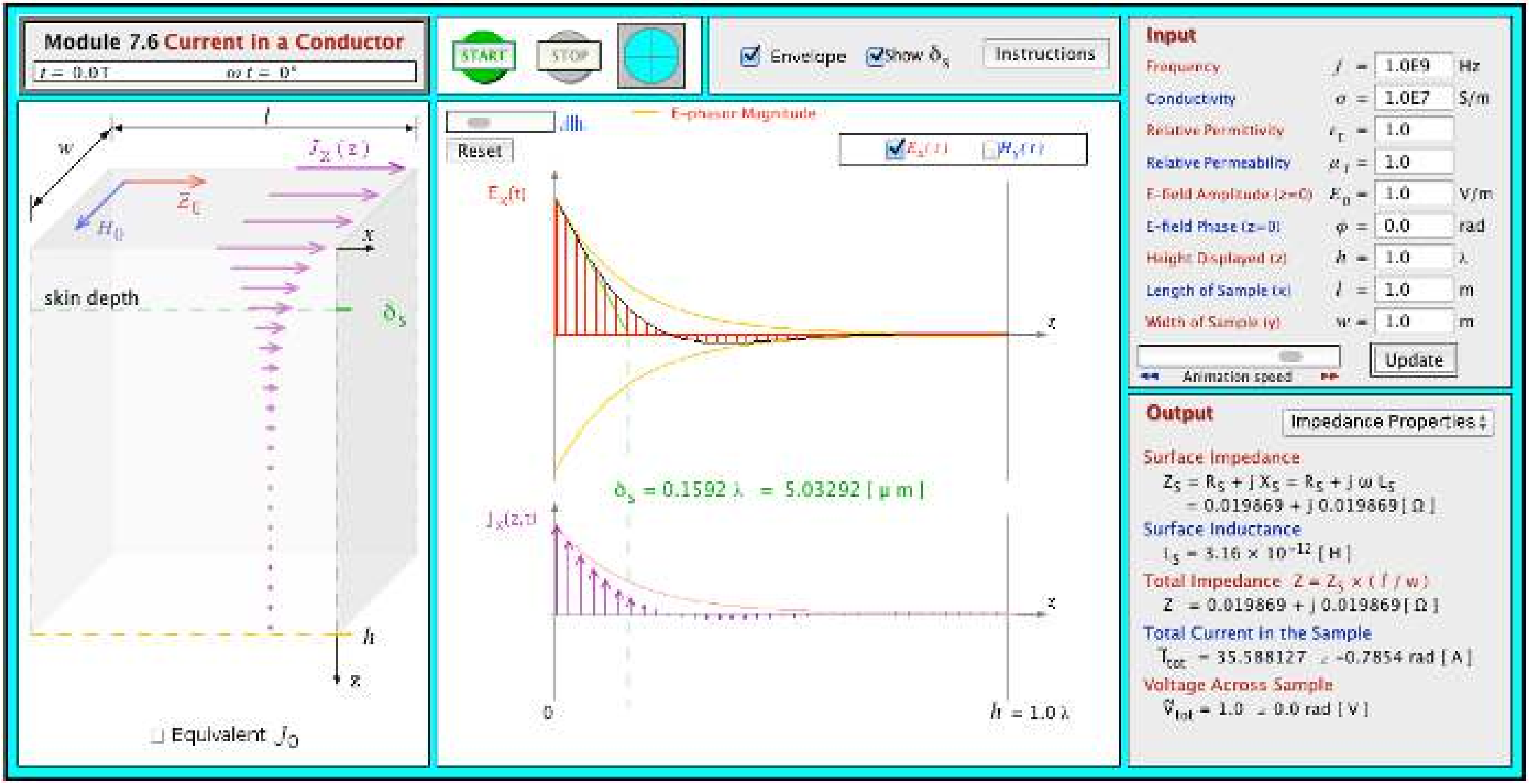


Figure 7-16: The inner conductor of the coaxial cable in (a) is represented in (b) by a planar conductor of width $2\pi a$ and depth δ_s, as if its skin has been cut along its length on the bottom side and then unfurled into a planar geometry.



Module 7.6 Current in a Conductor
START
STOP
Envelope
Show δs
Instructions
Reset
E-phasor Magnitude
skin depth
Equivalent J0
Input
Frequency f = 1.0E9 Hz
Conductivity σ = 1.0E7 S/m
Relative Permittivity εr = 1.0
Relative Permeability μr = 1.0
E-field Amplitude (z=0) E0 = 1.0 V/m
E-field Phase (z=0) φ = 0.0 rad
Height Displayed (z) h = 1.0 λ
Length of Sample (x) l = 1.0 m
Width of Sample (y) w = 1.0 m
Update
Animation speed
δs = 0.1592 λ = 5.03292 [μ m]
h = 1.0 λ
Output
Impedance Properties
Surface Impedance
Zs = Rs + j Xs = Rs + j ω Ls
= 0.019869 + j 0.019869 [Ω]
Surface Inductance
Ls = 3.16 × 10^-12 [H]
Total Impedance Z = Zs × (l / w)
Z = 0.019869 + j 0.019869 [Ω]
Total Current in the Sample
Itot = 35.588127 ∠ -0.7854 rad [A]
Voltage Across Sample
Vtot = 1.0 ∠ 0.0 rad [V]

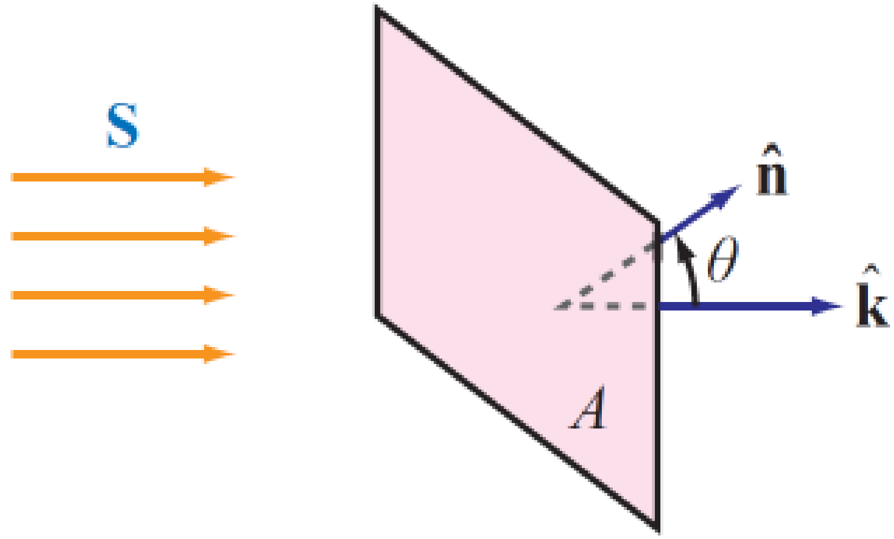
Power Density

Poynting vector:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \qquad (\text{W/m}^2).$$

Total power intercepted by A:

$$P = \int_A \mathbf{S} \cdot \hat{\mathbf{n}} \, dA \qquad (\text{W}).$$



Time-average power density:

$$\mathbf{S}_{\text{av}} = \frac{1}{2} \mathfrak{Re} \left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^* \right] \qquad (\text{W/m}^2).$$

Power Density Carried by Plane Wave

For a plane wave with **E** field :

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\,\widetilde{E}_x(z) + \hat{\mathbf{y}}\,\widetilde{E}_y(z)$$
$$= (\hat{\mathbf{x}}\,E_{x0} + \hat{\mathbf{y}}\,E_{y0})e^{-jkz},$$

the average power density carried by the wave is:

$$\mathbf{S}_{\text{av}} = \hat{\mathbf{z}}\,\frac{1}{2\eta}(|E_{x0}|^2 + |E_{y0}|^2)$$
$$= \hat{\mathbf{z}}\,\frac{|\widetilde{\mathbf{E}}|^2}{2\eta} \qquad (\text{W/m}^2),$$

Plane Wave in Lossy Medium

For a plane wave travelling in a lossy medium:

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\,\widetilde{E}_x(z) + \hat{\mathbf{y}}\,\widetilde{E}_y(z)$$

$$= (\hat{\mathbf{x}}\,E_{x0} + \hat{\mathbf{y}}\,E_{y0})e^{-\alpha z}e^{-j\beta z},$$

$$\widetilde{\mathbf{H}}(z) = \frac{1}{\eta_c}(-\hat{\mathbf{x}}\,E_{y0} + \hat{\mathbf{y}}\,E_{x0})e^{-\alpha z}e^{-j\beta z},$$

the power density is :

$$\mathbf{S}_{av}(z) = \frac{1}{2}\,\mathfrak{Re}\left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^*\right]$$

$$= \frac{\hat{\mathbf{z}}(|E_{x0}|^2 + |E_{y0}|^2)}{2}e^{-2\alpha z}\,\mathfrak{Re}\left(\frac{1}{\eta_c^*}\right).$$

By expressing η_c in polar form as

$$\eta_c = |\eta_c|e^{j\theta_\eta},$$

$$\mathbf{S}_{av}(z) = \hat{\mathbf{z}}\,\frac{|\widetilde{E}(0)|^2}{2|\eta_c|}\,e^{-2\alpha z}\cos\theta_\eta \quad (\text{W/m}^2)$$

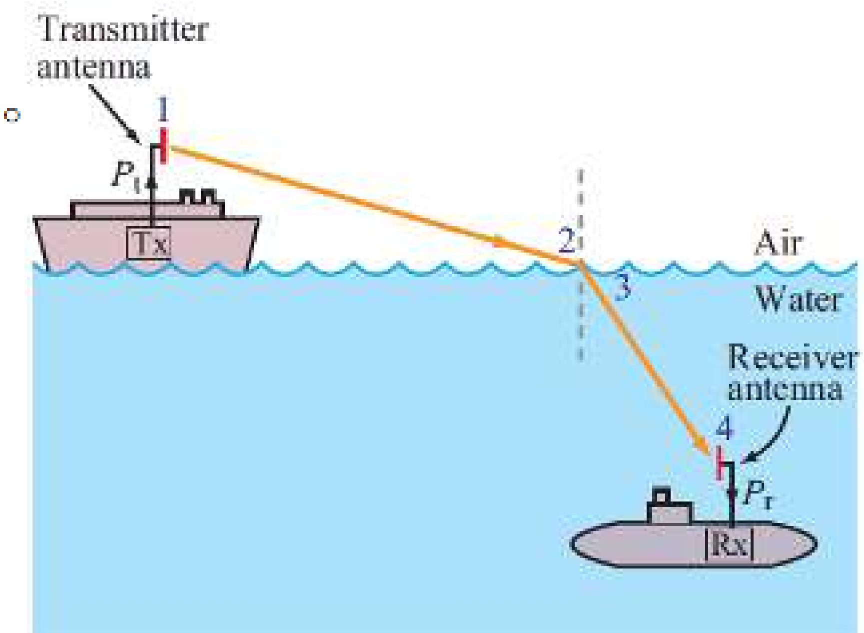
Whereas the fields $\widetilde{\mathbf{E}}(z)$ and $\widetilde{\mathbf{H}}(z)$ decay with z as $e^{-\alpha z}$, the power density $\mathbf{S}_{av}$ decreases as $e^{-2\alpha z}$.

Example 7-6: Power Received by a Submarine Antenna

A submarine at a depth of 200 m below the sea surface uses a wire antenna to receive signal transmissions at 1 kHz. Determine the power density incident upon the submarine antenna due to the EM wave of Example 7-4.

Solution: From Example 7-4, $|\vec{E}(0)| = |E_{x0}| = 4.44$ (mV/m), $\alpha = 0.126$ (Np/m), and $\eta_c = 0.044\angle 45^\circ$ (Ω). Application of Eq. (7.109) gives

$$\begin{aligned} \mathbf{S}_{\text{av}}(z) &= \hat{\mathbf{z}}\,\frac{|E_0|^2}{2|\eta_c|}\,e^{-2\alpha z}\cos\theta_\eta \\ &= \hat{\mathbf{z}}\,\frac{(4.44\times 10^{-3})^2}{2\times 0.044}\,e^{-0.252z}\cos 45^\circ \\ &= \hat{\mathbf{z}}\,0.16e^{-0.252z} \qquad (\text{mW/m}^2). \end{aligned}$$



At $z = 200$ m, the incident power density is

$$\begin{aligned} \mathbf{S}_{\text{av}} &= \hat{\mathbf{z}}\,(0.16\times 10^{-3}e^{-0.252\times 200}) \\ &= 2.1\times 10^{-26} \qquad (\text{W/m}^2). \end{aligned}$$

Summary

Chapter 7 Relationships

Complex Permittivity

$$\varepsilon_c = \varepsilon' - j\varepsilon''$$
$$\varepsilon' = \varepsilon$$
$$\varepsilon'' = \frac{\sigma}{\omega}$$

Lossless Medium

$$k = \omega\sqrt{\mu\varepsilon}$$
$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \qquad (\Omega)$$
$$u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{(m/s)}$$
$$\lambda = \frac{2\pi}{k} = \frac{u_p}{f} \qquad \text{(m)}$$

Wave Polarization

$$\widetilde{\mathbf{H}} = \frac{1}{\eta}\,\hat{\mathbf{k}} \times \widetilde{\mathbf{E}}$$
$$\widetilde{\mathbf{E}} = -\eta\,\hat{\mathbf{k}} \times \widetilde{\mathbf{H}}$$

Maxwell's Equations for Time-Harmonic Fields

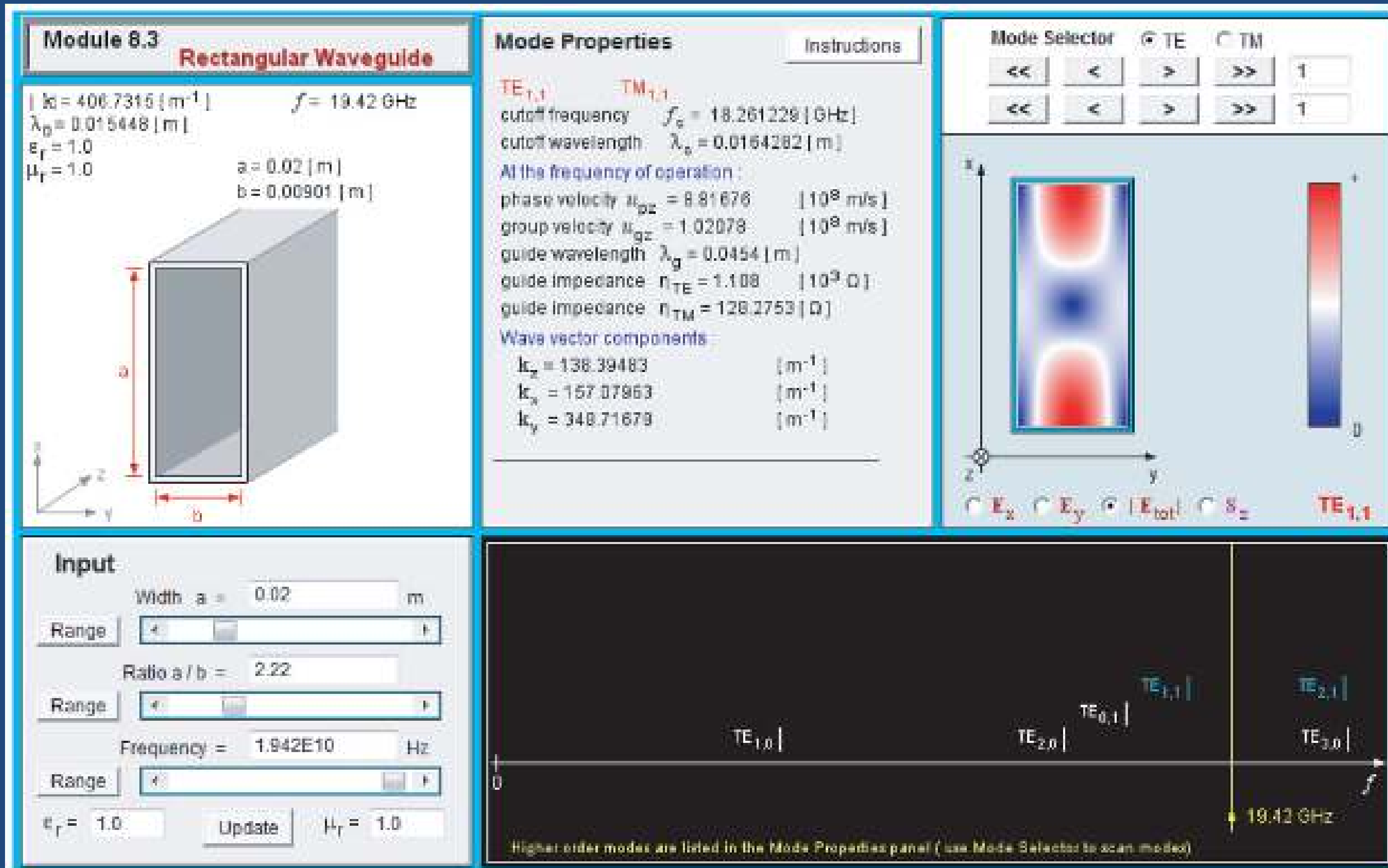
$$\nabla \cdot \widetilde{\mathbf{E}} = 0$$
$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$$
$$\nabla \cdot \widetilde{\mathbf{H}} = 0$$
$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon_c\widetilde{\mathbf{E}}$$

Lossy Medium

$$\alpha = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]\right\}^{1/2} \qquad \text{(Np/m)}$$
$$\beta = \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]\right\}^{1/2} \qquad \text{(rad/m)}$$
$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \qquad (\Omega)$$
$$\delta_s = \frac{1}{\alpha} \qquad \text{(m)}$$

Power Density

$$\mathbf{S}_{av} = \tfrac{1}{2}\,\mathfrak{Re}\left[\widetilde{\mathbf{E}} \times \widetilde{\mathbf{H}}^*\right] \qquad (\text{W/m}^2)$$



8. WAVE REFLECTION & TRANSMISSION

7e Applied EM by Ulaby and Ravaioli

Overview

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Characterize the reflection and transmission behavior of plane waves incident upon plane boundaries, for both normal and oblique incidence.
2. Calculate the transmission properties of optical fibers.
3. Characterize wave propagation in a rectangular wave-guide.
4. Determine the behavior of resonant modes inside a rectangular cavity.

Signal Refraction at Boundaries

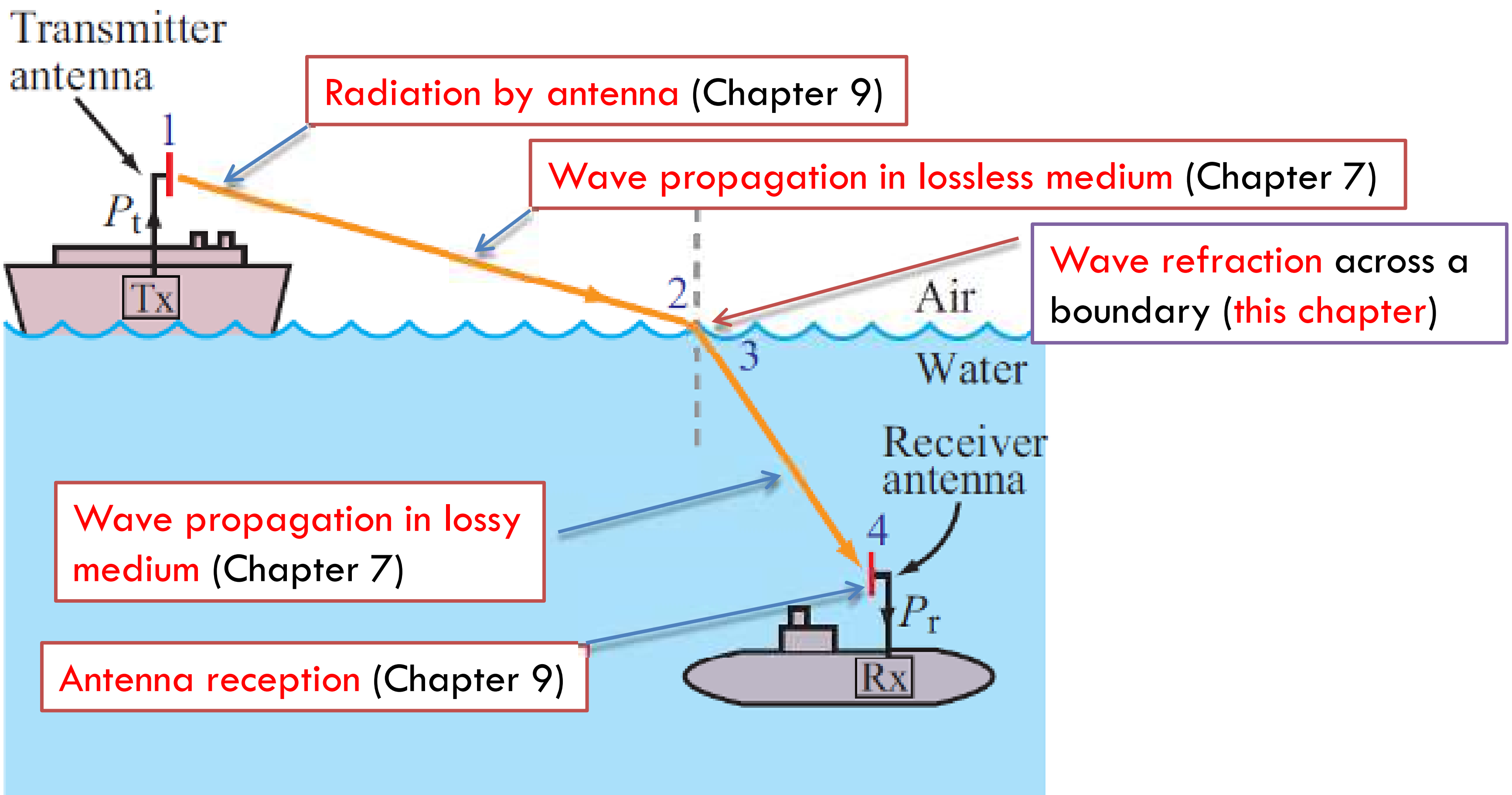
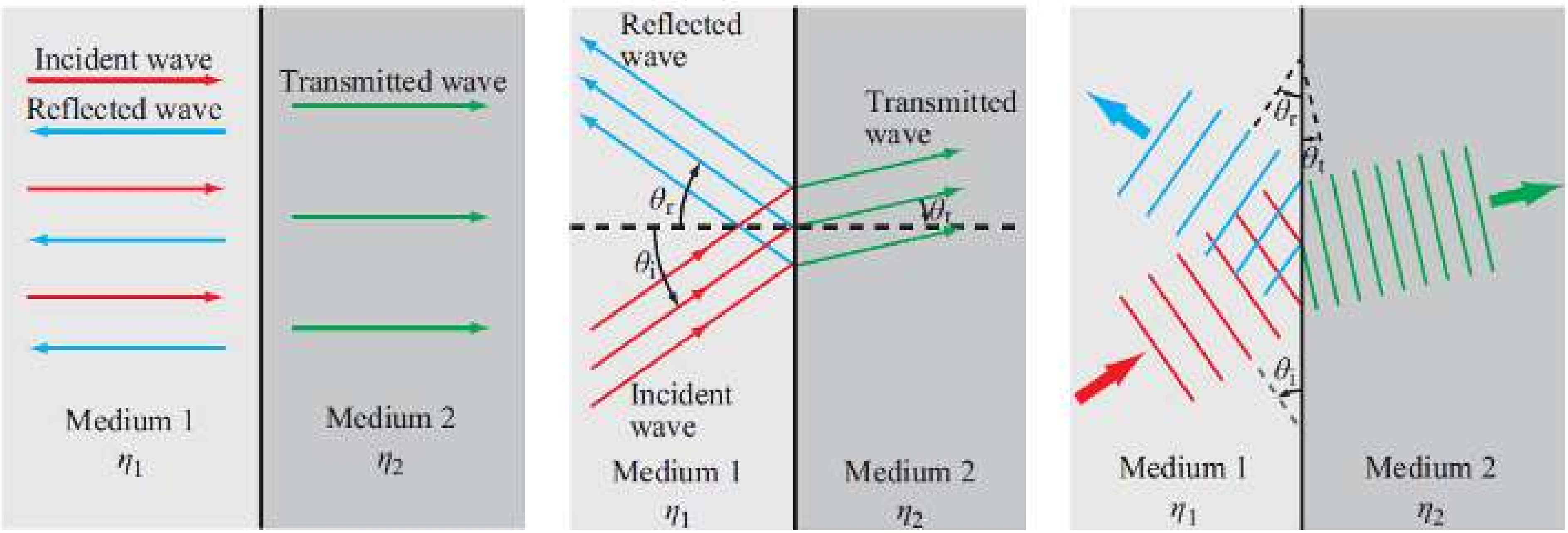


Figure 8-1: Signal path between a shipboard transmitter (Tx) and a submarine receiver (Rx).

Normal and Oblique Incidence



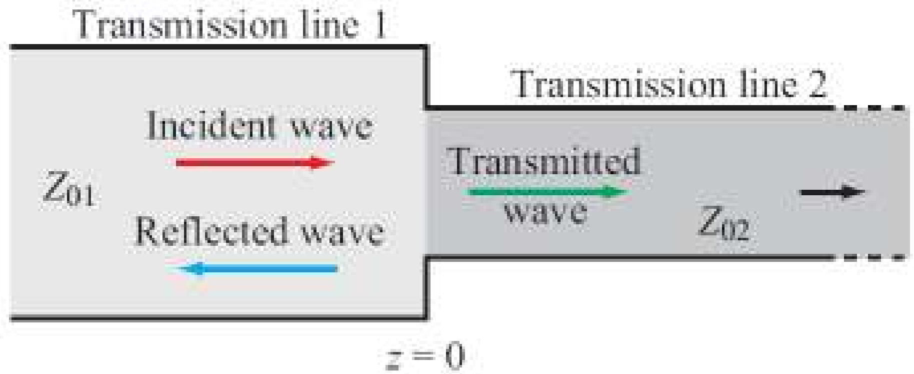
(a) Normal incidence

(b) Ray representation of oblique incidence

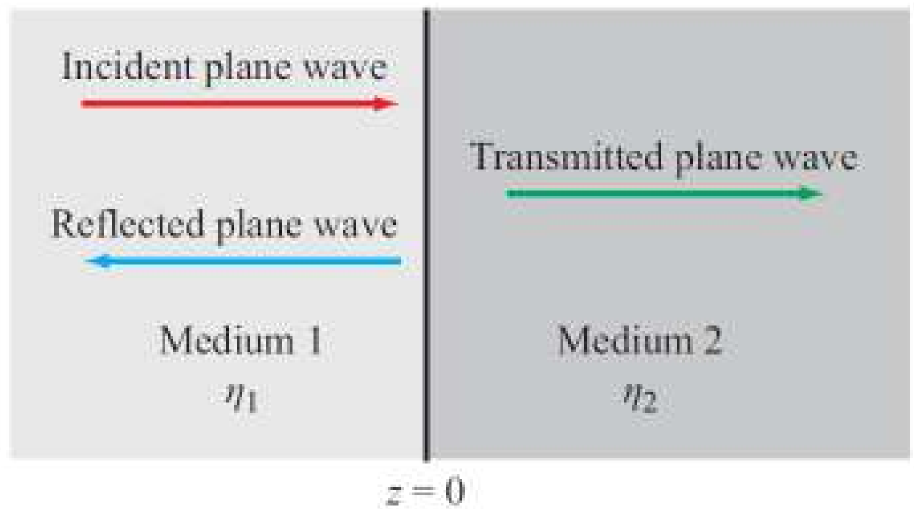
(c) Wavefront representation of oblique incidence

Normal Incidence

We can use all the transmission-line concepts and techniques of Chapter 2 to analyze plane wave reflection and transmission at interfaces between dissimilar media



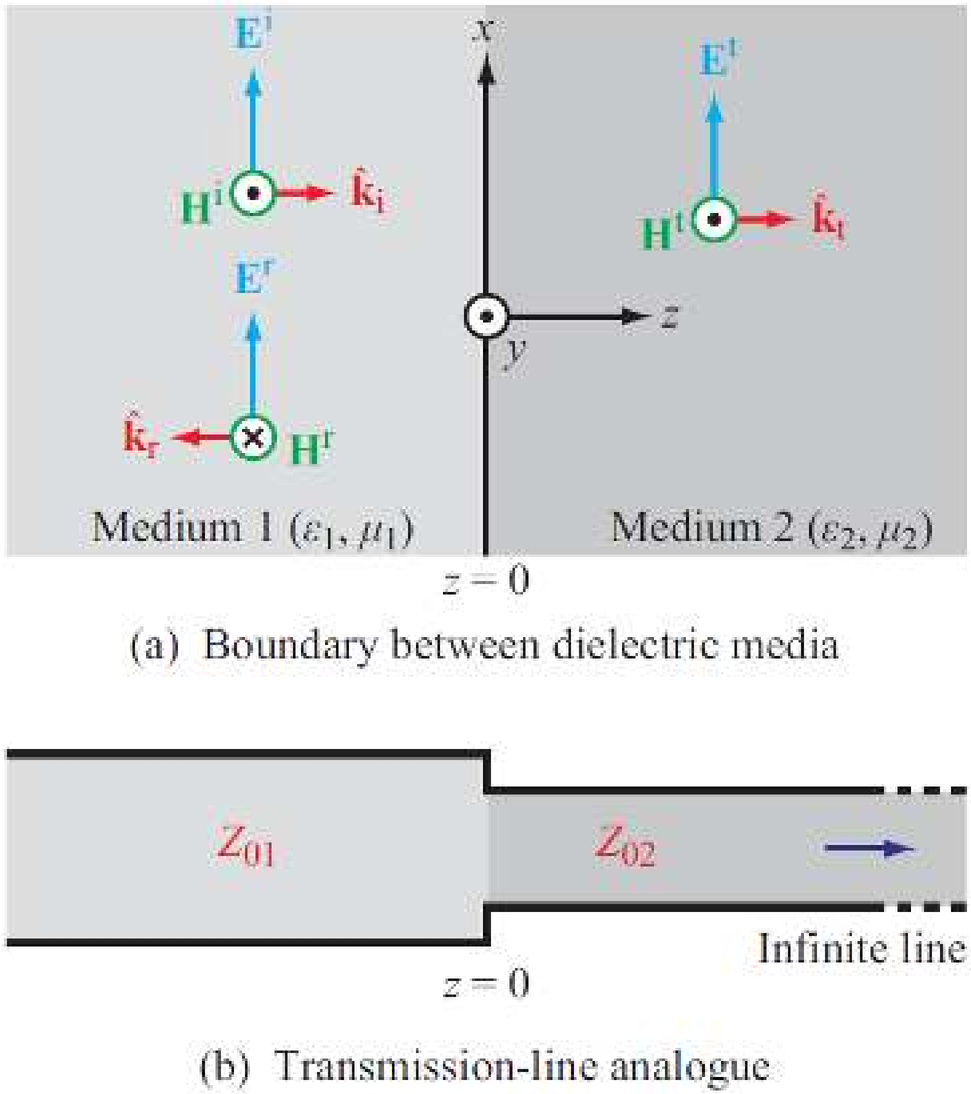
(a) Boundary between transmission lines



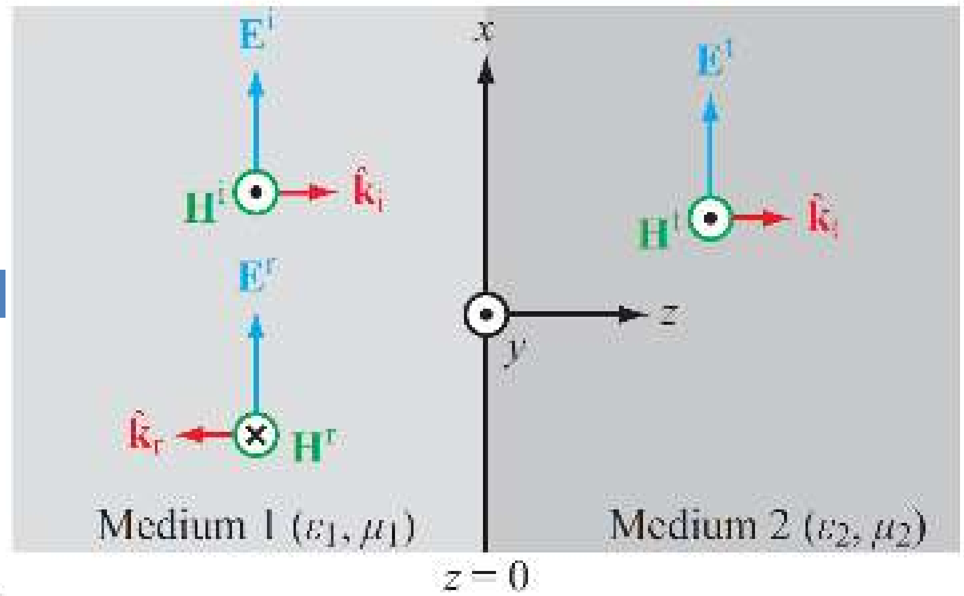
(b) Boundary between different media

Figure 8-2: Discontinuity between two different transmission lines is analogous to that between two dissimilar media.

Modeling Normal Incidence



Individual Waves



Incident Wave

Note – sign

$$\widetilde{\mathbf{E}}^i(z) = \hat{\mathbf{x}} E_0^i e^{-jk_1 z},$$

$$\widetilde{\mathbf{H}}^i(z) = \hat{\mathbf{z}} \times \frac{\widetilde{\mathbf{E}}^i(z)}{\eta_1} = \hat{\mathbf{y}} \frac{E_0^i}{\eta_1} e^{-jk_1 z}.$$

Reflected Wave

Note + sign

$$\widetilde{\mathbf{E}}^r(z) = \hat{\mathbf{x}} E_0^r e^{jk_1 z},$$

$$\widetilde{\mathbf{H}}^r(z) = (-\hat{\mathbf{z}}) \times \frac{\widetilde{\mathbf{E}}^r(z)}{\eta_1} = -\hat{\mathbf{y}} \frac{E_0^r}{\eta_1} e^{jk_1 z}.$$

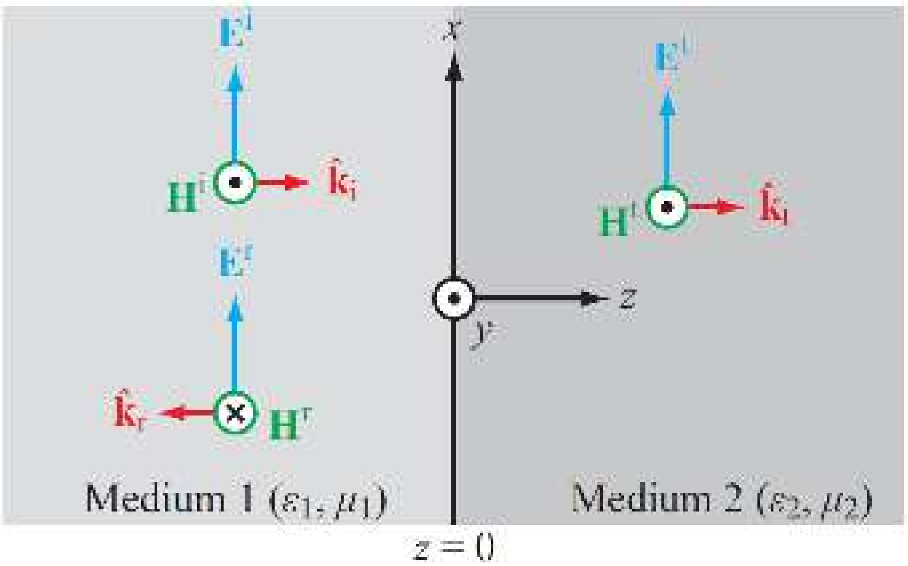
Transmitted Wave

Note sign

$$\widetilde{\mathbf{E}}^t(z) = \hat{\mathbf{x}} E_0^t e^{-jk_2 z},$$

$$\widetilde{\mathbf{H}}^t(z) = \hat{\mathbf{z}} \times \frac{\widetilde{\mathbf{E}}^t(z)}{\eta_2} = \hat{\mathbf{y}} \frac{E_0^t}{\eta_2} e^{-jk_2 z}.$$

Boundary Conditions



Total fields

Medium 1

$$\widetilde{\mathbf{E}}_1(z) = \widetilde{\mathbf{E}}^{i}(z) + \widetilde{\mathbf{E}}^{r}(z) = \hat{\mathbf{x}}(E_0^{i} e^{-jk_1 z} + E_0^{r} e^{jk_1 z}), \tag{8.4a}$$

$$\widetilde{\mathbf{H}}_1(z) = \widetilde{\mathbf{H}}^{i}(z) + \widetilde{\mathbf{H}}^{r}(z) = \hat{\mathbf{y}}\, \frac{1}{\eta_1} (E_0^{i} e^{-jk_1 z} - E_0^{r} e^{jk_1 z}). \tag{8.4b}$$

With only the transmitted wave present in medium 2, the total fields are

Medium 2

$$\widetilde{\mathbf{E}}_2(z) = \widetilde{\mathbf{E}}^{t}(z) = \hat{\mathbf{x}} E_0^{t} e^{-jk_2 z}, \tag{8.9a}$$

$$\widetilde{\mathbf{H}}_2(z) = \widetilde{\mathbf{H}}^{t}(z) = \hat{\mathbf{y}}\, \frac{E_0^{t}}{\eta_2} e^{-jk_2 z}. \tag{8.9b}$$

At the boundary $z = 0$:

$$\widetilde{\mathbf{E}}_1(0) = \widetilde{\mathbf{E}}_2(0) \quad \text{or} \quad E_0^{i} + E_0^{r} = E_0^{t},$$

$$\widetilde{\mathbf{H}}_1(0) = \widetilde{\mathbf{H}}_2(0) \quad \text{or} \quad \frac{E_0^{i}}{\eta_1} - \frac{E_0^{r}}{\eta_1} = \frac{E_0^{t}}{\eta_2}.$$

Solution gives:

$$E_0^{r} = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) E_0^{i} = \Gamma E_0^{i},$$

$$E_0^{t} = \left(\frac{2\eta_2}{\eta_2 + \eta_1}\right) E_0^{i} = \tau E_0^{i},$$

where

$$\Gamma = \frac{E_0^{r}}{E_0^{i}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{(normal incidence)},$$

$$\tau = \frac{E_0^{t}}{E_0^{i}} = \frac{2\eta_2}{\eta_2 + \eta_1} \quad \text{(normal incidence)}.$$

Reflection and Transmission Coefficients

$$\Gamma = \frac{E_0^{\mathrm{r}}}{E_0^{\mathrm{i}}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{(normal incidence)},$$

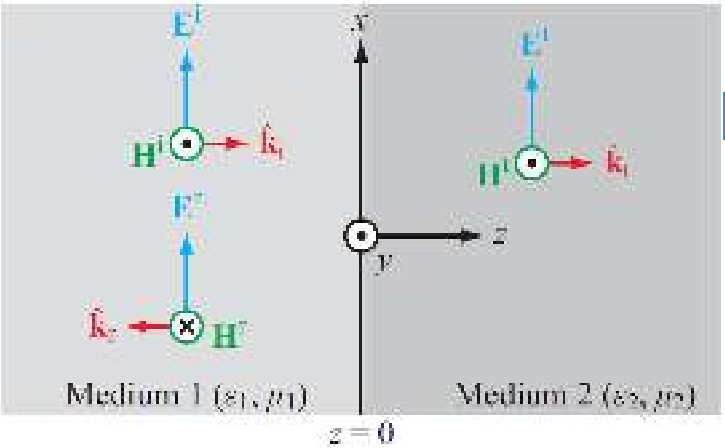
$$\tau = \frac{E_0^{\mathrm{t}}}{E_0^{\mathrm{i}}} = \frac{2\eta_2}{\eta_2 + \eta_1} \quad \text{(normal incidence)}.$$

Similar form as for transmission lines

$$\tau = 1 + \Gamma \quad \text{(normal incidence)}.$$

$$\Gamma = \frac{\sqrt{\varepsilon_{\mathrm{r}_1}} - \sqrt{\varepsilon_{\mathrm{r}_2}}}{\sqrt{\varepsilon_{\mathrm{r}_1}} + \sqrt{\varepsilon_{\mathrm{r}_2}}} \quad \text{(nonmagnetic media)}.$$

Analogy of Normal Incidence to Transmission Lines



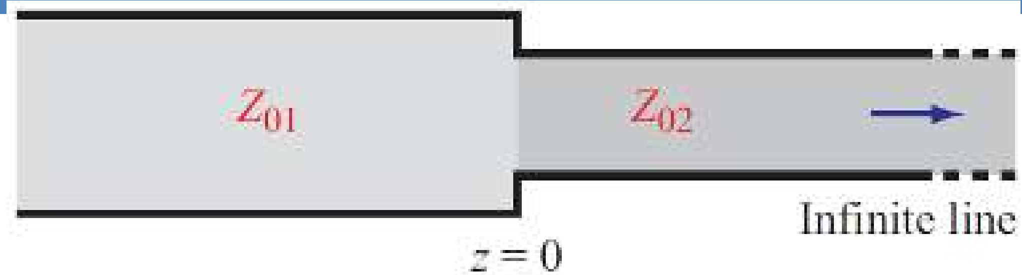
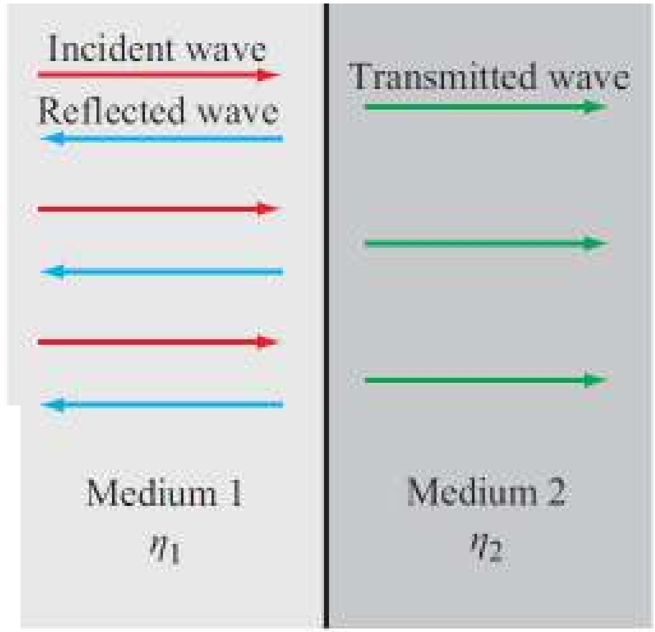


Table 8-1: Analogy between plane-wave equations for normal incidence and transmission-line equations, both under lossless conditions.

Plane Wave [Fig. 8-4(a)]		Transmission Line [Fig. 8-4(b)]	
$\widetilde{\mathbf{E}}_1(z) = \hat{\mathbf{x}} E_0^i (e^{-jk_1 z} + \Gamma e^{jk_1 z})$	(8.5a)	$\widetilde{V}_1(z) = V_0^+ (e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z})$	(8.5b)
$\widetilde{\mathbf{H}}_1(z) = \hat{\mathbf{y}} \dfrac{E_0^i}{\eta_1} (e^{-jk_1 z} - \Gamma e^{jk_1 z})$	(8.6a)	$\widetilde{I}_1(z) = \dfrac{V_0^+}{Z_{01}} (e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z})$	(8.6b)
$\widetilde{\mathbf{E}}_2(z) = \hat{\mathbf{x}} \tau E_0^i e^{-jk_2 z}$	(8.7a)	$\widetilde{V}_2(z) = \tau V_0^+ e^{-j\beta_2 z}$	(8.7b)
$\widetilde{\mathbf{H}}_2(z) = \hat{\mathbf{y}} \tau \dfrac{E_0^i}{\eta_2} e^{-jk_2 z}$	(8.8a)	$\widetilde{I}_2(z) = \tau \dfrac{V_0^+}{Z_{02}} e^{-j\beta_2 z}$	(8.8b)
$\Gamma = (\eta_2 - \eta_1)/(\eta_2 + \eta_1)$		$\Gamma = (Z_{02} - Z_{01})/(Z_{02} + Z_{01})$	
$\tau = 1 + \Gamma$		$\tau = 1 + \Gamma$	
$k_1 = \omega\sqrt{\mu_1 \varepsilon_1}$, $\quad k_2 = \omega\sqrt{\mu_2 \varepsilon_2}$		$\beta_1 = \omega\sqrt{\mu_1 \varepsilon_1}$, $\quad \beta_2 = \omega\sqrt{\mu_2 \varepsilon_2}$	
$\eta_1 = \sqrt{\mu_1/\varepsilon_1}$, $\quad \eta_2 = \sqrt{\mu_2/\varepsilon_2}$		Z_{01} and Z_{02} depend on transmission-line parameters	

Power Transfer



Medium 1

$$\mathbf{S}_{\mathrm{av}_1}(z) = \tfrac{1}{2}\mathfrak{Re}[\widetilde{\mathbf{E}}_1(z) \times \widetilde{\mathbf{H}}_1^*(z)]$$

$$= \tfrac{1}{2}\mathfrak{Re}\left[\hat{\mathbf{x}} E_0^{\mathrm{i}}(e^{-jk_1z} + \Gamma e^{jk_1z}) \times \hat{\mathbf{y}}\, \frac{E_0^{\mathrm{i}*}}{\eta_1}(e^{jk_1z} - \Gamma^* e^{-jk_1z})\right]$$

$$= \hat{\mathbf{z}}\, \frac{|E_0^{\mathrm{i}}|^2}{2\eta_1}\,(1 - |\Gamma|^2),$$

$$\mathbf{S}_{\mathrm{av}_1} = \mathbf{S}_{\mathrm{av}}^{\mathrm{i}} + \mathbf{S}_{\mathrm{av}}^{\mathrm{r}},$$

$$\mathbf{S}_{\mathrm{av}}^{\mathrm{i}} = \hat{\mathbf{z}}\, \frac{|E_0^{\mathrm{i}}|^2}{2\eta_1},$$

$$\mathbf{S}_{\mathrm{av}}^{\mathrm{r}} = -\hat{\mathbf{z}}|\Gamma|^2\, \frac{|E_0^{\mathrm{i}}|^2}{2\eta_1} = -|\Gamma|^2 \mathbf{S}_{\mathrm{av}}^{\mathrm{i}}.$$

Medium 2

$$\mathbf{S}_{\mathrm{av}_2}(z) = \tfrac{1}{2}\mathfrak{Re}[\widetilde{\mathbf{E}}_2(z) \times \widetilde{\mathbf{H}}_2^*(z)]$$

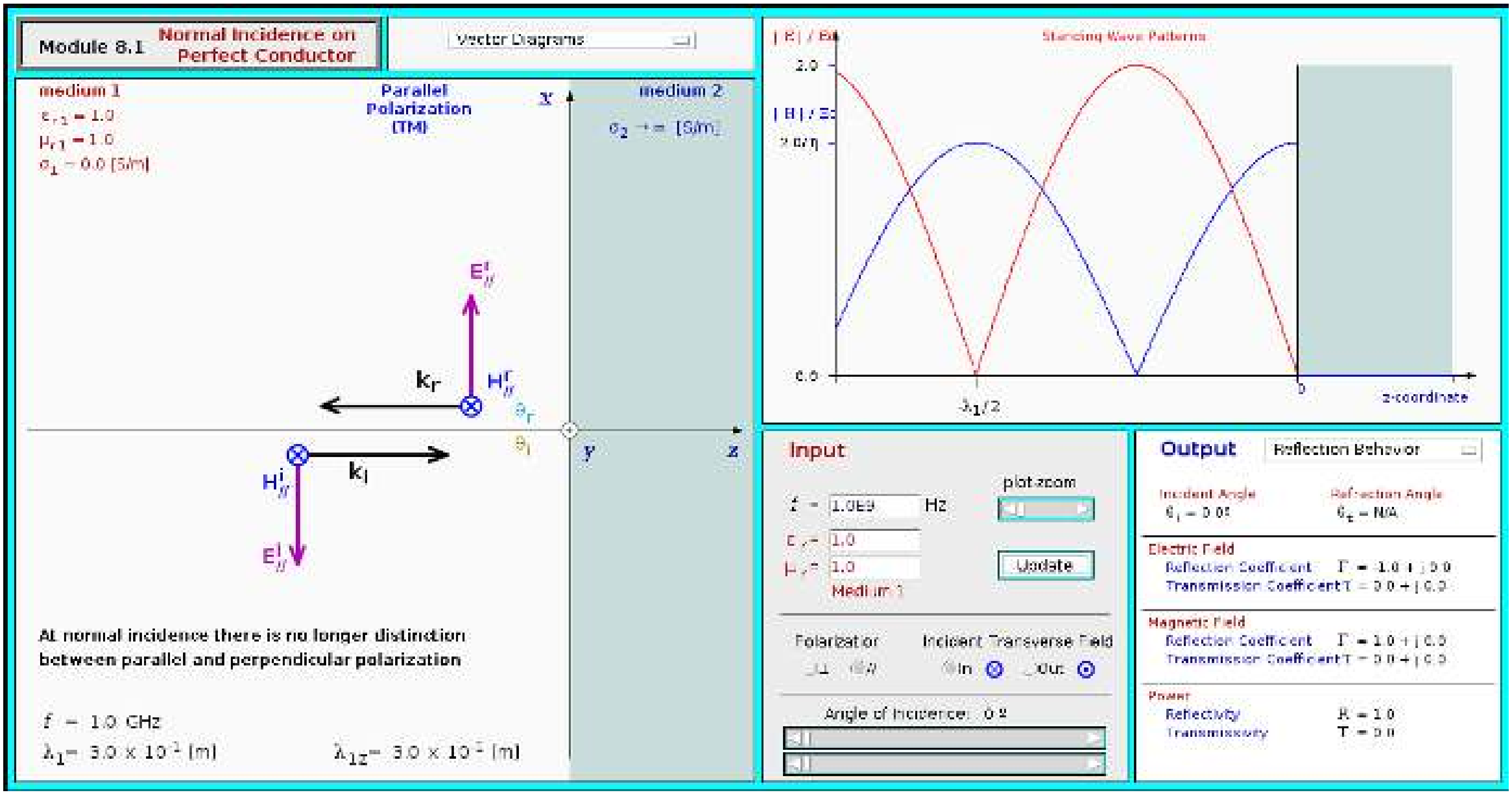
$$= \tfrac{1}{2}\mathfrak{Re}\left[\hat{\mathbf{x}}\tau E_0^{\mathrm{i}} e^{-jk_2z} \times \hat{\mathbf{y}}\tau^*\, \frac{E_0^{\mathrm{i}*}}{\eta_2}\, e^{jk_2z}\right]$$

$$= \hat{\mathbf{z}}|\tau|^2\, \frac{|E_0^{\mathrm{i}}|^2}{2\eta_2}.$$

$$\boxed{\frac{\tau^2}{\eta_2} = \frac{1 - \Gamma^2}{\eta_1} \qquad \text{(lossless media),}}$$

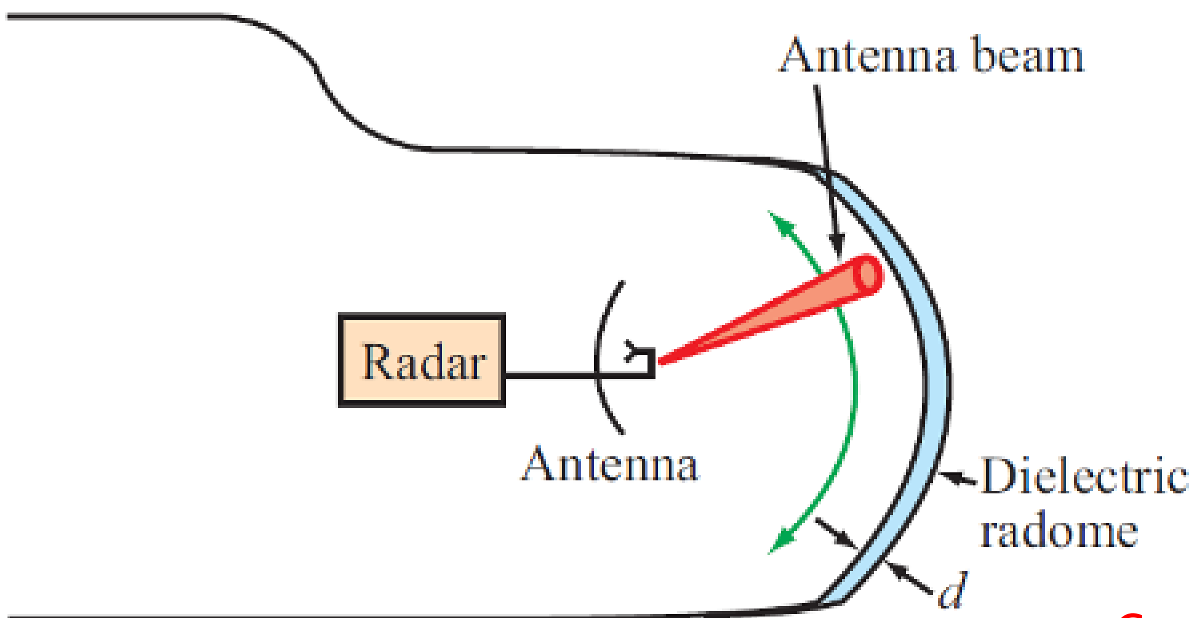
leads to

$$\mathbf{S}_{\mathrm{av}_1} = \mathbf{S}_{\mathrm{av}_2}.$$


Module 8.1
Normal Incidence on Perfect Conductor
Vector Diagrams
medium 1
Parallel Polarization (TM)
medium 2
At normal incidence there is no longer distinction between parallel and perpendicular polarization
f = 1.0 GHz
Standing Wave Patterns
z-coordinate
Input
plot zoom
Hz
Update
Medium 1
Polarization
Incident Transverse Field
Angle of Incidence:
Output
Reflection Behavior
Incident Angle
Refraction Angle
Electric Field
Reflection Coefficient
Transmission Coefficient
Magnetic Field
Reflection Coefficient
Transmission Coefficient
Power
Reflectivity
Transmissivity

Example 8-1: Radar Radome Design

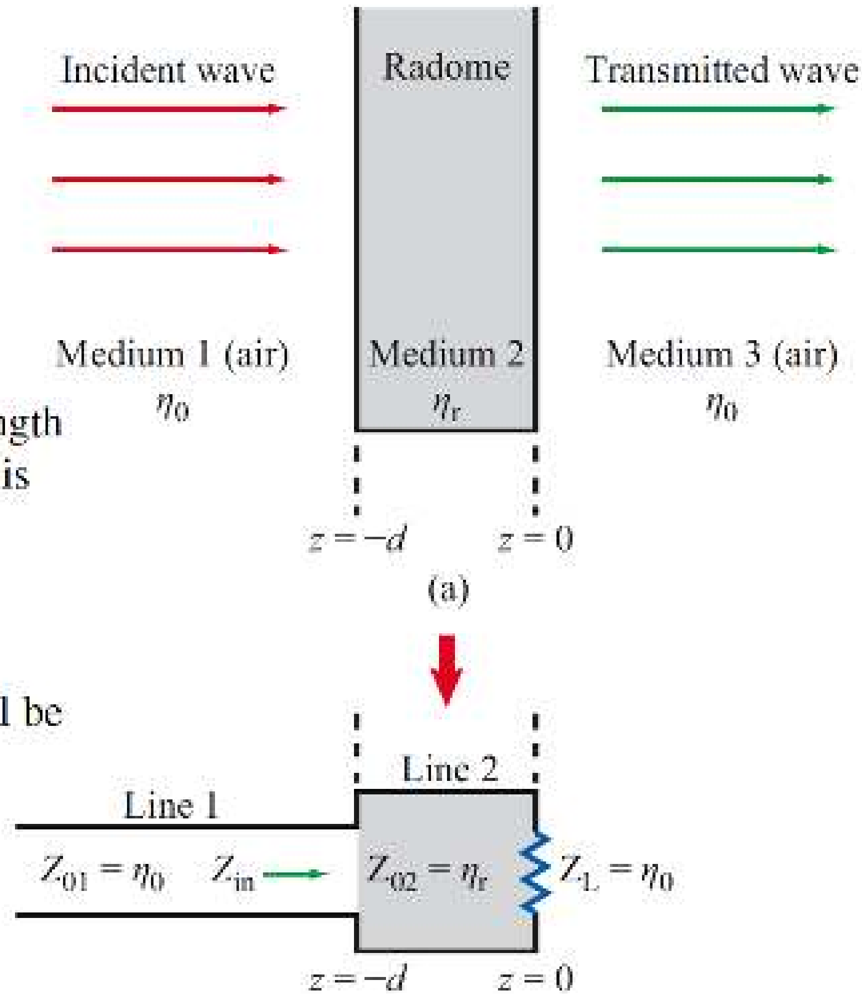
A 10-GHz aircraft radar uses a narrow-beam scanning antenna mounted on a gimbal behind a dielectric radome, as shown in Fig. 8-5. Even though the radome shape is far from planar, it is approximately planar over the narrow extent of the radar beam. If the radome material is a lossless dielectric with $\varepsilon_r = 9$ and $\mu_r = 1$, choose its thickness d such that the radome appears transparent to the radar beam. Structural integrity requires d to be greater than 2.3 cm.



Cont.

Example 8-1: Radar Radome

From transmission lines, since Media 1 and 3 are the same (air), no net reflection will occur at $z = -d$ if the radome thickness is an integer multiple of $\lambda_2/2$



At 10 GHz, the wavelength in air is $\lambda_0 = c/f = 3$ cm, while in the radome material it is

$$\lambda_2 = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{3 \text{ cm}}{3} = 1 \text{ cm}.$$

Hence, by choosing $d = 5\lambda_2/2 = 2.5$ cm, the radome will be non-reflecting and structurally stable.

Example 8-2: Yellow Light Incident upon a Glass Surface

A beam of yellow light with wavelength 0.6 μm is normally incident in air upon a glass surface. If the surface is situated in the plane $z = 0$ and the relative permittivity of glass is 2.25, determine:

(a) the locations of the electric field maxima in medium 1 (air),

(b) the standing-wave ratio, and

(c) the fraction of the incident power transmitted into the glass medium.

Solution: **(a)** We begin by determining the values of η_1, η_2, and Γ:

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 120\pi \ (\Omega),$$

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{1}{\sqrt{\varepsilon_r}} \simeq \frac{120\pi}{\sqrt{2.25}} = 80\pi \ (\Omega),$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{80\pi - 120\pi}{80\pi + 120\pi} = -0.2.$$

Cont.

Example 8-2 cont.

Hence, $|\Gamma| = 0.2$ and $\theta_r = \pi$. From Eq. (8.16), the electric-field magnitude is maximum at

$$l_{max} = \frac{\theta_r \lambda_1}{4\pi} + n\frac{\lambda_1}{2}$$
$$= \frac{\lambda_1}{4} + n\,\frac{\lambda_1}{2} \qquad (n = 0, 1, 2, \ldots)$$

with $\lambda_1 = 0.6\ \mu$m.

(b)

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.2}{1 - 0.2} = 1.5.$$

(c) The fraction of the incident power transmitted into the glass medium is equal to the ratio of the transmitted power density, given by Eq. (8.20), to the incident power density, $S^i_{av} = |E^i_0|^2/2\eta_1$:

$$\frac{S_{av_2}}{S^i_{av}} = \tau^2\,\frac{|E^i_0|^2}{2\eta_2} \Bigg/ \left[\frac{|E^i_0|^2}{2\eta_1}\right] = \tau^2\,\frac{\eta_1}{\eta_2}\,.$$

In view of Eq. (8.21),

$$\frac{S_{av_2}}{S^i_{av}} = 1 - |\Gamma|^2 = 1 - (0.2)^2 = 0.96, \text{ or } 96\%.$$

Example 8-3: Normal Incidence on a Metal Surface

A 1-GHz x-polarized plane wave traveling in the $+z$-direction is incident from air upon a copper surface. The air-to-copper interface is at $z = 0$ and copper has $\varepsilon_r = 1$, $\mu_r = 1$, and $\sigma = 5.8 \times 10^7$ S/m. If the amplitude of the electric field of the incident wave is 12 (mV/m), obtain expressions for the instantaneous electric and magnetic fields in the air medium. Assume the metal surface to be several skin depths deep.

Solution: In medium 1 (air), $\alpha = 0$,

$$\beta = k_1 = \frac{\omega}{c} = \frac{2\pi \times 10^9}{3 \times 10^8} = \frac{20\pi}{3} \qquad \text{(rad/m)},$$

$$\eta_1 = \eta_0 = 377\ (\Omega), \qquad \lambda = \frac{2\pi}{k_1} = 0.3\ \text{m}.$$

At $f = 1$ GHz, copper is an excellent conductor because

$$\frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega \varepsilon_r \varepsilon_0} = \frac{5.8 \times 10^7}{2\pi \times 10^9 \times (10^{-9}/36\pi)} = 1 \times 10^9 \gg 1.$$

Cont.

Example 8-3 (cont.)

$$\eta_{c_2} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}$$

$$= (1+j)\left[\frac{\pi \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}\right]^{1/2}$$

$$= 8.25(1+j) \qquad (\mathrm{m}\Omega).$$

Since η_{c_2} is so small compared to $\eta_0 = 377$ (Ω) for air, the copper surface acts, in effect, like a short circuit. Hence,

$$\Gamma = \frac{\eta_{c_2} - \eta_0}{\eta_{c_2} + \eta_0} \simeq -1.$$

Cont.

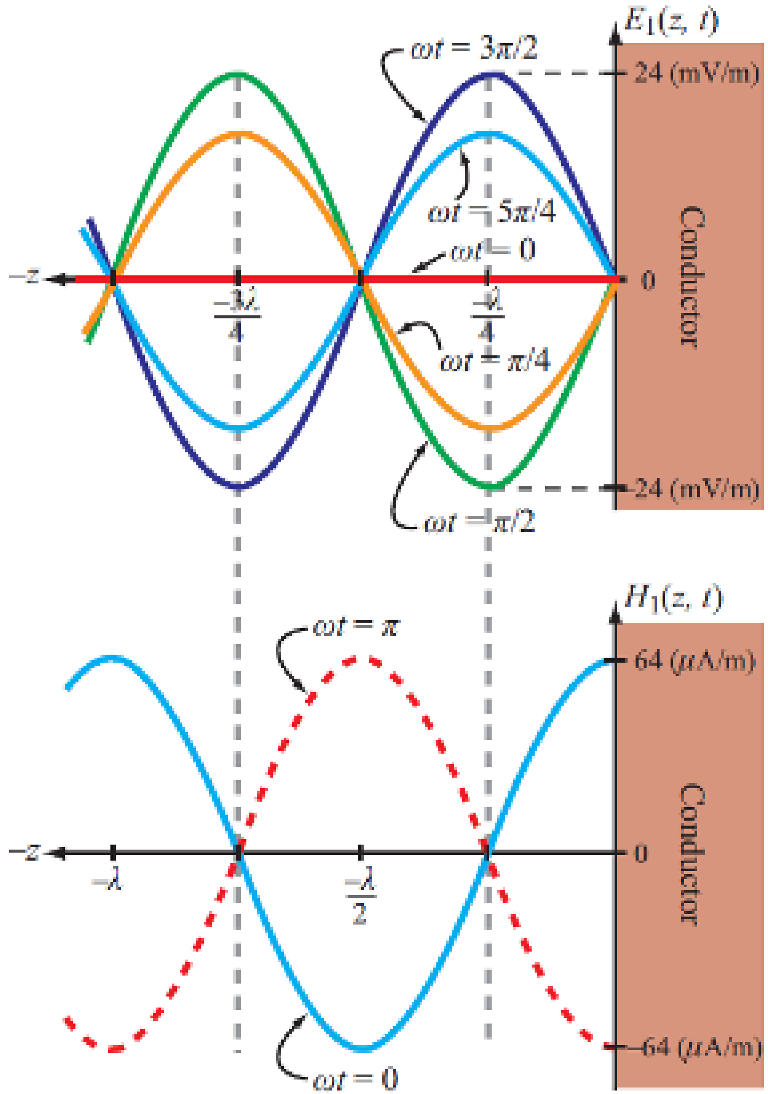
Example 8-3 (cont.)

Upon setting $\Gamma = -1$

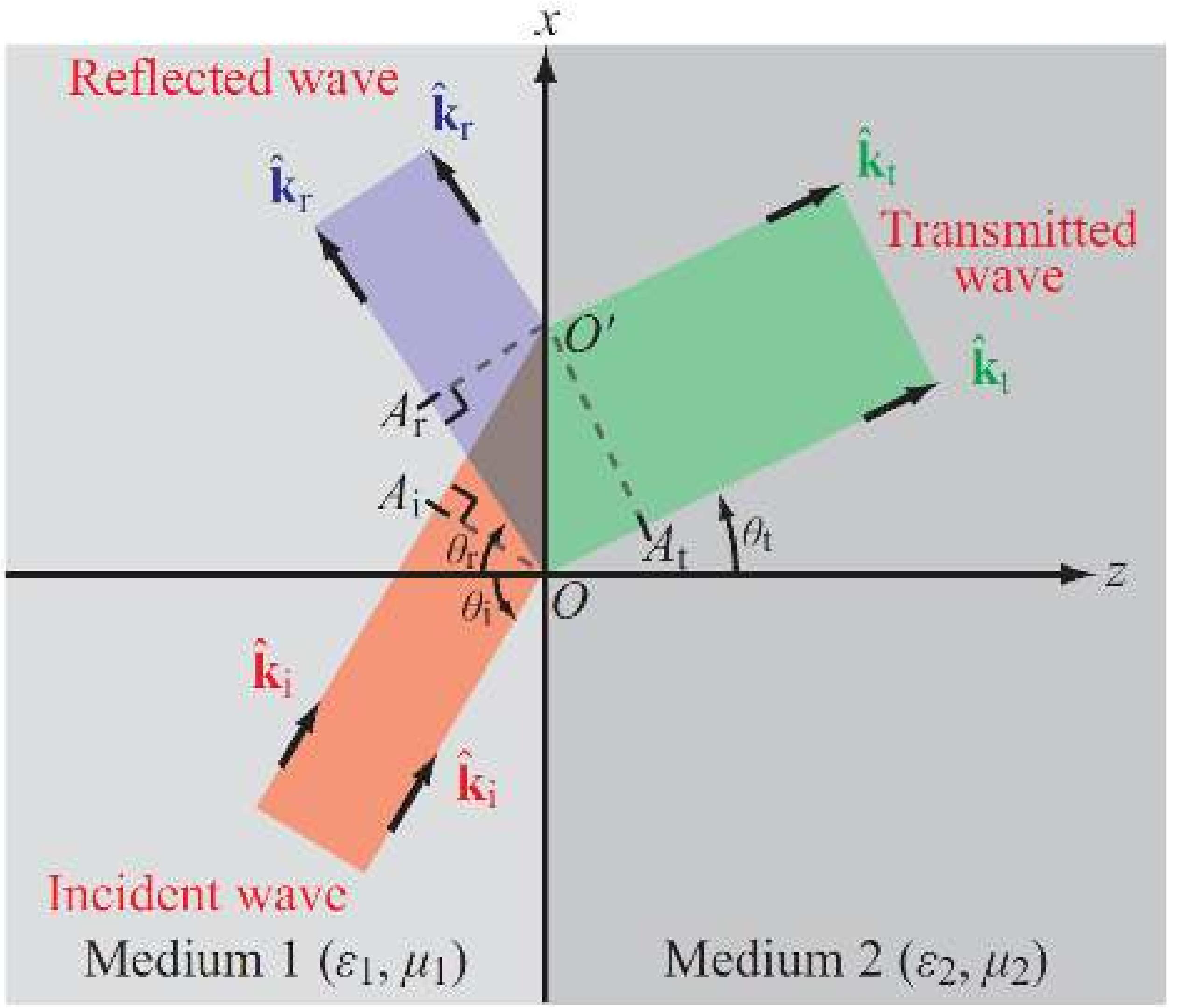
$$\widetilde{\mathbf{E}}_1(z) = \hat{\mathbf{x}} E_0^{\mathrm{i}} (e^{-jk_1 z} - e^{jk_1 z})$$
$$= -\hat{\mathbf{x}} j 2 E_0^{\mathrm{i}} \sin k_1 z,$$
$$\widetilde{\mathbf{H}}_1(z) = \hat{\mathbf{y}} \frac{E_0^{\mathrm{i}}}{\eta_1} (e^{-jk_1 z} + e^{jk_1 z})$$
$$= \hat{\mathbf{y}} 2 \frac{E_0^{\mathrm{i}}}{\eta_1} \cos k_1 z.$$

With $E_0^{\mathrm{i}} = 12$ (mV/m), the instantaneous fields associated with these phasors are

$$\mathbf{E}_1(z,t) = \mathfrak{Re}[\widetilde{\mathbf{E}}_1(z)\, e^{j\omega t}]$$
$$= \hat{\mathbf{x}}\, 2 E_0^{\mathrm{i}} \sin k_1 z \sin \omega t$$
$$= \hat{\mathbf{x}}\, 24 \sin(20\pi z/3) \sin(2\pi \times 10^9 t) \quad \text{(mV/m)},$$
$$\mathbf{H}_1(z,t) = \mathfrak{Re}[\widetilde{\mathbf{H}}_1(z)\, e^{j\omega t}]$$
$$= \hat{\mathbf{y}}\, 2 \frac{E_0^{\mathrm{i}}}{\eta_1} \cos k_1 z \cos \omega t$$
$$= \hat{\mathbf{y}}\, 64 \cos(20\pi z/3) \cos(2\pi \times 10^9 t) \quad (\mu\text{A/m}).$$



Snell's Laws



x
Reflected wave
$\hat{\mathbf{k}}_r$
$\hat{\mathbf{k}}_r$
$\hat{\mathbf{k}}_t$
Transmitted wave
O'
$\hat{\mathbf{k}}_t$
A_r
A_i
θ_r
A_t
θ_t
z
θ_i
O
$\hat{\mathbf{k}}_i$
$\hat{\mathbf{k}}_i$
Incident wave
Medium 1 (ε_1, μ_1)
Medium 2 (ε_2, μ_2)

Angles of Incidence, Reflection & Refraction

$$\theta_i = \theta_r \quad \text{(Snell's law of reflection)}, \qquad (8.28a)$$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}} \quad \text{(Snell's law of refraction)}. \qquad (8.28b)$$

Snell's law of reflection *states that the angle of reflection equals the angle of incidence, and* ***Snell's law of refraction*** *provides a relation between* $\sin\theta_t$ *and* $\sin\theta_i$ *in terms of the ratio of the phase velocities.*

Nonmagnetic Media

Index of refraction *n*:

$$\boxed{n = \frac{c}{u_p} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\mu_r\varepsilon_r}\,. \qquad (8.29)}$$

In view of Eq. (8.29), Eq. (8.28b) may be rewritten as

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} = \sqrt{\frac{\mu_{r_1}\varepsilon_{r_1}}{\mu_{r_2}\varepsilon_{r_2}}}\,. \qquad (8.30)$$

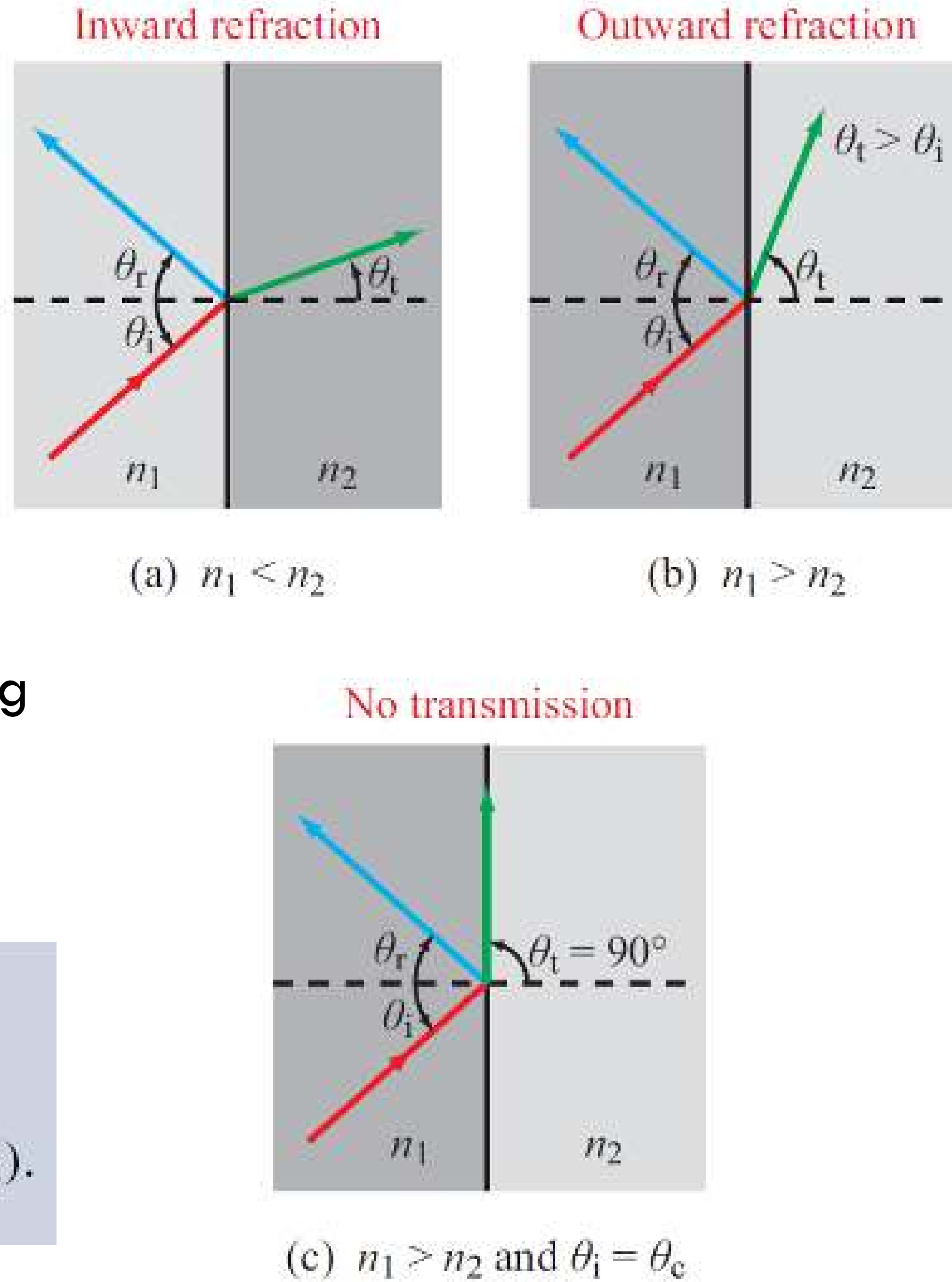
For nonmagnetic materials, $\mu_{r_1} = \mu_{r_2} = 1$, in which case

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} = \sqrt{\frac{\varepsilon_{r_1}}{\varepsilon_{r_2}}} = \frac{\eta_2}{\eta_1} \qquad (\text{for } \mu_1 = \mu_2). \qquad (8.31)}$$

Refraction

When the refraction angle is 90 degrees, the corresponding incidence angle is called the critical angle.

$$\sin\theta_c = \frac{n_2}{n_1}\sin\theta_t\Big|_{\theta_t=\pi/2} = \frac{n_2}{n_1}$$
$$= \sqrt{\frac{\varepsilon_{r_2}}{\varepsilon_{r_1}}} \qquad (\text{for } \mu_1 = \mu_2).$$



Example 8-4: Light Beam Passing through a Slab

A dielectric slab with index of refraction n_2 is surrounded by a medium with index of refraction n_1, as shown in Fig. 8-11. If $\theta_i < \theta_c$, show that the emerging beam is parallel to the incident beam.

Solution: At the slab's upper surface, Snell's law gives

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1$$

and, similarly, at the slab's lower surface,

$$\sin\theta_3 = \frac{n_2}{n_3}\sin\theta_2 = \frac{n_2}{n_1}\sin\theta_2.$$

Substituting Eq. (8.33) into Eq. (8.34) gives

$$\sin\theta_3 = \left(\frac{n_2}{n_1}\right)\left(\frac{n_1}{n_2}\right)\sin\theta_1 = \sin\theta_1.$$

Hence, $\theta_3 = \theta_1$. The slab displaces the beam's position, but the beam's direction remains unchanged.

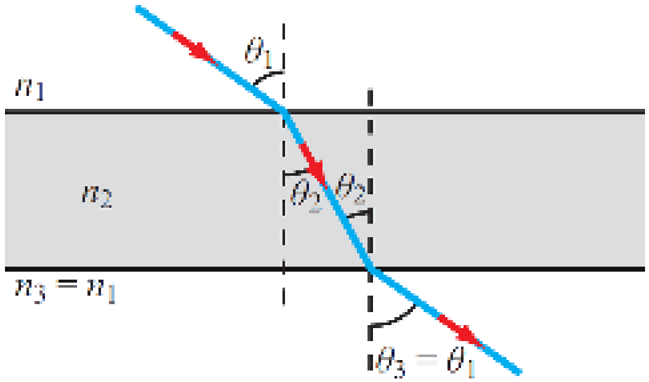
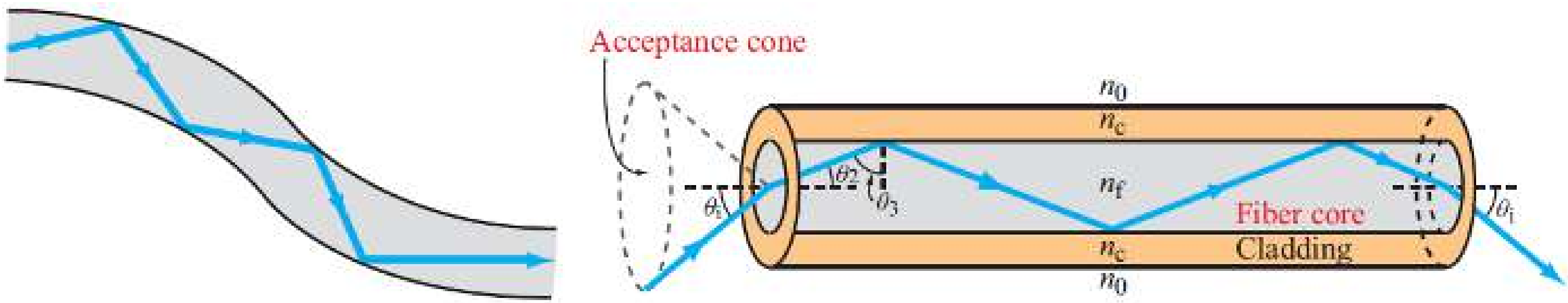


Figure 8-11: The exit angle θ_3 is equal to the incidence angle θ_1 if the dielectric slab has parallel boundaries and is surrounded by media with the same index of refraction on both sides (Example 8-4).

Optical Fiber



(a) Optical fiber (b) Successive internal reflections

Figure 8-12: Waves can be guided along optical fibers as long as the reflection angles exceed the critical angle for total internal reflection.

The ***acceptance angle*** θ_a is defined as the maximum value of θ_i for which the condition of total internal reflection remains satisfied:

$$\boxed{\sin\theta_a = \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2}. \qquad (8.40)}$$

Modal Dispersion

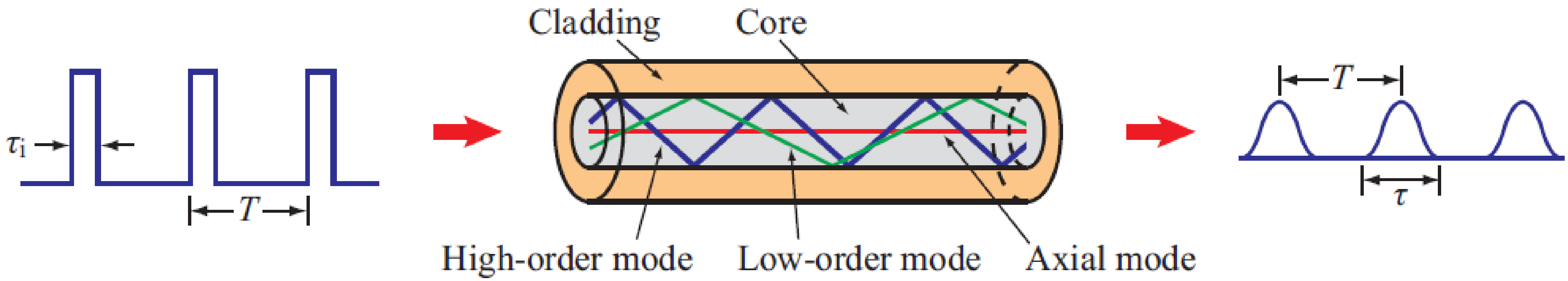
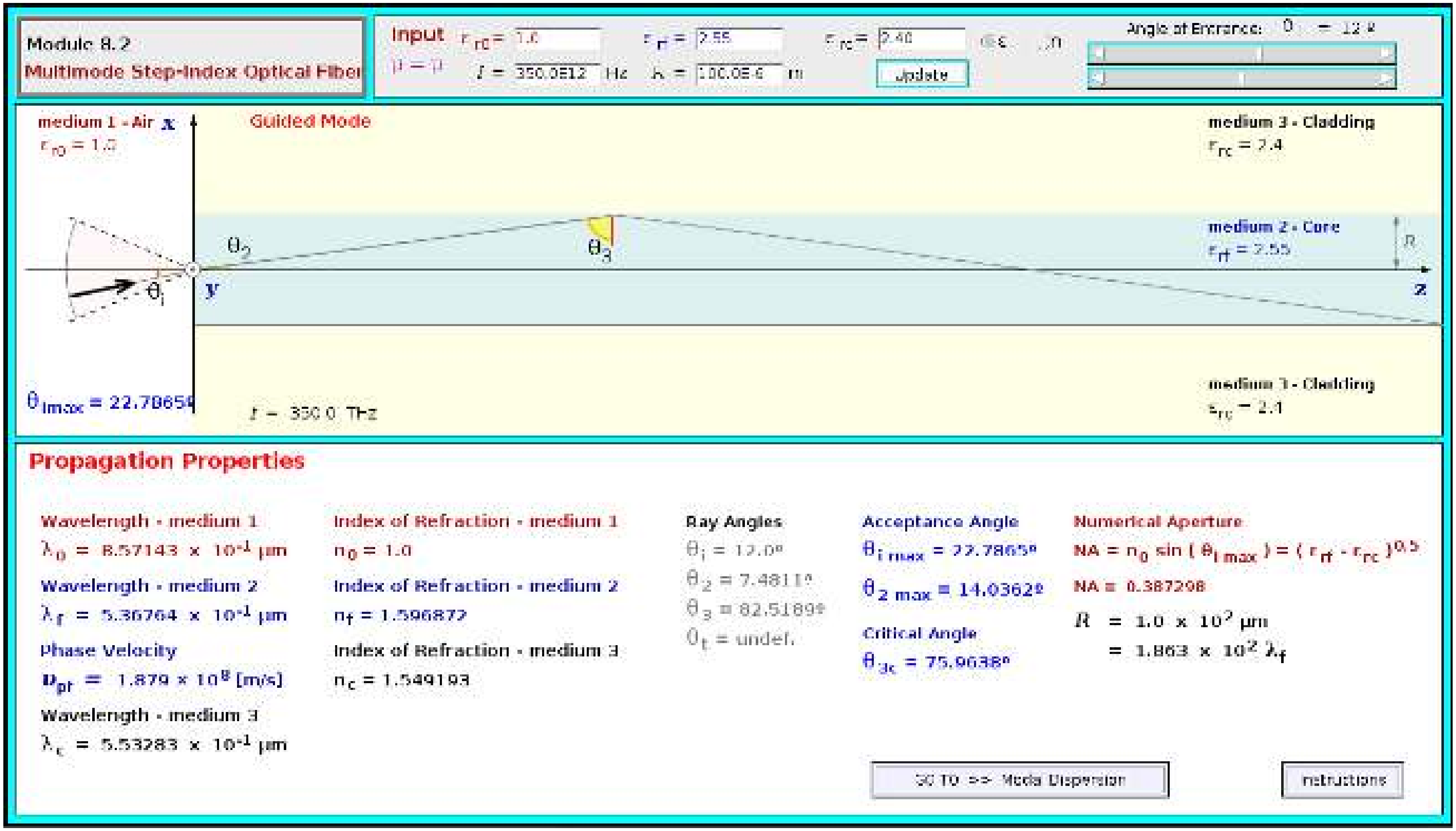


Figure 8-13: Distortion of rectangular pulses caused by modal dispersion in optical fibers.

Highest data rate:

$$f_p = \frac{1}{T} = \frac{1}{2\tau} = \frac{cn_c}{2ln_f(n_f - n_c)} \qquad \text{(bits/s)}$$



Module 8.2
Multimode Step-Index Optical Fiber
Input
Update
Angle of Entrance:
medium 1 - Air
Guided Mode
medium 3 - Cladding
medium 2 - Core
medium 3 - Cladding
θimax = 22.7865°
f = 350.0 THz
Propagation Properties
Wavelength - medium 1
λ0 = 8.57143 x 10-1 µm
Wavelength - medium 2
λf = 5.36764 x 10-1 µm
Phase Velocity
vpf = 1.879 x 10^8 [m/s]
Wavelength - medium 3
λc = 5.53283 x 10-1 µm
Index of Refraction - medium 1
n0 = 1.0
Index of Refraction - medium 2
nf = 1.596872
Index of Refraction - medium 3
nc = 1.549193
Ray Angles
θi = 12.0°
θ2 = 7.4811°
θ3 = 82.5189°
θt = undef.
Acceptance Angle
θi max = 22.7865°
θ2 max = 14.0362°
Critical Angle
θ3c = 75.9638°
Numerical Aperture
NA = n0 sin (θi max) = (εrf - εrc)^0.5
NA = 0.387298
R = 1.0 x 10^2 µm
= 1.863 x 10^2 λf
GO TO >> Mode Dispersion
Instructions

Example 8-5: Transmission Data Rate on Optical Fibers

A 1-km-long optical fiber (in air) is made of a fiber core with an index of refraction of 1.52 and a cladding with an index of refraction of 1.49. Determine

(a) the acceptance angle θ_a, and

(b) the maximum usable data rate of signals that can be transmitted through the fiber.

Solution: **(a)** From Eq. (8.40),

$$\sin\theta_a = \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2} = [(1.52)^2 - (1.49)^2]^{1/2} = 0.3,$$

which corresponds to $\theta_a = 17.5°$.

(b) From Eq. (8.45),

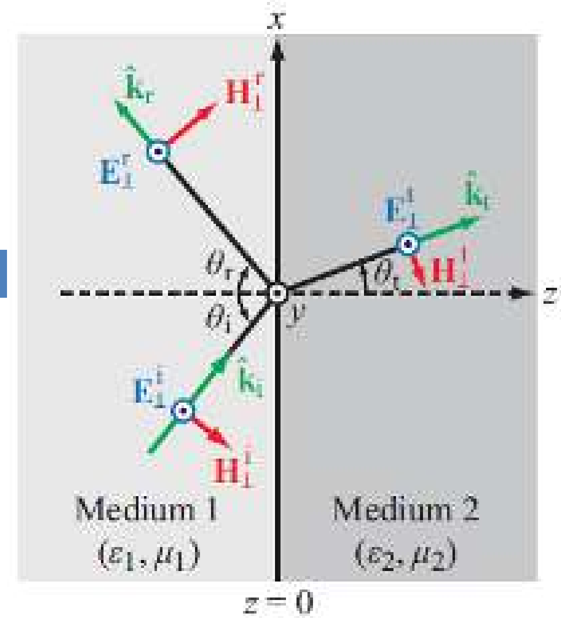
$$f_p = \frac{cn_c}{2ln_f(n_f - n_c)}$$

$$= \frac{3 \times 10^8 \times 1.49}{2 \times 10^3 \times 1.52(1.52 - 1.49)} = 4.9 \text{ (Mb/s)}.$$

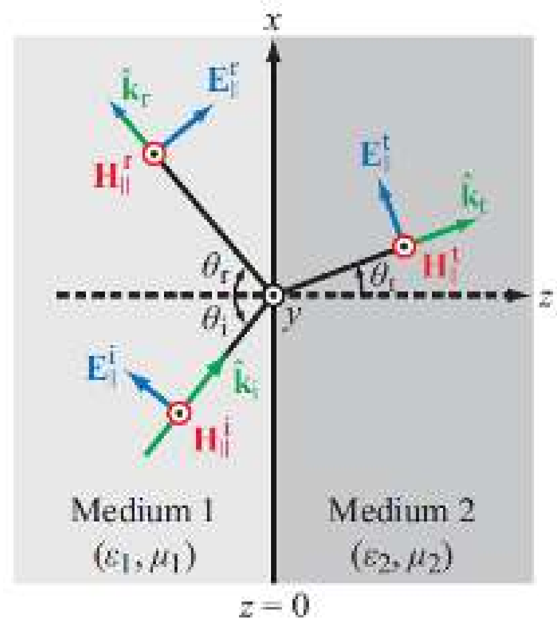
Oblique Incidence

***Plane of incidence** is defined as the plane containing the normal to the boundary and the direction of propagation of the incident wave (x-y plane in the figure).*

A wave of arbitrary polarization may be described as the superposition of two orthogonally polarized waves, one with its electric field parallel to the plane of incidence (***parallel polarization***) and the other with its electric field perpendicular to the plane of incidence (***perpendicular polarization***).



(a) Perpendicular polarization



(b) Parallel polarization

Perpendicular Polarization

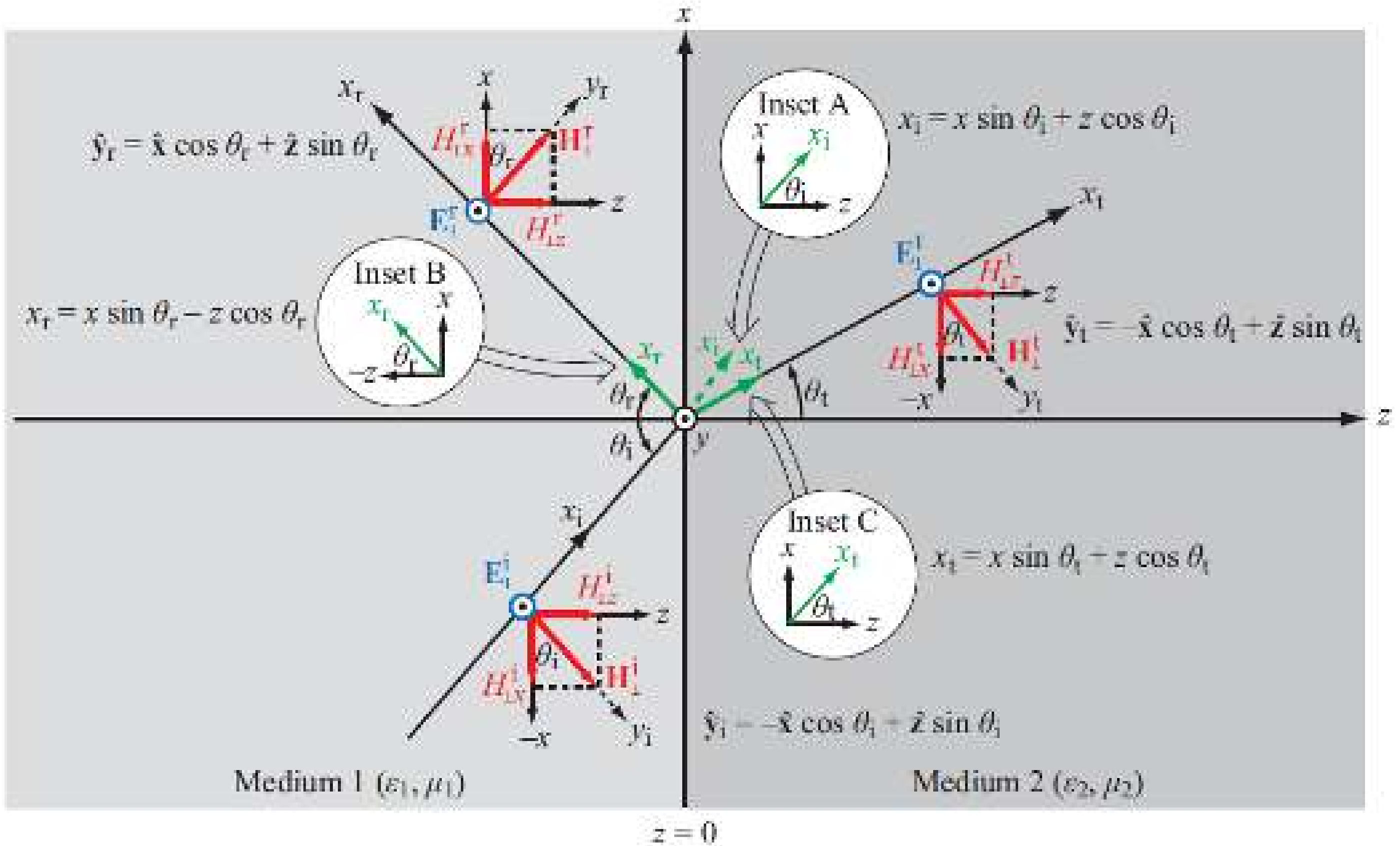


Figure 8-15: Perpendicularly polarized plane wave incident at an angle θ_i upon a planar boundary.

Incident Wave

For a wave incident along the propagation direction $\hat{\mathbf{x}}_i$

$$\widetilde{\mathbf{E}}^i_\perp = \hat{\mathbf{y}} E^i_{\perp 0} e^{-jk_1 x_i},$$

$$\widetilde{\mathbf{H}}^i_\perp = \hat{\mathbf{y}}_i \frac{E^i_{\perp 0}}{\eta_1} e^{-jk_1 x_i},$$

From the figure:

$$x_i = x\sin\theta_i + z\cos\theta_i,$$

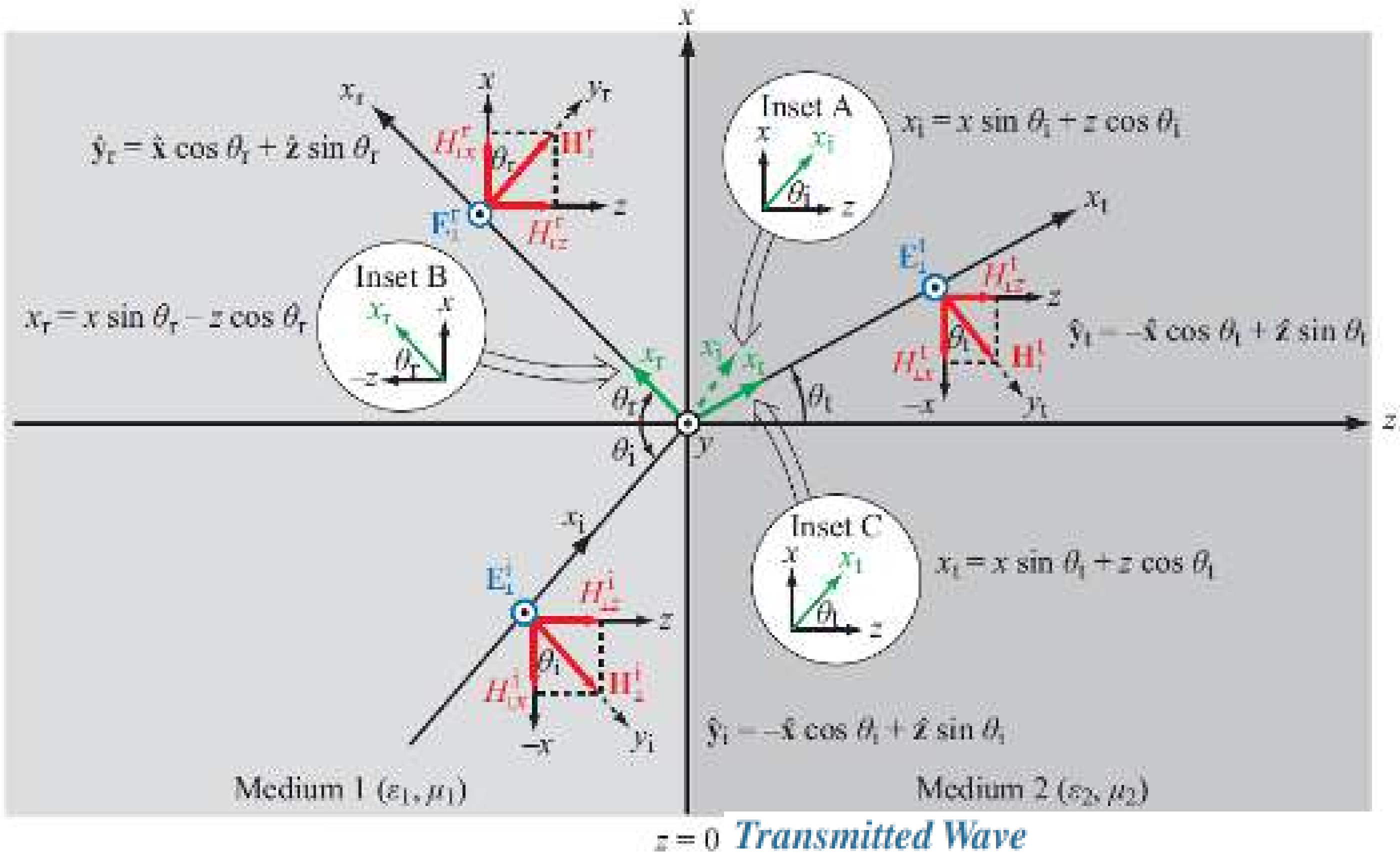
$$\hat{\mathbf{y}}_i = -\hat{\mathbf{x}}\cos\theta_i + \hat{\mathbf{z}}\sin\theta_i.$$

Hence:

$$\widetilde{\mathbf{E}}^i_\perp = \hat{\mathbf{y}} E^i_{\perp 0} e^{-jk_1(x\sin\theta_i + z\cos\theta_i)},$$

$$\widetilde{\mathbf{H}}^i_\perp = (-\hat{\mathbf{x}}\cos\theta_i + \hat{\mathbf{z}}\sin\theta_i) \times \frac{E^i_{\perp 0}}{\eta_1} e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}.$$

Perpendicular Polarization



Reflected Wave

$$\widetilde{\mathbf{E}}^{\mathrm{r}}_{\perp} = \hat{\mathbf{y}} E^{\mathrm{r}}_{\perp 0} e^{-jk_1 x_{\mathrm{r}}}$$

$$= \hat{\mathbf{y}} E^{\mathrm{r}}_{\perp 0} e^{-jk_1 (x \sin\theta_{\mathrm{r}} - z\cos\theta_{\mathrm{r}})},$$

$$\widetilde{\mathbf{H}}^{\mathrm{r}}_{\perp} = \hat{\mathbf{y}}_{\mathrm{r}} \frac{E^{\mathrm{r}}_{\perp 0}}{\eta_1} e^{-jk_1 x_{\mathrm{r}}}$$

$$= (\hat{\mathbf{x}}\cos\theta_{\mathrm{r}} + \hat{\mathbf{z}}\sin\theta_{\mathrm{r}}) \times \frac{E^{\mathrm{r}}_{\perp 0}}{\eta_1} e^{-jk_1(x\sin\theta_{\mathrm{r}} - z\cos\theta_{\mathrm{r}})}$$

Transmitted Wave

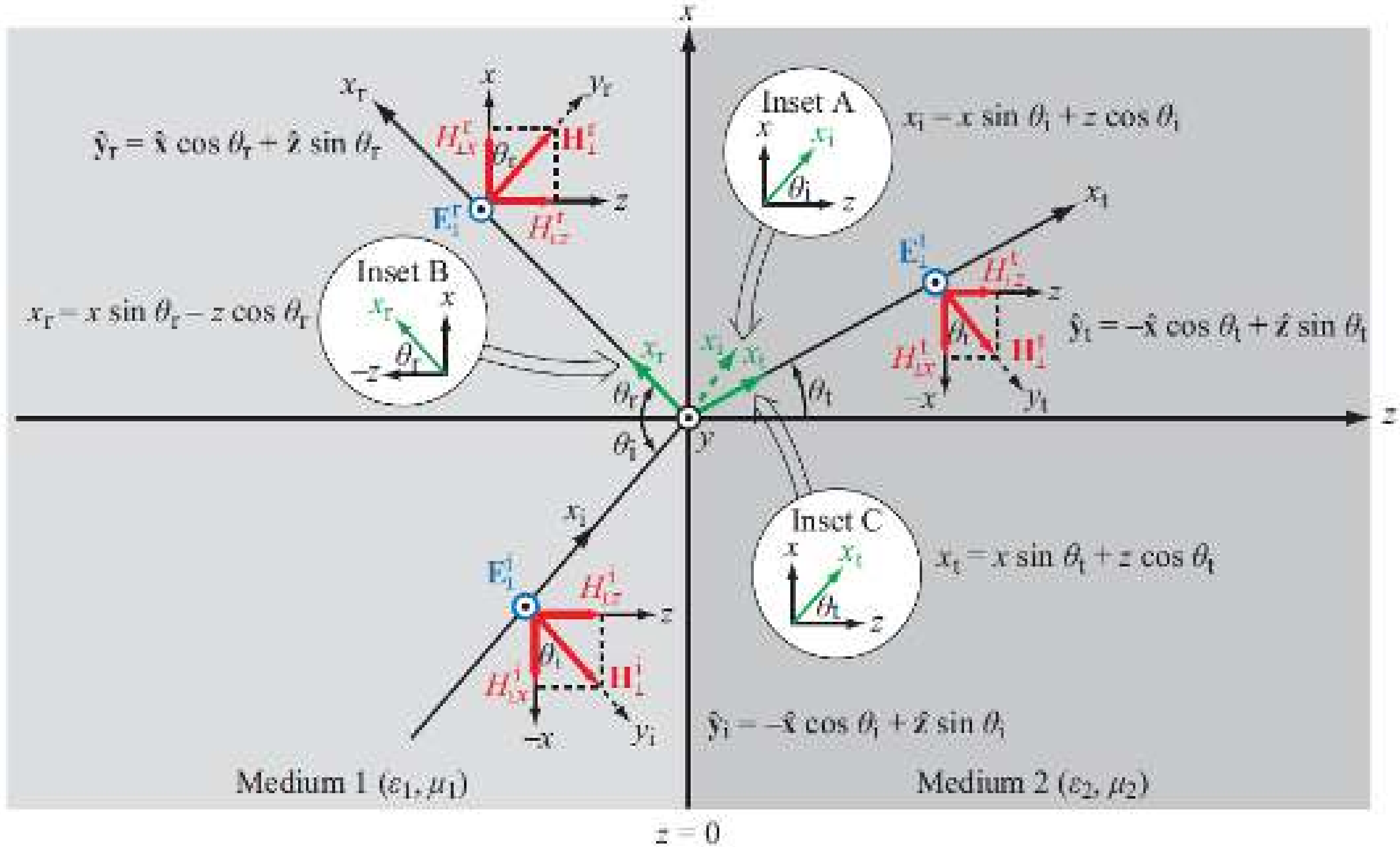
$$\widetilde{\mathbf{E}}^{\mathrm{t}}_{\perp} = \hat{\mathbf{y}} E^{\mathrm{t}}_{\perp 0} e^{-jk_2 x_{\mathrm{t}}}$$

$$= \hat{\mathbf{y}} E^{\mathrm{t}}_{\perp 0} e^{-jk_2 (x \sin\theta_{\mathrm{t}} + z\cos\theta_{\mathrm{t}})},$$

$$\widetilde{\mathbf{H}}^{\mathrm{t}}_{\perp} = \hat{\mathbf{y}}_{\mathrm{t}} \frac{E^{\mathrm{t}}_{\perp 0}}{\eta_2} e^{-jk_2 x_{\mathrm{t}}}$$

$$= (-\hat{\mathbf{x}}\cos\theta_{\mathrm{t}} + \hat{\mathbf{z}}\sin\theta_{\mathrm{t}}) \times \frac{E^{\mathrm{t}}_{\perp 0}}{\eta_2} e^{-jk_2(x\sin\theta_{\mathrm{t}} + z\cos\theta_{\mathrm{t}})}$$

Perpendicular Polarization



Applying Boundary Conditions

Tangential *E* continuous

$$(\widetilde{E}^{i}_{\perp y} + \widetilde{E}^{r}_{\perp y})\Big|_{z=0} = \widetilde{E}^{t}_{\perp y}\Big|_{z=0}$$

$$E^{i}_{\perp 0}e^{-jk_1 x\sin\theta_i} + E^{r}_{\perp 0}e^{-jk_1 x\sin\theta_r} = E^{t}_{\perp 0}e^{-jk_2 x\sin\theta_t}$$

Tangential *H* continuous

$$(\widetilde{H}^{i}_{\perp x} + \widetilde{H}^{r}_{\perp x})\Big|_{z=0} = \widetilde{H}^{t}_{\perp x}\Big|_{z=0}.$$

or

$$-\frac{E^{i}_{\perp 0}}{\eta_1}\cos\theta_i\, e^{-jk_1 x\sin\theta_i} + \frac{E^{r}_{\perp 0}}{\eta_1}\cos\theta_r\, e^{-jk_1 x\sin\theta_r} = -\frac{E^{t}_{\perp 0}}{\eta_2}\cos\theta_t\, e^{-jk_2 x\sin\theta_t}.$$

Solution of Boundary Equations

1. Exponents have to be equal for all values of *x*. Hence,

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$$

2. Consequently, remaining terms become

$$E^i_{\perp 0} + E^r_{\perp 0} = E^t_{\perp 0},$$

$$\frac{\cos\theta_i}{\eta_1}(-E^i_{\perp 0} + E^r_{\perp 0}) = -\frac{\cos\theta_t}{\eta_2}\, E^t_{\perp 0}$$

3. Solution gives expressions for reflection and transmission coefficients:

$$\Gamma_\perp = \frac{E^r_{\perp 0}}{E^i_{\perp 0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t},$$

$$\tau_\perp = \frac{E^t_{\perp 0}}{E^i_{\perp 0}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}.$$

$$\tau_\perp = 1 + \Gamma_\perp.$$

Example 8-6: Wave Incident Obliquely on a Soil Surface

Using the coordinate system of Fig. 8-15, a plane wave radiated by a distant antenna is incident in air upon a plane soil surface located at $z = 0$. The electric field of the incident wave is given by

$$\mathbf{E}^{\mathrm{i}} = \hat{\mathbf{y}}100\cos(\omega t - \pi x - 1.73\pi z) \qquad \text{(V/m)}, \qquad (8.61)$$

and the soil medium may be assumed to be a lossless dielectric with a relative permittivity of 4.

(a) Determine k_1, k_2, and the incidence angle θ_{i}.

(b) Obtain expressions for the total electric fields in air and in the soil.

(c) Determine the average power density carried by the wave traveling in soil.

Cont.

Example 8-6 (cont.)

Solution: (a) We begin by converting Eq. (8.61) into phasor form, akin to the expression given by Eq. (8.46a):

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = \hat{\mathbf{y}}100e^{-j\pi x - j1.73\pi z} = \hat{\mathbf{y}}100e^{-jk_1x_{\mathrm{i}}} \quad \text{(V/m)}, \tag{8.62}$$

where x_{i} is the axis along which the wave is traveling, and

$$k_1x_{\mathrm{i}} = \pi x + 1.73\pi z. \tag{8.63}$$

Using Eq. (8.47a), we have

$$k_1x_{\mathrm{i}} = k_1x\sin\theta_{\mathrm{i}} + k_1z\cos\theta_{\mathrm{i}}. \tag{8.64}$$

Hence,

$$k_1\sin\theta_{\mathrm{i}} = \pi,$$
$$k_1\cos\theta_{\mathrm{i}} = 1.73\pi,$$

which together give

$$k_1 = \sqrt{\pi^2 + (1.73\pi)^2} = 2\pi \quad \text{(rad/m)},$$
$$\theta_{\mathrm{i}} = \tan^{-1}\left(\frac{\pi}{1.73\pi}\right) = 30^\circ.$$

Cont.

Example 8-6 (cont.)

The wavelength in medium 1 (air) is

$$\lambda_1 = \frac{2\pi}{k_1} = 1 \text{ m},$$

and the wavelength in medium 2 (soil) is

$$\lambda_2 = \frac{\lambda_1}{\sqrt{\varepsilon_{r_2}}} = \frac{1}{\sqrt{4}} = 0.5 \text{ m}.$$

The corresponding wave number in medium 2 is

$$k_2 = \frac{2\pi}{\lambda_2} = 4\pi \qquad \text{(rad/m)}.$$

(b) Given that $\theta_i = 30°$, the transmission angle θ_t is obtained with the help of Eq. (8.56):

$$\sin\theta_t = \frac{k_1}{k_2}\sin\theta_i = \frac{2\pi}{4\pi}\sin 30° = 0.25$$

or

$$\theta_t = 14.5°.$$

With $\varepsilon_1 = \varepsilon_0$ and $\varepsilon_2 = \varepsilon_{r_2}\varepsilon_0 = 4\varepsilon_0$, the reflection and transmission coefficients for perpendicular polarization are determined with the help of Eqs. (8.59) and (8.60),

$$\Gamma_\perp = \frac{\cos\theta_i - \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}}{\cos\theta_i + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_i}} = -0.38,$$

$$\tau_\perp = 1 + \Gamma_\perp = 0.62.$$

Cont.

Example 8-6 (cont.)

Using Eqs. (8.48a) and (8.49a) with $E^{\mathrm{i}}_{\perp 0} = 100$ V/m and $\theta_{\mathrm{i}} = \theta_{\mathrm{r}}$, the total electric field in medium 1 is

$$\begin{aligned}
\widetilde{\mathbf{E}}^1_{\perp} &= \widetilde{\mathbf{E}}^{\mathrm{i}}_{\perp} + \widetilde{\mathbf{E}}^{\mathrm{r}}_{\perp} \\
&= \hat{\mathbf{y}} E^{\mathrm{i}}_{\perp 0} e^{-jk_1(x \sin\theta_{\mathrm{i}} + z\cos\theta_{\mathrm{i}})} \\
&\quad + \hat{\mathbf{y}} \Gamma E^{\mathrm{i}}_{\perp 0} e^{-jk_1(x \sin\theta_{\mathrm{i}} - z\cos\theta_{\mathrm{i}})} \\
&= \hat{\mathbf{y}} 100 e^{-j(\pi x + 1.73\pi z)} - \hat{\mathbf{y}} 38 e^{-j(\pi x - 1.73\pi z)},
\end{aligned}$$

and the corresponding instantaneous electric field in medium 1 is

$$\begin{aligned}
\mathbf{E}^1_{\perp}(x, z, t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}^1_{\perp} e^{j\omega t}\right] \\
&= \hat{\mathbf{y}}[100\cos(\omega t - \pi x - 1.73\pi z) \\
&\quad - 38\cos(\omega t - \pi x + 1.73\pi z)] \qquad \text{(V/m)}.
\end{aligned}$$

Cont.

Example 8-6 (cont.)

In medium 2, using Eq. (8.49c) with $E^{\mathrm{t}}_{\perp 0} = \tau_\perp E^{\mathrm{i}}_{\perp 0}$ gives

$$\begin{aligned}\widetilde{\mathbf{E}}^{\mathrm{t}}_{\perp} &= \hat{\mathbf{y}}\tau E^{\mathrm{i}}_{\perp 0} e^{-jk_2(x\sin\theta_{\mathrm{t}} + z\cos\theta_{\mathrm{t}})} \\ &= \hat{\mathbf{y}}62e^{-j(\pi x + 3.87\pi z)}\end{aligned}$$

and, correspondingly,

$$\begin{aligned}\mathbf{E}^{\mathrm{t}}_{\perp}(x, z, t) &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}^{\mathrm{t}}_{\perp} e^{j\omega t}\right] \\ &= \hat{\mathbf{y}}62\cos(\omega t - \pi x - 3.87\pi z) \qquad \text{(V/m)}.\end{aligned}$$

(c) In medium 2, $\eta_2 = \eta_0/\sqrt{\varepsilon_{\mathrm{r}_2}} \simeq 120\pi/\sqrt{4} = 60\pi$ (Ω), and the average power density carried by the wave is

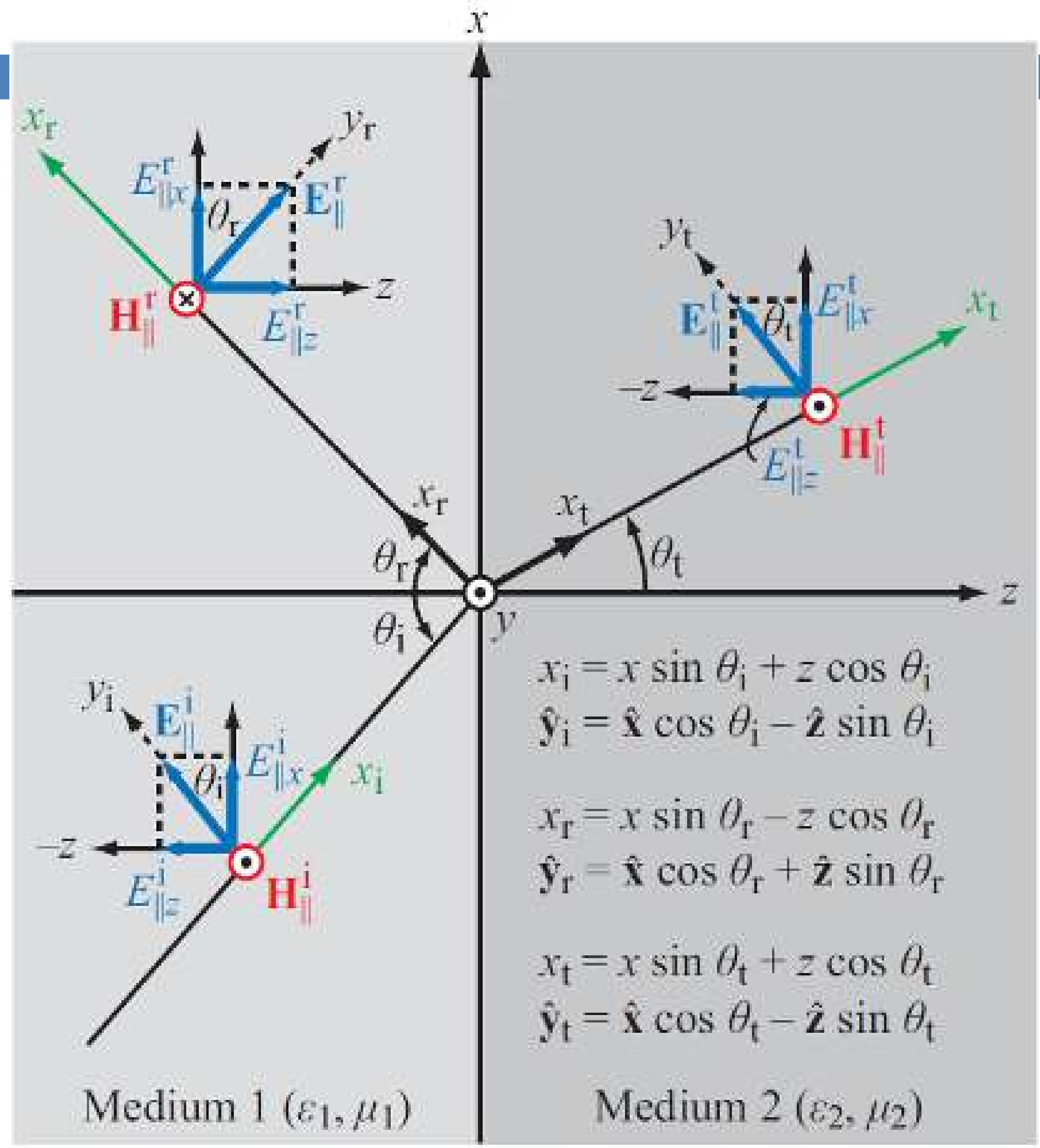
$$S^{\mathrm{t}}_{\mathrm{av}} = \frac{|E^{\mathrm{t}}_{\perp 0}|^2}{2\eta_2} = \frac{(62)^2}{2 \times 60\pi} = 10.2 \qquad (\mathrm{W/m^2}).$$

Parallel Polarization

$$\Gamma_{\parallel} = \frac{E^{\mathrm{r}}_{\parallel 0}}{E^{\mathrm{i}}_{\parallel 0}} = \frac{\eta_2 \cos\theta_{\mathrm{t}} - \eta_1 \cos\theta_{\mathrm{i}}}{\eta_2 \cos\theta_{\mathrm{t}} + \eta_1 \cos\theta_{\mathrm{i}}} ,$$

$$\tau_{\parallel} = \frac{E^{\mathrm{t}}_{\parallel 0}}{E^{\mathrm{i}}_{\parallel 0}} = \frac{2\eta_2 \cos\theta_{\mathrm{i}}}{\eta_2 \cos\theta_{\mathrm{t}} + \eta_1 \cos\theta_{\mathrm{i}}} .$$

$$\tau_{\parallel} = (1 + \Gamma_{\parallel}) \frac{\cos\theta_{\mathrm{i}}}{\cos\theta_{\mathrm{t}}} .$$



Reflection Coefficient vs. Angle

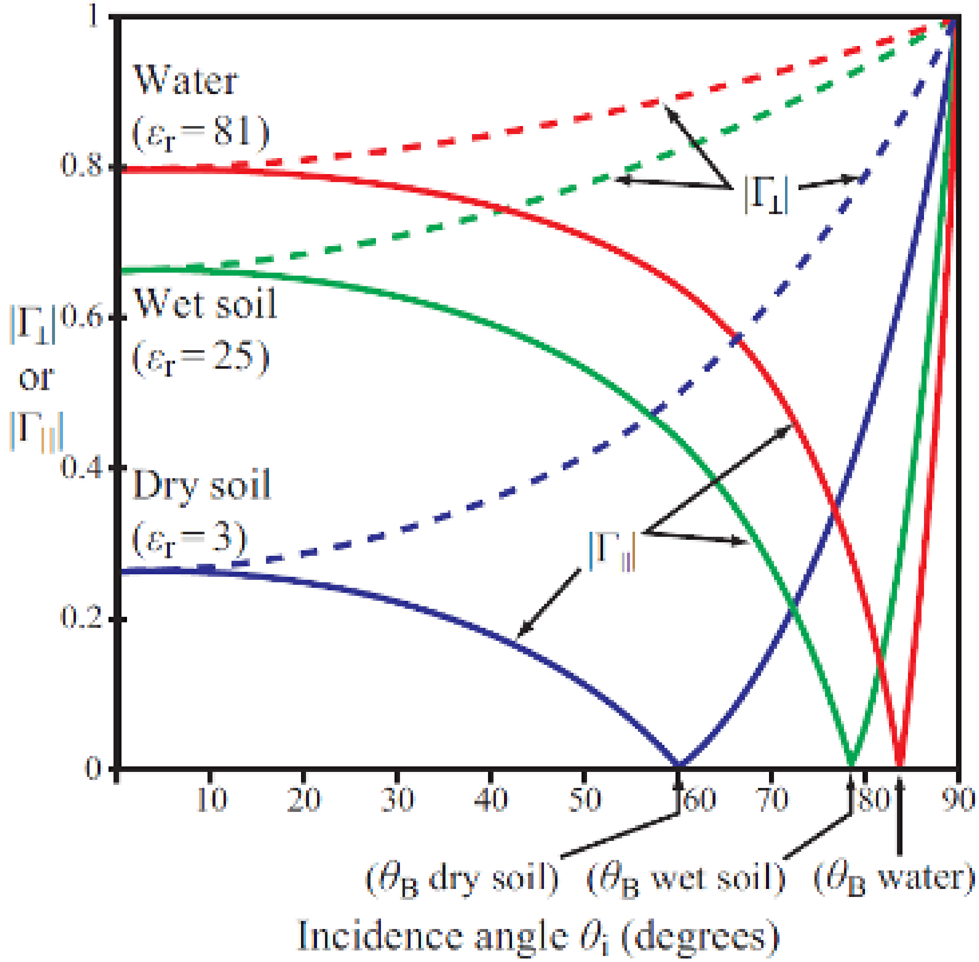


Figure 8-17: Plots for $|\Gamma_\perp|$ and $|\Gamma_\parallel|$ as a function of θ_i for a dry soil surface, a wet-soil surface, and a water surface. For each surface, $|\Gamma_\parallel| = 0$ at the Brewster angle.

Brewster Angle

The Brewster angle θ_{B} is defined as the incidence angle θ_{i} at which the Fresnel reflection coefficient $\Gamma = 0$.

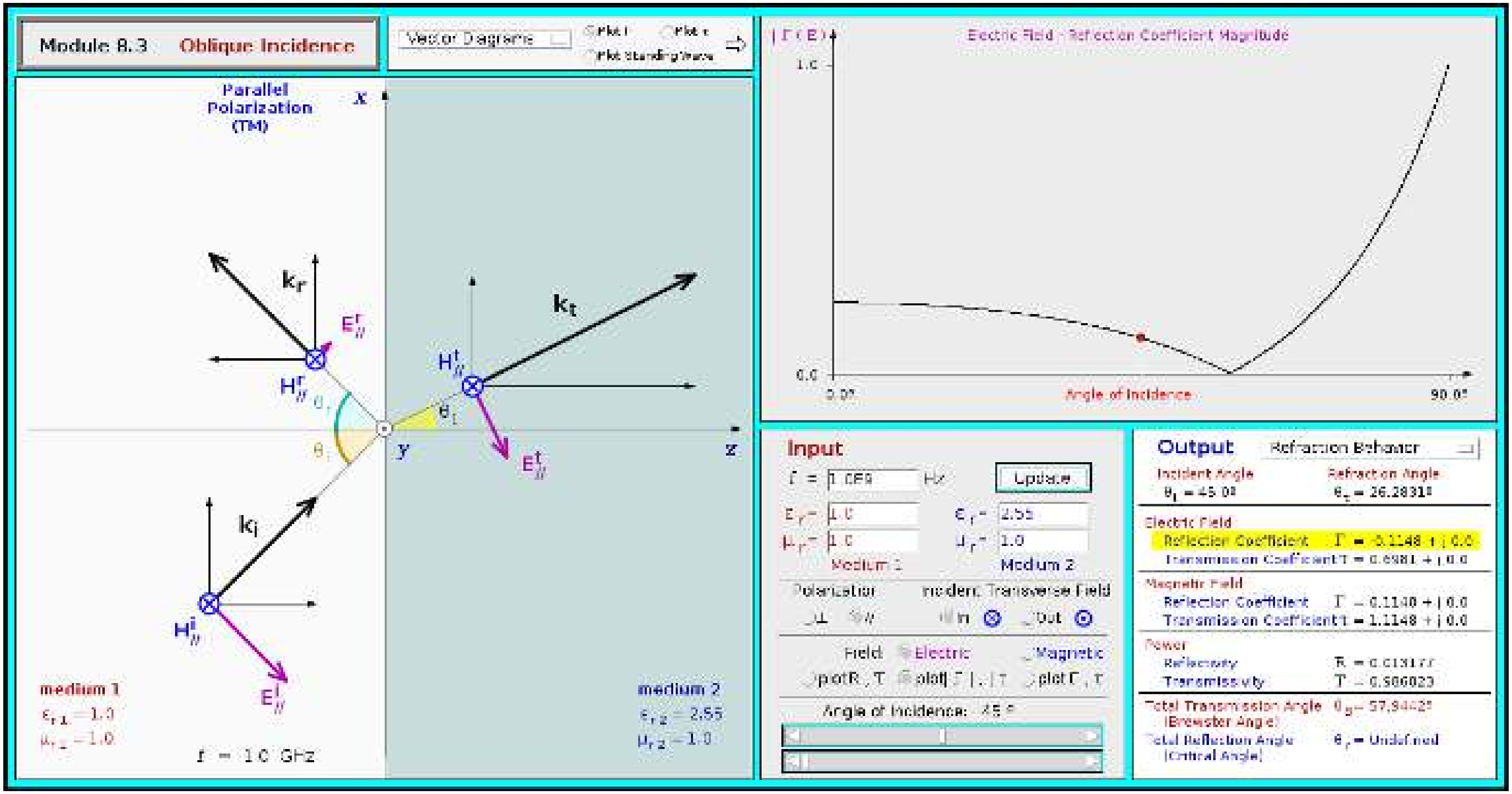
Perpendicular Polarization

$\theta_{\mathrm{B}\perp}$ ***does not exist for nonmagnetic materials.***

Parallel Polarization

$$\theta_{\mathrm{B}\parallel} = \sin^{-1}\sqrt{\frac{1}{1+(\varepsilon_1/\varepsilon_2)}}$$

$$= \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad (\text{for } \mu_1 = \mu_2).$$

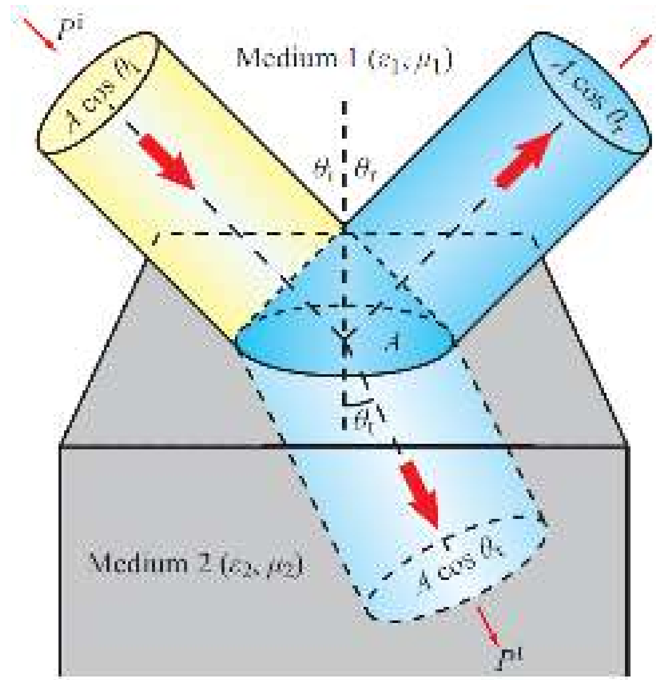

Module 8.3 Oblique Incidence
Vector Diagrams
Parallel Polarization (TM)
medium 1
medium 2
f = 1.0 GHz
Electric Field - Reflection Coefficient Magnitude
Angle of Incidence
Input
Update
Medium 1
Medium 2
Polarization
Incident Transverse Field
Field
Electric
Magnetic
Angle of Incidence: 45 °
Output
Refraction Behavior
Incident Angle
Refraction Angle
Electric Field
Reflection Coefficient
Magnetic Field
Reflection Coefficient
Power
Total Transmission Angle (Brewster Angle)
Total Reflection Angle (Critical Angle)

Power Reflectivity and Transmissivity

$$T_\perp = \frac{P^t_\perp}{P^i_\perp} = \frac{|E^t_{\perp 0}|^2}{|E^i_{\perp 0}|^2}\,\frac{\eta_1}{\eta_2}\,\frac{A\cos\theta_t}{A\cos\theta_i}$$

$$= |\tau_\perp|^2 \left(\frac{\eta_1\cos\theta_t}{\eta_2\cos\theta_i}\right), \qquad (8.79a)$$

$$T_\parallel = \frac{P^t_\parallel}{P^i_\parallel} = |\tau_\parallel|^2 \left(\frac{\eta_1\cos\theta_t}{\eta_2\cos\theta_i}\right). \qquad (8.79b)$$



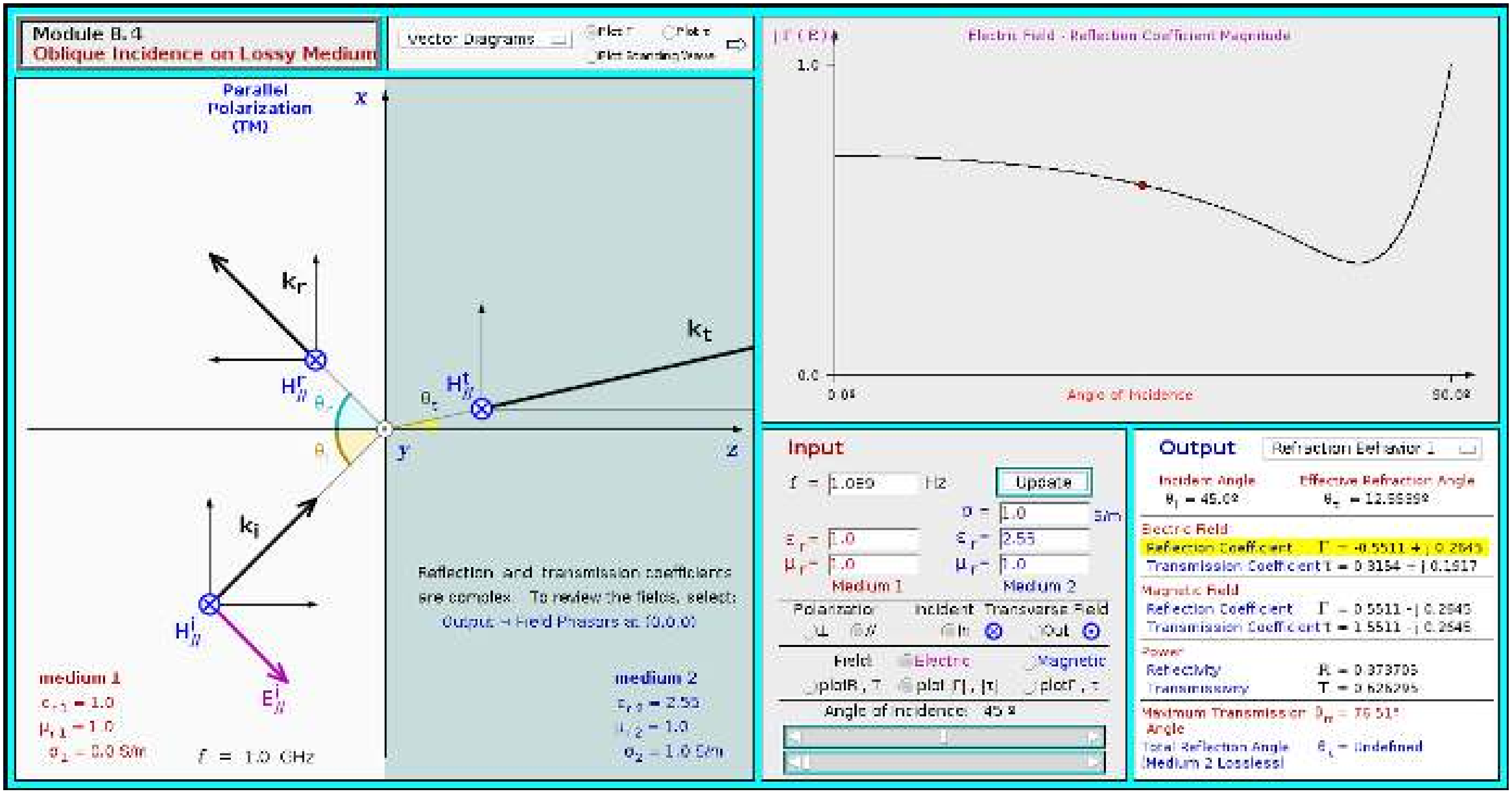
The incident, reflected, and transmitted waves do not have to obey any such laws as conservation of electric field, conservation of magnetic field, or conservation of power density, but they do have to obey the law of conservation of power.

Summary For Reflection and Transmission

Table 8-2: Expressions for Γ, τ, R, and T for wave incidence from a medium with intrinsic impedance η_1 onto a medium with intrinsic impedance η_2. Angles θ_i and θ_t are the angles of incidence and transmission, respectively.

Property	Normal Incidence $\theta_i = \theta_t = 0$	Perpendicular Polarization	Parallel Polarization
Reflection coefficient	$\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$	$\Gamma_\perp = \dfrac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$	$\Gamma_\parallel = \dfrac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$
Transmission coefficient	$\tau = \dfrac{2\eta_2}{\eta_2 + \eta_1}$	$\tau_\perp = \dfrac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$	$\tau_\parallel = \dfrac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$
Relation of Γ to τ	$\tau = 1 + \Gamma$	$\tau_\perp = 1 + \Gamma_\perp$	$\tau_\parallel = (1 + \Gamma_\parallel) \dfrac{\cos\theta_i}{\cos\theta_t}$
Reflectivity	$R = \lvert\Gamma\rvert^2$	$R_\perp = \lvert\Gamma_\perp\rvert^2$	$R_\parallel = \lvert\Gamma_\parallel\rvert^2$
Transmissivity	$T = \lvert\tau\rvert^2 \left(\dfrac{\eta_1}{\eta_2}\right)$	$T_\perp = \lvert\tau_\perp\rvert^2 \dfrac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i}$	$T_\parallel = \lvert\tau_\parallel\rvert^2 \dfrac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i}$
Relation of R to T	$T = 1 - R$	$T_\perp = 1 - R_\perp$	$T_\parallel = 1 - R_\parallel$

Notes: (1) $\sin\theta_t = \sqrt{\mu_1\varepsilon_1/\mu_2\varepsilon_2}\,\sin\theta_i$; (2) $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$; (3) $\eta_2 = \sqrt{\mu_2/\varepsilon_2}$; (4) for nonmagnetic media, $\eta_2/\eta_1 = n_1/n_2$.



Module 8.4
Oblique Incidence on Lossy Medium
Vector Diagrams
Parallel Polarization (TM)
x
y
z
k_r
k_i
k_t
$H^r_{//}$
$H^t_{//}$
$H^i_{//}$
$E^i_{//}$
medium 1
ε_{r1} = 1.0
μ_{r1} = 1.0
σ_1 = 0.0 S/m
f = 1.0 GHz
medium 2
ε_{r2} = 2.55
μ_{r2} = 1.0
σ_2 = 1.0 S/m
Reflection and transmission coefficients are complex. To review the fields, select:
Output → Field Phasors at (0,0,0)
Electric Field - Reflection Coefficient Magnitude
1.0
0.0
0.0°
90.0°
Angle of Incidence
Input
Update
Medium 1
Medium 2
Polarization
Incident Transverse Field
Field
Electric
Magnetic
Angle of Incidence: 45 °
Output
Refraction Behavior 1
Incident Angle
θ_i = 45.0°
Effective Refraction Angle
Electric Field
Reflection Coefficient
Transmission Coefficient
Magnetic Field
Reflection Coefficient
Transmission Coefficient
Power
Reflectivity
Transmissivity
Maximum Transmission Angle
Total Reflection Angle
(Medium 2 Lossless)

Tech Brief 16: Bar-Code Reader

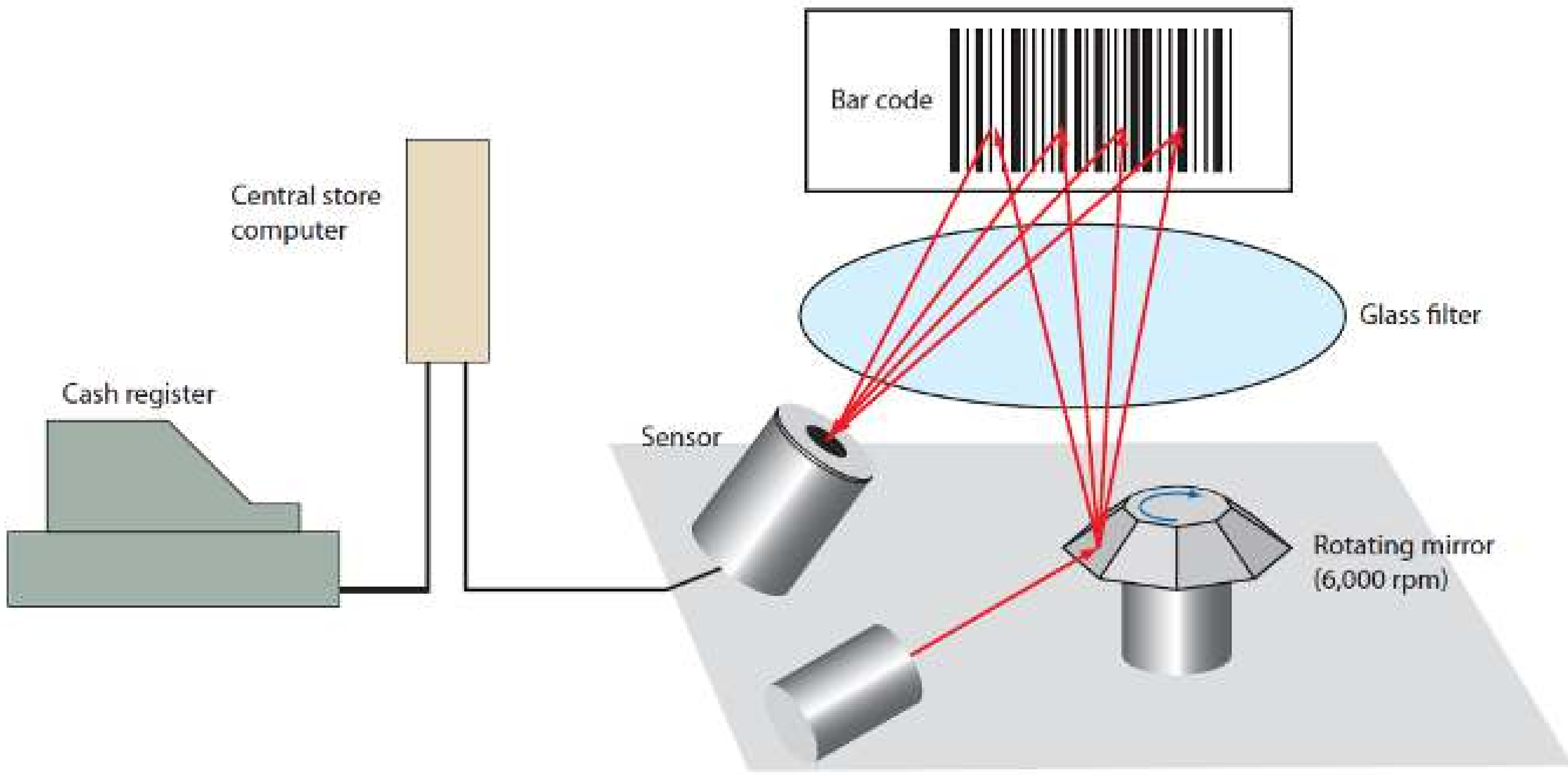


Figure TF16-1: Elements of a bar-code reader.

Tech Brief 16: Bar-Code Reader

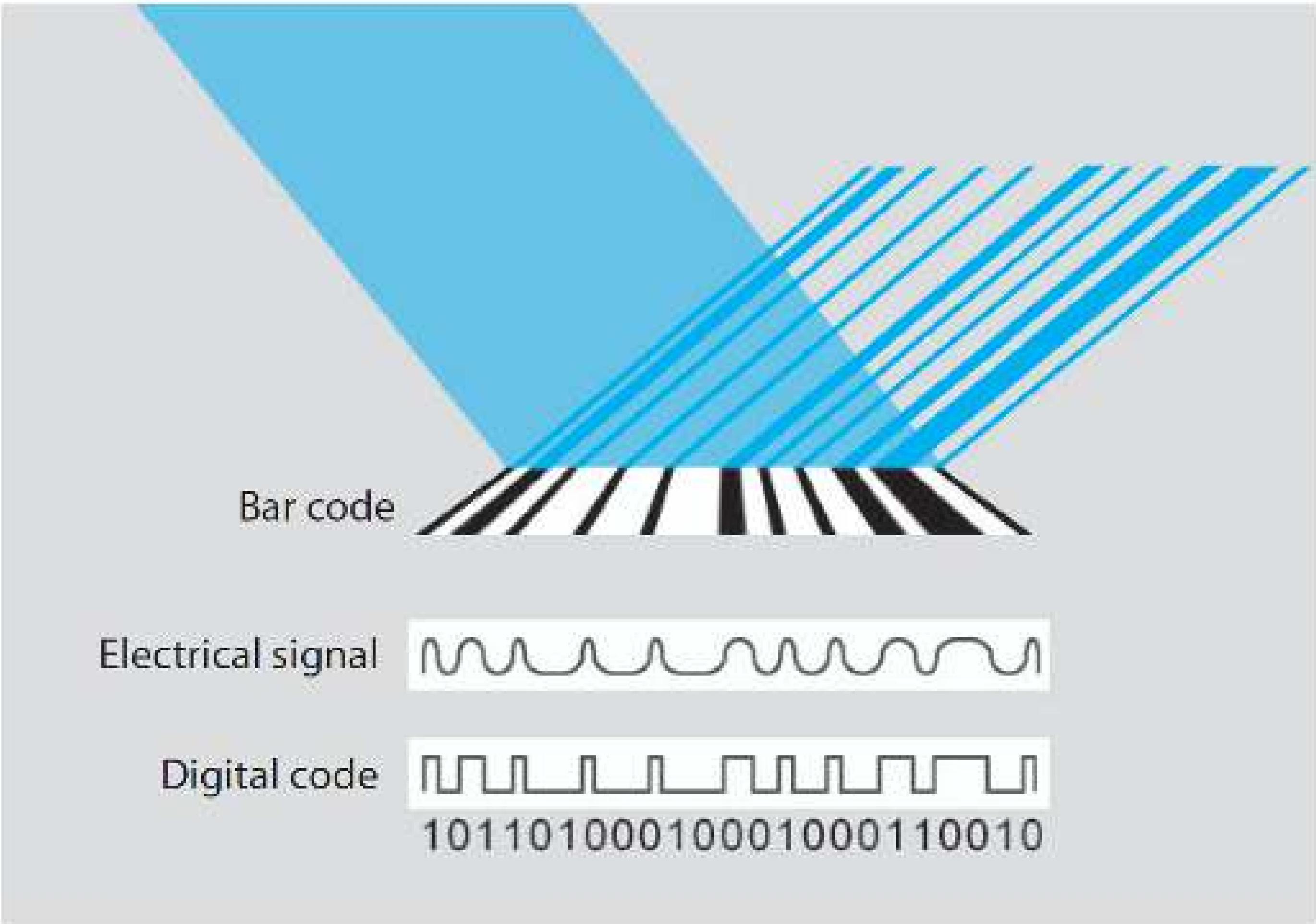


Figure TF16-2: Bar code contained in reflected laser beam.

Waveguides

- Examples of non-TEM transmission lines
- We covered the basics of wave propagation in an optical fiber earlier
- We will now examine wave propagation in a rectangular waveguide with metal surfaces
- The energy is carried by Transverse Electric or Transverse Magnetic modes, or a combination of both

If **E** *is transverse to* **k** *but* **H** *is not, we call it a* ***transverse electric*** *(TE) mode, and if* **H** *is transverse to* $\hat{\mathbf{k}}$ *but* **E** *is not, we call it a* ***transverse magnetic*** *(TM) mode.*

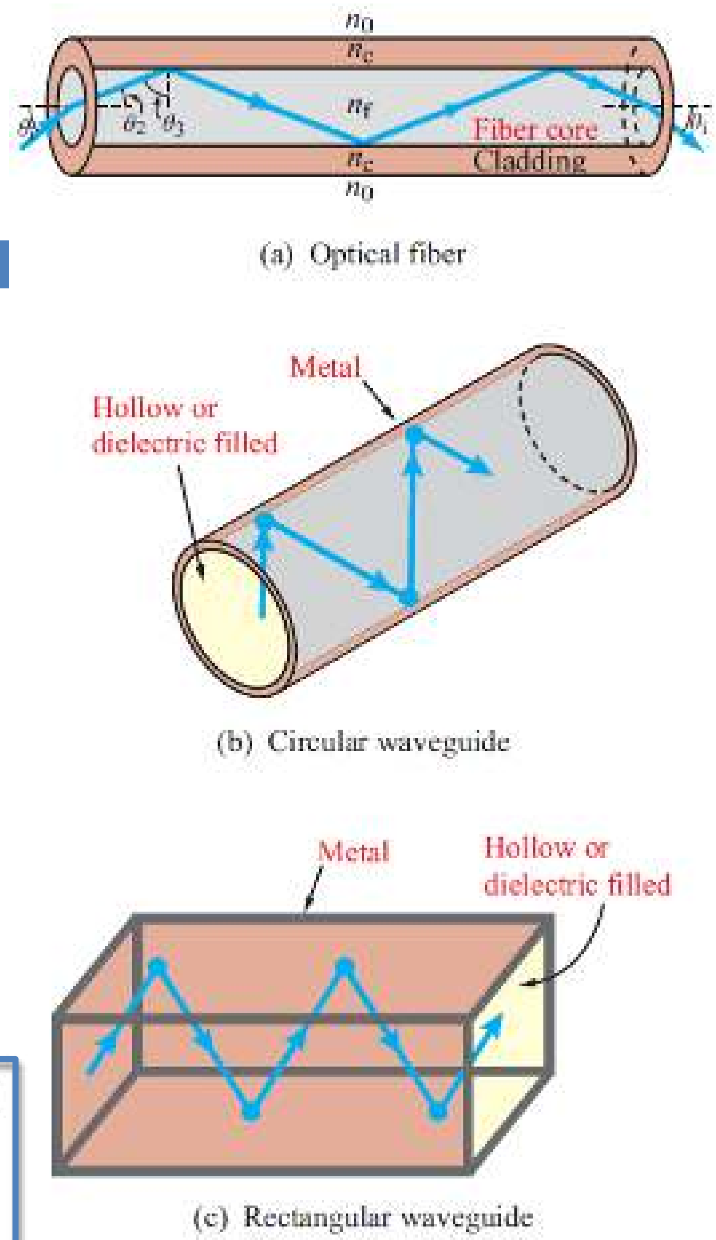


Figure 8-20: Wave travel by successive reflections in (a) an optical fiber, (b) a circular metal waveguide, and (c) a rectangular metal waveguide.

Coax-to-Waveguide Connection

An extended section of the inner conductor of a coaxial cable can serve to couple energy into a waveguide or from the waveguide

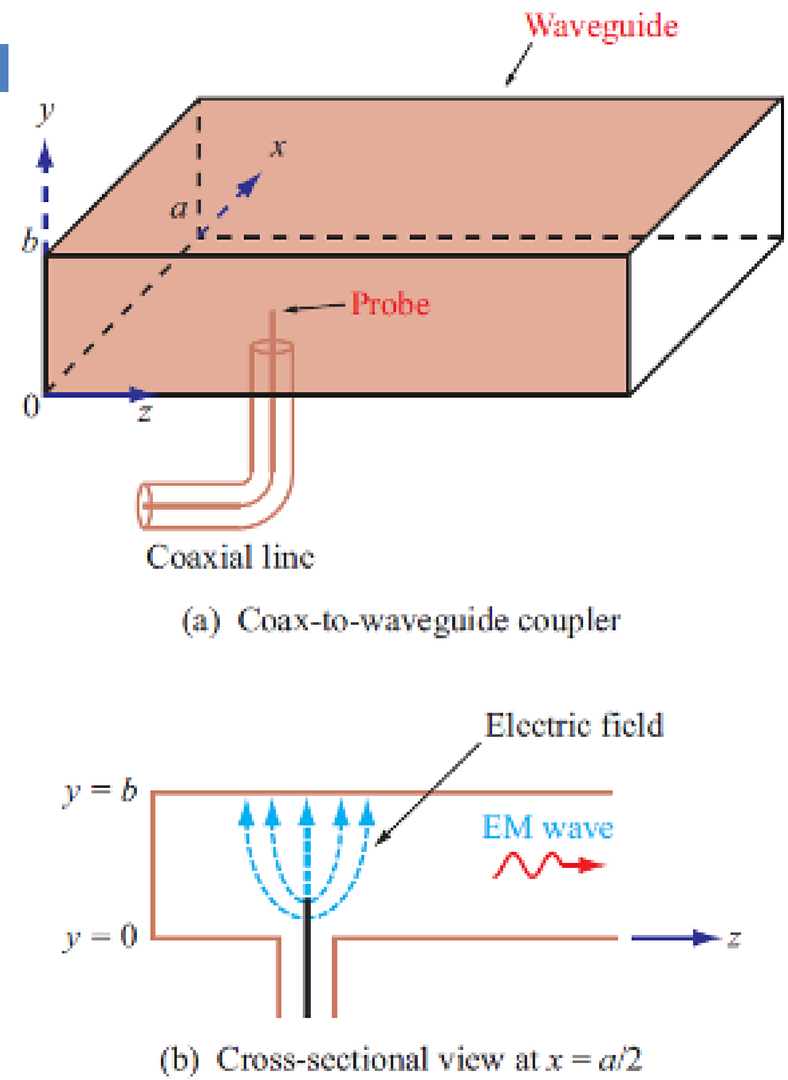


Figure 8-21: The inner conductor of a coaxial cable can excite an EM wave in the waveguide.

Transverse Magnetic (TM) Mode

*Applying Maxwell's equations to a wave propagating in the z-direction with its Hz = 0 (**for the TM Mode**) leads to:*

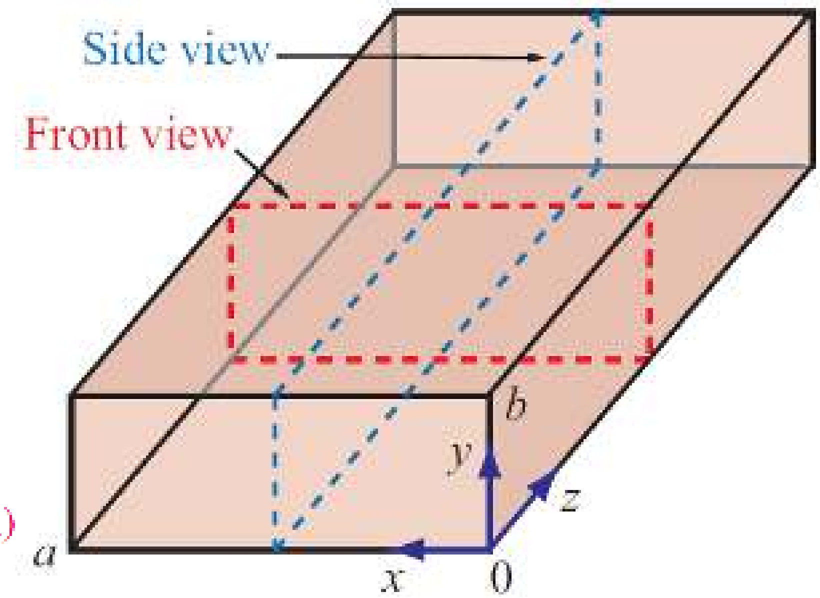
$$\widetilde{E}_z = \widetilde{e}_z e^{-j\beta z} = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z},$$

$$\widetilde{E}_x = \frac{-j\beta}{k_c^2}\left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}, \tag{8.104a}$$

$$\widetilde{E}_y = \frac{-j\beta}{k_c^2}\left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z}, \tag{8.104b}$$

$$\widetilde{H}_x = \frac{j\omega\varepsilon}{k_c^2}\left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z}, \tag{8.104c}$$

$$\widetilde{H}_y = \frac{-j\omega\varepsilon}{k_c^2}\left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z}.$$

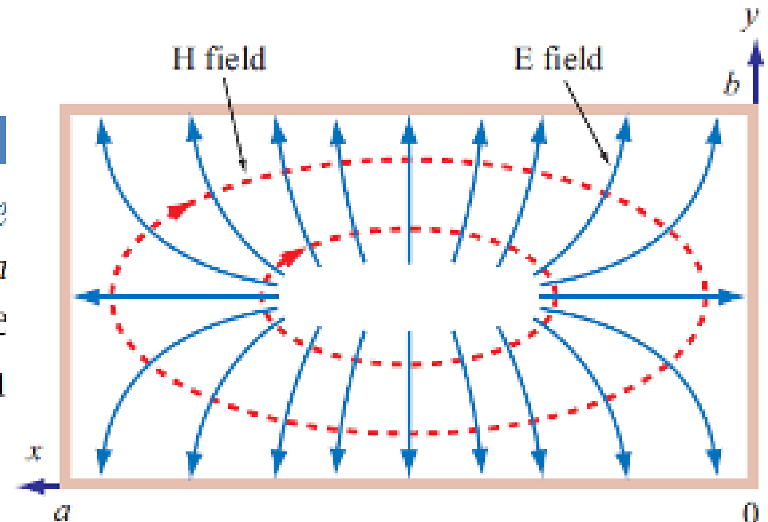


(a) Cross-sectional planes

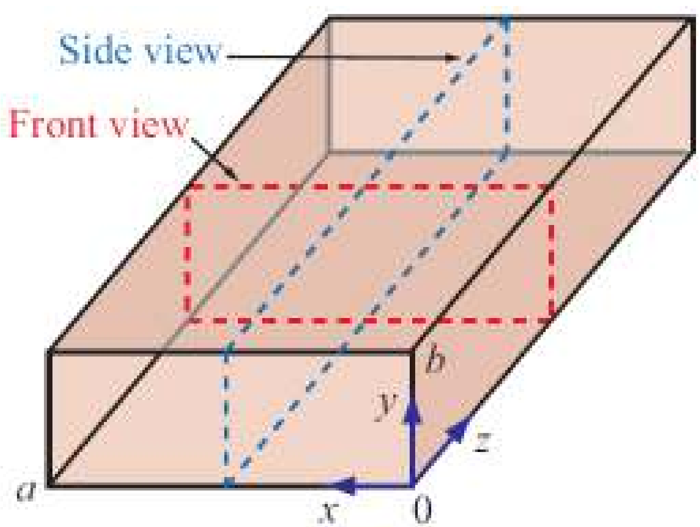
m and *n* are positive integers

Transverse Magnetic (TM) Mode

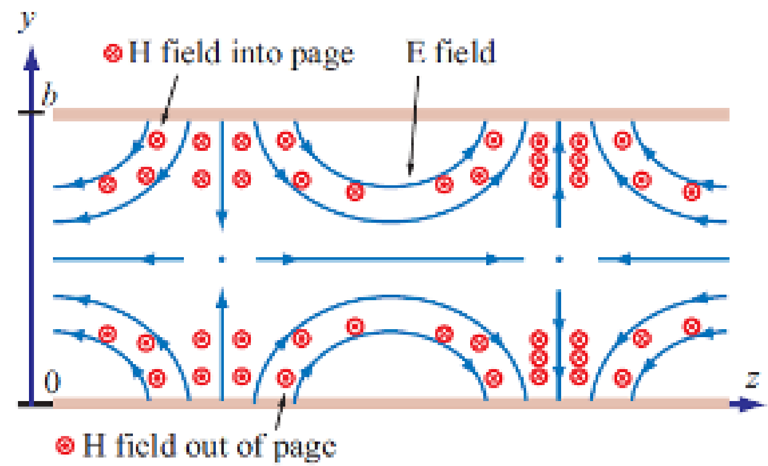
Each combination of the integers m and n represents a viable solution, or a mode, denoted TM_{mn}. Associated with each mn mode are specific field distributions for the region inside the guide. Figure 8-23 depicts the **E** and **H** field lines for the TM_{11} mode across two different cross sections of the guide.



(b) Field lines for front view



(a) Cross-sectional planes



(c) Field lines for side view

Figure 8-23: TM_{11} electric and magnetic field lines across two cross-sectional planes.

Properties of TM Modes

1. Phase constant

For a wave travelling inside the guide along the z-direction, its phase factor is e−jβz with:

$$\beta = \sqrt{k^2 - k_c^2}$$
$$= \sqrt{\omega^2 \mu \varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad \text{(TE and TM).}$$

A wave, in a given mode, can propagate through the guide only if its frequency f > fmn, as only then β = real.

2. Cutoff Frequency

Corresponding to each mode (*m*, *n*), *there is a* ***cutoff frequency fmn at which β = 0.*** ***By*** setting *β = 0 in Eq. (8.105) and then solving for f, we have*

$$f_{mn} = \frac{u_{p0}}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \text{(TE and TM)}$$

3. Phase Velocity

$$u_p = \frac{\omega}{\beta} = \frac{u_{p0}}{\sqrt{1 - (f_{mn}/f)^2}} . \qquad \text{(TE and TM)}$$

4. Wave Impedance in the Guide

$$Z_{TM} = \frac{\widetilde{E}_x}{\widetilde{H}_y} = -\frac{\widetilde{E}_y}{\widetilde{H}_x} = \frac{\beta\eta}{k} = \eta \sqrt{1 - \left(\frac{f_{mn}}{f}\right)^2}$$

Whereas properties 1 to 3 are common to both modes, property 4 is specific to TM modes.

Example 8-8: Mode Properties

A TM wave propagating in a dielectric-filled waveguide of unknown permittivity has a magnetic field with y-component given by

$$H_y = 6\cos(25\pi x)\sin(100\pi y) \times \sin(1.5\pi \times 10^{10}t - 109\pi z) \qquad \text{(mA/m)}.$$

If the guide dimensions are $a = 2b = 4$ cm, determine: (a) the mode numbers, (b) the relative permittivity of the material in the guide, (c) the phase velocity, and (d) obtain an expression for E_x.

Solution: **(a)** By comparison with the expression for $\tilde{H}_y$ given by Eq. (8.104d), we deduce that the argument of x is $(m\pi/a)$ and the argument of y is $(n\pi/b)$. Hence,

$$25\pi = \frac{m\pi}{4 \times 10^{-2}},$$

$$100\pi = \frac{n\pi}{2 \times 10^{-2}},$$

which yield $m = 1$ and $n = 2$. Therefore, the mode is TM_{12}.

Example 8-8: Mode Properties (cont.)

(b) The second sine function in the expression for H_y represents $\sin(\omega t - \beta z)$, which means that

$$\omega = 1.5\pi \times 10^{10} \text{ (rad/s), or } f = 7.5 \text{ GHz},$$

$$\beta = 109\pi \text{ (rad/m)}.$$

By rewriting Eq. (8.105) so as to obtain an expression for $\varepsilon_\mathrm{r} = \varepsilon/\varepsilon_0$ in terms of the other quantities, we have

$$\varepsilon_\mathrm{r} = \frac{c^2}{\omega^2}\left[\beta^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right],$$

where c is the velocity of light. Inserting the available values,

$$\varepsilon_\mathrm{r} = \frac{(3 \times 10^8)^2}{(1.5\pi \times 10^{10})^2} \cdot \left[(109\pi)^2 + \left(\frac{\pi}{4 \times 10^{-2}}\right)^2 + \left(\frac{2\pi}{2 \times 10^{-2}}\right)^2\right] = 9.$$

Example 8-8: Mode Properties (cont.)

(c)

$$u_{\mathrm{p}} = \frac{\omega}{\beta} = \frac{1.5\pi \times 10^{10}}{109\pi} = 1.38 \times 10^{8} \text{ m/s},$$

(d) From Eq. (8.109),

$$Z_{\mathrm{TM}} = \eta\sqrt{1 - (f_{12}/f)^2}$$

Application of Eq. (8.106) yields $f_{12} = 5.15$ GHz for the TM_{12} mode. Using that in the expression for Z_{TM}, in additon to $f = 7.5$ GHz and $\eta = \sqrt{\mu/\varepsilon} = (\sqrt{\mu_0/\varepsilon_0})/\sqrt{\varepsilon_{\mathrm{r}}} = 377/\sqrt{9} = 125.67\ \Omega$, gives

$$Z_{\mathrm{TM}} = 91.3\ \Omega.$$

Hence,

$$\begin{aligned} E_x &= Z_{\mathrm{TM}} H_y \\ &= 91.3 \times 6\cos(25\pi x)\sin(100\pi y) \\ &\quad \times \sin(1.5\pi \times 10^{10} t - 109\pi z) \qquad (\mathrm{mV/m}) \\ &= 0.55\cos(25\pi x)\sin(100\pi y) \\ &\quad \times \sin(1.5\pi \times 10^{10} t - 109\pi z) \qquad (\mathrm{V/m}). \end{aligned}$$

Transverse Electric Mode

For the TE mode with $E_z = 0$,

$$\widetilde{E}_y = \frac{-j\omega\mu}{k_c^2}\left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\, e^{-j\beta z}, \tag{8.110b}$$

$$\widetilde{H}_x = \frac{j\beta}{k_c^2}\left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\, e^{-j\beta z}, \tag{8.110c}$$

$$\widetilde{H}_y = \frac{j\beta}{k_c^2}\left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\, e^{-j\beta z}, \tag{8.110d}$$

$$\widetilde{H}_z = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\, e^{-j\beta z}, \tag{8.110e}$$

TE mode properties are the same as TM , except for wave impedance:

$$\boxed{Z_{\mathrm{TE}} = \frac{\widetilde{E}_x}{\widetilde{H}_y} = -\frac{\widetilde{E}_y}{\widetilde{H}_x} = \frac{\eta}{\sqrt{1-(f_{mn}/f)^2}}.}$$

Example 8-9: Cutoff Frequencies

For a hollow rectangular waveguide with dimensions $a = 3$ cm and $b = 2$ cm, determine the cutoff frequencies for all modes, up to 20 GHz. Over what frequency range will the guide support the propagation of a single dominant mode?

Solution: A hollow guide has $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$. Hence, $u_{p0} = 1/\sqrt{\mu_0\varepsilon_0} = c$. Application of Eq. (8.106) gives the cutoff frequencies shown in Fig. 8-24, which start at 5 GHz for the TE_{10} mode. To avoid all other modes, the frequency of operation should be restricted to the 5–7.5 GHz range.

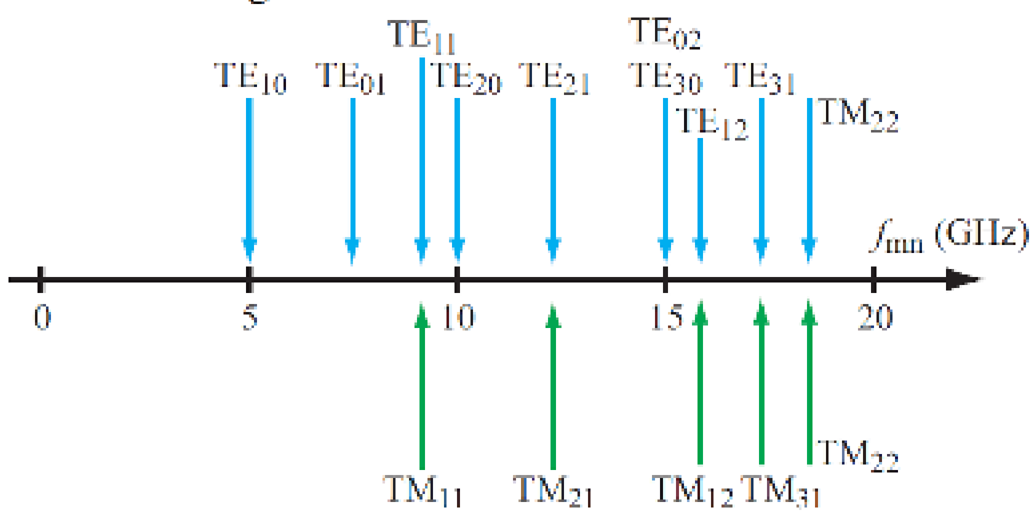


Figure 8-24: Cutoff frequencies for TE and TM modes in a hollow rectangular waveguide with $a = 3$ cm and $b = 2$ cm (Example 8-9).

Properties of TE and TM Modes

Table 8-3: Wave properties for TE and TM modes in a rectangular waveguide with dimensions $a \times b$, filled with a dielectric material with constitutive parameters ε and μ. The TEM case, shown for reference, pertains to plane-wave propagation in an unbounded medium.

Rectangular Waveguides		Plane Wave
TE Modes	**TM Modes**	**TEM Mode**
$\widetilde{E}_x = \frac{j\omega\mu}{k_c^2}\left(\frac{n\pi}{b}\right) H_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\ e^{-j\beta z}$	$\widetilde{E}_x = \frac{-j\beta}{k_c^2}\left(\frac{m\pi}{a}\right) E_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\ e^{-j\beta z}$	$\widetilde{E}_x = E_{x0} e^{-j\beta z}$
$\widetilde{E}_y = \frac{-j\omega\mu}{k_c^2}\left(\frac{m\pi}{a}\right) H_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)\ e^{-j\beta z}$	$\widetilde{E}_y = \frac{-j\beta}{k_c^2}\left(\frac{n\pi}{b}\right) E_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)\ e^{-j\beta z}$	$\widetilde{E}_y = E_{y0} e^{-j\beta z}$
$\widetilde{E}_z = 0$	$\widetilde{E}_z = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)\ e^{-j\beta z}$	$\widetilde{E}_z = 0$
$\widetilde{H}_x = -\widetilde{E}_y / Z_{\rm TE}$	$\widetilde{H}_x = -\widetilde{E}_y / Z_{\rm TM}$	$\widetilde{H}_x = -\widetilde{E}_y / \eta$
$\widetilde{H}_y = \widetilde{E}_x / Z_{\rm TE}$	$\widetilde{H}_y = \widetilde{E}_x / Z_{\rm TM}$	$\widetilde{H}_y = \widetilde{E}_x / \eta$
$\widetilde{H}_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)\ e^{-j\beta z}$	$\widetilde{H}_z = 0$	$\widetilde{H}_z = 0$
$Z_{\rm TE} = \eta / \sqrt{1 - (f_c/f)^2}$	$Z_{\rm TM} = \eta \sqrt{1 - (f_c/f)^2}$	$\eta = \sqrt{\mu/\varepsilon}$
Properties Common to TE and TM Modes		
$f_c = \frac{u_{p_0}}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$		$f_c =$ not applicable
$\beta = k\sqrt{1 - (f_c/f)^2}$		$k = \omega\sqrt{\mu\varepsilon}$
$u_p = \frac{\omega}{\beta} = u_{p_0} / \sqrt{1 - (f_c/f)^2}$		$u_{p_0} = 1/\sqrt{\mu\varepsilon}$

Propagation Velocities

1. Phase Velocity

The phase velocity is defined as the velocity of the sinusoidal pattern of the wave

$$u_p = \frac{\omega}{\beta} = \frac{u_{p0}}{\sqrt{1-(f_{mn}/f)^2}} .$$

2. Group Velocity

The velocity with which the envelope—or equivalently the wave group—travels through the medium is called the ***group velocity*** u_g. As such, u_g is the velocity of the energy carried by the wave-group, and of the information encoded in it. Depending on whether or not the propagation medium is dispersive, u_g may or may not be equal to the phase velocity u_p.

$$u_g = \frac{1}{d\beta/d\omega} = u_{p0}\sqrt{1-(f_{mn}/f)^2}$$

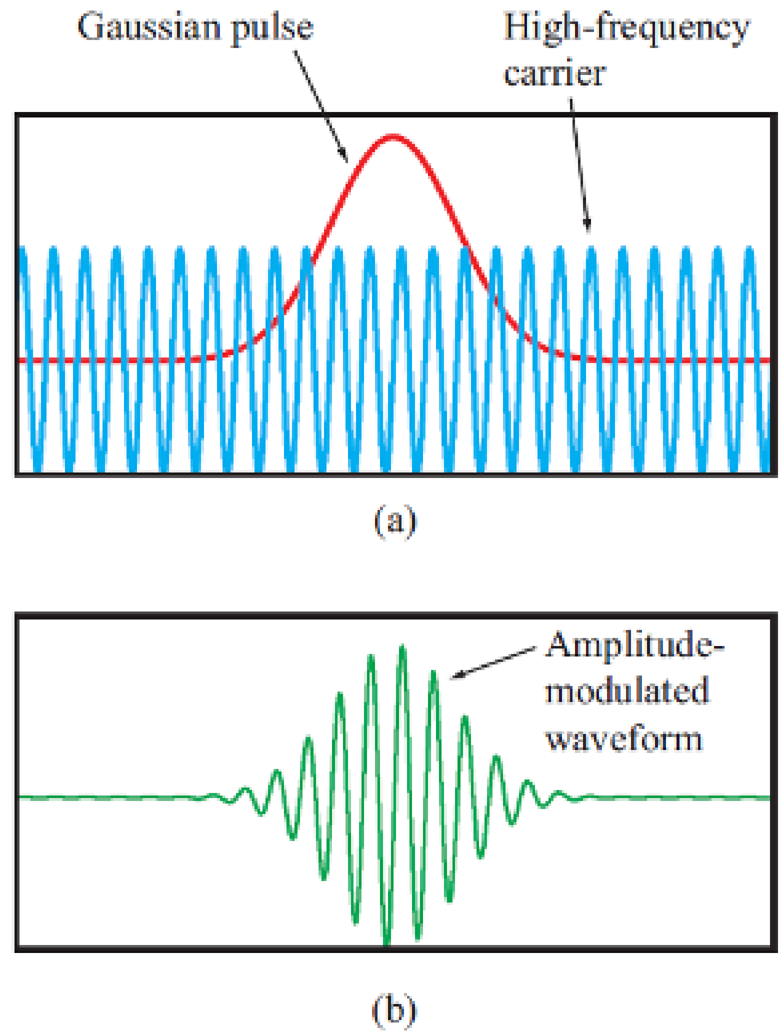


Figure 8-25: The amplitude-modulated high-frequency waveform in (b) is the product of the Gaussian-shaped pulse with the sinusoidal high-frequency carrier in (a).

ω-β Diagram for TE and TM Modes

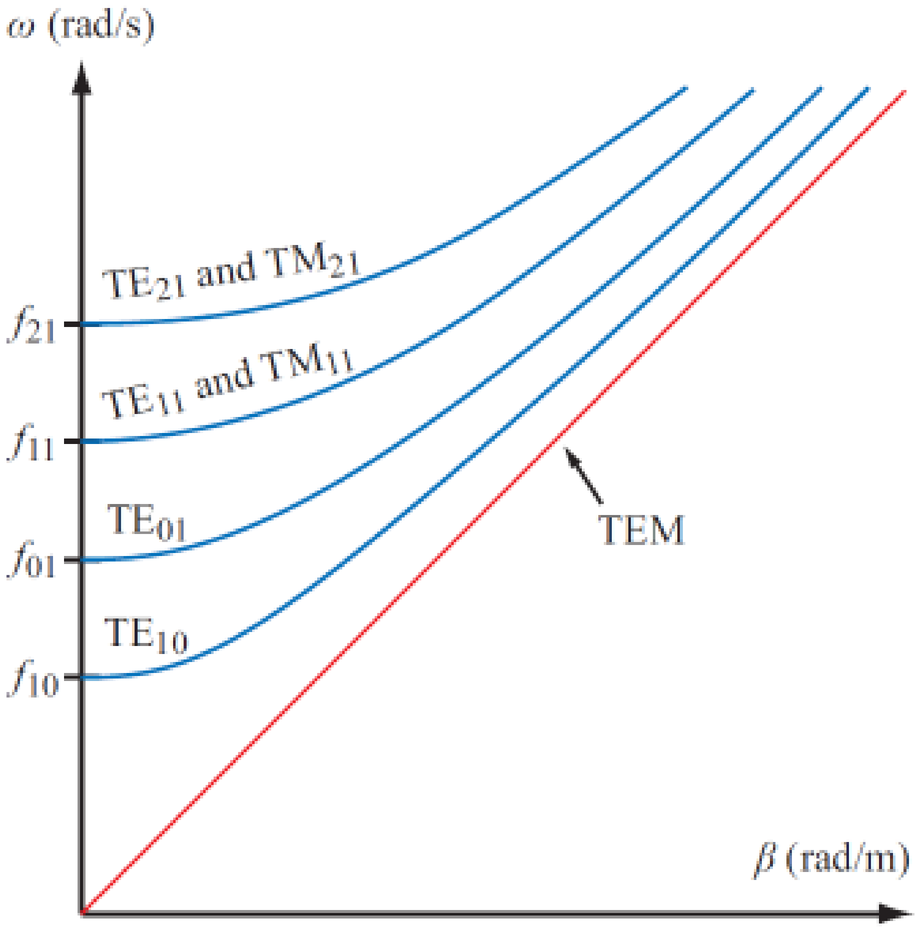
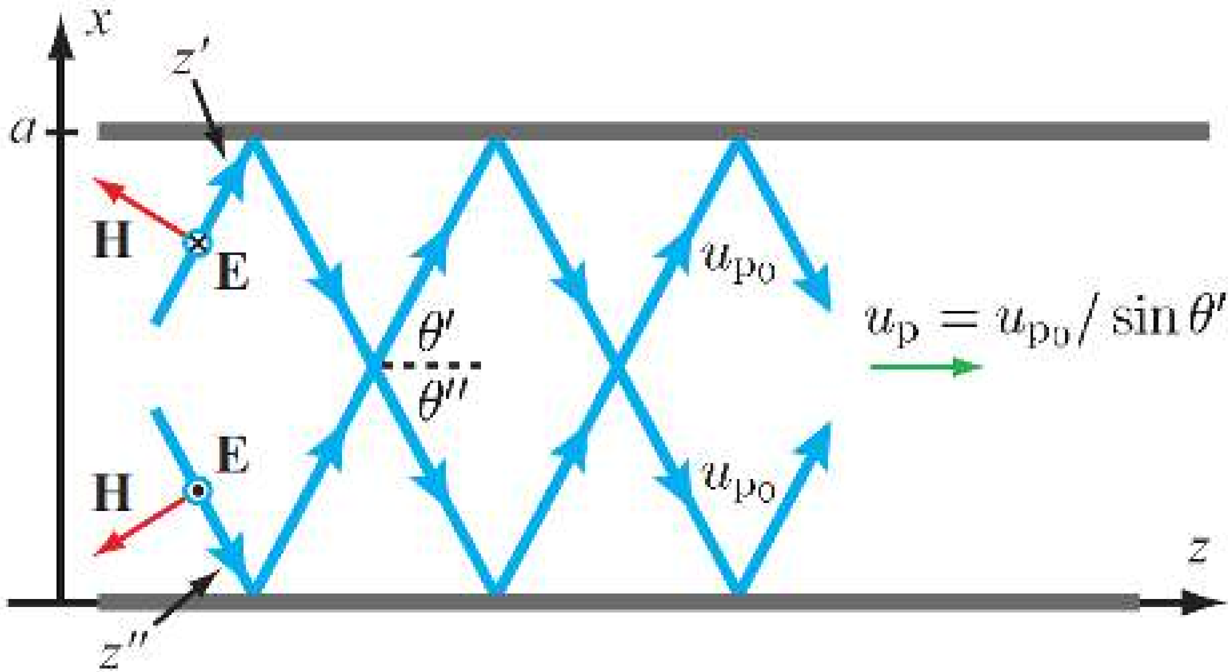


Figure 8-26: ω-β diagram for TE and TM modes in a hollow rectangular waveguide. The straight line pertains to propagation in an unbounded medium or on a TEM transmission line.

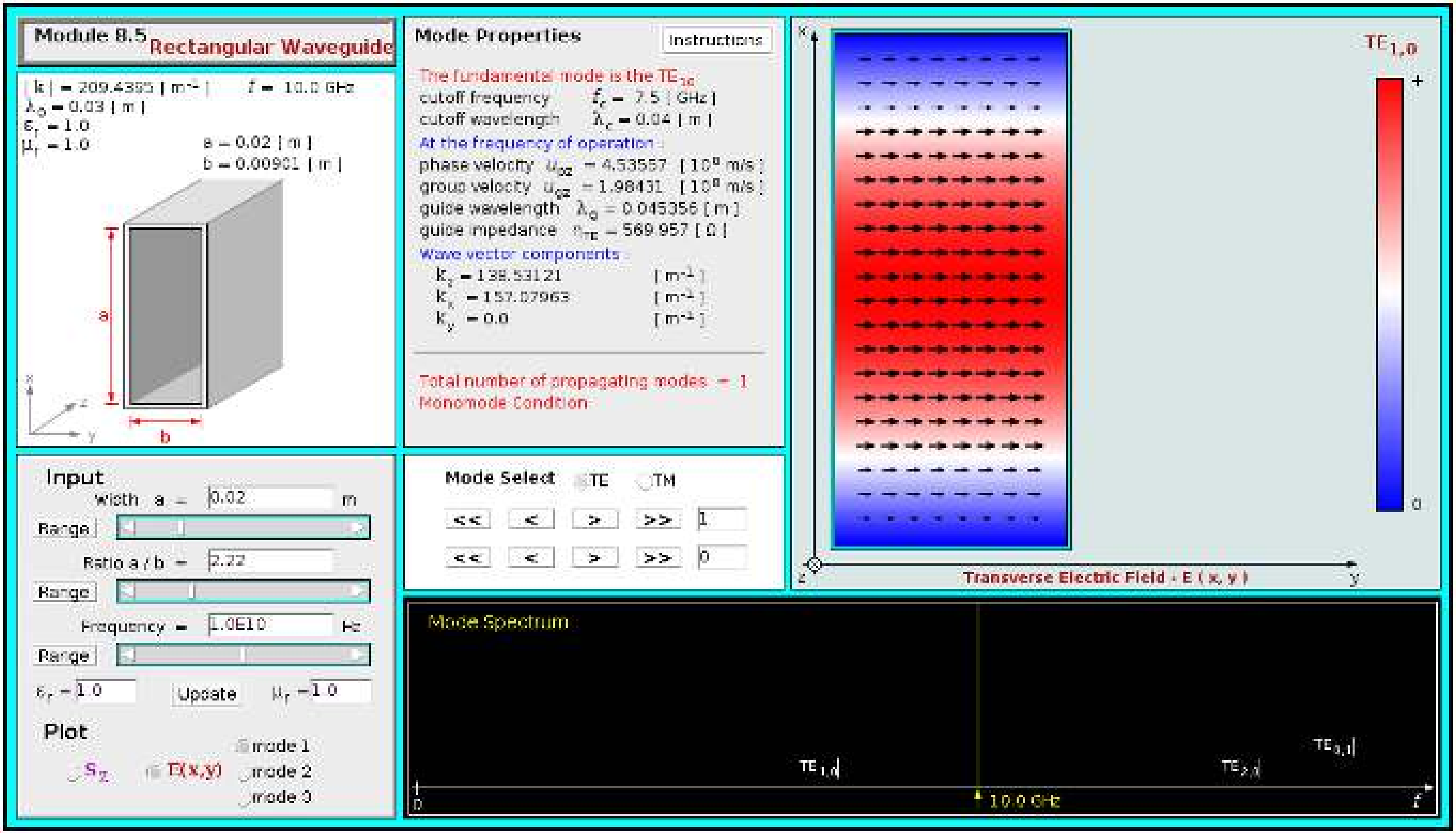
1. Note cutoff frequencies along vertical axis.
2. The ratio of the value of ω to that of β defines $u_p = \omega/\beta$, whereas it is the slope $d\omega/d\beta$ of the curve at that point that defines the group velocity u_g.
3. For all modes, as f becomes much larger than the cutoff frequency, the ω-β curve approaches the TEM case, for which $u_p = u_g$.
4. $$u_p u_g = u_{p0}^2.$$

Zigzag Reflections



(b) TEM waves

Figure 8-27: The TE_{10} mode can be constructed as the sum of two TEM waves.



Module 8.5 Rectangular Waveguide
|k| = 209.4395 [m⁻¹]   f = 10.0 GHz
λ0 = 0.03 [m]
εr = 1.0
μr = 1.0
a = 0.02 [m]
b = 0.00901 [m]
a
b
x
y
z
Mode Properties
Instructions
The fundamental mode is the TE10
cutoff frequency fc = 7.5 [GHz]
cutoff wavelength λc = 0.04 [m]
At the frequency of operation :
phase velocity upz = 4.53557 [10⁸ m/s]
group velocity ugz = 1.98431 [10⁸ m/s]
guide wavelength λg = 0.045356 [m]
guide impedance ηTE = 569.957 [Ω]
Wave vector components :
kz = 138.53121 [m⁻¹]
kx = 157.07963 [m⁻¹]
ky = 0.0 [m⁻¹]
Total number of propagating modes = 1
Monomode Condition
Input
width a = 0.02 m
Range
Ratio a / b = 2.22
Range
Frequency = 1.0E10 Hz
Range
εr = 1.0
Update
μr = 1.0
Plot
Sz
E(x,y)
mode 1
mode 2
mode 3
Mode Select
TE
TM
<< < > >> 1
<< < > >> 0
TE 1,0
+
0
Transverse Electric Field - E (x, y)
Mode Spectrum
TE1,0
TE2,0
TE0,1
0
10.0 GHz
f

Resonant Cavities

A rectangular waveguide has metal walls on four sides. When the two remaining sides are terminated with conducting walls, the waveguide becomes a cavity. By designing cavities to *resonate at specific frequencies, they can be used as circuit elements in microwave oscillators, amplifiers, and bandpass filters.*

Resonant Frequency

$$f_{mnp} = \frac{u_{p0}}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

Quality factor

$$Q \approx \frac{f_{mnp}}{\Delta f}$$

The quality factor is defined in terms of the ratio of the energy stored in the cavity volume to the energy dissipated in the cavity walls through conduction.

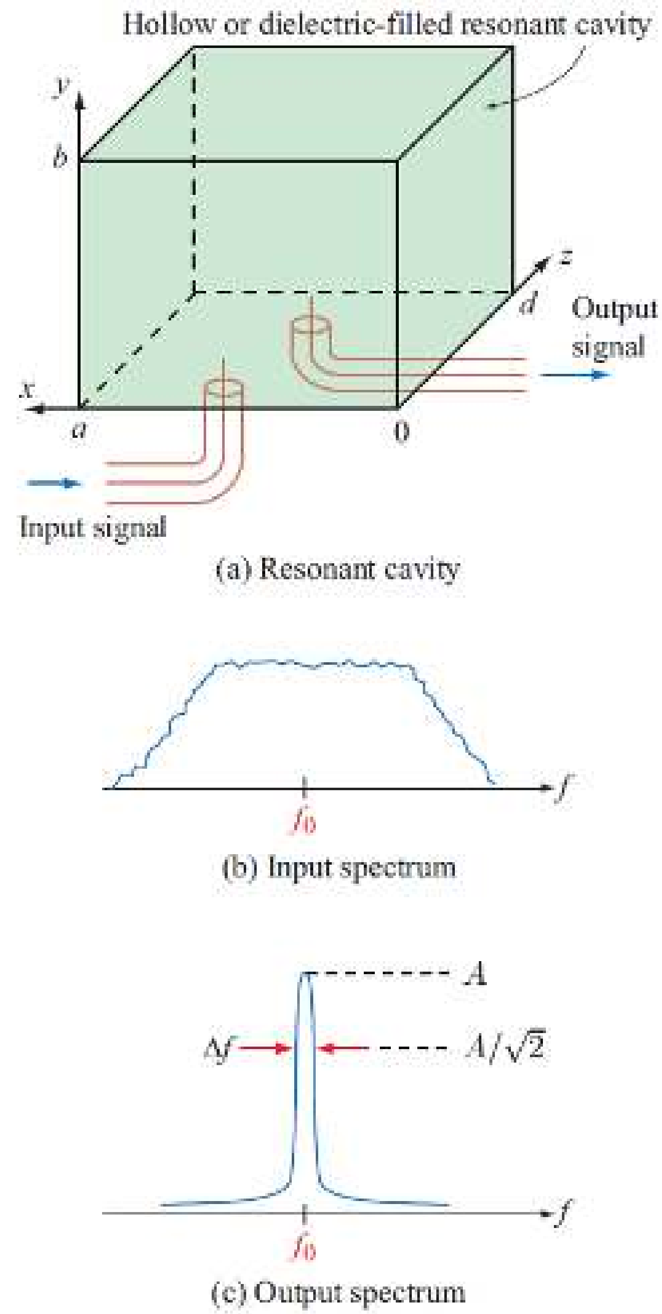


Figure 8-28: A resonant cavity supports a very narrow bandwidth around its resonant frequency f_0.

Example 8-11: Q of a Resonant Cavity

The quality factor for a hollow resonant cavity operating in the TE_{101} mode is

$$Q = \frac{1}{\delta_s} \frac{abd(a^2 + d^2)}{[a^3(d + 2b) + d^3(a + 2b)]}, \qquad (8.124)$$

where $\delta_s = 1/\sqrt{\pi f_{mnp} \mu_0 \sigma_c}$ is the skin depth and σ_c is the conductivity of the conducting walls. Design a cubic cavity with a TE_{101} resonant frequency of 12.6 GHz and evaluate its bandwidth. The cavity walls are made of copper.

Solution: For $a = b = d$, $m = 1$, $n = 0$, $p = 1$, and $u_{p0} = c = 3 \times 10^8$ m/s, Eq. (8.122) simplifies to

$$f_{101} = \frac{3\sqrt{2} \times 10^8}{2a} \qquad \text{(Hz)},$$

which, for $f_{101} = 12.6$ GHz, gives

$$a = 1.68 \text{ cm}.$$

Cont.

Example 8-11: ***Q* of a Resonant Cavity**

At $f_{101} = 12.6$ GHz, the skin depth for copper (with $\sigma_c = 5.8 \times 10^7$ S/m) is

$$\delta_s = \frac{1}{[\pi f_{101}\mu_0\sigma_c]^{1/2}} = \frac{1}{[\pi \times 12.6 \times 10^9 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7]^{1/2}} = 5.89 \times 10^{-7} \text{ m.}$$

Upon setting $a = b = d$ in Eq. (8.124), the expression for Q of a cubic cavity becomes

$$Q = \frac{a}{3\delta_s} = \frac{1.68 \times 10^{-2}}{3 \times 5.89 \times 10^{-7}} \simeq 9,500.$$

Hence, the cavity bandwidth is

$$\Delta f \simeq \frac{f_{101}}{Q} \simeq \frac{12.6 \times 10^9}{9,500} \simeq 1.3 \text{ MHz.}$$

Chapter 8 Summary

Chapter 8 Relationships

Normal Incidence

$$\Gamma = \frac{E_0^r}{E_0^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\tau = \frac{E_0^t}{E_0^i} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$\tau = 1 + \Gamma$$

$$\Gamma = \frac{\sqrt{\varepsilon_{r_1}} - \sqrt{\varepsilon_{r_2}}}{\sqrt{\varepsilon_{r_1}} + \sqrt{\varepsilon_{r_2}}} \quad (\text{if } \mu_1 = \mu_2)$$

Snell's Laws

$$\theta_i = \theta_r$$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p_2}}{u_{p_1}} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}$$

Oblique Incidence

Perpendicular Polarization

$$\Gamma_\perp = \frac{E_{\perp 0}^r}{E_{\perp 0}^i} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\tau_\perp = \frac{E_{\perp 0}^t}{E_{\perp 0}^i} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

Parallel Polarization

$$\Gamma_\parallel = \frac{E_{\parallel 0}^r}{E_{\parallel 0}^i} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\tau_\parallel = \frac{E_{\parallel 0}^t}{E_{\parallel 0}^i} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

Brewster Angle

$$\theta_{B\parallel} = \sin^{-1}\sqrt{\frac{1}{1 + (\varepsilon_1/\varepsilon_2)}} = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

Waveguides

$$\beta = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$f_{mn} = \frac{u_{p_0}}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$u_p = \frac{\omega}{\beta} = \frac{u_{p_0}}{\sqrt{1 - (f_{mn}/f)^2}}$$

$$u_p u_g = u_{p_0}^2$$

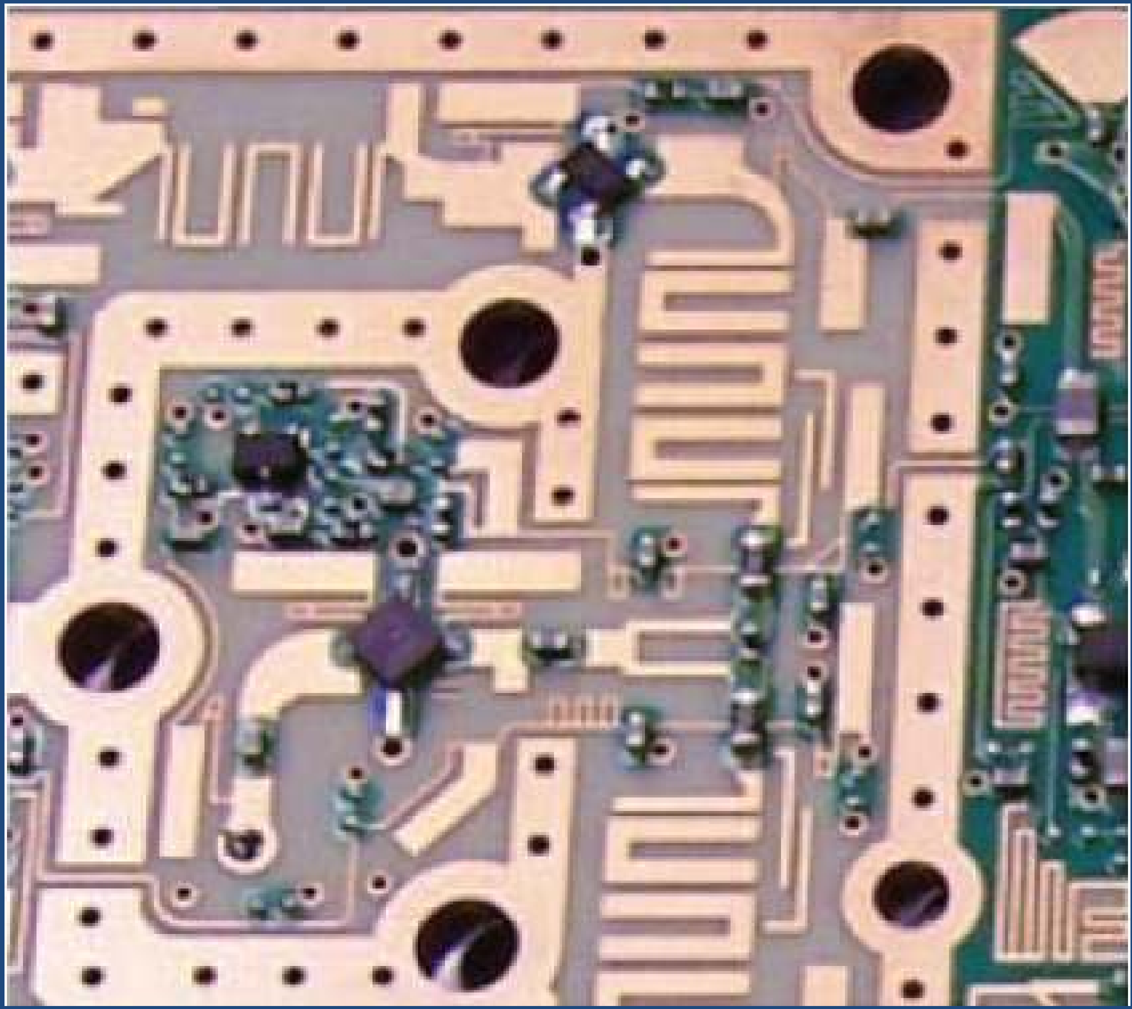
$$Z_{TE} = \frac{\eta}{\sqrt{1 - (f_{mn}/f)^2}}$$

$$Z_{TM} = \eta\sqrt{1 - \left(\frac{f_{mn}}{f}\right)^2}$$

Resonant Cavity

$$f_{mnp} = \frac{u_{p_0}}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

$$Q \approx \frac{f_{mnp}}{\Delta f}$$



2. TRANSMISSION LINES

7e Applied EM by Ulaby and Ravaioli

Chapter 2 Overview

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Calculate the line parameters, characteristic impedance, and propagation constant of coaxial, two-wire, parallel-plate, and microstrip transmission lines.
2. Determine the reflection coefficient at the load-end of the transmission line, the standing-wave pattern, and the locations of voltage and current maxima and minima.
3. Calculate the amount of power transferred from the generator to the load through the transmission line.
4. Use the Smith chart to perform transmission-line calculations.
5. Analyze the response of a transmission line to a voltage pulse.

Transmission Lines

A transmission line connects a generator to a load

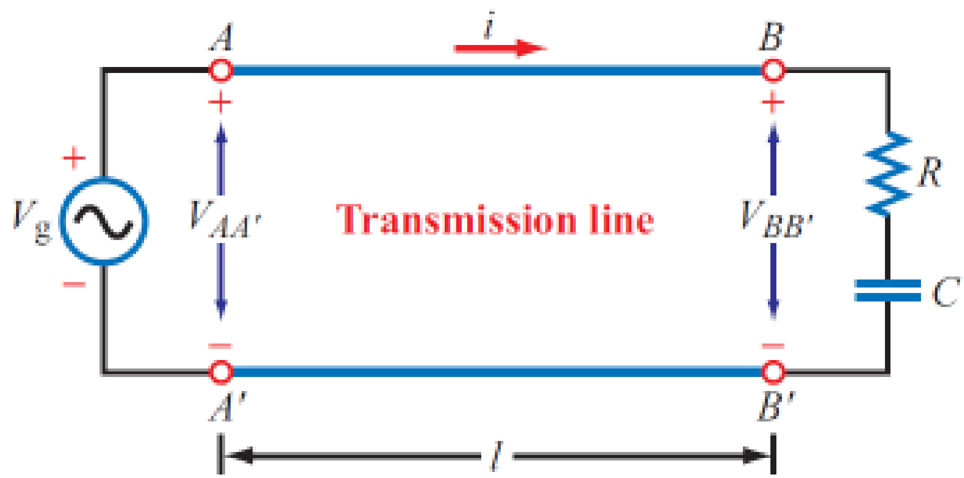
Figure 2-1 A transmission line is a two-port network connecting a generator circuit at the sending end to a load at the receiving end.

Transmission lines include:

- Two parallel wires
- Coaxial cable
- Microstrip line
- Optical fiber
- Waveguide
- etc.

Transmission Line Effects

Is the pair of wires connecting the voltage source to the RC load a transmission line? Yes.
The wires were ignored in circuits courses. Can we always ignore them? Not always.



$$V_{AA'} = V_g(t) = V_0 \cos \omega t \qquad \text{(V)}$$

$$V_{BB'}(t) = V_{AA'}(t - l/c) \qquad \text{Delayed by } l/c$$
$$= V_0 \cos[\omega(t - l/c)]$$
$$= V_0 \cos(\omega t - \phi_0),$$

At $t = 0$, and for $f = 1$ kHz , if:

(1) $l = 5$ cm:

$$V_{BB'} = V_0 \cos(2\pi f l/c) = 0.999999999998\ V_0$$

(2) But if $l = 20$ km:

$$V_{BB'} = 0.91 V_0$$

$$\phi_0 = \frac{\omega l}{c} = \frac{2\pi f l}{c} = 2\pi \frac{l}{\lambda} \quad \text{radians.} \qquad (2.4)$$

When l/λ is very small, transmission-line effects may be ignored, but when $l/\lambda \gtrsim 0.01$, it may be necessary to account not only for the phase shift due to the time delay, but also for the presence of reflected signals that may have been bounced back by the load toward the generator.

Dispersion

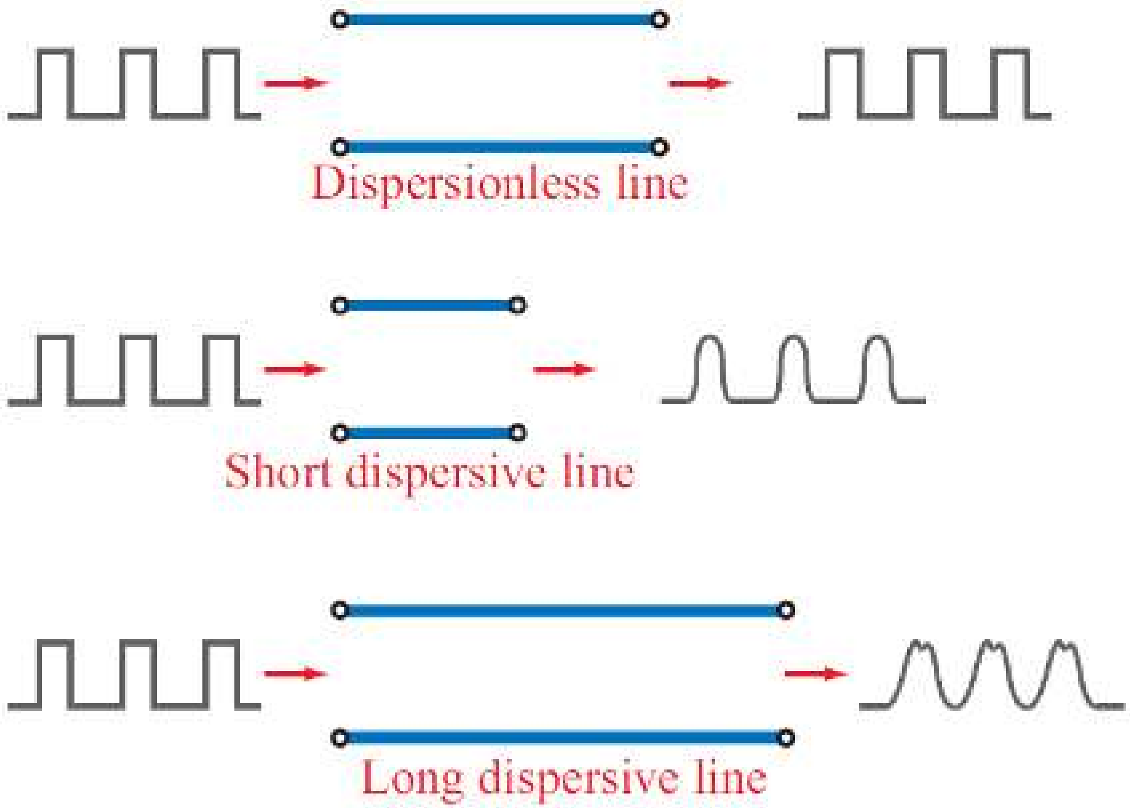
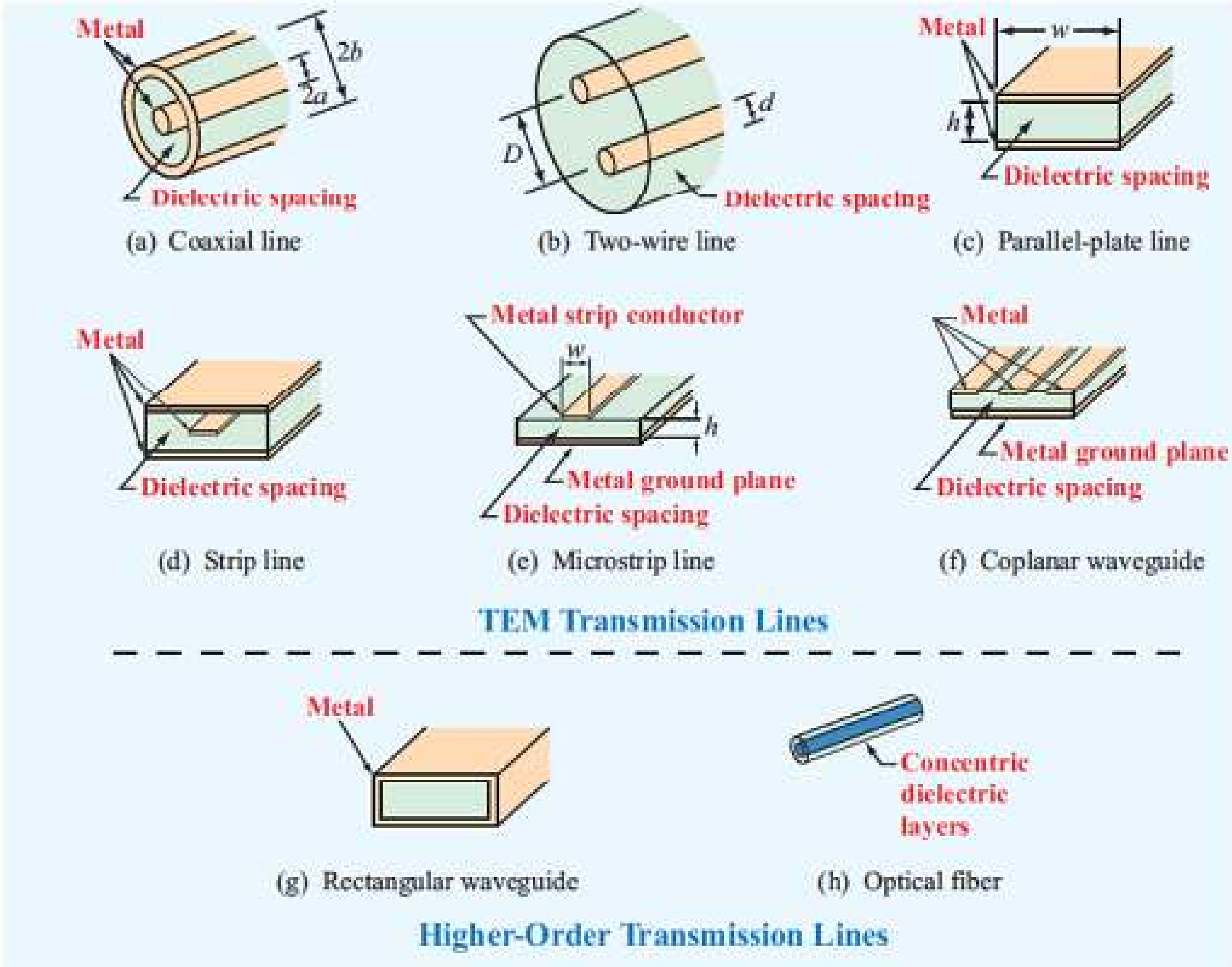


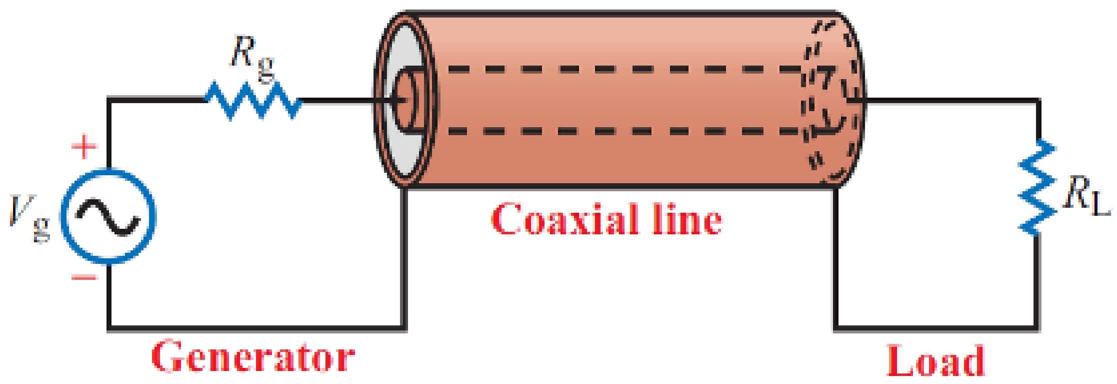
Figure 2-3: A dispersionless line does not distort signals passing through it regardless of its length, whereas a dispersive line distorts the shape of the input pulses because the different frequency components propagate at different velocities. The degree of distortion is proportional to the length of the dispersive line.

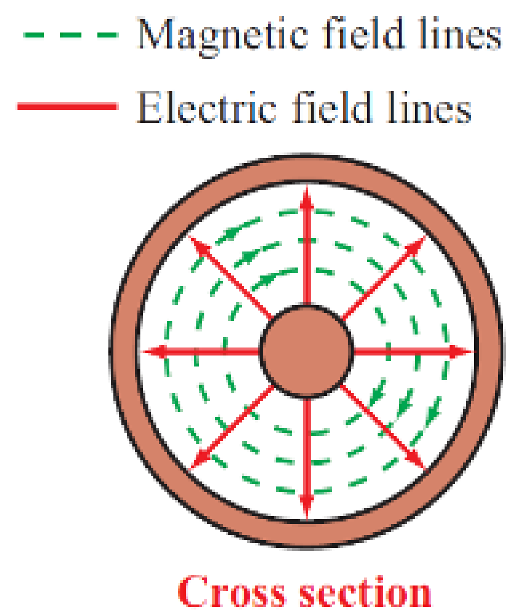
Types of Transmission Modes

TEM (Transverse Electromagnetic): Electric and magnetic fields are orthogonal to one another, and both are orthogonal to direction of propagation



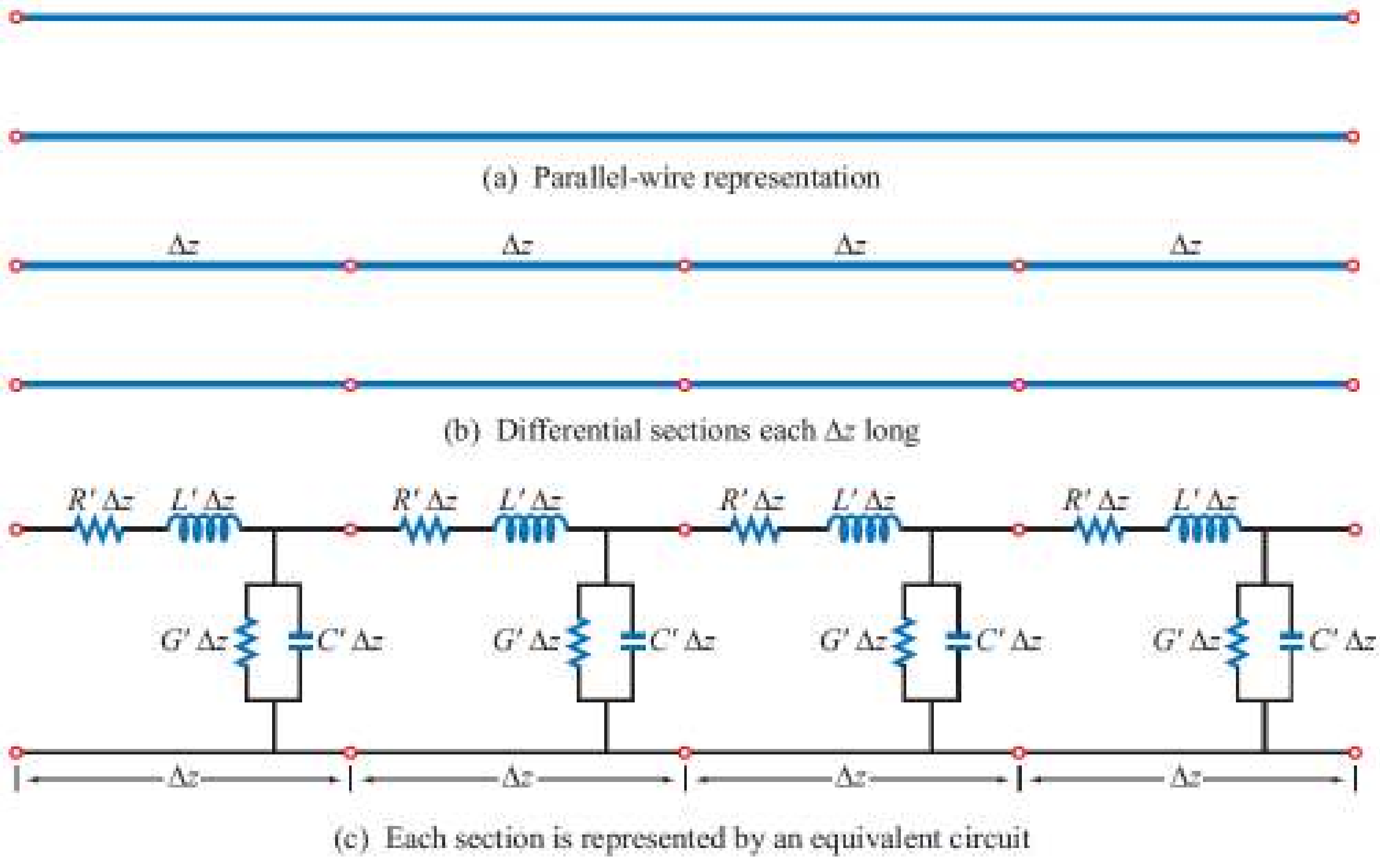
Example of TEM Mode





Electric Field E is radial
Magnetic Field H is azimuthal
Propagation is into the page

Transmission Line Model



- R': The combined ***resistance*** of both conductors per unit length, in Ω/m,
- L': The combined ***inductance*** of both conductors per unit length, in H/m,
- G': The ***conductance*** of the insulation medium between the two conductors per unit length, in S/m, and
- C': The ***capacitance*** of the two conductors per unit length, in F/m.

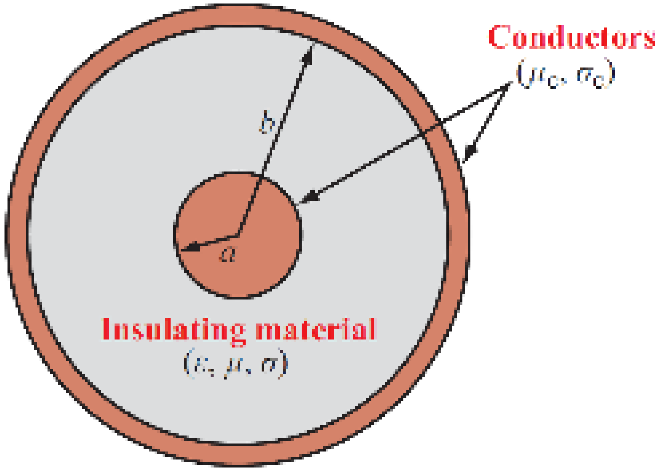
Table 2-1: Transmission-line parameters R', L', G', and C' for three types of lines.

Parameter	Coaxial	Two-Wire	Parallel-Plate	Unit
R'	$\frac{R_s}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2R_s}{\pi d}$	$\frac{2R_s}{w}$	Ω/m
L'	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]$	$\frac{\mu h}{w}$	H/m
G'	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\sigma w}{h}$	S/m
C'	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right]}$	$\frac{\varepsilon w}{h}$	F/m

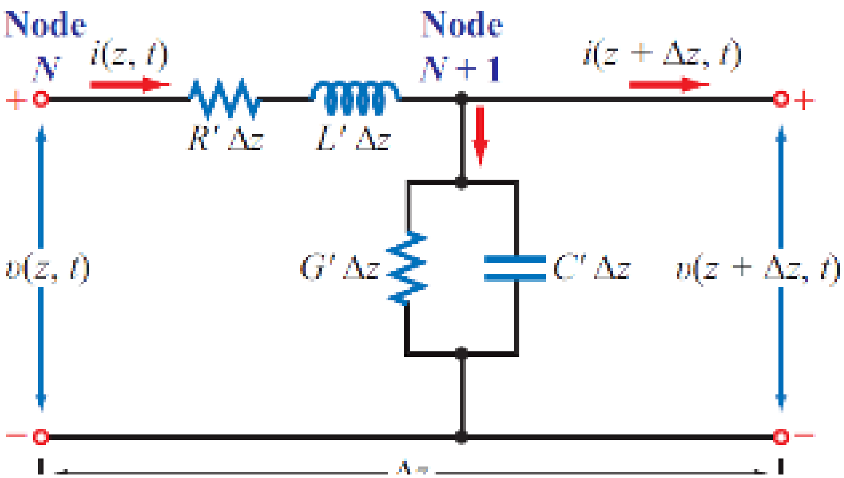
Notes: (1) Refer to Fig. 2-4 for definitions of dimensions. (2) μ, ε, and σ pertain to the insulating material between the conductors. (3) $R_s = \sqrt{\pi f \mu_c/\sigma_c}$. (4) μ_c and σ_c pertain to the conductors. (5) If $(D/d)^2 \gg 1$, then $\ln\left[(D/d)+\sqrt{(D/d)^2-1}\right] \simeq \ln(2D/d)$.

Expressions will be derived in later chapters

*The pertinent **constitutive parameters** apply to all three lines and consist of two groups: (1) μ_c and σ_c are the magnetic permeability and electrical conductivity of the conductors, and (2) ε, μ, and σ are the electrical permittivity, magnetic permeability, and electrical conductivity of the insulation material separating them.*



Transmission-Line Equations



$$\upsilon(z,t) - R'\Delta z\, i(z,t) - L'\Delta z \frac{\partial i(z,t)}{\partial t} - \upsilon(z+\Delta z, t) = 0.$$

Upon dividing all terms by Δz and rearranging them, we obtain

$$-\left[\frac{\upsilon(z+\Delta z,t) - \upsilon(z,t)}{\Delta z}\right] = R' i(z,t) + L'\frac{\partial i(z,t)}{\partial t}. \qquad (2.13)$$

In the limit as $\Delta z \to 0$, Eq. (2.13) becomes a differential equation:

$$-\frac{\partial \upsilon(z,t)}{\partial z} = R'\, i(z,t) + L'\frac{\partial i(z,t)}{\partial t}. \qquad (2.14)$$

$$i(z,t) - G'\Delta z\, \upsilon(z+\Delta z,t) - C'\Delta z \frac{\partial \upsilon(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0. \qquad (2.15)$$

Upon dividing all terms by Δz and taking the limit $\Delta z \to 0$, Eq. (2.15) becomes a second-order differential equation:

$$-\frac{\partial i(z,t)}{\partial z} = G'\, \upsilon(z,t) + C'\frac{\partial \upsilon(z,t)}{\partial t}. \qquad (2.16)$$

ac signals: use phasors

$$\upsilon(z,t) = \mathfrak{Re}[\tilde{V}(z)\, e^{j\omega t}],$$
$$i(z,t) = \mathfrak{Re}[\tilde{I}(z)\, e^{j\omega t}],$$

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\,\tilde{I}(z),$$
$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\,\tilde{V}(z).$$

Telegrapher's equations

Derivation of Wave Equations

$$-\frac{d\widetilde{V}(z)}{dz} = (R' + j\omega L')\,\widetilde{I}(z),$$

$$-\frac{d\widetilde{I}(z)}{dz} = (G' + j\omega C')\,\widetilde{V}(z).$$

Combining the two equations leads to:

$$\frac{d^2\widetilde{V}(z)}{dz^2} - (R' + j\omega L')(G' + j\omega C')\,\widetilde{V}(z) = 0,$$

$$\boxed{\frac{d^2\widetilde{V}(z)}{dz^2} - \gamma^2\,\widetilde{V}(z) = 0, \qquad (2.21)}$$

Second-order differential equation

$$\boxed{\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}, \qquad (2.22)}$$

complex propagation constant

attenuation constant

Phase constant

$$\gamma = \alpha + j\beta$$

$$\alpha = \mathfrak{Re}(\gamma) = \mathfrak{Re}\left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) \quad \text{(Np/m)}, \qquad (2.25a)$$

$$\beta = \mathfrak{Im}(\gamma) = \mathfrak{Im}\left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) \quad \text{(rad/m)}. \qquad (2.25b)$$

Solution of Wave Equations (cont.)

$$\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\,\tilde{V}(z) = 0, \qquad (2.21)$$

$$\frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\,\tilde{I}(z) = 0. \qquad (2.23)$$

Proposed form of solution:

$$\tilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \qquad \text{(V)},$$

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \qquad \text{(A)}.$$

Using:

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\,\tilde{I}(z),$$

It follows that:

$$\tilde{I}(z) = \frac{\gamma}{R' + j\omega L'}\left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right]$$

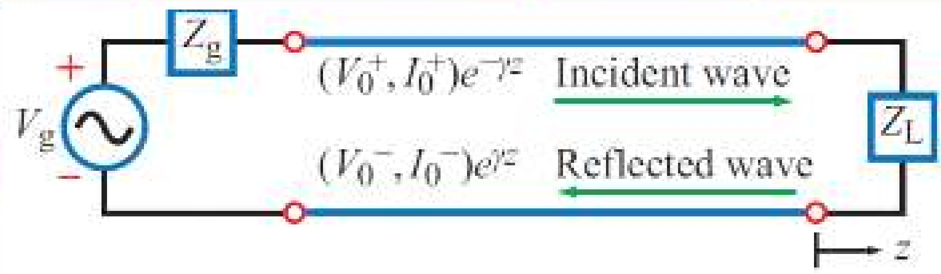


Figure 2-9: In general, a transmission line can support two traveling waves, an incident wave [with voltage and current amplitudes (V_0^+, I_0^+)] traveling along the $+z$-direction (towards the load) and a reflected wave [with (V_0^-, I_0^-)] traveling along the $-z$-direction (towards the source).

Comparison of each term with the corresponding term in Eq. (2.26b) leads us to conclude that

$$\frac{V_0^+}{I_0^+} = Z_0 = \frac{-V_0^-}{I_0^-}, \qquad (2.28)$$

where

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \qquad (\Omega), \qquad (2.29)$$

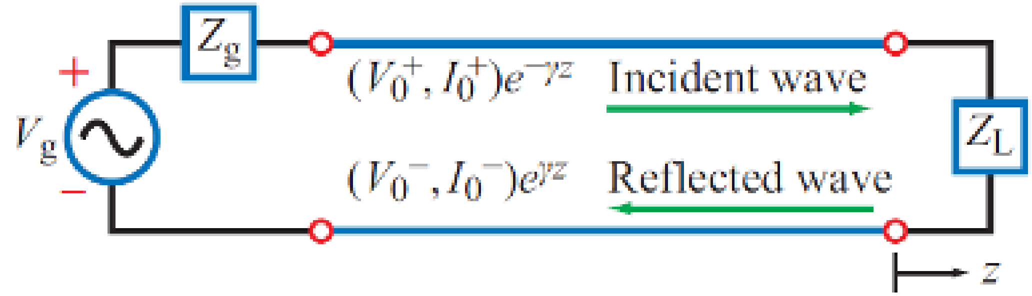
is called the ***characteristic impedance*** of the line.

Solution of Wave Equations (cont.)

In general:

$$V_0^+ = |V_0^+|e^{j\phi^+},$$

$$V_0^- = |V_0^-|e^{j\phi^-}.$$



The presence of two waves on the line propagating in opposite directions produces a ***standing wave.***

$$\begin{aligned}
\upsilon(z,t) &= \Re\mathfrak{e}(\widetilde{V}(z)e^{j\omega t}) \\
&= \Re\mathfrak{e}\left[\left(V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}\right) e^{j\omega t}\right] \\
&= \Re\mathfrak{e}[|V_0^+|e^{j\phi^+} e^{j\omega t} e^{-(\alpha+j\beta)z} \\
&\qquad + |V_0^-|e^{j\phi^-} e^{j\omega t} e^{(\alpha+j\beta)z}] \\
&= |V_0^+|e^{-\alpha z}\cos(\omega t - \beta z + \phi^+) \\
&\qquad + |V_0^-|e^{\alpha z}\cos(\omega t + \beta z + \phi^-)
\end{aligned}$$

wave along +z because coefficients of *t* and *z* have opposite signs

wave along –z because coefficients of *t* and *z* have the same sign

Example 2-1: Air Line

An *air line* is a transmission line in which air separates the two conductors, which renders $G' = 0$ because $\sigma = 0$. In addition, assume that the conductors are made of a material with high conductivity so that $R' \simeq 0$. For an air line with a characteristic impedance of 50 Ω and a phase constant of 20 rad/m at 700 MHz, find the line inductance L' and the line capacitance C'.

Solution: The following quantities are given:

$$Z_0 = 50\ \Omega, \qquad \beta = 20 \text{ rad/m},$$
$$f = 700 \text{ MHz} = 7 \times 10^8 \text{ Hz}.$$

With $R' = G' = 0$, Eqs. (2.25b) and (2.29) reduce to

$$\beta = \mathfrak{Im}\left[\sqrt{(j\omega L')(j\omega C')}\right] = \mathfrak{Im}\left(j\omega\sqrt{L'C'}\right) = \omega\sqrt{L'C'},$$

$$Z_0 = \sqrt{\frac{j\omega L'}{j\omega C'}} = \sqrt{\frac{L'}{C'}}.$$

The ratio of β to Z_0 is

$$\frac{\beta}{Z_0} = \omega C',$$

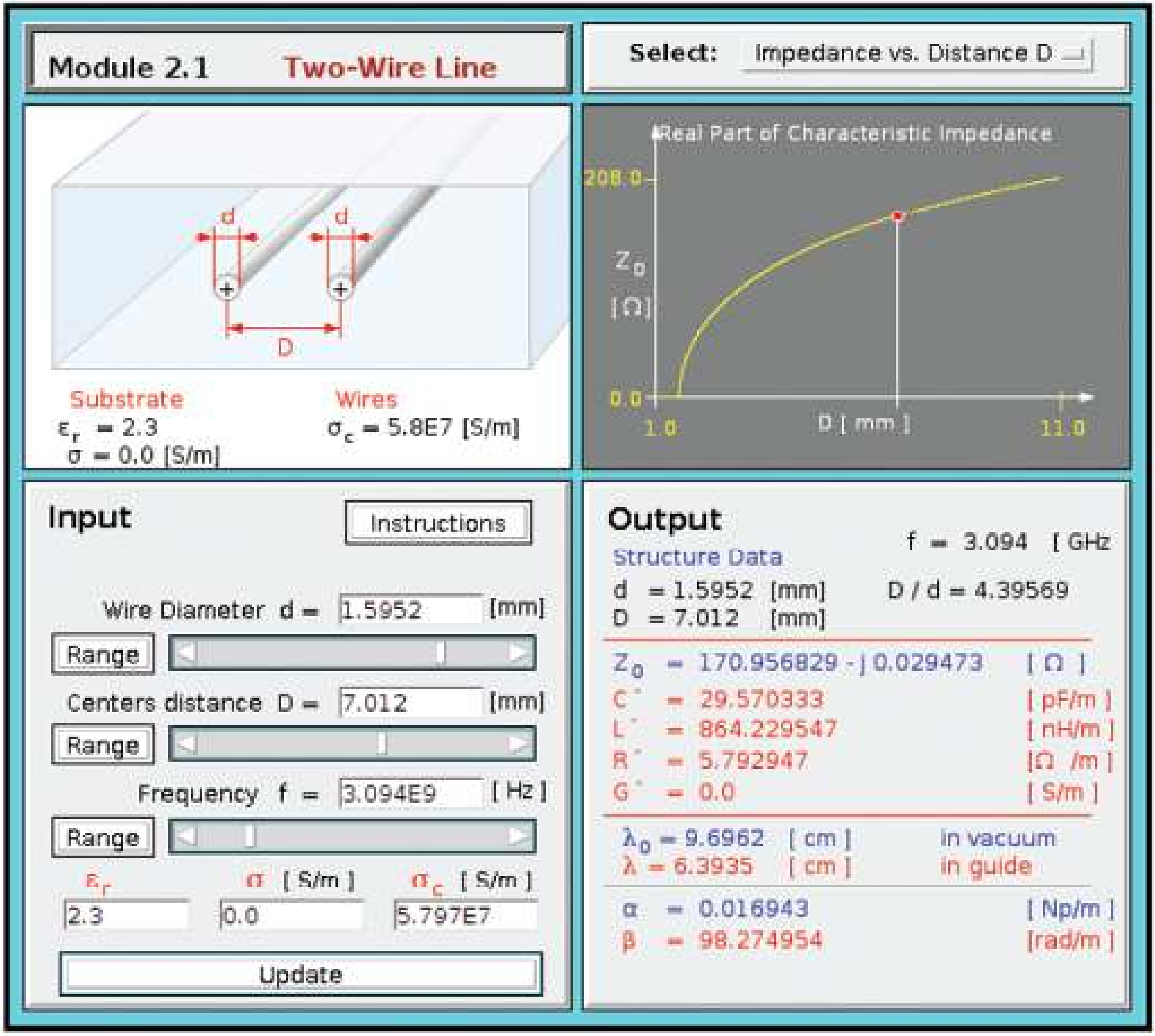
or

$$C' = \frac{\beta}{\omega Z_0} = \frac{20}{2\pi \times 7 \times 10^8 \times 50} = 9.09 \times 10^{-11} \text{ (F/m)} = \boxed{90.9 \text{ (pF/m)}.}$$

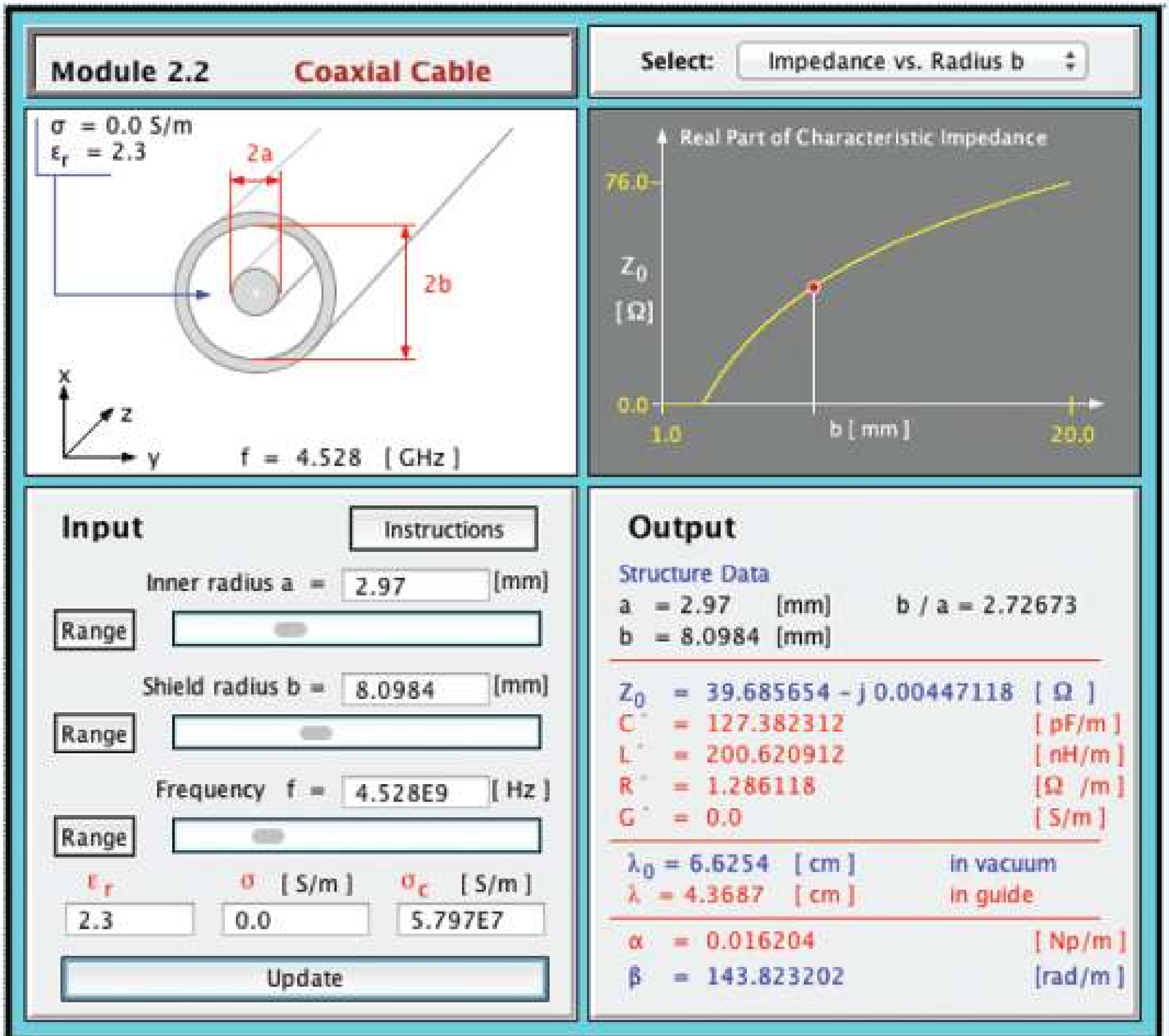
From $Z_0 = \sqrt{L'/C'}$, it follows that

$$L' = Z_0^2 C' = (50)^2 \times 90.9 \times 10^{-12} = 2.27 \times 10^{-7} \text{ (H/m)} = \boxed{227 \text{ (nH/m)}.}$$

Module 2.1 Two-Wire Line The input data specifies the geometric and electric parameters of a two-wire transmission line. The output includes the calculated values for the line parameters, characteristic impedance Z_0, and attenuation and phase constants, as well as plots of Z_0 as a function of d and D.



Module 2.2 Coaxial Cable Except for changing the geometric parameters to those of a coaxial transmission line, this module offers the same output information as Module 2.1.



Lossless Microstrip Line

Phase velocity in dielectric: $u_p = \dfrac{c}{\sqrt{\varepsilon_r}}$

Phase velocity for microstrip: $u_p = \dfrac{c}{\sqrt{\varepsilon_{eff}}}$

$$\varepsilon_{eff} = \frac{\varepsilon_r + 1}{2} + \left(\frac{\varepsilon_r - 1}{2}\right)\left(1 + \frac{10}{s}\right)^{-xy}, \qquad (2.36)$$

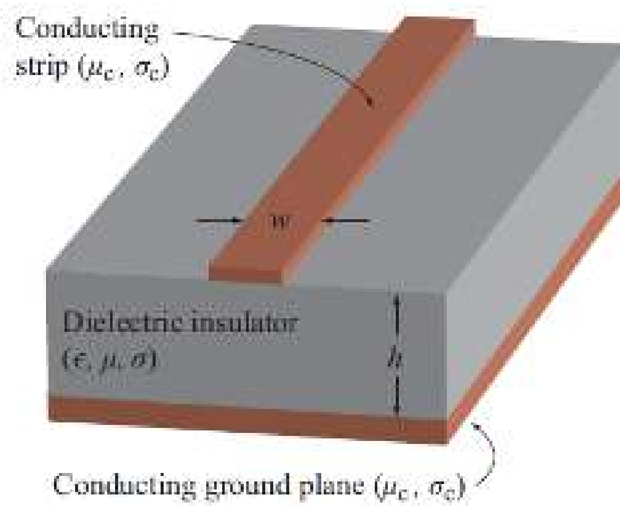
where s is the width-to-thickness ratio,

$$s = \frac{w}{h}, \qquad (2.37)$$

and x and y are intermediate variables given by

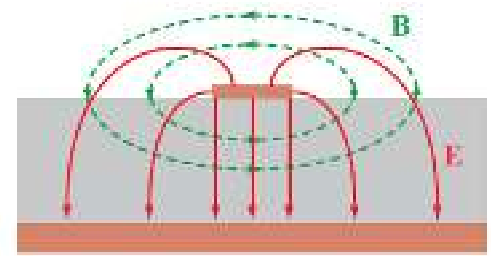
$$x = 0.56\left[\frac{\varepsilon_r - 0.9}{\varepsilon_r + 3}\right]^{0.05}, \qquad (2.38a)$$

$$y = 1 + 0.02\ln\left(\frac{s^4 + 3.7 \times 10^{-4}s^2}{s^4 + 0.43}\right) + 0.05\ln(1 + 1.7 \times 10^{-4}s^3). \qquad (2.38b)$$

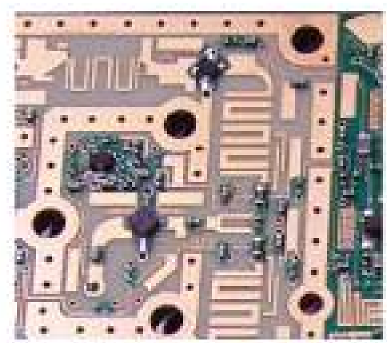


(a) Longitudinal view

Quasi-TEM



(b) Cross-sectional view with E and B field lines



(c) Microwave circuit

Microstrip (cont.)

The characteristic impedance of the microstrip line is given by

$$Z_0 = \frac{60}{\sqrt{\varepsilon_{\rm eff}}} \ln \left\{ \frac{6 + (2\pi - 6)e^{-t}}{s} + \sqrt{1 + \frac{4}{s^2}} \right\}, \qquad (2.39)$$

$$t = \left(\frac{30.67}{s} \right)^{0.75}$$

$$R' = 0 \qquad (\text{because } \sigma_c = \infty),$$

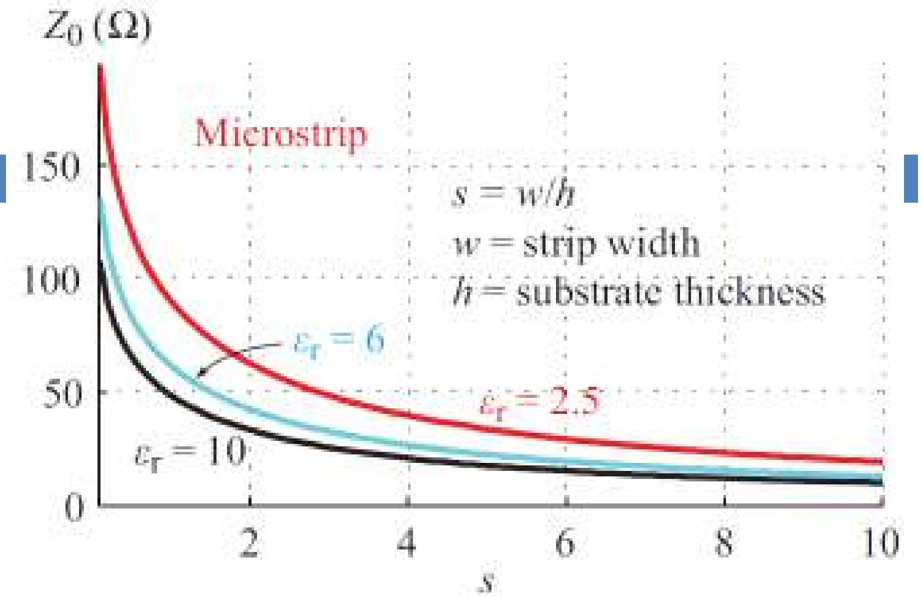
$$G' = 0 \qquad (\text{because } \sigma = 0),$$

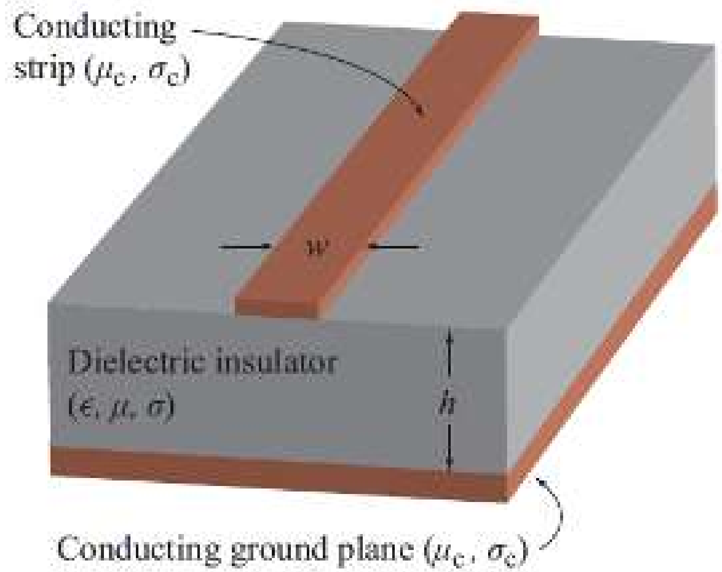
$$C' = \frac{\sqrt{\varepsilon_{\rm eff}}}{Z_0 c},$$

$$L' = Z_0^2 C',$$

$$\alpha = 0 \qquad (\text{because } R' = G' = 0),$$

$$\beta = \frac{\omega}{c} \sqrt{\varepsilon_{\rm eff}}.$$





(a) Longitudinal view

Microstrip (cont.)

Inverse process:

Given Z_0, find s

The solution formulas are based on two numerical fits, defined in terms of the value of Z_0 relative to that of the effective permittivity.

(a) For $Z_0 \leq (44 - 2\varepsilon_r)\ \Omega$,

$$s = \frac{w}{h} = \frac{2}{\pi}\left\{(q-1) - \ln(2q-1) + \frac{\varepsilon_r - 1}{2\varepsilon_r}\left[\ln(q-1) + 0.29 - \frac{0.52}{\varepsilon_r}\right]\right\} \quad (2.42a)$$

with

$$q = \frac{60\pi^2}{Z_0\sqrt{\varepsilon_r}}, \quad (2.42b)$$

and

(b) for $Z_0 \geq (44 - 2\varepsilon_r)\ \Omega$,

$$s = \frac{w}{h} = \frac{8e^p}{e^{2p} - 2}, \quad (2.43a)$$

with

$$p = \sqrt{\frac{\varepsilon_r + 1}{2}}\frac{Z_0}{60} + \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\right)\left(0.23 + \frac{0.12}{\varepsilon_r}\right). \quad (2.43b)$$

Example 2-2: Microstrip Line

A 50-Ω microstrip line uses a 0.5-mm–thick sapphire substrate with $\varepsilon_r = 9$. What is the width of its copper strip?

Solution: Since $Z_0 = 50 > 44 - 18 = 32$, we should use Eq. (2.43):

$$p = \sqrt{\frac{\varepsilon_r + 1}{2}} \times \frac{Z_0}{60} + \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\right)\left(0.23 + \frac{0.12}{\varepsilon_r}\right)$$
$$= \sqrt{\frac{9+1}{2}} \times \frac{50}{60} + \left(\frac{9-1}{9+1}\right)\left(0.23 + \frac{0.12}{9}\right)$$
$$= 2.06,$$
$$s = \frac{w}{h}$$
$$= \frac{8e^p}{e^{2p} - 2}$$
$$= \frac{8e^{2.06}}{e^{4.12} - 2}$$
$$= 1.056.$$

Hence,

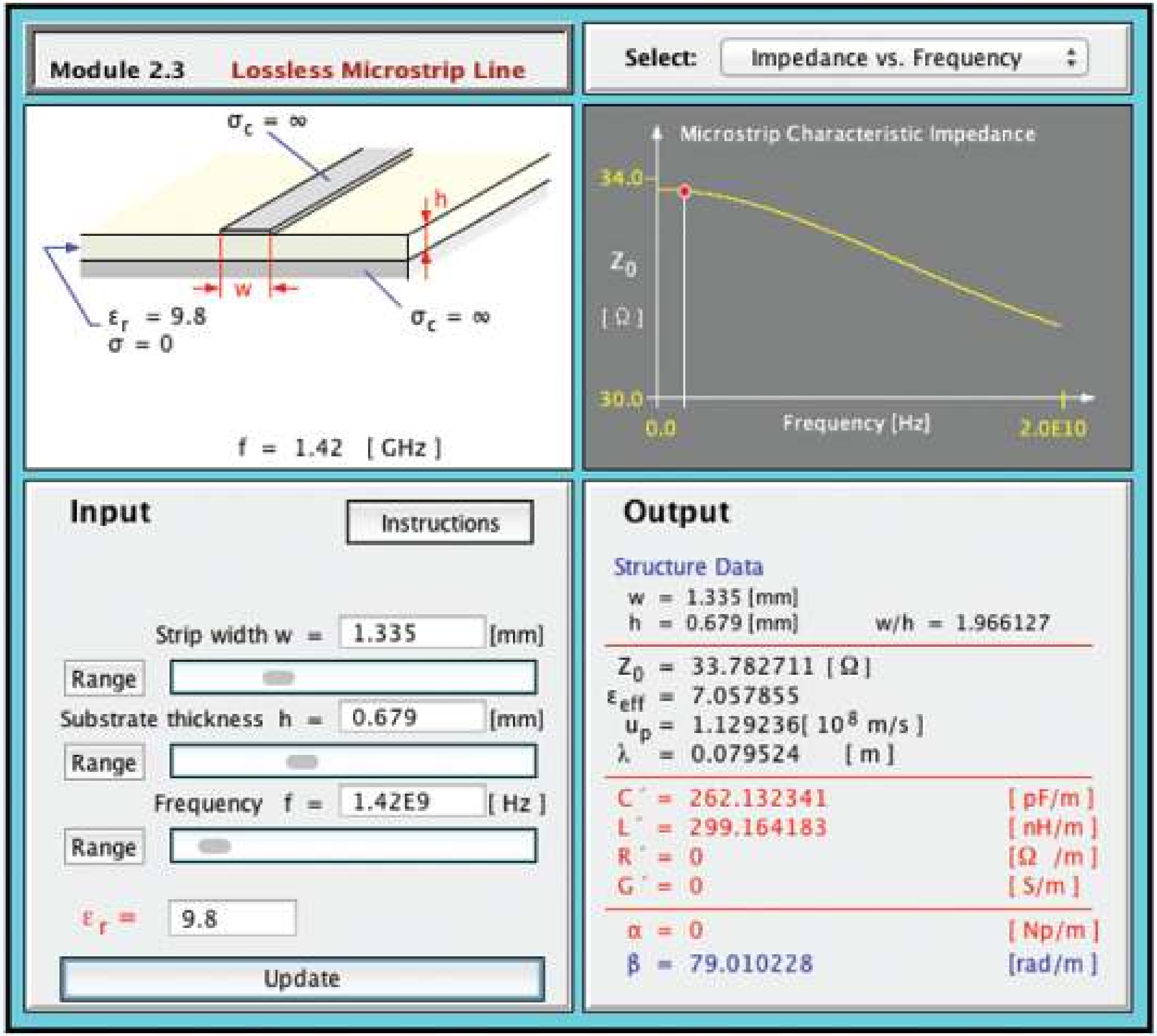
$$w = sh$$
$$= 1.056 \times 0.5 \text{ mm}$$
$$\boxed{= 0.53 \text{ mm.}}$$

To check our calculations, we will use $s = 1.056$ to calculate Z_0 to verify that the value we obtained is indeed equal or close to 50 Ω. With $\varepsilon_r = 9$, Eqs. (2.36) to (2.40) yield

$$x = 0.55,$$
$$y = 0.99,$$
$$t = 12.51,$$
$$\boxed{\varepsilon_{\text{eff}} = 6.11,}$$
$$Z_0 = 49.93\ \Omega.$$

The calculated value of Z_0 is, for all practical purposes, equal to the value specified in the problem statement.

Module 2.3 Lossless Microstrip Line The output panel lists the values of the transmission-line parameters and displays the variation of Z_0 and ϵ_{eff} with h and w.



Lossless Transmission Line

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}.$$

If $R' \ll \omega L'$ and $G' \ll \omega C'$

Then:

$$\gamma = \alpha + j\beta = j\omega\sqrt{L'C'}, \qquad (2.44)$$

which in turn implies that

$$\alpha = 0 \qquad \text{(lossless line)},$$
$$\beta = \omega\sqrt{L'C'} \qquad \text{(lossless line)}. \qquad (2.45)$$

For the characteristic impedance, application of the lossless line conditions to Eq. (2.29) leads to

$$Z_0 = \sqrt{\frac{L'}{C'}} \qquad \text{(lossless line)}, \qquad (2.46)$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{L'C'}},$$

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}.$$

$$\beta = \omega\sqrt{\mu\varepsilon} \qquad \text{(rad/m)}, \qquad (2.49)$$
$$u_p = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \text{(m/s)}, \qquad (2.50)$$

If sinusoidal waves of different frequencies travel on a transmission line with the same phase velocity, the line is called ***nondispersive***.

$$\lambda = \frac{u_p}{f} = \frac{c}{f}\frac{1}{\sqrt{\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$$

Table 2-2: Characteristic parameters of transmission lines.

	Propagation Constant $\gamma = \alpha + j\beta$	Phase Velocity u_p	Characteristic Impedance Z_0
General case	$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$	$u_p = \omega/\beta$	$Z_0 = \sqrt{\dfrac{(R' + j\omega L')}{(G' + j\omega C')}}$
Lossless $(R' = G' = 0)$	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = \sqrt{L'/C'}$
Lossless coaxial	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = \left(60/\sqrt{\varepsilon_r}\right)\ln(b/a)$
Lossless two-wire	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = \left(120/\sqrt{\varepsilon_r}\right) \cdot \ln[(D/d) + \sqrt{(D/d)^2 - 1}]$ $Z_0 \simeq \left(120/\sqrt{\varepsilon_r}\right)\ln(2D/d)$, if $D \gg d$
Lossless parallel-plate	$\alpha = 0,\ \beta = \omega\sqrt{\varepsilon_r}/c$	$u_p = c/\sqrt{\varepsilon_r}$	$Z_0 = \left(120\pi/\sqrt{\varepsilon_r}\right)(h/w)$

Notes: (1) $\mu = \mu_0$, $\varepsilon = \varepsilon_r\varepsilon_0$, $c = 1/\sqrt{\mu_0\varepsilon_0}$, and $\sqrt{\mu_0/\varepsilon_0} \simeq (120\pi)\ \Omega$, where ε_r is the relative permittivity of insulating material. (2) For coaxial line, a and b are radii of inner and outer conductors. (3) For two-wire line, d = wire diameter and D = separation between wire centers. (4) For parallel-plate line, w = width of plate and h = separation between the plates.

Voltage Reflection Coefficient

$$\widetilde{V}_L = \widetilde{V}(z=0) = V_0^+ + V_0^-,$$

$$\widetilde{I}_L = \widetilde{I}(z=0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}.$$

At the load ($z = 0$):

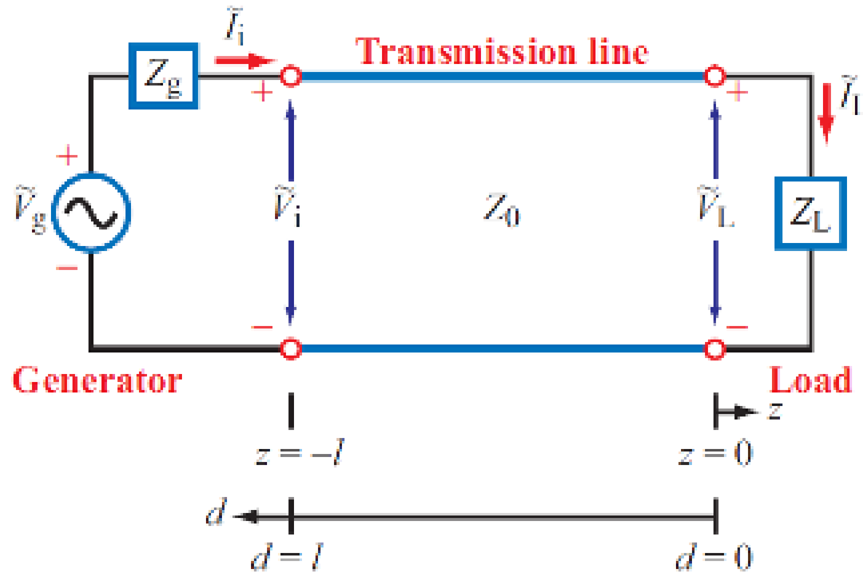
$$Z_L = \frac{\widetilde{V}_L}{\widetilde{I}_L}$$

$$\widetilde{V}_L = \widetilde{V}(z=0) = V_0^+ + V_0^-,$$

$$\widetilde{I}_L = \widetilde{I}(z=0) = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}.$$

Using these expressions in Eq. (2.55), we obtain

$$Z_L = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right) Z_0.$$



$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{z_L - 1}{z_L + 1} \quad \text{(dimensionless)}, \qquad (2.59)$$

Reflection coefficient

$$z_L = \frac{Z_L}{Z_0}$$

Normalized load impedance

Voltage Reflection Coefficient

$$\Gamma = |\Gamma| e^{j\theta_r}$$

Reflection Coefficient $\Gamma = |\Gamma| e^{j\theta_r}$

Load	$\lvert\Gamma\rvert$	θ_r
Z_0 $Z_L = (r + jx)Z_0$	$\left[\dfrac{(r-1)^2 + x^2}{(r+1)^2 + x^2}\right]^{1/2}$	$\tan^{-1}\left(\dfrac{x}{r-1}\right) - \tan^{-1}\left(\dfrac{x}{r+1}\right)$
Z_0 Z_0	0 (no reflection)	irrelevant
Z_0 (short)	1	$\pm 180°$ (phase opposition)
Z_0 (open)	1	0 (in-phase)
Z_0 $jX = j\omega L$	1	$\pm 180° - 2\tan^{-1} x$
Z_0 $jX = \dfrac{-j}{\omega C}$	1	$\pm 180° + 2\tan^{-1} x$

$$z_L = Z_L / Z_0 = (R + jX)/Z_0 = r + jx$$

Current Reflection Coefficient

$$\frac{I_0^-}{I_0^+} = -\frac{V_0^-}{V_0^+} = -\Gamma. \qquad (2.61)$$

We note that whereas the ratio of the voltage amplitudes is equal to Γ, *the ratio of the current amplitudes is equal to* $-\Gamma$.

Example 2-3: Reflection Coefficient of a Series *RC* Load

A 100-Ω transmission line is connected to a load consisting of a 50-Ω resistor in series with a 10-pF capacitor. Find the reflection coefficient at the load for a 100-MHz signal.

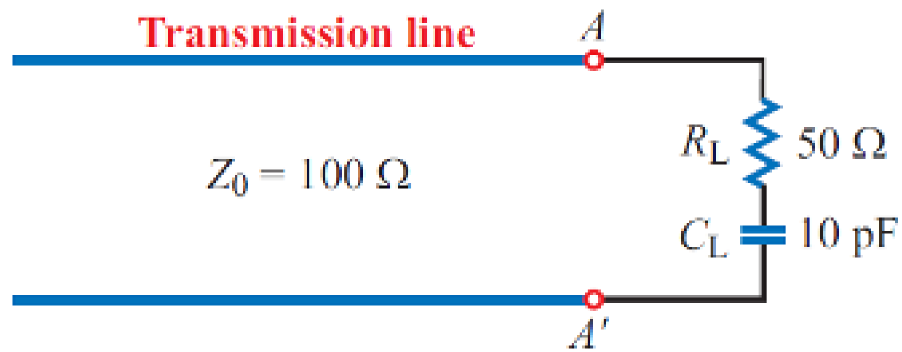
Solution: The following quantities are given (Fig. 2-13):

$$R_L = 50\ \Omega, \qquad C_L = 10\text{ pF} = 10^{-11}\text{ F},$$

$$Z_0 = 100\ \Omega, \qquad f = 100\text{ MHz} = 10^8\text{ Hz}.$$

The normalized load impedance is

$$\begin{aligned} z_L &= \frac{Z_L}{Z_0} = \frac{R_L - j/(\omega C_L)}{Z_0} \\ &= \frac{1}{100}\left(50 - j\frac{1}{2\pi \times 10^8 \times 10^{-11}}\right) \\ &= (0.5 - j1.59)\ \Omega. \end{aligned}$$



$$\begin{aligned} \Gamma &= \frac{z_L - 1}{z_L + 1} \\ &= \frac{0.5 - j1.59 - 1}{0.5 - j1.59 + 1} \\ &= \frac{-0.5 - j1.59}{1.5 - j1.59} = \frac{-1.67e^{j72.6^\circ}}{2.19e^{-j46.7^\circ}} = -0.76e^{j119.3^\circ}. \end{aligned}$$

This result may be converted into the form of Eq. (2.62) by replacing the minus sign with e^{-j180°. Thus,

$$\Gamma = 0.76e^{j119.3^\circ}e^{-j180^\circ} = 0.76e^{-j60.7^\circ} = 0.76\angle{-60.7^\circ},$$

or

$$\boxed{|\Gamma| = 0.76, \qquad \theta_r = -60.7^\circ.}$$

Standing Waves

Using the relation $V_0^- = \Gamma V_0^+$:
yields

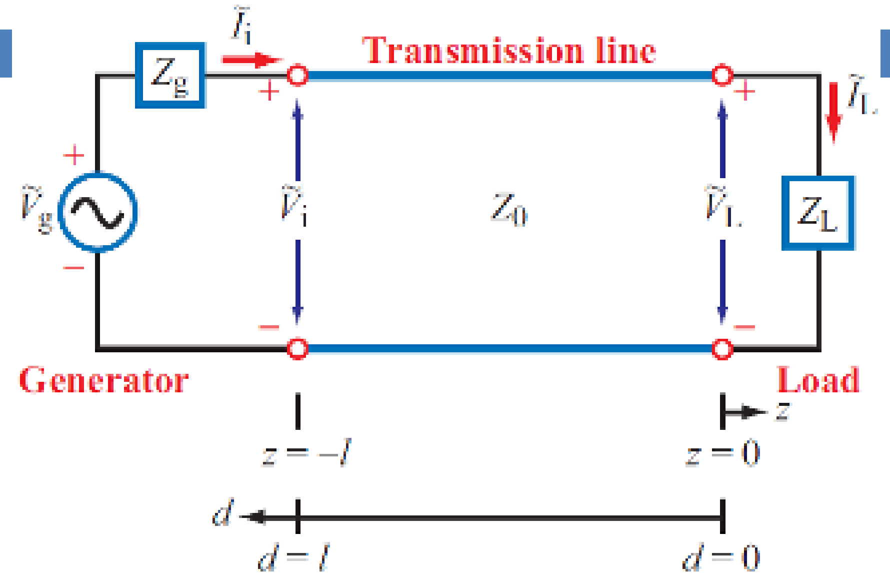
$$\widetilde{V}(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z}),$$

$$\widetilde{I}(z) = \frac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z}).$$

$$|\widetilde{V}(z)| = \left\{ \left[V_0^+ (e^{-j\beta z} + |\Gamma| e^{j\theta_r} e^{j\beta z}) \right] \cdot \left[(V_0^+)^* (e^{j\beta z} + |\Gamma| e^{-j\theta_r} e^{-j\beta z}) \right] \right\}^{1/2}$$

$$= |V_0^+| \left[1 + |\Gamma|^2 + |\Gamma| (e^{j(2\beta z + \theta_r)} + e^{-j(2\beta z + \theta_r)}) \right]^{1/2}$$

$$= |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta z + \theta_r) \right]^{1/2}, \qquad (2.64)$$



To express the magnitude of $\widetilde{V}$ as a function of d instead of z, we replace z with $-d$ on the right-hand side of Eq. (2.64):

$$|\widetilde{V}(d)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta d - \theta_r) \right]^{1/2}. \qquad (2.66)$$

voltage magnitude

By applying the same steps to Eq. (2.63b), a similar expression can be derived for $|\widetilde{I}(d)|$, the magnitude of the current $\widetilde{I}(d)$:

$$|\widetilde{I}(d)| = \frac{|V_0^+|}{Z_0} [1 + |\Gamma|^2 - 2|\Gamma| \cos(2\beta d - \theta_r)]^{1/2}. \qquad (2.67)$$

current magnitude

Standing-Wave Pattern

Whereas the repetition period is λ for the incident and reflected waves considered individually, the repetition period of the standing-wave pattern is λ/2.

$$|\tilde{V}(d)| = |V_0^+|\left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)\right]^{1/2}. \quad (2.66)$$

$$|\tilde{I}(d)| = \frac{|V_0^+|}{Z_0}\left[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta d - \theta_r)\right]^{1/2}. \quad (2.67)$$

Voltage magnitude is maximum

when $(2\beta d_{\min} - \theta_r) = (2n+1)\pi$

When voltage is a maximum, current is a minimum, and vice versa

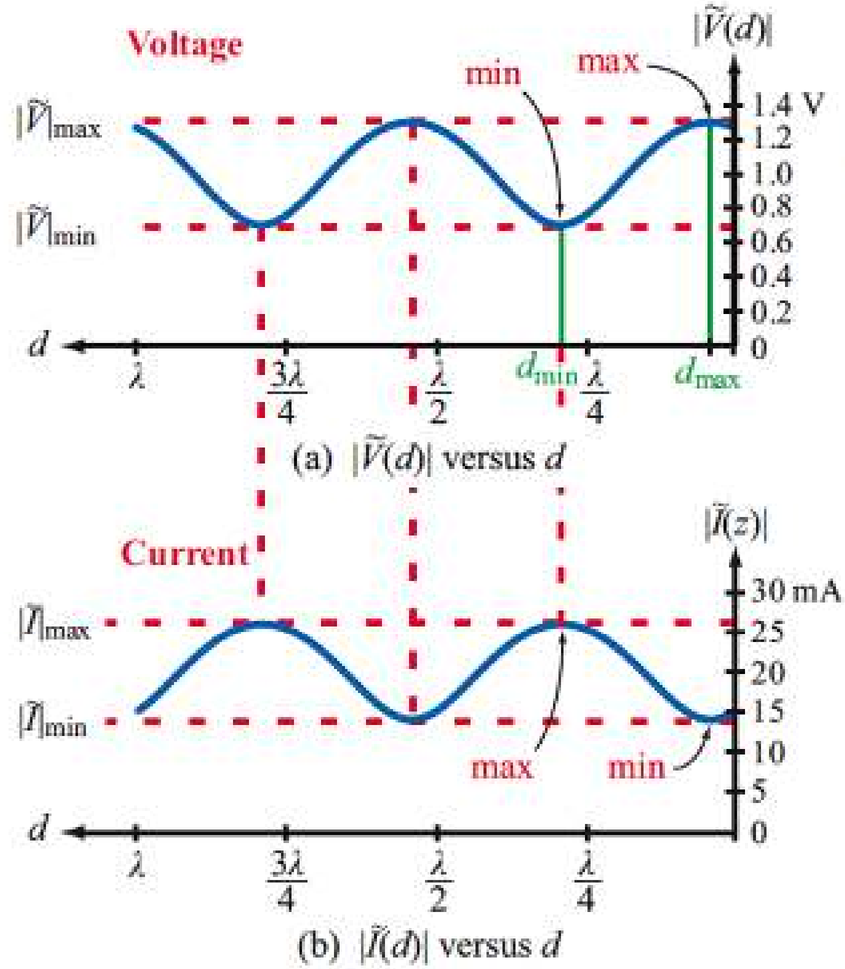
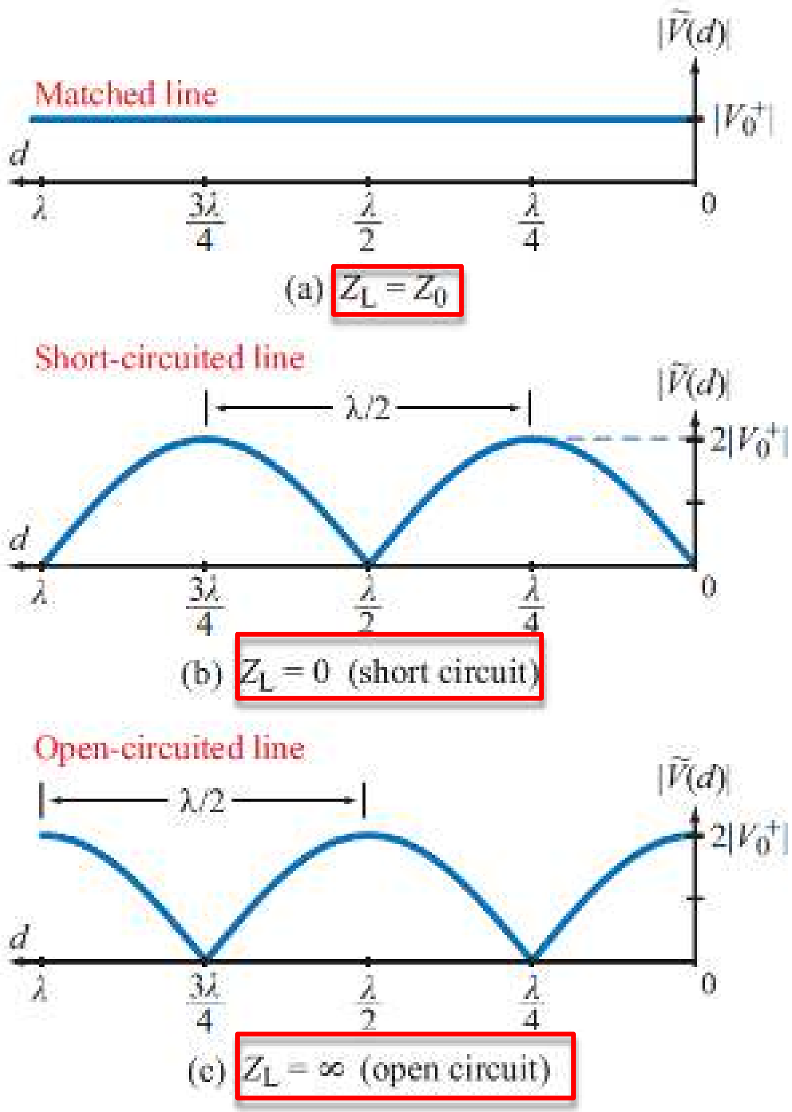


Figure 2-14: Standing-wave pattern for (a) $|\tilde{V}(d)|$ and (b) $|\tilde{I}(d)|$ for a lossless transmission line of characteristic impedance $Z_0 = 50\ \Omega$, terminated in a load with a reflection coefficient $\Gamma = 0.3e^{j30°}$. The magnitude of the incident wave $|V_0^+| = 1$ V. The standing-wave ratio is $S = |\tilde{V}|_{max}/|\tilde{V}|_{min} = 1.3/0.7 = 1.86$.

Standing Wave Patterns for 3 Types of Loads



With no reflected wave present, there will be no interference and no standing waves.

Example 2-4: $|\Gamma|$ for Purely Reactive Load

Show that $|\Gamma| = 1$ for a lossless line connected to a purely reactive load.

Solution: The load impedance of a purely reactive load is

$$Z_L = jX_L.$$

From Eq. (2.59), the reflection coefficient is

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0}$$

$$= \frac{-(Z_0 - jX_L)}{(Z_0 + jX_L)} = \frac{-\sqrt{Z_0^2 + X_L^2}\, e^{-j\theta}}{\sqrt{Z_0^2 + X_L^2}\, e^{j\theta}} = -e^{-j2\theta},$$

where $\theta = \tan^{-1} X_L / Z_0$. Hence

$$|\Gamma| = |-e^{-j2\theta}| = [(e^{-j2\theta})(e^{-j2\theta})^*]^{1/2} = 1.$$

Maxima & Minima

Standing-Wave Pattern

Let us denote $d_{\max}$ as the distance from the load at which $|\widetilde{V}(d)|$ is a maximum. It then follows that

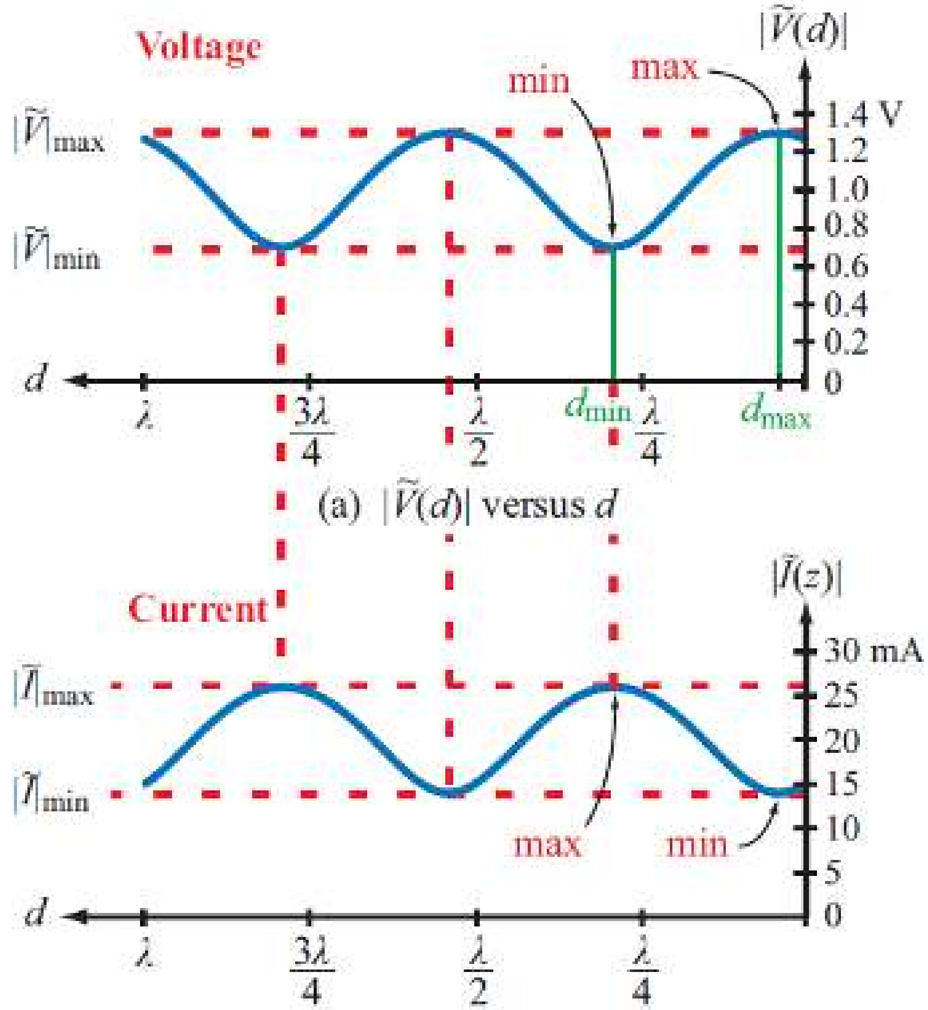
$$|\widetilde{V}(d)| = |\widetilde{V}|_{\max} = |V_0^+|[1 + |\Gamma|], \qquad (2.68)$$

when

$$2\beta d_{\max} - \theta_r = 2n\pi, \qquad (2.69)$$

with $n = 0$ or a positive integer. Solving Eq. (2.69) for $d_{\max}$, we have

$$d_{\max} = \frac{\theta_r + 2n\pi}{2\beta} = \frac{\theta_r\lambda}{4\pi} + \frac{n\lambda}{2}, \qquad \begin{cases} n = 1, 2, \ldots & \text{if } \theta_r < 0, \\ n = 0, 1, 2, \ldots & \text{if } \theta_r \geq 0, \end{cases} \qquad (2.70)$$



(a) $|\widetilde{V}(d)|$ versus d

(b) $|\widetilde{I}(d)|$ versus d

Maxima & Minima (cont.)

$$|\widetilde{V}|_{\min} = |V_0^+|[1 - |\Gamma|],$$

$$\text{when } (2\beta d_{\min} - \theta_r) = (2n+1)\pi$$

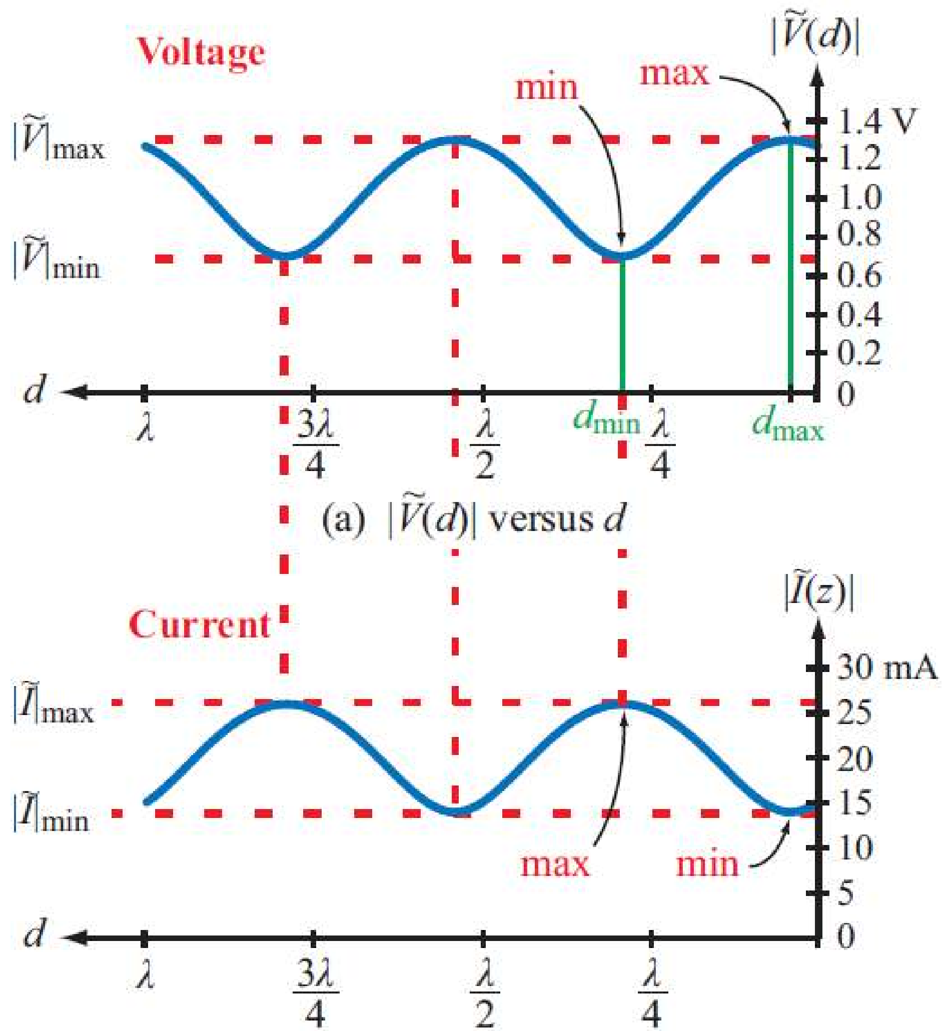
$$S = \frac{|\widetilde{V}|_{\max}}{|\widetilde{V}|_{\min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \quad \text{(dimensionless)}$$

S = Voltage Standing Wave Ratio

For a matched load: $S = 1$

For a short, open, or purely reactive load:

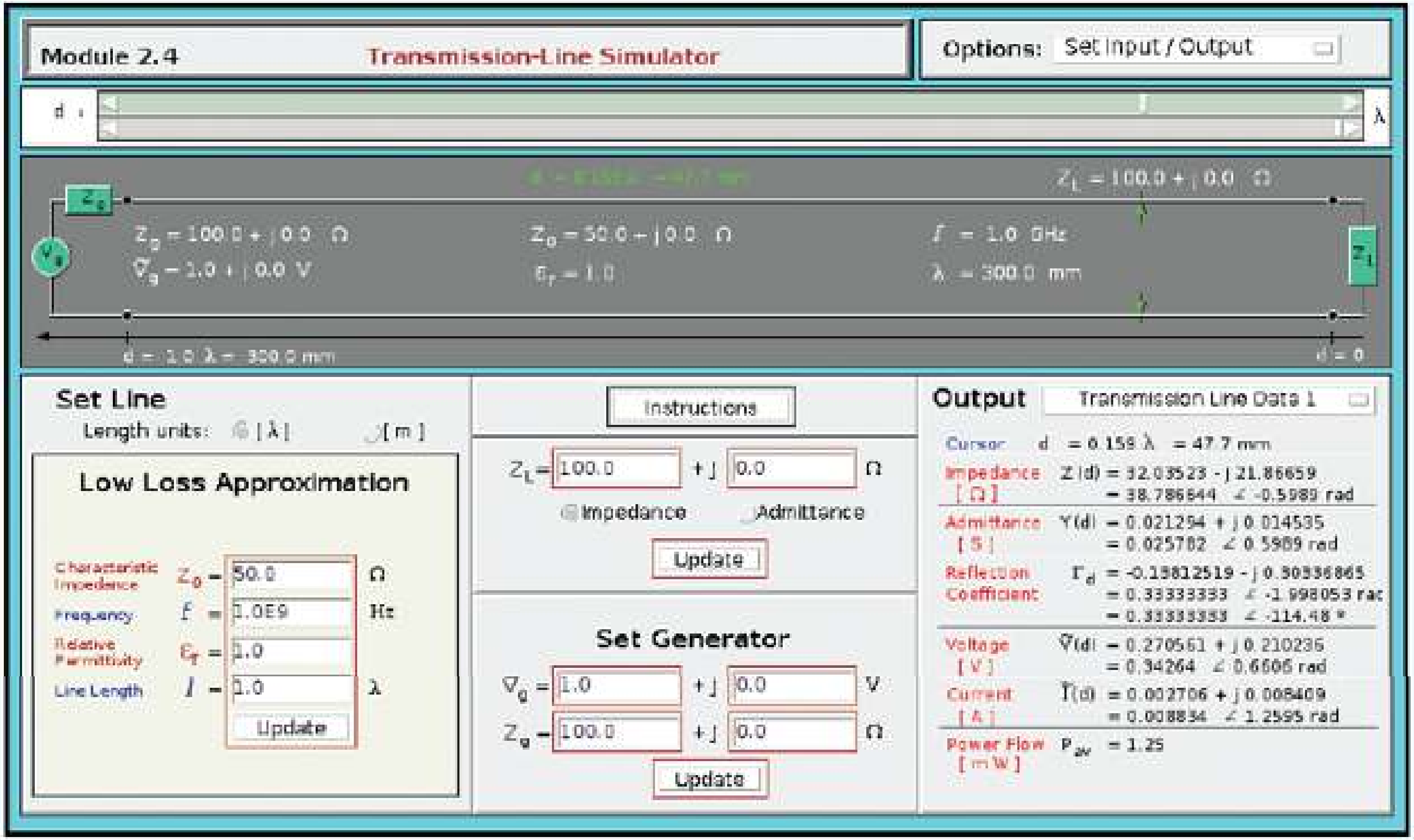
$S = \infty$



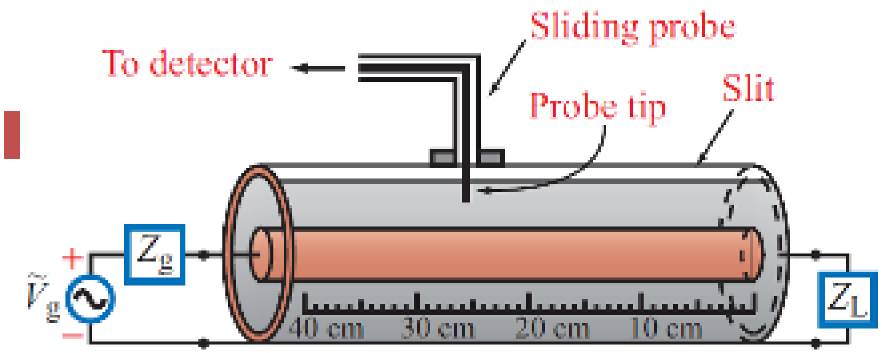
(a) $|\widetilde{V}(d)|$ versus d

(b) $|\widetilde{I}(d)|$ versus d

Module 2.4 Transmission-Line Simulator Upon specifying the requisite input data—including the load impedance at $d = 0$ and the generator voltage and impedance at $d = l$—this module provides a wealth of output information about the voltage and current waveforms along the trasmission line. You can view plots of the standing wave patterns for voltage and current, the time and spatial variations of the instantaneous voltage $v(d, t)$ and current $i(d, t)$, and other related quantities.



Example 2-6: Measuring Z_L with a Slotted Line



Solution: The following quantities are given:

$$Z_0 = 50\ \Omega,$$
$$S = 3,$$
$$d_{\min} = 12\text{ cm}.$$

Since the distance between successive voltage minima is $\lambda/2$,

$$\lambda = 2 \times 0.3 = 0.6\text{ m},$$

and

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.6} = \frac{10\pi}{3} \qquad \text{(rad/m)}.$$

From Eq. (2.73), solving for $|\Gamma|$ in terms of S gives

$$|\Gamma| = \frac{S-1}{S+1} = \frac{3-1}{3+1} = 0.5.$$

Next, we use the condition given by Eq. (2.71) to find θ_r:

$$2\beta d_{\min} - \theta_r = \pi, \qquad \text{for } n = 0 \text{ (first minimum)},$$

which gives

$$\theta_r = 2\beta d_{\min} - \pi = 2 \times \frac{10\pi}{3} \times 0.12 - \pi = -0.2\pi \text{ (rad)} = -36^\circ.$$

Hence,

$$\Gamma = |\Gamma| e^{j\theta_r} = 0.5 e^{-j36^\circ} = 0.405 - j0.294.$$

Solving Eq. (2.59) for Z_L, we have

$$Z_L = Z_0 \left[\frac{1+\Gamma}{1-\Gamma}\right] = 50\left[\frac{1 + 0.405 - j0.294}{1 - 0.405 + j0.294}\right] \boxed{= (85 - j67)\ \Omega.}$$

Wave Impedance

At a distance *d* from the load:

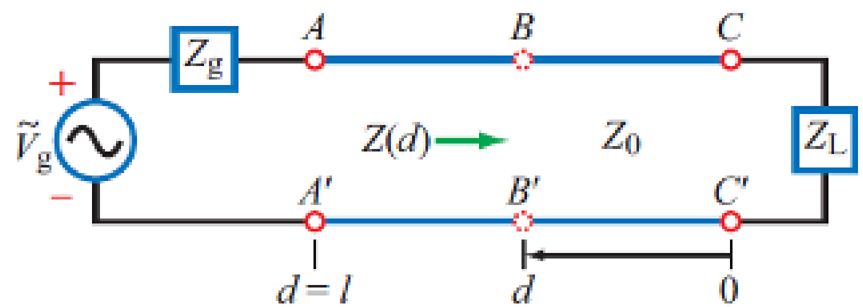
$$Z(d) = \frac{\tilde{V}(d)}{\tilde{I}(d)} = \frac{V_0^+[e^{j\beta d} + \Gamma e^{-j\beta d}]}{V_0^+[e^{j\beta d} - \Gamma e^{-j\beta d}]} Z_0 = Z_0 \left[\frac{1+\Gamma e^{-j2\beta d}}{1-\Gamma e^{-j2\beta d}}\right] = Z_0 \left[\frac{1+\Gamma_d}{1-\Gamma_d}\right] \quad (\Omega),$$

where we define

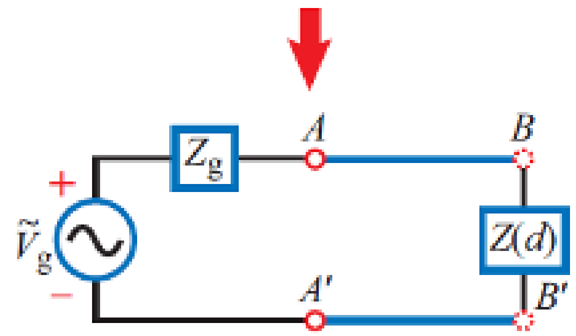
$$\Gamma_d = \Gamma e^{-j2\beta d} = |\Gamma| e^{j\theta_r} e^{-j2\beta d} = |\Gamma| e^{j(\theta_r - 2\beta d)}$$

as the ***phase-shifted voltage reflection coefficient***,

$Z(d)$ is the ratio of the total voltage (incident- and reflected-wave voltages) to the total current at any point d on the line, in contrast with the characteristic impedance of the line Z_0, which relates the voltage and current of each of the two waves individually ($Z_0 = V_0^+/I_0^+ = -V_0^-/I_0^-$).

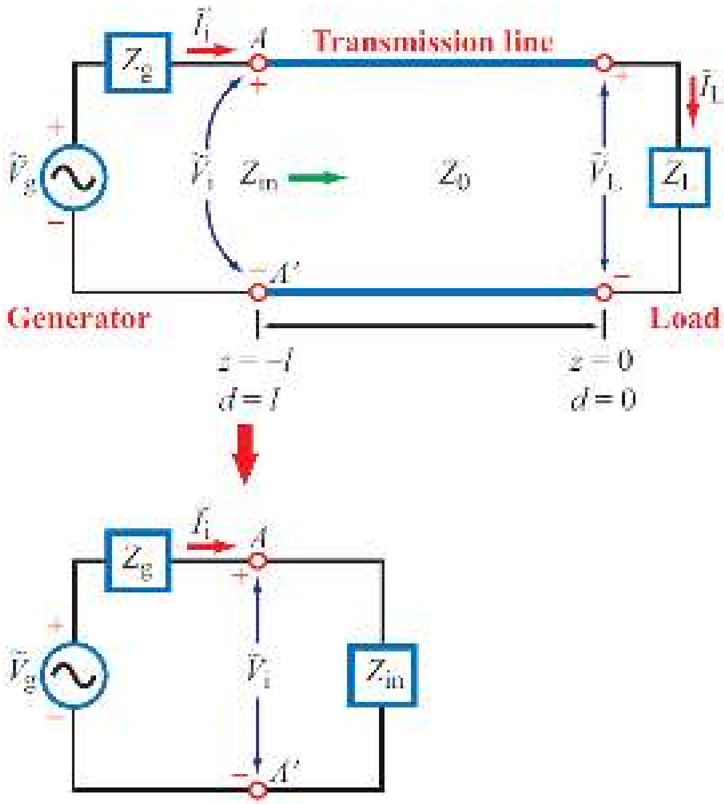


(a) Actual circuit



(b) Equivalent circuit

Input Impedance



$$Z_{in} = Z_0 \left(\frac{z_L \cos\beta l + j \sin\beta l}{\cos\beta l + j z_L \sin\beta l} \right)$$
$$= Z_0 \left(\frac{z_L + j \tan\beta l}{1 + j z_L \tan\beta l} \right). \qquad (2.79)$$

Fig. 2-18. The phasor voltage across Z_{in} is given by

$$\tilde{V}_i = \tilde{I}_i Z_{in} = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}, \qquad (2.80)$$

Simultaneously, from the standpoint of the transmission line, the voltage across it at the input of the line is given by Eq. (2.63a) with $z = -l$:

$$\tilde{V}_i = \tilde{V}(-l) = V_0^+[e^{j\beta l} + \Gamma e^{-j\beta l}]. \qquad (2.81)$$

Equating Eq. (2.80) to Eq. (2.81) and then solving for V_0^+ leads to

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \right) \left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}} \right). \qquad (2.82)$$

At input, $d = l$: $$Z_{in} = Z(l) = Z_0 \left[\frac{1 + \Gamma_l}{1 - \Gamma_l} \right].$$

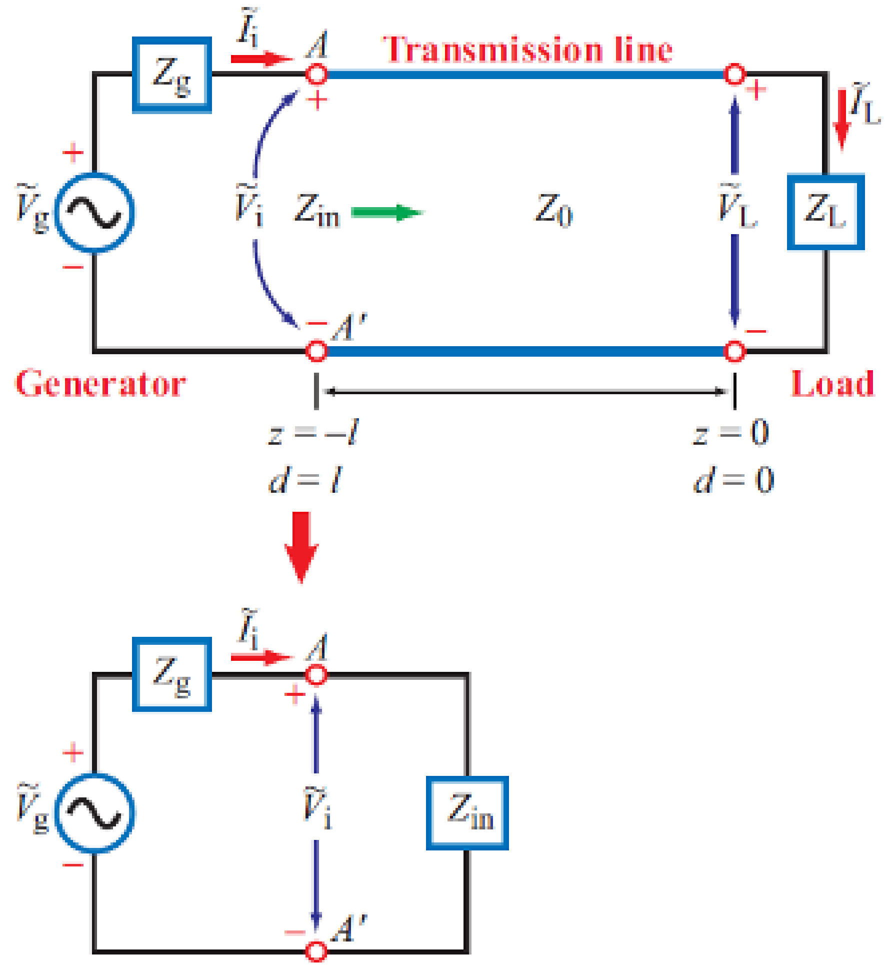
$$\Gamma_l = \Gamma e^{-j2\beta l} = |\Gamma| e^{j(\theta_r - 2\beta l)}.$$

Example 2-7: Complete Solution for $v(z,t)$ and $i(z,t)$

A 1.05-GHz generator circuit with series impedance $Z_g = 10\ \Omega$ and voltage source given by

$$v_g(t) = 10\sin(\omega t + 30°) \qquad \text{(V)}$$

is connected to a load $Z_L = (100 + j50)\ \Omega$ through a 50-Ω, 67-cm long lossless transmission line. The phase velocity of the line is 0.7c, where c is the velocity of light in a vacuum. Find $v(z,t)$ and $i(z,t)$ on the line.



Solution: From the relationship $u_p = \lambda f$, we find the wavelength

$$\lambda = \frac{u_p}{f} = \frac{0.7 \times 3 \times 10^8}{1.05 \times 10^9} = 0.2 \text{ m},$$

and

$$\beta l = \frac{2\pi}{\lambda}\, l = \frac{2\pi}{0.2} \times 0.67 = 6.7\pi = 0.7\pi = 126°,$$

where we have subtracted multiples of 2π. The voltage reflection coefficient at the load is

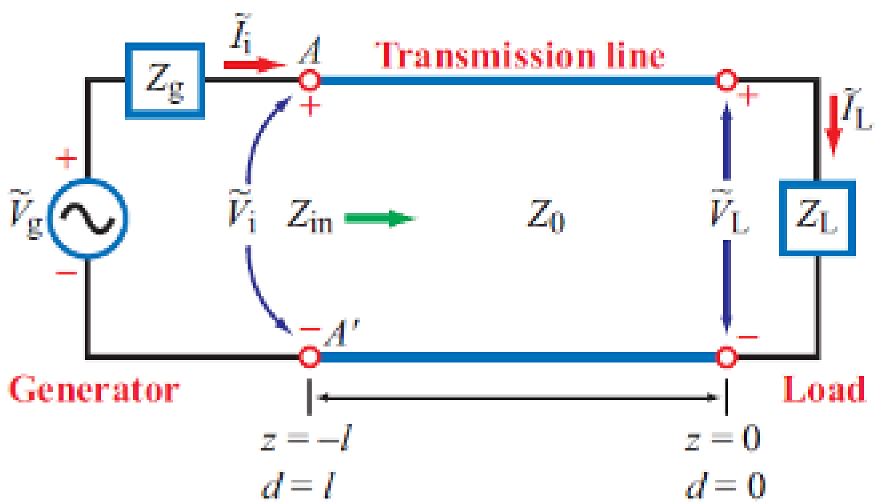
$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(100 + j50) - 50}{(100 + j50) + 50} = 0.45e^{j26.6°}.$$

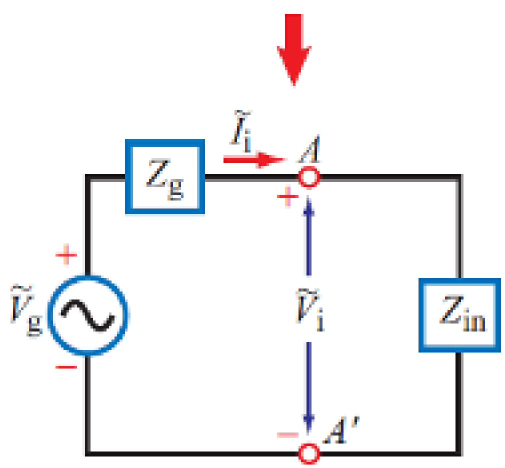
Cont.

Example 2-7: Complete Solution for $v(z,t)$ and $i(z,t)$ (cont.)

A 1.05-GHz generator circuit with series impedance $Z_g = 10\ \Omega$ and voltage source given by

$$v_g(t) = 10 \sin(\omega t + 30^\circ) \qquad \text{(V)}$$





$$Z_{in} = Z_0 \left(\frac{1 + \Gamma_l}{1 - \Gamma_l} \right)$$

$$= Z_0 \left(\frac{1 + \Gamma e^{-j2\beta l}}{1 - \Gamma e^{-j2\beta l}} \right)$$

$$= 50 \left(\frac{1 + 0.45 e^{j26.6^\circ} e^{-j252^\circ}}{1 - 0.45 e^{j26.6^\circ} e^{-j252^\circ}} \right) = (21.9 + j17.4)\ \Omega.$$

Rewriting the expression for the generator voltage with the cosine reference, we have

$$v_g(t) = 10 \sin(\omega t + 30^\circ)$$

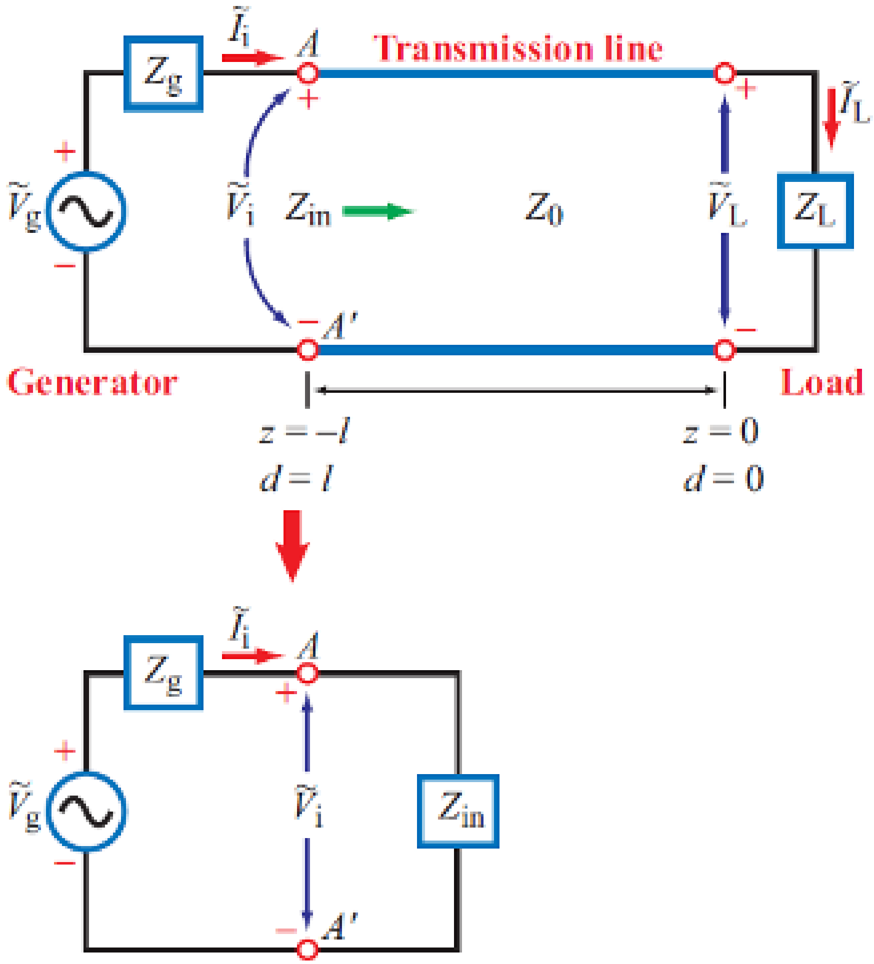
$$= 10 \cos(90^\circ - \omega t - 30^\circ)$$

$$= 10 \cos(\omega t - 60^\circ)$$

$$= \Re\mathfrak{e}[10 e^{-j60^\circ} e^{j\omega t}] = \Re\mathfrak{e}[\tilde{V}_g e^{j\omega t}] \qquad \text{(V)}.$$

Cont.

Example 2-7: Complete Solution for $v(z,t)$ and $i(z,t)$ (cont.)



Hence, the phasor voltage $\tilde{V}_g$ is given by

$$\tilde{V}_g = 10\, e^{-j60^\circ} = 10\angle{-60^\circ} \quad \text{(V)}.$$

Application of Eq. (2.82) gives

$$\begin{aligned} V_0^+ &= \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) \\ &= \left[\frac{10e^{-j60^\circ}(21.9 + j17.4)}{10 + 21.9 + j17.4}\right] \\ &\quad \cdot (e^{j126^\circ} + 0.45e^{j26.6^\circ}e^{-j126^\circ})^{-1} \\ &= 10.2e^{j159^\circ} \quad \text{(V)}. \end{aligned}$$

Using Eq. (2.63a) with $z = -d$, the phasor voltage on the line is

$$\begin{aligned} \tilde{V}(d) &= V_0^+(e^{j\beta d} + \Gamma e^{-j\beta d}) \\ &= 10.2e^{j159^\circ}(e^{j\beta d} + 0.45e^{j26.6^\circ}e^{-j\beta d}), \end{aligned}$$

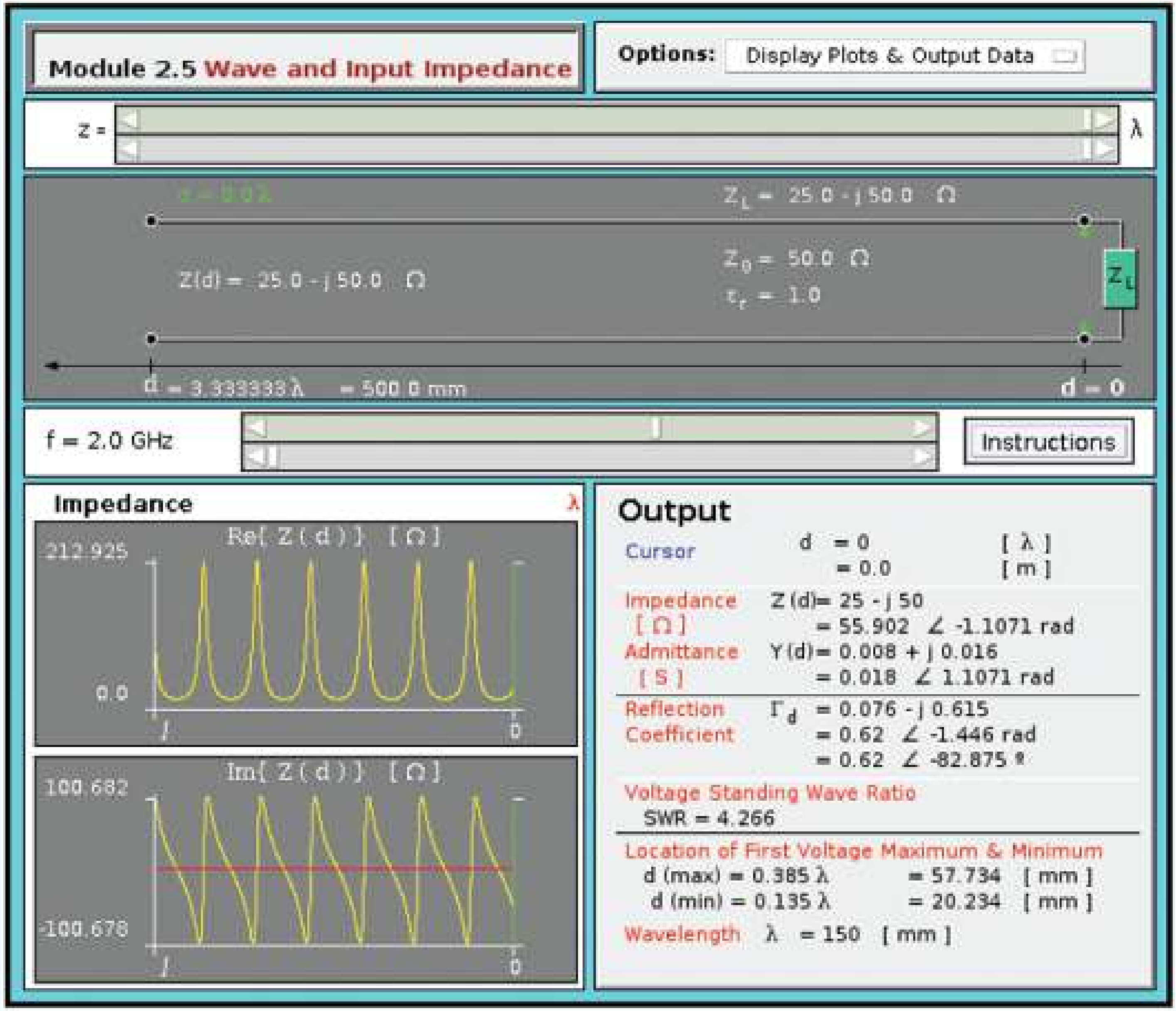
and the corresponding instantaneous voltage $v(d,t)$ is

$$\boxed{\begin{aligned} v(d,t) &= \Re\mathfrak{e}[\tilde{V}(d)\, e^{j\omega t}] \\ &= 10.2\cos(\omega t + \beta d + 159^\circ) \\ &\quad + 4.55\cos(\omega t - \beta d + 185.6^\circ) \quad \text{(V)}. \end{aligned}}$$

Similarly, Eq. (2.63b) leads to

$$\boxed{\begin{aligned} \tilde{I}(d) &= 0.20e^{j159^\circ}(e^{j\beta d} - 0.45e^{j26.6^\circ}e^{-j\beta d}), \\ i(d,t) &= 0.20\cos(\omega t + \beta d + 159^\circ) \\ &\quad + 0.091\cos(\omega t - \beta d + 185.6^\circ) \quad \text{(A)}. \end{aligned}}$$

Module 2.5 Wave and Input Impedance The wave impedance, $Z(d) = \tilde{V}(d)/\tilde{I}(d)$, exhibits a cyclical pattern as a function of position along the line. This module displays plots of the real and imaginary parts of $Z(d)$, specifies the locations of the voltage maximum and minimum nearest to the load, and provides other related information.



Short-Circuited Line

For the short-circuited line: $\Gamma = -1$

$$\tilde{V}_{sc}(d) = V_0^+[e^{j\beta d} - e^{-j\beta d}] = 2jV_0^+ \sin\beta d,$$

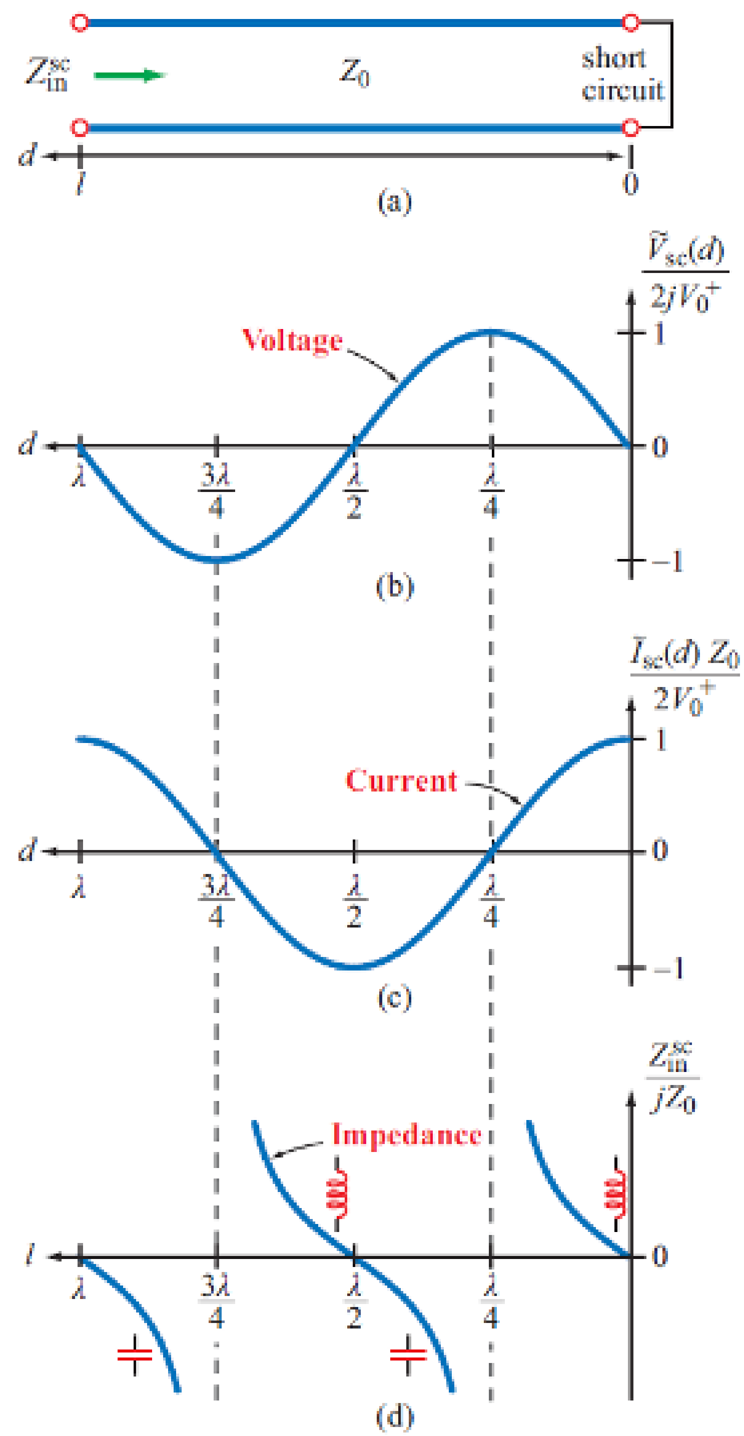
$$\tilde{I}_{sc}(d) = \frac{V_0^+}{Z_0}[e^{j\beta d} + e^{-j\beta d}] = \frac{2V_0^+}{Z_0} \cos\beta d,$$

$$Z_{sc}(d) = \frac{\tilde{V}_{sc}(d)}{\tilde{I}_{sc}(d)} = jZ_0 \tan\beta d.$$

At its input, the line appears like an inductor or a capacitor depending on the sign of $\tan\beta d$

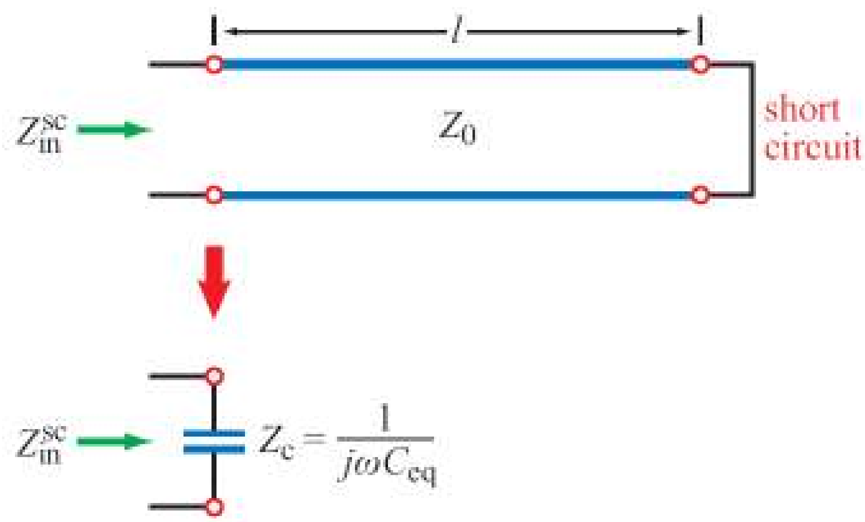
$$j\omega L_{eq} = jZ_0 \tan\beta l, \qquad \text{if } \tan\beta l \geq 0$$

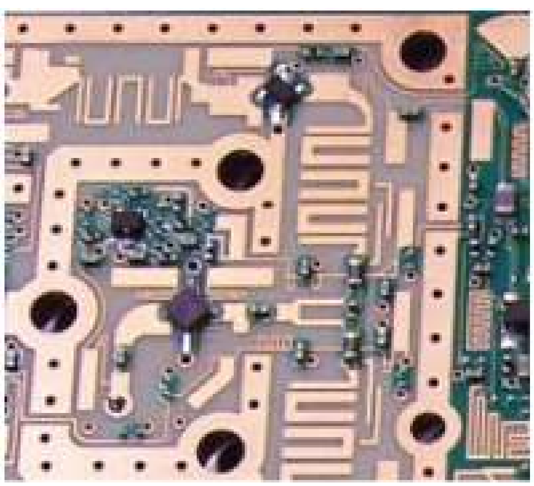
$$\frac{1}{j\omega C_{eq}} = jZ_0 \tan\beta l, \qquad \text{if } \tan\beta l \leq 0$$



Example 2-8: Equivalent Reactive Elements

Choose the length of a shorted 50-Ω lossless transmission line (Fig. 2-20) such that its input impedance at 2.25 GHz is identical to that of a capacitor with capacitance $C_{eq} = 4$ pF. The wave velocity on the line is $0.75c$.





Solution: We are given

$$u_p = 0.75c = 0.75 \times 3 \times 10^8 = 2.25 \times 10^8 \text{ m/s},$$

$$Z_0 = 50\ \Omega,$$

$$f = 2.25 \text{ GHz} = 2.25 \times 10^9 \text{ Hz},$$

$$C_{eq} = 4 \text{ pF} = 4 \times 10^{-12} \text{ F}.$$

The phase constant is

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{u_p} = \frac{2\pi \times 2.25 \times 10^9}{2.25 \times 10^8} = 62.8 \qquad \text{(rad/m)}.$$

From Eq. (2.89), it follows that

$$\begin{aligned} \tan \beta l &= -\frac{1}{Z_0 \omega C_{eq}} \\ &= -\frac{1}{50 \times 2\pi \times 2.25 \times 10^9 \times 4 \times 10^{-12}} \\ &= -0.354. \end{aligned}$$

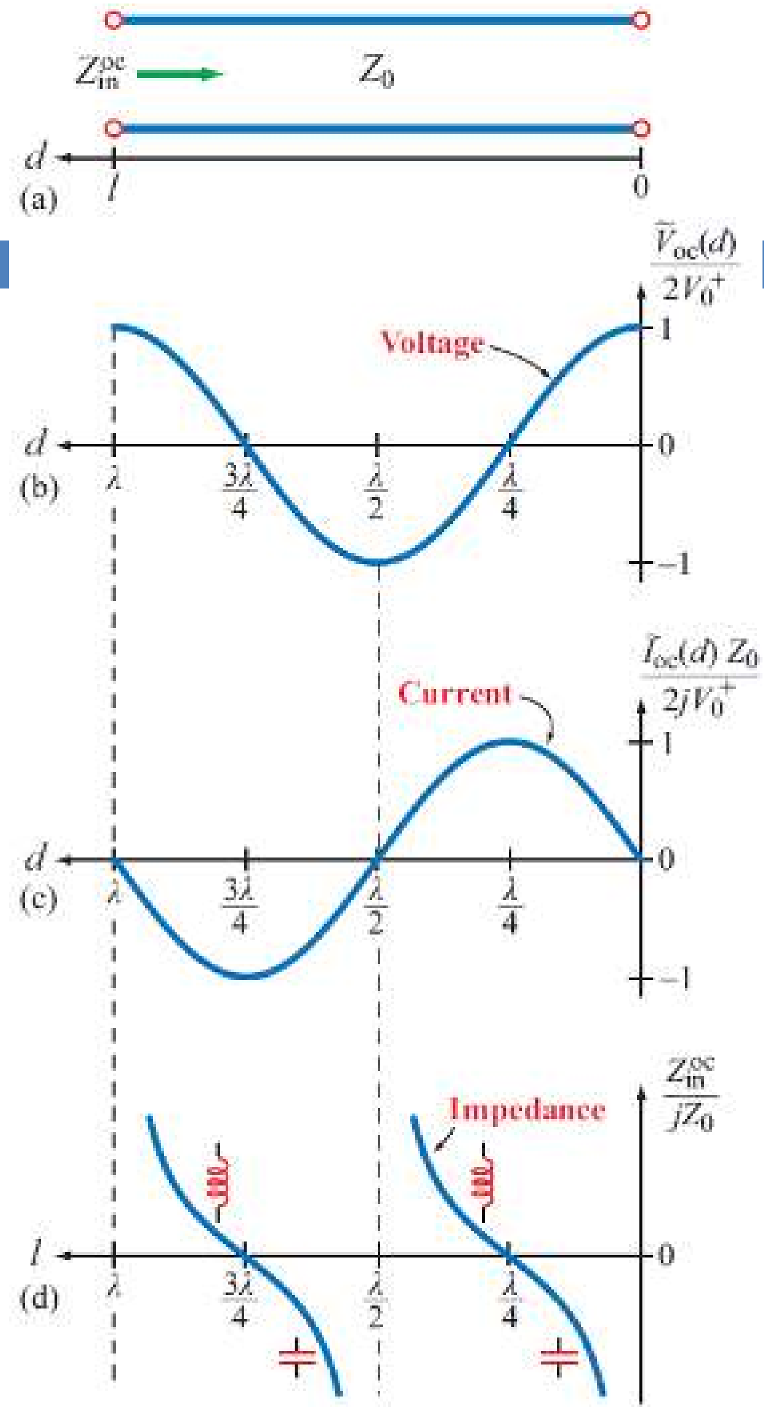
The tangent function is negative when its argument is in the second or fourth quadrants. The solution for the second quadrant is

$$\beta l_1 = 2.8 \text{ rad} \quad \text{or} \quad \boxed{l_1 = \frac{2.8}{\beta} = \frac{2.8}{62.8} = 4.46 \text{ cm},}$$

and the solution for the fourth quadrant is

$$\beta l_2 = 5.94 \text{ rad} \quad \text{or} \quad \boxed{l_2 = \frac{5.94}{62.8} = 9.46 \text{ cm}.}$$

Open-Circuited Line



$$\widetilde{V}_{oc}(d) = V_0^+[e^{j\beta d} + e^{-j\beta d}] = 2V_0^+ \cos\beta d,$$

$$\widetilde{I}_{oc}(d) = \frac{V_0^+}{Z_0}[e^{j\beta d} - e^{-j\beta d}] = \frac{2jV_0^+}{Z_0} \sin\beta d,$$

$$Z_{in}^{oc} = \frac{\widetilde{V}_{oc}(l)}{\widetilde{I}_{oc}(l)} = -jZ_0 \cot\beta l. \qquad (2.93)$$

Short-Circuit/Open-Circuit Method

- For a line of known length l, measurements of its input impedance, one when terminated in a short and another when terminated in an open, can be used to find its characteristic impedance Z_0 and electrical length βl

$$Z_{\text{in}}^{\text{sc}} = \frac{\widetilde{V}_{\text{sc}}(l)}{\widetilde{I}_{\text{sc}}(l)} = jZ_0 \tan \beta l.$$

$$Z_{\text{in}}^{\text{oc}} = \frac{\widetilde{V}_{\text{oc}}(l)}{\widetilde{I}_{\text{oc}}(l)} = -jZ_0 \cot \beta l.$$

$$Z_0 = +\sqrt{Z_{\text{in}}^{\text{sc}}\, Z_{\text{in}}^{\text{oc}}}\,,$$

$$\tan \beta l = \sqrt{\frac{-Z_{\text{in}}^{\text{sc}}}{Z_{\text{in}}^{\text{oc}}}}\,.$$

2-8.4 Lines of Length $l = n\lambda/2$

If $l = n\lambda/2$, where n is an integer,

$$\tan \beta l = \tan[(2\pi/\lambda)(n\lambda/2)] = \tan n\pi = 0.$$

Consequently, Eq. (2.79) reduces to

$$Z_{\text{in}} = Z_{\text{L}}, \qquad \text{for } l = n\lambda/2, \qquad (2.96)$$

which means that a half-wavelength line (or any integer multiple of $\lambda/2$) does not modify the load impedance.

2-8.5 Quarter-Wavelength Transformer

$$Z_{\text{in}} = \frac{Z_0^2}{Z_{\text{L}}}, \qquad \text{for } l = \lambda/4 + n\lambda/2.$$

Example 2-10: $\lambda/4$ Transformer

A 50-Ω lossless transmission line is to be matched to a resistive load impedance with $Z_{\text{L}} = 100\ \Omega$ via a quarter-wave section as shown in Fig. 2-22, thereby eliminating reflections along the feedline. Find the required characteristic impedance of the quarter-wave transformer.

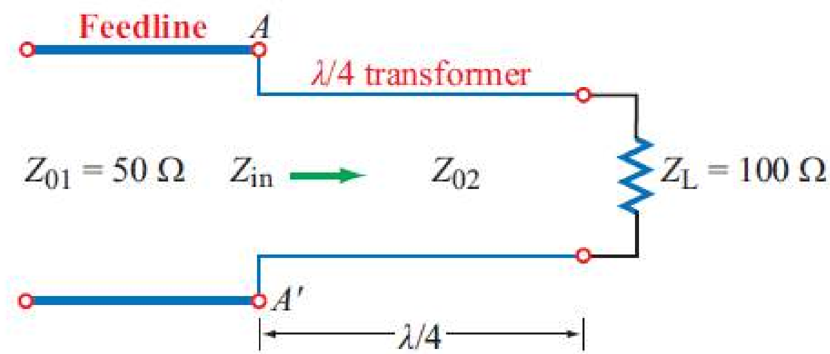


Figure 2-22: Configuration for Example 2-10.

Solution: To eliminate reflections at terminal AA', the input impedance Z_{in} looking into the quarter-wave line should be equal to Z_{01}, the characteristic impedance of the feedline. Thus, $Z_{\text{in}} = 50\ \Omega$. From Eq. (2.97),

$$Z_{\text{in}} = \frac{Z_{02}^2}{Z_{\text{L}}},$$

or

$$Z_{02} = \sqrt{Z_{\text{in}} Z_{\text{L}}} = \sqrt{50 \times 100} = 70.7\ \Omega.$$

Whereas this eliminates reflections on the feedline, it does not eliminate them on the $\lambda/4$ line.

Table 2-4: Properties of standing waves on a lossless transmission line.

Voltage Maximum **Voltage Minimum**	$\lvert\tilde{V}\rvert_{max} = \lvert V_0^+\rvert[1+\lvert\Gamma\rvert]$ $\lvert\tilde{V}\rvert_{min} = \lvert V_0^+\rvert[1-\lvert\Gamma\rvert]$
Positions of voltage maxima (also positions of current minima) Position of first maximum (also position of first current minimum)	$d_{max} = \frac{\theta_r\lambda}{4\pi} + \frac{n\lambda}{2}, \quad n = 0, 1, 2, \ldots$ $d_{max} = \begin{cases} \frac{\theta_r\lambda}{4\pi}, & \text{if } 0 \le \theta_r \le \pi \\ \frac{\theta_r\lambda}{4\pi} + \frac{\lambda}{2}, & \text{if } -\pi \le \theta_r \le 0 \end{cases}$
Positions of voltage minima (also positions of current maxima) Position of first minimum (also position of first current maximum)	$d_{min} = \frac{\theta_r\lambda}{4\pi} + \frac{(2n+1)\lambda}{4}, \quad n = 0, 1, 2, \ldots$ $d_{min} = \frac{\lambda}{4}\left(1 + \frac{\theta_r}{\pi}\right)$
Input Impedance	$Z_{in} = Z_0\left(\frac{z_L + j\tan\beta l}{1 + jz_L\tan\beta l}\right) = Z_0\left(\frac{1+\Gamma_l}{1-\Gamma_l}\right)$
Positions at which Z_{in} is real	at voltage maxima and minima
Z_{in} at voltage maxima	$Z_{in} = Z_0\left(\frac{1+\lvert\Gamma\rvert}{1-\lvert\Gamma\rvert}\right)$
Z_{in} at voltage minima	$Z_{in} = Z_0\left(\frac{1-\lvert\Gamma\rvert}{1+\lvert\Gamma\rvert}\right)$
Z_{in} of short-circuited line	$Z_{in}^{sc} = jZ_0\tan\beta l$
Z_{in} of open-circuited line	$Z_{in}^{oc} = -jZ_0\cot\beta l$
Z_{in} of line of length $l = n\lambda/2$	$Z_{in} = Z_L, \quad n = 0, 1, 2, \ldots$
Z_{in} of line of length $l = \lambda/4 + n\lambda/2$ Z_{in} of matched line	$Z_{in} = Z_0^2/Z_L, \quad n = 0, 1, 2, \ldots$ $Z_{in} = Z_0$
$\lvert V_0^+\rvert$ = amplitude of incident wave; $\Gamma = \lvert\Gamma\rvert e^{j\theta_r}$ with $-\pi < \theta_r < \pi$; θ_r in radians; $\Gamma_l = \Gamma e^{-j2\beta l}$.	

Instantaneous Power Flow

$$v(d,t) = \Re\mathfrak{e}[\widetilde{V} e^{j\omega t}]$$
$$= \Re\mathfrak{e}[|V_0^+| e^{j\phi^+} (e^{j\beta d} + |\Gamma| e^{j\theta_r} e^{-j\beta d}) e^{j\omega t}]$$
$$= |V_0^+|[\cos(\omega t + \beta d + \phi^+)$$
$$+ |\Gamma| \cos(\omega t - \beta d + \phi^+ + \theta_r)], \quad (2.99a)$$

$$i(d,t) = \frac{|V_0^+|}{Z_0}[\cos(\omega t + \beta d + \phi^+)$$
$$- |\Gamma| \cos(\omega t - \beta d + \phi^+ + \theta_r)], \quad (2.99b)$$

$$P(d,t) = v(d,t)\, i(d,t)$$
$$= |V_0^+|[\cos(\omega t + \beta d + \phi^+)$$
$$+ |\Gamma| \cos(\omega t - \beta d + \phi^+ + \theta_r)]$$
$$\times \frac{|V_0^+|}{Z_0}[\cos(\omega t + \beta d + \phi^+)$$
$$- |\Gamma| \cos(\omega t - \beta d + \phi^+ + \theta_r)]$$
$$= \frac{|V_0^+|^2}{Z_0}[\cos^2(\omega t + \beta d + \phi^+)$$
$$- |\Gamma|^2 \cos^2(\omega t - \beta d + \phi^+ + \theta_r)]$$

$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{Z_0} \cos^2(\omega t + \beta d + \phi^+) \qquad \text{(W)},$$

$$P^{\mathrm{r}}(d,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{Z_0} \cos^2(\omega t - \beta d + \phi^+ + \theta_r)$$

Using the trigonometric identity

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x),$$

the expressions in Eq. (2.101) can be rewritten as

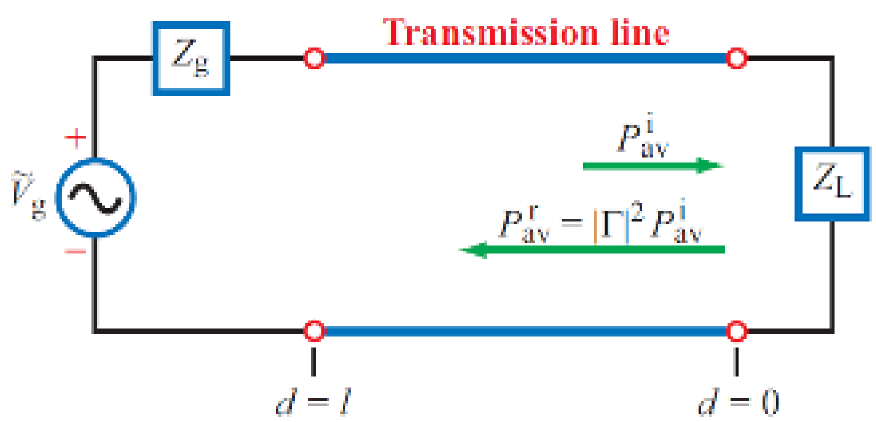
$$\boxed{\begin{aligned} P^{\mathrm{i}}(d,t) &= \frac{|V_0^+|^2}{2Z_0}[1 + \cos(2\omega t + 2\beta d + 2\phi^+)], \\ P^{\mathrm{r}}(d,t) &= -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}[1 + \cos(2\omega t - 2\beta d \\ &\qquad + 2\phi^+ + 2\theta_r)]. \end{aligned}}$$

The power oscillates at twice the rate of the voltage or current.

Average Power

$$P^{\mathrm{i}}(d,t) = \frac{|V_0^+|^2}{2Z_0}[1 + \cos(2\omega t + 2\beta d + 2\phi^+)],$$

$$P^{\mathrm{r}}(d,t) = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0}[1 + \cos(2\omega t - 2\beta d + 2\phi^+ + 2\theta_{\mathrm{r}})].$$



$$P^{\mathrm{i}}_{\mathrm{av}}(d) = \frac{1}{T}\int_0^T P^{\mathrm{i}}(d,t)\,dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} P^{\mathrm{i}}(d,t)\,dt. \quad (2.103)$$

Upon inserting Eq. (2.102a) into Eq. (2.103) and performing the integration, we obtain

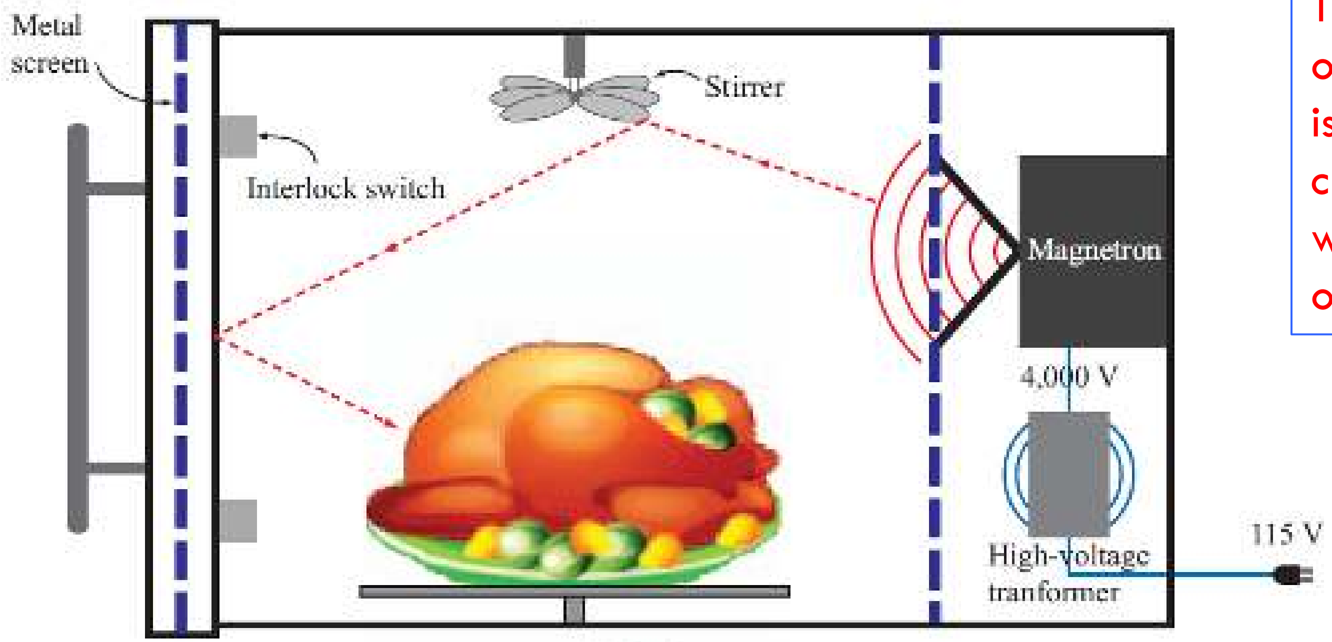
$$\boxed{P^{\mathrm{i}}_{\mathrm{av}} = \frac{|V_0^+|^2}{2Z_0} \quad \text{(W)},} \quad (2.104)$$

which is identical with the dc term of $P^{\mathrm{i}}(d,t)$ given by Eq. (2.102a). A similar treatment for the reflected wave gives

$$\boxed{P^{\mathrm{r}}_{\mathrm{av}} = -|\Gamma|^2 \frac{|V_0^+|^2}{2Z_0} = -|\Gamma|^2 P^{\mathrm{i}}_{\mathrm{av}}.} \quad (2.105)$$

The average reflected power is equal to the average incident power, diminished by a multiplicative factor of $|\Gamma|^2$.

Tech Brief 3: Standing Waves in Microwave Oven



(a) Cavity

The stirrer or rotation of the food platform is used to constantly change the standing wave pattern in the oven cavity

Tech Brief 3: Role of Frequency

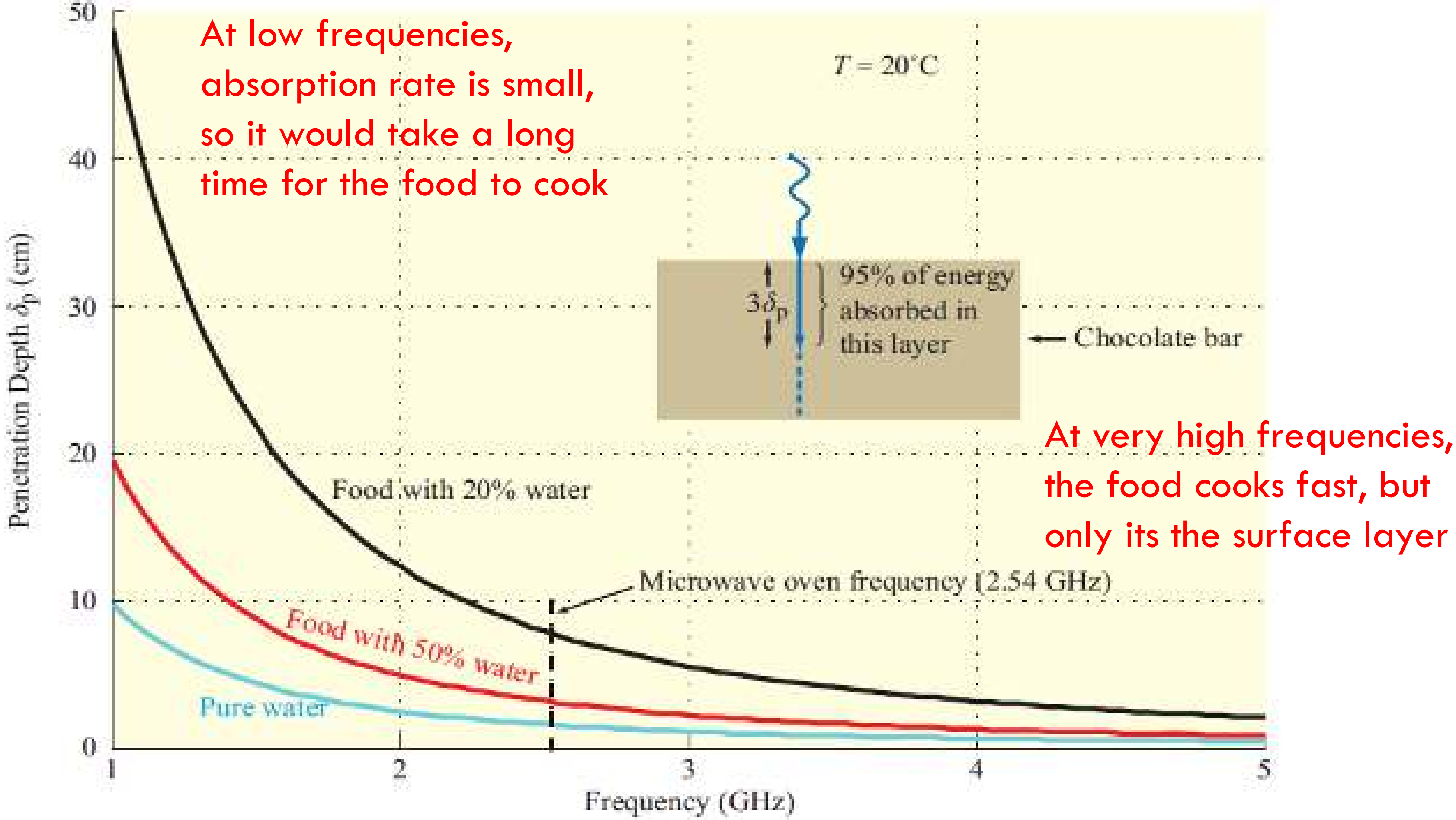
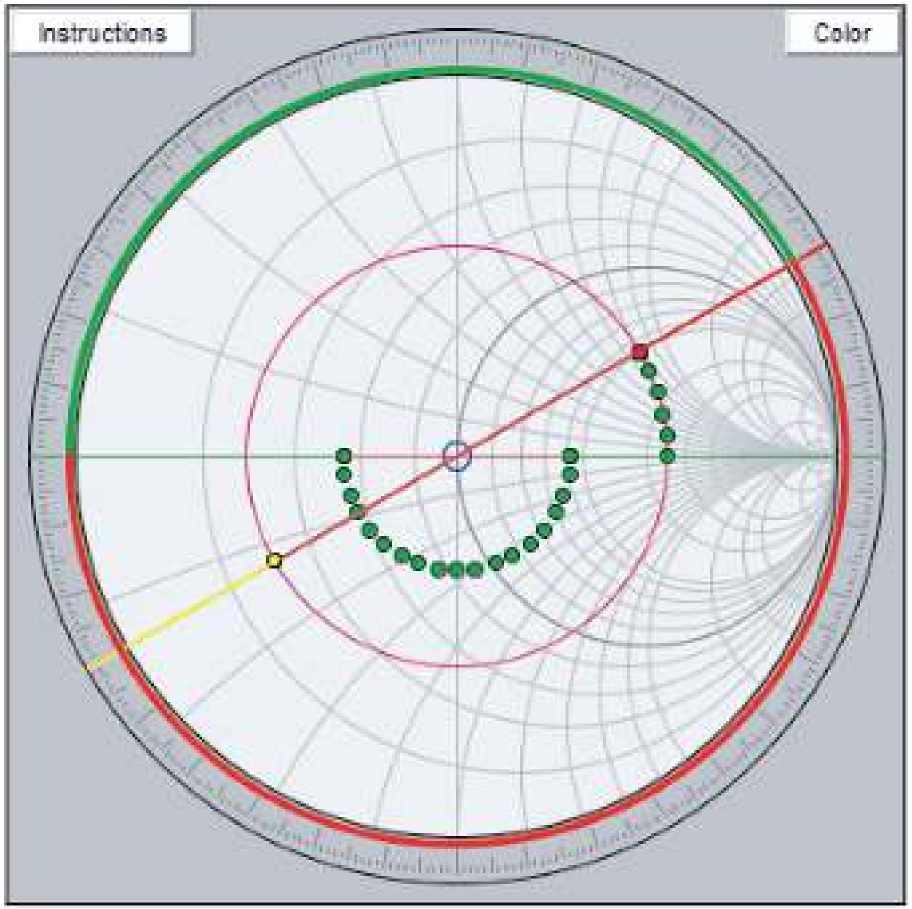


Figure TF3-1: Penetration depth as a function of frequency (1–5 GHz) for pure water and two foods with different water contents.

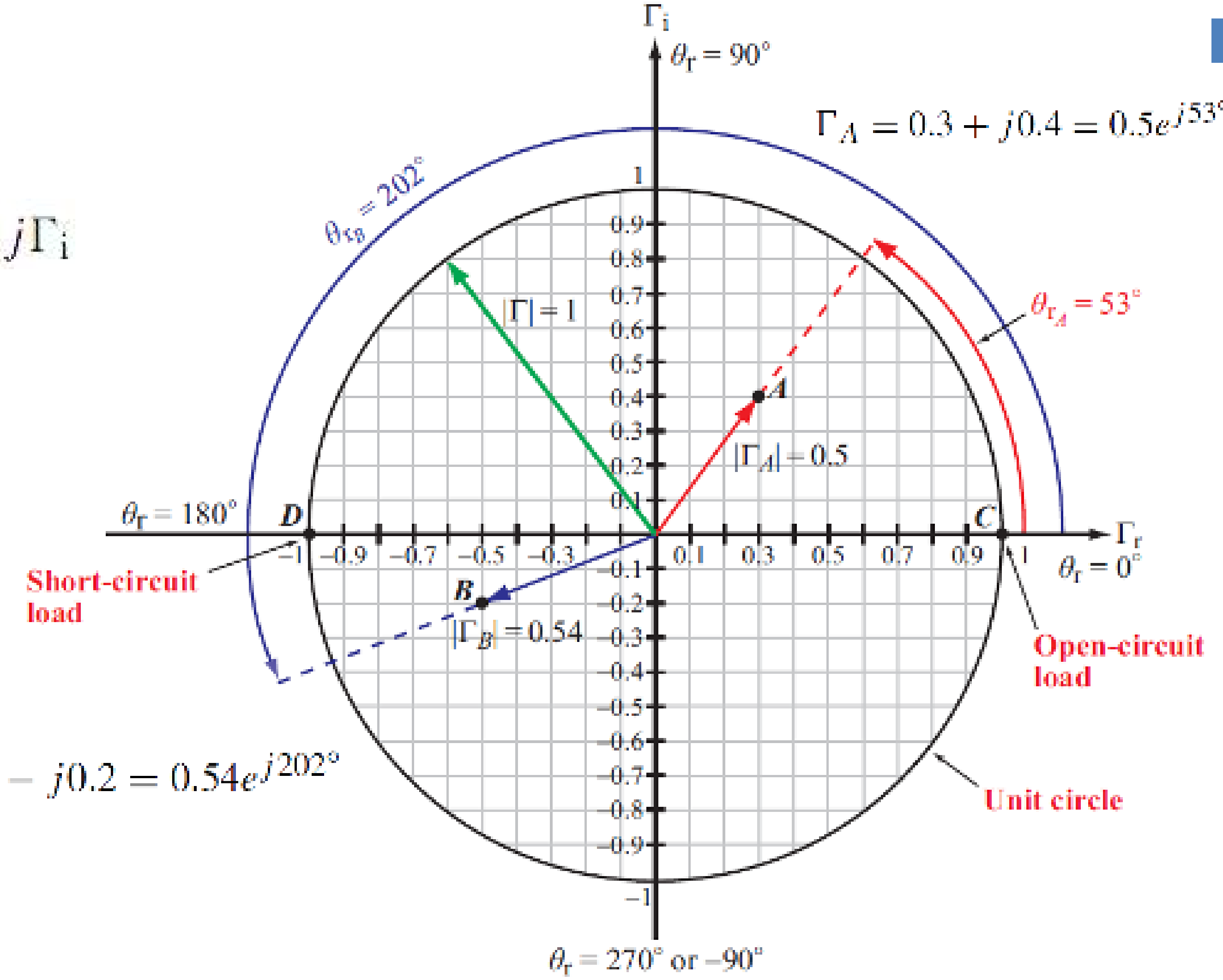
The Smith Chart

- Developed in 1939 by P. W. Smith as a graphical tool to analyze and design transmission-line circuits
- Today, it is used to characterize the performance of microwave circuits



Complex Plane

$$\Gamma = |\Gamma| e^{j\theta_r} = \Gamma_r + j\Gamma_i$$



$$\Gamma_B = -0.5 - j0.2 = 0.54e^{j202°}$$

Smith Chart Parametric Equations

$$\Gamma = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1} = \frac{z_L - 1}{z_L + 1}$$

$$r_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

$$x_L = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

$$\boxed{z_L = \frac{1 + \Gamma}{1 - \Gamma}. \qquad (2.112)}$$

Equation for a circle

$$\left(\Gamma_r - \frac{r_L}{1 + r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1 + r_L}\right)^2. \qquad (2.116)$$

$$z_L = r_L + j x_L.$$

The standard equation for a circle in the x–y plane with center at (x_0, y_0) and radius a is

$$(x - x_0)^2 + (y - y_0)^2 = a^2. \qquad (2.117)$$

A similar manipulation of the expression for x_L given by Eq. (2.115b) leads to

$$r_L + j x_L = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i}$$

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2, \qquad (2.118)$$

Smith Chart Parametric Equations

$$\left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r_L}\right)^2, \qquad (2.116)$$

r_L circles

r_L circles are contained inside the unit circle

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2, \qquad (2.118)$$

x_L circles

Only parts of the x_L circles are contained within the unit circle

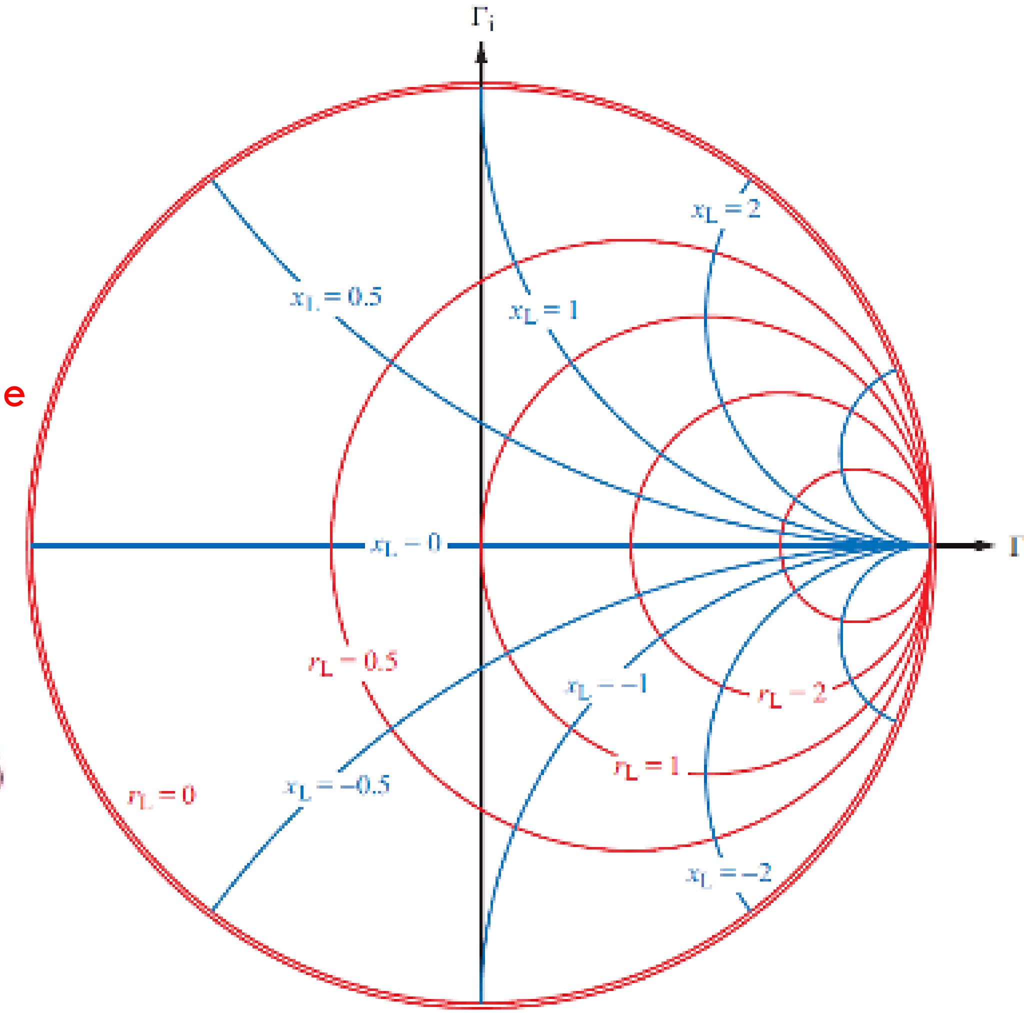
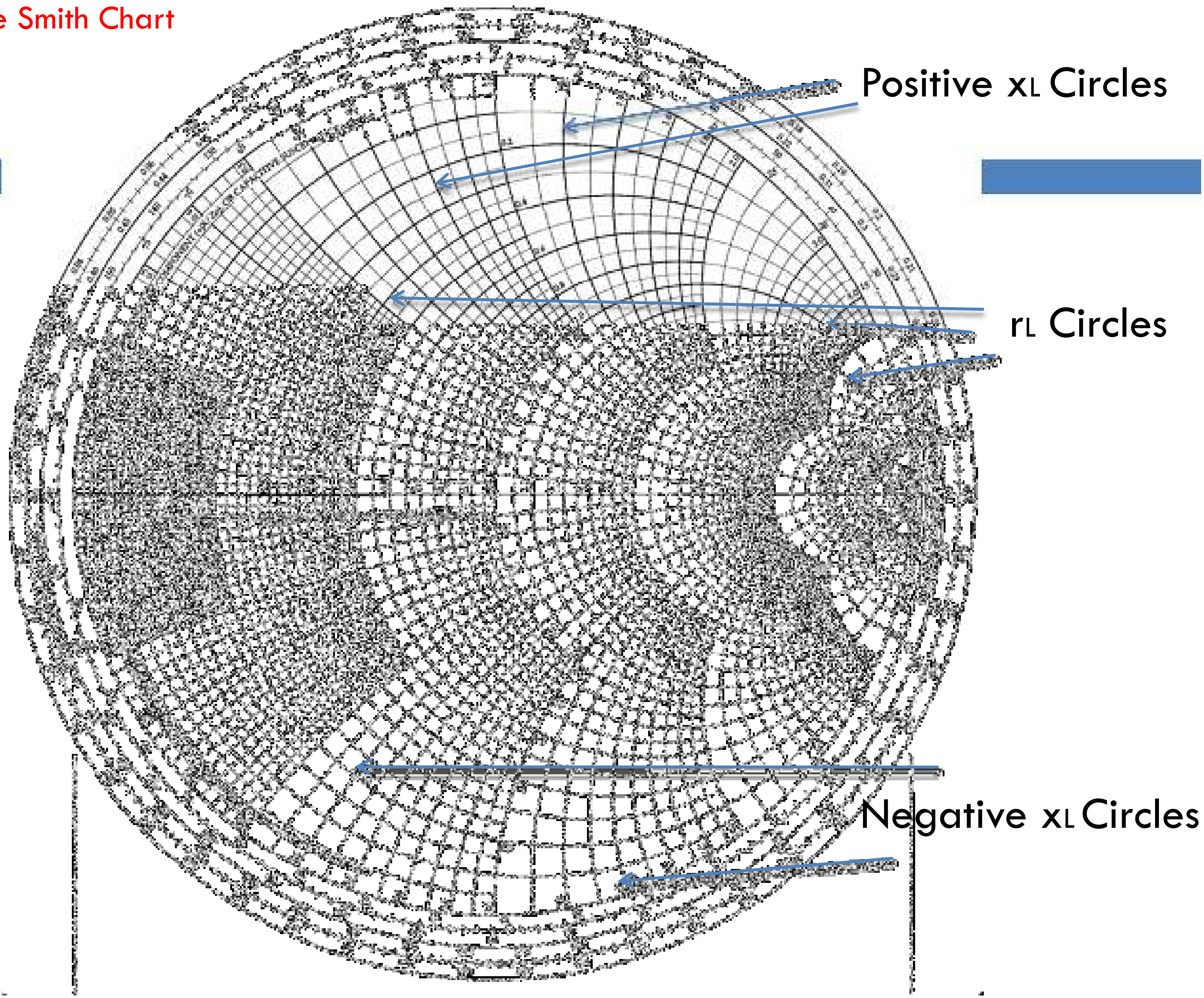


Figure 2-25: Families of r_L and x_L circles within the domain $|\Gamma| \le 1$.

Complete Smith Chart



Reflection coefficient at the load

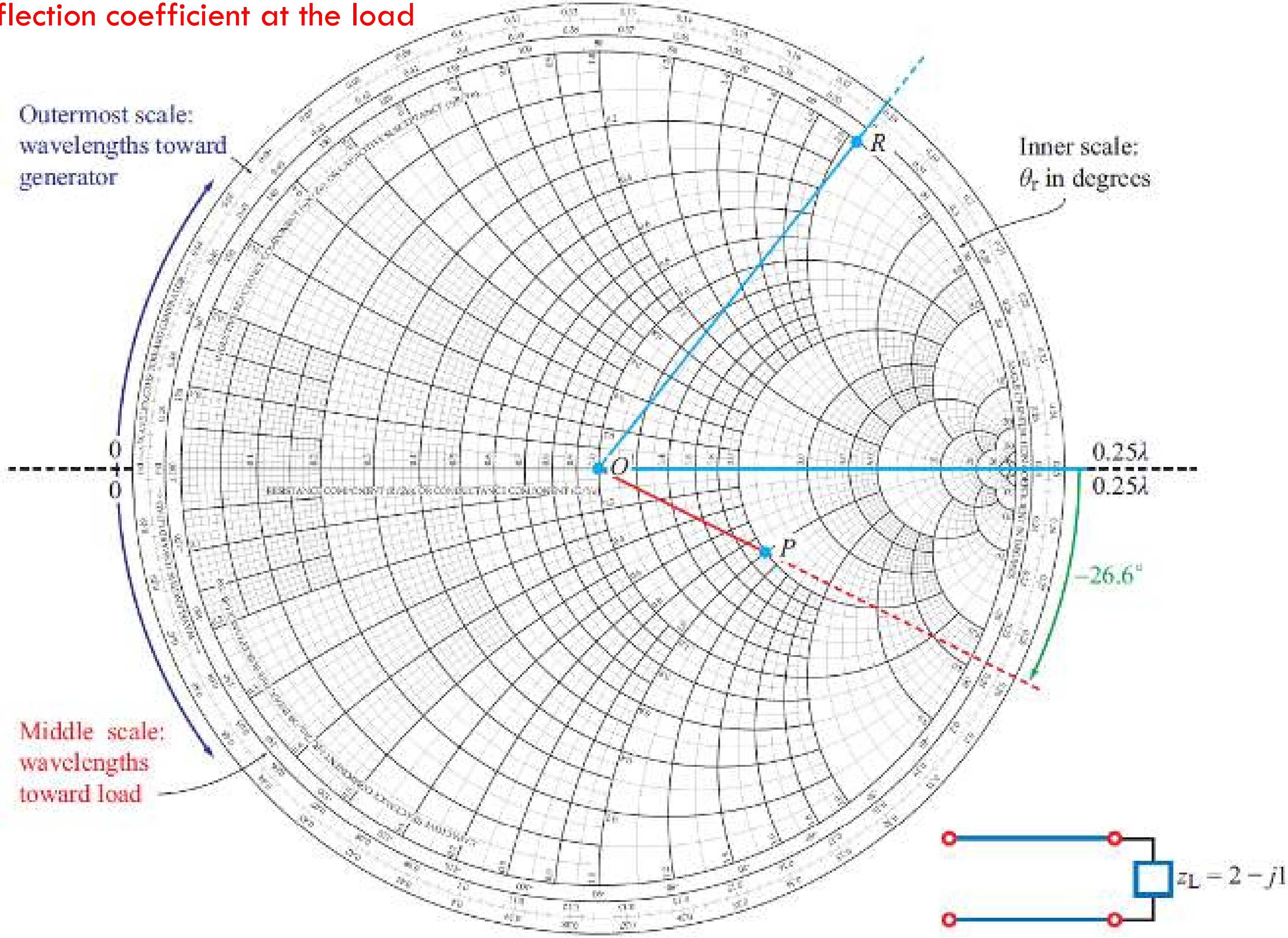


Figure 2-26: Point P represents a normalized load impedance $z_L = 2 - j1$. The reflection coefficient has a magnitude $|\Gamma| = \overline{OP}/\overline{OR} = 0.45$ and an angle $\theta_r = -26.6°$. Point R is an arbitrary point on the $r_L = 0$ circle (which also is the $|\Gamma| = 1$ circle).

Input Impedance

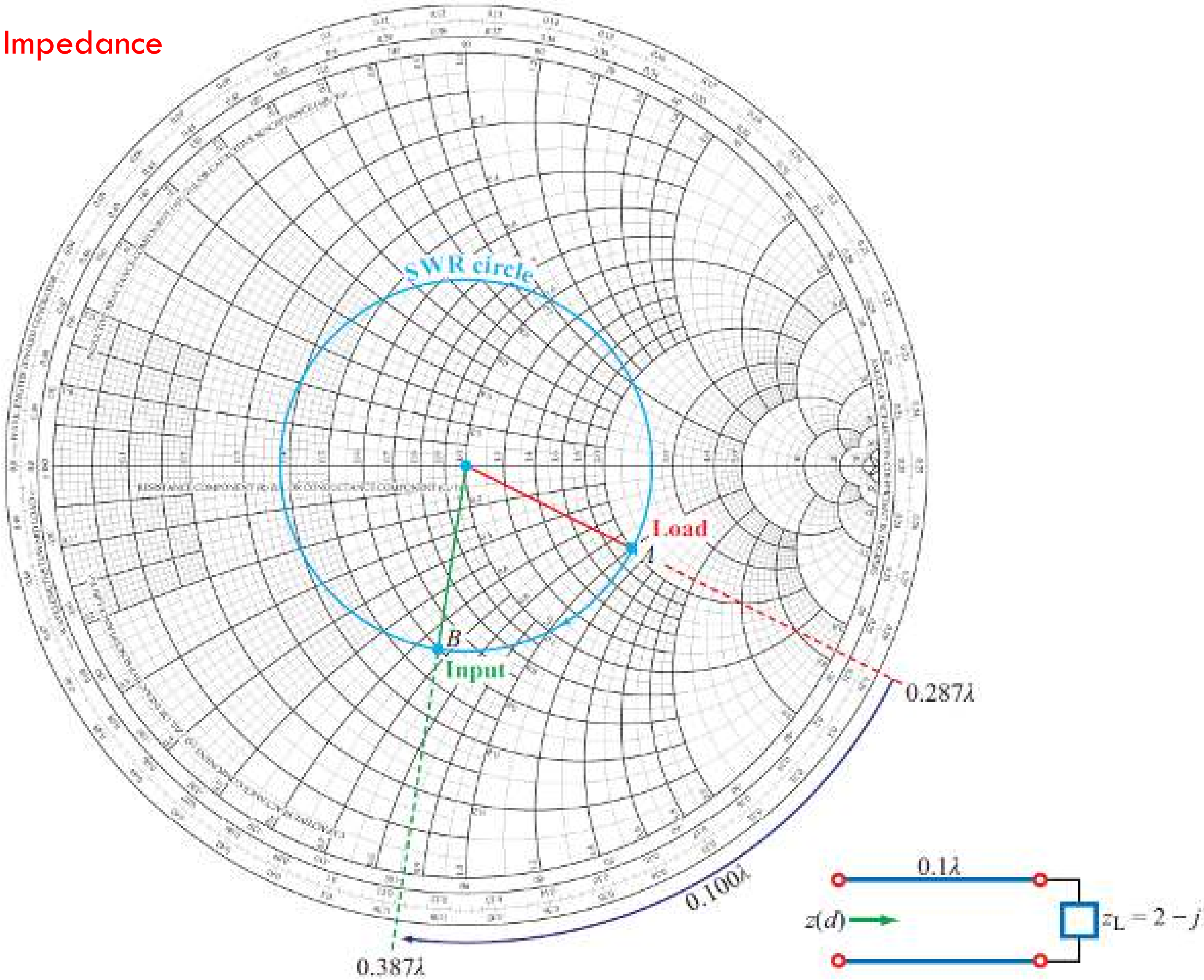


Figure 2-27: Point A represents a normalized load $z_L = 2 - j1$ at 0.287λ on the WTG scale. Point B represents the line input at $d = 0.1\lambda$ from the load. At B, $z(d) = 0.6 - j0.66$.

Maxima and Minima

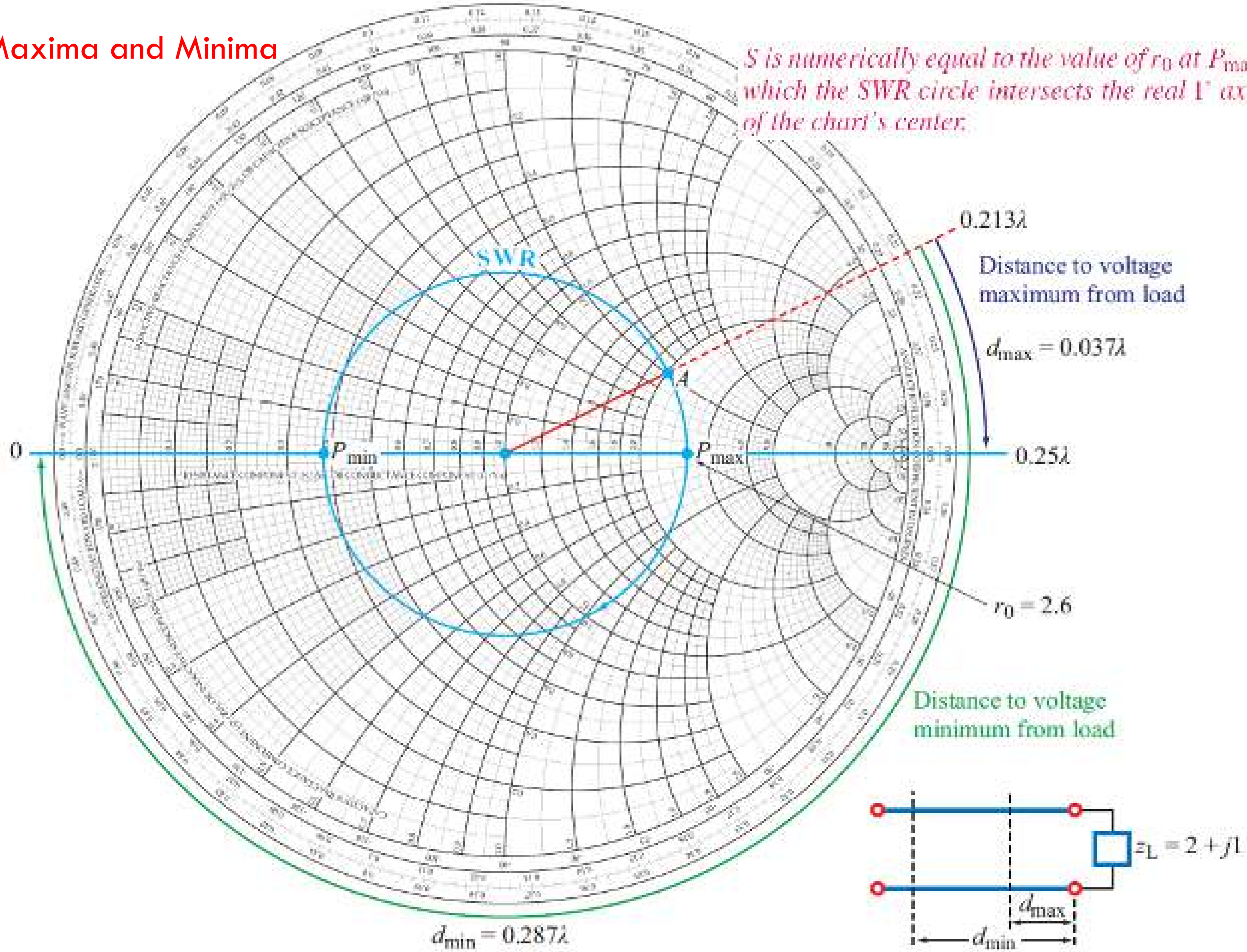
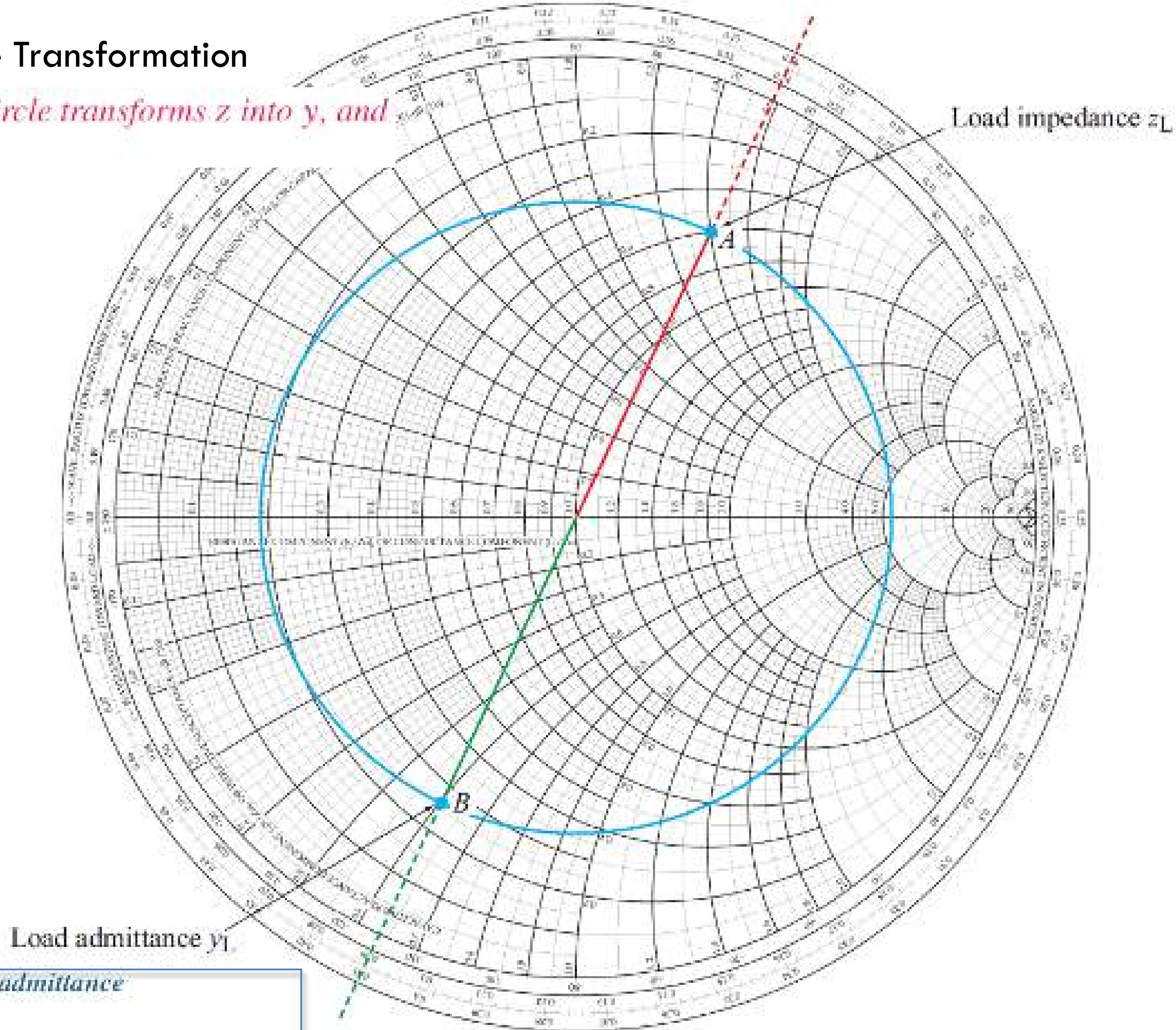


Figure 2-28: Point A represents a normalized load with $z_L = 2 + j1$. The standing wave ratio is $S = 2.6$ (at P_{max}), the distance between the load and the first voltage maximum is $d_{max} = (0.25 - 0.213)\lambda = 0.037\lambda$, and the distance between the load and the first voltage minimum is $d_{min} = (0.037 + 0.25)\lambda = 0.287\lambda$.

Impedance to Admittance Transformation

Rotation by λ/4 on the SWR circle transforms z into y, and vice versa.



normalized admittance

$$y = \frac{Y}{Y_0} = \frac{G}{Y_0} + j\frac{B}{Y_0} = g + jb \qquad \text{(dimensionless)}, \quad (2.132)$$

where $Y_0 = 1/Z_0$ is the ***characteristic admittance of the line***

Example 2-11: Smith Chart Calculations

A 50-Ω lossless transmission line of length 3.3λ is terminated by a load impedance $Z_L = (25 + j50)$ Ω.

$$z_L = \frac{Z_L}{Z_0} = \frac{25 + j50}{50} = 0.5 + j1$$

(a)

$$\Gamma = 0.62\angle 83^\circ$$

(b)

$$\boxed{S = 4.26}$$

(c)

$$d_{max} = (0.25 - 0.135)\lambda = 0.115\lambda$$

$$d_{min} = (0.5 - 0.135)\lambda = 0.365\lambda$$

(d)

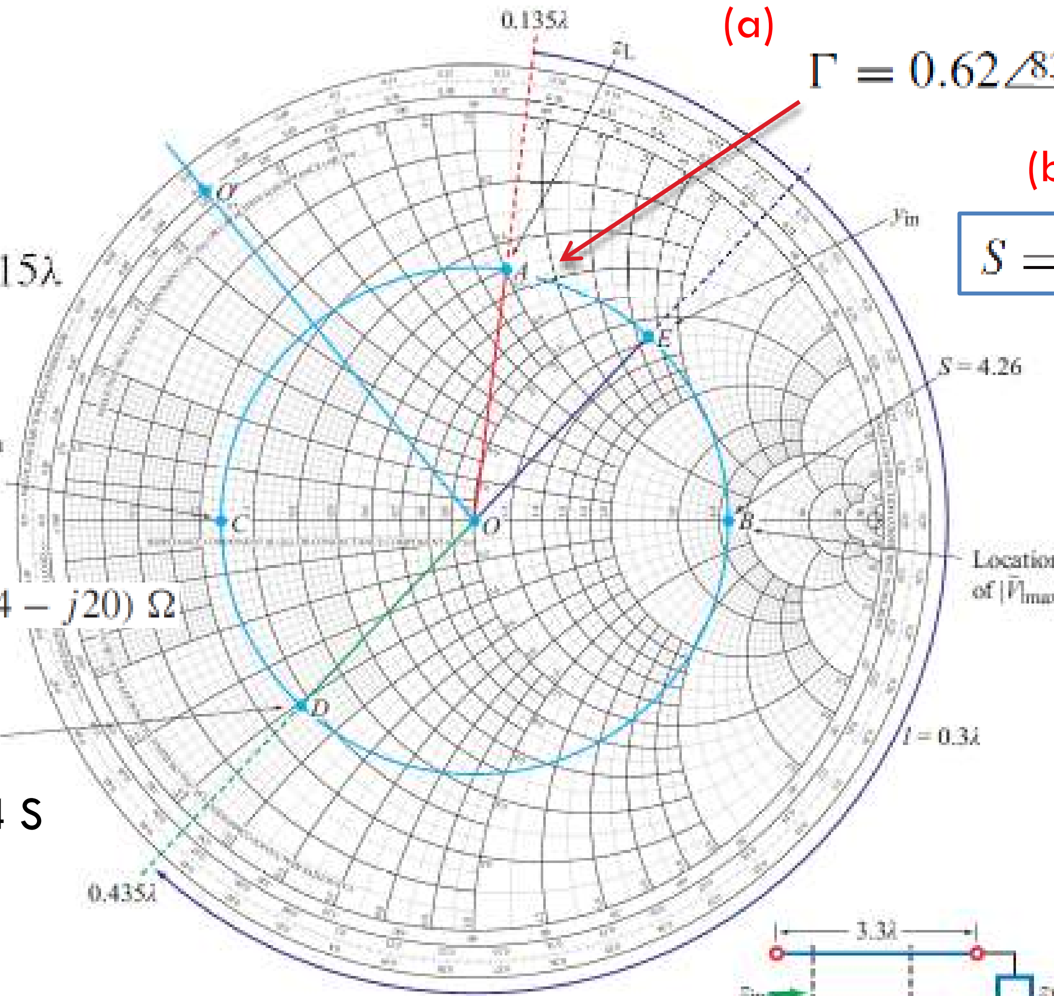
$$z_{in} = 0.28 - j0.40$$

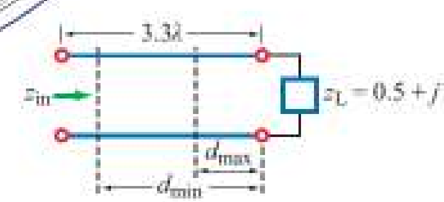
$$Z_{in} = z_{in} Z_0 = (0.28 - j0.40)50 = (14 - j20)\ \Omega$$

(e)

y_{in} = 1.15+j1.7

Yin = y_{in} Y_0 = 0.023+j0.034 S





Example 2-12: Determining Z_L Using the Smith Chart

Given:

$S = 3$

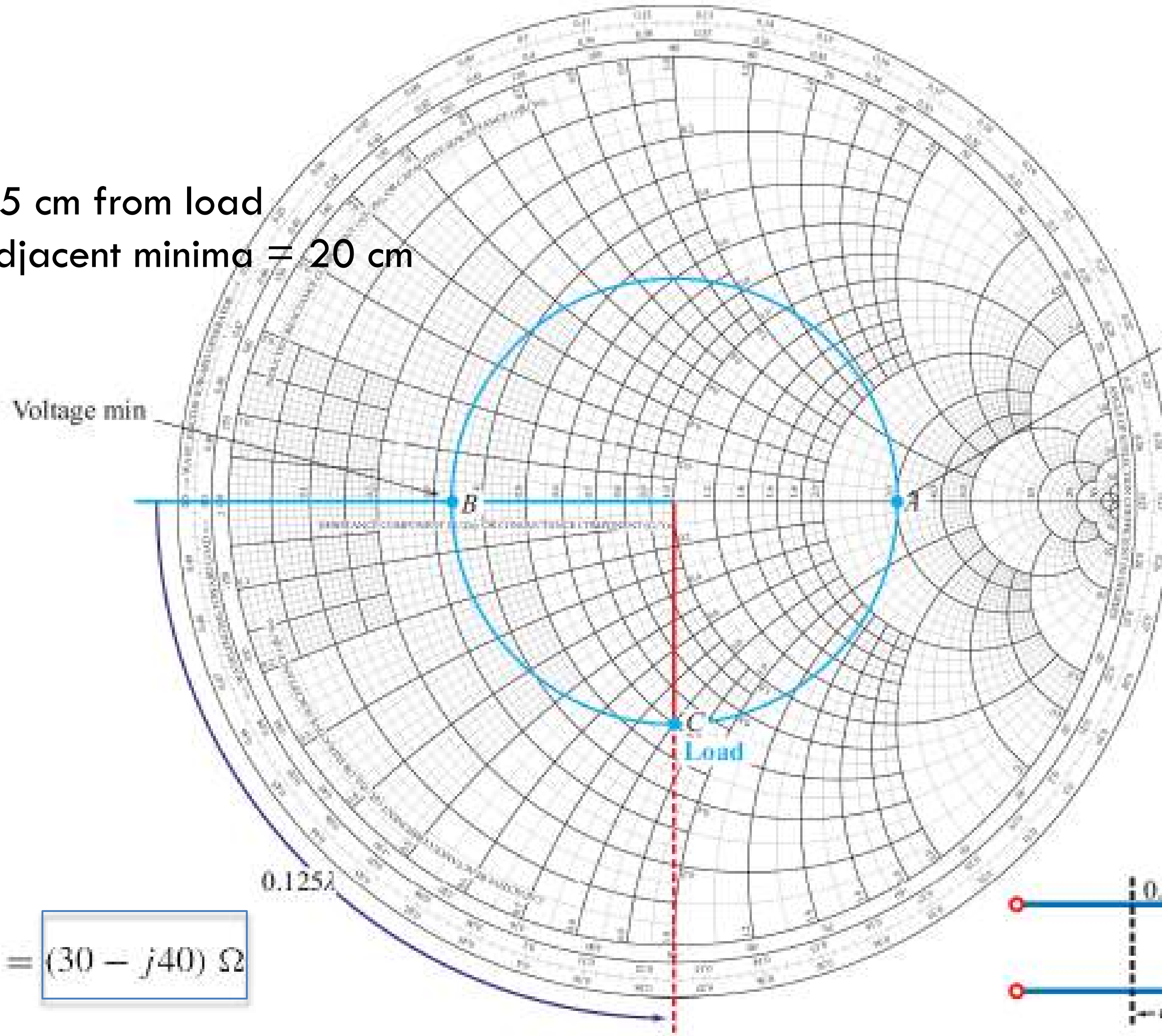
$Z_0 = 50\ \Omega$

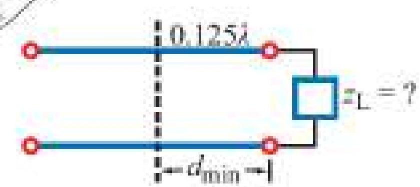
first voltage min @ 5 cm from load

Distance between adjacent minima = 20 cm

Determine: Z_L

$$d_{\min} = \frac{5}{40} = 0.125\lambda$$



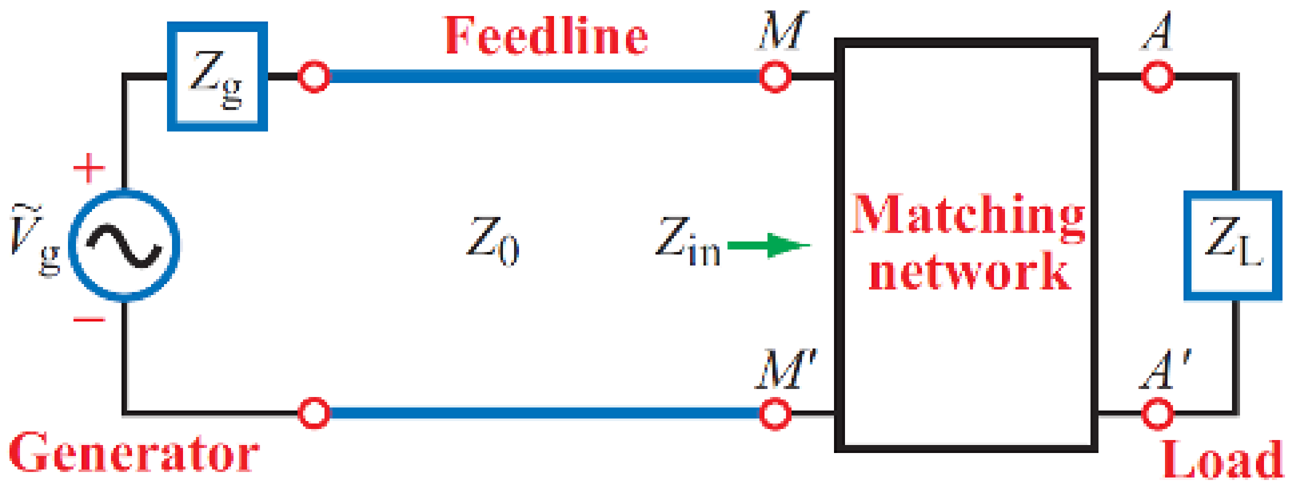


$$z_L = 0.6 - j0.8$$

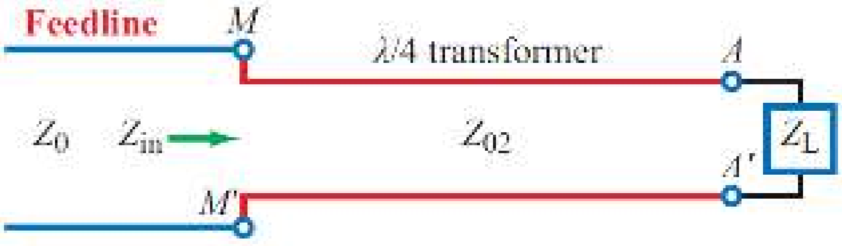
$$Z_L = 50(0.6 - j0.8) = \boxed{(30 - j40)\ \Omega}$$

Matching Networks

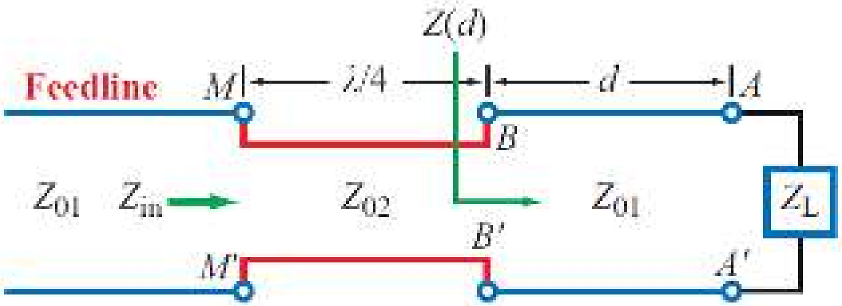
The purpose of the matching network is to eliminate reflections at terminals MM' for waves incident from the source. Even though multiple reflections may occur between AA' and MM', only a forward traveling wave exists on the feedline.



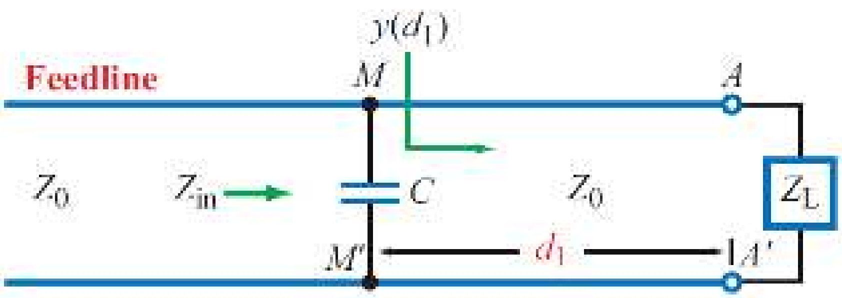
Examples of Matching Networks



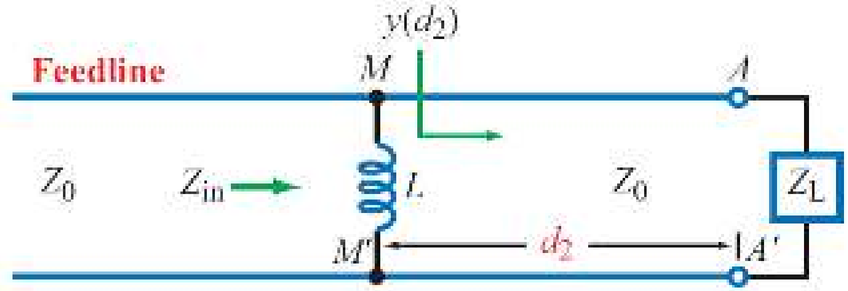
(a) In-series $\lambda/4$ transformer inserted at AA'



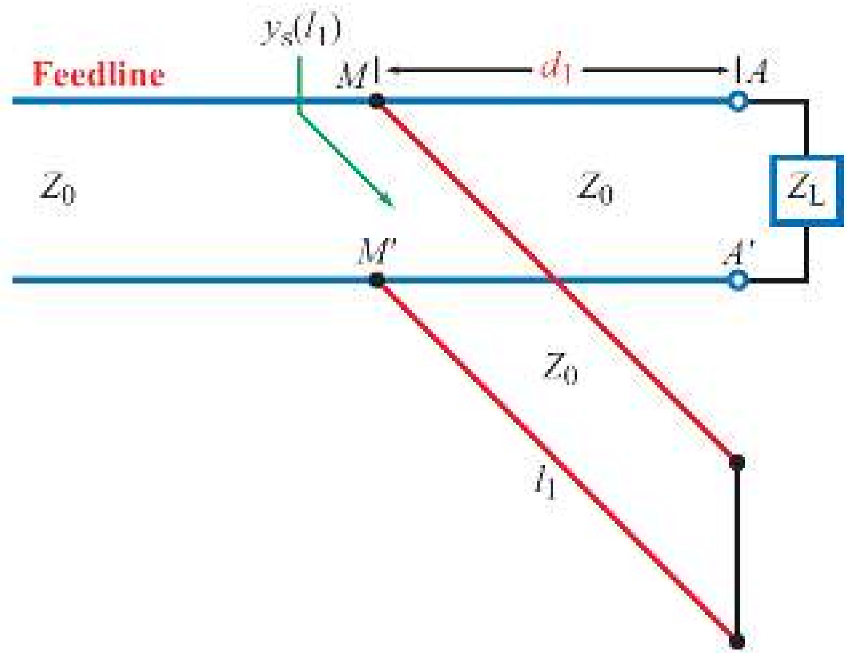
(b) In-series $\lambda/4$ transformer inserted at $d = d_{max}$ or $d = d_{min}$



(c) In-parallel insertion of capacitor at distance d_1



(d) In-parallel insertion of inductor at distance d_2



(e) In-parallel insertion of a short-circuited stub

Lumped-Element Matching

Choose d and Ys to achieve a match at MM'

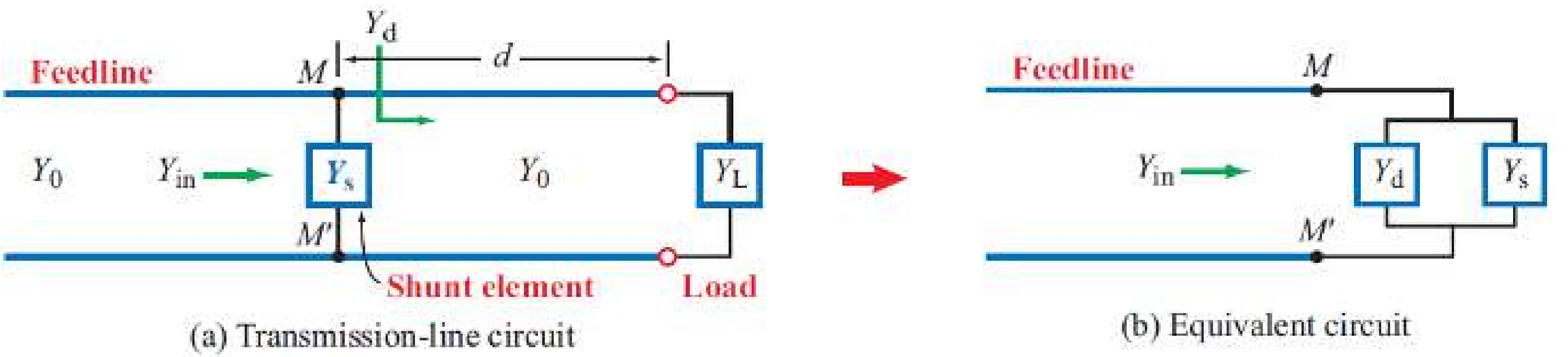


Figure 2-34: Inserting a reactive element with admittance Y_s at MM' modifies Y_d to Y_{in}.

$$y_{in} = g_d + j(b_d + b_s). \qquad (2.140)$$

$$Y_{in} = Y_d + Y_s$$

$$Y_{in} = (G_d + jB_d) + jB_s$$
$$= G_d + j(B_d + B_s).$$

To achieve a matched condition at MM', it is necessary that $y_{in} = 1 + j0$, which translates into two specific conditions, namely

$$g_d = 1 \quad \text{(real-part condition)}, \qquad (2.141a)$$

$$b_s = -b_d \quad \text{(imaginary-part condition)}. \qquad (2.141b)$$

Example 2-13: Lumped Element

A load impedance $Z_L = 25 - j50\ \Omega$ is connected to a 50-Ω transmission line. Insert a shunt element to eliminate reflections towards the sending end of the line. Specify the insert location d (in wavelengths), the type of element, and its value, given that $f = 100$ MHz.

$$z_L = \frac{Z_L}{Z_0} = \frac{25 - j50}{50} = 0.5 - j1$$

$$y_L = 0.4 + j0.8$$

Solution for Point C (Fig. 2-36): At C,

$$y_d = 1 + j1.58,$$

which is located at 0.178λ on the WTG scale. The distance between points B and C is

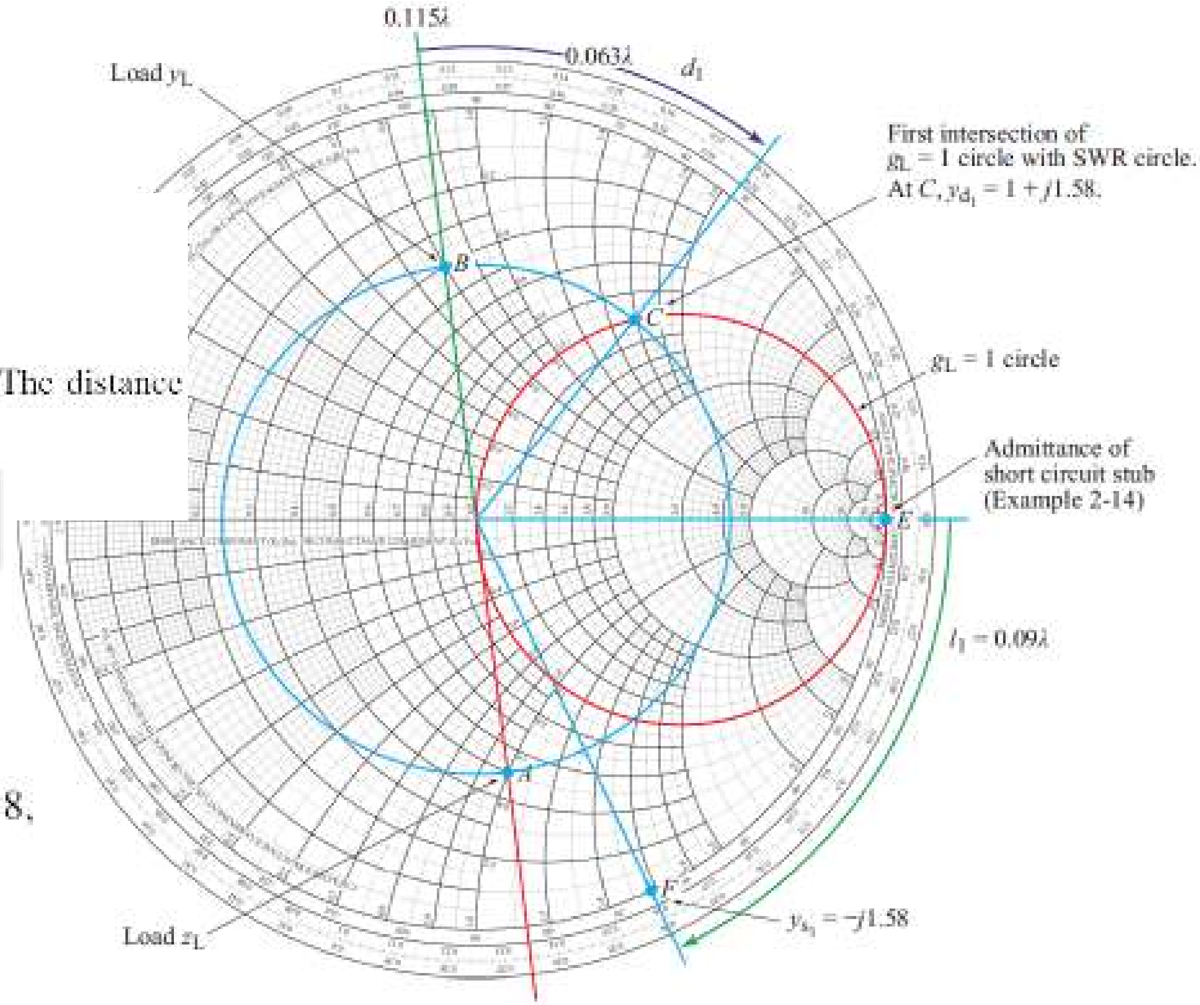
$$\boxed{d_1 = (0.178 - 0.115)\lambda = 0.063\lambda.}$$

we need $y_{in} = 1 + j0$. Thus,

$$1 + j0 = y_s + 1 + j1.58,$$

or

$$\boxed{y_s = -j1.58.}$$



Example 2-13: Lumped Element Cont.

A load impedance $Z_L = 25 - j50\ \Omega$ is connected to a 50-Ω transmission line. Insert a shunt element to eliminate reflections towards the sending end of the line. Specify the insert location d (in wavelengths), the type of element, and its value, given that $f = 100$ MHz.

$$z_L = \frac{Z_L}{Z_0} = \frac{25 - j50}{50} = 0.5 - j1$$

$$y_L = 0.4 + j0.8$$

Solution for Point C (Fig. 2-36): At C,

$$y_d = 1 + j1.58,$$

which is located at 0.178λ on the WTG scale. The distance between points B and C is

$$d_1 = (0.178 - 0.115)\lambda = 0.063\lambda.$$

we need $y_{in} = 1 + j0$. Thus,

$$1 + j0 = y_s + 1 + j1.58,$$

or

$$y_s = -j1.58.$$

The corresponding impedance of the lumped element is

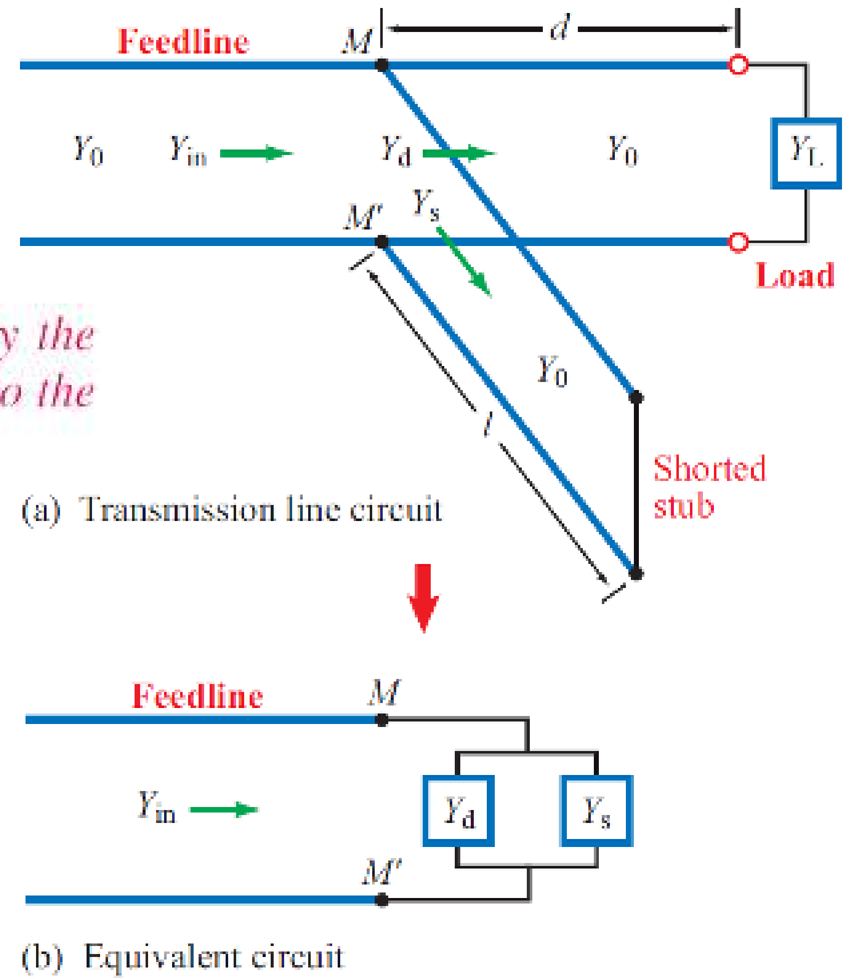
$$Z_{s_1} = \frac{1}{Y_{s_1}} = \frac{1}{y_{s_1} Y_0} = \frac{Z_0}{jb_{s_1}} = \frac{Z_0}{-j1.58} = \frac{jZ_0}{1.58} = j31.62\ \Omega.$$

Since the value of Z_{s_1} is positive, the element to be inserted should be an inductor and its value should be

$$\boxed{L = \frac{31.62}{\omega} = \frac{31.62}{2\pi \times 10^8} = 50\ \text{nH}.}$$

Single-Stub Matching

The required two degrees of freedom are provided by the length l of the stub and the distance d from the load to the stub position.



Example 2-14: Single-Stub Matching

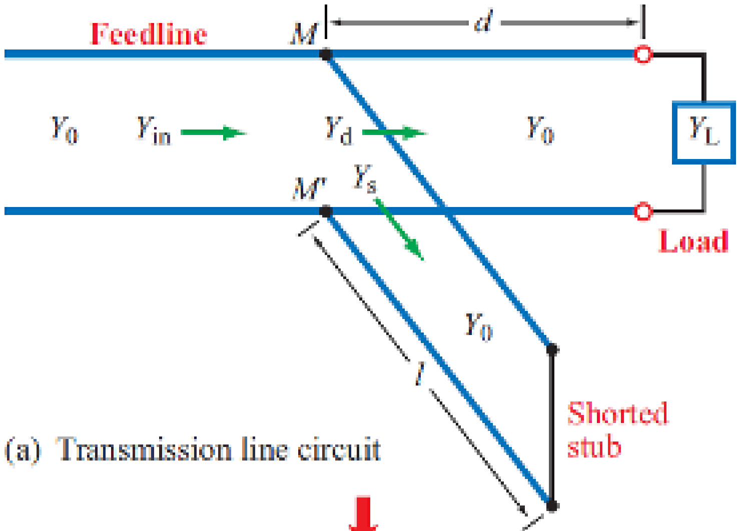
Repeat Example 2-13, but use a shorted stub (instead of a lumped element) to match the load impedance $Z_L = (25 - j50)\ \Omega$ to the 50-Ω transmission line.

Solution: In Example 2-13, we demonstrated that the load can be matched to the line via either of two solutions:

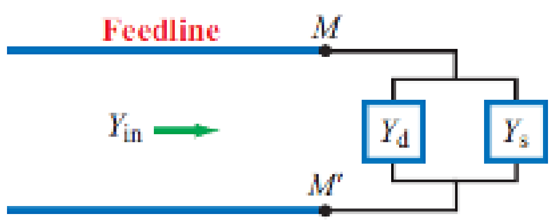
(1) $\boxed{d_1 = 0.063\lambda,}$ and $y_{s_1} = jb_{s_1} = -j1.58,$

(2) $d_2 = 0.207\lambda,$ and $y_{s_2} = jb_{s_2} = j1.58.$

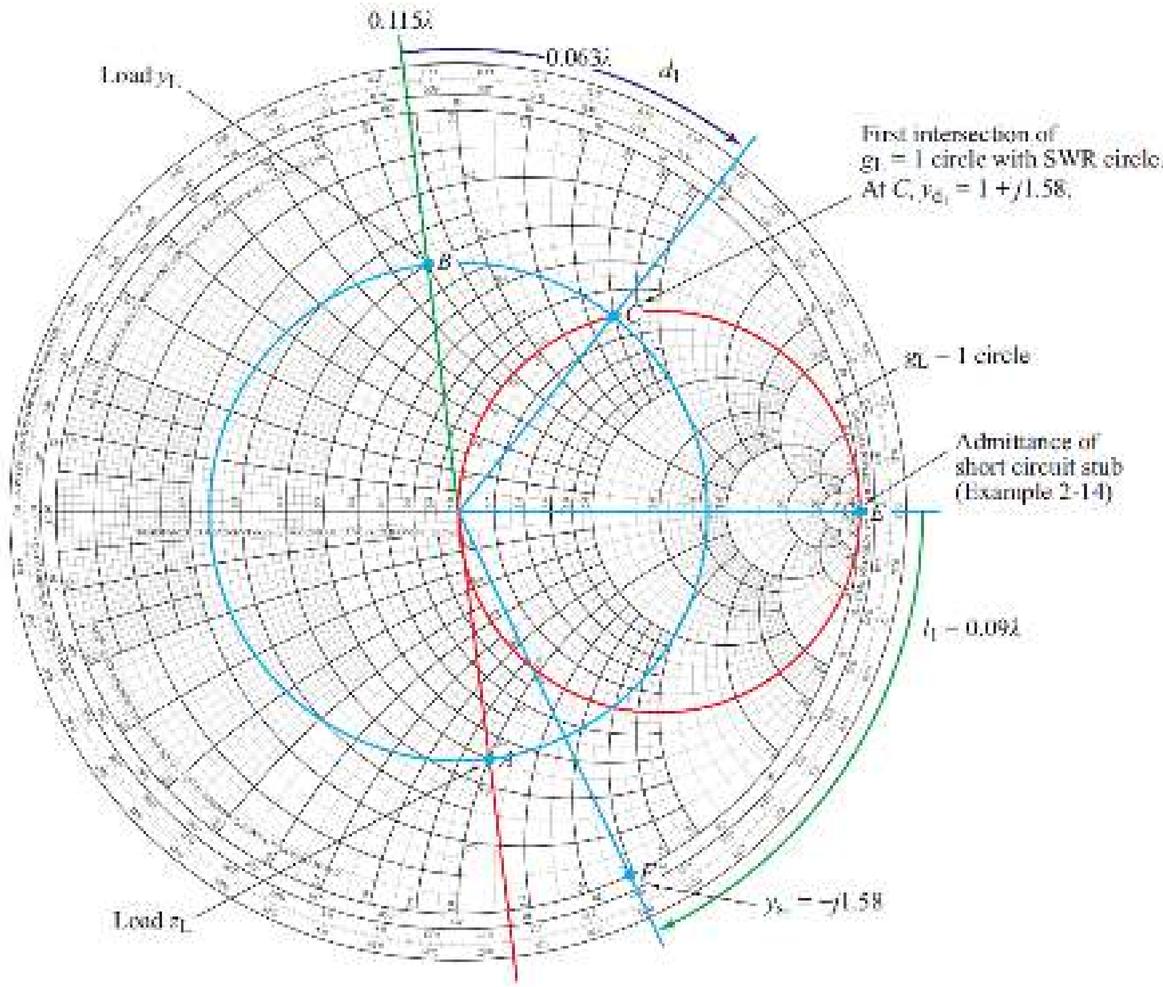
$$\boxed{l_1 = (0.34 - 0.25)\lambda = 0.09\lambda}$$



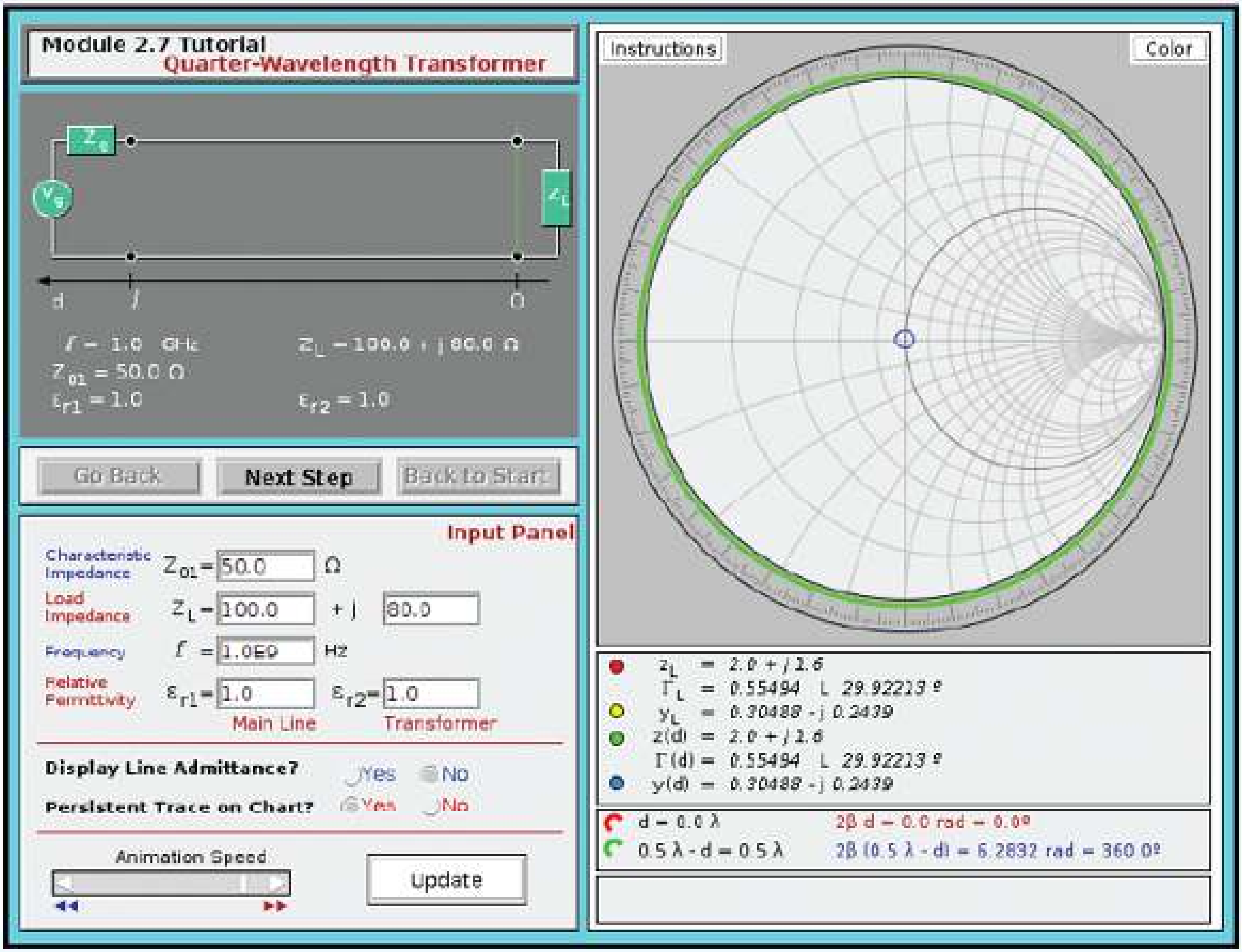
(a) Transmission line circuit



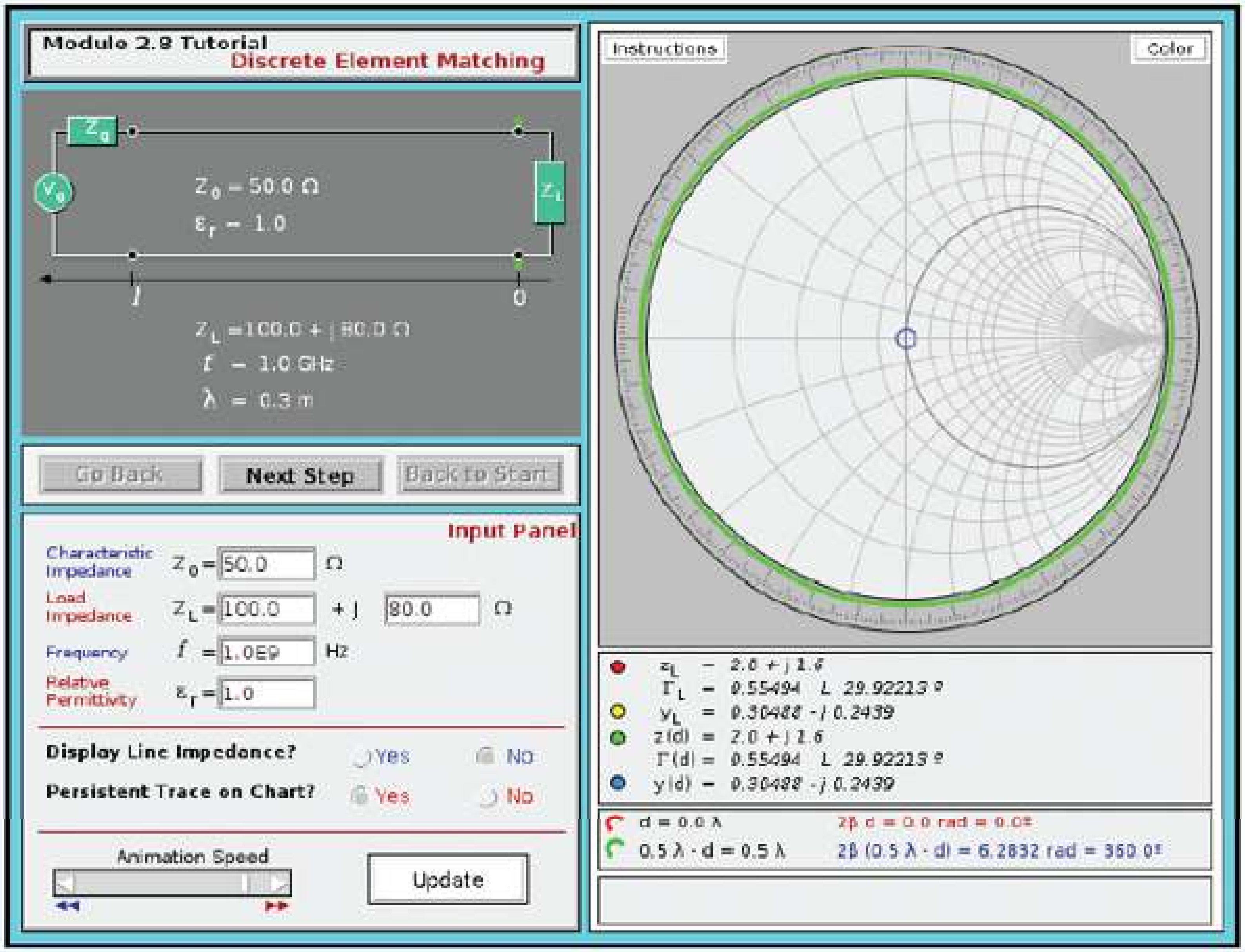
(b) Equivalent circuit



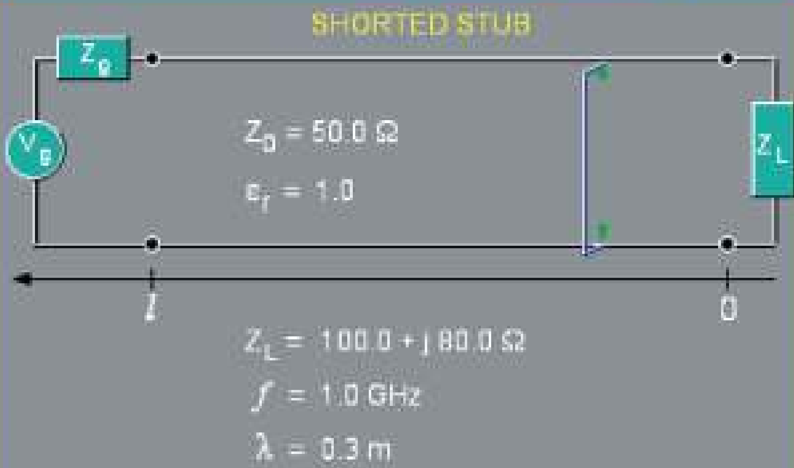
Module 2.7 Quarter-Wavelength Transformer This module allows you to go through a multi-step procedure to design a quarter-wavelength transmission line that, when inserted at the appropriate location on the original line, presents a matched load to the feedline.



Module 2.8 Discrete Element Matching For each of two possible solutions, the module guides the user through a procedure to match the feedline to the load by inserting a capacitor or an inductor at an appropriate location along the line.



Module 2.9 **Single-Stub Tuning**



Go Back | Next Step | Back to Start

Step 4 - First Solution

To cancel the imaginary part of the line admittance we add a stub with:

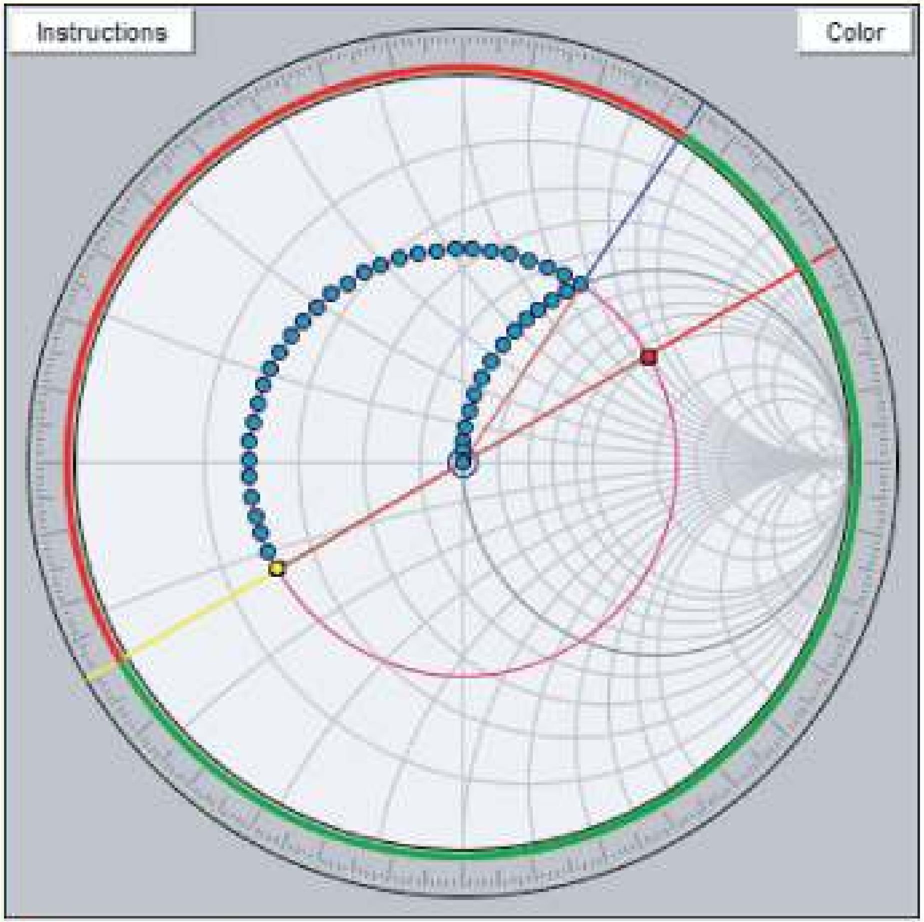
Length: $l_1 = 0.1024\ \lambda$

Admittance:

$Y_1 = -j\,0.02668$ S (actual)

$y_1 = -j\,1.33417$ (normalized)

Instructions | Color

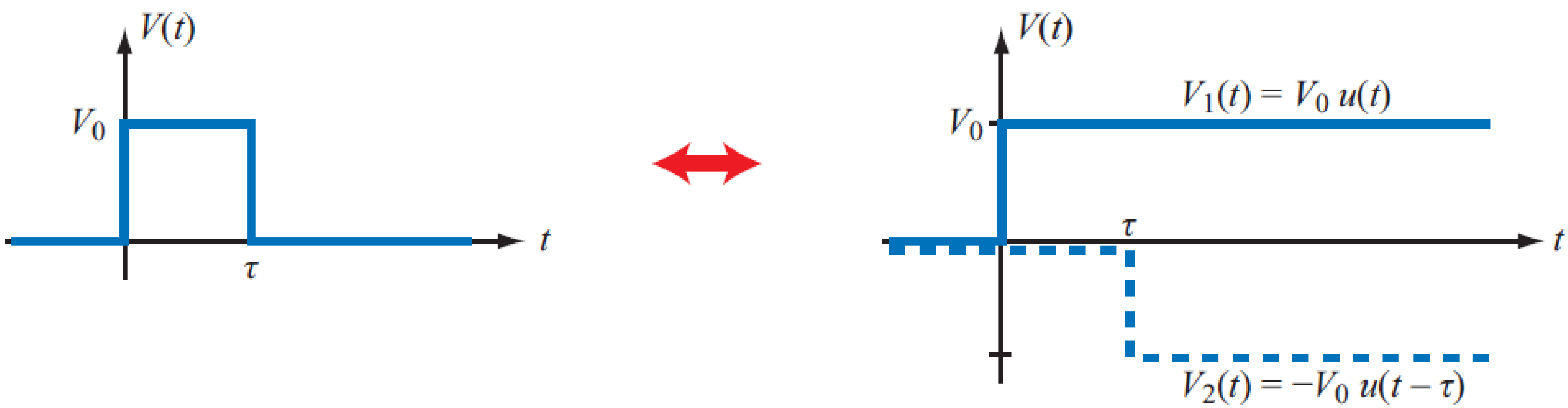


z_L = 2.0 + j 1.6
Γ_L = 0.55494 ∠ 29.92213°
y_L = 0.30488 - j 0.2439
z(d) = 1.0 + j 0.0
Γ(d) = 0.0 ∠ 0.0°
y(d) = 1.0 + j 0.0

d = 0.2134 λ — 2β d = 2.6813 rad = 153.6287°
0.5 λ - d = 0.2866 λ — 2β (0.5 λ - d) = 3.6019 rad = 206.3713°

Transients

*The **transient response** of a voltage pulse on a transmission line is a time record of its back and forth travel between the sending and receiving ends of the line, taking into account all the multiple reflections (echoes) at both ends.*

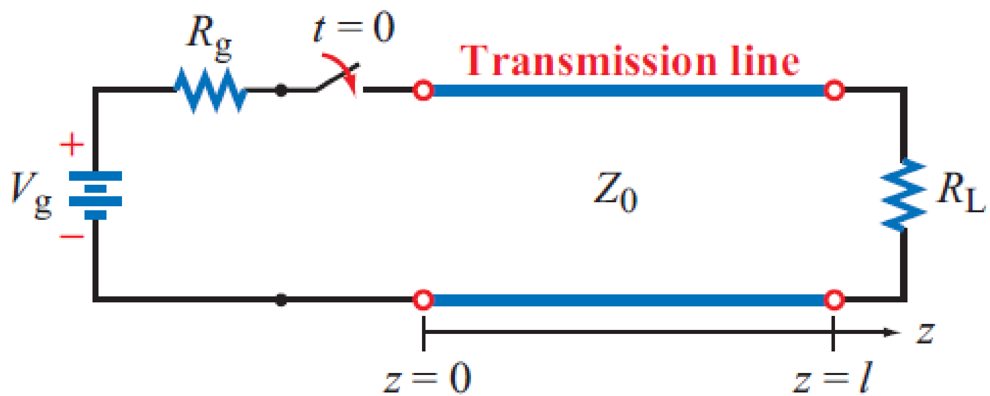


(a) Pulse of duration τ

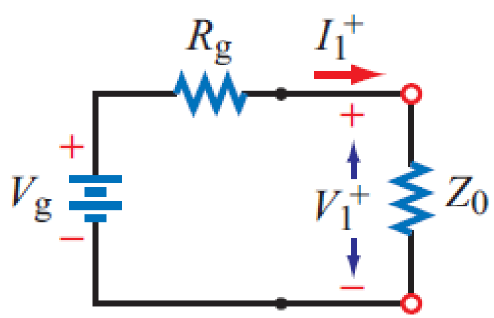
(b) $V(t) = V_1(t) + V_2(t)$

Rectangular pulse is equivalent to the sum of two step functions

Transient Response



(a) Transmission-line circuit



(b) Equivalent circuit at $t = 0^+$

Initial current and voltage

$$I_1^+ = \frac{V_g}{R_g + Z_0},$$

$$V_1^+ = I_1^+ Z_0 = \frac{V_g Z_0}{R_g + Z_0}$$

Reflection at the load

$$V_1^- = \Gamma_L V_1^+,$$

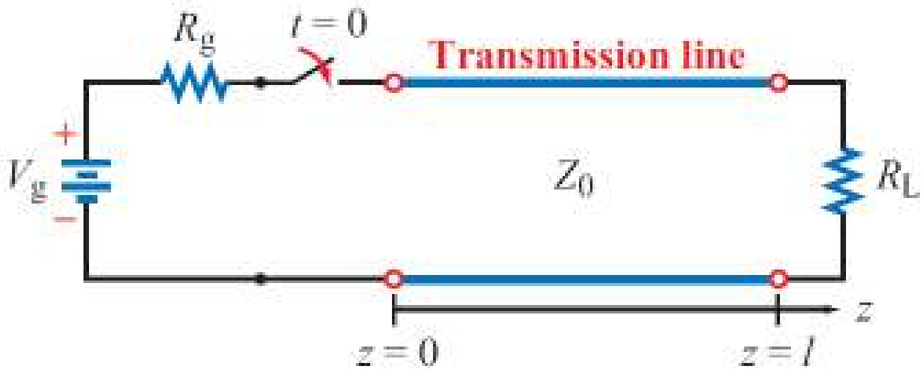
Load reflection coefficient $\Gamma_L = \dfrac{R_L - Z_0}{R_L + Z_0}$

Second transient

$$V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+$$

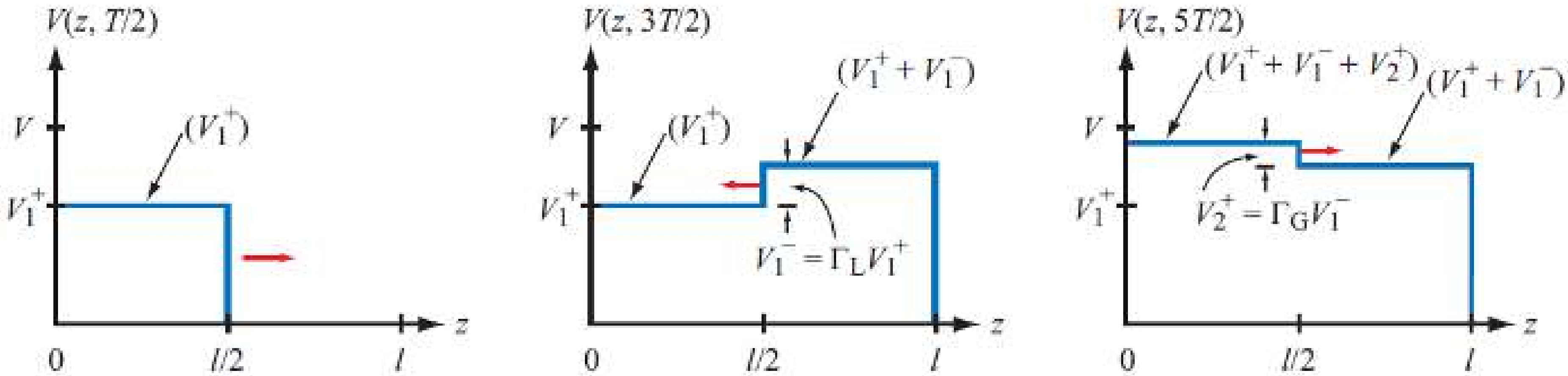
Generator reflection coefficient $\Gamma_g = \dfrac{R_g - Z_0}{R_g + Z_0}$

Voltage Wave



(a) Transmission-line circuit

$T = l/u_p$ is the time it takes the wave to travel the full length of the line



(a) $V(z)$ at $t = T/2$ (b) $V(z)$ at $t = 3T/2$ (c) $V(z)$ at $t = 5T/2$

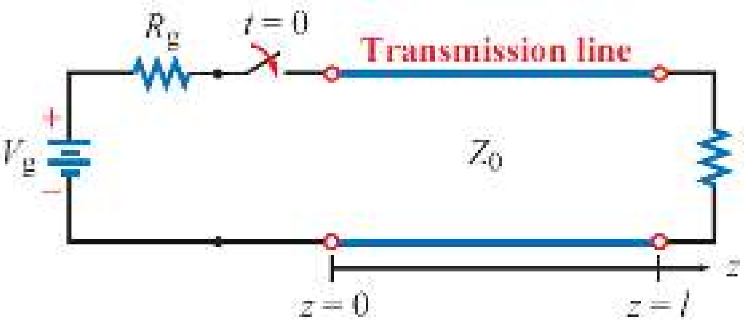
$R_g = 4Z_0$ and $R_L = 2Z_0$. The corresponding reflection coefficients are $\Gamma_L = 1/3$ and $\Gamma_g = 3/5$.

Current Wave

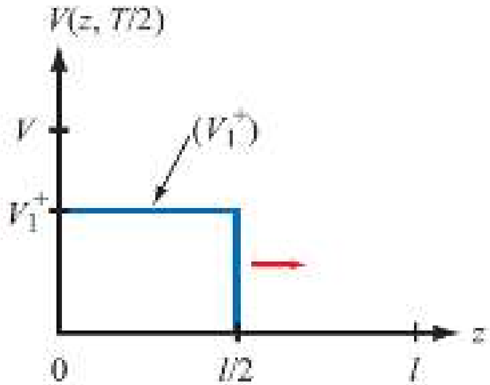
Reflection coefficient for current is the negative of that for voltage

$$I_1^- = -\Gamma_L I_1^+,$$

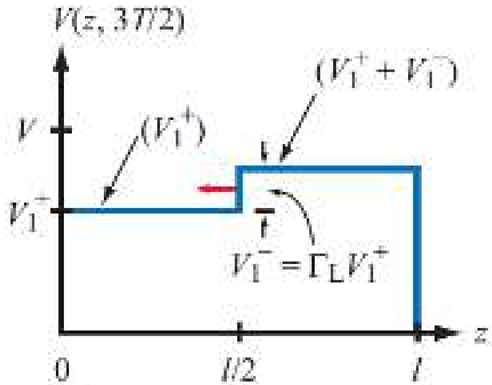
$$I_2^+ = -\Gamma_g I_1^- = \Gamma_g \Gamma_L I_1^+$$



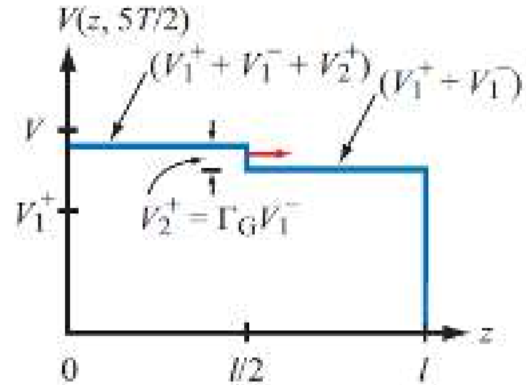
(a) Transmission-line circuit



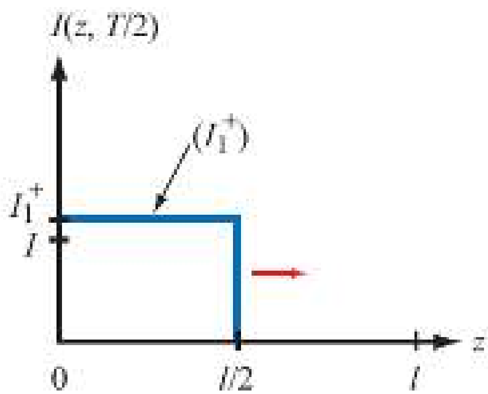
(a) $V(z)$ at $t = T/2$



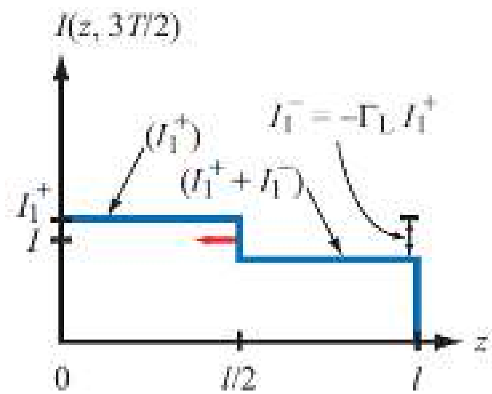
(b) $V(z)$ at $t = 3T/2$



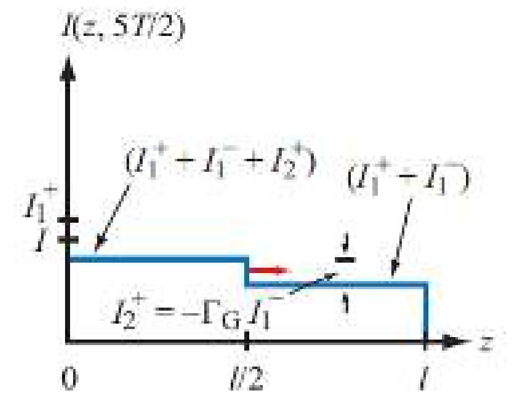
(c) $V(z)$ at $t = 5T/2$



(d) $I(z)$ at $t = T/2$

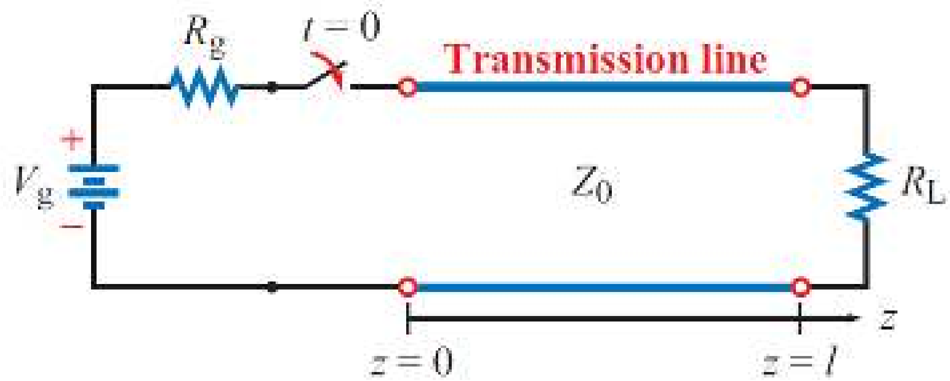


(e) $I(z)$ at $t = 3T/2$



(f) $I(z)$ at $t = 5T/2$

Steady State Response



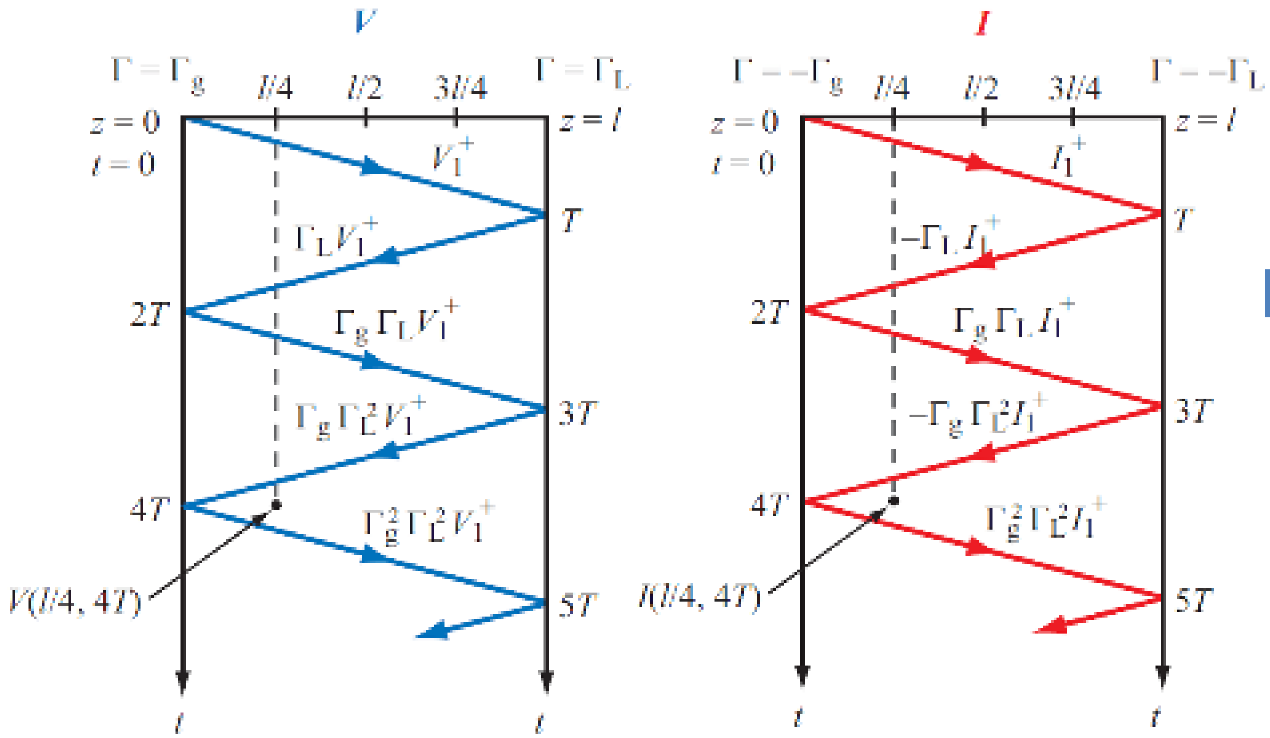
(a) Transmission-line circuit

The multiple-reflection process continues indefinitely, and the ultimate value that $V(z, t)$ reaches as t approaches $+\infty$ is the same at all locations on the transmission line.

$$V_\infty = \frac{V_g R_L}{R_g + R_L}$$

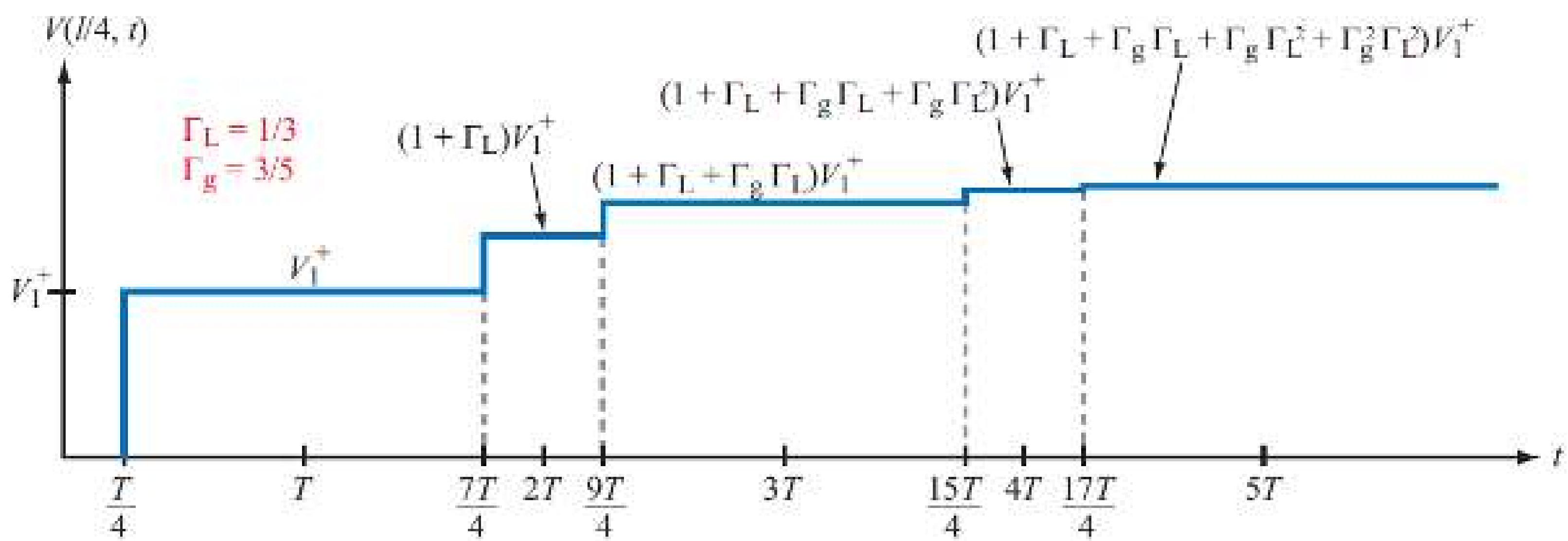
$$I_\infty = \frac{V_\infty}{R_L} = \frac{V_g}{R_g + R_L}$$

Bounce Diagrams



(a) Voltage bounce diagram

(b) Current bounce diagram



(c) Voltage versus time at $z = l/4$

Example 2-15: Pulse Propagation

The transmission-line circuit of Fig. 2-43(a) is excited by a rectangular pulse of duration $\tau = 1$ ns that starts at $t = 0$. Establish the waveform of the voltage response at the load, given that the pulse amplitude is 5 V, the phase velocity is c, and the length of the line is 0.6 m.

Solution: The one-way propagation time is

$$T = \frac{l}{c} = \frac{0.6}{3 \times 10^8} = 2 \text{ ns}.$$

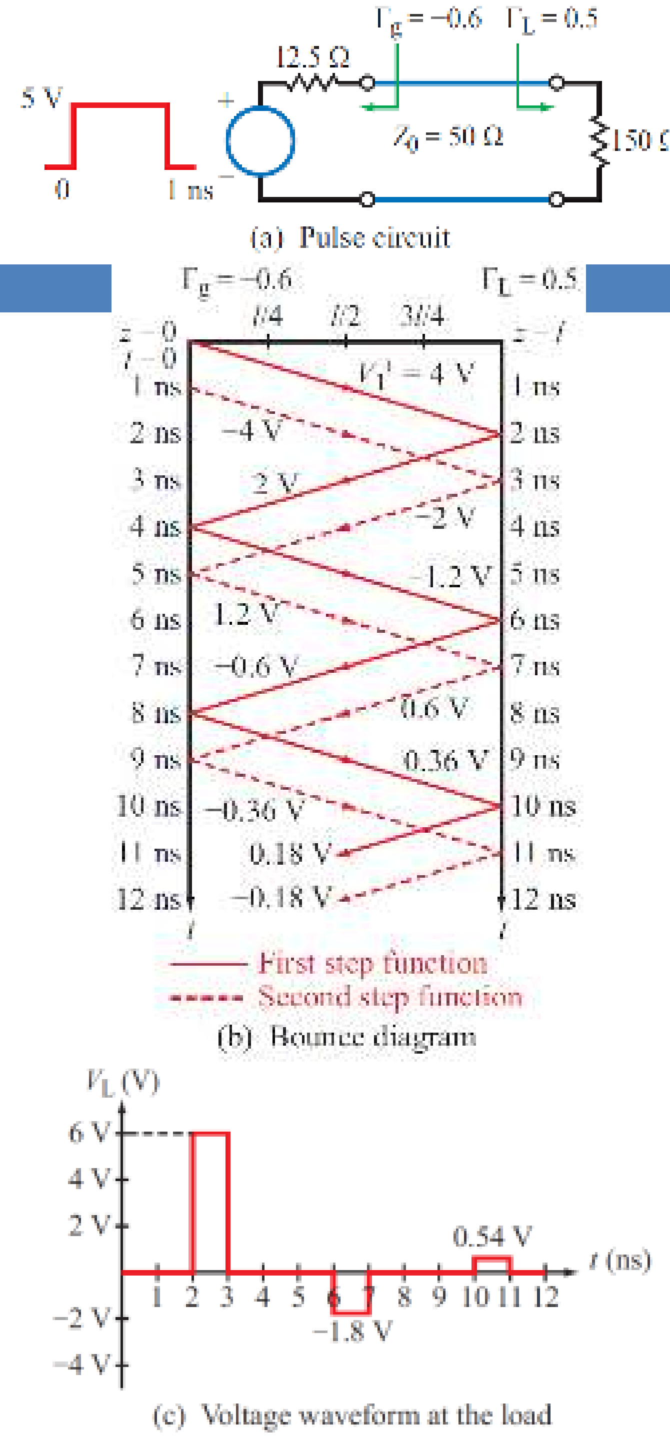
The reflection coefficients at the load and the sending end are

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{150 - 50}{150 + 50} = 0.5,$$

$$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} = \frac{12.5 - 50}{12.5 + 50} = -0.6.$$

By Eq. (2.147), the pulse is treated as the sum of two step functions, one that starts at $t = 0$ with an amplitude $V_{10} = 5$ V and a second one that starts at $t = 1$ ns with an amplitude $V_{20} = -5$ V. Except for the time delay of 1 ns and the sign reversal of all voltage values, the two step functions will generate identical bounce diagrams, as shown in Fig. 2-43(b). For the first step function, the initial voltage is given by

$$V_1^+ = \frac{V_{01} Z_0}{R_g + Z_0} = \frac{5 \times 50}{12.5 + 50} = 4 \text{ V}.$$



(a) Pulse circuit

(b) Bounce diagram

(c) Voltage waveform at the load


Module 2.10
Transient Response
Options:
Transient Plots
Details
z = 48.93 [mm]
Rg
Vg
Vgmax
0
τ
t
RL
0
l
z
Set Load
RL = 10.0 [Ω] Update
Resistance
Conductance
Set Generator
Vgmax = 1.0 [V]
Rg = 100.0 [Ω]
Update
Transient Plots
Voltage V(t)
Step
Pulse
Pulse Width 160.0 ps
V(t) [mV]
Cursor Location
200.0
-200.0
0
4T
8T
12T
Data
Pulse Input τ = 160.0 [ps]
Cursor z = 48.93 [mm]
Transit Time: input to load
T = 333.33333 [ps]
Transit Time: input to cursor position
TC = 163.1 [ps]
Transit Time: input to load and back to cursor
TLC = 503.56667 [ps]
Phase velocity up = 3.0 [10^8 m/s]
Reflection coefficients
Γg = 0.6
ΓL = -0.6667

Technology Brief 4: EM Cancer Zapper

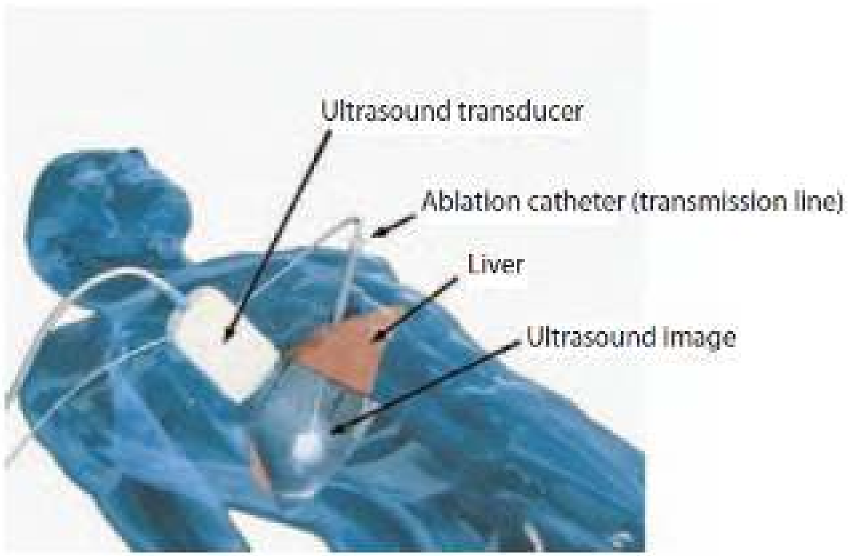


Figure TF4-1: Microwave ablation for liver cancer treatment.

Figure TF4-2: Photograph of the setup for a percutaneous microwave ablation procedure in which three single microwave applicators are connected to three microwave generators. (Courtesy of *RadioGraphics*, October 2005 pp. 569–583.)

Technology Brief 4: High Voltage Pulses

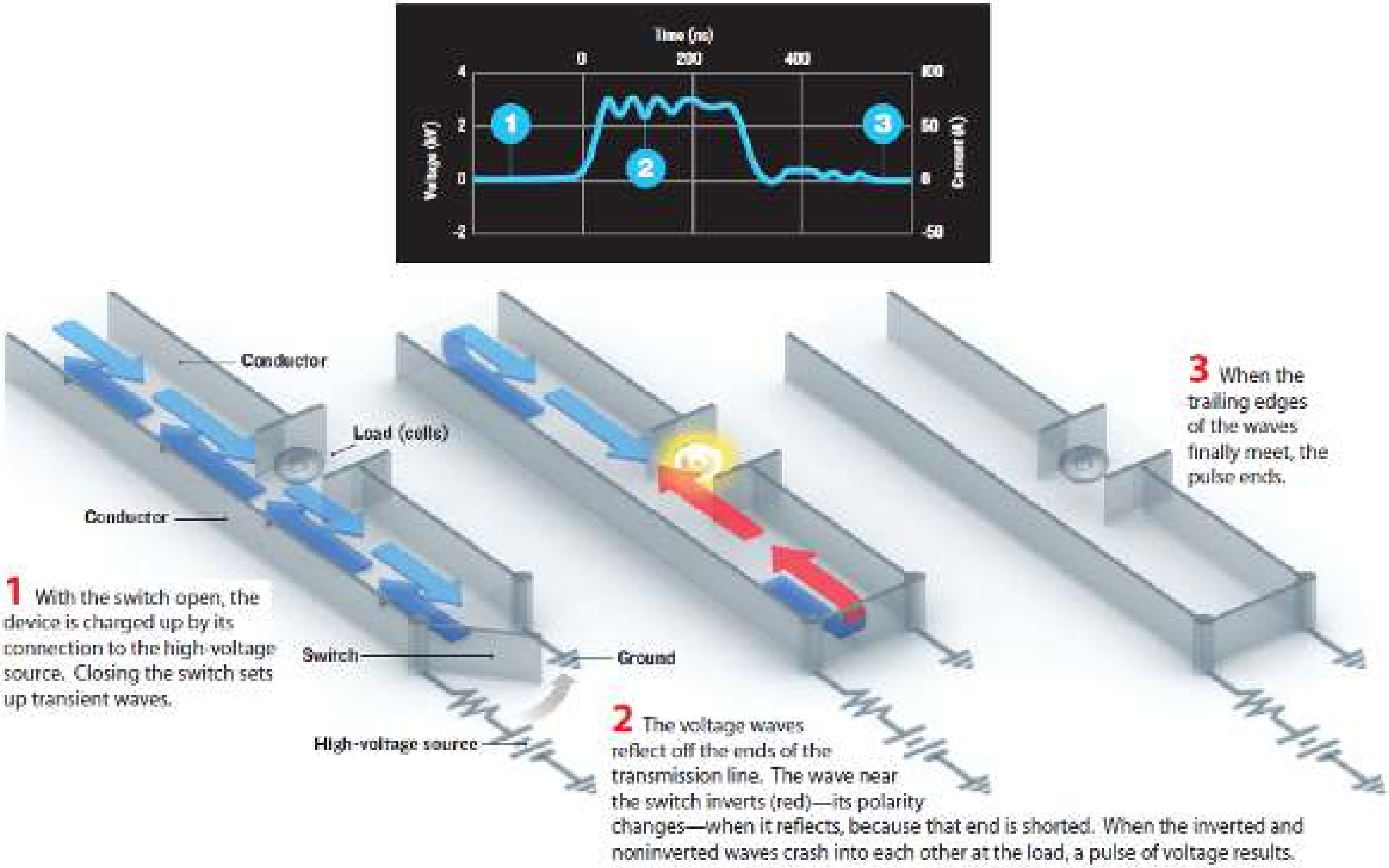


Figure TF4-3: High-voltage nanosecond pulse delivered to tumor cells via a transmission line. The cells to be shocked by the pulse sit in a break in one of the transmission-line conductors. (Courtesy of *IEEE Spectrum*, August 2006.)

Summary

Chapter 2 Relationships

TEM Transmission Lines

$$L'C' = \mu\varepsilon$$

$$\frac{G'}{C'} = \frac{\sigma}{\varepsilon}$$

$$\alpha = \mathfrak{Re}(\gamma) = \mathfrak{Re}\left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) \quad \text{(Np/m)}$$

$$\beta = \mathfrak{Im}(\gamma) = \mathfrak{Im}\left(\sqrt{(R' + j\omega L')(G' + j\omega C')}\right) \quad \text{(rad/m)}$$

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \quad (\Omega)$$

$$\Gamma = \frac{z_L - 1}{z_L + 1}$$

Step Function Transient Response

$$V_1^+ = \frac{V_g Z_0}{R_g + Z_0}$$

$$V_\infty = \frac{V_g R_L}{R_g + R_L}$$

$$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0}$$

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

Lossless Line

$$\alpha = 0$$

$$\beta = \omega\sqrt{L'C'}$$

$$Z_0 = \sqrt{\frac{L'}{C'}}$$

$$u_p = \frac{1}{\sqrt{\mu\varepsilon}} \quad \text{(m/s)}$$

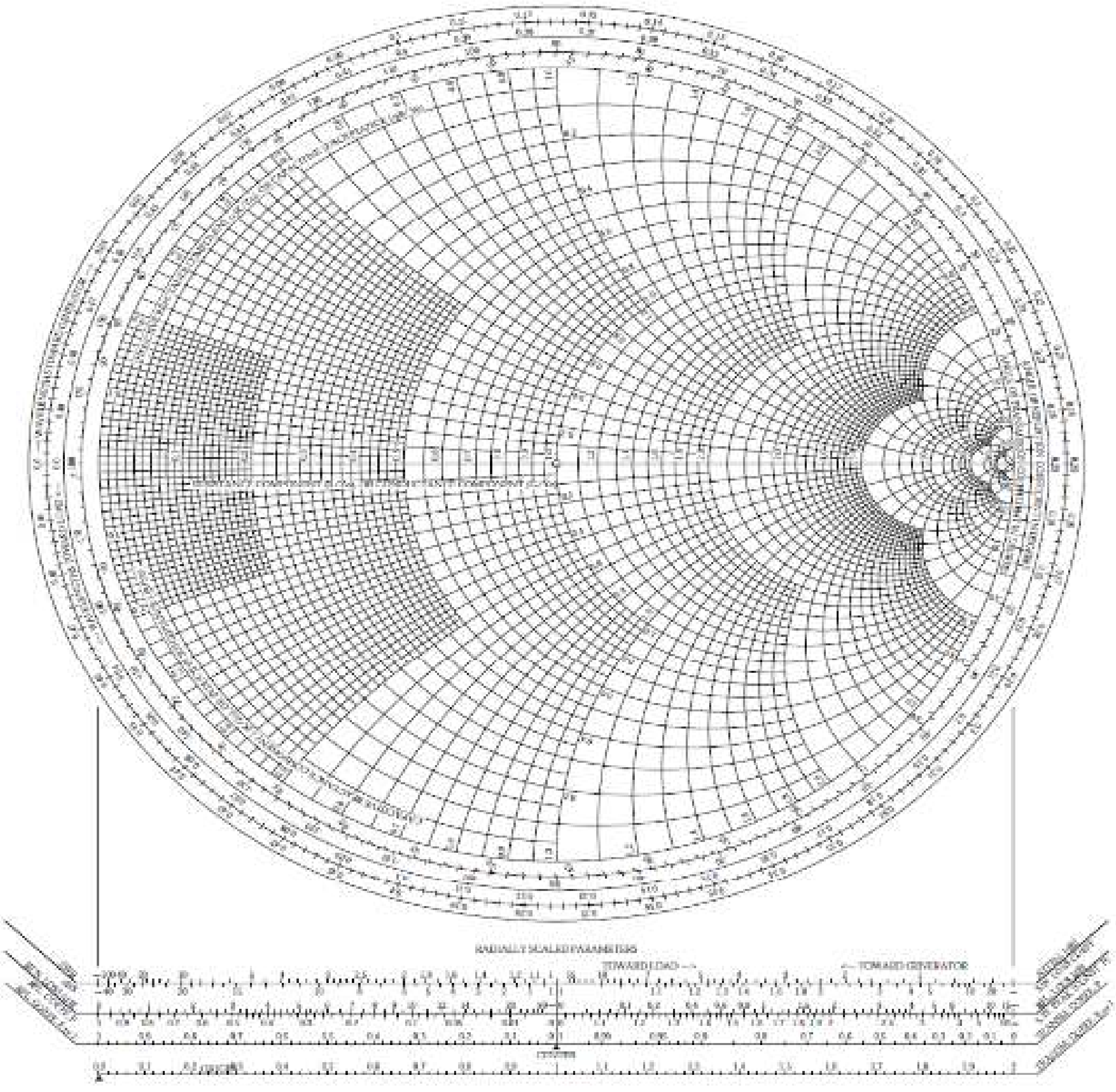
$$\lambda = \frac{u_p}{f} = \frac{c}{f}\,\frac{1}{\sqrt{\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$$

$$d_{max} = \frac{\theta_r\lambda}{4\pi} + \frac{n\lambda}{2}$$

$$d_{min} = \frac{\theta_r\lambda}{4\pi} + \frac{(2n+1)\lambda}{4}$$

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$$P_{av} = \frac{|V_0^+|^2}{2Z_0}\,[1 - |\Gamma|^2]$$


RADIALLY SCALED PARAMETERS
TOWARD LOAD —>
<— TOWARD GENERATOR
CENTER