مادة إشـارات وأنظمة

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إرادة - ثقـة - تغييـر

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Ch 1 : introduction to signals

- definition

* the **signal** is a function which represents the time variation of physical variable.

$i(t) \rightarrow$ current

$v(t) \rightarrow$ voltage

$T(t) \rightarrow$ Temperature

$p(t) \rightarrow$ power or prressure

$e(t) \rightarrow$ Energy

$I(x,y) \rightarrow$ Image

* the **System** is a process that generates response (output / input effect) for agiven cause (effect relationship / response)

$$x(t) \xrightarrow{\text{input}} \boxed{T} \xrightarrow{\text{output}} y(t) \qquad \therefore y(t) = T(x(t))$$

T : operator Transformation

Ex:

$$x(t) \rightarrow \boxed{\text{buffer}} \rightarrow y(t) = x(t)$$

↳ identity system • Amp with Gain = 1

$$x(t) \rightarrow \boxed{(\ \ )^2} \rightarrow y(t) = x^2(t)$$

$$x(t) \rightarrow \boxed{\int_{-\infty}^{t}} \rightarrow y(t) \qquad y(t) = \int_{-\infty}^{t} x(t)\, dt$$

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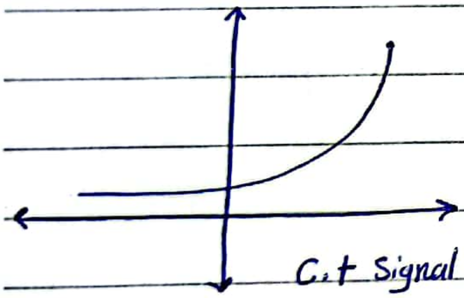
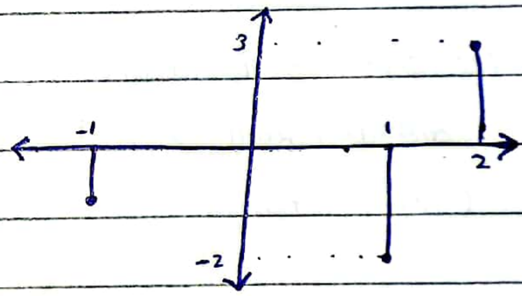
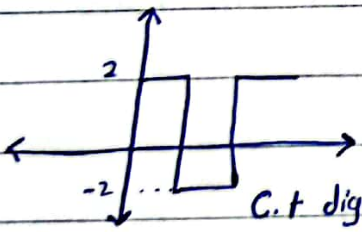
note :- C.t : Continuous time
D.t : Discrete time
Analog : نأخذ عدد كبير من القيم (محور الصادات)
digital : نأخذ عدد محدود من القيم

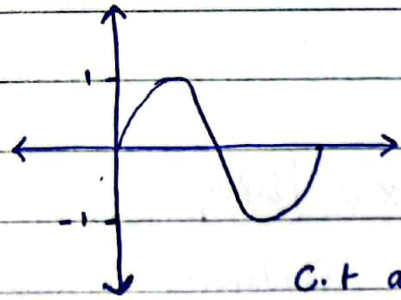
* broadly system can be classified as :-
1] C.t system : both input and output are C.t signals
2] D.t system : both input and output are D.t signals
3] Hybrid system : either input or output is c.t and the other is D.t

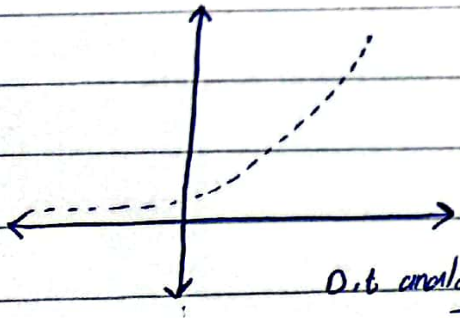
Ex: music / CD / DVD ...

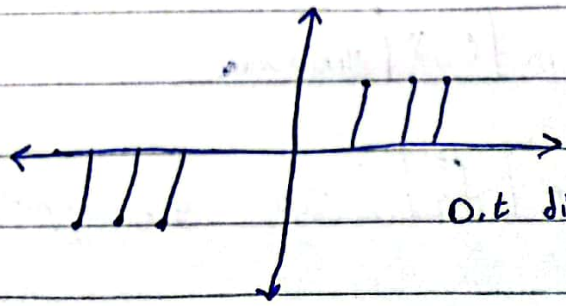

C.t Signal




C.t digital


C.t analog


D.t analog


D.t digital

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* Types of signals :

1] C.t and D.t signals

2] Analog and digital signals

3] power and energy signals

- power signal : x(t) is power signal if $0 < P_x < \infty$

where * $P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$ *

- energy signal : x(t) is energy signal if $0 < E_x < \infty$

where * $E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$

* Energy signal has zero power.
* power signal has ∞ energy.

- Some signals are neither.

Ex: $x(t) = \begin{cases} 1 & t>0 \\ -1 & t<0 \end{cases}$

① $P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} 1 \, dt = 0$

2) $E_x = \int_{-\infty}^{\infty} 1 \, dt = \infty$ neither *.

Ex: $x(t) = A\cos\omega t$

① $E_x = \int_{-\infty}^{\infty} A^2 \cos^2\omega t \, dt = \frac{A^2}{2}\int_{-\infty}^{\infty} 1 + \cos 2\omega t \, dt = \infty + 0 = \infty$

② $P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2 \cos^2\omega t \, dt = \frac{A^2}{4T}\int_{-T}^{T} 1 + \cos 2\omega t \, dt = \frac{2TA^2}{\cancel{4T}\,2T}$

$= \frac{A^2}{2}$ ✓ $\boxed{\text{power}}$

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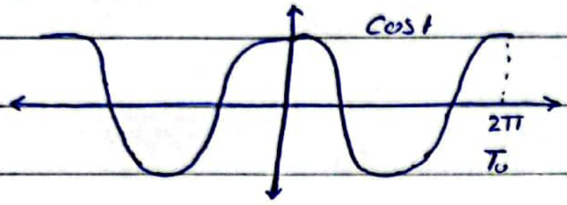
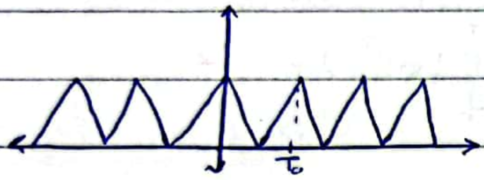
4] periodic and aperiodic [non periodic] Signals :

- periodic : if $x(t) = x(t + KT)$ for $T > 0$.

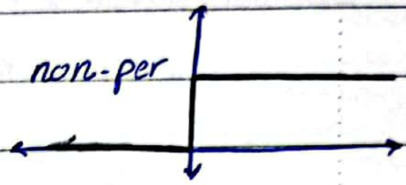


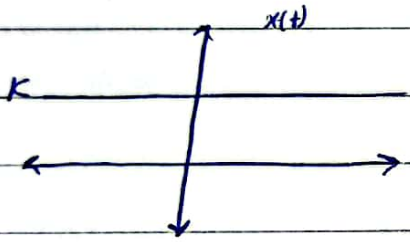
$\cos t = \cos(t + K2\pi)$



periodic

non-per



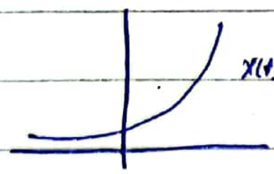


$x(t) = K$ non periodic

5] deterministic and random Signals :

- deterministic : If the signal take specific values described either by equation or graph

$x(t)$ $\cos(t)$



- random : if the signal takes random values [non deterministic]

Ex: noise, thermal noise, quantum noise, voice.

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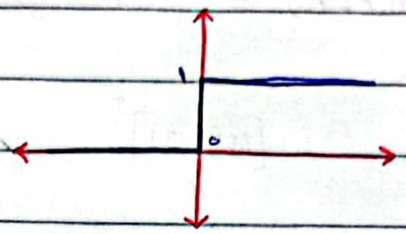
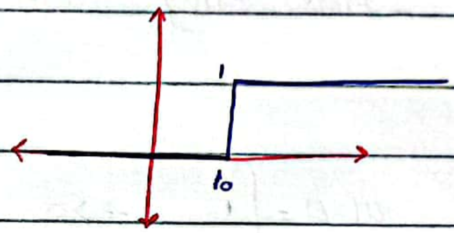
⇒ common signals in engineering:-

1] unit-step signals:-

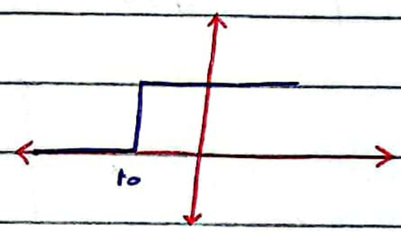
$$u(t) = \begin{cases} 1 & t>0 \\ 0 & t<0 \\ \frac{1}{2} & t=0 \end{cases}$$



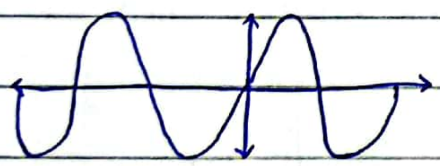
$$u(t-t_0) = \begin{cases} 1 & t>t_0 \\ 0 & t<t_0 \\ \frac{1}{2} & t=t_0 \end{cases}$$

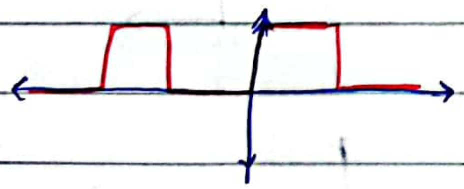


$$u(t+t_0) = \begin{cases} 1 & t>t_0 \\ 0 & t<t_0 \\ \frac{1}{2} & t=t_0 \end{cases}$$

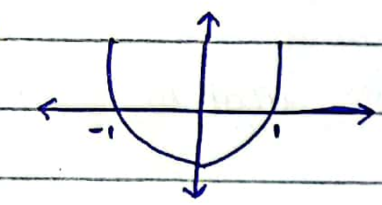


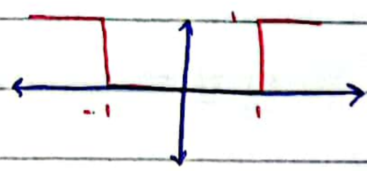
$u(\sin t)$



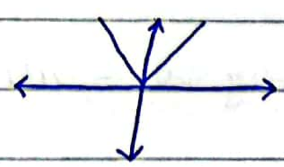


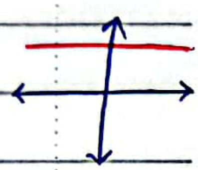
$u(t^2-1)$





$u(|t|)$





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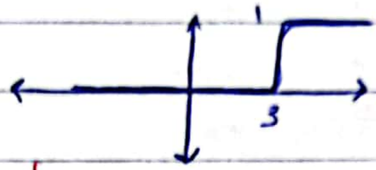
* properties of u(t) :

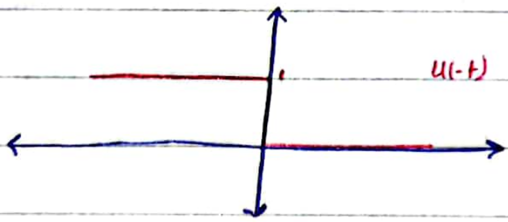
- $u(t-t_0) = [u(t-t_0)]^k$ k is integer

Ex : $[u(t-3)]^3$



- time Scaling $u(at) = \begin{cases} u(t) & a>0 \\ u(-t) & a<0 \end{cases}$ $a \in \mathbb{R}$

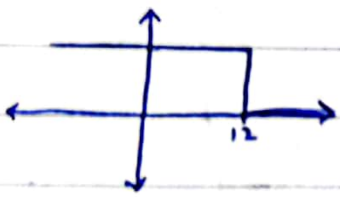
$u(-t) = \begin{cases} 1 & -t>0 \\ 0 & -t<0 \end{cases} \rightarrow u(-t) = \begin{cases} 1 & t<0 \\ 0 & t>0 \end{cases}$



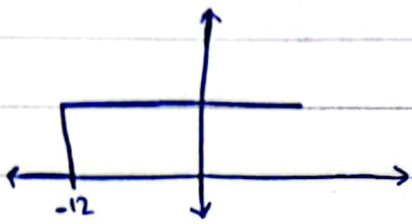
u(-t) time invers

- $u(at+b) = \begin{cases} u(t+\frac{b}{a}) & a>0 \\ u(-t+\frac{b}{a}) & a<0 \end{cases}$

$u(-\frac{t}{3}+4) = u(-t+12)$



$u(\frac{t}{3}+4) = u(t+12)$



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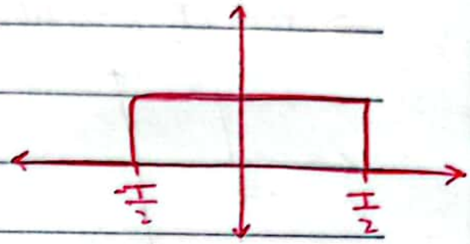
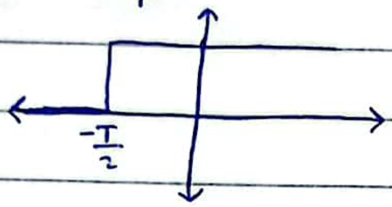
2] Rectangular pulse

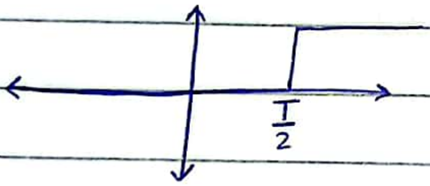
$$\text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1 & -\frac{T}{2} < t < \frac{T}{2} \\ 0 & t < -\frac{T}{2} \,,\ t > \frac{T}{2} \end{cases}$$



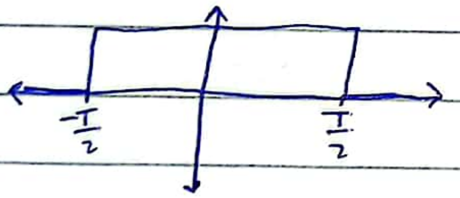
→ $\text{rect}\left(\frac{t}{T}\right) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$



$u\left(t + \frac{T}{2}\right)$

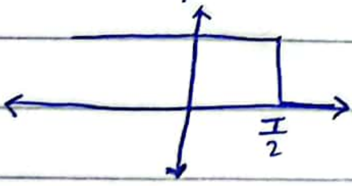


$u\left(t - \frac{T}{2}\right)$

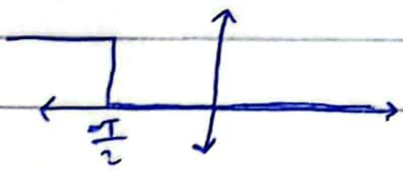


$u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$

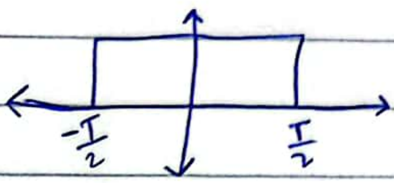
→ $\text{rect}\left(\frac{t}{T}\right) = \left(-t + \frac{T}{2}\right) - u\left(-t - \frac{T}{2}\right)$



$u\left(-t + \frac{T}{2}\right)$



$u\left(-t - \frac{T}{2}\right)$



$u\left(-t + \frac{T}{2}\right) - u\left(-t - \frac{T}{2}\right)$

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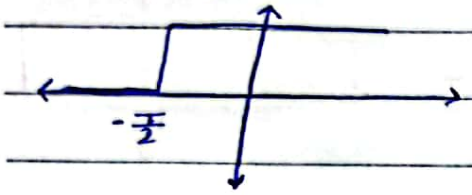
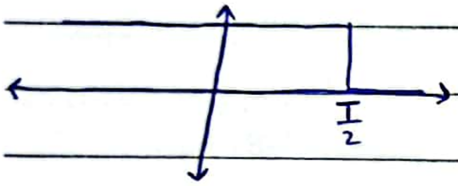
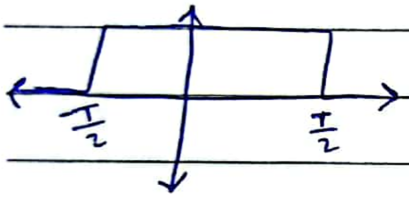
→ $\text{rect}\left(\frac{t}{T}\right) = u\left(t + \frac{T}{2}\right) \cdot u\left(-t + \frac{T}{2}\right)$



$u\left(t + \frac{T}{2}\right)$



$u\left(-t + \frac{T}{2}\right)$



$u\left(t + \frac{T}{2}\right) \cdot u\left(-t + \frac{T}{2}\right)$

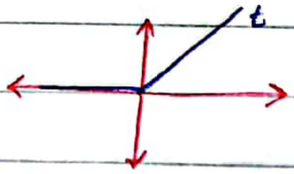
3] unit ramp signals :-

$r(t) = \int_{-\infty}^{t} u(\tau)\, d\tau$

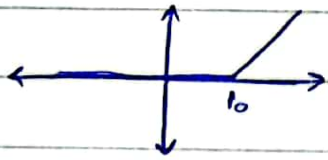
• when $t<0$ $\quad r(t) = \int_{-\infty}^{t} 0\, dt = 0$

• when $t>0$ $\quad r(t) = \int_{0}^{t} 1\, dt = t$

$$\Rightarrow r(t) = \begin{cases} t & t>0 \\ 0 & t<0 \end{cases}$$



* $r(t-t_0)$



$r(t-t_0) = (t-t_0)\, u(t-t_0)$

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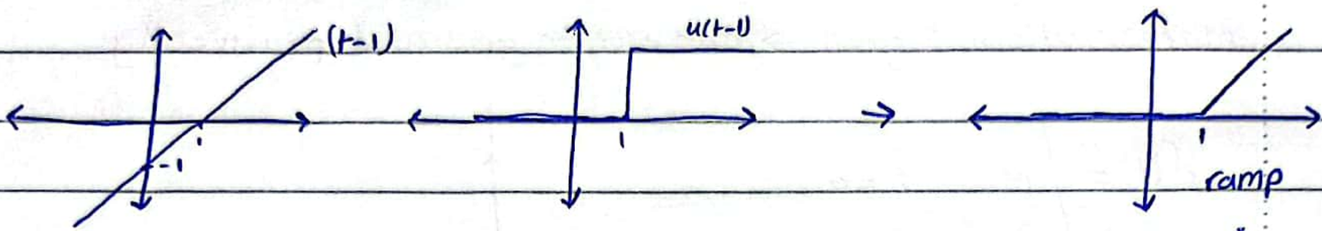
$r(t) = (t-1)\, u(t-1)$



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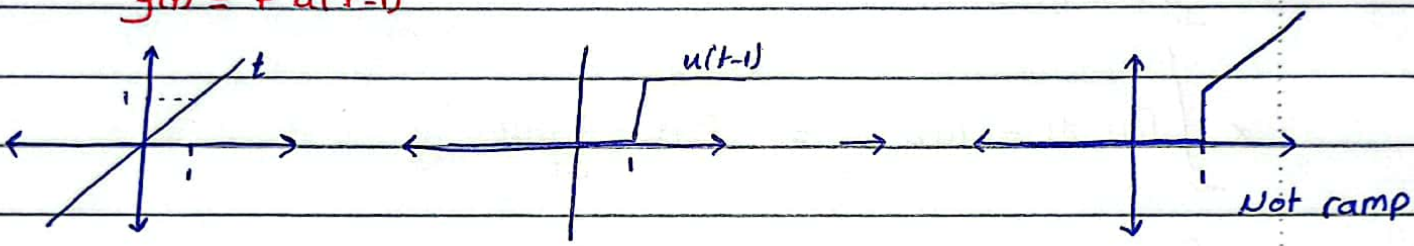
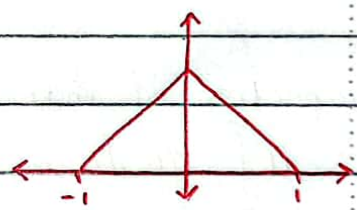
$y(t) = t\, u(t-1)$

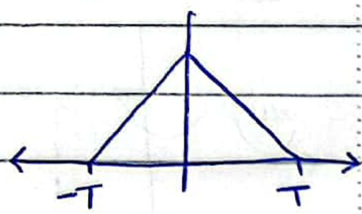


4] triangular pulse:

$$\Delta(t) = tri(t) = \begin{cases} 1-|t| & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$



$$tri\left(\frac{t}{T}\right) = \begin{cases} 1-\left|\frac{t}{T}\right| & \left|\frac{t}{T}\right| < 1 \\ 0 & \text{oth} \end{cases}$$



$$\left|\frac{t}{T}\right| < 1 \rightarrow -1 < \frac{t}{T} < 1$$

$$\boxed{-T < t < T}$$

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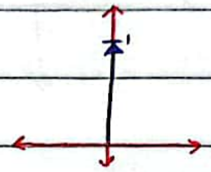
5] unit impulse (Dirac - delta) function:

mathematical signal → can Not be generated practically

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$



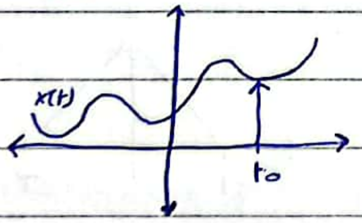
* $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

* $\int_{-\infty}^{t} \delta(t)\, dt = u(t) \Rightarrow \delta(t) = \frac{d}{dt} u(t)$

$\delta(t)$ → infinte energy and infinte power

*properties of $\delta(t)$:

1] $\int_{-\infty}^{\infty} x(t)\, \delta(t-t_0)\, dt = x(t_0)$

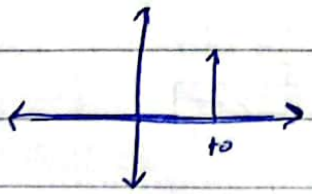


at $t = t_0$

$x(t_0)\, \delta(t-t_0)$

$\int_{-\infty}^{\infty} x(t_0)\, \delta(t-t_0)\, dt$

$= x(t_0) \times 1 = x(t_0)$



* $\int_{a}^{b} x(t)\, \delta(t-t_0)\, dt = \begin{cases} 0 & t_0 \notin [a,b] \\ x(t_0) & t_0 \in [a,b] \end{cases}$

2] $\int_{-\infty}^{\infty} x(t-t_0)\,\delta(t)\,dt = x(-t_0)$

3] $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

4] $\delta(t-t_0) = \frac{d}{dt}u(t-t_0) \Rightarrow u(t-t_0) = \int_{-\infty}^{t}\delta(t-t_0)\,dt$

5] $\delta(at) = \frac{1}{|a|}\delta(t)$

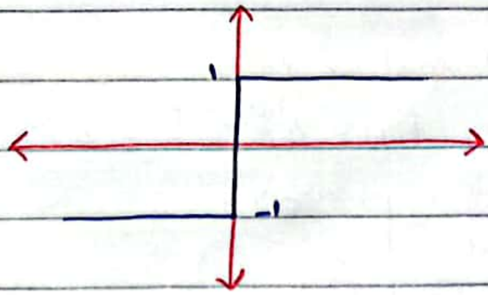
6] $\delta(at+b) = \frac{1}{|a|}\delta\left(t+\frac{b}{a}\right)$

7] $\int_{-\infty}^{\infty}\delta(at-t_0)\,dt = \frac{1}{|a|}$

8] $\delta(-t) = \delta(t)$ [$\delta(t)$ is even function]

9] $\int_{-\infty}^{\infty} x(t)\,\frac{d^n}{dt^n}\delta(t-t_0)\,dt = (-1)^n\frac{d^n}{dt^n}x(t_0)$

6] Signum Signal :-



$$Sgn(t) = \begin{cases} -1 & t<0 \\ 0 & t=0 \\ 1 & t>0 \end{cases}$$

$$= \begin{cases} 2u(t)-1 & , t>0 \\ u(t)-u(-t) & , t<0 \end{cases}$$

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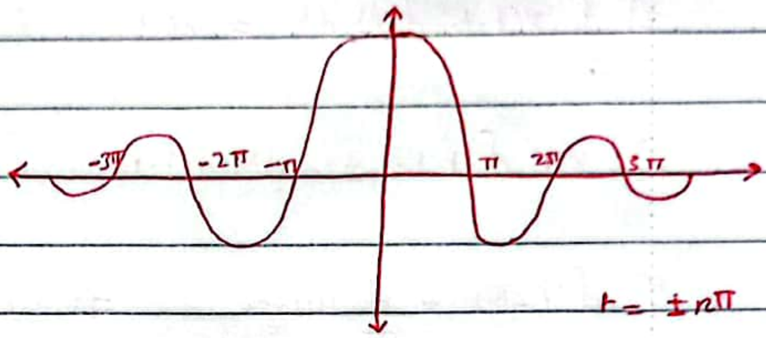
7] Sinc signal

$$x(t) = \text{sinc}(t) = \frac{\sin t}{t}$$

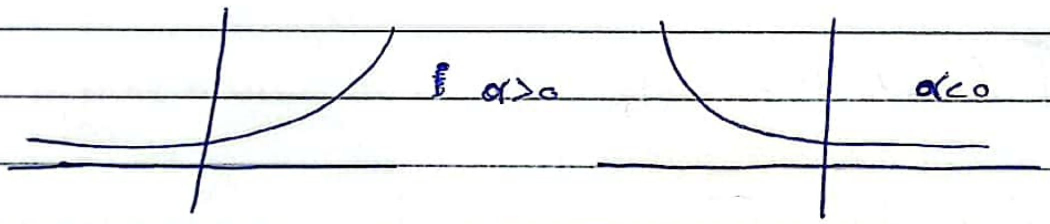


$t = \pm n\pi$

8] exponantial Signal :

$$f(t) = Ae^{(\alpha + j\omega)t}$$

1] Real exp $\omega = 0$ $f(t) = Ae^{\alpha t}$ $A, \alpha \in \mathbb{R}$

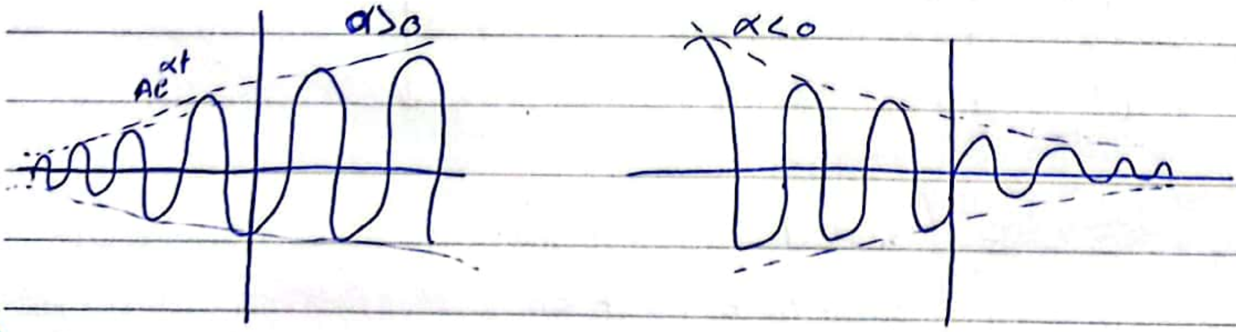


2] Sinusoidal $\alpha = 0$ $f(t) = e^{j\omega t} = \cos \omega t + j \sin \omega t$

3] DC $\alpha = 0$, $\omega = 0 \rightarrow Ae^{0} = A$

4] damped or raised sinusoidal

$$f(t) = Ae^{(\alpha + j\omega)t} = Ae^{\alpha t}(\cos \omega t + j \sin \omega t)$$



N O T E B O O K

Subject Date No.

* Transformation of C.t signals:

- time transformation. - Amplitude transformation.

- time transformation

• reversal:

$y(t) = x(-t)$ [$y(t)$ is a mirror image of $x(t)$ about y-axis]

Ex: CD backward

• time scaling:

$y(t) = x(at)$

- if $|a| > 1 \Rightarrow y(t)$ is a compressed version of $x(t)$ by factor $\underline{\underline{a}}$ [speeded by factor of a]

- if $|a| < 1 \Rightarrow y(t)$ is expanded (dilated) version of $x(t)$ by factor $\frac{1}{a}$

• time shifting:

$y(t) = x(t - t_0)$ shifit to the right [delay by t_0]

$y(t) = x(t + t_0)$ shifit to the left [advanced by t_0]

• general time transformation:

$y(t) = x(at + b)$ $a, b \in R$

$y(t) = x(a(t + \frac{b}{a}))$

→ if $a > 0$ scaling by $\underline{a} \Rightarrow$ ~~s~~ shift left by $\frac{b}{a}$

→ if $a < 0$ → [invert + scale] → if $\frac{b}{a} > 0$ shift left by $|\frac{b}{a}|$

→ [scale + invert] → if $\frac{b}{a} < 0$ shift right by $|\frac{b}{a}|$

N O T E B O O K

- Amplitude transformation:

- **Amplitude reversal**

  $y(t) = -x(t)$ [reflection about x-axis]

- **Amplitude Scaling**

  $y(t) = A[x(t)]$ → if $|A| > 1$ amplification by A
  → if $|A| < 1$ attention by A

- **Amplitude shifting**

  $y(t) = x(t) + B$ [B > 0 shift up] [B < 0 shift down]

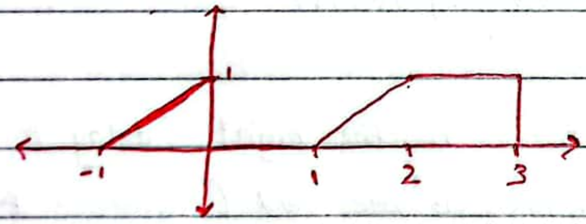
- **general amplitude transformation:**

  $y(t) = Ax(t) + B \qquad A, B \in \mathbb{R}$

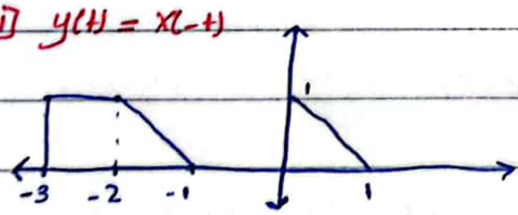
- **general time and Amplitude transformation**

  $y(t) = A\,x(at+b) + B$

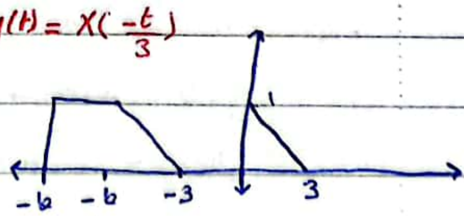
Ex:



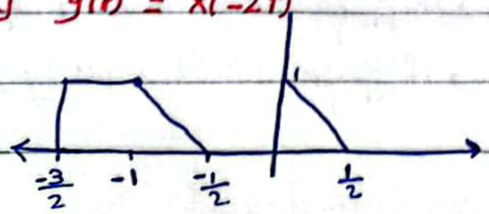
1] $y(t) = x(-t)$



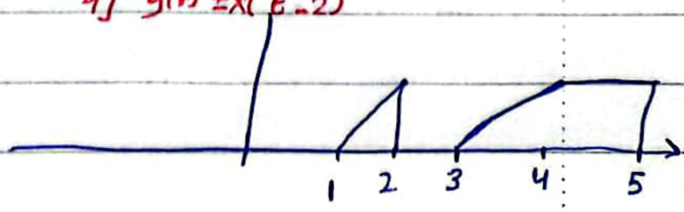
3] $y(t) = x\left(\frac{-t}{3}\right)$



2] $y(t) = x(-2t)$

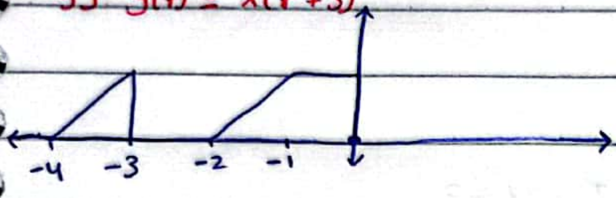


4] $y(t) = x(t-2)$



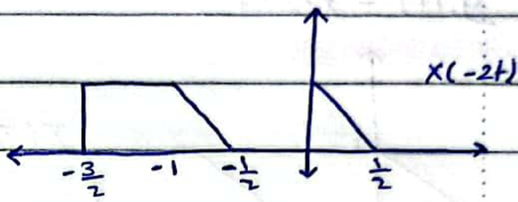
Subject Date No.

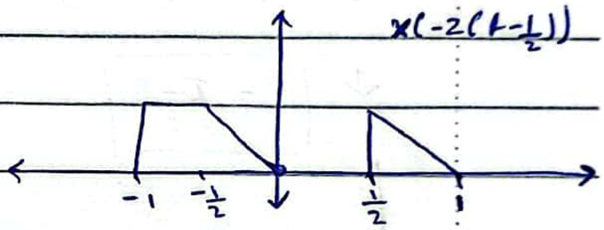
5] $y(t) = x(t+3)$



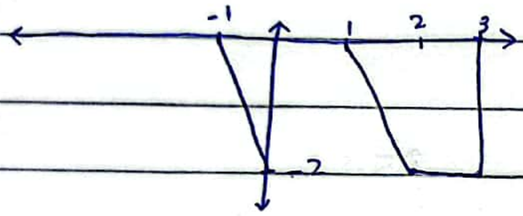
6] $y(t) = x(1-2t)$

$= x\left(-2\left(t-\frac{1}{2}\right)\right)$

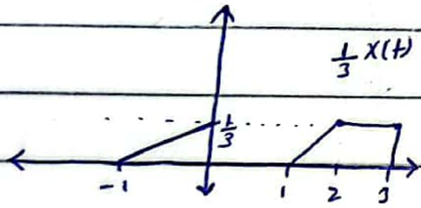


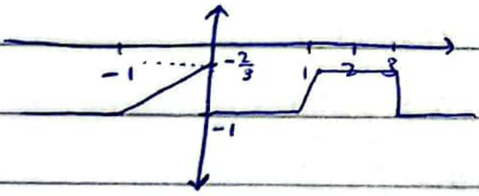


7] $y(t) = -2x(t)$



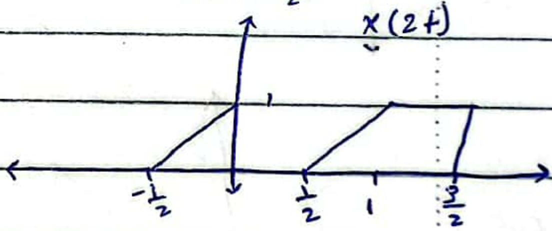
8] $y(t) = \frac{1}{3}x(t) - 1$

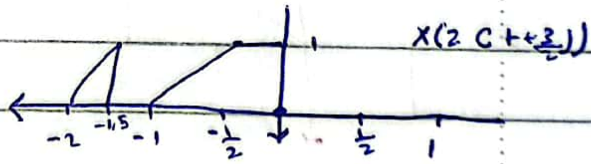


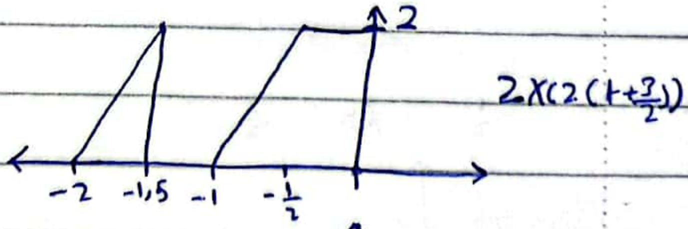


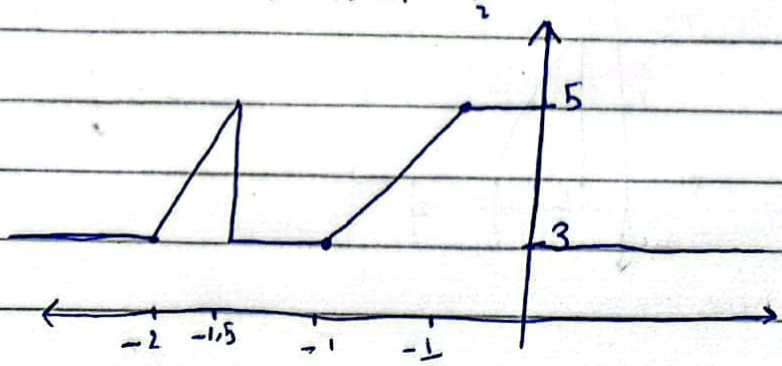
9] $y(t) = 2x(2t+3) + 3$

$x\left(2\left(t+\frac{3}{2}\right)\right)$







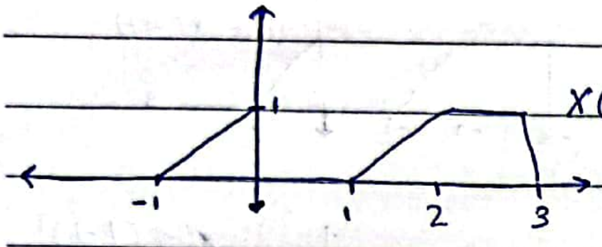


Subject Date No.

* Alternatively :-

$y(t) = 2\,x(2t+3) + 3$

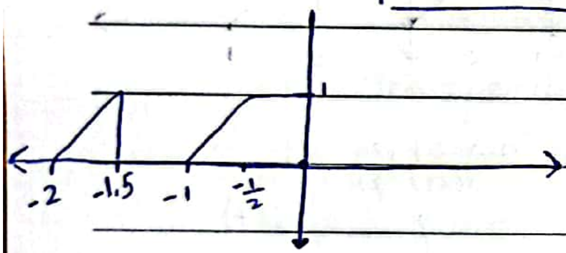
$y_1(t) = x(2t+3)$

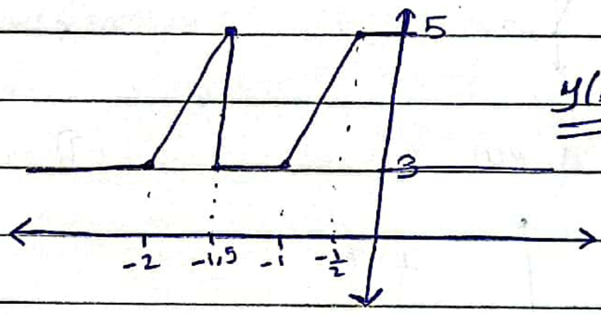


$\tau = 2t+3$

$t = \frac{\tau - 3}{2}$

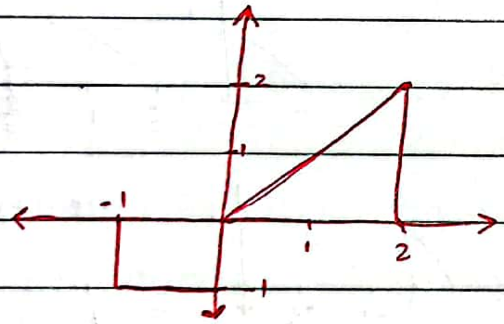
$$\boxed{t = \frac{\tau - 3}{2}}$$





$\underline{\underline{y(t)}}$

Ex:

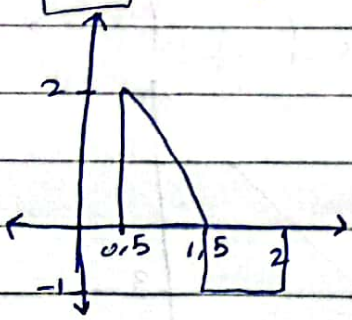


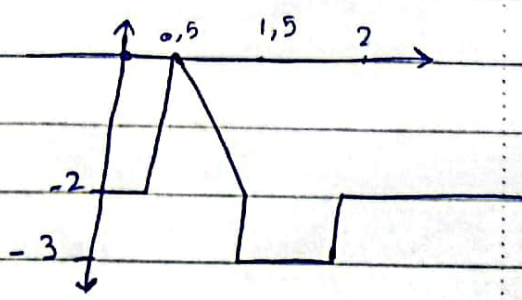
① $y(t) = x(-2t+3) - 2$

$y_1(t) = x(-2t+3)$

$\tau = -2t+3$

$$\boxed{t = \frac{3-\tau}{2}}$$



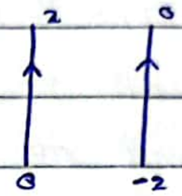


N O T E B O O K

Subject Date No.

② write $y(t)$ in terms of $x(t)$, assume that $y(t) = A\,x(at+b) + B$

a, b, A, B ?



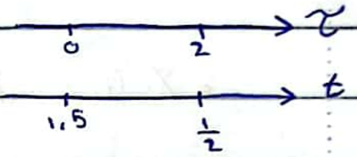
- amplitude function $y(t) = A\,x(t) + B$

$-2 = A \times 0 + B$

$B = -2$

$0 = A \times 2 + B \rightarrow A = 1$

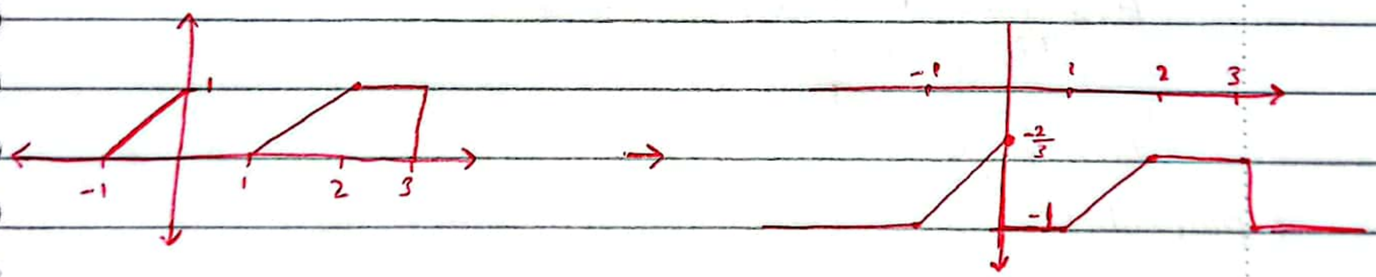
- time $y(t) = x(at+b)$



$$\left.\begin{array}{l} 0 = a \times 1.5 + b \\ 2 = a \times 0.5 + b \end{array}\right] \rightarrow a = -2$$

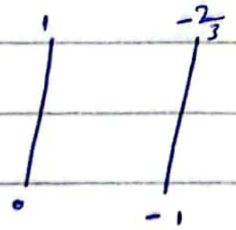
$b = 3$

$\Rightarrow$ general time and amplitude equation $y(t) = x(-2t+3) - 2$



$y(t) = A\,x(at+b) + B$

* $y(t) = A\,x(t) + B$



$-1 = A \times 0 + B$

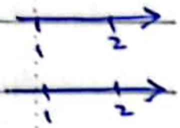
$\boxed{B = -1}$

$-\frac{2}{3} = A \cdot 1 - 1$

$A = -\frac{2}{3} + 1$

$A = \frac{1}{3}$

* $y(t) = x(at+b)$



$a = 1,\ b = 0$

$\Rightarrow$ general time and amplitude equation

$y(t) = \frac{1}{3}\,x(t) - 1$

NOTEBOOK

Subject Date No.

⇒ even and odd Signal :

- even if $x(-t) = x(t)$
- odd if $x(-t) = -x(t)$

- any signal $x(t)$ can be written as:

$$X(t) = X_e(t) + X_o(t)$$

where

- $X_e(t) = \frac{x(t) + [x(-t)]}{2}$

- $X_o(t) = \frac{x(t) - x(-t)}{2}$

* the avarage value of $x(t)$ is

$$A_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt = A_{xe}$$

$$A_{xo} = 0$$

* properties :-

1] even + even = even

2] even + odd = neither

3] odd + odd = odd

4] even * even = even

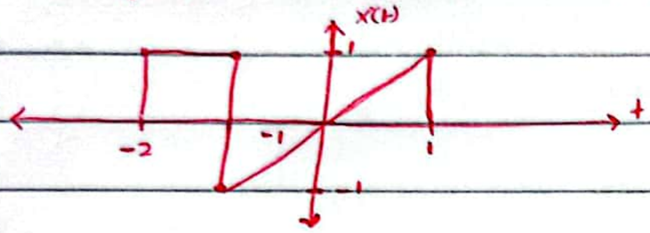
5] odd * odd = even

6] odd * even = odd

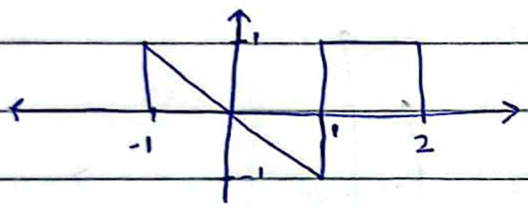
CamScanner

Subject Date No.

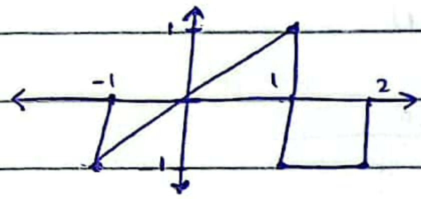
Ex: plot $x_e(t)$, $x_o(t)$, A_x, A_{xe}, A_{xo}



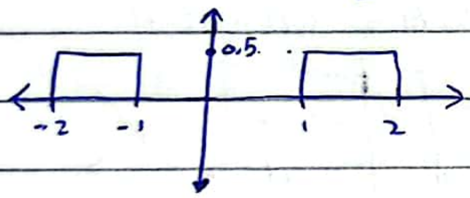
$\Rightarrow x(-t)$



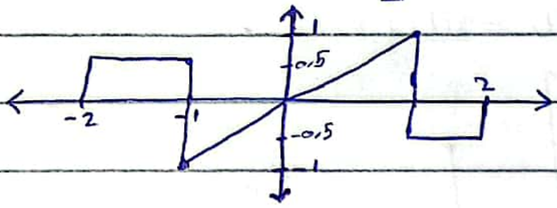
$\Rightarrow -x(-t)$



① $x_e(t) = \dfrac{x(t) + x(-t)}{2}$



② $x_o(t) = \dfrac{x(t) - x(-t)}{2}$



③ $A_x = \lim\limits_{T\to\infty} \dfrac{1}{2T} \displaystyle\int_{-T}^{T} x(t)\,dt$

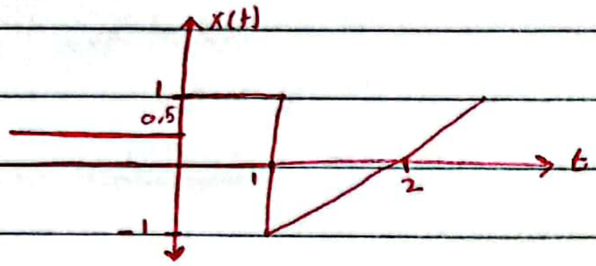
$$\lim_{T\to\infty} \frac{1}{2T}\left[\int_{-T}^{T} dt + \int_{-T}^{T} t\,dt\right] = \lim_{T\to\infty} \frac{1}{2T}\left[2T + 0\right] = \underline{1}$$

$A_{xe} = 1$, $A_{xo} = 0$ [always for any odd signal]

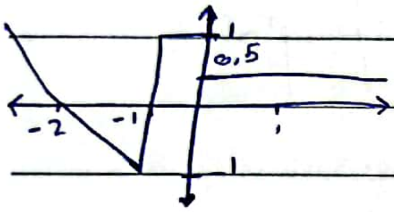
note: $x_e(0) = x(0)$ always

NOTEBOOK

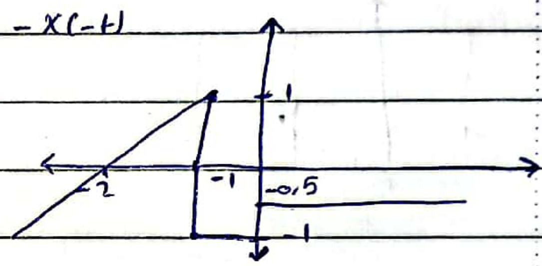
Ex: Find x_e, x_o, A_x, A_{xe}, A_{xo}



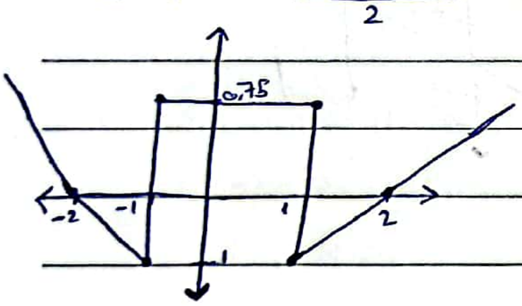
$\Rightarrow x(-t)$



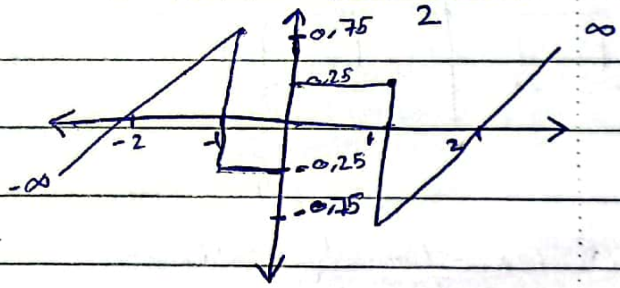
$\Rightarrow -x(-t)$



① $x_e(t) = \dfrac{x(t) + x(-t)}{2}$



② $x_o(t) = \dfrac{x(t) - x(-t)}{2}$



$(-\infty, \infty)$ not limited

③ A_x (calculaition of Area)

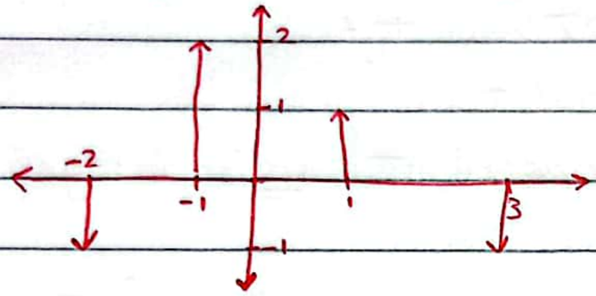
$$A_x = \int_{-\infty}^{0} \tfrac{1}{2}\,dt + \int_{0}^{1} dt + \int_{1}^{\infty} (t-2)\,dt = \infty$$

$A_{xe} = \infty$ $\quad A_{xo} = 0$

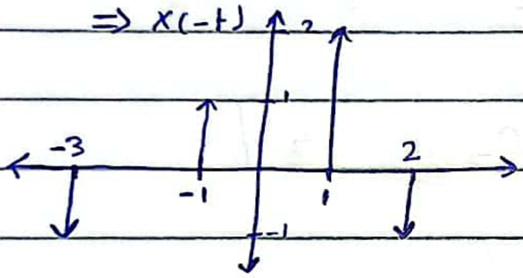
Subject Date No.

Ex: Find $x_e, x_o, A_x, A_{xe}, A_{xo}$ for this descrete signal

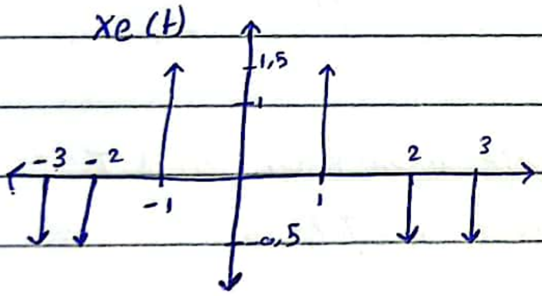
- this signal contains only impulses



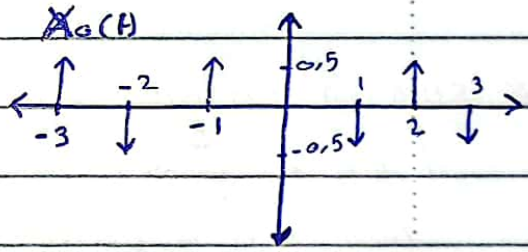
$\Rightarrow x(-t)$



$x_e(t)$



$x_o(t)$

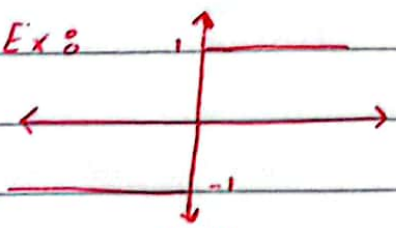


$$x(t) = -\delta(t+2) + 2\delta(t+1) + \delta(t-1) - \delta(t-3)$$

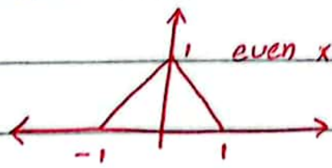
$= 1$

$A_{xe} = 1$ $A_{xo} = 0$

Ex:



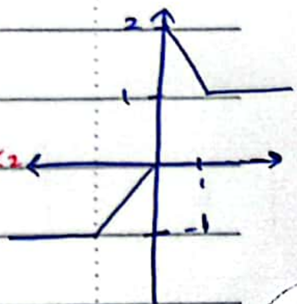
odd x_1



even x_2

* plot $y(t) = x_1 + x_2$

neither



* plot $y(t) = x_1 \cdot x_2$

odd

N O T E B O O K

Subject Date No.

⇒ periodic Signals and aperiodic :

- c.t signal $x(t)$ is periodic of period T
  iff $x(t) = x(t+T)$ $\infty > T > 0$ $\forall t$

- if $x(t)$ is periodic with period $T>0$
  then it's also periodic with period nT , n is integer

- the fundemantal of $x(t)$ is the min value of $T>0$
  which satesfies denoted by T_0

- fundemental frequency of $x(t)$ is $\boxed{f_0 = \frac{1}{T_0} \text{ HZ}}$

and fundemental radiun frequency of $x(t)$ is $\boxed{\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0 \text{ rad/sec}}$

* sum of periodic signals :

- if $y(t) = \sum_{i=1}^{n} x_i(t)$ where $x_i(t)$ is periodic with fund. period T_0
  then $y(t)$ is periodic iff all the ratios of $\frac{T_0 1}{T_0 i}$ are $\frac{\text{integr}}{\text{integr}}$
  otherwise $y(t)$ is aperiodic

- if $y(t)$ is periodic $\boxed{T_0 = k T_{01}}$ where $\boxed{k = LCM \Rightarrow \text{lest common}}$
  multiple

N O T E B O O K

Ex: check the following signals for periodicity:

1] $x(t) = e^{j\pi t}$

$x(t) = \cos(\pi t) + j\sin(\pi t)$

$\cos(\pi t) + j\sin(\pi t) = \cos\pi(t+T) + j\sin\pi(t+T)$

$\Rightarrow \cos(\pi t) = \cos\pi(t+T)$

$= \cos(\pi t + \pi T)$

$= \cos(\pi t)\cdot\underbrace{\cos(\pi T)}_{1} - \cancel{\sin(\pi t)\cdot\underbrace{\sin(\pi T)}_{0}}$

$= \cos(\pi t)$ $T_1 = 2$

$\Rightarrow \sin(\pi t) = \sin \pi t + \pi T \rightarrow T_2 = 2$

$\frac{T_2}{T_1} = \frac{2}{2} \Rightarrow \frac{int}{int}$

$k=1$ then $x(t)$ periodic by $T_0 = 2$

2] $x(t) = \cos 2t$

periodic by $T_0 = \pi$

note: $\cos(\omega t + \theta)$ periodic by $T_0 = \frac{2\pi}{\omega}$

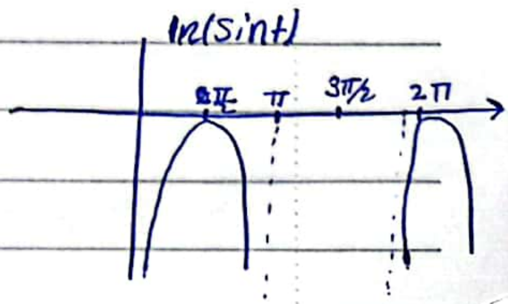
3] $x(t) = t\sin(t + \frac{\pi}{4})$

$t\sin(t+\frac{\pi}{4}) \overset{?}{=} (t+T)\sin(t+T+\frac{\pi}{4})$

aperiodic $t \neq (t+T)$ with $T>0$

4] $e^{\cos t} = x(t)$

$e^{\cos t} \overset{?}{=} e^{\cos(t+T)}$ $T = 2\pi$



5] $\ln(\sin t) = x(t)$

$= \ln(\sin(t+T))$ periodic by $T = 2\pi$

Subject Date No.

6] $x(t) = t + \cos 3t$

$t + \cos 3t = (t+T) + \cos(3t + 3T)$

aperiodic

7] $x(t) = \cos t - \frac{1}{2j}\sin(3t) + \cos(5t + \frac{\pi}{2})$

$\Rightarrow \cos t \quad T_1 = 2\pi$

$\Rightarrow \frac{-1}{2j}\sin(3t) \quad T_2 = \frac{2\pi}{3}$

$\Rightarrow \cos(5t + \frac{\pi}{2}) \quad T_3 = \frac{2\pi}{5}$

$\frac{T_1}{T_2} = 2\pi \cdot \frac{3}{2\pi} = 3$

$\frac{T_1}{T_3} = 2\pi \cdot \frac{5}{2\pi} = 5$

$\boxed{k = 15} \quad T_0 = T_1 k$

$= 30\pi$

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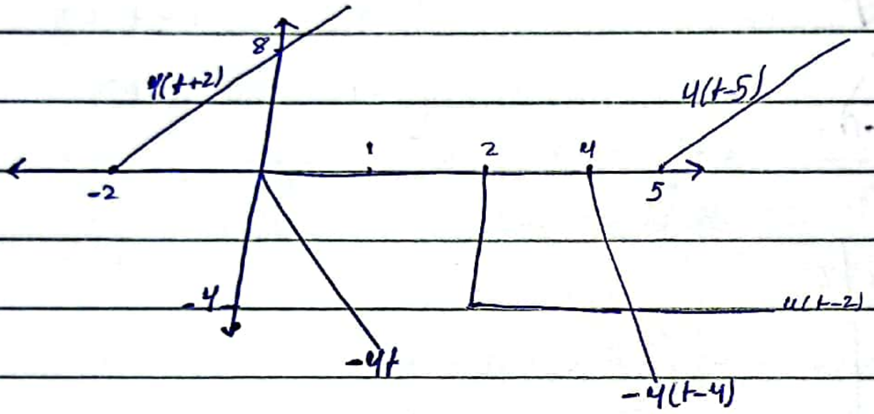
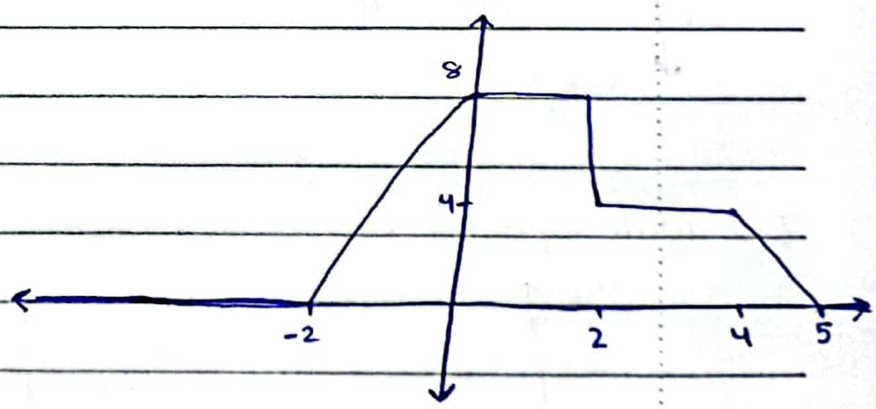
⇒ Mathimatical function of signals :

- given equation plot the graph
- given plot write equation

Ex: $x(t) = 4(t+2)\,u(t+2) - 4t\,u(t) - 4u(t-2) - 4(t-4)\,u(t-4) + 4(t-5)\,u(t-5)$



$$x(t) = \begin{cases} 0 & t < -2 \\ 4(t+2) & -2 < t < 0 \\ 8 & 0 < t < 2 \\ 4 & 2 < t < 4 \\ 20-4t & 4 < t < 5 \\ 0 & t > 5 \end{cases}$$



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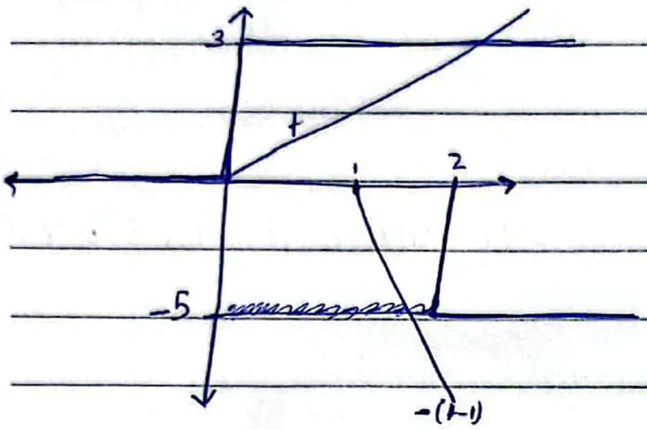
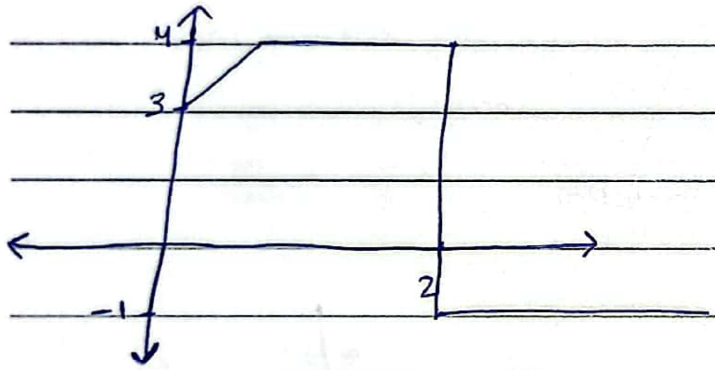
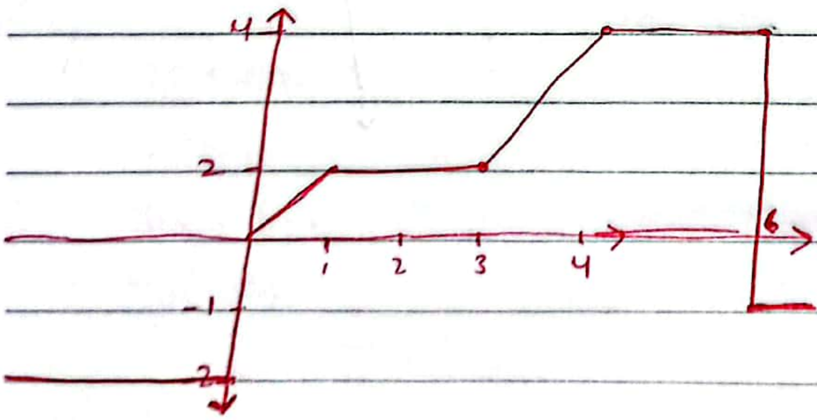
Ex: $x(t) = 3u(t) + tu(t) - (t-1)u(t-1) - 5u(t-2)$



$$x(t) = \begin{cases} 0 & t<0 \\ t+3 & 0<t<1 \\ 4 & 1<t<2 \\ -1 & t>2 \end{cases}$$



Ex: write equation of x(t) interms of unit-step?



$$x(t) = \begin{cases} -2 & t<0 \\ 2t & 0<t<1 \\ 2 & 1<t<3 \\ 2(t-2) & 3<t<4 \\ 4 & 4<t<6 \\ -1 & t>6 \end{cases}$$

$$x(t) = -2u(-t) + 2t[u(t) - u(t-1)] + 2[u(t-1) - u(t-3)]$$
$$+ (2t-4)[u(t-3) - u(t-4)] + 4[u(t-4) - u(t-6)]$$
$$- u(t-6)$$

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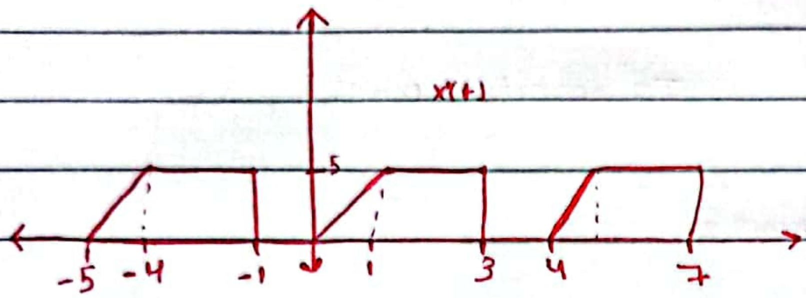
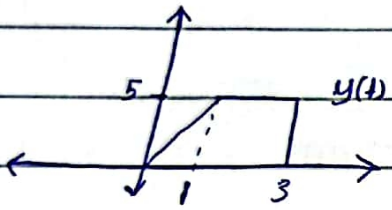
Ex: $x(t)$ is periodic with $T_0 = 4$





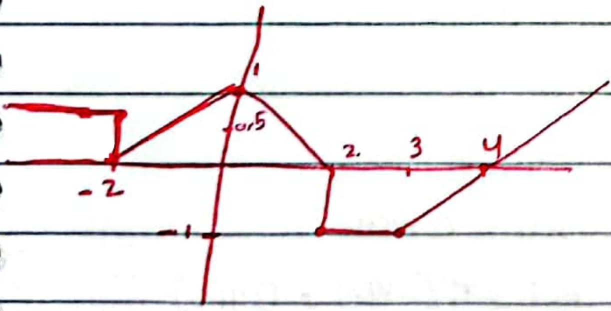
$$y(t) = \begin{cases} 5t & 0<t<1 \\ 5 & 1<t<3 \\ 0 & 3<t<4 \end{cases}$$

$$y(t) = 5t\left[u(t) - u(t-1)\right] + 5\left[u(t-1) - u(t-3)\right]$$

$$= 5t\,\mathrm{rect}\left[t - \tfrac{1}{2}\right] + 5\,\mathrm{rect}\left[\frac{t-2}{2}\right]$$

$$\Rightarrow x(t) = \sum_{-\infty}^{\infty} 5(t-4k)\,\mathrm{rect}\left[t-4k-\tfrac{1}{2}\right] + 5\,\mathrm{rect}\left[\frac{t-4k-2}{2}\right]$$

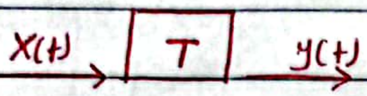
Ex: Find equation of $y(t)$



$$y(t) = \begin{cases} 0.5 & t<-2 \\ \tfrac{1}{2}t+1 & -2<t<0 \\ -\tfrac{1}{2}t+1 & 0<t<2 \\ -1 & 2<t<3 \\ t-4 & t>3 \end{cases}$$

$$y(t) = \tfrac{1}{2}u(-t-2) + \mathrm{tri}\left(\tfrac{t}{2}\right) - \left[u(t-2) - u(t-3)\right] + (t-4)\,u(t-3)$$

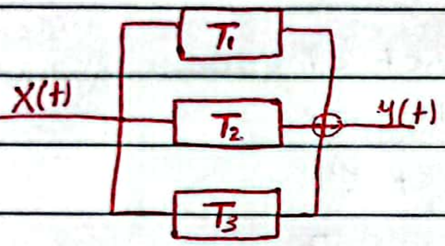
→ continuous systems:



T: operator trans

- Interconnecting of systems:

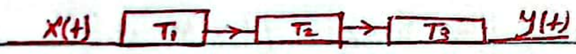
- parallel system:



$y(t) = y_1 + y_2 + y_3$

$y(t) = T_1 x(t) + T_2 x(t) + T_3 x(t)$

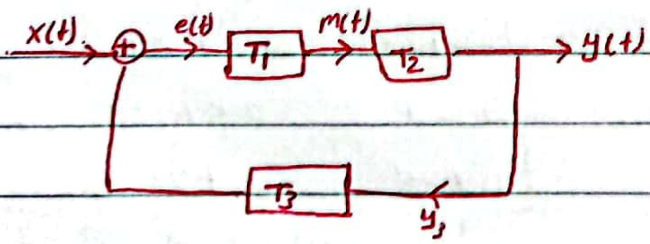
- series system:



$y(t) = T_3 [T_2 [T_1 x(t)]]$

$y(t) = T_{eq}\, x(t)$

- Feed back System:



$y(t) = T_2\, m(t)$

$m(t) = T_1 [\, x(t) + T_3 y(t)]$

$y_3 = T_3\, y(t)$

$y(t) = T_2 [T_1 [\, x(t) + T_3 y(t)]]$

$e(t) = x(t) - y_3(t)$

$= x(t) - T_3\, y(t)$

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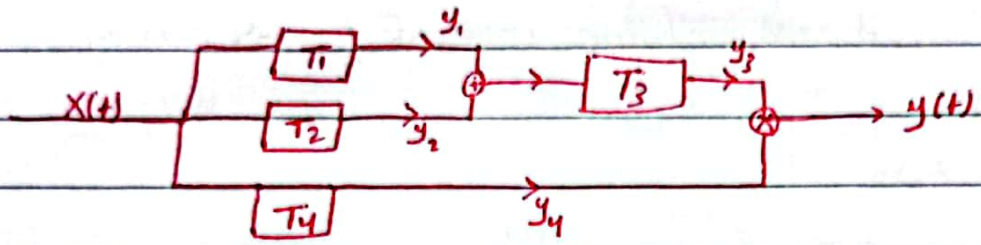
Ex: express $y(t)$ in the following system



$y(t) = y_3 \cdot y_4$

$y(t) = [T_4\, x(t)] \cdot [T_3 [T_1 x(t) + T_2 x(t)]]$

⇒ properties of cts systems:

1] memory or memoryless system:

- memoryless: if output $y(t)$ at $t = t_0$ depends only on input $x(t)$ at $t = t_0$
  - output depends on present value of input is called static system.

- memory: if output $y(t)$ at $t = t_0$ depends on values other than present values of input [pervious or futured is called dynamic system]

Ex: check the following systems for memoryless:

1] $y(t) = k\, x(t)$

$y(t_0) = k\, x(t_0)$ memoryless [احسبه نفس اللحظة] [depend on t_0]

2] $y(t) = x^2(t)$

$y(t_0) = x^2(t_0)$ memoryless [depend on t_0]

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3] $y(t) = \int_{0}^{t} x(2\tau)\, d\tau$

$y(t_0) = \int_{0}^{t_0} x(2\tau)\, d\tau$ depend on future value of t $\Rightarrow$ memory

4] $y(t) = x(t - t_0)$, $t_0 > 0$

$y(t_0) = x(0)$ and $t_0 \neq 0$ by the condition

y depends on another value of t_0 $\Rightarrow$ memory

5] $y(t) = x(t + t_0)$, $t_0 > 0$

$y(t_0) = x(2t_0)$ $\rightarrow$ $y(t)$ depends on future value of t_0 $\Rightarrow$ memory

6] $y(t) = x(-t)$

$y(t_0) = x(-t_0)$ $\Rightarrow$ memory

7] $y(t) = x(at + b)$

$y(t_0) = x(at_0 + b)$ $\Rightarrow$ memory

8] $y(t) = A\, x(t) + B$

$y(t_0) = A\, x(t_0) + B$ $\Rightarrow$ memoryless.

9] $y(t) = \sin(t) . x(t)$

memory less

10] $y(t) = \ln(x(t))$

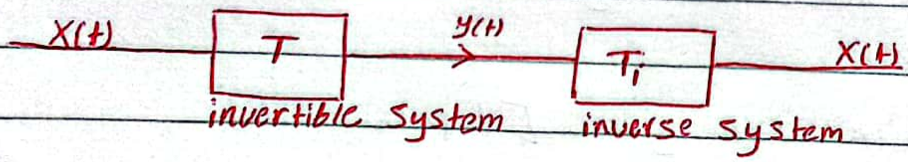
memoryless

11] $y(t) = A\, x(at+b) + B$

memory

$\Rightarrow$ invertibility :

- Invertible : distinct input leads to distinct output in other words if input can be determined uniquely for output by passing it through an inverse system other wise the system is non invertible



Ex:

1] $x(t) \rightarrow \boxed{k \cdot x(t)} \xrightarrow{y(t)} \boxed{y(t) \cdot \frac{1}{k}} \rightarrow x(t)$ invertable

2] $x(t) \rightarrow \boxed{[x(t)]^2} \xrightarrow{y(t)} \boxed{\sqrt{y(t)}} \rightarrow \begin{matrix} +x(t) \\ -x(t) \end{matrix}$ non - Invertable

3] $x(t) \rightarrow \boxed{\text{Scale by } 2} \xrightarrow{x(2t)} \boxed{\int_0^t} \xrightarrow{\int x(2t)} \boxed{\frac{d}{dt}} \xrightarrow{x(2t)} \boxed{\text{Scale by } t} \xrightarrow{x(t)}$ invertable

4] $x(t) = y(t+t_0)$ inv

5] $x(t) = y(t-t_0)$ inv

6] $x(t) = y(\frac{t}{a})$ inv

7] $x(t) = \frac{y(t) - B}{A}$ inv

8] $x(t) = \frac{y(t)}{\sin 2t}$ non - inv

9] $x(t) = \ln y(t)$ inv

10] $x(t) = e^{y(t)}$ inv

11] $x(t) = \cos^{-1} y(t)$ non - inv

⇒ Causality:

- Causal : if output at $t = t_0$
  $y(t)$ depend on input $x(t)$ for $t \leq t_0$

* output depend only on present or previous value of input.

- non causal : if output depend on future value of input

* static system are memoryless , causal

- not all dynemic system are non causal.

* all variable memory system must be causal system

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⇒ Stability :

- BIBO system : if the output is bounded for all bounded input.

- bounded signal : $x(t)$ is bounded if there exist $M \in \mathbb{R}$ such that $|x(t)| \leq M$ for $\forall t$

- marginally stable system if output is bounded for some input and unbounded for some other input

Ex: check BIBO stability of the following systems :

1] $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$x(t) = u(t)$ bond

$y(t) = \begin{cases} 0 & t<0 \\ t & t>0 \end{cases}$ unbounded → unstable

2] $|y(t)| = |\cos[x(t+1)]|$ → BIBO

3] $|y(t)| = |Ax(t) + B| \leq |A x(t)| + |B|$

$|A| M + |B|$ → BIBO

4] $|y(t)| = |x(at+b)| = M$ → BIBO

5] $|y(t)| = |x^n(t)| = M^n$ → BIBO

6] $|y(t)| = |e^t u(t+1)| \cdot |x(t-1)|$

$|e^t| \cdot M$ unstable

7] $|y(t)| = |e^{x(t)}| = |e^M|$ → BIBO

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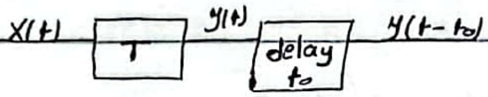
⇒ time invariance

System is [T.I] if time shift in the input signal by t_0 results in same shift in the output

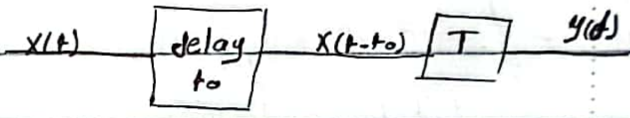
* othere wise system is tim varying [T.V]

Test for T.I



iff $y_d = y(t-t_0)$

T.I



if $y_d \neq y(t-t_0)$ T.V

Ex: check the follwing for T.I :-

1] $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$$y(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$$

$$y_d = \int_{-\infty}^{t} x(\tau - t_0)\, d\tau \rightarrow \int_{-\infty}^{t-t_0} x(u)\, du = y(t-t_0) \quad [T.I]$$

$u = \tau - t_0$

$du = d\tau$

2] $y(t) = \dfrac{d\,x(t)}{dt}$

$$y(t-t_0) = \left.\frac{d\,x(t)}{dt}\right|_{t-t_0}$$

$$y_d = \frac{d\,x(t-t_0)}{dt} = y(t-t_0)\ [T.I]$$

$t_0 = 1 \qquad x(t) = u(t)$

$$y(t-1) = \left.\frac{du(t)}{dt}\right|_{t-1} = \delta(t-1)$$

$$y_d = \frac{d\,u(t-1)}{dt} = \delta(t-1)$$

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3] $y(t) = A\,x(t) + B$

$y(t-t_0) = A\,x(t-t_0) + B$

$y_d = A\,x(t-t_0) + B$ [T.I]

4] $y(t) = x(at+b)$

$y(t-t_0) = x(a(t-t_0)+b)$

$y_d = x(at - t_0 + b)$ [T.V]

5] $y(t) = \int_0^t x(2\tau)\,d\tau$

$y(t-t_0) = \int_0^{t-t_0} x(2\tau)\,d\tau$

$y_d = \int_0^t x(2\tau - t_0)\,d\tau \Rightarrow \int_{-t_0}^{2t-t_0} x(u)\cdot\frac{1}{2}\,du$ [T.V]

$u = 2\tau - t_0$

$du = 2d\tau$

6] $y(t) = \ln(x(t))$

$y(t-t_0) = \ln x(t-t_0)$

$y_d = \ln(x(t-t_0))$ [T.I]

7] $y(t) = x(t)\cdot\cos t$

$y(t-t_0) = x(t-t_0)\cdot\cos(t-t_0)$

$y_d = x(t-t_0)\cos t$ [T.V]

8] $y(t) = |x(t)|$

$y(t-t_0) = |x(t-t_0)|$

$y_d = |x(t-t_0)|$ [T.I]

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⇒ linearity property :

- System is linear iff it is satisfies superposition principle, other wise non linear.

* Superposition principle :- mean two properties are satisfied

- additivity
$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t)$$
$$x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t)$$

- homogeneity

$$x_1(t) \to y_1(t)$$
$$a_1 x_1(t) \to a_1 y_1(t)$$

S.P.P $\quad a_1 x_1(t) + a_2 x_2(t) \to a_1 y_1(t) + a_2 y_2(t)$

Ex: check the following linearity :

1] $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$$y_1(t) = \int_{-\infty}^{t} x_1(\tau)\, d\tau \qquad y_2(t) = \int_{-\infty}^{t} x_2(\tau)\, d\tau$$

$$a_1 y_1 + a_2 y_2 = a_1 \int_{-\infty}^{t} x_1(\tau)\, d\tau + a_2 \int_{-\infty}^{t} x_2(\tau)\, d\tau$$

$$y^*(t) = \int_{-\infty}^{t} [a_1 x_1(\tau) + a_2 x_2(\tau)]\, d\tau$$

$$= a_1 \int_{-\infty}^{t} x_1(\tau)\, d\tau + a_2 \int_{-\infty}^{t} x_2(\tau)\, d\tau = a_1 y_1 + a_2 y_2 \quad ⁂ \quad \Rightarrow L$$

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2] $y(t) = \frac{d\,x(t)}{dt}$

$y_1(t) = \frac{d\,x_1(t)}{dt}$ $y_2(t) = \frac{d\,x_2(t)}{dt}$

$a_1 y_1 + a_2 y_2 = a_1 \frac{d x_1(t)}{dt} + a_2 \frac{d x_2(t)}{dt}$

$y^*(t) = \frac{d}{dt}[a_1 x_1(t) + a_2 x_2(t)] = a_1 \frac{d x_1(t)}{dt} + a_2 \frac{d x_2(t)}{dt}$ $\underline{\underline{L}}$

3] $y(t) = A\,x(t) + B$

$y_1(t) = A\,x_1(t) + B$ $y_2(t) = A\,x_2(t) + B$

$a_1 y_1 + a_2 y_2 = a_1[A x_1(t) + B] + a_2[A x_2(t) + B]$

$y^*(t) = A[a_1 x_1(t) + a_2 x_2(t)] + B$

$a_1 y_1 + a_2 y_2 \neq y^*(t)$ $\underline{\underline{\text{non-L}}}$

4] $y(t) = x(at+b)$

$a_1 y_1 + a_2 y_2 = a_1\, x_1(at+b) + a_2\, x_2(at+b)$

$y^*(t) = a_1\, x_1(at+b) + a_2\, x_2(at+b)$ $\underline{\underline{L}}$

5] $y(t) = \int_0^t x(2\tau)\, d\tau$

$y_1 = \int_0^t x_1(2\tau)\, d\tau$ $y_2 = \int_0^t x_2(2\tau)\, d\tau$

$a_1 y_1 + a_2 y_2 = a_1 \int_0^t x_1(2\tau) d\tau + a_2 \int_0^t x_2(2\tau)\, d\tau$

$y^*(t) = \int_0^t [a_1 x_1(2\tau) + a_2 x_2(2\tau)]\, d\tau$

$= a_1 \int_0^t x_1(2\tau)\, d\tau + a_2 \int_0^t x_2(2\tau)\, d\tau$ $\underline{\underline{L}}$

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6] $y(t) = \ln(x(t))$

$y_1 = \ln x_1(t) \qquad y_2 = \ln x_2(t)$

$a_1 y_1 + a_2 y_2 = a_1 \ln x_1(t) + a_2 \ln x_2(t)$

$y^x(t) = \ln[a_1 x_1(t) + a_2 x_2(t)] \qquad \Rightarrow$ non-L

7] $y(t) = x(t)\cos t$

$y_1 = x_1(t)\cos t \qquad y_2 = x_2(t)\cos t$

$a_1 y_1 + a_2 y_2 = a_1 x_1(t)\cos t + a_2 x_2(t)\cos t$

$y^x(t) = \cos t\,[a_1 x_1(t) + a_2 x_2(t)] \qquad \Rightarrow$ L

8] $y(t) = |x(t)|$

$y_1 = |x_1(t)| \qquad y_2 = |x_2(t)|$

$a_1 y_1 + a_2 y_2 = a_1 |x_1(t)| + a_2 |x_2(t)|$

$y^x(t) = |a_1 x_1(t) + a_2 x_2(t)| \qquad \Rightarrow$ non-L

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Ch3 "Continous time linear time invariant system" [L.T.I]

- Linear $a_1x_1(t) + a_2x_2(t) \longrightarrow \boxed{T} \longrightarrow a_1y_1(t) + a_2y_2(t)$

- time inveriant $x(t-t_0) \longrightarrow \boxed{T} \longrightarrow y(t-t_0)$

⇒ input relationship of an L.T.I System is defined by convolution integral [Convolution → $*$]

$$x(t) \longrightarrow \boxed{L.T.I} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

- where $h(t)$ is impulse response of L.T.I System

$T\{\delta(t)\} = h(t)$

$\delta(t) \longrightarrow \boxed{L.T.I} \longrightarrow h(t)$ $\Rightarrow h(t) = h(t) * \delta(t)$

$\delta(t-t_0) \longrightarrow \boxed{L.T.I} \longrightarrow h(t-t_0)$ $\Rightarrow h(t-t_0) = \delta(t-t_0) * h(t)$

$$\int_{-\infty}^{\infty} \delta(t-\tau)\, h(t-t_0)\, d\tau = h(t-t_0)$$

$*$

[⇒ In general $x(t) * \delta(t) = x(t)$]

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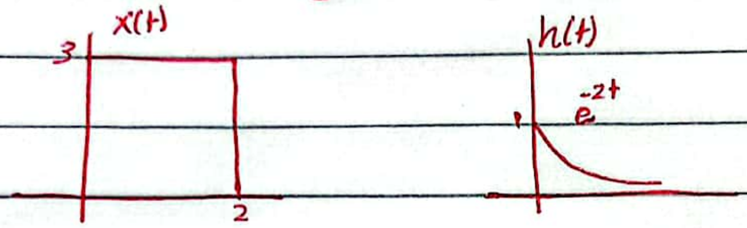
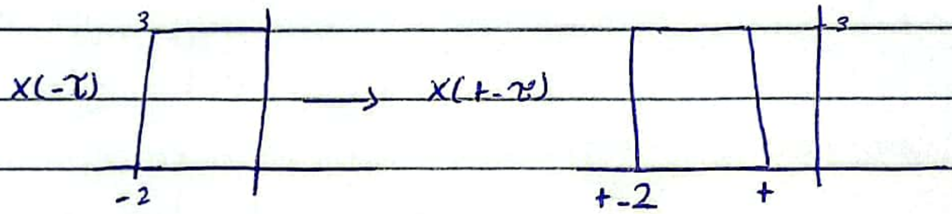
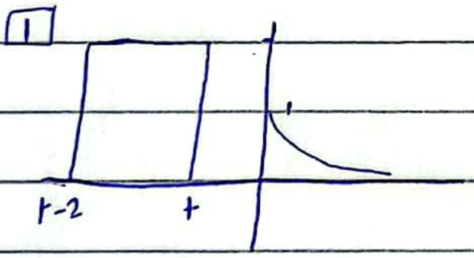
Ex.: an L.T.I system has input $x(t)$ and impulse response $h(t)$



$x(t) = 3\,\text{rect}\left(\frac{t-1}{2}\right)$, $h(t) = e^{-2t}u(t)$ Find and plot system response $y(t)$.

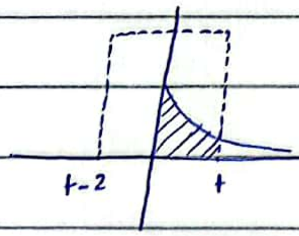


$\boxed{1}$



$t<0 \qquad y(t) = 0$

$\boxed{2}$

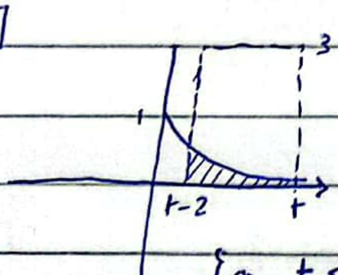


$t>0 \;,\; t-2<0$

$0<t<2 \qquad y(t) = \int_0^t 3 \cdot e^{-2\tau}\, d\tau = \frac{-3}{2} e^{-2\tau}\Big]_0^t = \frac{-3}{2}\left[e^{-2t}-1\right]$

الحدود [t ← 0] منطقة التداخل

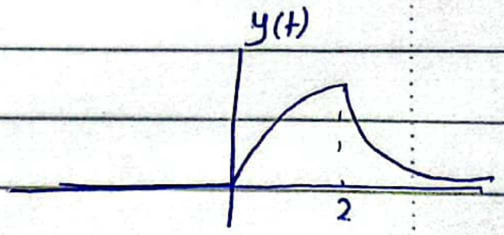
$\boxed{3}$



$t-2>0 \qquad [t>2]$

$y(t) = \int_{t-2}^{t} 3 e^{-2\tau} d\tau = \frac{-3}{2} e^{-2\tau}\Big]_{t-2}^{t} = \frac{-3}{2}\left[e^{-2t} - e^{4-2t}\right]$

$$y(t) = \begin{cases} 0 & t<0 \\ -\frac{3}{2}\left[e^{-2t}-1\right] & 0\le t<2 \\ -\frac{3}{2}\left[e^{-2t} - e^{4-2t}\right] & t>2 \end{cases}$$



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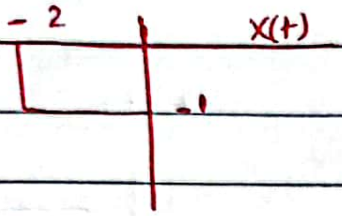
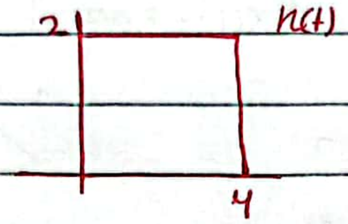
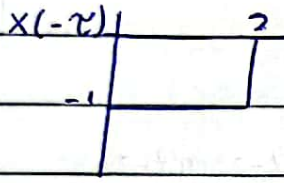
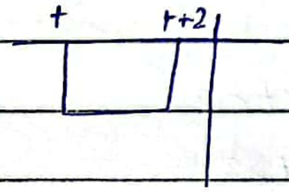
Ex: an L.T.I system $x(t)$, $h(t)$ Find $y(t)$



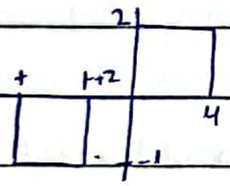




$\rightarrow x(t-\tau)$



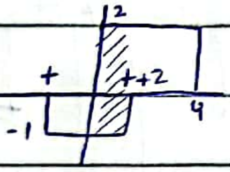
1]



$t+2<0 \Rightarrow \boxed{t<-2}$

$y(t)=0$

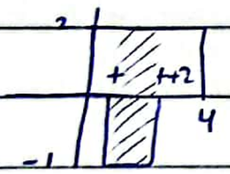
2]



$t+2>0 \rightarrow t>-2$, $t<0$

$\boxed{-2<t<0}$ $y(t)=\int_{0}^{t+2} -1*2\,d\tau = -2(t+2) = -2t-4$

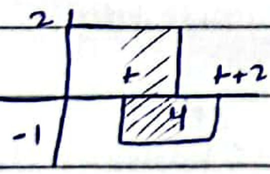
3]



$t+2<4 \rightarrow t<2$, $t>0$

$\boxed{0<t<2}$ $y(t)=\int_{t}^{t+2} -2\,d\tau = -2(t+2-t) = -4$

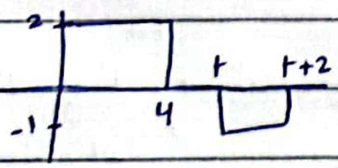
4]



$t+2>4 \rightarrow t>2$, $t<4$

$\boxed{2<t<4}$ $y(t)=\int_{t}^{4} -2\,d\tau = -2(4-t) = 2t-8$

5]



$t>4$ $y(t)=0$

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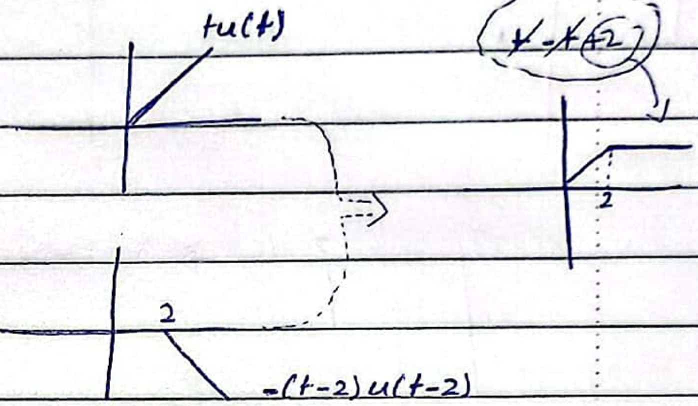
Ex: L.T.I system with $\begin{cases} h(t) = \delta(t) - \delta(t-2) \\ x(t) = t\,u(t) \end{cases}$, **Find** $y(t)$.

$y(t) = x(t) * h(t)$

$= t\,u(t)\,[\delta(t) - \delta(t-2)]$

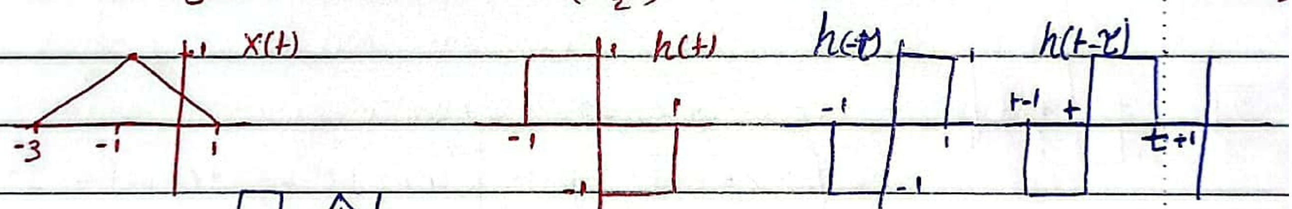
$= t\,u(t) - (t-2)\,u(t-2)$



$y(t) = \begin{cases} t & 0 < t < 2 \\ 2 & t > 2 \end{cases}$

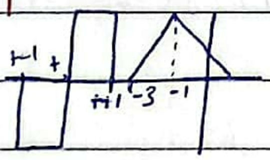
$y(t) = t\,\text{rect}\left(\frac{t-1}{2}\right) + 2\,u(t-2)$

Ex: L.T.I system with $x(t) = \text{tri}\left(\frac{t+1}{2}\right)$, $h(t) = \text{rect}(t+0.5) - \text{rect}(t-0.5)$



1] $t+1 < -3$

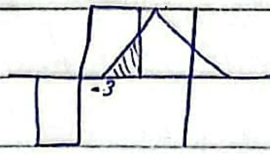
$[t < -4]$



$y(t) = 0$

2] $t+1 > -3 \rightarrow t > -4$

$t < -3$ $[-4 < t < -3]$

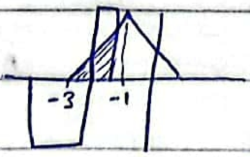


$y(t) = \int_{-3}^{t+1} 1 \times \frac{1}{2}(\tau+3)\,d\tau$

$= \left.\frac{1}{2}\tau^2 + 3\tau\right]_{-3}^{t+1}$

3] $t > -3$, $t+1 < -1 \rightarrow t < -2$

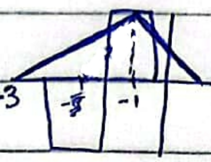
$[-3 < t < -2]$



$y(t) = \int_{-3}^{t} -1 \times \frac{1}{2}(\tau+3)\,d\tau + \int_{t}^{t+1} 1 \times \frac{1}{2}(\tau+3)\,d\tau$

4] $t+1 > -1 \rightarrow t > -2$, $t < -1$

$[-2 < t < -1]$

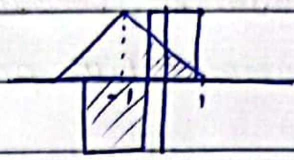


$y(t) = \int_{t-1}^{t} -1 \times \frac{1}{2}(\tau+3)\,d\tau + \int_{t}^{-1} 1 \times \frac{1}{2}(\tau+3)\,d\tau$

$+ \int_{-1}^{t+1} 1 \times -\frac{1}{2}(t-1)\,d\tau$

5] $t+1>0 \rightarrow t>-1$

$t-1<-1 \rightarrow t<0 \rightarrow [-1<t<0]$

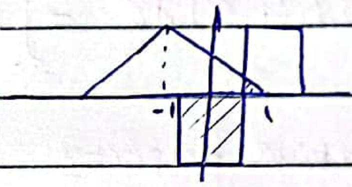


$$y(t) = \int_{t-1}^{-1} -1 \times \frac{1}{2}(\tau+3)\,d\tau + \int_{-1}^{t} (-1)(-\tfrac{1}{2})(\tau-1)\,d\tau$$

$$+\int_{t}^{t+1} 1 \times -\frac{1}{2}(\tau-1)\,d\tau$$

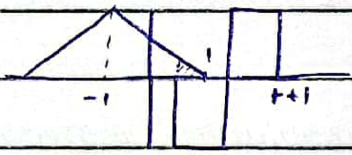
6] $t+1>1 \rightarrow t>0$

$t-1<0 \rightarrow t<1 \rightarrow [0<t<1]$



$$y(t) = \int_{t-1}^{t} (-1)\times(-\tfrac{1}{2})(\tau-1)\,d\tau + \int_{t}^{1} (1) \times -\frac{1}{2}(\tau-1)\,d\tau$$

7] $t>1$, $t-1<1 \rightarrow t<2$



$[1<t<2]$

$$y(t) = \int_{t-1}^{1} (-1)(-\tfrac{1}{2})(\tau-1)\,d\tau$$

⇒ properties of convolution:

- commutative property

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \quad = \quad h(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t)$$

$$\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

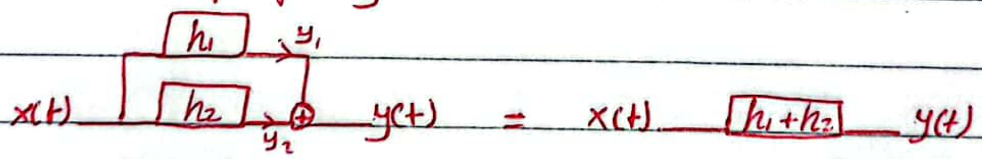
- associative property

Convolution of two or more signals input of order.

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

* in general the equivelant impulse response of n-systems in series is given by $h(t)_{eq} = * \, h(t)_i \qquad 1 \le i \le n$

- distributive property



$y_1 = x(t) * h_1$ $\qquad\qquad$ $y(t) = x(t) * [h_1 + h_2]$

$y_2 = x(t) * h_2$

$y(t) = y_1 + y_2$

* in general n-systems in parallel, ther equivalent implse respons $h(t)_{eq} = \sum_{1}^{n} h_i(t)$

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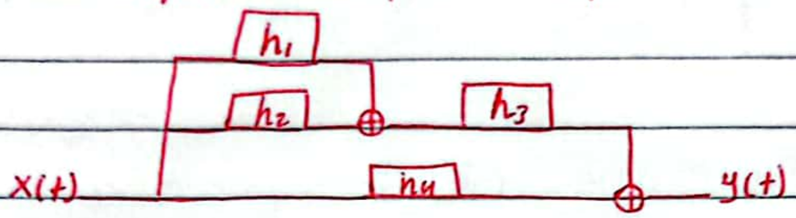
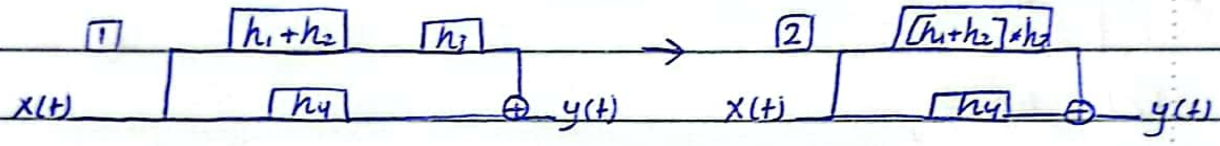
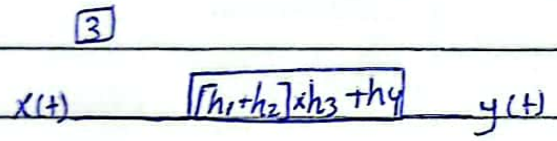
Ex: Find the equivalent impulse response of the system



sol:



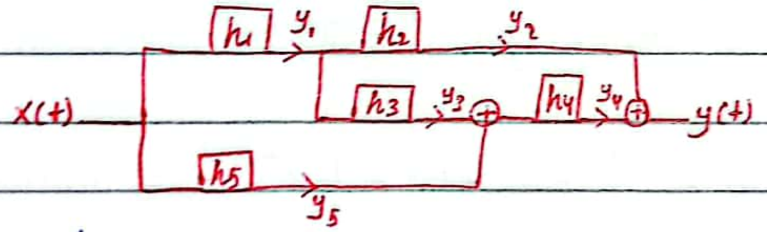


So $h(t)_{eq} = ([h_1+h_2]*h_3+h_4)$

Ex: Find the equ impulse response of the systems if $h_1 = h_4 = u(t)$

$h_2 = h_3 = 2\delta(t-1)$

$h_5 = e^{-t}u(t)$



$y_1 = x(t) * h_1$

$y_2 = y_1 * h_2$

$y_3 = y_1 * h_3$

$y_4 = h_4 * [y_3 + y_5]$

$y_5 = x(t) * h_5$

$\Rightarrow y(t) = y_2 + y_4$

$= [y_1 * h_2] + [h_4 * (y_3 + y_5)]$

$= [x(t) * h_1 * h_2] + [h_4 * [(y_1 * h_3) + (x(t) * h_5)]]$

$= [x(t) * h_1 * h_2] + [h_4 * [x(t) * h_1 * h_3 + x(t) * h_5]]$

$= x(t) [[h_1 * h_2] + [h_4 * h_1 * h_3] + [h_4 * h_5]]$ تعويض القيم الي بالسؤال

$= x(t) [\underbrace{2u(t-1)}_{h_A} + \underbrace{2u(t-1) * u(t)}_{h_B} + \underbrace{u(t) * e^{-t}u(t)}_{h_C}]$

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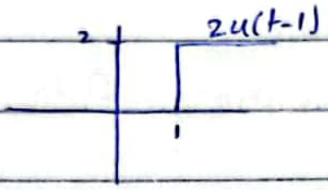
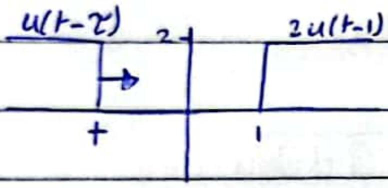
* $h_A = 2u(t-1)$



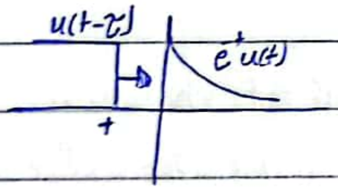
* $h_B = 2u(t-1) * u(t)$



$t<1, \; h_B = 0$

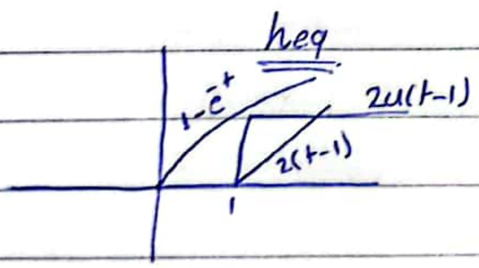
$t>1, \; h_B = \int_1^t 2 \, d\tau = 2(t-1)$

* $h_C = u(t) * e^{-t}u(t)$



$t<0, \; h_C = 0$

$t>0, \; h_C = \int_0^t e^{-\tau} d\tau = -e^{-\tau}\Big]_0^t = -e^{-t}+1 = 1-e^{-t}$



$y(t) = x(t) * h_{eq}$

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⇒ properties of L.T.I system:

1] memory less system:

L.T.I is memoryless if $h(t) = k\delta(t)$ otherwise memory

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$y(t_0) = \int_{-\infty}^{\infty} x(\tau)\, h(t_0-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, k\delta(t_0-\tau)\, d\tau = \underline{k\, x(t_0)}$$ ※ memoryless

2] invertibility:

all L.T.I system are invertible

$x(t)$ — $\boxed{T}$ — $y(t)$ — $\boxed{T_I}$ — $x(t)$

$x(t)$ — $\boxed{h(t)}$ — $y(t)$ — $\boxed{h(t)_I}$ — $x(t)$ = $x(t)$ — $\boxed{h_{eq}}$ — $x(t)$

$h_{eq} = h(t) * h(t)_I = \delta(t)$

3] Causality:

L.T.I System is Causal if $h(t)$ is causal $h(t) = 0 \;,\; \forall\, t<0$

4] stability:

L.T.I system is BIBO stable if $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

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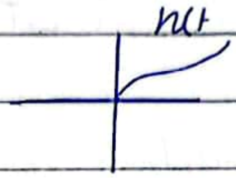
* response of causal L.T.I system



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

* unite step response of L.T.I

response of L.T.I system when $x(t) = u(t)$

$u(t) \longrightarrow \boxed{h(t)} \longrightarrow S(t)$ $\quad \therefore S(t) = y(t)$

$$S(t) = u(t) * h(t)$$

$$= \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

$$\Rightarrow S(t) = \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

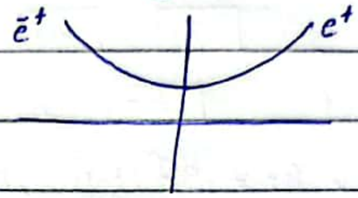
for causal L.T.I $\quad S(t) = \int_{-\infty}^{t} u(\tau)\, h(t-\tau)\, d\tau = \int_{0}^{t} h(\tau)\, d\tau$

N O T E B O O K

Ex: check the properties of L.T.I system:

1) $h(t) = e^{|t|}$

$$e^{|t|} = \begin{cases} e^{t} & t>0 \\ e^{-t} & t<0 \end{cases}$$

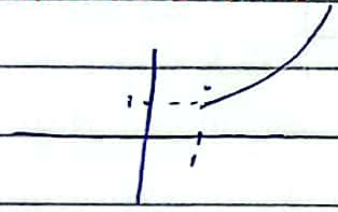


$\Rightarrow$ non-causal

stable or non $\Rightarrow \int_{-\infty}^{\infty} |h(t)| < \infty$ $\left[\int_{-\infty}^{0} e^{-t}\,dt + \int_{0}^{\infty} e^{t}\,dt = \infty \right]$ non stable

memory

2) $h(t) = e^{t}\, u(t-1)$

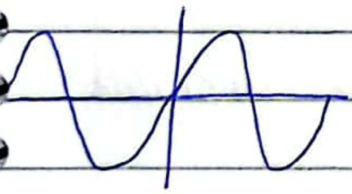


- causal

$\Rightarrow \int_{1}^{\infty} e^{t}\,dt = \infty$ un stable

memory

3) $h(t) = \sin 2t$

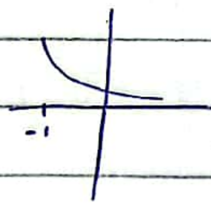


non causal

stable

memory

4) $h(t) = e^{-t}\, u(t+1)$

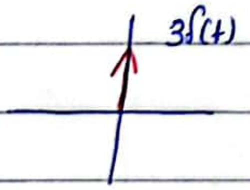


non causal

$\int_{-1}^{\infty} e^{-t}\,dt = -e^{-t}\Big]_{-1}^{\infty} = e^{1}$ stable

memory

5) $h(t) = \delta(t/3)$

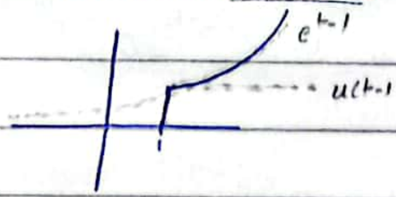


Causal

Stable

memoryless

6) $h(t) = e^{t-1}\, u(t-1)$



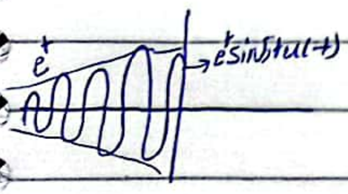
causal

$\Rightarrow \int_{1}^{\infty} e^{t-1}\,dt = e^{t-1}\Big]_{1}^{\infty} = \infty$

un stable

memory

7) $h(t) = e^{t} \sin 5t\, u(-t)$

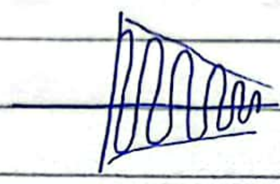


non causal

$\int_{-\infty}^{0} e^{t} \sin 5t\, dt \rightarrow$ stable

memory

8) $h(t) = e^{-t} \sin 5t\, u(t)$

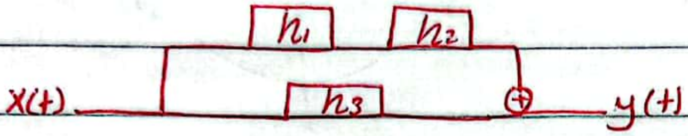


Causal

Stable

memory

Ex: Check system stability and causality, then Find $s(t)$

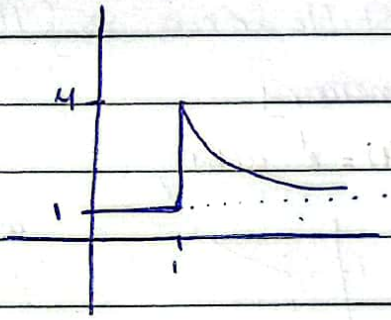


$h_1 = e^{-t}u(t)$, $h_2 = 3\delta(t-1)$, $h_3 = u(t)$

$h(t)_{eq} = h_1 * h_2 + h_3$

$= e^{-t}u(t) * 3\delta(t-1) + u(t)$

$= 3e^{-(t-1)}u(t-1) + u(t)$



$= \begin{cases} 1 & 0<t<1 \\ 1+3e^{-(t-1)} & t>1 \end{cases}$

$\Rightarrow$ causal

$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_0^1 1 \cdot dt + \int_1^{\infty} 1+3e^{-(t-1)}\, dt$

$= t\Big]_0^1 + t - 3e^{-(t-1)}\Big]_1^{\infty} = 1 + \infty - 0 - (1-3) = \infty$ unstable

$0<t<1 \quad s(t) = \int_0^t d\tau = t$

$t>1 \quad s(t) = \int_0^1 d\tau + \int_1^t 1+3e^{-(\tau-1)}\, d\tau$

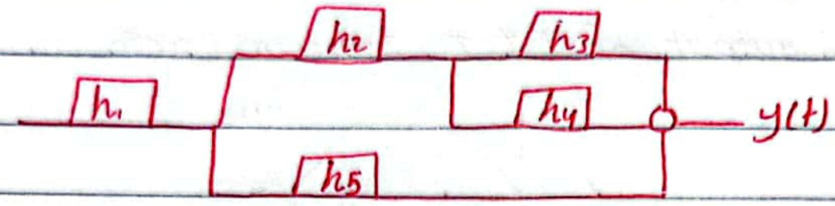
$= 1 + \Big[\tau - 3e^{-\tau+1}\Big]_1^t$

$\cancel{1} + (t - 3e^{-t+1}) - (\cancel{1} - 3)$

$\boxed{3 - 3e^{-t+1} + t}$

CamScanner

Ex:-



$h_3 = h_4 = u(t)$

$h_1 = h_2 = h_5 = 5\delta(t)$

1] Find heq

$$h_{eq} = [h_1 * h_2 * h_3] + [h_1 * h_2 * h_4] + [h_1 * h_5]$$

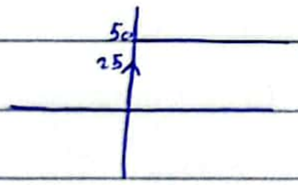
$$[5\delta(t) * 5\delta(t) * u(t)] + [5\delta(t) * 5\delta(t) * u(t)] + [5\delta(t) * 5\delta(t)]$$

$$[25\delta(t)\, u(t)] + [25\delta(t)\, u(t)] + [25\delta(t)]$$

$$[50\delta(t)\, u(t)] + [25\delta(t)]$$

$$h_{eq} = 50\, u(t) + 25\,\delta(t)$$

2] check stability, causality



Causal

$$\int_{-\infty}^{\infty} [50u(t) + 25\delta(t)]\, dt = 50t\Big]_{-\infty}^{\infty} + 25 = \infty \quad \text{unstable}$$

3] Find S(t)

$$S(t) = \int_0^t h_{eq}\, d\tau$$

$- \; t<0, \; S(t) = 0$

$- \; t>0, \; \int_0^t [50u(t) + 25\delta(\tau)]\, d\tau$

$$S(t) = 50t + 25u(t)$$

※ note

$$\int_c^a \delta(t-a)\, dt = 1$$

⇐ تكامل الـ $\delta(t)$ على ثابت يعطي واحد

$$\int_0^t \delta(t-a)\, dt = u(t)$$

⇐ تكاملها على اقتران يعطي $u(t)$

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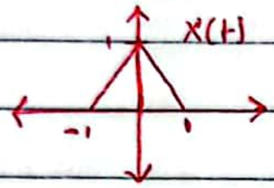
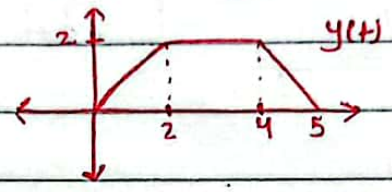
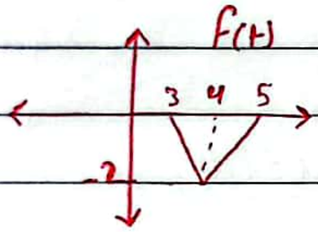
Ex: the input and output of L.T.I system are





what will be the output of new input is



$x(t) \; \boxed{L.T.I} \; y(t) = x(t) * h(t)$

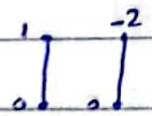
$f(t) \; \boxed{L.T.I} \; z(t) = f(t) * h(t)$

$a\,x(t) \rightarrow a\,y(t)$

$x(t-t_0) \rightarrow y(t-t_0)$

$f(t) = A\,x(at+b) + B$

• Amplitude $f(t) = A\,x(t) + B$

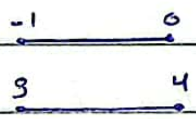


$0 = 0 + B \rightarrow B = 0$

$-2 = A + 0 \rightarrow A = -2$

$\Rightarrow f(t) = -2\,x(t)$

• time $f(t) = x(at+b)$



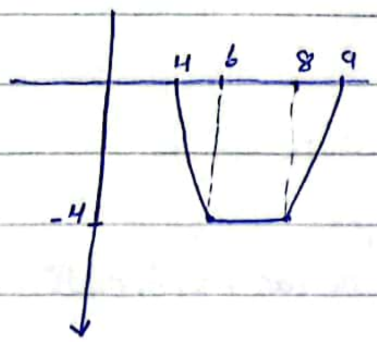
$-1 = 3a + b$

$0 = 4a + b$

$a = 1 \;, b = -4$

$f(t) = -2\,x(t-4)$

$z(t) = -2\,x(t-4)$



Ch4 : Fourier Series

- Fourier proved the set of function $\phi(t) = \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$

note $\Rightarrow \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{jn\omega_0 t} \cdot e^{-jm\omega_0 t}\, dt = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$

- any arbitrary periodic signal $x(t)$ at T_0 can be written as $x(t) = \sum_{-\infty}^{\infty} C_k e^{jk\omega_0 t}$ [complex F.S exp]

$\Rightarrow$ where C_k is Fourier series coefficients

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$ [and C_k in general is complex]

$$C_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt$$ [C_0 it's the average value of $x(t)$]

C_1 = coef of fund freq , C_k : $k \geq 2$ Harmonic coef

$$\Rightarrow P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt = |C_0|^2 + 2\sum_{k=1}^{\infty}|C_k|^2 = \sum_{-\infty}^{\infty}|C_k|^2$$

- proof :-

$x(t) = \sum_{-\infty}^{\infty} C_k e^{jk\omega t}$

$x^*(t) = \sum_{-\infty}^{\infty} C_k^* e^{-jk\omega t}$

$\Rightarrow x(t) \cdot x^{\boxed{*}}(t) = |x(t)|^2$ → conjugate : مرافق

$$P_x = \frac{1}{T_0}\int_{T_0} x(t) \cdot x^*(t)\, dt$$

$$P_x = \frac{1}{T_0}\int_{T_0} x(t) \sum_{-\infty}^{\infty} C_k^* e^{-jk\omega t}\, dt$$

$$P_x = \sum_{-\infty}^{\infty} C_k^* \cdot \underbrace{\frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega t}\, dt}_{C_k}$$

$$P_x = \sum_{-\infty}^{\infty} C_k^* \cdot C_k$$

$$P_x = \sum_{-\infty}^{\infty} |C_k|^2$$

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$* \; C_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$

→ we take [-k] $\quad C_{-k} = \frac{1}{T_0} \int_{T_0} x(t)\, e^{jk\omega_0 t}\, dt$

if we take the conjugate $\quad C_k^* = \frac{1}{T_0} \int_{T_0} x(t)\, e^{jk\omega_0 t}\, dt$

then $\quad C_{-k} = C_k^*$

⇒ Amplitude spectrum is even function $\quad C_{-k} = C_k^*$

⇒ phase spectrum is odd function $\quad \theta_{-k} = -\theta_k$

- $C_k = |C_k| \angle \theta_k$ or $|C_k|\, e^{j\theta_k}$ $\qquad -\pi < \theta \le \pi$

⇒ compact trigonometric Series :-

$$x(t) = \sum_{-\infty}^{\infty} C_k\, e^{jk\omega_0 t} = C_0 + \sum_{-\infty}^{-1} C_{-k}\, e^{-jk\omega_0 t} + \sum_{1}^{\infty} C_k\, e^{jk\omega_0 t}$$

$$= C_0 + \sum_{-\infty}^{-1} |C_{-k}|\, e^{-j\theta_k} \cdot e^{-jk\omega_0 t} + \sum_{1}^{\infty} |C_k|\, e^{j\theta_k} \cdot e^{jk\omega_0 t}$$

$$= C_0 + \sum_{1}^{\infty} C_k \; e^{-j[\theta_k + k\omega_0 t]} + e^{j[\theta_k + k\omega_0 t]}$$

$$x(t) = C_0 + \sum_{1}^{\infty} 2C_k \cos[k\omega_0 t + \theta_k]$$

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⇒ long trigonometric series :

$$x(t) = C_o + \sum_{1}^{\infty} 2|C_k| \cos(k\omega_o t + \theta_k)$$

$$= C_o + \sum_{1}^{\infty} 2|C_k| [\cos k\omega_o t . \cos\theta_k - \sin k\omega_o t . \sin\theta_k]$$

$$= C_o + \sum_{1}^{\infty} 2|C_k| \cos(k\omega_o t)\cos(\theta_k) - \sum_{1}^{\infty} 2|C_k| \sin(k\omega_o t) \sin(\theta_k)$$

$$= C_o + \sum_{1}^{\infty} A_k \cos k\omega_o t + \sum_{1}^{\infty} B_k \sin k\omega_o t$$

$$= C_o + \sum_{1}^{\infty} [A_k \cos k\omega_o t + B_k \sin k\omega_o t]$$

⇒ where A_k, B_k ~~are real~~

$$A_k = 2|C_k| \cos\theta_k \quad , \quad B_k = -2|C_k| \sin\theta_k$$

$$|C_k| = \frac{\sqrt{A_k^2 + B_k^2}}{2} \qquad \theta_k = \tan^{-1}\left(\frac{-B_k}{A_k}\right)$$

$$A_k = \frac{1}{T_o} \int_{T_o} x(t) \cos \omega k t \, dt$$

$$B_k = \frac{1}{T_o} \int_{T_o} x(t) \sin k\omega_o t \, dt$$

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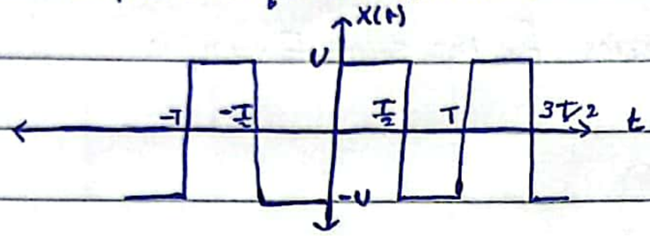
Example ⇒ Square wave :



$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\left[\int_0^{T/2} U e^{-jk\omega_0 t}\, dt + \int_{T/2}^{T} -U e^{-j\omega_0 k t}\, dt\right]$$

$$= \frac{U}{T_0}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\Big|_0^{T/2} - \frac{e^{-j\omega_0 k t}}{-jk\omega_0}\Big|_{T/2}^{T}\right]$$

$$\boxed{\omega_0 T_0 = 2\pi}$$

$$= \frac{U}{-jk\omega_0 T_0}\left[e^{-jk\frac{\omega_0 T_0}{2}} - 1 - e^{-j\omega_0 k T_0} + e^{-jk\frac{\omega_0 T_0}{2}}\right]$$

$$= \frac{jU}{k\omega_0 T_0}\left[e^{-jk\pi} - 1 - e^{-jk\frac{\pi}{2}2} + e^{-jk\pi}\right]$$

$$= \frac{jU}{k2\pi}\left[2e^{-jk\pi} - e^{-jk2\pi} - 1\right]$$

$$e^{-jk\pi} = [\cos(k\pi) - j\cancel{\sin(k\pi)}]\ k \begin{cases}\text{even} & = 1\\ \text{odd} & = -1\end{cases}$$

$$e^{-jk2\pi} = \cos(2k\pi) - j\cancel{\sin(2k\pi)} = 1$$

$$C_k = \frac{jU}{k2\pi}[2 - 1 - 1] = \text{zero} \quad \text{if } k \text{ even}$$

$$C_k = \frac{jU}{k2\pi}[-2 - 1 - 1] = \frac{jU}{k2\pi} * -4 = \frac{-2jU}{k\pi} \quad \text{if } k \text{ odd}$$

$$\boxed{C_k = |c_k| \angle \theta_k}$$

$$C_k = \frac{2U}{k\pi}\angle -90 = \frac{2U}{k\pi} e^{-j\frac{\pi}{2}}$$

$$x(t) = \sum_{-\infty}^{\infty} \frac{2U}{k\pi} e^{-j\frac{\pi}{2}} e^{jk\omega_0 t} \Rightarrow \underline{x(t)} = \sum_{1}^{\infty} \frac{4U}{k\pi} e^{j(k\omega_0 t - \frac{\pi}{2})}$$

$C_0 = 0$

the exponential form of Fourier Series

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the combined tri form of fourier series for the same Example

$$X(t) = \sum_{1}^{\infty} \frac{4V}{k\pi}\left(\cos k\omega_0 t - \frac{\pi}{2}\right)$$

the tri form of fourier Series

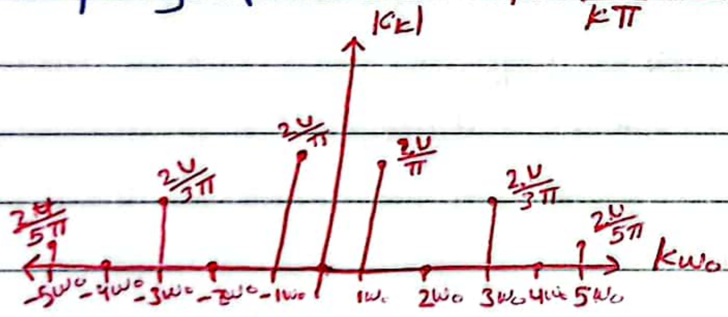
⇐ طريقتين يا اما بفك [Comotri] ب

او بإيجاد [Ak , Bk]

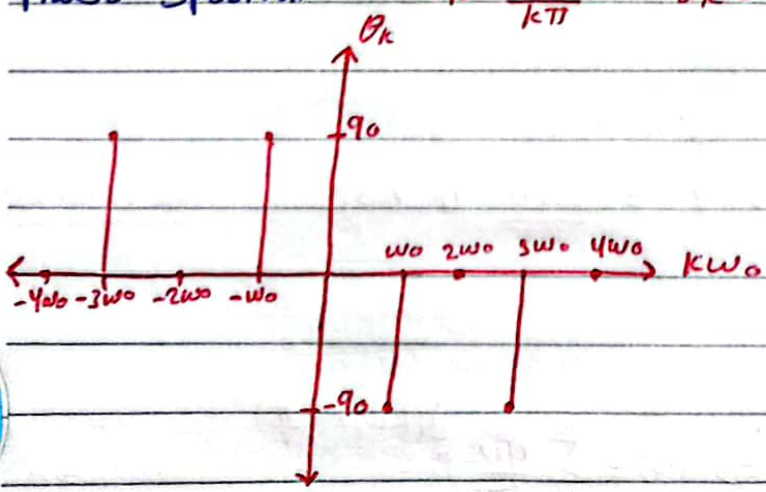
$$x(t) = \sum_{1}^{\infty} \frac{4V}{k\pi}\left[\cancel{\cos k\omega_0 t \cos \frac{\pi}{2}} + \sin k\omega_0 t \cdot \sin\frac{\pi}{2}\right]$$

$$X(t) = \sum_{1}^{\infty} \frac{4V}{k\pi} \sin k\omega_0 t$$

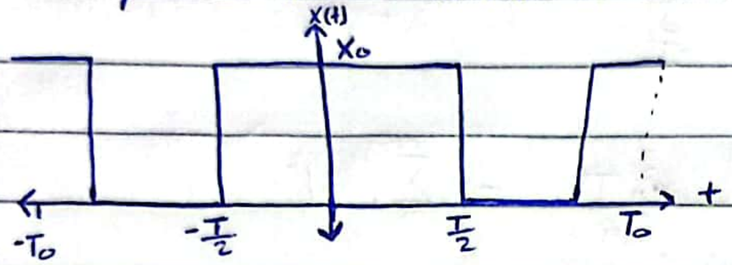
frequency spectrum $C_k = \frac{2V}{k\pi} \angle -90$ $|C_k| = \frac{2V}{k\pi}$ k is odd



phase spectrum $C_k = \frac{2V}{k\pi} \angle -90$ $\theta_k = -90$ k is odd



Example $\Rightarrow$ wher T is the width of the rectangular pulses



$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$* \Rightarrow \omega_0 T_0 = 2\pi$

$* \Rightarrow \frac{\omega_0}{2} = \frac{\pi}{T_0}$

$$C_k = \frac{1}{T_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} X_0\, e^{-jk\omega_0 t}\, dt$$

$$C_k = \frac{X_0}{T_0}\,\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\Bigg|_{-T/2}^{T/2} = \frac{X_0}{-jk\omega_0 T_0}\left[e^{-jk\omega_0 \frac{T}{2}} - e^{+jk\omega_0 \frac{T}{2}}\right]$$

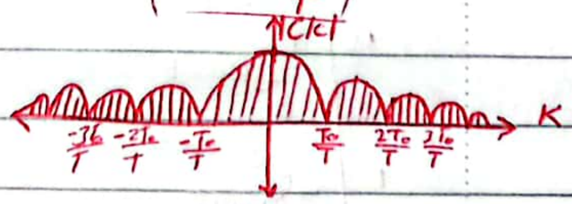
$$= \frac{X_0}{-jk2\pi}\left[e^{-jk\pi \frac{T}{T_0}} - e^{jk\pi\frac{T}{T_0}}\right] = \frac{X_0}{k\pi}\left[\frac{e^{jk\pi\frac{T}{T_0}} - e^{-jk\pi\frac{T}{T_0}}}{2j}\right]$$

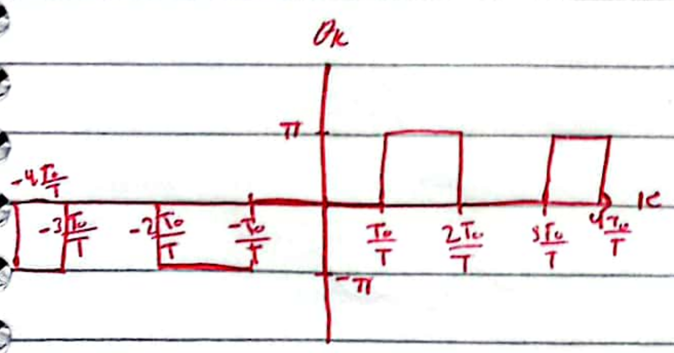
$$= \frac{X_0}{k\pi}\,\frac{\sin\left(\frac{k\pi T}{T_0}\right)}{\frac{T}{T_0}} * \frac{T}{T_0} \Rightarrow \frac{X_0 T}{T_0}\,\frac{\sin\left(k\pi\frac{T}{T_0}\right)}{k\pi\frac{T}{T_0}} \rightarrow \frac{\sin \pi t}{\pi t} = \text{sinc}\, t$$

$$\boxed{C_k = \frac{X_0 T}{T_0}\,\text{sinc}\left(\frac{kT}{T_0}\right)}$$

$k\pi\frac{T}{T_0} = n\pi$

$\boxed{k = \frac{nT_0}{T}}$





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$$P_x = \frac{1}{T_0}\int_{T_0} x^2(t)\,dt = \frac{1}{T_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_0^2\,dt$$

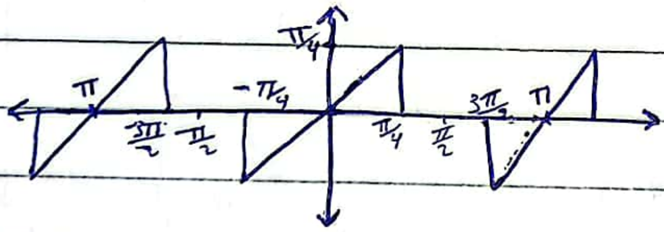
$$= \frac{1}{T_0}\cdot x_0^2\left[\frac{T}{2}+\frac{T}{2}\right] = \frac{x_0^2\,T}{T_0} = \sum_{-\infty}^{\infty}|c_k|^2$$

BW = الدورة × F

$= \frac{T_0}{T} \times F$

↳ width

Example ⇒ Find complex F.s exp ??



$$x(t) = \sum_{-\infty}^{\infty} c_k\, e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} t\, e^{-jk\omega_0 t}\,dt$$

u	dv
⊕ t	$e^{-jk\omega_0 t}$
⊖ 1	$\frac{e^{-jk\omega_0 t}}{-jk\omega_0}$
0	$\frac{e^{-jk\omega_0 t}}{-\omega_0^2 k^2}$

$T_0 = \pi$, $\omega_0 = \frac{2\pi}{\pi} = 2$

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$$C_k = \frac{1}{\pi}\left[\frac{-t\,e^{-jk\omega_0 t}}{j\omega_0 k} + \frac{e^{-jk\omega_0 t}}{\omega_0^2 k^2}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$C_k = \frac{1}{\pi}\left[\frac{-\pi e^{-jk\frac{2\pi}{4}}}{4j2k} + \frac{e^{-jk\frac{2\pi}{4}}}{4k^2}\right] - \frac{1}{\pi}\left[\frac{\pi e^{jk\frac{2\pi}{4}}}{4j2k} + \frac{e^{jk\frac{2\pi}{4}}}{4k^2}\right]$$

$$C_k = \frac{-e^{-jk\frac{\pi}{2}}}{4j2k} + \frac{e^{-jk\frac{\pi}{2}}}{\pi 4k^2} - \frac{e^{jk\frac{\pi}{2}}}{4j2k} - \frac{e^{jk\frac{\pi}{2}}}{\pi 4k^2}$$

$$\frac{-1}{4jk}\left[\frac{e^{jk\frac{\pi}{2}} + e^{-jk\frac{\pi}{2}}}{2}\right] + \frac{-j}{2\pi k^2}\left[\frac{e^{jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}}}{2j}\right]$$

$$C_k = \frac{-1}{4jk}\cos\left(\frac{k\pi}{2}\right) - \frac{j}{2\pi k^2}\sin\left(k\frac{\pi}{2}\right)$$

- $\cos\frac{k\pi}{2}$ when k is odd $\cos\frac{\pi}{2} = 0$, $\cos\frac{3\pi}{2} = 0 \ldots$
  when k is even $\underset{k=2}{\cos\pi = -1}$, $\underset{k=4}{\cos 2\pi = 1}$, $\underset{k=6}{\cos 3\pi = -1} \ldots$
  $(-1)^{\frac{k}{2}} \Rightarrow k$ is even.

- $\sin k\frac{\pi}{2}$ when k is even $\sin\pi = 0$, $\sin 2\pi = 0 \ldots$
  when k is odd $\sin\frac{3\pi}{2} = -1$, $\sin\frac{\pi}{2} = 1$, $\sin \ldots$
  $(-1)^{\frac{k-1}{2}}$ when k is odd

$$C_k = \begin{cases} \frac{-1}{4jk}\cdot(-1)^{\frac{k}{2}} & \text{, when } k \text{ is even} \\ \frac{-j}{2\pi k^2}\cdot(-1)^{\frac{k-1}{2}} & \text{, when } k \text{ is odd} \end{cases}$$

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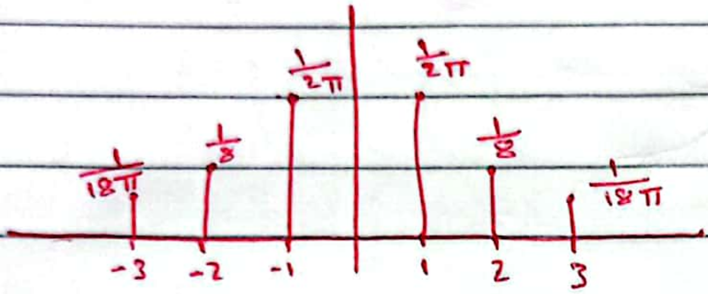
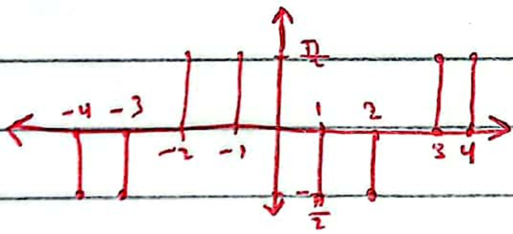
Frequency spectrum



phase

even $\Rightarrow \dfrac{(-1)^{\frac{k}{2}}}{4k} \cdot j = \dfrac{(-1)^{\frac{k}{2}}}{4k} e^{\frac{\pi}{2} j}$

odd $\Rightarrow \dfrac{(-1)^{\frac{k-1}{2}}}{2\pi k^2} \cdot -j = \dfrac{(-1)^{\frac{k-1}{2}}}{2\pi k^2} e^{-\frac{\pi}{2} j}$



$$P_x = \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} t^2\,dt = \frac{2}{\pi}\int_{0}^{\frac{\pi}{4}} t^2\,dt = \frac{2}{\pi}\left.\frac{t^3}{3}\right]_0^{\frac{\pi}{4}} = \frac{2}{\cancel{\pi}}\,\frac{\pi^{\cancel{3}\,2}}{3\times 4^3}$$

$$P_x = \frac{\pi^2}{96}$$

50 % of Power

$$\frac{P_x}{2} = \frac{\pi^2}{192} = 0{,}051$$

$P_0 = 0$, $P_1 = \left(\frac{\pi}{2}\right)^2 >> 0{,}051$ ✓

3db-BW $= 1 \times F = 1 * \frac{1}{\pi} = \frac{1}{\pi}$ HZ

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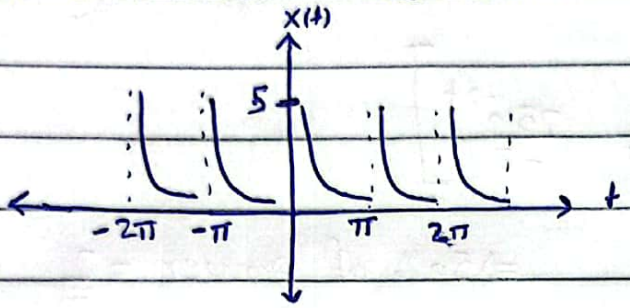
Ex: $x(t) = 5e^{-t} \quad 0 \leq t \leq \pi$



$$\boxed{T_0 = \pi}$$

$$\omega_0 T_0 = 2\pi$$

$$\boxed{\omega_0 = 2}$$

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k = \frac{1}{\pi}\int_0^{\pi} 5e^{-t}\, e^{-jk\omega_0 t}\, dt$$

$$C_k = \frac{5}{\pi}\int_0^{\pi} e^{-t(1+jk2)}\, dt$$

$$C_k = \frac{5}{\pi}\left[\frac{e^{-t(1+jk2)}}{-(1+jk2)}\right]_0^{\pi}$$

$$C_k = \frac{5}{\pi(1+jk2)} \cdot -e^{-t(1+jk2)}\Big|_0^{\pi}$$

$$C_k = \frac{5}{\pi(1+jk2)} - \left[e^{-\pi(1+jk2)} - 1\right]$$

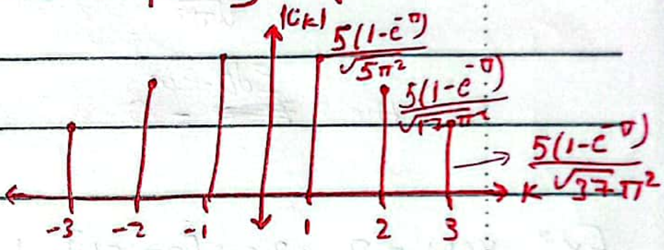
$$C_k = \frac{5}{\pi(1+jk2)}\left[1 - e^{-\pi} \cdot \overset{1}{\cancel{e^{-jk2\pi}}}\right]$$

$$* \; e^{-jk2\pi} = \overset{1}{\cancel{\cos 2k\pi}} - j\,\overset{0}{\cancel{\sin 2k\pi}}$$

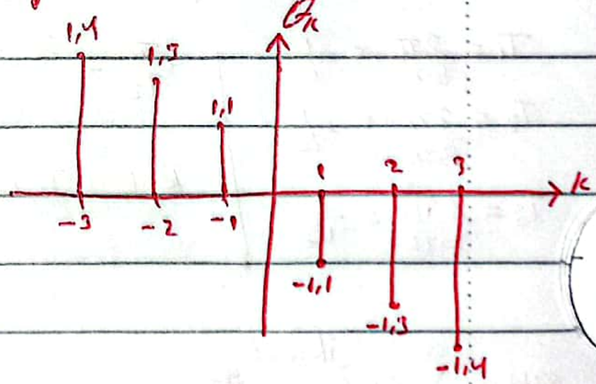
$$\Rightarrow C_k = \frac{5(1-e^{-\pi})}{\pi + jk2\pi}$$

$$|C_k| = \frac{5(1-e^{-\pi})}{\sqrt{\pi^2 + 4k^2\pi^2}}, \quad \theta_k = -\tan^{-1}\frac{2k\pi}{\pi} = -\tan^{-1} 2k$$

⇒ frequency spectrum



⇒ phase



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$\Rightarrow P_x = \frac{1}{T_0} \int_{T_0} (x(t))^2 \, dt$

$P_x = \frac{1}{\pi} \int_0^{\pi} 25 e^{-2t} \, dt = \frac{1}{\pi} \left. \frac{25 e^{-2t}}{-2} \right]_0^{\pi}$

$P_x = \frac{-12.5}{\pi} [e^{-2\pi} - 1] \simeq 4 \Rightarrow 50\%$ of power $= \underline{\underline{2}}$

$\rightarrow P_0 = |C_0|^2 = \left(\frac{5[1 - e^{-\pi}]}{\sqrt{\pi^2}} \right)^2 = \underline{\underline{2.31}} > 50\%$ of power

So the 3dB-Bw $= 0$.

Ex: $x(t) = 2 + 3\cos[(2\pi)(5t)] - \frac{1}{2}\sin[2\pi(10t) + \frac{\pi}{4}] + \frac{1}{j}\cos[2\pi(15t) + \frac{\pi}{2}]$

$\Rightarrow$ plote the amplitude and phase spectrum of $x(t)$

$\Rightarrow$ Bw of $x(t)$, dc power, Ac power

$T_1 = \frac{2\pi}{10\pi} = \frac{1}{5}$ | $\frac{T_1}{T_2} = 2$ $\quad \frac{T_1}{T_3} = 3$

$T_2 = \frac{2\pi}{20\pi} = \frac{1}{10}$

$T_3 = \frac{2\pi}{30\pi} = \frac{1}{15}$ | $k=1 \quad T_0 = kT_1 = \frac{1}{5}$

$x(t) = \sum_{-\infty}^{\infty} C_k e^{jkw_0t}$

$2 + \frac{3}{2} e^{j10\pi t} + \frac{3}{2} e^{-j10\pi t} - \frac{1}{4j} e^{j(20\pi t + \frac{\pi}{4})} + \frac{1}{4j} e^{-j(20\pi t + \frac{\pi}{4})}$

$+ \frac{1}{2j} e^{j(30\pi t + \frac{\pi}{2})} + \frac{1}{2j} e^{-j(30\pi t + \frac{\pi}{2})}$

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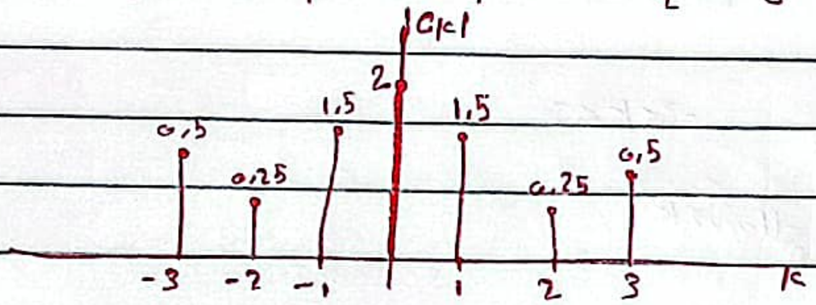
$$= 2 + \frac{3}{2} e^{j10\pi t} + \frac{3}{2} e^{-j10\pi t} - \frac{1}{4j} e^{j20\pi t} \cdot e^{\frac{\pi}{4} j} + \frac{1}{4j} e^{-j20\pi t} \cdot e^{-j\frac{\pi}{4}}$$

$$+ \frac{1}{2j} e^{j30\pi t} \cdot e^{j\frac{\pi}{2}} + \frac{1}{2j} e^{-j30\pi t} \cdot e^{-j\frac{\pi}{2}}$$

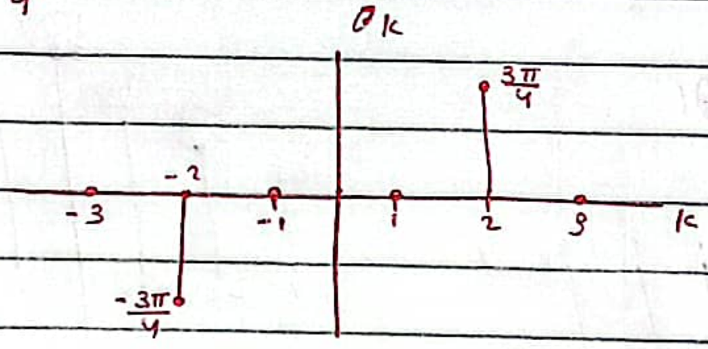
$$\boxed{j = e^{j\frac{\pi}{2}}}$$

$$= 2 + \frac{3}{2} e^{j10\pi t} + \frac{3}{2} e^{-j10\pi t} + \frac{1}{4} e^{\frac{3\pi}{4} j} e^{j20\pi t} + \frac{1}{4} e^{-j\frac{3\pi}{4}} e^{-j20\pi t} + \frac{1}{2} e^{j30\pi t} + \frac{1}{2} e^{-j30\pi t}$$

$C_0 = 2$ / $C_1 = \frac{3}{2}$, $C_{-1} = \frac{3}{2}$ / $C_2 = \frac{1}{4}$, $C_{-2} = \frac{1}{4}$ / $C_3 = \frac{1}{2}$, $C_3 = \frac{1}{2}$



$\theta_1 = 0$ / $\theta_2 = \frac{3\pi}{4}$ / $\theta_3 = 0$



$BW = 3 \cdot f = 3 \times \frac{1}{T_0} = 3 \times 5 = 15$

$D_{cp} = C_0^2 = 2^2 = 4 \Rightarrow 3.db = 0$

$A_{cp} = 2[C_1^2 + C_2^2 + C_3^2] = 2[(1,5)^2 + (0,25)^2 + (0,5)^2] = 5,125$

Ex: $x(t) = \sum_{-3}^{3} \frac{2}{1+j2k} e^{j200\pi k t}$

⇒ Find funda. freq.

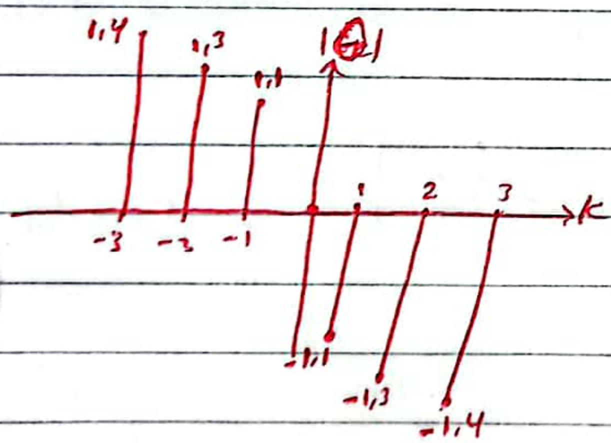
$f_o = \frac{\omega_o}{2\pi} = \frac{200\pi}{2\pi} = 100$ HZ

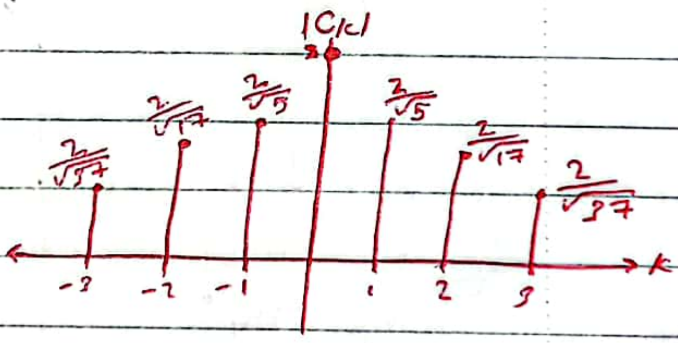
⇒ plot ampl. and phas. spectrum

$C_k = \frac{2}{1+j2k}$ $\quad -3 \le k \le 3$

$C_k = \frac{2}{\sqrt{1+4k^2}} e^{-j\tan^{-1} 2k}$

$|C_k| = \frac{2}{\sqrt{1+4k^2}}$, $\theta_k = -\tan^{-1} 2k$





⇒ Find P_{DC}, P_{AC}, 3db, Bw

$P_{DC} = |C_0|^2 = 4$ watt

$P_{AC} = 2\left[|C_1|^2 + |C_2|^2 + |C_3|^2\right] = 2{,}05$ watt

$P_x = P_{DC} + P_{AC} = 6{,}05$ watt

3db = 0 * F = 0

Bw = 3 * F = 300

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$x(t) = \cos(2t + \frac{\pi}{4}) \cdot \sin 3t$

⇒ Find coefs and plot amp and phas spectrum.

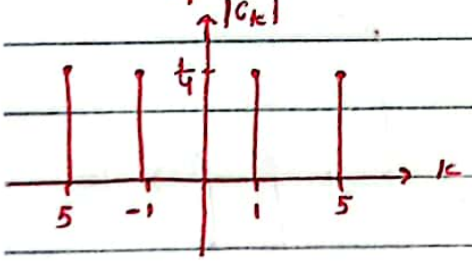
$$\frac{e^{j(2t+\frac{\pi}{4})} + e^{-j(2t+\frac{\pi}{4})}}{2} \cdot \frac{e^{j3t} - e^{-j3t}}{2j}$$

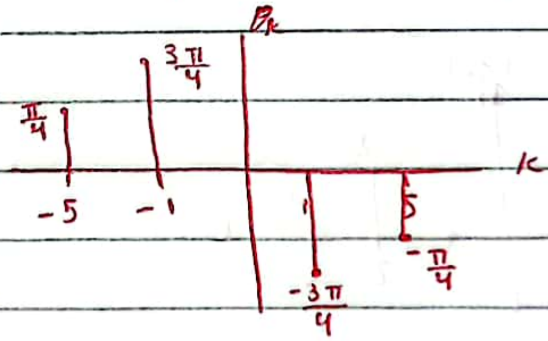
$$\frac{1}{4j}\left[\left(e^{j2t} \cdot e^{j\frac{\pi}{4}} + e^{-j2t} \cdot e^{-j\frac{\pi}{4}}\right) \cdot \left(e^{j3t} - e^{-j3t}\right)\right]$$

$$\frac{-j}{4}\left[e^{j5t} e^{j\frac{\pi}{4}} - e^{-jt} e^{j\frac{\pi}{4}} + e^{jt} e^{-j\frac{\pi}{4}} - e^{-j5t} e^{-j\frac{\pi}{4}}\right]$$

$$\frac{1}{4} e^{-j\frac{3\pi}{4}} e^{jt} + \frac{1}{4} e^{j\frac{3\pi}{4}} e^{-jt} + \frac{1}{4} e^{-j\frac{\pi}{4}} e^{j5t} + \frac{1}{4} e^{j\frac{\pi}{4}} e^{-j5t}$$

$c_1 = \frac{1}{4} e^{-j\frac{3\pi}{4}}$, $c_{-1} = \frac{1}{4} e^{j\frac{3\pi}{4}}$, $c_5 = \frac{1}{4} e^{-j\frac{\pi}{4}}$, $c_{-5} = \frac{1}{4} e^{j\frac{\pi}{4}}$





⇒ Find P_x, 3db

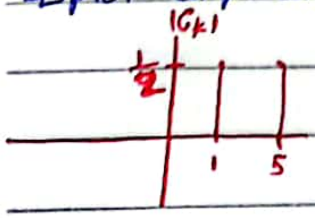
$P_x = \sum |c_k|^2 = 2\left[(\frac{1}{4})^2 + (\frac{1}{4})^2\right] = \frac{1}{4}$ watt

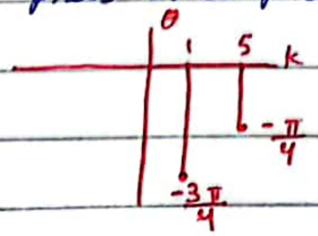
$3db = 1 \times f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}$ HZ

↳ ?? $c_1 = \frac{1}{4}$

$P_1 = \frac{1}{8}$ ⇒ 50% of power

⇒ plot amp and phas of compact F.S and write $x(t)$.

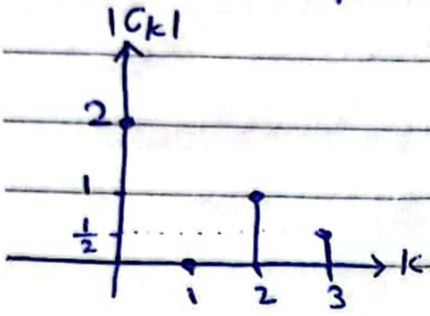


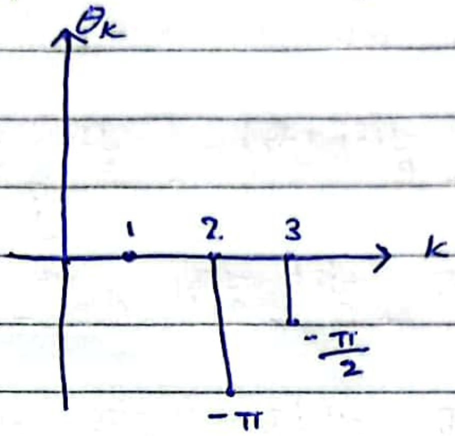


$x(t) = c_0 + \sum_{1}^{\infty} 2|c_k| \cos(k\omega_0 t + \theta_k)$

$x(t) = \frac{1}{2}\cos(t - \frac{3\pi}{4}) + \frac{1}{2}\cos(5t - \frac{\pi}{4})$

Ex: the ampl. and phas spectrum of compact trigonometric





⇒ write $x(t)$:

general equ of compact $x(t) = C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(k\omega_0 t + \theta_k)$

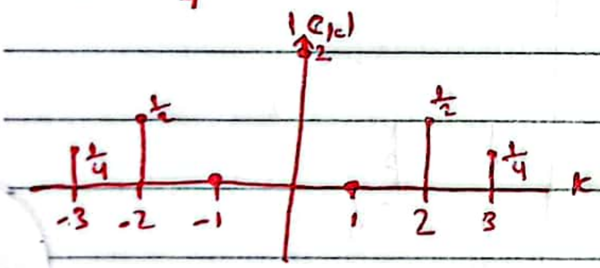
$C_0 = 2$

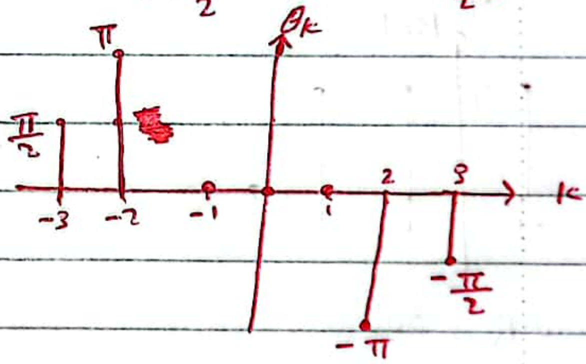
$C_1 = 0$

$C_2 = \frac{1}{2} e^{-j\pi}$

$C_3 = \frac{1}{4} e^{-j\frac{\pi}{2}}$

$\Rightarrow x(t) = 2 + \cos(\omega_0 t 2 - \pi) + \frac{1}{2}\cos(3\omega_0 t - \frac{\pi}{2})$





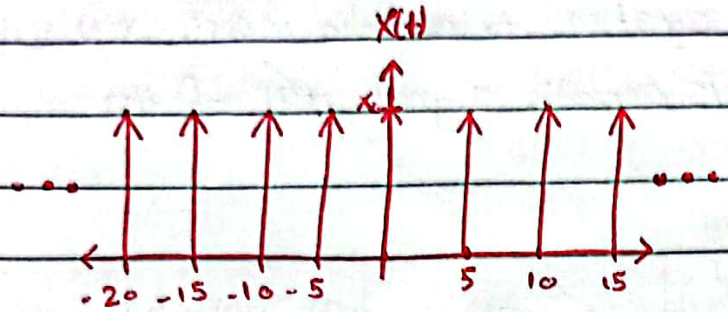
$BW = 3f$

$P_x = C_0^2 + 2\left[|C_2|^2 + |C_3|^2\right]$

$= 4 + 2\left[\frac{1}{4} + \frac{1}{16}\right] = 4.625$ watt

Subject   Date   No.

Ex: $x(t) = \sum_{-\infty}^{\infty} x_o \delta(t-5n)$ Find coef of F.S and plot amp and phas



if $n=-1$ $\quad x_o\delta(t+1)$

if $n=0$ $\quad x_o\delta(t)$

if $n=1$ $\quad x_o\delta(t-5)$

if $n=2$ $\quad x_o\delta(t-10)$

$\vdots$

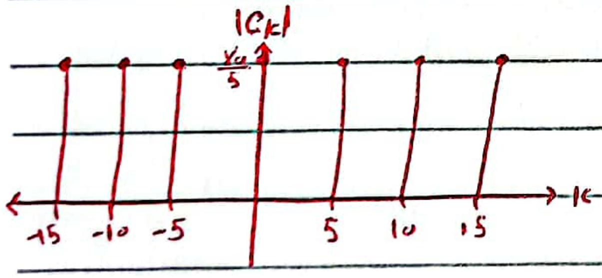
$T_o = 5$

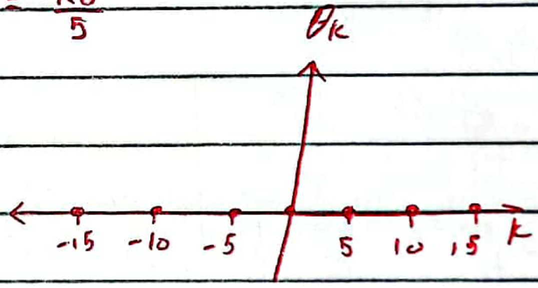
$$C_k = \frac{1}{5}\int_{-2.5}^{2.5} x_o \delta(t)\, e^{-jk\omega_o t}\, dt \qquad k=0$$

$$\frac{1}{5}\int_{-2.5}^{2.5} x_o \delta(t)\, dt$$

$$= \frac{x_o}{5}\int_{-2.5}^{2.5} \delta(t)\, dt \Rightarrow C_k = \frac{x_o}{5}$$

$$\boxed{\text{not } \int_a^b \delta(t)\, dt = 1}$$





System analysis

the response of a stable LTI system to periodic input $x(t)$ of fund. period T_0 is periodic signal $y(t)$ of the same period

$$x(t) \longrightarrow \underset{h(t)}{\boxed{L.T.I}} \longrightarrow y(t) \qquad y(t) = x(t) * h(t)$$

$$x(t) = \sum_{-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

then $y(t) = \sum_{-\infty}^{\infty} H(k\omega_0) C_k e^{jk\omega_0 t} = \sum C_{ky} e^{jk\omega_0 t}$

when $H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau$ system frequency response

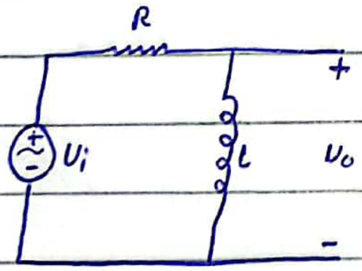
$$= \frac{y(t)}{x(t)}\Big|_{x(t) = e^{j\omega t}}$$

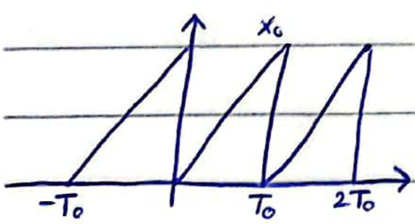
$\Rightarrow C_{ky} = H(k\omega_0) C_{kx}$

$Z_c = \frac{1}{j\omega C}$

$Z_L = j\omega L$

Ex: Find V_o





$-V_i + IR + V_o = 0$

$V_i = V_o + IR$

$V_i = L \frac{dI}{dt} + IR$

$V_i = \cancel{L} \frac{d}{dt}\left[\frac{1}{\cancel{L}} \int V_o dt\right] + R \cdot \frac{1}{L} \int V_o dt$

$V_i = V_o + \frac{R}{L} \int V_o dt$

$e^{j\omega t} = e^{j\omega t} H(\omega) + \frac{R}{L} \int V_o dt$

$j\omega e^{j\omega t} = j\omega e^{j\omega t} H(\omega) + \frac{R}{L} V_o$

$j\omega \cancel{e^{j\omega t}} = j\omega \cancel{e^{j\omega t}} H(\omega) + \frac{R}{L} \cancel{e^{j\omega t}} H(\omega)$

$j\omega = \left[j\omega + \frac{R}{L}\right] H(\omega) \Rightarrow H(\omega) = \frac{j\omega}{j\omega + \frac{R}{L}}$ $\boxed{= \frac{j\omega L}{j\omega L + R}}$

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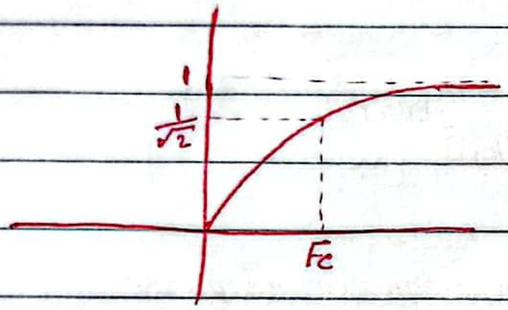
$$H(\omega) = \frac{j\omega L}{j\omega L + R}$$

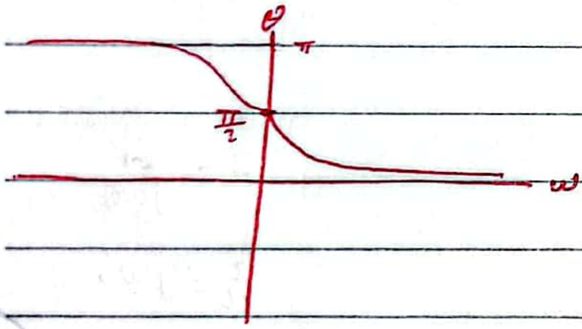
$$V_o = H(\omega)\, e^{j\omega t}$$

$$V_o = \frac{j\omega L}{j\omega L + R}\, e^{j\omega t}$$

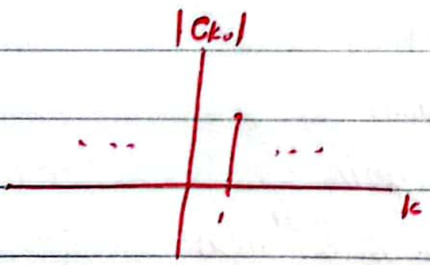
$$\text{Ampl} \Rightarrow \frac{\omega L}{\sqrt{(\omega L)^2 + R^2}}$$

$$\text{phas} \Rightarrow \frac{e^{\frac{\pi}{2} j}}{e^{\tan^{-1}\frac{\omega L}{R}\, j}} = e^{j\left[\frac{\pi}{2} - \tan^{-1}\frac{\omega L}{R}\right]}$$





$$C_{k0} = \begin{cases} \frac{x_0}{2} H(0) = 0 & k = 0 \\ \frac{j x_0}{2\pi k} \cdot \frac{k\omega_0 L}{\sqrt{R^2 + (k\omega_0 L)^2}} \cdot e^{j\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{k\omega_0 L}{R}\right)\right)} & k \neq 0 \end{cases}$$



Subject Date No.

Fourier series transformation :-

⇒ Amplitude transformation -

$$y(t) = A\,x(t) + B$$

$$x(t) = C_{0x} + \sum_{\substack{-\infty \\ k\neq 0}}^{\infty} C_{kx}\, e^{jk\omega_0 t}$$

$$y(t) = A\Big[C_{0x} + \sum_{\substack{-\infty \\ k\neq 0}}^{\infty} C_{kx}\, e^{jk\omega_0 t}\Big] + B$$

$$y(t) = A C_{0x} + \sum_{\substack{-\infty \\ k\neq 0}}^{\infty} A\, C_{kx}\, e^{jk\omega_0 t} + B$$

$$y(t) = A C_{0x} + B + \sum_{\substack{-\infty \\ k\neq 0}}^{\infty} A\, C_{kx}\, e^{jk\omega_0 t}$$

$$y(t) = C_{0y} + \sum_{\substack{-\infty \\ k\neq 0}}^{\infty} C_{ky}\, e^{jk\omega_0 t}$$

where $\left[C_{0y} = A C_{0x} + B\right]$, $\left[\underset{k\neq 0}{C_{ky} = A\, C_{kx}}\right]$

⇒ time transformation -

• time reversal:

$$y(t) = x(-t)$$

$\boxed{C_{ky} = C_{kx}^{*}}$ ⇒ $|C_{ky}| = |C_{kx}|$ and $\theta_{ky} = -\theta_{kx}$

Prof

$$x(t) = \sum_{-\infty}^{\infty} C_{kx}\, e^{jk\omega_0 t}$$

$$x(-t) = \sum_{-\infty}^{\infty} C_{kx}\, e^{-jk\omega_0 t} = \sum C_{-k_x}\, e^{jk\omega_0 t}$$

So $C_{-k} = C_{k}^{*}$

$C_{-kx} = C_{kx}^{*}$

then $x(-t) = \sum C_{k_x}^{*}\, e^{jk\omega_0 t} = \sum C_{ky}\, e^{jk\omega_0 t} = y(t)$

Subject Date No.

- time shift

$y(t) = x(t-t_0)$

$C_{ky} = C_{kx} e^{-jk w_0 t_0} \Rightarrow |C_{ky}| = |C_{kx}|$ and $\theta_{ky} = \theta_{kx} - k w_0 t_0$

prof $x(t) = \sum C_{kx} e^{jk w_0 t}$

$x(t-t_0) = \sum C_{kx} e^{jk w_0 (t-t_0)}$

$= \sum C_{kx} e^{jk w_0 t} e^{-jk w_0 t_0}$

$= \sum C_{ky} e^{jk w_0 t}$

- delay in time by $t_0 \Rightarrow$ delay phase be $e^{-jk w_0 t_0}$ in freq.

- time scaling

$y(t) = x(at)$

$$C_{ky} = \begin{cases} C_{kx} & a > 0 \\ C^*_{kx} & a < 0 \end{cases}$$

$$\boxed{T_{0y} = \frac{T_{0x}}{a} \quad w_{0y} = \frac{2\pi}{T_{0y}} = \frac{2\pi}{T_{0x}} \cdot a \quad w_{0y} = a w_{0x}}$$

prof $y(t) = x(at) = \sum C_{kx} e^{jk w_0 a t}$

$= \sum C_{ky} e^{jk w_{0y} t}$

$C_{ky} = \frac{1}{T_{0y}} \int y(t) e^{-jk w_{0y} t} dt$

$= \frac{1}{\frac{T_{0x}}{a}} \int x(at) e^{-jk a w_{0x} t} dt$

$\tau = at \to d\tau = a\,dt \to dt = \frac{d\tau}{a}$

$= \frac{a}{T_{0x}} \int x(\tau) e^{-jk w_{0x} \tau} \frac{d\tau}{\cancel{a}}$

$= \frac{1}{T_{0x}} \int x(\tau) e^{-jk w_{0x} \tau} d\tau$

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Subject Date No.

⇒ general time transformation

$y(t) = x(at+b)$ $a, b \in \mathbb{R}$

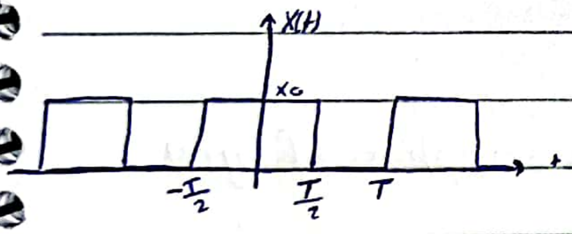
$$C_{ky} = \begin{cases} C_{kx}\, e^{jk\omega_0 b} & a>0 \\ \left[C_{kx}\, e^{jk\omega_0 b}\right]^* & a<0 \end{cases}$$

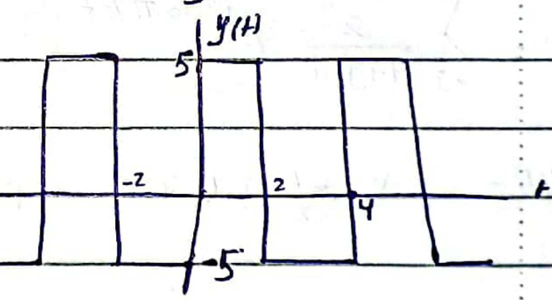
⇒ general time and Amplitude transformation

$y(t) = A X(at+b) + B$

$$C_{ky} = \begin{cases} A C_{0x} + B & k_y = 0,\ a>0 \\ A C_{0x}^* + B & k_y = 0,\ a<0 \\ A\, C_{kx}\, e^{jk\omega_0 b} & k_y \neq 0,\ a>0 \\ A \left[C_{kx}\, e^{jk\omega_0 b}\right]^* & k_y \neq 0,\ a<0 \end{cases}$$

Ex: plot the amplitude and phas spectrum of $y(t)$





$C_{kx} = \frac{T X_0}{T_0} \operatorname{Sinc}\left(\frac{kT}{T_0}\right)$ $\Rightarrow \frac{T}{T_0} = \frac{1}{2}$

$y(t) = A x(at+b) + B$

⇒ amp. transf.

$y(t) = A x + B$

$X_0 \to 5$, $0 \to -5$

$-5 = A \cdot 0 + B \Rightarrow B = -5$

$5 = A X_0 - 5 \Rightarrow A = \frac{10}{X_0}$

⇒ time transf.

$-\frac{T}{2}$ —— $\frac{T}{2}$

0 —— 2

$y(t) = x(at+b)$

$\tau = at + b$

$-\frac{T}{2} = a(0) + b$

$\Rightarrow b = -\frac{T}{2} = -1$

$\frac{T}{2} = 2a - 1$

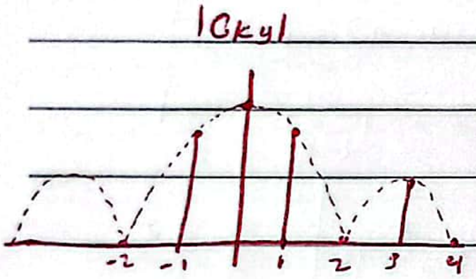
$1 = 2a - 1 \Rightarrow a = 1$

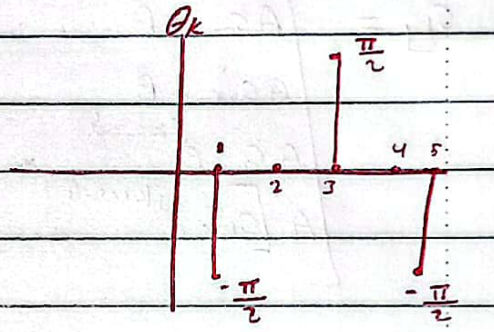
NOTEBOOK

$\Rightarrow y(t) = \frac{10}{x_0} x(t-1) - 5$

$\left[C_{ky} = A\, C_{kx}\, e^{jk\omega_0 b}\right]$

$C_{ky} = \frac{10}{x_0} \frac{x_0}{2} \text{sinc} \frac{k}{2} \; e^{-jk\frac{\pi}{2}} \qquad k \neq 0$



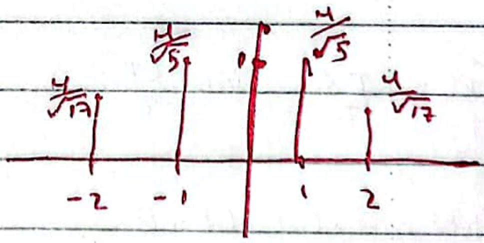


Ex:

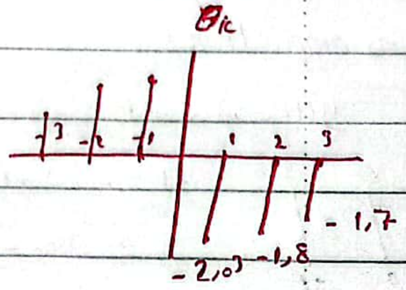
$x(t) = \sum_{-3}^{3} \frac{2}{1+j2k} e^{j200\pi k t}$

$y(t) = -2x\left(\frac{-t}{2}+1\right)+3$ plot amp and phas of $y(t)$

$A=-2, \; B=3 \quad a = \frac{-1}{2}, \; b=1$



$C_{ky} = \begin{cases} A C_{0x} + B & k=0 \\ \left[A C_{kx} e^{j\omega k b}\right]^* & k \neq 0 \end{cases}$



$C_{ky} = \begin{cases} -1 & k=0 \\ \frac{-4}{\sqrt{1+4k^2}} e^{-j\tan^{-1} 2k} \cdot e^{j200\pi k} & -3 \leq k \leq 3 \end{cases}$

Subject Date No.

Ex: an LTI system if described by $2y' + 3y = x(t)$

Find:-

1] system freq. response $H(\omega)$

2] plot amp. and phase spectrum of $H(\omega)$

3] BW of $H(\omega)$

4] type of filter

5] if $x(t)$ is 2 in table 4.3 find response $y(t)$ and plot amp. and phas of $y(t)$

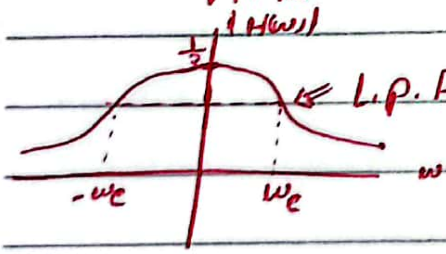
$$H(\omega) = \frac{y(t)}{x(t)}\bigg|_{x(t) = e^{j\omega t}} \Rightarrow y(t) = H(\omega)e^{j\omega t}$$

$$2H(\omega) j\omega e^{j\omega t} + 3H(\omega)e^{j\omega t} = e^{j\omega t}$$

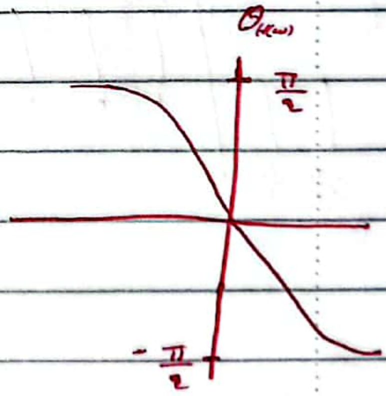
$$2H(\omega) j\omega + 3H(\omega) = 1$$

$$H(\omega)[2j\omega + 3] = 1 \Rightarrow H(\omega) = \frac{1}{2j\omega + 3}$$

$$= \frac{1}{\sqrt{9 + 4\omega^2}} e^{-j\tan^{-1}\frac{2\omega}{3}}$$



⇐ L.P.F [low pass filter]



$$H(\omega_c) = \frac{1}{\sqrt{2}} H(0)$$

$$|H(\omega)|^2 = \frac{1}{2} \cdot \frac{1}{9}$$

$$\frac{1}{9 + 4\omega_c^2} = \frac{1}{18} \Rightarrow 9 + 4\omega_c^2 = 18$$

$$\omega_c^2 = \frac{9}{4}$$

$$\omega_c = \pm\frac{3}{2} \text{ rad/sec}$$

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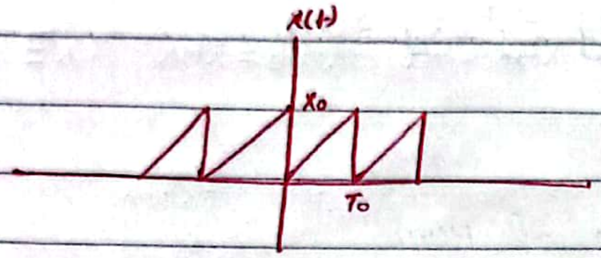
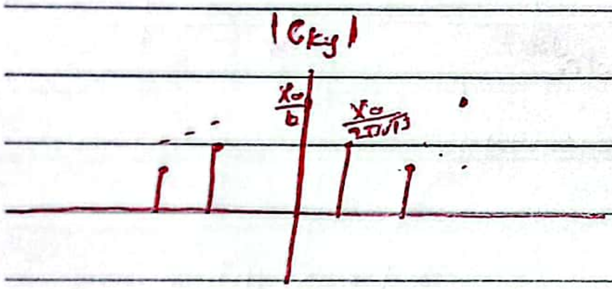
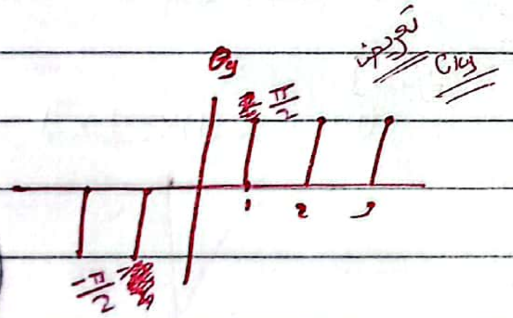


$$C_{k_x} = \begin{cases} \frac{X_0}{2} & k = 0 \\ \frac{jX_0}{2\pi k} & k \neq 0 \end{cases}$$

$$y(t) = \sum_{-\infty}^{\infty} C_{k_y} e^{jk\omega_0 t}$$

$$C_{k_y} = C_{k_x} H(k\omega_0) \Rightarrow C_{k_y} = \begin{cases} \frac{X_0}{2} \times \frac{1}{3} & k = 0 \\ \frac{jX_0}{2\pi k} \frac{1}{\sqrt{9+4k^2\omega_0^2}} e^{-j\tan^{-1}\frac{2k\omega_0}{3}} & k \neq 0 \end{cases}$$





Ch5 : Fourier transform :-

⇒ defenition : if $x(t)$ is phisically rezabile signal [energy or power] then it has fourier transform (F.T) given by

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = F[x(t)]$$

$$\Rightarrow x(t) \longleftrightarrow X(f) \quad * \text{F.T pair} *$$

$$x(t) = F^{-1}[X(f)] = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, dt$$

in general $X(f)$ is complex function $X(f) = |X(f)|\, e^{j\theta f}$

$|X(f)|$ and f ⇒ Amplitude Spectrum

θ_f and f ⇒ phase spectrum

if $x(f)$ is real ⇒ $X(-f) = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi f t}\, dt = X^*(f) = |X(F)|\, e^{-j\theta F}$

* Amplitude ⇒ even * phase ⇒ odd

- Note

Energy spectral density (ESD) $\psi(f) = |X(f)|^2$

$$E_x = \int_{-\infty}^{\infty} |X(f)|^2\, df = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

⇒ 3db-Bw of $x(t)$

$$|X(f)| = \frac{1}{\sqrt{2}} X(0) \rightarrow |X(f)|^2 = \frac{1}{2} X(0)$$

• اي سيجنال بضغطها على ال time بتزيد بال f لأن $F = \frac{1}{T}$

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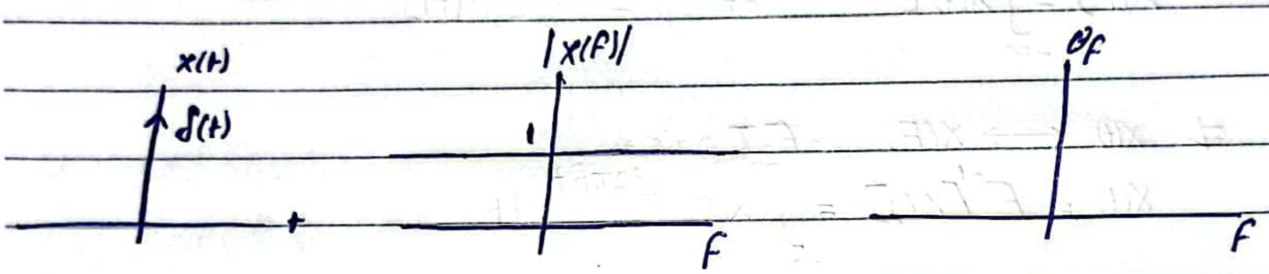
Ex: $x(t) = \delta(t)$, Find $X(f)$ and plot the Amp and phase spectrum

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

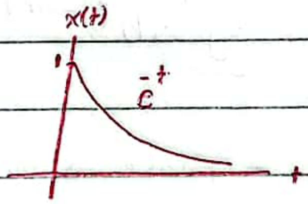
$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt = 1$$



$\Rightarrow BW = \infty$

$* \ \delta(t) \longleftrightarrow 1 \ *$

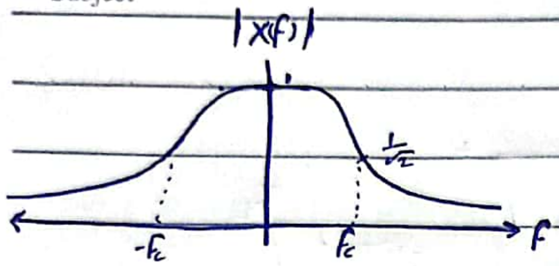
Ex: $x(t) = e^{-t} u(t)$

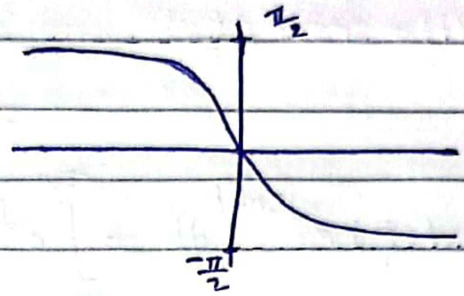


$$X(f) = \int_{-\infty}^{\infty} e^{-t} u(t)\, e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-t} e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-t(1+j2\pi f)}\, dt$$

$$= \left. \frac{e^{-t(1+j2\pi f)}}{-(1+j2\pi f)} \right]_0^{\infty} = 0 + \frac{1}{1+j2\pi f} = \frac{1}{1+j2\pi f}$$

$$\Rightarrow |X(f)| = \frac{1}{\sqrt{1+4\pi^2 f^2}}, \quad \theta_f = -\tan^{-1} 2\pi f$$





$\Rightarrow BW = \infty$ f-axis

$\Rightarrow$ 3db-BW $\quad \frac{1}{\sqrt{2}} x(0) = |X(f_c)|$

$$\frac{1}{\sqrt{2}} \times 1 = \frac{1}{\sqrt{1+4\pi^2 f^2}} \Rightarrow 2 = 1 + 4\pi^2 f^2 \Rightarrow f = \frac{1}{2\pi} \text{ HZ}$$

$$\Rightarrow 3db = \frac{1}{2\pi} \quad \text{So} \int_{-\frac{1}{2\pi}}^{\frac{1}{2\pi}} |X(f)|^2 df \text{ must be } = 50\% \text{ of } E$$

$$\left[E_x = \int_0^\infty e^{-2t} dt = \frac{1}{2} \text{ J} \right]$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\frac{1}{2\pi}}^{\frac{1}{2\pi}} \frac{1 \times 2\pi}{1 + \underbrace{4\pi^2 f^2}_{(2\pi f)^2}} df = \frac{1}{2\pi} \tan^{-1} 2\pi f \Big]_{-\frac{1}{2\pi}}^{\frac{1}{2\pi}}$$

$$\frac{1}{2\pi} * \tan^{-1}(1) - \tan^{-1}(-1)$$

$$\frac{1}{2\pi} * \frac{\pi}{2} = \frac{1}{4} \Rightarrow 50\% \text{ of } E \checkmark$$

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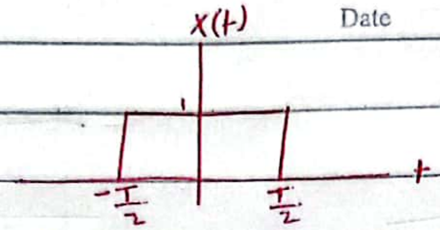
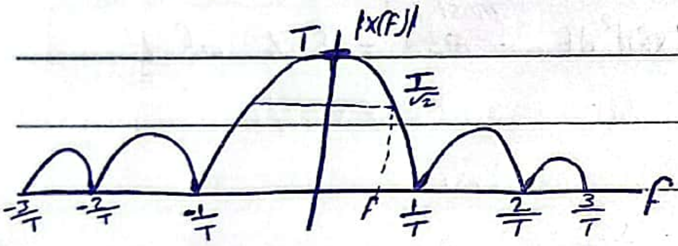
Ex: $x(t) = \mathrm{rect}\left(\frac{t}{T}\right)$



$$X(f) = \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t}{T}\right) e^{-j2\pi f t}\, dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi f t}\, dt = \left. \frac{e^{-j2\pi f t}}{-j2\pi f} \right]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= \frac{e^{-j2\pi f \frac{T}{2}} - e^{j2\pi f \frac{T}{2}}}{-j2\pi f} = \frac{1}{\pi f} * \frac{e^{j\pi f T} - e^{-j\pi f T}}{j2} = T \cdot \frac{\sin \pi f T}{\pi f \cdot T}$$

$\Rightarrow T\,\mathrm{sinc} fT$



$\Rightarrow$ BW [first null] $= \frac{1}{T}$

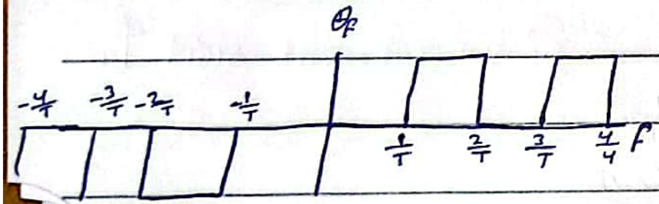
$\Rightarrow$ 3db - BW $= f_1$

$$\frac{T}{\sqrt{2}} = T\,\mathrm{sinc} f T$$

$$1 = \sqrt{2}\,\mathrm{sinc} f_1 T$$

$$\vdots$$

$$f_1 = \frac{\sin^{-1} \frac{\pi f_1 T}{\sqrt{2}}}{\pi T}$$



Subject　　　　　　　　　　　　　　　　Date　　　　　　No.

properties of fourier transform :

1] linearity　　$F[\alpha x(t) + \beta y(t)] = \alpha X(f) + \beta Y(f)$

⟹ Ex: $x(t) = e^{-|t|} = \begin{cases} e^{-t} & t>0 \\ e^{t} & t<0 \end{cases}$

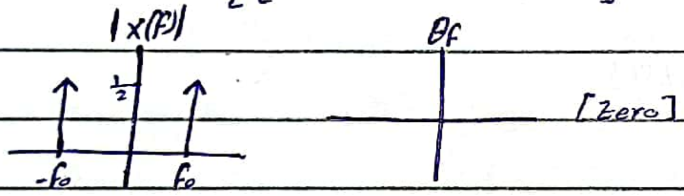
$= e^{-t}u(t) + e^{t}u(-t)$

$X(f) = \dfrac{1}{1+j2\pi f} + \dfrac{1}{1-j2\pi f} = \dfrac{2}{1+(2\pi f)^2}$

⟹ Ex: $x(t) = \cos 2\pi f t$

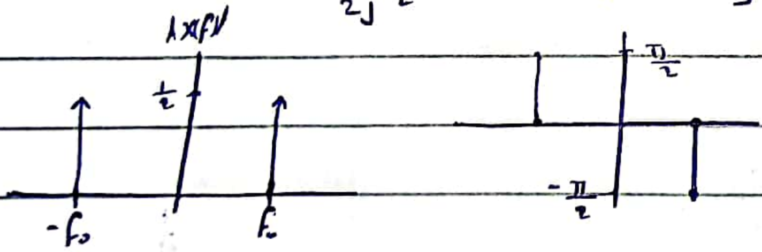
$= \frac{1}{2}\left[e^{j2\pi f t} + e^{-j2\pi f t}\right]$

$X(f) = \frac{1}{2}\left[\delta(f-f_o) + \delta(f+f_o)\right]$



⟹ Ex: $x(t) = \sin 2\pi f t$

$X(f) = \frac{1}{2j}\left[\delta(f-f_o) - \delta(f+f_o)\right]$



2] time scaling　　$F[x(at)] = \frac{1}{|a|} \cdot X(\frac{f}{a})$　,　$F[x(-t)] = X(-f) = X^{*}(f)$

Ex: $y(t) = e^{-\alpha t}u(t)$ ⟹ $Y(f) = \frac{1}{\alpha} \cdot \dfrac{1}{1+j2\pi\frac{f}{\alpha}}$

Ex: $rect(t) \rightarrow \operatorname{sinc} F$

$rect(\frac{t}{T}) \rightarrow T \operatorname{sinc} FT$

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3] duality :-

$$x(t) \longleftrightarrow y(f)$$

$$y(t) \longleftrightarrow x(-f)$$

Ex: $x(t) = \text{sinc}(tT)$

$$X(f) = \frac{1}{T} \text{rect}\left(\frac{-f}{T}\right) = \frac{1}{T} \text{rect}\left(\frac{f}{T}\right)$$

Ex: $x(t) = k \qquad X(f) = k\,\delta(f)$

4] time shift :

$$x(t-t_0) \longleftrightarrow e^{-j2\pi f t_0} X(f)$$

$$x(t+t_0) \longleftrightarrow e^{j2\pi f t_0} X(f)$$

Ex: $x(t) = 5\,\text{rect}\left(\frac{t-1.5}{3}\right)$

$$X(f) = 15 \text{ sinc } 3F \cdot e^{-j2\pi(1.5f)}$$

5] time transformation :

$$x(at+b) \longleftrightarrow \frac{1}{|a|} \cdot X\left(\frac{f}{a}\right) \cdot e^{j2\pi \frac{f}{a} \cdot b}$$

6] Frequency shift [modulation]

$$x(t)\, e^{j2\pi f_0 t} \longrightarrow X(f-f_0)$$

$$x(t)\, e^{-j2\pi f_0 t} \longrightarrow X(f+f_0)$$

Ex: $x(t) = \text{rect}\left(\frac{t}{T}\right) \cos 2\pi f_0 t$

$$= \text{rect}\left(\frac{t}{T}\right) \cdot \frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right]$$

$$X(f) = \frac{1}{2}\left[T \text{sinc } T(f-f_0) + T \text{sinc} T(f+f_0)\right]$$

not :- $x(t) \cos 2\pi f_0 t \longleftrightarrow \frac{1}{2}\left[X(f-f_0) + X(f+f_0)\right]$

$$x(t) \sin 2\pi f_0 t \longleftrightarrow \frac{1}{2j}\left[X(f-f_0) - X(f+f_0)\right]$$

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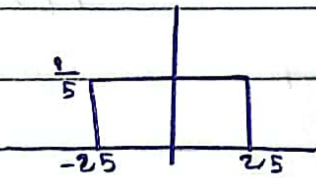
7] Area under x(t):

$$A_x = \int_{-\infty}^{\infty} x(t)\,dt = X(f)\Big|_{f=0}$$

Ex:- $x(t) = \text{sinc}\,5t$

$$A_x = \int_{-\infty}^{\infty} \text{sinc}\,5t\,dt = X(f)\Big|_{f=0}$$

$$X(f) = \frac{1}{5}\,\text{rect}\,\frac{f}{5}$$



$$X(0) = \frac{1}{5}$$

$$A_x = \frac{1}{5}$$

$\Rightarrow$ note: $\int_{-\infty}^{\infty} \text{sinc}\,t\,dt = 1$

8] Area under X(f):

$$A_x = \int_{-\infty}^{\infty} X(f)\,df = x(t)\Big|_{t=0}$$

Ex: $x(t) = e^{-|t|}$

$\Rightarrow X(f) = \dfrac{2}{1+(2\pi f)^2}$ $\qquad A_x = \displaystyle\int_{-\infty}^{\infty} \frac{2}{1+(2\pi f)^2}\,df$

$$A_x = x(t)\Big|_{t=0} = 1$$

9] differentiation in time:-

$$x'(t) \longrightarrow j2\pi f\,X(f)$$

$$x^{(n)}(t) \longrightarrow (j2\pi f)^n\,X(f)$$

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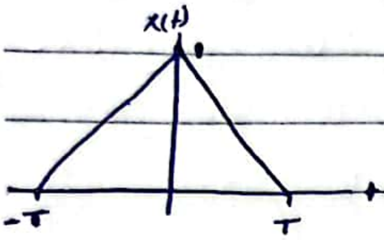
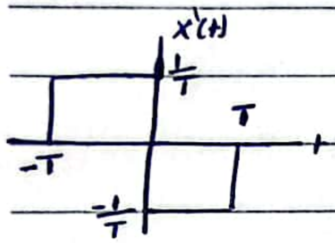
Ex: $x(t) = \text{tri}(\frac{t}{T})$, Find $X(f)$?



$$x'(t) = -\frac{1}{T}[u(t) - u(t-T)] + \frac{1}{T}[u(t+T) - u(t)]$$

$$x'(t) = \frac{1}{T}\,\text{rect}\left(\frac{t+\frac{T}{2}}{T}\right) - \frac{1}{T}\,\text{rect}\left(\frac{T+\frac{T}{2}}{T}\right)$$



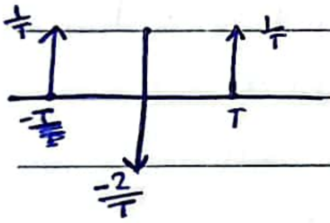
$$F[x'(t)] = \text{sinc}(FT)\,.\,e^{\frac{j2\pi FT}{2}} - \text{Sinc}\,FF\,e^{\frac{-j2\pi FT}{2}}$$

$$= j\,2\,\text{sinc}(FT)\,\sin(\pi FT)$$

$$= j2\,\pi FT\,\text{sinc}^2 FT$$

$$= j2\pi f\,X(f)$$

$\ast$ $X(f) = T\,\text{sinc}^2 FT$



or $x''(t) = \frac{1}{T}\delta(t+T) - \frac{2}{T}\delta(t) + \frac{1}{T}\delta(t-T)$

$$F(x''(t)) = \frac{1}{T}e^{J2\pi FT} - \frac{2}{T} + \frac{1}{T}e^{-J\pi FT}$$

$$= \frac{1}{T}[2\cos 2\pi FT - 2]$$

$$= \frac{1}{T}[2(1 - 2\sin^2\pi FT) - 2]$$

$$= -\frac{4}{T}\,\frac{\sin^2(\pi FT)}{(\pi FT)^2} \ast (\pi FT)^2$$

$$F[x''(t)] = -4\pi^2F^2T\,\text{sinc}^2(FT)$$

$\ast$ $X(f) = T\,\text{sinc}^2 FT$

Ex: $y' + 2y = \delta(2t)$ Find the response $y(t)$?

$$J2\pi f\,y(f) + 2y(f) = \frac{1}{2}$$

$$y(f) = \frac{1}{4 + j4\pi f}$$

$$y(t) = F^{-1}[y(f)] = \frac{1}{2}e^{-2t}u(t)$$

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[6] differentiation in frequency:

$$g'(f) \longrightarrow -j2\pi t\, g(t)$$

$$g^{(n)}(f) \longrightarrow (-j2\pi t)^n g(t)$$

- $t\,g(t) = \frac{j}{2\pi} g'(f)$
- $f\,g(f) = \frac{x'(t)}{j2\pi}$

Ex: $x(t) = e^{-\alpha t^2}$, $x(f)$?

$$x'(t) = -2\alpha t\, x(t)$$

$$j2\pi f X(f) = -2\alpha t\, x(t)$$

$$j2\pi f X(f) = -2\alpha \frac{j\, X'(f)}{2\pi}$$

$$\int \frac{X'(f)}{X(f)} df = \int \frac{2\pi^2 f}{-\alpha} df$$

$$\ln X(f) = \frac{-\pi^2 f^2}{\alpha} \Rightarrow X(f) = e^{-\frac{\pi^2 f^2}{\alpha}}$$

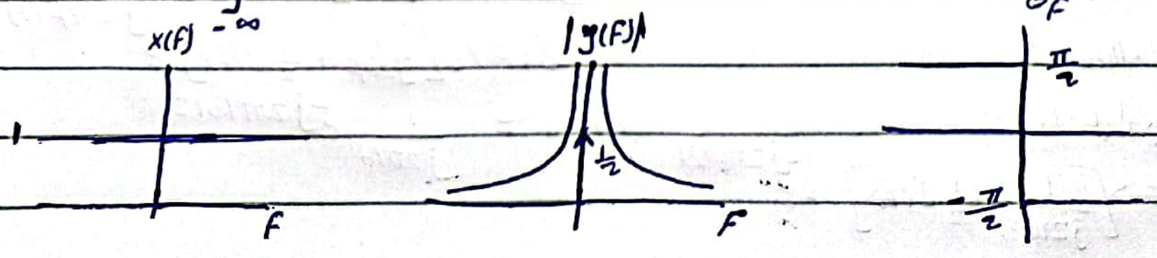
[7] integration in time:

$$\int_{-\infty}^{t} x(\tau)\, d\tau \longrightarrow \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\, \delta(f)$$

Example:

$$y(t) = u(t) \rightarrow y(f) = \frac{1}{j2\pi f}(1) + \frac{1}{2}\delta(f)$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$



$\theta_F = -\frac{\pi}{2}\, sgn(F)$

sgn is odd

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Ex: $y(t) = \delta(t)$, $y(f)$?

$y(t) = u'(t)$

$$F[u'(t)] = j2\pi f\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right]$$

$$= 1 + \frac{1}{2}\delta(f)\, j2\pi f$$

$$= 1$$

$\Rightarrow$ where $y'(t) = j2\pi F\, u(f)$

[12] Conjugation :-

$x(t) \longleftrightarrow x(f)$

then $x^*(t) \longleftrightarrow x(-f) = x^*(f)$

$x(t) = \alpha(t) + j\,\beta(t)$

$$F[x^*(t)] = F[\alpha(t) - j\beta(t)]$$

$$= \alpha(f) - j\beta(f)$$

[13] Convolution :

$x(t) * h(t) = x(f) \cdot H(f)$

$x(t) \longrightarrow \boxed{L.T.I} \longrightarrow y(t) = x(t) * h(t)$

$y(f) = x(f) \cdot H(f)$

$y(t) = F^{-1}[y(f)]$

Ex: $x(t) = e^{j2\pi f_0 t}$ $\quad x(t) \longrightarrow \underset{L.T.I}{\boxed{u(t-2)}} \longrightarrow y(t)$ Find the respons $y(t)$??

$y(t) = x(t) * h(t)$

$y(f) = x(f) \cdot H(f)$

$e^{j2\pi f_0 t} \xleftrightarrow{F} \delta(F - f_0)$

$u(t-2) \xleftrightarrow{F} \left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] * e^{-j2\pi f(2)}$

$y(f) = F^{-1}[y(t)]$

$\delta(F - f_0)\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] e^{-j2\pi f(2)}$

$$= \frac{1}{j2\pi f_0}\, e^{-j2\pi f_0(2)}$$

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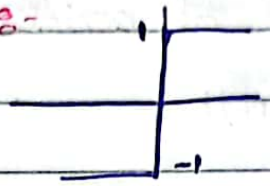
note for time function sgn(t) :-

$$sgn(t) = 2u(t) - 1$$



$$F[sgn(t)] = F[2u(t) - 1]$$

$$= 2\left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] - \delta(f)$$

$$= \frac{1}{j\pi f}$$

$\boxed{14}$ multiplication in time :-

$$x(t)\, h(t) \longleftrightarrow X(f) * h(f)$$

$$x(t) * h(t) \longleftrightarrow X(f) \cdot H(f)$$

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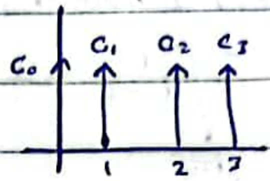
• Fourier transform of peridic signals:-

$$x(t) = \sum_{-\infty}^{\infty} C_k e^{jk(2\pi f)t}$$

$$X(f) = \sum_{-\infty}^{\infty} C_k \delta(f - kf_0)$$



$$C_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk(2\pi f_0)t}\, dt$$

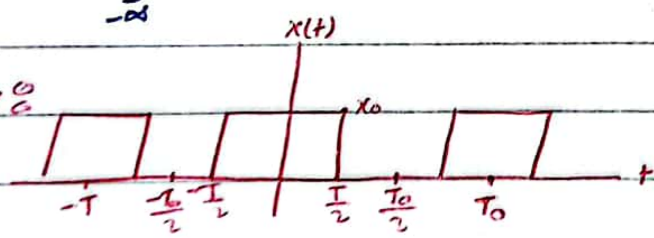
let $y(t) = \begin{cases} x(t) & -\frac{T_0}{2} < t < \frac{T_0}{2} \\ 0 & o.w \end{cases}$

$$C_k = \frac{1}{T_0} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)\, e^{-jk\,2\pi f_0 t}\, dt$$

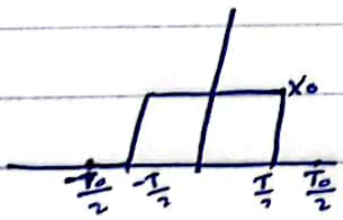
$$C_k = \frac{1}{T_0}\, y(kf_0)$$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \ldots \quad \text{where } f = kf_0$$

Ex: Find $X(f)$, plot Amp and phas



$$X_p(f) = \sum_{-\infty}^{\infty} \frac{1}{T_0} \underbrace{X(kf_0)}_{X(f)} \delta(f - f_0)$$



$\underbrace{x(t)}_{\text{one periode}} = \begin{cases} x_0\, \text{rect}\left(\frac{t}{T}\right) & -\frac{T_0}{2} < t < \frac{T_0}{2} \\ 0 & o.w \end{cases}$

$$X_p(f) = \sum_{-\infty}^{\infty} \frac{1}{T_0}\, T x_0\, \text{Sinc}\, kf_0 T\, \delta(f - kf_0)$$

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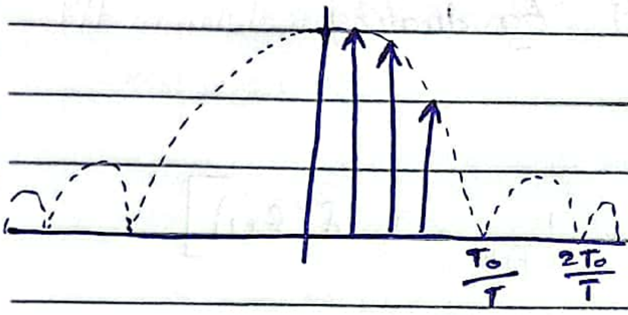
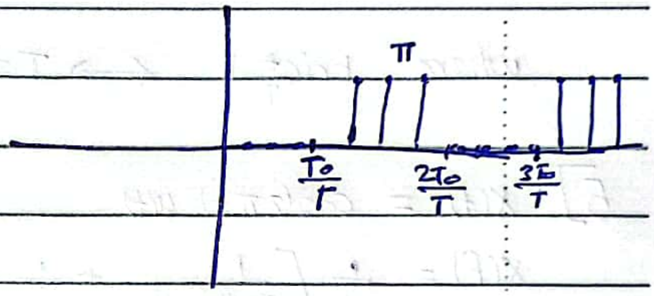
$$x_p(t) = \sum_{-\infty}^{\infty} c_k e^{jk2\pi f_0 t}$$

$$X_p(f) = \sum c_k \delta(f - f_0 k) = \sum \frac{1}{T_0} X(kf_0)\, \delta(f - kf_0)$$

$$k f_0 T = n$$

$$k = \frac{n}{f_0 T} \Rightarrow k = \frac{n T_0}{T}$$





Ex: Find F.T for the following:

1] $x(t) = |t|$

$$x(t) = \begin{cases} t & t > 0 \\ -t & t < 0 \end{cases}$$

$$x'(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases} = \operatorname{sgn}(t)$$

$$j2\pi f\, X(f) = \frac{1}{j\pi f} \Rightarrow X(f) = \frac{-1}{2\pi^2 f^2}$$

2] $x(t) = \ln t$

$$x'(t) = \frac{1}{t}$$

$$\cancel{j}2\cancel{\pi} f\, X(f) = -\cancel{j}\,\cancel{\pi} \operatorname{sgn}(f)$$

$$X(f) = \frac{-\operatorname{sgn}(f)}{2f}$$

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3] $x(t) = e^{-j5t} \sin 2\pi t$

$$X(f) = \frac{1}{2j}\left[\delta\left(f-1+\frac{5}{2\pi}\right) - \delta\left(f+1+\frac{5}{2\pi}\right)\right.$$

4] $x(t) = \operatorname{sinc}^2 5t$

$$\frac{1}{5}\operatorname{tri}\left(\frac{-f}{5}\right) = \frac{1}{5}\operatorname{tri}\left(\frac{f}{5}\right) \quad \underline{\underline{\text{even}}}$$

where $\operatorname{tri}\left(\frac{t}{T}\right) \longleftrightarrow T\operatorname{sinc}^2(fT)$ by duality.

5] $x(t) = \cos(2\pi t)\, u(t)$

$$X(f) = \frac{1}{2}\left[\frac{1}{j2\pi(f-1)} + \frac{1}{2}\delta(f-1) + \frac{1}{j2\pi(f+1)} + \frac{1}{2}\delta(f+1)\right]$$

use $y(t)\cos 2\pi f_o t \longleftrightarrow \frac{1}{2}\left[y(f-f_o) + y(f+f_o)\right]$

where ~~y(t)~~ $u(t) \longleftrightarrow \frac{1}{2}\left[\delta(f) + \frac{2}{j2\pi f}\right]$

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Find inverse F.T for the following :-

1] $x(f) = \frac{1}{2+4\pi^2 f^2}$

$x(f) = \frac{1}{(\sqrt{2})^2+(2\pi f)^2} \longrightarrow x(t) = \frac{1}{2\sqrt{2}} e^{\sqrt{2}|t|}$

2] $X(f) = \frac{1}{j}\left[\delta(f-2) - \delta(f+2)\right]$

$x(t) = 2\sin 2\pi(2)t$

3] $X(f) = \text{sinc}^2\, 3f$

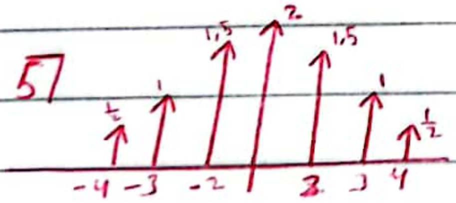
$x(t) = \frac{1}{3}\,\text{tri}\left(\frac{t}{3}\right)$

4] $X(f) = \delta(f) + \frac{1}{j\pi f}$

$x(t) = 2u(t)$

or $1+\text{sgn}(t)$

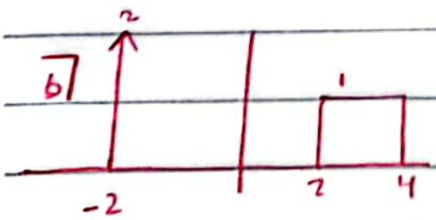
5]



$x(t) = 4\cos 2\pi(2)t + 2\cos 2\pi(3)t + \cos 2\pi 4t$

where $\frac{1}{2}\left[\delta(f+f_0) + \delta(f-f_0)\right] \longrightarrow x_0 \cos 2\pi f_0 t$

6]



$x(f) = 2\delta(f+2) + \text{rect}\,\frac{f-3}{2}$

$x(t) = 2\, e^{-j2\pi(2)t} + 2\,\text{sinc}\,2t \cdot e^{j2\pi(3)t}$

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parseval's Energy and power theorems :-

1] parseval's Energy theorem :-

- if $x(t)$ is an energy signal, then it's energy can be found in time domain and in frequency.

$$\Rightarrow E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

$|X(f)|^2$ is called Energy spectral density [ESD] and denoted by $\Psi_x(f)$

$$E_x = \int_{-\infty}^{\infty} \Psi_x(f) \, df$$

Ex: $x(t) = e^{-\alpha t} u(t)$ verify parseval's Energy theorem:

$$\Rightarrow E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{0}^{\infty} e^{-2\alpha t} dt = \left.\frac{e^{-2\alpha t}}{-2\alpha}\right]_0^{\infty} = \frac{1}{2\alpha} \; J$$

$$\Psi_x(f) = |X(f)|^2 \Rightarrow X(f) = \frac{1}{\alpha + j2\pi f}, \quad |X(f)| = \frac{1}{\sqrt{\alpha^2 + 4\pi^2 f^2}}$$

$$\Psi_x(f) = \frac{1}{\alpha^2 + (2\pi f)^2}$$

$$\Rightarrow E_x = \int_{-\infty}^{\infty} \Psi_x(f) \, df = \int_{-\infty}^{\infty} \frac{1}{\alpha^2 + (2\pi f)^2} df = \frac{1}{2\pi\alpha} \left. \tan^{-1} \frac{2\pi f}{\alpha} \right]_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi\alpha}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{1}{2\alpha}$$

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2] parseval's power theorem :-

- if $x(t)$ is power signal, then it's power can be found in time domain and frequency domain

$$\Rightarrow P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |X_T(f)|^2 df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2 df$$

where $\lim_{T\to\infty} \frac{1}{T}|X_T(f)|^2$ is called power spectral density [PSD] and denoted by $S_{x(f)}$

$$\Rightarrow P_x = \int_{-\infty}^{\infty} S_{x(f)}\, df$$

note $X_T(f) = F[x_T(t)] = F[x(t)\, rect(\frac{t}{T})]$

- For periodic signal $x(t)$ of fund. period T_0

Since $x(t) = \sum_{-\infty}^{\infty} C_k e^{jk2\pi f t} \Rightarrow X(f) = \sum_{-\infty}^{\infty} C_k \delta(f - kf_0)$

$$\cdot P_x = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt = \sum_{-\infty}^{\infty} C_k^2$$

$$\Rightarrow S_{x(f)} = |X(f)|^2 = \sum_{-\infty}^{\infty} |C_k|^2 \delta(f - f_k)$$

$$P_x = \int_{-\infty}^{\infty} S_{x(f)}\, df = \int_{-\infty}^{\infty} \sum_{-\infty}^{\infty} |C_k|^2 \delta(f - f_k)\, df = \sum_{-\infty}^{\infty} |C_k|^2 \int_{-\infty}^{\infty} \delta(f - kf)\, df = \sum_{-\infty}^{\infty} |C_k|^2$$

So $$P_x = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt = \int_{-\infty}^{\infty} S_{x(f)}\, df$$

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Ex: $X(t) = \cos 2\pi t$ verify parseval's power theorem:-

- $x(t)$ periodic with fun. period $T_0 = 1$

$$\Rightarrow P_x = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^2 2\pi t\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2}[1 + \cos 4\pi t]\, dt = \left[\frac{1}{2}t + \frac{1}{8\pi}\sin 4\pi t\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{2} \text{ watt}$$

$$x(t) = \cos 2\pi t = \frac{e^{j2\pi t} + e^{-j2\pi t}}{2} = \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-2j\pi t}$$

$$\Rightarrow C_1 = \frac{1}{2}, \quad C_{-1} = \frac{1}{2}$$

$$X(f) = \sum c_k\, \delta(f - kf) = \frac{1}{2}\delta(f-1) + \frac{1}{2}\delta(f+1)$$

$$\Rightarrow P_x = \int_{-\infty}^{\infty} S_{X(f)}\, df \quad , \quad S_{X(f)} = |X(f)|^2 = \frac{1}{4}\delta(f-1) + \frac{1}{4}\delta(f+1)$$

$$P_x = \int_{-\infty}^{\infty} \left[\frac{1}{4}\delta(f-1) + \frac{1}{4}\delta(f+1)\right] df$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2} \text{ watt}$$

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Ex: $x(t) = k$, $\forall t$. Verify parseval's power theorem:

$x(t)$ is aperiodic signal

$$\Rightarrow P_x = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\, dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} k^2\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\, k^2 t \Big]_{-\frac{T}{2}}^{\frac{T}{2}} = \lim_{T\to\infty} \frac{1}{T} k^2 T = \underline{\underline{k^2 \text{ watt}}}$$

$$\Rightarrow P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} X_T(f)\, df$$

- $x_T(t) = x(t)\, \text{rect}(\frac{t}{T}) = k\, \text{rect}(\frac{t}{T})$

$X_T(f) = kT\, \text{sinc}\, fT$

$|X_T(f)|^2 = k^2 T^2\, \text{sinc}^2 fT$

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} k^2 T^2\, \text{sinc}^2 fT\, df$$

$$= \lim_{T\to\infty} \frac{1}{T} k^2 T \int_{-\infty}^{\infty} T\, \text{sinc}^2 fT\, df$$

$\Rightarrow$ * using area under $h(f)$ property of F.T

$$\int_{-\infty}^{\infty} h(f)\, df = g(t)\Big|_{t=0}$$

So $\lim_{T\to\infty} \frac{1}{T} k^2 T \times 1 = \underline{\underline{k^2 \text{ watt}}}$

$$\Rightarrow \int_{-\infty}^{\infty} T\, \text{sinc}^2 fT\, df = \text{tri}(\tfrac{t}{T})\Big|_{t=0} = 1$$

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