EICoM
ELECTRICAL COMPUTER MECHATRONICS

تلخيص

تفاضل وتكامل 2

للطالـب المبدع

محمود مجدلاني

إرادة - ثقـة - تغيير

CHAPTER (7)

Integration by parts

- Factor the denominator Q(x) : $(ax + b)$, $(ax^2 + bx + c)$

$(ax + b) \Downarrow$ Linear

$(ax^2 + bx + c) \Downarrow$ Irreducible quadratric

$(b^2 - 4ac < 0)$

Q(x)

CASE (1) : Linear factor

- $$\frac{1}{x(x-1)(x+5)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+5}$$

CASE (2) : Repeated Linear factor

- $$\frac{1}{(x-3)^2\,(2x+1)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{2x+1} + \frac{D}{(2x+1)^2} + \frac{E}{(2x+1)^3}$$

CASE (3) : Irreducible quadratric factor

- $$\frac{1}{(x^2+1)\,(x^2+x+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+x+1}$$

CASE (4) : Repeded irreducible quadratric factor

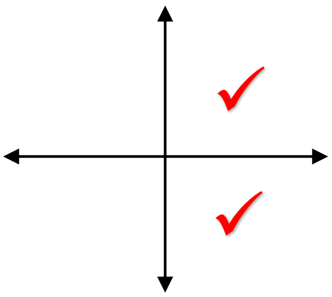
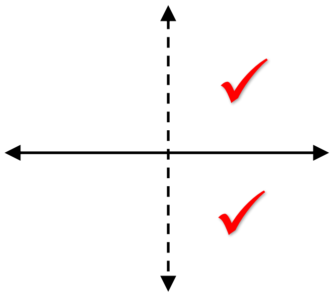
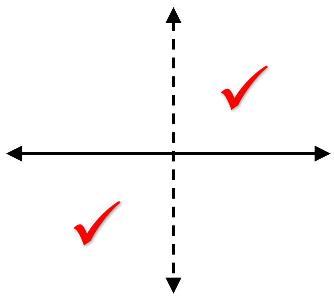
- $$\frac{1}{(x^2+4)^2\,(x^2+1)^3} = \frac{A_1x+B_1}{x^2+4} + \frac{A_2x+B_2}{(x^2+4)^2} + \frac{A_3x+B_3}{x^2+1} + \frac{A_4x+B_4}{(x^2+1)^2} + \frac{A_5x+B_5}{(x^2+1)^3}$$

Trigonometric Integrals

$$\int \underbrace{\text{polynomial}}_{u} . \underbrace{\begin{matrix} e^{(ax+b)} \\ cos(ax+b) \\ sin(ax+b) \\ \sqrt{ax+b} \end{matrix}}_{dv} . \; dx$$

- $\int sin^n x \; .dx = \frac{-1}{n} sin^{n-1}x \; . \; cosx + \frac{n-1}{n} \int sin^{n-2}x . dx$
- $\int cos^n x \; .dx = \frac{1}{n} cos^{n-1}x \; . \; sinx + \frac{n-1}{n} \int cos^{n-2}x . dx$
- $\int tan^n x \; .dx = \frac{tan^{n-1}x}{n-1} - \int tan^{n-2}x . dx$
- $\int sec^n x \; .dx = \frac{sec^{n-1}x - tanx}{n-1} + \frac{n-2}{n-1} \int sec^{n-2}x . dx$

Trigonometric substitution

Trigonometric substitution		
$\sqrt{a^2 - x^2}$	$\sqrt{a^2 + x^2}$	$\sqrt{x^2 - a^2}$
$x = a \; sin\theta$	$x = a \; tan\theta$	$x = a \; sec\theta$
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$0 \le \theta < \frac{\pi}{2}$ $\pi \le \theta < \frac{3\pi}{2}$
		

Integral of rotional functions by partial fraction

$f(x)=\frac{p(x)}{Q(x)}$

- $\deg(p) \geq \deg(Q) \longrightarrow \begin{array}{r|l} & s(x) \\ \hline Q(x) & p(x) \\ & R(x) \end{array} \longrightarrow s(x)+\frac{R(x)}{Q(x)} \longrightarrow \deg(R) < \deg(Q)$ → Partial fractions
- $\deg(p) < \deg(Q)$ → Partial fractions

Strategy for integration

$\int \sin^m x \cos^n x \, dx$

➢ اذا كان (m , n)
- كلاهما فردي → التعويض : إفرض (y) ما تحت لاس الاكبر
- احدهما زوجي والأخر فردي → التعويض : إفرض (y) ما تحت لاس الاكبر
- كلاهما زوجي ← تحل على المطابقة $\left(\frac{1}{2}(1+\cos 2x)\right)$

$\int \sec^m x \tan^n x \, dx$

$\int \csc^m x \cot^n x \, dx$

➢ اذا كان (m , n)
- كلاهما فردي ← التعويض : $y=\csc x$, $y=\sec x$
- كلاهما زوجي ← التعويض : $y=\cot x$, $y=\tan x$
- اذا كانت (m) زوجي و (n) فردي ← أي وحدة بنفرض مش أي محدد
- اذا كانت (m) فردي و (n) زوجي
  - أجزاء
  - عامل مشترك $(\sec^2 x , \csc^2 x)$

Improper integrals

➢ $$\int_{1}^{+\infty} \frac{1}{x^{p}}\,.dx = \begin{cases} p > 1 \;, & \text{convergent to} = \frac{1}{p-1} \\ p \leq 1 \;, & \text{divergent} \end{cases}$$

✓ $$\int_{a}^{+\infty} f(x)\,.dx = \lim_{t\to\infty} \int_{a}^{t} f(x)\,.dx$$

✓ $$\int_{-\infty}^{a} f(x)\,.dx = \lim_{t\to-\infty} \int_{t}^{a} f(x)\,.dx$$

Called convergent if limit exist

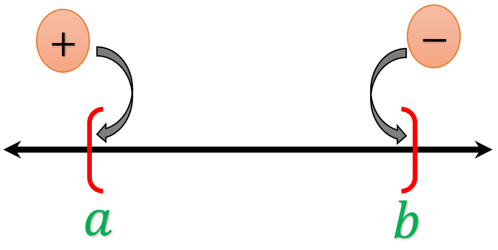
Called divergent if limit d.n.e

❖ $$\int_{-\infty}^{+\infty} f(x)\,.dx = \int_{-\infty}^{a} f(x)\,.dx + \int_{a}^{+\infty} f(x)\,.dx$$

في حال كان بينهم فترة غير متصلة (مفتوحة)

✚ $$\int_{a}^{b} f(x)\,.dx = \lim_{t\to b^{-}} \int_{a}^{t} f(x)\,.dx$$

✚ $$\int_{a}^{b} f(x)\,.dx = \lim_{t\to a^{+}} \int_{t}^{b} f(x)\,.dx$$



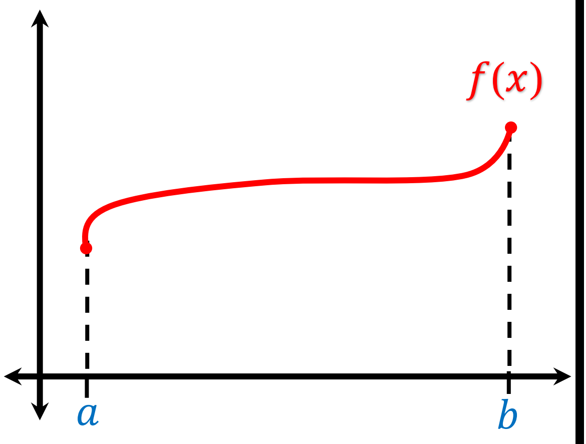
Note:

The improper integral is called convergent if limit exists and divergent if the limit d.n.e

CHAPTER (8)

Arc length

The Arc length formula



- If $f'(x)$ is CTS on $[a, b]$ then the length of the ($y = f(x)$) $\Longrightarrow$ ($a \le x \le b$) is :

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ .dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ .dx$$

- If $g'(x)$ is CTS on $[c, d]$ then the length of the ($x = g(y)$) $\Longrightarrow$ ($c \le y \le d$) is :

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\ .dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ .dy$$

اطراف التكامل هي مجال الاقترانات

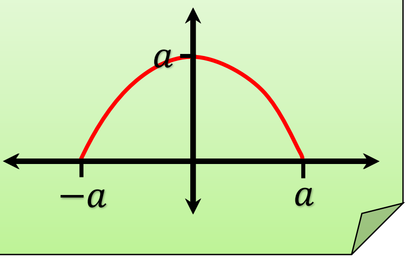
اذا طلب set up يعني بده شكل التكامل

Note:

✓ $f(x) = \sqrt{a - x^2}$

$-\sqrt{a} \le x \le \sqrt{a}$ (*radius*) (*semicircle*)

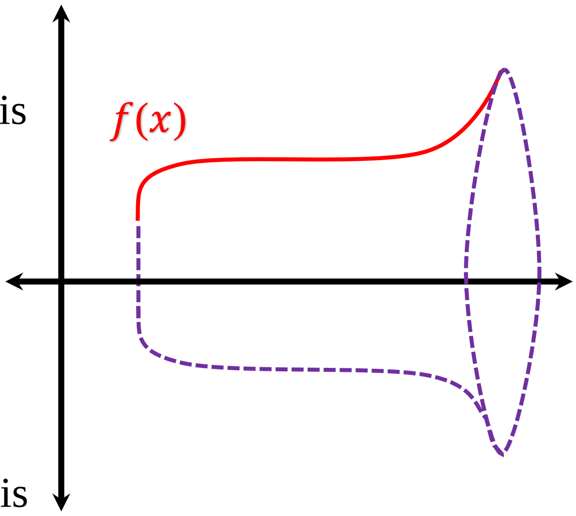
✓ $L = \pi r$



Surface

- $f(x)$ is *positive* and has a CTS derivative we define the surface curve $(\ y = f(x)\ ) \Longrightarrow (\ a \le x \le b\ )$ about the x-axis

$$S = \int_a^b 2\pi\, f(x) \sqrt{1 + (f'(x))^2}\ .dy$$

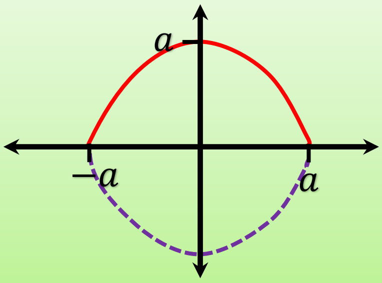


- $g(y)$ is *positive* and has a CTS derivative we define the surface curve $(\ x = g(y)\ ) \Longrightarrow (\ c \le y \le d\ )$ about the y-axis

$$S = \int_c^d 2\pi\, y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ .dy$$

Note:

- ✓ $f(x) = \sqrt{a - x^2}$
  $-\sqrt{a} \le x \le \sqrt{a}$ (*radius*) (كرة)
- ✓ $S = 4\pi r^2$



لما بجدي اوجد ($f(x) = \sqrt{a - x^2}$) :

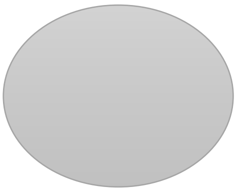
الطول : بيكون شكلها قوس يعني بطلع طول القوس

المساحة : بيكون شكلها كرة يعني بطلع مساحة الكرة

CHAPTER (10)

Curves defined by parametric equations

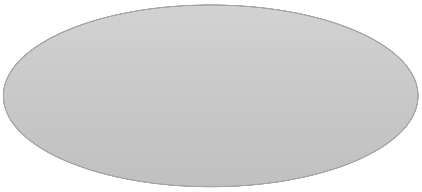
Circle equation



- $x^2 + y^2 = 1$
- ✓ Circle center = (0 , 0)
- ✓ Radius = 1

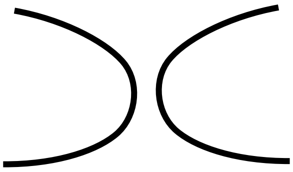
Ellipse equation

- $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$



Hyperbolic equation

- $x^2 - y^2 = 1$



Note :

- اذا جاء تربيع بالمعادلة فتكون (parabola)
- اذا جاء بدون تربيع أي انه خطي (Line)

Line segment

$(x_1 , y_1) \Longrightarrow (x_2 , y_2)$

- ✓ $x = x_1 + (x_2 - x_1)\, t$
- ✓ $y = y_1 + (y_2 - y_1)\, t$

$0 \leq t \leq 1$

Note :

$$\frac{d^2y}{dx^2}\begin{cases}\text{Concave up} \longrightarrow \dfrac{d^2y}{dx^2} > 0\\[2ex] \text{Concave down} \longrightarrow \dfrac{d^2y}{dx^2} < 0\end{cases}$$

Circle

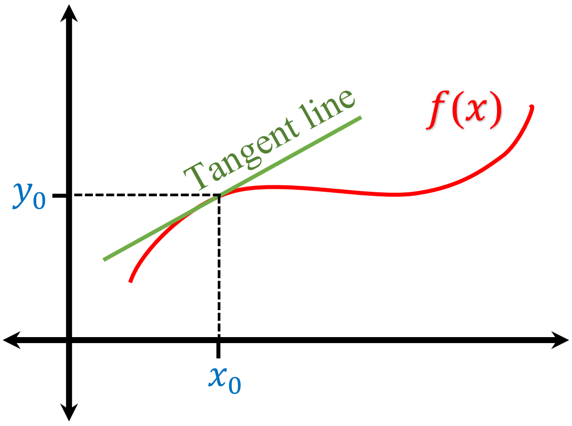
- Circle center (h , k)
- Radius = r

$$0 \le t \le 2\pi$$

- ✓ $x = h + r\cos t$
- ✓ $y = k + r\sin t$

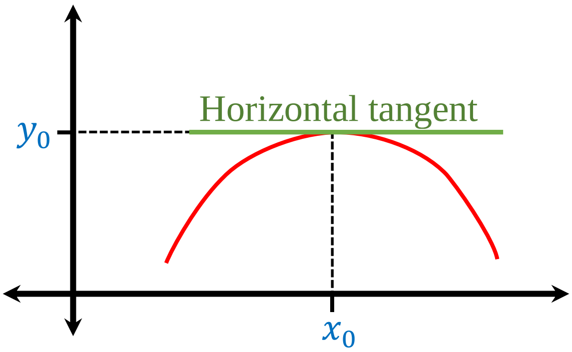
Equation of tangent Line

- m = slope = $\left.\dfrac{dy}{dx}\right|_{(x_0\,,\,y_0)}$
- $y - y_0 = m\,(x - x_0)$



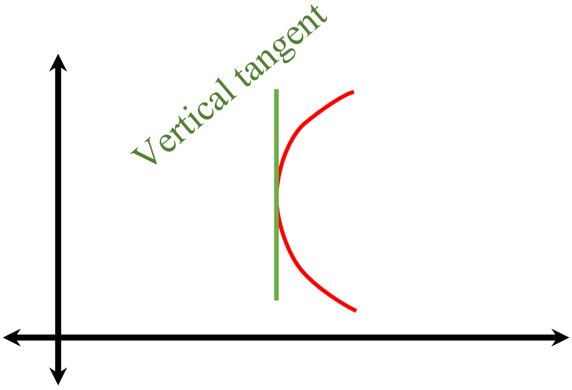
Curve has horizontal tangent

- $m = 0$
- $$\frac{dy}{dx} = \frac{\frac{dy}{dt} = 0}{\frac{dx}{dt} \ne 0}$$



Curve has vertical tangent

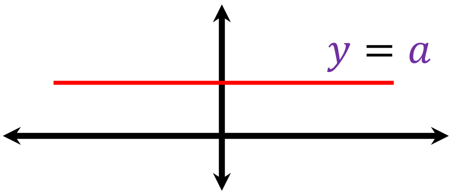
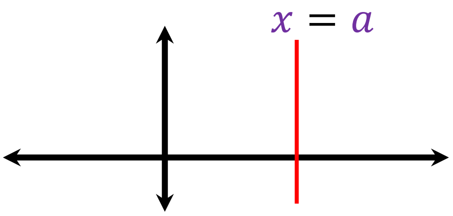
- $$\frac{dy}{dx} = \frac{\frac{dy}{dt} \ne 0}{\frac{dx}{dt} = 0}$$



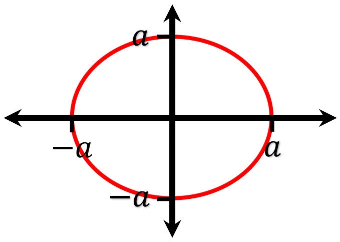
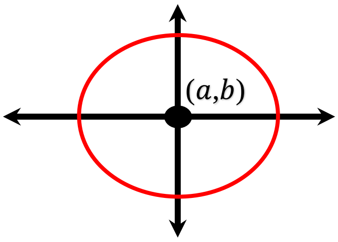
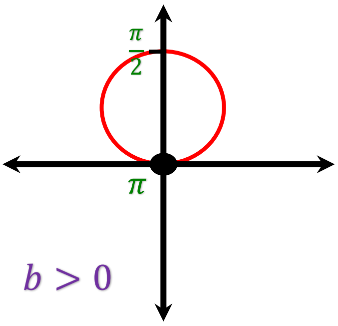
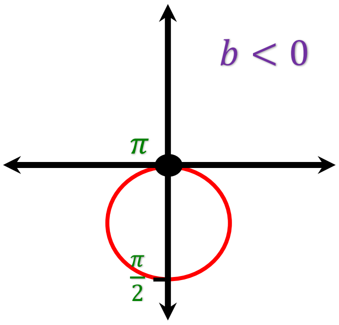
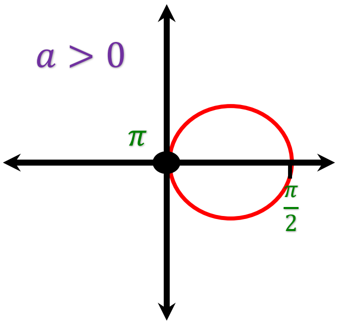
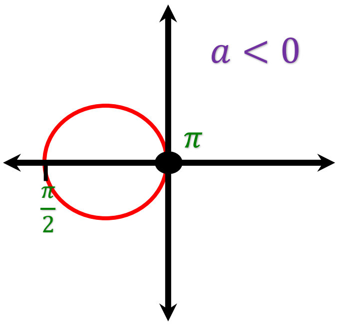
Polar curves

- Lines
- Circle
- Cardioid

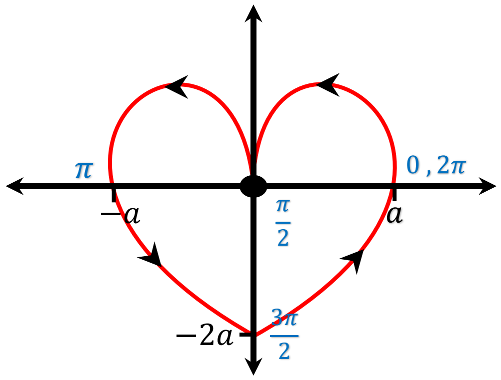
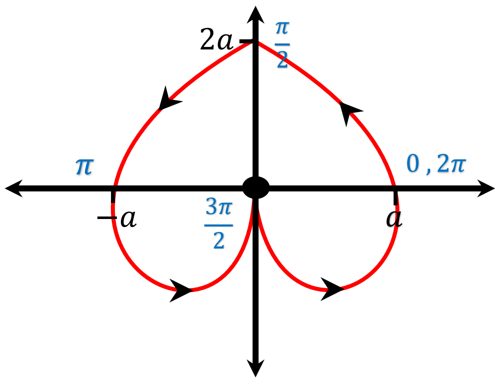
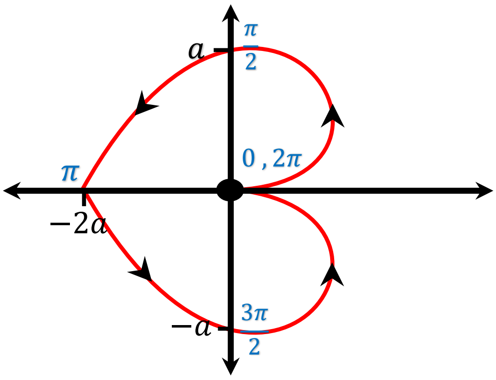
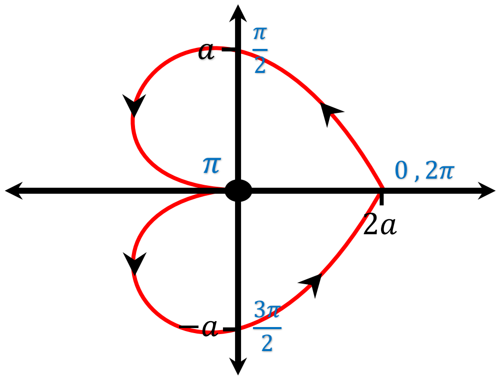
LINES

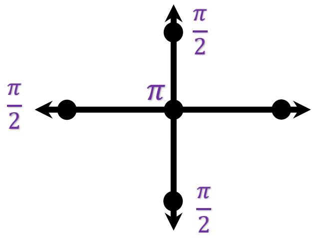
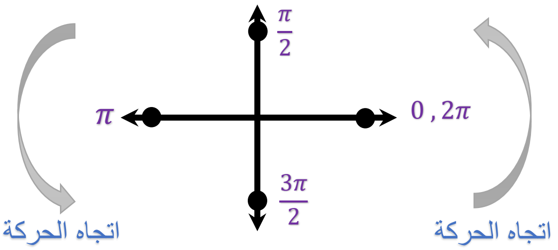
Lines	
Horzantal line	Vertical line
$r = y\ csc\theta$	$r = x\ sec\theta$
	

CIRCLE

circle	
$a^2 = x^2 + y^2$	$r = 2a\ cos\theta + 2b\ sin\theta$
Circle centre $(0{,}0)$ Radius $= \lvert a \rvert$	Circle centre $(a{,}b)$ Radius $= \sqrt{a^2 + b^2}$ $0 \leq \theta \leq \pi$
	
$r = 2b\ sin\theta$	$r = 2a\ cos\theta$
Circle centre $(0{,}b)$ Radius $= \lvert b \rvert$ $0 \leq \theta \leq \pi$	Circle centre $(a{,}0)$ Radius $= \lvert a \rvert$ $0 \leq \theta \leq \pi$
 	 

CARDIOID

Cardioid	
$r = a\,(1 - sin\theta)$	$r = a\,(1 + sin\theta)$
$0 \le \theta \le 2\pi$	$0 \le \theta \le 2\pi$
	
$r = a\,(1 - cos\theta)$	$r = a\,(1 + cos\theta)$
$0 \le \theta \le 2\pi$	$0 \le \theta \le 2\pi$
	

لتسهيل حفظ الزوايا	
circle	Cardioid
 دائما π بالمركز و $\frac{\pi}{2}$ لبرا	

CHAPTER (11)

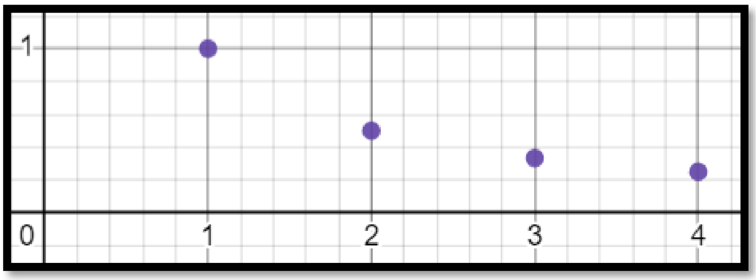
Sequences

Graph of sequences

- $a_n = \frac{1}{n}$

$a_1 = 1 \quad , \quad a_2 = \frac{1}{2}$

$a_3 = \frac{1}{3} \quad , \quad a_4 = \frac{1}{4}$



conv

$\lim_{n\to\infty} \frac{1}{n} = 0$

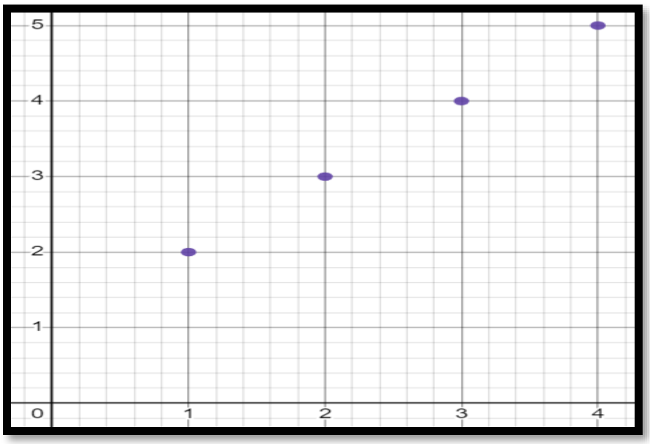
to

0

- $a_n = n + 1$

$a_1 = 2 \quad , \quad a_2 = 3$

$a_3 = 4 \quad , \quad a_4 = 5$



Div

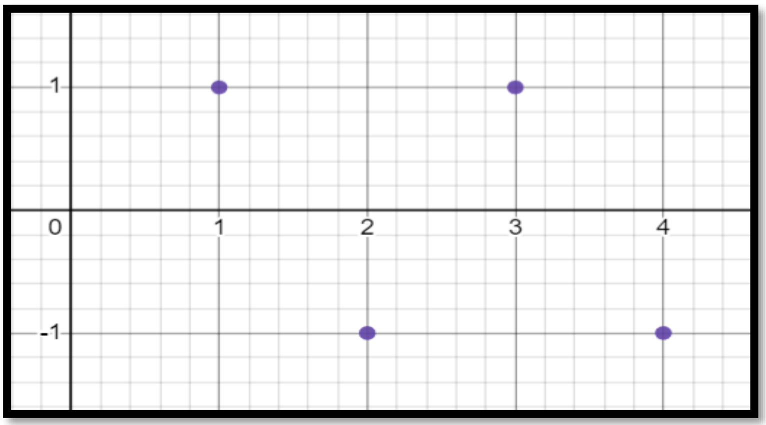
$\lim_{n\to\infty} n + 1 = \infty$

d.n.e

- $a_n = (-1)^{n+1}$

$a_1 = 1 \quad , \quad a_2 = -1$

$a_3 = 3 \quad , \quad a_4 = -1$



Div

$\lim_{n\to\infty} (-1)^{n+1} = \infty$

d.n.e

- Theorem (1)

If $\lim\limits_{n\to\infty}|a_n| = 0 \xrightarrow{\text{then}} \lim\limits_{n\to\infty} a_n = 0$ يجب ان تساوي 0 حتى تكون conv

- Theorem (2)

$$a_{2n} = \lim_{n\to\infty} a_n = L$$
$$a_{2n+1} = \lim_{n\to\infty} a_n = L$$

→ يجب ان يتساويا حتى تكون conv

Note :

$$\lim_{n\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}$$

❖ For what values of (r) is the *seq* (r^n) converged

$$r^n = \begin{cases} \text{Conv} , & -1 < r \le 1 \\ \text{Div} , & \text{other wise} \end{cases} \longrightarrow \begin{cases} -1 < r < 1 , & \text{conv to } 0 \\ r = 1 , & \text{conv to } 1 \end{cases}$$

$a_1 = 1$ $\quad a_{n+2} = a_n + a_{n+1}$

$a_2 = 1$ $\quad a_{n+2} = \{1,1,2,3,5,8,\ldots..$

$\pi \cong 3{,}14$ // $e \cong 2{,}7$

- Theorem (3)

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ such that ($a_n \le b_n \le c_n$)

If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L \xrightarrow[\text{then}]{} \lim\limits_{n\to\infty} b_n = L$

✓ Ex : $a_n = \dfrac{\sin(n)}{n}$

$$-1 \le \sin(n) \le 1$$

$$\frac{-1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}$$

ندخل $\lim\limits_{n\to\infty}$

$$0 \le \frac{\sin(n)}{n} \le 0$$

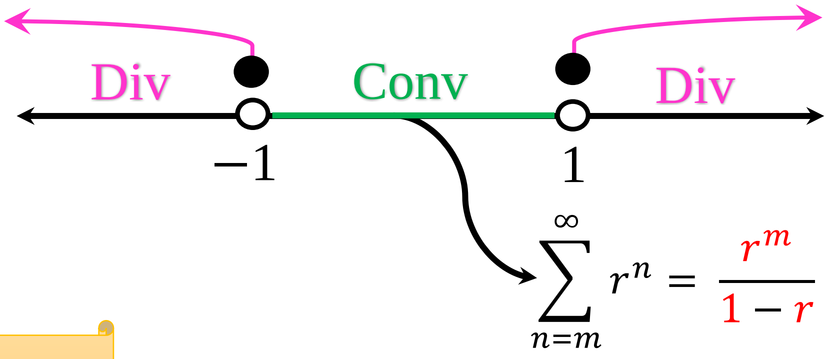
conv to 0

Series

الرسمات غير مطلوبة

Geometric series

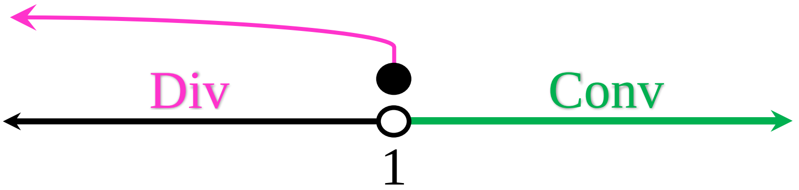
- $$\sum_{n=1}^{\infty} r^n = \begin{cases} \text{Conv} \Longrightarrow |r| < 1 \Longrightarrow -1 < r < 1 \\ \text{Div} \Longrightarrow |r| \geq 1 \Longrightarrow r \geq 1 \,,\, r \leq -1 \end{cases}$$



$$\sum_{n=m}^{\infty} r^n = \frac{r^m}{1-r}$$

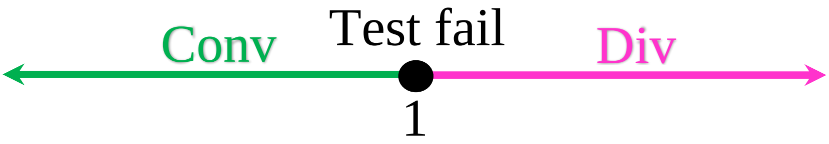
P - series

- $$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{Conv} \Longrightarrow p > 1 \\ \text{Div} \Longrightarrow p \leq 1 \end{cases}$$



Ratio test	Root test
$C = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$	$C = \lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert}$
لما اشوفهم بحل على طول على ratio test : $n!\,,\,5!\,,\,n^n\,,\,3^n$	لما اشوفه بحل على طول على root test : $(b_n)^n$

1) $C < 1 \Longrightarrow \sum a_n$ conv
2) $C > 1 \,,\, C = \infty \Longrightarrow \sum a_n$ div
3) $C = 1 \Longrightarrow$ test fail



Alternating series test

في حال ما نجح الاختبار على طول
بروح احل على Divergence test

- $\sum_{n=1}^{\infty} (-1)^n a_n \quad , \quad a_n > 0$

1) $\lim_{n\to\infty} a_n = 0$
2) a_n decreasing

$\Big\} \longrightarrow$ Then $\sum_{n=1}^{\infty} (-1)^n a_n \Longrightarrow$ conv

Divergence test

$$\lim_{n\to\infty} a_n \begin{cases} = 0 \begin{cases} \longrightarrow \text{Conv} \\ \longrightarrow \text{Div} \end{cases} \\ \neq 0 \Longrightarrow \text{Div} \end{cases}$$

Limit comparison test

- $C = \lim_{n\to\infty} \frac{a_n}{b_n}$

1) $C > 0$
2) $C \neq \infty$

❖ Then both series are conv or both series div

$\frac{\text{جذر}}{\text{كثير حدود}}$ ، $\frac{\text{كثير حدود}}{\text{كثير حدود}}$

Integral test

- $\sum_{n=1}^{\infty} a_n \quad , \quad f(n) = a_n$

$$f(n) \begin{cases} \longrightarrow \text{Positive } (+) \\ \longrightarrow \text{Continuous} \\ \longrightarrow \text{Decreasing} \end{cases} \Bigg\} \text{then} \begin{cases} \int_m^{\infty} f(x).dx \ \text{conv} \Longleftrightarrow \sum_{n=m}^{\infty} a_n \ \text{conv} \\ \int_m^{\infty} f(x).dx \ \text{div} \Longleftrightarrow \sum_{n=m}^{\infty} a_n \ \text{div} \end{cases}$$

Telescoping sum

- If $\lim\limits_{n\to\infty} s_n = L \xrightarrow[\text{then}]{} \sum\limits_{n=1}^{\infty} a_n = L$

❖ Ex: $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)}$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{A}{n} \quad \frac{B}{n+1} \Longrightarrow A = 1 \;,\; B = -1$$

كسور جزئية

$$\left.\begin{aligned} s_1 &= 1 + \frac{1}{2} \\ s_2 &= 1 + \frac{1}{3} \\ s_4 &= 1 + \frac{1}{4} \end{aligned}\right\} \Longrightarrow s_n = 1 + \frac{1}{n+1} \Longrightarrow \lim_{n\to\infty} 1 + \frac{1}{n+1} = 1$$

Algebraic properties of infinite series

- If $\sum a_n$ and $\sum b_n$ are conv then $\sum a_n + \sum b_n$ conv
- $\sum a_n \pm \sum b_n = \sum a_n \mp \sum b_n$
- If C is a *non-zero constant* , then :

1) If $\sum a_n$, conv $\longleftrightarrow \sum C\,a_n$, conv

2) If $\sum a_n$, div $\longleftrightarrow \sum C\,a_n$, div

الخلاصة :

اذا كانوا :

conv + conv = conv

conv + div = div

$$\text{Div} + \text{div} = \begin{cases} \text{conv} \\ \text{div} \end{cases}$$

$\sum a_n$	$\sum \lvert a_n \rvert$	
Conv	Conv	$\sum a_n$ Abs.conv
Conv	Div	$\sum a_n$ C.C
Div	Div	$\sum a_n$ Div

Theorem:

$$\sum a_n \ \underline{\text{Abs.conv}} \Longrightarrow \sum a_n \ \underline{\text{conv}}$$

Power series

Power series centered at a

- $$\sum_{n=0}^{\infty} c_n \, (x-a)^n = c_0 + c_1 \, (x-a) + \ c_2 \, (x-a)^2 + \ldots\ldots$$

	Radius of conv	Interval of conv
The series conv only when $x = a$	$R = 0$ $\lim\limits_{n\to\infty} \ldots = \infty$	$\{a\}$
The series conv for all x	$R = +\infty$ $\lim\limits_{n\to\infty} \ldots = 0$	$(-\infty, \infty)$
There is a positive number R such that series conv $\Longrightarrow \lvert x-a \rvert < R$ div $\Longrightarrow \lvert x-a \rvert > R$	R	$(a-R\,,\,a+R)$ $[a-R\,,\,a+R)$ $[a-R\,,\,a+R]$ $[a-R\,,\,a+R)$

❖ Theorem:

$$f(x) = \sum_{n=1}^{\infty} c_n \, x^n$$

بنشتغل على :

Ratio test , Root test

❖ Domain:

The set of all x at the series conv

يعني بده المجال لما تكون conv

Taylor and Maclaurin series

- Taylor : $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

- Maclaurin : $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

Maclaurian series

1) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ R = 1

2) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ R = ∞

3) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ R = ∞

4) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ R = ∞

5) $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ R = 1

6) $ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ R = 1