ElCoM
ELECTRICAL COMPUTER MECHATRONICS

تفاضل وتكامل 2

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للطالـب المبدع

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إرادة - ثقـة - تغيير

Subject Lec 1 Date No.

* Integration by substituation التكامل بالتعويض

- $\int f'(x) \cdot g(f(x))\, dx$, $u = f(x) \Rightarrow du = f'(x) \cdot dx$

e.g) $\int x^2 \cos(x^3)\, dx$

$x^3 = u$

$3x^2 dx = du$

$dx = \frac{du}{3x^2}$

$\Rightarrow I = \int \cancel{x^2} \cos(u) \cdot \frac{du}{3\cancel{x^2}}$

$= \frac{1}{3} \int \cos(u) \cdot du$

$= \frac{1}{3} \sin(u) + C = \frac{1}{3} \sin(x^3) + C$

e.g) $\int 3^{\sin x} \cdot \cos x\, dx$

$\sin x = u$

$\cos x\, dx = du$

$dx = \frac{du}{\cos x}$

$\Rightarrow I = \int 3^u \cancel{\cos x} \frac{du}{\cancel{\cos x}}$

$= \int 3^u du$

$= \frac{3^u}{\ln 3} + C = \frac{3^{\sin x}}{\ln 3} + C$

e.g) $\int x \cdot \sin(x^2)\, dx$

$x^2 = u$

$2x\, dx = du$

$dx = \frac{du}{2x}$

$\Rightarrow I = \int x \cdot \sin(u) \frac{du}{2x}$

$= \frac{1}{2} \int \sin(u)\, du$

$= -\frac{1}{2} \cos(u)\, du + C = -\frac{1}{2} \cos(x^2) + C$

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e.g) $\int 2x\sqrt{1+x^2}\,dx$

$1+x^2 = u$

$2x\,dx = du$

$dx = \frac{du}{2x}$

$\Rightarrow I = \int \cancel{2x}\sqrt{u}\,\frac{du}{\cancel{2x}}$

$= \int u^{\frac{1}{2}}\,du$

$= \frac{2u^{\frac{3}{2}}}{3} + C = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$

e.g) $\int \frac{4x}{\sqrt{2x^2+1}}\,dx$

$2x^2+1 = u$

$4x\,dx = du$

$dx = \frac{du}{4x}$

$\Rightarrow I = \int \frac{\cancel{4x}}{\sqrt{u}}\,\frac{du}{\cancel{4x}}$

$= \int (u)^{-\frac{1}{2}}\,du$

$= \frac{2(u)^{\frac{1}{2}}}{1} + C = 2\sqrt{2x^2+1} + C$

e.g) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

$\sqrt{x} = u$

$\frac{dx}{2\sqrt{x}} = du$

$dx = 2\sqrt{x}\,du$

$\Rightarrow I = \int \frac{\sin(u)}{\cancel{\sqrt{x}}}\,2\cancel{\sqrt{x}}\,du$

$= 2\int \sin(u)\,du$

$= -2\cos(u) + C = -2\cos(\sqrt{x}) + C$

e.g) $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,.\,\sin x\,dx$

$\cos x = u$

$-\sin x\,dx = du$

$dx = \frac{du}{-\sin x}$

$\Rightarrow I = \int_1^0 \sqrt{u}\,.\,\cancel{\sin x}\,\frac{du}{-\cancel{\sin x}}$

$= -\int_1^0 (u)^{\frac{1}{2}}$

$= -\frac{2}{3}\,u^{\frac{3}{2}}\Big|_0^1$

$= -\frac{2}{3}(1-0) = -\frac{2}{3}$

if $x = 0 \rightarrow u = \cos(0) = 1$

if $x = \frac{\pi}{2} \rightarrow u = \cos(\frac{\pi}{2}) = 0$

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e.g) $\int \frac{\sqrt{1+\tan x}}{\cos^2 x}\, dx$

$1+\tan x = u$

$\sec^2 x\, dx = du$

$dx = \frac{du}{\sec^2 x}$

$\rightarrow I = \int \sqrt{u}\, \frac{\cancel{\sec^2 x}\, du}{\cancel{\sec^2 x}}$

$= \int (u)^{\frac{1}{2}}\, du$

$= \frac{2}{3}(u)^{\frac{3}{2}} + c = \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + c$

e.g) $\int \frac{dx}{x(\ln x)^2}$

$\ln x = u$

$\frac{dx}{x} = du$

$dx = x\, du$

$I = \int \frac{\cancel{x}\, du}{\cancel{x}\,(u)^2}$

$= \int (u)^{-2}\, du$

$= \frac{(u)^{-1}}{-1} + c = \frac{-1}{\ln x} + c$

e.g) $\int \frac{\cos^2(\ln x)}{x}\, dx$

$\ln x = u$

$\frac{dx}{x} = du$

$dx = x\, du$

$I = \int \frac{\cos^2(u).\cancel{x}\,du}{\cancel{x}}$

$= \int \cos^2(u)\, du$

$= \int \frac{1}{2} + \frac{1}{2}\cos(2u)\, du$

$= \frac{1}{2}u + \frac{1}{2} \times \frac{\sin(2u)}{2} + c = \frac{1}{2}\ln x + \frac{1}{4}\sin(2\ln x) + c$

e.g) $\int_0^1 x\sqrt{1-x}\, dx$

$1-x = u \rightarrow x = 1-u$

$-dx = du$

$dx = -du$

$I = \int_1^0 (1-u)\sqrt{u}.-du$

$= -\int_1^0 u^{\frac{1}{2}} - u^{\frac{3}{2}}\, du$

$= -\left(\frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5}\right)\Big|_1^0 = -2\left((0-\frac{1}{3})-(0-\frac{1}{5})\right)$

if $x=0 \rightarrow u = 1-0 = 1$

$= -2\left(\frac{5\times 1}{5\times 3} - \frac{1\times 3}{5\times 3}\right) = -2 \times \frac{-2}{15} = \frac{4}{15}$

if $x=1 \rightarrow u = 1-1 = 0$

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e.g) $\int \frac{e^{-x}}{\sqrt{1-e^{-x}}}\,dx$

$1-e^{-x} = u$

$e^{-x}dx = du$

$dx = \frac{du}{e^{-x}}$

$dx = e^{x}\cdot du$

$I = \int \frac{\cancel{e^{-x}}\,\cancel{e^{x}}}{\sqrt{u}}\,du$

$= \int u^{-\frac{1}{2}}\,du$

$= \frac{2u^{\frac{1}{2}}}{1} + C = 2\sqrt{1-e^{-x}} + C$

e.g) $\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx$

$e^{-x} = u$

$-e^{-x}dx = du$

$dx = \frac{du}{-e^{-x}}$

$I = \int \frac{e^{-x}}{\sqrt{1-(e^{-x})^2}}\,dx$

$= \int \frac{\cancel{e^{-x}}}{\sqrt{1-u^2}}\,\frac{du}{-\cancel{e^{-x}}}$

$= -\int \frac{1}{\sqrt{1-u^2}}\,du$

$= -\sin^{-1}u + C = -\sin^{-1}(e^{-x}) + C$

e.g) $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$

$\ln x = u$

$\frac{dx}{x} = du$

$dx = x\,du$

$I = \int \frac{\cancel{x}\,du}{\cancel{x}\sqrt{1-u^2}}$

$= \int \frac{1}{\sqrt{1-u^2}}\,du$

$= \sin^{-1}(u) + C = \sin^{-1}(\ln x) + C$

e.g) $\int \frac{dx}{\sqrt{x}(1+x)}$

$\sqrt{x} = u \rightarrow u^2 = x$

$\frac{dx}{2\sqrt{x}} = du$

$dx = 2\sqrt{x}\,du$

$I = \int \frac{2\cancel{\sqrt{x}}\,du}{\cancel{\sqrt{x}}(1+x)}$

$= 2\int \frac{du}{1+u^2}$

$= 2\tan^{-1}(u) + C = 2\tan^{-1}(\sqrt{x}) + C$

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* Integration by parts التكامل بالأجزاء

$\triangle$ $(f(x)\cdot g(x))' = f(x)\cdot g'(x) + g(x)\cdot f'(x)$

$\int (f(x)\cdot g(x))'\,dx = \int f(x)\cdot g'(x)\,dx + \int g(x)\cdot f'(x)\,dx$

$f(x)\cdot g(x) = \int f(x)\cdot g'(x)\,dx + \int g(x)\cdot f'(x)\,dx$

$\therefore \int f(x)\cdot g'(x) = f(x)\cdot g(x) - \int g(x)\cdot f'(x)\,dx$ #

$\triangle$ $u = f(x) \Rightarrow \dfrac{du}{dx} = f'(x)$, $v = g(x) \Rightarrow \dfrac{dv}{dx} = g'(x)$

$\therefore \int u\,dv = u.v - \int v\,du$

$$\begin{array}{cc} u & dv \\ \downarrow & \downarrow \\ du & v \end{array} \quad \oplus,\ \ominus$$

e.g) $\int x \ln x\,dx$

$$\begin{array}{cc} u & dv \\ \ln x & x \\ \downarrow & \downarrow \\ \frac{1}{x} & \frac{x^2}{2} \end{array} \quad \oplus,\ \ominus$$

$\Rightarrow I = \frac{x^2}{2}\ln x - \int \frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{2}}}{2}\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c$

e.g) $\int (\ln x)^2\,dx$

1) $$\begin{array}{cc} u & dv \\ (\ln x)^2 & 1 \\ \downarrow & \downarrow \\ \frac{2\ln x}{x} & x \end{array} \quad \oplus,\ \ominus$$

2) $$\begin{array}{cc} u & dv \\ \ln x & 1 \\ \downarrow & \downarrow \\ \frac{1}{x} & x \end{array} \quad \oplus,\ \ominus$$

$\Rightarrow I = x(\ln x)^2 - \int \frac{2\ln x}{\cancel{x}}\cdot\cancel{x}\,dx$

$= x(\ln x)^2 - 2\int \ln x\,dx$

$= x(\ln x)^2 - 2\left[x\ln x - \int \frac{1}{x}\cdot x\,dx\right]$

$= x\ln x^2 - 2\,[x\ln x - x] + c$

مهم تكامل Lnx

• Note that:

* $(\ln x)^n \neq \ln x^n$

* $\int \ln x^n = n\int \ln x\,dx$

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e.g) $\int \ln x^2 \, dx = 2\int \ln x \, dx$

$= 2[x\ln x - x] + c$

$= 2x\ln x - 2x + c$

e.g) $\int \sin^{-1}x \, dx$

$\Rightarrow I = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}} dx$ $\rightarrow$ $u = 1-x^2$, $du = -2x\,dx$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{u}} \cdot \frac{du}{-2x}$

$= x\sin^{-1}x + \frac{1}{2}\int \frac{1}{\sqrt{u}} du$

$= x\sin^{-1}x + \frac{1}{2}\int u^{-\frac{1}{2}} du$

$= x\sin^{-1}x + \frac{1}{2} * \frac{2}{1} u^{\frac{1}{2}} + c = x\sin^{-1}x + \sqrt{1-x^2} + c$

u	dv
$\sin^{-1}x$	1
$\frac{1}{\sqrt{1-x^2}}$	x

e.g) $\int x \tan^{-1}x \, dx$

$\Rightarrow I = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2} dx$ $\boxed{\pm 1}$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[\int \frac{x^2+1-1}{x^2+1} dx\right.$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[\int \frac{\cancel{x^2+1}}{\cancel{x^2+1}} - \frac{1}{x^2+1} dx\right]$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[x - \tan^{-1}x\right] + c$

u	dv
$\tan^{-1}x$	x
$\frac{1}{1+x^2}$	$\frac{x^2}{2}$

e.g) $\int x^2 e^x \, dx$

u	dv
x^2	e^x
$2x$	e^x
2	e^x
0	e^x

$\Rightarrow I = x^2e^x - 2xe^x + 2e^x + c$

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e.g) $\int 2x^3 \cos x \, dx$

u		dv
$2x^3$		$\cos x$
$6x^2$	$+$	$\sin x$
$12x$	$-$	$-\cos x$
12	$+$	$-\sin x$
0	$-$	$\cos x$

$\Rightarrow I = 2x^3 \sin x + 6x^2 \cos x$

$-12x \sin x - 12 \cos x + c$

e.g) $\int x \sec^2 x \tan x \, dx$

u		dv
x		$\sec^2 \tan x$
1	$+$	$\frac{1}{2} \tan^2 x$
0	$-$	$\frac{1}{2} \tan x - \frac{x}{2}$

• $\int \sec^2 \tan x \, dx \longrightarrow u = \tan x$, $du = \sec^2 x \, dx$, $dx = \frac{du}{\sec^2 x}$

$= u \, du = \frac{1}{2} \tan^2 x + c$

• $\int \frac{1}{2} \tan^2 x = \frac{1}{2} \int \sec^2 x - 1 = \frac{1}{2} [\tan x - x] + c$

$\Rightarrow I = \frac{1}{2} x \tan^2 x - \frac{1}{2} \tan x + \frac{x}{2} + C$

e.g) $\int e^{\sqrt{x}} \, dx$

$\rightarrow \sqrt{x} = u$

$\frac{dx}{2\sqrt{x}} = du$

$dx = 2\sqrt{x} \, du$

u		dv
$2u$		e^u
2	$+$	e^u
0	$-$	e^u

$\Rightarrow I = \int e^u \cdot 2\sqrt{x} \, du$

$= \int 2u \cdot e^u \, du$

$= 2u e^u - 2e^u + c$

$= 2\sqrt{x} \, e^{\sqrt{x}} - 2e^{\sqrt{x}} + c$

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e.g) $\int e^x \sin x \, dx$ → أجزاء دوري

$$\begin{array}{ccc} u & & dv \\ e^x & \oplus & \sin x \\ e^x & \ominus & -\cos x \\ e^x & \oplus & -\sin x \end{array}$$

بكامل و بشتق لحد ما يصير الأول نفس الأخير

$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

$2\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$

$\int e^x \sin x \, dx = \frac{1}{2}\left[-e^x \cos x + e^x \sin x\right] + c$

e.g) $\int \sin(\ln x) \, dx$ → $u = \ln x$ → $e^u = e^{\ln x}$ → $\frac{u}{e^u}$　$\frac{dv}{\sin x}$

$du = \frac{dx}{x}$　　$e^u = x$

$x\,du = dx$

$I = \int \sin(u)\, x \, du$

$= \int e^u \sin(u) \, du$ → نفس حل الي فوق بس بعوض u

$= \frac{1}{2}\left[-e^u \cos u + e^u \sin u\right] + c$

$= \frac{1}{2} e^{\ln x}\left[-\cos(\ln x) + \sin(\ln x)\right] + c$

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* Integration of Rational Functions by Partial Fractions

التكامل بالكسور الجزئية

$\int \frac{Poly}{Poly} = \int \frac{f(x)}{g(x)}$

Case I : if $\deg f(x) \geq \deg g(x) \Rightarrow$ Long devision

e.g) $\int \frac{x^3+2x}{x-1}\,dx$

$$\begin{array}{r} x^2+x+3 \\ x-1 \,\big)\, x^3+2x \\ \ominus x^3 \oplus x^2 \\ \hline x^2+2x \\ \ominus x^2 \oplus x \\ \hline 3x \\ \ominus 3x \oplus 3 \\ \hline +3 \end{array}$$

$= \int$ ناتج $+ \int \frac{\text{باقي}}{\text{المقسوم عليه}}$

$= \int x^2+x+3 + \int \frac{3}{x-1}$

$= \frac{x^3}{3} + \frac{x^2}{2} + 3x + 3\ln|x-1| + c$

Case II : if $\deg f(x) < \deg g(x)$

① $\int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$

eg) $\int \frac{x^2}{x^3+1}dx = \frac{1}{3}\int \frac{3x^2}{x^3+1} = \frac{1}{3}\ln|x^3+1| + c$

② $\int \frac{f(x)}{g(x)} = \int \frac{f(x)}{(a_1x+b_1)(a_2x+b_2)(a_3x+b_3)\dots} = \int \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \frac{C}{a_3x+b_3} \dots$

قابل للتحليل

eg) $\int \frac{5}{x^2-1}dx = \int \frac{5}{(x-1)(x+1)}dx = \int \frac{A}{x-1} + \frac{B}{x+1}\,dx$

أصفار المقامات

Now, $5 = A(x+1) + B(x-1)$

if $x=-1 \rightarrow 5 = -2B \rightarrow \boxed{B = \frac{-5}{2}}$

if $x = 1 \rightarrow 5 = 2A \rightarrow \boxed{A = \frac{5}{2}}$

$I = \int \frac{5/2}{x-1} + \frac{-5/2}{x+1}\,dx$

$= \frac{5}{2}\ln|x-1| - \frac{5}{2}\ln|x+1| + c = \ln|x-1|^{\frac{5}{2}} - \ln|x+1|^{\frac{5}{2}} + c = \ln\left|\frac{x+1}{x-1}\right|^{\frac{5}{2}} + c$

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5 = 3 + 2 معاملات

③ $\int \frac{F(x)}{(a_1x+b_1)^{2}(a_2x+b_2)^{3}\ldots} = \int \frac{A}{(a_1x+b_1)} + \frac{B}{(a_1x+b_1)^2} + \frac{C}{(a_2x+b_2)} + \frac{D}{(a_2x+b_2)^2} + \frac{E}{(a_2x+b_2)^3}$

قابل للتحليل لاقترانات خطية مكررة

e.g) find the form of the partial fraction decomposition of:

$\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

$\frac{x+1}{x^3+3x^2} = \frac{x+1}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$

$\frac{3}{(x^2+2x+1)^2(x-1)^3} = \frac{3}{((x+1)^2)^2(x-1)^3} = \frac{3}{(x+1)^4(x-1)^3}$

$= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^4} + \frac{E}{x-1} + \frac{F}{(x-1)^2} + \frac{g}{(x-1)^3}$

④ $\int \frac{F(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\ldots} = \int \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{a_2x^2+b_2x+c_2}$

غير قابل للتحليل لاقترانات تربيعية غير قابلة للتحليل

لا يتحلل $\rightarrow D = b^2 - 4ac = \ominus$

⑤ $\int \frac{F(x)}{(a_1x^2+b_1x+c_1)^2(a_2x^2+b_2x+c_2)^3\ldots} = \int \frac{A_1x+B_1}{(a_1x^2+b_1x+c_1)} + \frac{A_2x+B_2}{(a_1x^2+b_1x+c_1)^2}$

$+ \frac{A_3x+B_3}{(a_2x^2+b_2x+c_2)} + \frac{A_4x+B_4}{(a_2x^2+b_2x+c_2)^2} + \frac{A_5x+B_5}{(a_2x^2+b_2x+c_2)^3} + \ldots$

غير قابل للتحليل لاقترانات تربيعية غير قابلة للتحليل لكن مكررة

e.g) $\int \frac{\cos x}{\sin^3 x + \sin x}\, dx$

$u = \sin x$
$du = \cos x\, dx$
$dx = \frac{du}{\cos x}$

$I = \int \frac{\cancel{\cos x}}{u^3+u} \cdot \frac{du}{\cancel{\cos x}} = \int \frac{du}{u^3+u} = \int \frac{du}{u(u^2+1)} = \int \frac{A}{u} + \frac{Bu+C}{u^2+1}$

$\Rightarrow 1 = A(u^2+1) + u(Bu+C)$

$* u = 0 \quad \boxed{A = 1}$

$* u = 1 \quad 1 = 2A + B + C \rightarrow 1 = 2 + B + C \rightarrow \boxed{B + C = -1}$

$* u = -1 \quad 1 = 2A + B - C \rightarrow 1 = 2 + B - C \rightarrow \boxed{B - C = -1}$

$\rightarrow B + C = -1$
$B - C = -1$
$2B = -2 \rightarrow \boxed{B = -1} \rightarrow \boxed{C = 0}$

$I = \int \frac{1}{u} - \frac{u}{u^2+1}$

$= \ln|u| - \frac{1}{2}\ln|u^2+1| + C$

$= \ln|\sin x| - \frac{1}{2}\ln|\sin^2 x + 1| + C$

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e.g) $\displaystyle\int \frac{2x+4}{x^3-2x^2}\,dx = \int \frac{2x+4}{x^2(x-2)}\,dx = \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$

$\rightarrow 2x+4 = Ax(x-2) + B(x-2) + C(x^2)$

- $x=0 \rightarrow \boxed{B=-2}$
- $x=2 \rightarrow \boxed{C=2}$
- $x=1 \rightarrow \boxed{A=-2}$

$\rightarrow I = \displaystyle\int \frac{-2}{x} - \frac{2}{x^2} + \frac{2}{x-2}\,dx$

$= -2\ln|x| - \frac{2}{x} + 2\ln|x-2| + C$

- Note that

$\displaystyle\int \frac{-2}{x^2} = -2\int x^{-2}$

$-\frac{2x^{-1}}{-1} + c = \frac{2}{x} + c$

e.g) Find the form of the partial fraction decomposition:

① $\dfrac{x+1}{x^3+4x} = \dfrac{x+1}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$

② $\dfrac{1}{x^2-9} = \dfrac{1}{(x-3)(x+3)} = \dfrac{A}{(x-3)} + \dfrac{B}{(x+3)}$

③ $\dfrac{x^2+1}{x^3(x+1)(x^2+1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{d}{x+1} + \dfrac{ex+f}{x^2+1} + \dfrac{gx+h}{(x^2+1)^2}$

④ $\dfrac{x+5}{(x^2+x+1)^2(x-2)(x^2-1)^2} = \dfrac{Ax+B}{(x^2+x+1)} + \dfrac{Cx+D}{(x^2+x+1)^2} + \dfrac{F}{x-2} + \dfrac{g}{x-1} + \dfrac{h}{(x-1)^2} + \dfrac{I}{x+1} + \dfrac{J}{(x+1)^2}$

$((x-1)(x+1))^2$

$(x-1)^2(x+1)^2 =$

⑤ $\dfrac{1}{x^4+4x^2+3} = \dfrac{1}{(x^2+3)(x^2+1)} = \dfrac{Ax+B}{x^2+3} + \dfrac{Cx+D}{x^2+1}$

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Trigonometric Integrals

- $\int \sin^n x \, dx = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

- $\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$

- to solve $\int \sin^m x \cos^n x \, dx$:-

△ if n odd:

① split off a factor of $\cos x$

② apply the identity $\cos^2 x = 1 - \sin^2 x$

③ make the substitution $u = \cancel{\cos x}\ \sin x$

△ if m odd:

① split off a factor of $\sin x$

② apply the identity $\sin^2 x = 1 - \cos^2 x$

③ make the substitution $u = \cos x$

△ if m & n even:

Use the identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

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e.g) evaluate the following integrals

① $\int \sin^2 x \, dx = \int \frac{1}{2}\left[1 - \cos 2x\right] dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C$

② $\int \cos^2 x \, dx = \frac{1}{2} \cos x \sin x + \frac{1}{2}\int 1 \, dx = \frac{\sin 2x}{2} + \frac{x}{2} + C$

③ $\int \cos^3(2x) \, dx =$

Let $u = 2x \Rightarrow du = 2\,dx$

$= \frac{1}{2}\int \cos^3 u \, du = \frac{1}{2}\left[\frac{1}{3}\cos^2 u \sin u + \frac{2}{3}\int \cos u \, du\right]$

$= \frac{1}{2}\left[\frac{1}{3}\cos^2 u \sin u + \frac{2}{3}\sin u\right] + C$

$= \frac{1}{6}\cos^2(2x)\sin(2x) + \frac{1}{3}\sin(2x) + C$

④ $\int \sin^5 x \, dx = \frac{-1}{5}\cos x \sin^4 x + \frac{4}{5}\int \sin^3 x \, dx$

$= \frac{-1}{5}\cos x \sin^4 x + \frac{4}{5}\left[\frac{-1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin x \, dx\right]$

$= \frac{-1}{5}\cos x \sin^4 x + \frac{4}{5}\left[\frac{-1}{3}\cos x \sin^2 x + \frac{2}{3}(-\cos x)\right] + C$

⑤ $\int \sin^3 x \sqrt{\cos x} \, dx$; $\cos x = u \Rightarrow -\sin x \, dx = du$

$= -\int (1 - u^2)\sqrt{u} \, du$

$= -\left[\frac{2}{3}(\cos x)^{3/2} - \frac{2}{7}(\cos x)^{7/2}\right] + C$

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⑥ $\int \cos^4 x \sin x \, dx \quad ; \quad u = \cos x \Rightarrow du = -\sin x \, dx$

$= -\int u^4 \, du = -\dfrac{(\cos x)^5}{5} + C$

⑦ $\int \sin^{3/2} x \cos^3 x \, dx = \int \sin^{3/2} x \cos x \cos^2 x \, dx$

$= \int \sin^{3/2} x \cos x [1 - \sin^2 x] \, dx \quad , \quad \boxed{\sin x = u}$

$= \int u^{3/2} [1 - u^2] \, du = \dfrac{2}{5} \sin^{5/2} x - \dfrac{2}{9} \sin^{9/2} x + C$

⑧ $\int \sin^2(2x) \cos^3(2x) \, dx \quad ; \quad 2x = u \Rightarrow 2dx = du$

$= \dfrac{1}{2} \int \sin^2 u \cos^3 u \, du = \dfrac{1}{2} \int \cos^2 u \cdot \cos u \cdot \sin^2 u \, du$

$= \dfrac{1}{2} \int (1 - \sin^2 u) \cos u \sin^2 u \, du \quad ; \quad \boxed{z = \sin u}$:
:

⑨ $\int \sin^3 x \cos^4 x \, dx$

$= \int \sin x \sin^2 x \cos^4 x \, dx$

$= \int \sin x (1 - \cos^2 x) \cos^4 x \, dx \quad , \quad \boxed{u = \cos x}$

$= \int \sin x [\cos^4 x - \cos^6 x] \, dx$

$= \int u^6 - u^4 \, du = \dfrac{\cos^7 x}{x} - \dfrac{\cos^5 x}{5} + C$

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(10) $\int \sin^3 x \cos^3 x \, dx = \int \sin x (1-\cos^2 x) \cos^3 x \, dx$

$= \int \sin x \left[\cos^3 x - \cos^5 x\right] dx$

$u = \cos x$;
⋮

(11) $\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}(1-\cos 2x) * \frac{1}{2}(1+\cos 2x) \, dx$

$= \frac{1}{4} \int 1 - \cos^2(2x) \, dx$

$= \frac{1}{4} \int 1 - \frac{1}{2}\left[1 + \cos 4x\right] dx$;
⋮

or $\int (\sin x \cos x)^2 = \int \left(\frac{\sin 2x}{2}\right)^2 dx$

$= \frac{1}{4} \int \frac{1}{2}(1 - \cos 4x) \, dx$

$= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + C$

(12) $\int \sqrt{\sin x} \, \cos^5 x \, dx$; $\boxed{u = \sin x}$

$= \int \sqrt{u} \, (1-u^2)^2 \, du$

$= \frac{2}{3}(\sin x)^{3/2} - \frac{4}{7}(\sin x)^{7/2} + \frac{2}{11}(\sin x)^{11/2} + C$

H.w (1) $\int \sin^7 x \, dx$

(2) $\int \sin^3(2x) \cos^3(2x) \, dx$

(3) $\int \sin^2 x \cos^4 x \, dx$

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$$= -\frac{1}{2}\left[\sin(ax - bx) + \sin(ax - bx)\right]$$

- How to integrate :-

① $\int \sin(\alpha x)\cos(\beta x)\,dx = \frac{1}{2}\int \left[\sin(\alpha x - \beta x) + \sin(\alpha x + \beta x)\right] dx$

② $\int \sin(\alpha x)\sin(\beta x)\,dx = \frac{1}{2}\int \left[\cos(\alpha x - \beta x) - \cos(\alpha x + \beta x)\right] dx$

③ $\int \cos(\alpha x)\cos(\beta x)\,dx = \frac{1}{2}\int \left[\cos(\alpha x - \beta x) + \cos(\alpha x + \beta x)\right] dx$

④ $\int \cos(\alpha x)\sin(\beta x)\,dx = \frac{1}{2}\int \left[\sin(\alpha x + \beta x) - \sin(\alpha x - \beta x)\right] dx$

e.g. $\int \sin(7x)\cos(3x)\,dx$

$$= \frac{1}{2}\int \left(\sin(4x) + \sin(10x)\right) dx = \frac{1}{2}\left[\frac{\cos(4x)}{-4} - \frac{\cos(10x)}{10}\right] + C$$

e.g. $\int \sin\left(\frac{x}{2}\right)\cos x\,dx$

$$= \frac{1}{2}\int \left[\sin\left(\frac{-x}{2}\right) + \sin\left(\frac{3x}{2}\right)\right] dx$$

$$= \frac{1}{2}\left[2\cos\left(\frac{x}{2}\right) - \frac{2}{3}\cos\left(\frac{3x}{2}\right)\right] + C$$

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- $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$

- $\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$

- to integrate $\int \tan^m x \sec^n x \, dx$

① if n even :
① split off a factor of $\sec^2 x$
② apply the identity $\sec^2 x = \tan^2 x + 1$
③ make the substitution $u = \tan x$

② if m odd :
① split off a factor of $\sec x \tan x$
② apply the identity $\tan^2 x = \sec^2 x - 1$
③ make the substitution $u = \sec x$

③ if m even , n odd
Use the identities to reduce the integral to powers of $\sec x$ alone , then use the reduction formula for powers of $\sec x$

- to integrate $\int \cot^m x \csc^m x \, dx$

Use $\cot^2 x = \csc^2 x - 1$

مستر/ تفاعل

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① $\int \tan x \, dx = -\ln|\cos x| + C$

② $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + C$

③ $\int \sec^2 x \, dx = \tan x + C$

④ $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

⑤ $\int \sec^3 x \, dx = \int \sec^2 x \sec x \, dx$

$\underline{u}$		$\underline{dv}$
$\sec x$	$\xrightarrow{+}$	$\sec^2 x$
$\downarrow$		$\downarrow$
$\sec x \tan x$	$\xrightarrow{-\int}$	$\tan x$

$= \sec x \tan x - \int \sec x \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$

$= \sec x \tan x - \int \sec^3 x + \int \sec x \, dx$

$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$

$\therefore \int \sec^3 x \, dx = \frac{1}{2} \left[\sec x \tan x + \ln|\sec x + \tan x| \right] + C$

⑥ $\int \tan^4 \left(\frac{x}{2} \right) dx \quad ; \quad u = x/2$

$= 2 \int \tan^4 u \, du = 2 \left[\frac{\tan^3 u}{3} - \int \tan^2 u \right] = 2 \left[\frac{\tan^3 (x/2)}{3} - \tan(x/2) + x/2 \right] + C$

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⑦ $\int_0^{\pi/4} \sec^3 x \, dx$

$= \frac{\sec x \tan x}{2} \Big|_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \sec x \, dx$

$= \frac{\sqrt{2}}{2} + \frac{1}{2} \left[\ln|\sec x + \tan x| \Big|_0^{\pi/4} \right]$

$= \frac{\sqrt{2}}{2} + \frac{1}{2} \ln(\sqrt{2} + 1)$

⑧ $\int \sec^4(2x) \tan^6(2x) \, dx$

$2x = u \Rightarrow 2dx = du$

$= \frac{1}{2} \int \sec^4(u) \tan^6(u) \, du$

$= \frac{1}{2} \int \sec^2(u) \cdot \sec^2 u \tan^6 u \, du$

$= \frac{1}{2} \int \sec^2 u \left[\tan^2 u + 1 \right] \tan^6 u \, du$

$v = \tan u \Rightarrow dv = \sec^2 u \, du$

$= \frac{1}{2} \int (\tan^2 u + 1) \tan^6 u \, du$

$= \frac{1}{2} \int (v^2 + 1) v^6 \, dv$

$= \frac{1}{2} \left[\int (v^8 + v^6) \, dv \right.$

$= \frac{1}{2} \left[\frac{(\tan 2x)^9}{9} + \frac{(\tan 2x)^7}{7} \right] + C$

مسعد/ أناناس

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لمن تكون عنا مطلوبين
آكترس أمر متعرض للمضاعف
المشترك الأصغر بيطلع
المقرب الأصلي

⑨ $\int \tan^5 x \sec^4 x \, dx$

$= \int \tan^5 x (\tan^2 x + 1) \sec^2 x \, dx$

$u = \tan x$;

(or) $\int \tan x \sec x (\sec^2 x - 1) \sec^3 x \, dx$

$u = \sec x$ ⋮

⑩ $\int \sqrt{\tan x} \sec^4 x \, dx$

$= \int \sqrt{\tan x} \sec^2 x (\tan^2 x + 1)$

$u = \tan x$

$= \int \sqrt{u} (u^2 + 1) \, du$ ⋮

⑪ $\int \cot^3 x \csc x \, dx$

$= \int \cot^2 x \cot x \csc x \, dx$

$= \int (\csc^2 x - 1) \csc x \cot x \, dx$

$u = \csc x \Rightarrow du = -\csc x \cot x \, dx$

$= -\int (u^2 - 1) \, du$

⋮

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(12) $\int \tan^2 x \sec x \, dx$

$= \int (\sec^2 x - 1) \sec x \, dx$

$= \int \sec^3 x - \sec x \, dx$

$= \dfrac{\sec x \tan x}{2} + \dfrac{1}{2} \ln |\sec x \tan x| - \ln |\sec x + \tan x| + C$

$= \dfrac{\sec x \tan x}{2} - \dfrac{1}{2} \ln |\sec x \tan x| + C$

تصميم/ آلاء فضل

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Subject Lec 5 Date No.

* Trigonometric Substitutions

expression	Trig. sub	θ
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 \leq \theta \leq \frac{\pi}{2}$ if $x \geq a$
		$\frac{\pi}{2} \leq \theta \leq \pi$ if $x \leq a$

- Note that: $\sqrt{a^2-x^2}$, $x\ a\sin\theta$

- $\sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$

$= a\sqrt{\cos^2\theta}$

$= a\cos\theta$ since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

- $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta}$

$= a\sqrt{1+\tan^2\theta}$

$= a\sec\theta$

- $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$

$= a\sqrt{\sec^2\theta - 1}$

$= a\tan\theta$

- $\sqrt{a^2-b^2x^2} \longrightarrow bx = a\sin\theta$
- $\sqrt{a^2+b^2x^2} \longrightarrow bx = a\tan\theta$
- $\sqrt{b^2x^2-a^2} \longrightarrow bx = a\sec\theta$

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Subject Date No.

Ex. write the Trigonometric substitution :-

1) $\int \sqrt{4+x^2} \longrightarrow x = 2\tan\theta$

2) $\int \sqrt{x^2-5} \longrightarrow x = \sqrt{5}\sec\theta$

3) $\int \sqrt{1+(x+2)^2} \longrightarrow x+2 = \tan\theta$

4) $\int \frac{dx}{(9-x^2)^{\frac{3}{2}}} \longrightarrow x = 3\sin\theta$

5) $\int \frac{dx}{(2x^2-3)^{\frac{5}{2}}} \longrightarrow \sqrt{2}\,x = \sqrt{3}\sec\theta$

Ex. $\int \frac{\sqrt{x^2-16}}{2x}\,dx$

$\boxed{x = 4\sec\theta} \Rightarrow \boxed{dx = 4\sec\theta\tan\theta\, d\theta}$

$\boxed{\sec\theta = \frac{x}{4}}$

$\sec^{-1}\left(\frac{x}{4}\right) = \sec^{-1}(\sec\theta)$

$\boxed{\sec^{-1}\left(\frac{x}{4}\right) = \theta}$

$\Rightarrow I = \int \frac{\sqrt{16\sec^2\theta-16}}{2(4\sec\theta)} \cdot 4\sec\theta\tan\theta\, d\theta$

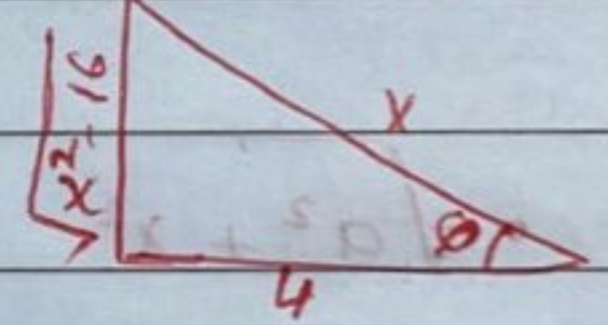
$= \int \frac{\sqrt{16(\sec^2\theta-1)}}{2} \cdot \tan\theta\, d\theta$

$= \int \frac{4}{2}\sqrt{\sec\theta-1} \cdot \tan\theta\, d\theta$

$= \int \frac{4}{2}\tan\theta \cdot \tan\theta\, d\theta$

$= 2\int \tan^2\theta\, d\theta$

$= 2\int \sec^2\theta - 1\, d\theta$



مقابل / مجاور من △ $= 2[\tan\theta - \theta] + C \longrightarrow$ نرجع للفرضية

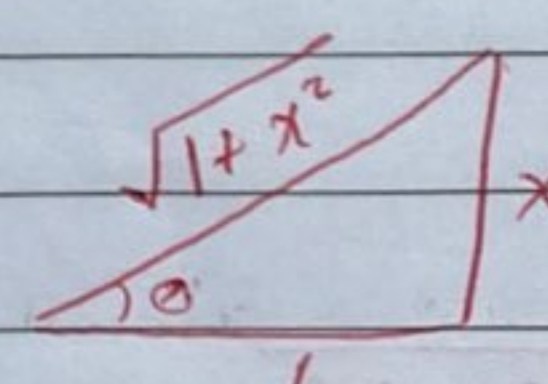
$\Longrightarrow = 2\left[\frac{\sqrt{x^2-16}}{4} - \sec^{-1}\left(\frac{x}{4}\right)\right] + C$

Ex. $\int \frac{\sqrt{x^2+1}}{x}\,dx$

$\boxed{x = \tan\theta} \Rightarrow \boxed{dx = \sec^2\theta\, d\theta}$

$= \int \frac{\sqrt{\tan^2\theta+1}}{\tan\theta} \cdot \sec^2\theta\, d\theta$

$= \int \frac{\sqrt{\sec^2\theta}}{\tan\theta} \cdot \sec^2\theta\, d\theta$



$= \int \frac{\sec\theta}{\tan\theta} \cdot (1+\tan^2\theta)\, d\theta$

$= \int \frac{\sec\theta}{\tan\theta} + \int \frac{\sec\theta\tan^2\theta}{\tan\theta}$

$= \int \frac{1}{\cos\theta} \cdot \frac{1}{\tan\theta} + \int \sec\theta \cdot \tan\theta\, d\theta = \int \frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta} + \int \sec\theta\tan\theta\, d\theta$

$= \int \csc\theta\, d\theta + \int \sec\theta\tan\theta\, d\theta = -\ln|\csc\theta + \cot\theta| + \sec\theta + C$ نرجع للفرضية

وتر / مقابل   مجاور / مقابل   وتر / مجاور

$\Longrightarrow = -\ln\left|\frac{\sqrt{x^2+1}}{x} + \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C$

NOTEBOOK

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Subject Date No.

Ex. $\int_{1}^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}$

$\boxed{x = 2\sin} \Rightarrow \boxed{dx = 2\cos\theta\, d\theta}$

$\boxed{\theta = \sin^{-1}(\frac{x}{2})}$ if $x=1 \to \theta = \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$

if $x=\sqrt{2} \to \theta = \sin^{-1}(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$

$I = \int_{\pi/6}^{\pi/4} \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}}$

$= \int_{\pi/6}^{\pi/4} \frac{2\cos\theta\, d\theta}{4\sin^2\theta \cdot 2\sqrt{1-\sin^2\theta}}$

$= \int_{\pi/6}^{\pi/4} \frac{2\cos\theta\, d\theta}{4\sin^2\theta \cdot 2\cos\theta}$

$= \frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta\, d\theta$

$= \frac{1}{4}\left(-\cot\theta \Big|_{\pi/6}^{\pi/4}\right)$

$= \frac{1}{4}\left(-\cot(\frac{\pi}{4}) + \cot(\frac{\pi}{6})\right)$

$= \frac{1}{4}\left(-1+\sqrt{3}\right)$

Ex. $\int \frac{dx}{\sqrt{5-4x-2x^2}}$

شروط اكمال المربع :

- معامل x^2 موجب

- نضيف ونطرح $(\frac{\text{معامل } x}{2})^2$

$= \int \frac{dx}{\sqrt{-2(x^2+2x+1-1-\frac{5}{2})}}$

$= \int \frac{dx}{\sqrt{-2(x^2+2x-\frac{5}{2})}}$

$= \int \frac{dx}{\sqrt{-2((x+1)^2-\frac{7}{2})}}$

$= \int \frac{dx}{\sqrt{7-2(x+1)^2}}$

Let $\sqrt{2}(x+1) = \sqrt{7}\sin\theta$

$x+1 = \frac{\sqrt{7}}{\sqrt{2}}\sin\theta$

$dx = \frac{\sqrt{7}}{\sqrt{2}}\cos\theta\, d\theta$

$= \int \frac{\frac{\sqrt{7}}{\sqrt{2}}\cos\theta\, d\theta}{\sqrt{7-7\sin^2\theta}}$

$= \int \frac{\sqrt{7}\cos\theta\, d\theta}{\sqrt{2}\sqrt{7}\sqrt{1-\sin^2\theta}}$

$= \int \frac{1}{\sqrt{2}}\frac{\cos\theta}{\cos\theta}\, d\theta$

$= \frac{1}{\sqrt{2}}\int d\theta$

$= \frac{1}{\sqrt{2}}\theta + C$

Now, $\sqrt{2}(x+1) = \sqrt{7}\sin\theta$

$\frac{\sqrt{2}}{\sqrt{7}}(x+1) = \sin\theta$

$\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{7}}(x+1)\right) = \sin^{-1}(\sin\theta)$

$\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{7}}(x+1)\right) = \theta$

$\Rightarrow I = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{7}}(x+1)\right) + C$

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Ex. $\int \frac{x}{x^2-4x+8}\,dx \xrightarrow{\text{غيرقابل للتحليل}}$

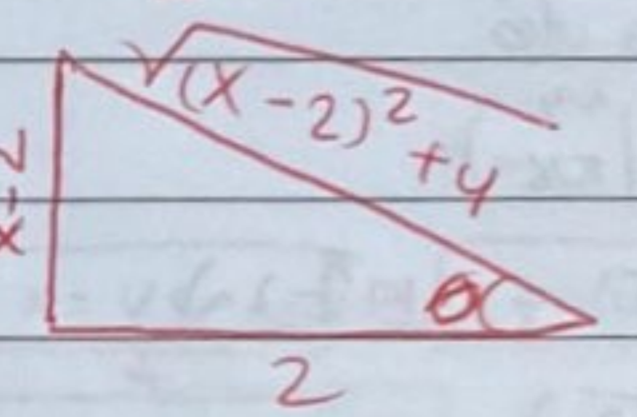
$= \int \frac{x}{(x^2-4x+4-4+8)}\,dx$ Let $x-2 = 2\tan\theta$

$= \int \frac{x}{(x-2)^2+4}\,dx$ $dx = 2\sec^2\theta\,d\theta$

$= \int \frac{2\tan\theta+2}{4\tan^2\theta+4}\cdot 2\sec^2\theta\,d\theta$

$= \int \frac{2(\tan\theta+1)}{4(\tan^2\theta+1)}\cdot 2\sec^2\theta\,d\theta$ Now, $\frac{x-2}{2} = \tan\theta$

$= \int \frac{2(\tan\theta+1)}{4(\sec^2\theta)}\cdot 2\sec^2\theta\,d\theta$



$= \int \tan\theta+1\ d\theta$

$= -\ln|\cos\theta| + \theta + c$

$= \ln|\sec\theta| + \theta + c$

$= \ln\left|\frac{\sqrt{(x-2)^2+4}}{2}\right| + \tan^{-1}\left(\frac{x-2}{2}\right) + c$

Ex. $\int \frac{\cos x}{\sin x\sqrt{4\sin^2 x-1}}\,dx$ $\Rightarrow \sin x = u$

$= \int \frac{\cos x}{u\sqrt{4u^2-1}}\cdot\frac{du}{\cos x}$ $\cos x\,dx = du$

$= \int \frac{du}{u\sqrt{4u^2-1}}$ Let $2u = \sec\theta$

$= \int \frac{\sec\theta\tan\theta\,d\theta}{2(\frac{\sec\theta}{2})\sqrt{\sec^2\theta-1}}$ $2du = \sec\theta\tan\theta\,d\theta$

$= \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\,d\theta$ Now $\theta = \sec^{-1}(2u)$

$= \int d\theta$

$= \theta + c$

$= \sec^{-1}(2u) + c$

$= \sec^{-1}(2\sin x) + c$

Ex. The most proper trigonometric substituable to solve $\int \frac{dx}{\sqrt{x^2-4x+5}}$ is :- $x = \tan\theta + 2$

الحل مربع

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Subject Lec 7 Date No.

* Improper Integral

- ${}_a\int^{\infty} f(x)dx$, ${}_{-\infty}\int^{b} f(x)dx$, ${}_{-\infty}\int^{\infty} f(x)\,dx$, ${}_a\int^{b} f(x)\,dx$, f(x) is cont

$\boxed{\text{Case I}}$ Let f(x) cont on $[a, \infty)$, then:

$${}_a\int^{\infty} f(x)\,dx = \underbrace{\lim_{t\to\infty} {}_a\int^{t} f(x)\,dx}$$

exist, equal to M ⇓ ${}_a\int^{\infty} f(x)$ conv to M ↳ تقترب

d.n.e ⇓ ${}_a\int^{\infty} f(x)$ diverge ↳ متباعد

$\boxed{\text{Case II}}$ Let f(x) cont on $(-\infty, b]$, then:

$${}_{-\infty}\int^{b} f(x)\,dx = \underbrace{\lim_{t\to-\infty} {}_t\int^{b} f(x)\,dx}$$

exist, equal to M ⇓ ${}_{-\infty}\int^{b} f(x)\,dx$ converge to M

d.ne ⇓ ${}_{-\infty}\int^{b} f(x)\,dx$ diverge

$\boxed{\text{Case III}}$ Let f(x) cont on $(-\infty, \infty)$, then:

$${}_{-\infty}\int^{\infty} f(x)\,dx = \underbrace{{}_{-\infty}\int^{a} f(x)\,dx}_{I_1} + \underbrace{{}_a\int^{\infty} f(x)\,dx}_{I_2}$$

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Subject Date No.

- if $I_1 + I_2$ conv $\Rightarrow \int_{-\infty}^{\infty}$ conv to $M+L$
  " s.t $I_1 \rightarrow M$ "
  $I_2 \rightarrow L$

- if I_1 <u>or</u> I_2 div $\Rightarrow \int_{-\infty}^{\infty} f(x)$ div

Examples:

1) $\int_0^{\infty} e^{-4x} dx = \lim_{t\to\infty} \int_0^t e^{-4x} dx$

$= \lim_{t\to\infty} \frac{e^{-4x}}{-4} \Big|_0^t$ $\qquad e^{-\infty} = 0$

$\qquad e^{\infty} = \infty$

$= \lim_{t\to\infty} \frac{-1}{4} [e^{-4t} - 1]$ $\qquad \tan \infty = \frac{\pi}{2}$

$\qquad \tan -\infty = -\frac{\pi}{2}$

$= \frac{-1}{4} [0 - 1] = \frac{1}{4}$

$\therefore$ converge to $\frac{1}{4}$

2) $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \underbrace{\int_{-\infty}^{0}}_{I_1} + \underbrace{\int_0^{\infty}}_{I_2}$

- $I_1 = \int_{-\infty}^{0} \frac{dx}{x^2+1} = \lim_{t\to-\infty} \int_t^0 \frac{dx}{x^2+1} = \lim_{t\to-\infty} \tan^{-1}x \Big|_t^0 = \lim_{t\to-\infty} [\tan^{-1}0 - \tan^{-1}t]$

$= 0 - (-\frac{\pi}{2}) = \frac{\pi}{2}$

$\therefore I_1$ conv to $\frac{\pi}{2}$

- $I_2 = \int_0^{\infty} \frac{dx}{x^2+1} = \lim_{t\to\infty} \int_0^t \frac{dx}{x^2+1} = \lim_{t\to\infty} \tan^{-1}x \Big|_0^t = \lim_{t\to\infty} [\tan^{-1}t - \tan^{-1}0]$

$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$\therefore I_2$ conv to $\frac{\pi}{2}$

$\Rightarrow I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ conv to $\frac{\pi}{2} + \frac{\pi}{2} = \pi$

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$\boxed{\text{Case IV}}$ if $f(x)$ dis cont at $x=a$, then

$$\int_a^b f(x)\,dx = \underbrace{\lim_{t\to a^+}\int_t^b f(x)\,dx}$$

exist = conv d.n.e = div

$\boxed{\text{Case V}}$ if $f(x)$ dis cont at $x=b$, then

$$\int_a^b f(x)\,dx = \underbrace{\lim_{t\to b^-}\int_a^t f(x)\,dx}$$

exist = conv d.n.e = div

$\boxed{\text{Case VI}}$ if $f(x)$ discont at $x=c \in (a,b)$, then

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{I_1} + \underbrace{\int_c^b f(x)\,dx}_{I_2}$$

- if $I_1 + I_2$ conv $\Rightarrow \int_a^b f(x)\,dx$ conv
- if I_1 or I_2 div $\Rightarrow \int_a^b f(x)\,dx$ div

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$\ln 0^+ = -\infty$

$\ln\infty = \infty$

$\frac{\text{عدد}}{\infty} = 0$

$\frac{\infty}{\text{عدد}} = \infty$

Examples:

1) $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \underbrace{\int_0^1 \frac{dx}{\sqrt{x}(x+1)}}_{I_1} + \underbrace{\int_1^\infty \frac{dx}{\sqrt{x}(x+1)}}_{I_2}$

- $I_1 = \int_0^1 \frac{dx}{\sqrt{x}(x+1)} = \lim_{t\to0^+} \int_t^1 \frac{dx}{\sqrt{x}(x+1)} = \lim_{t\to0^+} 2\tan^{-1}\sqrt{x}\Big|_t^1$
  $= \lim_{t\to0^+}\left[2\tan^{-1}\sqrt{1} - 2\tan^{-1}\sqrt{t}\right] = 2\cdot\frac{\pi}{4} - 0 = \frac{\pi}{2}$

- $I_2 = \int_1^\infty \frac{dx}{\sqrt{x}(x+1)} = \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}(x+1)} = \lim_{t\to\infty} 2\tan^{-1}\sqrt{x}\Big|_1^t$
  $= \lim_{t\to\infty}\left[\tan^{-1}\sqrt{t} - 2\tan^{-1}\sqrt{1}\right] = 2\cdot\frac{\pi}{2} - \frac{2\pi}{4} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

- $I = \int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \frac{\pi}{2} + \frac{\pi}{2} = \pi \longrightarrow$ conv to π

أول اشي $\int \frac{dx}{\sqrt{x}(x+1)} = \int \frac{2\sqrt{x}\,du}{\sqrt{x}(u^2+1)} = 2\tan^{-1}(\sqrt{x}) + c \qquad u = \sqrt{x} \to dx = 2\sqrt{x}\,du$

2) $\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty} \frac{-1}{x}\Big|_1^t = \lim_{t\to\infty}\left[\frac{-1}{t}+1\right] = 1$
$\therefore$ conv to 1

3) $\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty} \ln|x|\Big|_1^t = \lim_{t\to\infty}\left[\ln|t| - \ln|1|\right] = \infty$
$\therefore$ div

* Note that: $\int_1^\infty \frac{1}{x^p}dx = \begin{cases} \text{converge to } \frac{1}{p-1}, & p > 1 \\ \text{diverge}, & 0 < p \le 1 \end{cases}$

Examples:

1) $\int_1^\infty \frac{1}{x}dx =$ div

2) $\int_1^\infty \frac{1}{\sqrt{x}}dx =$ div

3) $\int_1^\infty \frac{1}{x^5}dx =$ conv to $\frac{1}{4}$

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Ex: Find the values of P for which the integral converge.

$$\int_e^{\infty} \frac{1}{x(\ln x)^{2P}}$$

$u = \ln x$

$x\,du = dx$

if $x = e \to u = 1$

$x = \infty \to u = \infty$

$$\therefore \int_1^{\infty} \frac{x\,du}{x(u)^{2P}} = \int_1^{\infty} \frac{1}{u^{2P}}\,du$$

conv if $2P > 1 \Rightarrow \boxed{P > \frac{1}{2}}$

Ex: Find all values of P such that the following integrals are improper.

1) $\int_{-1}^{2} \frac{1}{x+P}\,dx$

$x + P = 0 \to x = -P$

$-1 \leq x \leq 2$

$-1 \leq -P \leq 2$

$1 \geq P \geq -2$ $\therefore P \in [-2, 1]$

2) $\int_{-1}^{3} \frac{dx}{2x-P}$

$2x - P = 0 \to x = \frac{P}{2}$

$-1 \leq x \leq 3$

$-1 \leq \frac{P}{2} \leq 3$

$-2 \leq P \leq 6$ $\therefore P \in [-2, 6]$

3) $\int_{-3}^{2} \frac{dx}{x^2 - P}$

$x^2 - P = 0 \to x^2 = P$

$-3 \leq x \leq 2$

$0 \leq x^2 \leq 9$

$0 \leq P \leq 9$ $\therefore P \in [0, 9]$

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Ex: find a positive value of p such that $\int_0^\infty \frac{dx}{x^2+p^2}$ converge to 1

$I = \int_0^\infty \frac{dx}{x^2+p^2} = \lim_{t\to\infty} \frac{1}{p} \tan^{-1}(\frac{x}{p}) \Big|_0^t = 1$

$\Rightarrow \frac{1}{p} [\tan^{-1}(\infty) - \tan^{-1}(0)] = 1$

$\frac{1}{p} \cdot \frac{\pi}{2} - 0 = 1$

$\frac{1}{p} \frac{\pi}{2} = 1$ $\boxed{\therefore p = \frac{\pi}{2}}$

Ex: $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ conv or div

$= \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \sin^{-1}x \Big|_0^t = \lim_{t\to 1^-} [\sin^{-1}t - \sin^{-1}0]$

$= \lim \frac{\pi}{2} - 0 = \frac{\pi}{2}$ $\therefore$ conv to $\frac{\pi}{2}$

Ex: $\int_{-\infty}^0 x e^x \, dx$ conv or div

u	dv
x	e^x
1	e^x
0	e^x

$I = \lim_{t\to-\infty} \int_t^0 x e^x dx$

$= \lim_{t\to-\infty} [xe^x - e^x] \Big|_t^0$

$= \lim_{t\to-\infty} [0 \cdot e^0 - e^0 - te^t + e^t]$

$= \lim_{t\to-\infty} [-1 - te^t]$

$= -1 - \lim_{t\to-\infty} te^t$ $\quad -\infty \cdot 0$ L'H

$\Rightarrow -1 - \lim_{t\to-\infty} \frac{t}{e^{-t}}$

$= -1 - \lim_{t\to-\infty} \frac{1}{-e^{-t}}$

$= -1 - 0$

$= -1$

$\therefore$ conv to -1

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Ex: $\int_0^1 \ln x \, dx$ conv or div

$= \lim_{t\to 0^+} \int_t^1 \ln x \, dx$

$= \lim_{t\to 0^+} x\ln x - x \Big|_t^1$

$= \lim_{t\to 0^+} [1.\ln 1 - 1] - [t\ln t - t]$

$0 \cdot -\infty$

$= \lim_{t\to 0^+} -1 - \lim_{t\to 0^+} t\ln t$

$= -1 - \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}}$

$= -1 - \lim_{t\to 0^+} \frac{\frac{1}{t}}{-1/t^2}$

$= -1 - \lim_{t\to 0^+} -t$

$= -1 + 0$

$= -1$ $\therefore$ conv to -1

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Subject Lec 8 Date No.

* Arc Length ... Length of a plane curve

Suppose that $y = f(x)$ is a smooth curve on $[a,b]$ "i.e $f'(x)$ cont on $[a,b]$", then the arc length is defined as:

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

Ex: Find the arc length of the curve $y = x^{\frac{3}{2}}$ from $(1,1)$ to $(2, 2\sqrt{2})$

① $y' = \frac{3}{2} x^{\frac{1}{2}}$

② $(y')^2 = \frac{9}{4} x$

③ $L = \int_1^2 \sqrt{1 + \frac{9}{4}x}\; dx$

⋮

Ex: set up the integral for the arc length of the curve $y = \frac{x^2}{4} - \frac{1}{2}\ln x$ on $[1,2]$ about x-axis :-

① $y' = \frac{2x}{4} - \frac{1}{2x}$

② $(y')^2 = \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = \frac{x^2}{4} - 2 \cdot \frac{x}{2} \cdot \frac{1}{2x} + \frac{1}{4x^2} = \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}$

③ $(y')^2 + 1 = \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} + 1 = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$

④ $L = \int_1^2 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2} = \int_1^2 \frac{x}{2} + \frac{1}{2x}\, dx$ - - - .

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- Area of surface of revolution

Let $f(x)$ smooth, nonnegative on $[a,b]$, then the surfaces are S of the surface of revolution generated by revolving the portion of the curve $y=f(x)$ between $x=a$, $x=b$ about x-axis:-

$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\, dx$$

Ex: Find the area of the surface generated by revolving $y=\sqrt{x}$ about x-axis on $[1,4]$

① $y' = \frac{1}{2\sqrt{x}}$ ② $(y')^2 = \frac{1}{4x}$

③ $S = \int_1^4 2\pi \cdot \sqrt{x} \cdot \sqrt{1+\frac{1}{4x}}\, dx$

$= \int_1^4 2\pi \sqrt{x+\frac{1}{4}}$ ...

Ex: Find the area of the surface generated by revolving $y=\sqrt{4-x^2}$ on $[-1,1]$ about x-axis:-

① $y' = \frac{-2x}{2\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}}$ ② $(y')^2 = \frac{x^2}{4-x^2}$

③ $(y')^2+1 = \frac{x^2}{4-x^2} + 1 = \frac{x^2+4-x^2}{4-x^2} = \frac{4}{4-x^2}$

④ $S = \int_{-1}^{1} 2\pi \cdot \sqrt{4-x^2} \cdot \sqrt{\frac{4}{4-x^2}} = \int_{-1}^{1} 2\pi \sqrt{4}\, dx$ ...

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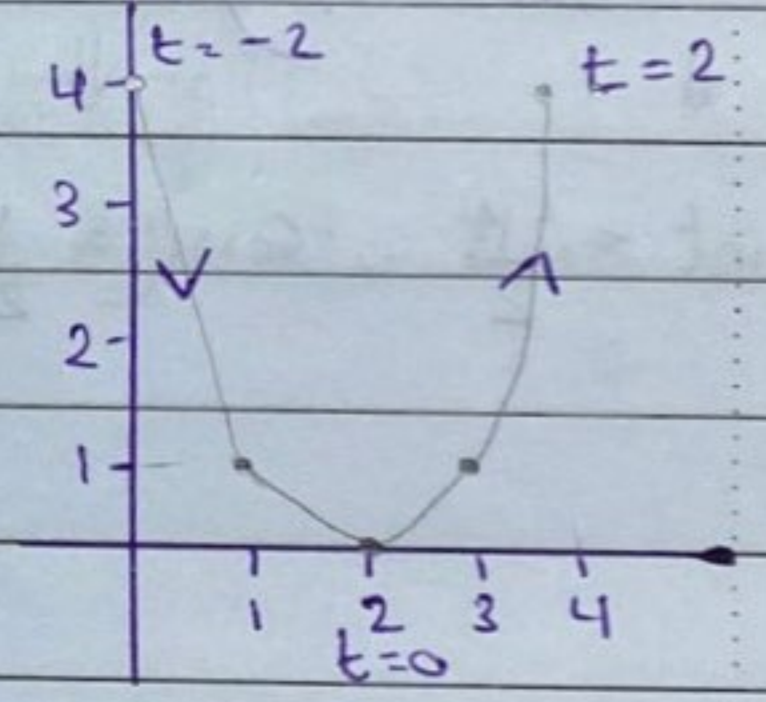
• Curves Defined by Parametric Equations

- Defn: the parametric equation of $y=f(x)$ is:-

$x = x(t)$, $y = y(t)$, t: independent variable
x, y: dependent variable

Example: Identify the parametric equations:-
$x = t+2$, $y = t^2$, $-2 \leq t \leq 2$

t	$x(t)$	$y(t)$	(x,y)
-2	0	4	$(0,4)$
-1	1	1	$(1,1)$
0	2	0	$(2,0)$
1	3	1	$(3,1)$
2	4	4	$(4,4)$



or Find the Cartesian equation:-

① $t = x+2$
② $y = t^2 \Rightarrow y = (x-2)^2$
$t = -2 \Rightarrow (x,y) = (0,4)$
$t = 2 \Rightarrow (x,y) = (4,4)$

parabola ← قطع مكافئ *
hyperbola ← قطع زائد *
ellipse ← قطع ناقص *

$\therefore y = (x-2)^2$ from $(0,4)$ to $(4,4)$

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Example: Find the Cartesian equation for:-

1) $x = 2t$, $y = t^2+1$, $t \in \mathbb{R}$

$\Downarrow$

$t = \frac{x}{2} \rightarrow y = \left(\frac{x}{2}\right)^2 + 1 \Longrightarrow y = \frac{x^2}{4} + 1$ parabola

2) $x = t^2+1$, $y = t^4 - 3$

$\Downarrow$

$t^2 = x-1 \Rightarrow y = (x-1)^2 - 3$ parabola

3) $x = 2\sin t$, $y = 2\cos t$, $0 \leq t \leq 2\pi$

$\Downarrow$ $\Downarrow$

$\sin t = \frac{x}{2}$, $\cos t = \frac{y}{2} \Longrightarrow \sin^2 t + \cos^2 t = 1$

$\frac{x^2}{4} + \frac{y^2}{4} = 1$

$x^2 + y^2 = 4$ circle

4) $x = 3\sec t$, $y = 4\tan t$, $0 \leq t \leq 2\pi$

$\Downarrow$ $\Downarrow$

$\sec t = \frac{x}{3}$, $\tan t = \frac{y}{4} \Longrightarrow \sec^2 t - \tan^2 t = 1$

$\frac{x^2}{9} - \frac{y^2}{16} = 1$ hyperbola

5) $x = \cos t + 1$, $y = \sin^2 t$

$\Downarrow$

$\cos t = x-1 \Rightarrow \sin^2 t = 1 - \cos^2 t$

$y = 1 - (x-1)^2$ parabola

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6> $x = \sec t$, $y = \sin^2 t - 2$

$x = \frac{1}{\cos t}$

$\cos t = \frac{1}{x} \Rightarrow y = \sin^2 t - 2$

$y = 1 - \cos^2 t - 2$

$y = -1 - \cos^2 t \Rightarrow y = -1 - \frac{1}{x^2}$

7> $x = 2\sin^2 t$, $y = 3\cos^2 t$, $0 \leq t \leq 2\pi$

$\sin^2 t = \frac{x}{2}$, $\cos^2 t = \frac{3}{y} \Longrightarrow \sin^2 t + \cos^2 t = 1$

$\frac{x}{2} + \frac{y}{3}$

$3x + 2y = 6$ Line

8> $x = \cosh t$, $y = \sinh t$

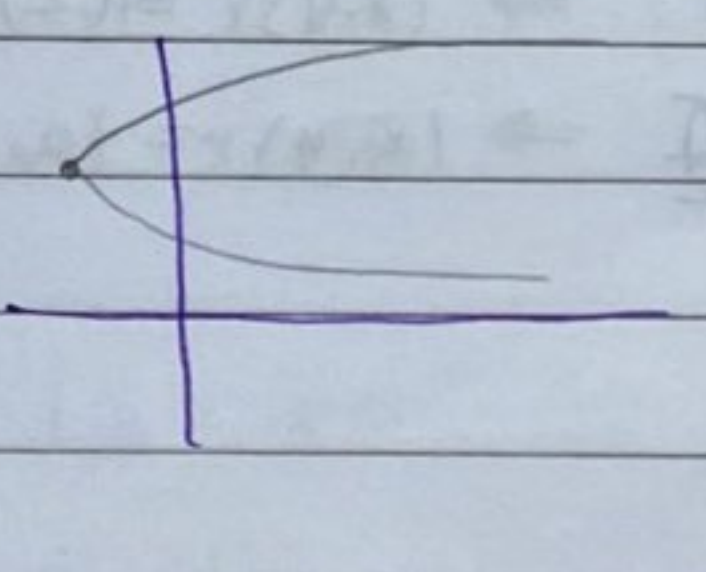
$\cosh^2 t - \sinh^2 t = 1$

$x^2 - y^2 = 1$ hyperbola

9> $x = t^2 - 2t$, $y = t + 1$, $t \in \mathbb{R}$

$x = (y-1)^2 - 2(y-1) \Leftarrow t = y - 1$

$x = y^2 - 4y - 3$



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10> $x = e^{-t}$, $y = e^{2t}$, $0 \leq t \leq \ln 8$

$\Downarrow$

$\ln x = -t$

$t = -\ln x \Rightarrow y = e^{-2\ln x} \Rightarrow y = e^{\ln x^{-2}} \Rightarrow y = \frac{1}{x^2}$

$t = 0 \Rightarrow x = e^0 = 1$, $y = e^0 = 1 \longrightarrow (1,1)$

$t = \ln 8 \Rightarrow x = e^{-\ln 8} = \frac{1}{8}$, $y = e^{2\ln 8} = 64 \rightarrow (\frac{1}{8}, 64)$

$\therefore y = \frac{1}{x^2}$ from $(1,1)$ to $(\frac{1}{8}, 64)$

Example: sketch the curve

a> $x = 2\cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$

$\Downarrow$ $\Downarrow$

$\cos t = \frac{x}{2}$, $\sin t = \frac{y}{2}$

$\Rightarrow \sin^2 t + \cos^2 t = 1$

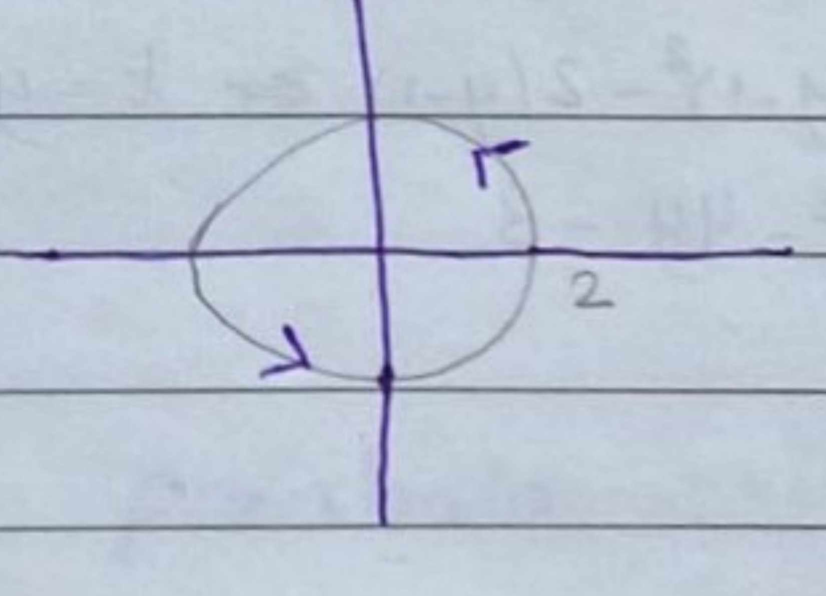
$\frac{x^2}{4} + \frac{y^2}{4} = 1$

$x^2 + y^2 = 4$ circle centered by $(0,0)$ with $r=2$

, counter clock wise

$t = 0 \Rightarrow (x,y) = (2,0)$

$t = \frac{\pi}{2} \Rightarrow (x,y) = (0,2)$



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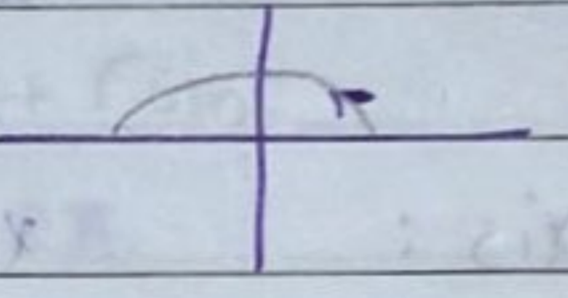
b> $x = 2\cos t$, $y = 2\sin t$, $0 \leq t \leq \pi$

↳ semi circle

$t = 0 \Rightarrow (x, y) = (2, 0)$

$t = \frac{\pi}{2} \Rightarrow (x, y) = (0, 2)$

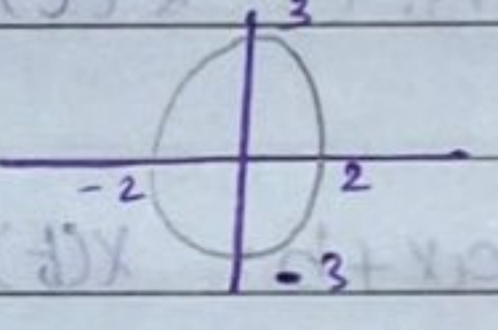
$t = \pi \Rightarrow (x, y) = (-2, 0)$



Example:- sketch the curve

$x = 2\cos t$, $y = 3\sin t$, $0 \leq t \leq 2\pi$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ ellipse



Example :- Find the parametric equation for $9x^2 + 4y^2 = 36$

$$\frac{9x^2}{36} + \frac{4y^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow x = 2\cos t$$

$$y = 3\sin t$$

$$0 \leq t \leq 2\pi$$

* $\frac{x^2}{a} + \frac{y^2}{b} = 1 \longrightarrow x = \cos t \times \sqrt{a}$

$y = \sin t \times \sqrt{b}$

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Subject Date No.

- Note that : The parametric equation for lines

1> x-axis: $x(t)=t$, $y(t)=0$, $t\in\mathbb{R}$

2> y-axis: $x(t)=0$, $y(t)=t$, $t\in\mathbb{R}$

3> the line $x=a$: $x(t)=a$, $y(t)=t$, $t\in\mathbb{R}$

4> the line $y=b$: $x(t)=t$, $y(t)=b$, $t\in\mathbb{R}$

5> the line $y=ax+b$: $x(t)=t$, $y(t)=at+b$, $t\in\mathbb{R}$

6> the line from (x_0,y_0) to (x_1,y_1) is:

$$x = x_0 + (x_1 - x_0)t$$
$$y = y_0 + (y_1 - y_0)t$$
$$0 \leq t \leq 1$$

Example; Find the parametric equation for the lines:-

1> From $(-5,-3)$ to $(0,2)$

$x(t) = 0 + (-5-0)t \rightarrow x(t) = -5t$

$y(t) = 2 + (-3-2)t \rightarrow y(t) = 2-5t$

$0 \leq t \leq 1$

2> $y=3x-2$, $0\leq x\leq 1$

$x(t)=t$, $y(t)=3t-2$, $0\leq t\leq 1$

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- The parametric equation for the circle

$(x-a)^2 + (y-b)^2 = r^2$ $\quad x(t) = a + r\cos t$

$y(t) = b + r\sin t$

$0 \leq t \leq 2\pi$

- The parametric equation for the semi circle

$x(t) = a + r\cos t$

$y(t) = b + r\sin t$

$0 \leq t \leq \pi$

Example :- Find the parametric equation for

$(x+1)^2 + (y-2)^2 = 25$

$x(t) = -1 + 5\cos t$ $\quad$ center $= (-1, 2)$

$y(t) = 2 + 5\sin t$ $\quad$ $r = 5$

$0 \leq t \leq 2\pi \Rightarrow$ circle

$0 \leq t \leq \pi \Rightarrow$ semicircle

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Subject Lec 10 Date No.

- Calculus with parametric equation (curves)

Remark:

$\triangle$ For the parametric curve $x(t) = x$, $y(t) = y$

the slope is given by:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

Example: Find the slope of the tangent line for

$x(t) = 2\cos t + \sin 2t$, $y(t) = 2\sin t + \cos 2t$ at $t=0$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \left.\frac{2\cos t - 2\sin 2t}{2\sin t + 2\cos 2t}\right|_{t=0} = \frac{2-0}{0+2} = \frac{2}{2} = 1$$

Example: Find the slope for $x(t) = t - t^4$, $y(t) = t^2 - t^3$ at $(0,0)$

$\triangle$ $(0,0) \Rightarrow$ $x(t) = 0 \Rightarrow t - t^4 = 0$

$t(1-t^3) = 0$ $t=0$, $t=1$

$\triangle$ $y(t) = 0 \Rightarrow t^2 - t^3 = 0$

$t^2(1-t) = 0$ $t=0$, $t=1$

$\therefore$ $t=0$, $t=1$

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Subject Date No.

③ slope $\Rightarrow \frac{dy}{dx} = \frac{y'}{x'} = \frac{2t+3t^2}{1-4t^3}$

① slope$\Big|_{t=0} = 0$ ② slope$\Big|_{t=1} = \frac{2+3}{1-4} = \frac{5}{-3}$

Example, Find the equation of the tangent line for

$x(t) = e^t \quad , \quad y(t) = t + e^{-t} \quad$ at $\quad t=0$

1) $x_0 = e^0 = 1$

2) $y_0 = 0 + e^0 = 1$

3) slope$\Big|_{t=0} = \frac{y'}{x'}\Big|_{t=0} = \frac{1+(-e^{-t})}{e^t}\Big|_{t=0} = \frac{1-1}{1} = 0$

$\therefore \; y - y_0 = m\,(x - x_0)$

$y - 1 = 0\,(x-1) \Rightarrow \boxed{y=1}$

△ Recall that the slope of $x = x(t)$, $y = y(t)$ is

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}\Big|_{t=t_0}$ then:-

ⓐ the curve has horizantal tangent line if $y'(t) = 0$, $x'(t) \neq 0$

ⓑ the curve has vertical tangent line if $y'(t) \neq 0$, $x'(t) = 0$

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Example: Find the value of t such that the curve parametrize by $x(t) = t^2 - 1$, $y(t) = t^4 - 4t^2$ has:

ⓐ horizantal tangent line

ⓑ vertical tangent line

1) $x'(t) = 2t$, $y'(t) = 4t^3 - 8t$

2) $x'(t) = 0 \Rightarrow 2t = 0 \Rightarrow t = 0$

$y'(t) = 0 \Rightarrow 4t^3 - 8t = 0 \rightarrow 4t(t^2 - 2) = 0 \Rightarrow t = 0$

$t = \pm\sqrt{2}$

a) HTL (if $y'(t) = 0$ but $x'(t) \neq 0$)

① $t = 0 \Rightarrow x'(0) = 0$ ✗ [illegible]

② $t = \sqrt{2} \Rightarrow x'(\sqrt{2}) = 2\sqrt{2}$ ✓

③ $t = -\sqrt{2} \Rightarrow x'(-\sqrt{2}) = -2\sqrt{2}$ ✓

∴ the curve has horizantal tangent line at $t = 2\sqrt{2}$, $t = -2\sqrt{2}$

b) VTL (if $x'(t) = 0$ but $y'(t) \neq 0$)

$t = 0 \Rightarrow y'(0) = 4(0) - 8(0) = 0$

so there is no vertical Line tangent

N O T E B O O K

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Subject Date No.

Example: Find the value of $t \in [0,\pi]$ such that the curve parametrize by $x = \cos(2t)$, $y = \sin(4t)$ has:

ⓐ horizontal tangent

ⓑ vertical tangent

Δ $x'(t) = -2\sin 2t$

Δ $y'(t) = 4\cos 4t$

ⓐ $y'(t) = 0 \Longrightarrow 4\cos 4t = 0$ $\quad 4t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

$\Rightarrow t = \underbrace{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}}_{\in [0,\pi]}, \frac{9\pi}{8} \notin [0,\pi]$

Now, $x'(t) \neq 0$?

$x'(\frac{\pi}{8}) \neq 0 \neq -2\sin(\frac{\pi}{4})$

$x'(\frac{3\pi}{8}) \neq 0$, $x'(\frac{5\pi}{8}) \neq 0$, $x'(\frac{7\pi}{8}) \neq 0$

$\therefore t = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

ⓑ $x'(t) = 0 \Rightarrow -2\sin 2t = 0$ $\quad 2t = 0, \pi, 2\pi$

$\Rightarrow t = 0, \frac{\pi}{2}, \pi \rightarrow \in [0,\pi]$

Now, $y'(t) \neq 0$?

$y'(0) = 4\cos 4(0) \neq 0$

$y'(\frac{\pi}{2}) = 4\cos 4(\frac{\pi}{2}) \neq 0$

$y(\pi) = 4\cos 4\pi \neq 0$

$\therefore t = 0, \frac{\pi}{2}, \pi$

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Example: the parametric curve $x(t) = t^3 - 3t$

$y(t) = t^2 + 6t$ has horizantal tangent at the point:

1> $x'(t) = 3t^2 - 3$

2> $y'(t) = 2t + 6$

3> $y'(t) = 0 \rightarrow 2t + 6 = 0 \rightarrow t = -3$

$x'(-3) = 3(-3)^2 - 3 = 24 \neq 0$

$\therefore t = -3 \Longrightarrow$ point $(x(-3), y(-3)) = (-18, -9)$

⚠ Let $C : x = x(t)$, $y = y(t)$, $t \in [a,b]$ then the arc length of C is:

$$\rightarrow L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Example: Find the arc length of the curve

$C: x(t) = \cos t$, $y(t) = \sin t$, $0 \leq t \leq 2\pi$

1> $x'(t) = -\sin t$

2> $y'(t) = \cos t$

3> $L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$

$= \int_0^{2\pi} \sqrt{1}\, dt$

$= 2\pi$

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Example: Find the arc length of the curve

C: $x(t) = e^t - t$, $y(t) = 4e^{\frac{1}{2}t}$, $t \in [0, 1]$

1) $x'(t) = e^t - 1$

2) $y'(t) = \frac{4}{2} e^{\frac{1}{2}t} = 2e^{\frac{1}{2}t}$

3) $(x')^2 + (y')^2 = (e^t - 1)^2 + (2e^{\frac{1}{2}t})^2$

$= e^{2t} - 2e^t + 1 + 4e^t$

$= e^{2t} + 2e^t + 1 \Rightarrow (e^t + 1)^2$

$L = \int_0^1 \sqrt{(e^t + 1)^2}\, dt$

$= \int_0^1 |e^t + 1|\, dt$

$= e^t + t \Big|_0^1$

$= (e + 1) - (1 + 0)$

$= e$

4) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$, $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave up

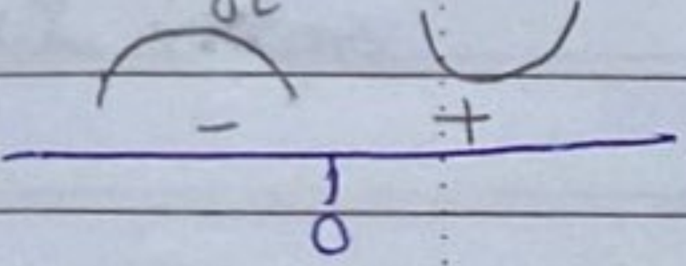
$\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave down

Example: consider the curve C: $x(t) = t^2$, $y = t^3 - 3t$, determine whether the curve concave up or down.

1) $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2 - 3}{2t}$

2) $\frac{d^2y}{dx} = \frac{\frac{(2t)(6t) - (3t^2 - 3)(2)}{4t^2}}{2t} = \frac{12t^2 - 6t^2 + 6}{8t^3} = \frac{6t^2 + 6}{8t^3}$

Concave: up $\rightarrow (0, \infty)$ down $\rightarrow (-\infty, 0)$



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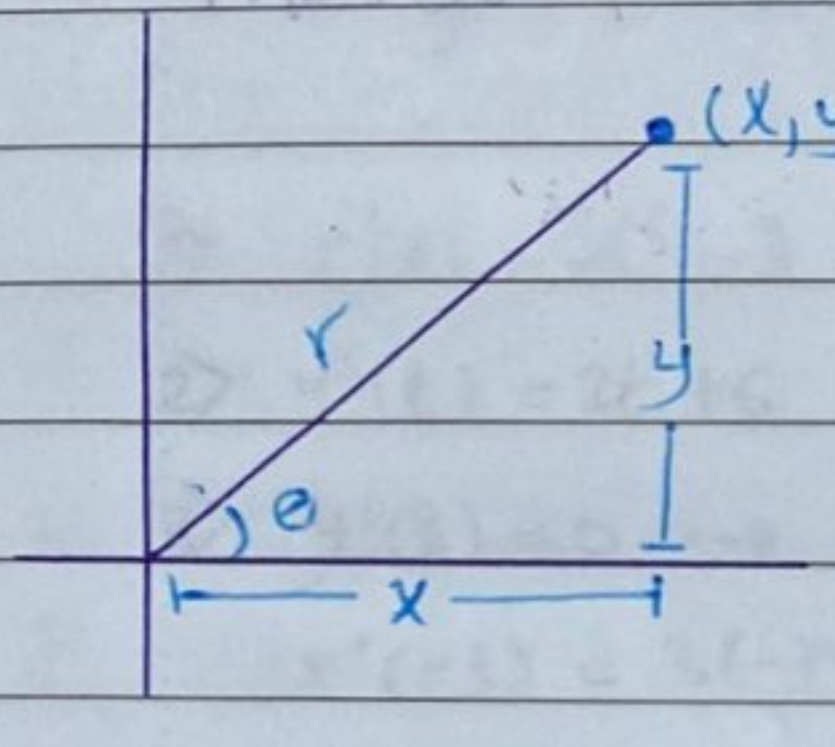
- Polar coordinates



(x,y): rectangular coordinate
"Cartesian"

(r,θ): Polar coordinate

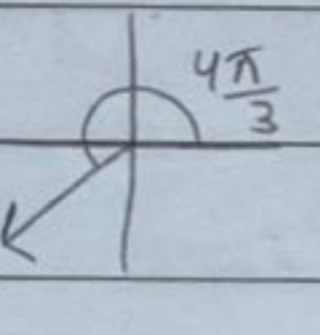
① $(r,\theta) \longrightarrow (x,y)$

$\cos\theta = \dfrac{x}{r} \longrightarrow x = r\cos\theta$

$\sin\theta = \dfrac{y}{r} \longrightarrow y = r\sin\theta$

Example: Given that the polar coordinate $(6, \frac{4\pi}{3})$. Find the rectangular coordinate?

① $r = 6$, $\theta = \dfrac{4\pi}{3}$



② $x = 6\cos\left(\dfrac{4\pi}{3}\right) = 6 \cdot \dfrac{-1}{2} = -3$

③ $y = 6\sin\left(\dfrac{4\pi}{3}\right) = \dfrac{6 \cdot -\sqrt{3}}{2} = -3\sqrt{3}$

$\therefore (x,y) = (-3, -3\sqrt{3})$

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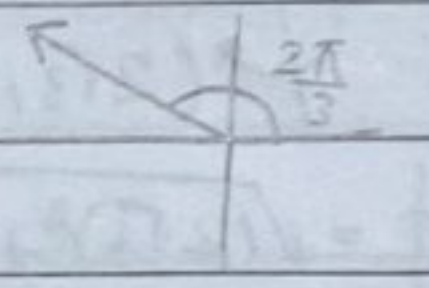
Example: Find the cartesian coordinate (x,y) of the point $(2, \frac{2\pi}{3})$?

① $r = 2$, $\theta = \frac{2\pi}{3}$



② $x = 2\cos(\frac{2\pi}{3}) = 2 \cdot \frac{-1}{2} = -1$

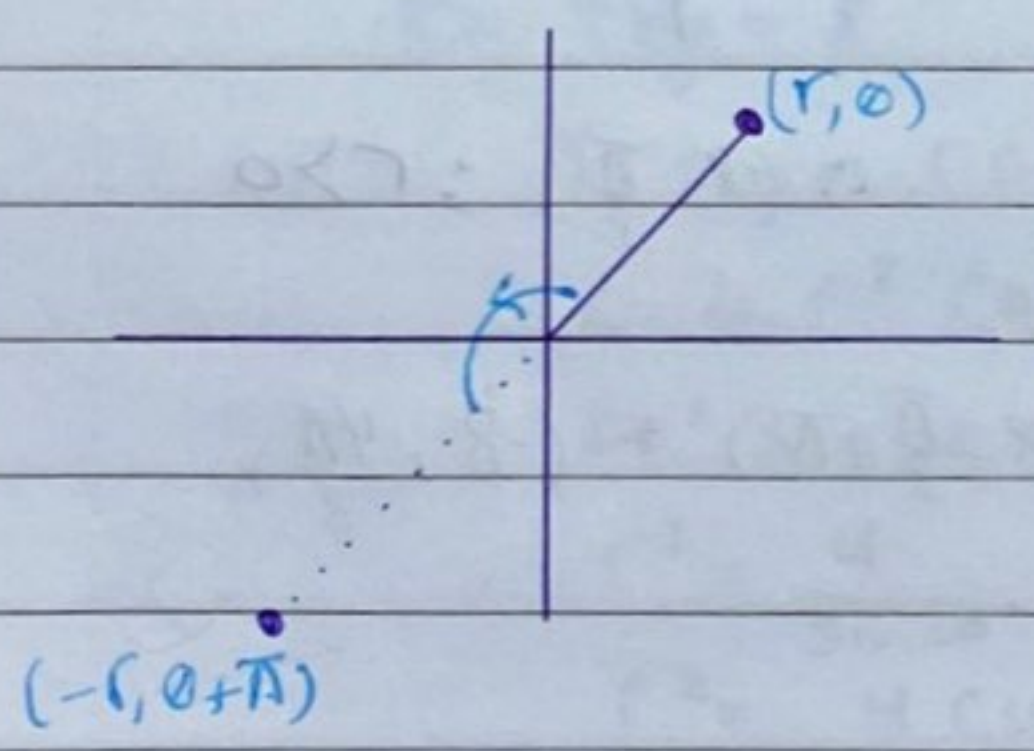
③ $y = 2\sin(\frac{2\pi}{3}) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$

$\therefore (x,y) = (-1, \sqrt{3})$

② $(x,y) \longrightarrow (r,\theta)$

$r^2 = x^2 + y^2 \longrightarrow r = \pm\sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x} \longrightarrow \theta = \tan^{-1}(\frac{y}{x})$



all Polar coordinates:

if $r > 0$:

$(r,\theta) = (r, \theta + 2\pi n)$

$n \in \mathbb{Z}$ integer

if $r < 0$:

$(-r, \theta+\pi) = (-r, \theta + 2\pi n)$

$n \in \mathbb{Z}$ integer

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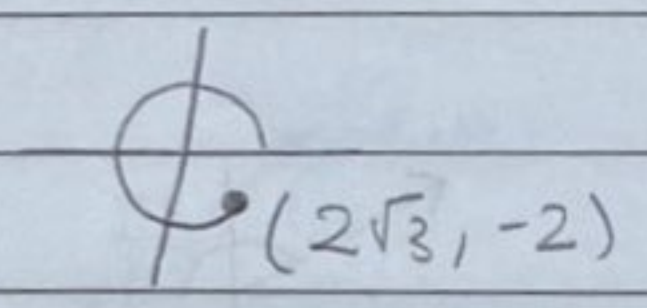
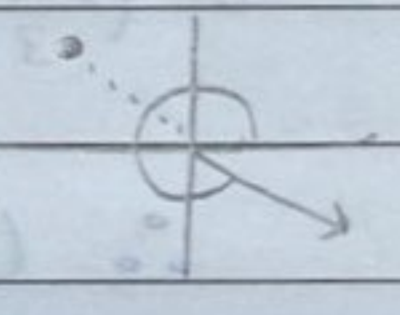
Example: Find all polar coordinates for $(2\sqrt{3}, -2)$

① 

② $r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = 4$

③ $\theta = \tan^{-1}\left(\frac{-2}{2\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{-\pi}{6}$

$\therefore r, \theta = \left(4, \frac{-\pi}{6}\right)$



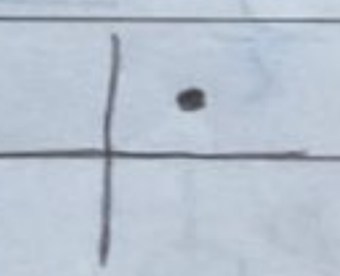
all polar

④ if $r > 0$: $\left(4, \frac{-\pi}{6} + 2n\pi\right)$, $n \in \mathbb{Z}$

if $r < 0$: $\left(-4, \frac{-\pi}{6} + \pi + 2n\pi\right) = \left(-4, \frac{5\pi}{6} + 2n\pi\right)$, $n \in \mathbb{Z}$

Example: Find the polar coordinates (r, θ) of the point $(4, 4\sqrt{3})$, where $r < 0$, and $0 \leq \theta \leq 2\pi$.

① $r = \sqrt{(4)^2 + (4\sqrt{3})^2} = 8$



② $\theta = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \frac{\pi}{3}$ $\therefore \left(8, \frac{\pi}{3}\right)$; $r > 0$

③ $r < 0$, $0 \leq \theta \leq 2\pi$ $\therefore \left(-8, \frac{\pi}{3} + \pi\right) = \left(-8, \frac{4\pi}{3}\right)$

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Example: The polar coordinates (r,θ) of the cartesian point $(-3,-3)$, $r<0$, $\theta \in [0,2\pi]$ is given by:

① $r = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$

② $\theta = \tan^{-1}\left(\frac{-3}{-3}\right) = \tan^{-1}(1) = \frac{5\pi}{4}$ $\therefore \left(3\sqrt{2}, \frac{5\pi}{4}\right)$ $r>0$

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$\rightarrow 5\pi + \pi \notin [0, 2\pi]$

③ $r<0$ $\therefore -3\sqrt{2}, \left(\frac{5\pi}{4} - \pi\right) = \left(-3\sqrt{2}, \frac{\pi}{4}\right)$

Example: Express the following equation in polar coordinates.

1> $x^2 + y^2 = 4$

$r^2 = 4 \rightarrow r = \pm 2$

2> $x^2 + y^2 + 7y = 0$

$r^2 + 7r\sin\theta = 0$

3> $4xy = 8$

$4r\cos\theta \cdot r\sin\theta = 8$

$2 \cdot \underline{2} \cdot r^2 \underline{\cos\theta \sin\theta} = 8$

$2r^2 \sin 2\theta = 8$

$r^2 = \frac{4}{\sin 2\theta}$

$r^2 = 4\csc 2\theta$

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4> $(x^2+y^2)^2 = x^2-y^2$

$(r^2)^2 = r^2\cos^2\theta - r^2\sin^2\theta$

$r^2 r^2 = r^2[\cos^2\theta - \sin^2\theta]$

$r^2 = \cos 2\theta$, Find domain θ ??

$r = \pm\sqrt{\cos 2\theta}$ $\therefore \cos 2\theta \geq 0$

$-\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2}$

$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

5> $y = \frac{1}{\sqrt{3}}x$

$\frac{y}{x} = \frac{1}{\sqrt{3}} \longrightarrow \tan\theta = \frac{1}{\sqrt{3}} \longrightarrow \theta = \frac{\pi}{6}$

Example : Find the cartesian equation for :

1> $(r = \cos\theta) \times r$

$r^2 = r\cos\theta$

$x^2+y^2 = x$

2> $r^2 = \sin 2\theta$

$(r^2 = 2\sin\theta\cos\theta) \times r^2$

$(r^2)^2 = 2.r\sin\theta . r\cos\theta$

$(x^2+y^2)^2 = 2y.x$

3> $r = \frac{6}{3-\sin\theta}$

$3r - r\sin\theta = 6$

$(3r = 6 + r\sin\theta)^2$

$9r^2 = (6+r\sin\theta)^2$

$x^2+y^2 = \frac{(6+y)^2}{9}$

4> $r = \sec\theta . \tan\theta$

$r = \frac{1}{\cos\theta} . \frac{\sin\theta}{\cos\theta}$

$r = \frac{\sin\theta}{\cos^2\theta}$

$(r\cos^2\theta = \sin\theta) \times r$

$r^2\cos^2\theta = r\sin\theta$

$x^2 = y$

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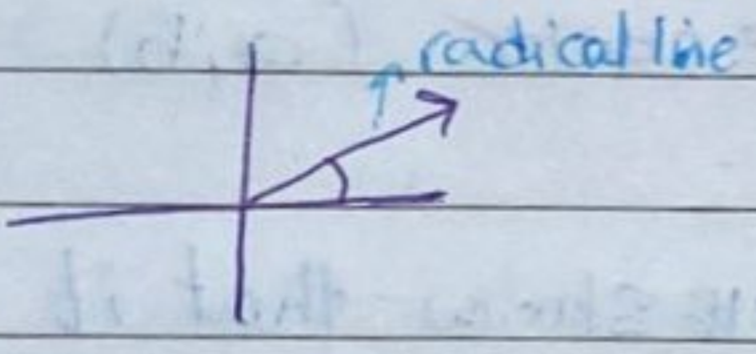
Subject Lec 12 : Area in polar coordinates Date No.

- Basic polar curves:

△1 Radical line

$\theta = \theta_0$ Ex: $\theta = \frac{\pi}{4}$



△2 General line

$r = \frac{c}{a\cos\theta + b\sin\theta}$, a, b, c constants

$ar\cos\theta + br\sin\theta = c$

$ax + by = c$

Example: Find the slope of $r = \frac{3}{2\cos\theta + 4\sin\theta}$

① $2r\cos\theta + 4r\sin\theta = 3$

$2x + 4y = 3$

$4y = 3 - 2x$

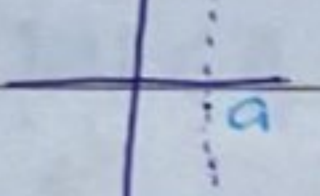
$y = \frac{3}{4} - \frac{\cancel{4}}{2}x$

② slope $= \frac{-2}{4}$

$= \frac{-1}{2}$

Note that:

△1 $r = \frac{a}{\cos\theta} = a\sec\theta$ "vertical line $x = a$"



△2 $r = \frac{b}{\sin\theta} = b\csc\theta$ "Horizantal line $y = b$"

General line ⇓ ⇑

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3 Circles

$r = 2a\cos\theta + 2b\sin\theta$

circle center (a,b) , radius $r = \sqrt{a^2+b^2}$

* show that it is acircle equation:

$(r = 2a\cos\theta + 2b\sin\theta) \times r$

$r^2 = 2ar\cos\theta + 2br\sin\theta$

$x^2+y^2 = 2ax + 2by$

$(x^2-2ax) + (y^2-2by) = 0 \Rightarrow$ إكمال مربع لكل قوس

$(x^2-2ax+a^2) + (y^2-2by+b^2) = a^2+b^2$

$(x-a)^2 + (y-b)^2 = a^2+b^2$

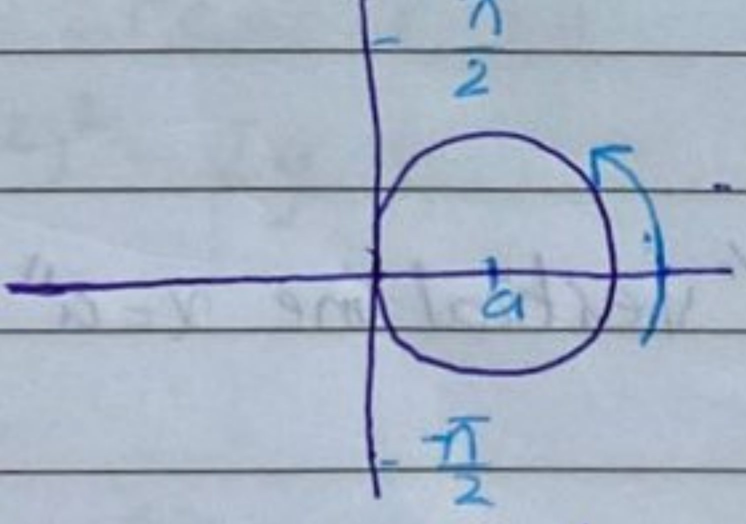
circlecenter (a,b) , radius $= \sqrt{a^2+b^2}$

Example: The center of the circle $r = 4\sin\theta - 3\cos\theta$ is:

$(-\frac{3}{2}, 2)$

Note that:

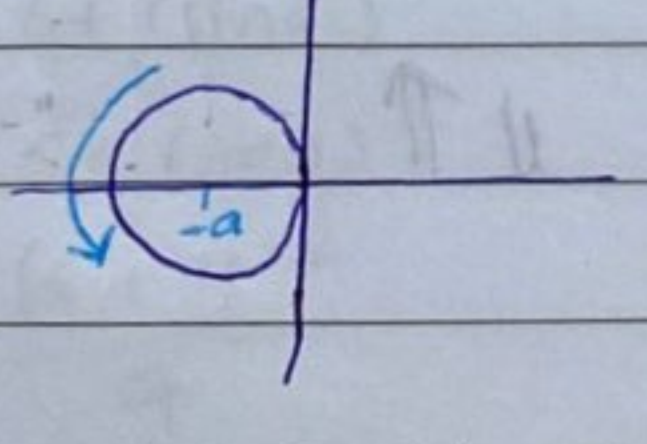
1. $r = 2a\cos\theta$ circle center $(a,0)$, $r = a$



$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

* counterclockwise *

2. $r = -2a\cos\theta$ circle center $(-a,0)$, $r = a$



$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

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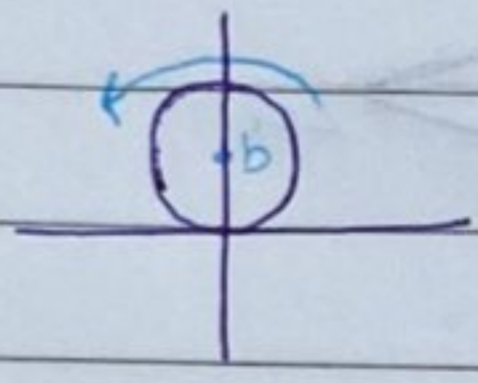
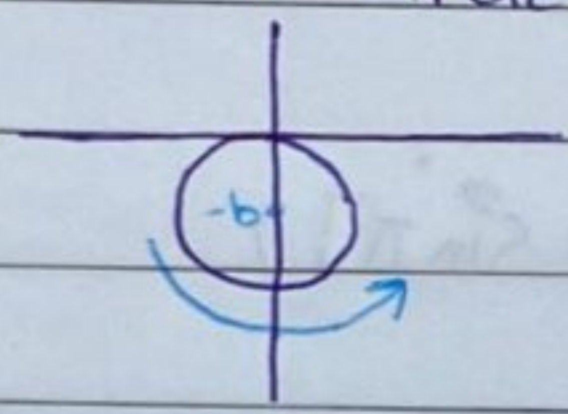
③ $r = 2b\sin\theta$ circle center $(0, b)$, $r = b$



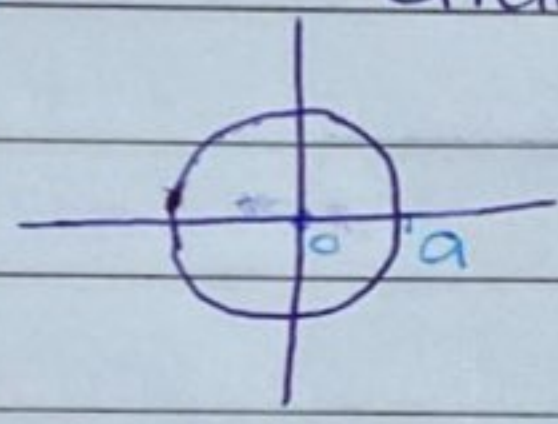
$0 \leq \theta \leq \pi$

④ $r = 2b\sin\theta$ circle center $(0, -b)$, $r = b$



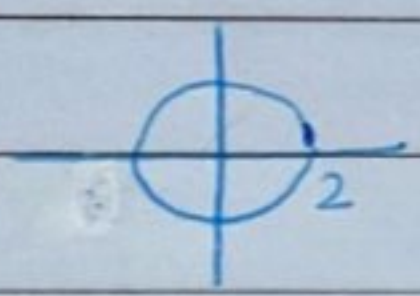
$\pi \leq \theta \leq 2\pi$

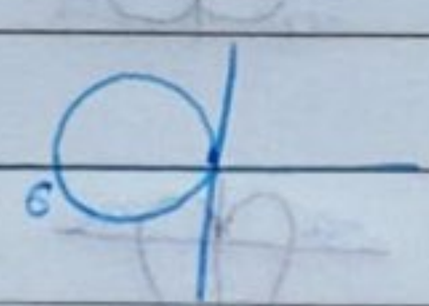
⑤ $r = a$ circle center $(0, 0)$, $r = a$

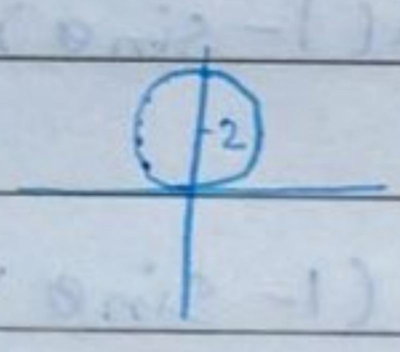


$0 \leq \theta \leq 2\pi$

Example: what is the equation for each one?





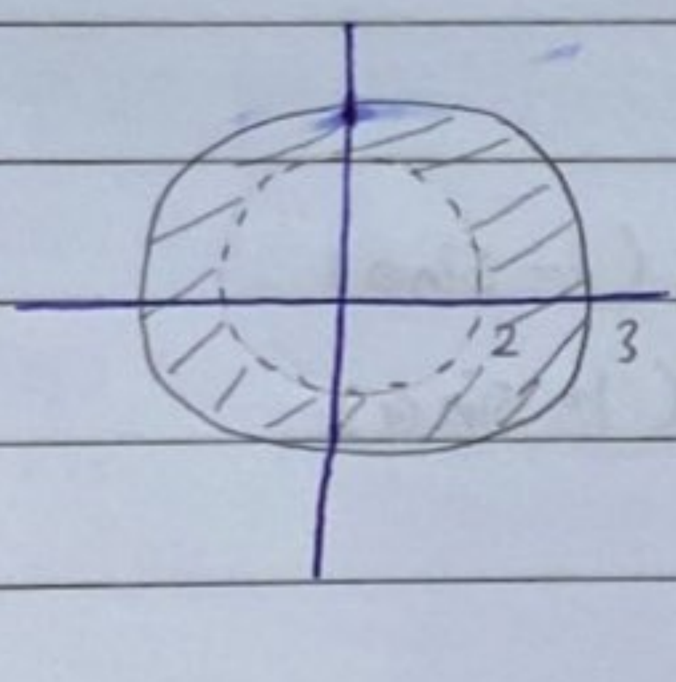


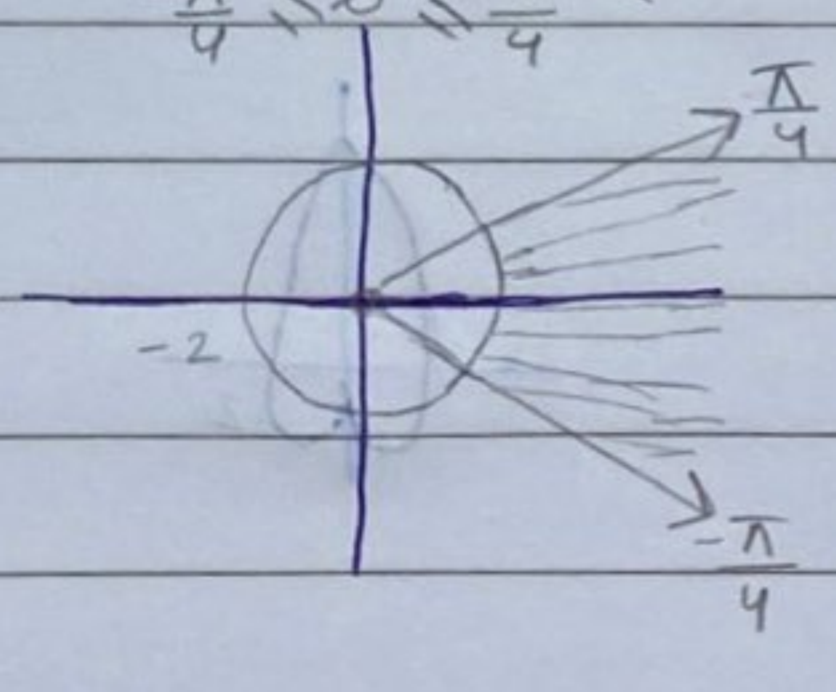
$r = 2$ $\quad$ $r = -6\cos\theta$ $\quad$ $r = 4\sin\theta$

Example: sketch the region

1> $2 < r \leq 3$ $\quad$ 2> $r \geq 2$, $|\theta| \leq \frac{\pi}{4}$

$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$





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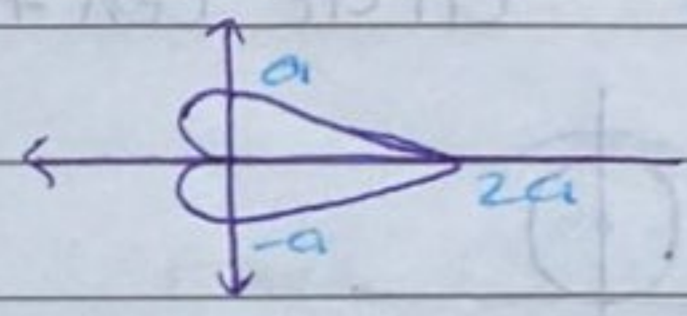
(1) Cardiod

(1) $r = a(1+\cos\theta)$

$= -a(1-\cos\theta)$



↳ How? $(r,\theta) = (-r, \theta+\pi)$

* $r = a(1+\cos\theta)$

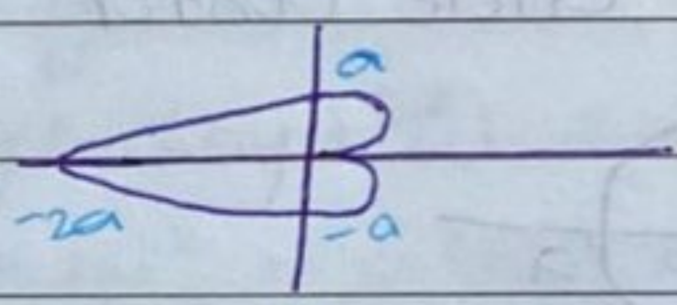
$-r = a(1+\cos(\theta+\pi))$

$r = -a(1+[\cos\theta\cos\pi - \sin\theta \cancel{\sin\pi}])$

$r = -a(1-\cos\theta)$

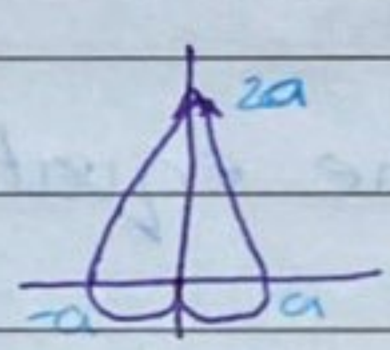
(2) $r = a(1-\cos\theta)$

$= -a(1+\cos\theta)$



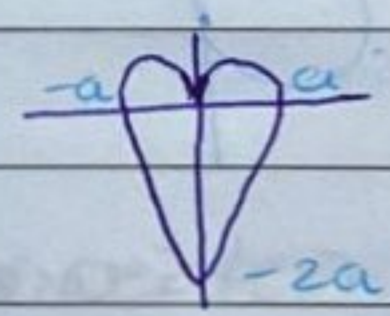
(3) $r = a(1+\sin\theta)$

$= -a(1-\sin\theta)$

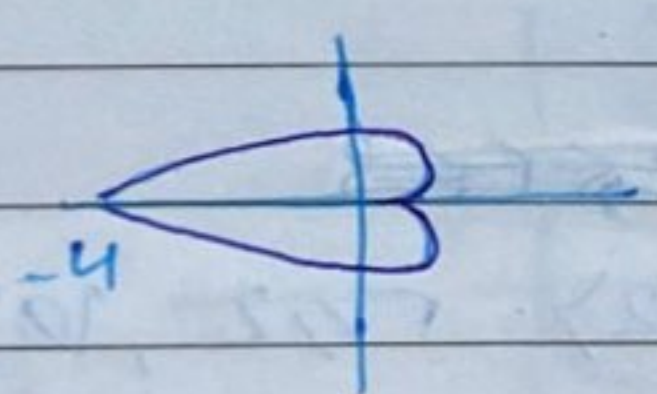


(4) $r = a(1-\sin\theta)$

$= -a(1+\sin\theta)$

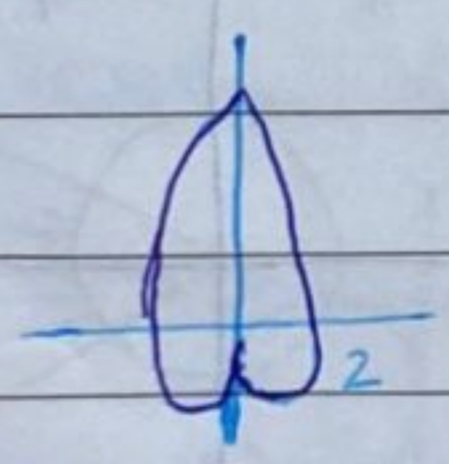


Example:



$r = -2(1+\cos\theta)$

$= 2(1-\cos\theta)$



$r = -2(1-\sin\theta)$

$= 2(1+\sin\theta)$

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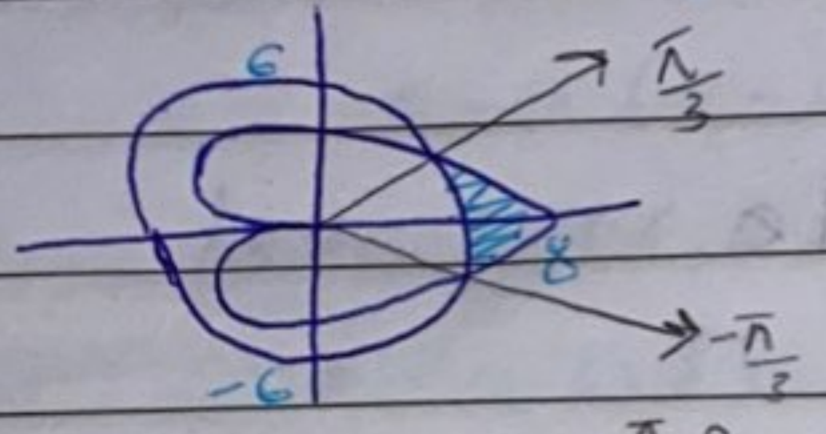
Example: find the area of the region that is inside $r = 4+4\cos\theta$ and outside $r = 6$



$4+4\cos\theta = 6$

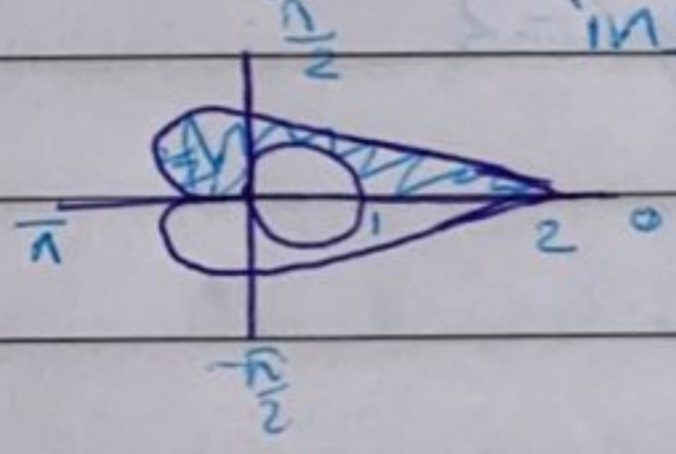
$4\cos\theta = 2 \qquad \theta = \pm\frac{\pi}{3}$

$\cos\theta = \frac{1}{2}$

$\Rightarrow A = 2 * \frac{1}{2}\int_0^{\frac{\pi}{3}} (4+4\cos\theta)^2 - (6)^2 \, d\theta$

$= 18\sqrt{3} - 4\pi$

Example: find the area of the region in the upper half plane outside the circle $r=\cos\theta$ and inside the cardiod $r = 1+\cos\theta$

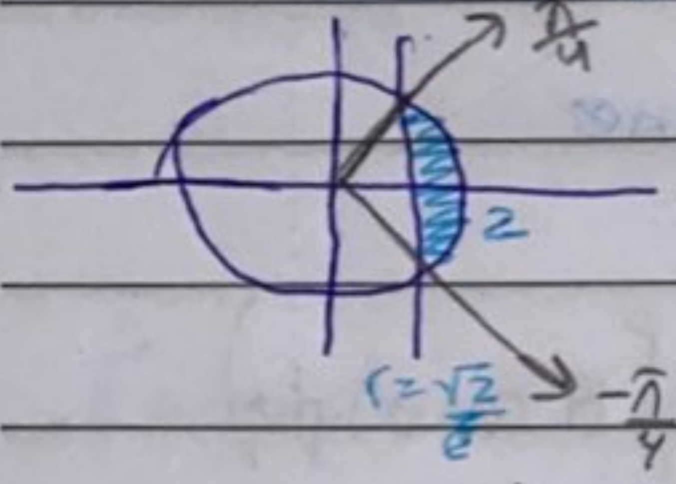


$1+\cos\theta = \cos\theta$

$1 \neq 0$

$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} [(1+\cos\theta)^2 - (\cos\theta)^2] d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1+\cos\theta)^2 d\theta$

Example: find the area of the region that is inside $r=2$ and to right of $r\cos = \sqrt{2}$



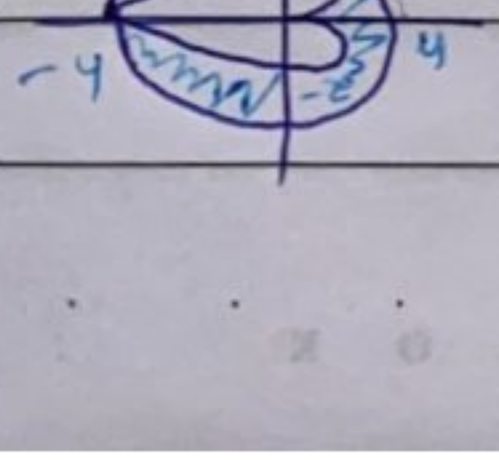
$2 = \frac{\sqrt{2}}{\cos\theta}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\theta = \pm\frac{\pi}{4}$

$A = 2 * \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[(2)^2 - \left(\frac{\sqrt{2}}{\cos\theta}\right)^2\right] d\theta = \pi - 2$

Example: find the area of region outside $r = 2-2\cos\theta$ and inside $r=4$.



$4 = 2-2\cos\theta \qquad \cos\theta = -1$

$2 = 1-\cos\theta \qquad \theta = \pi$

$A = 2 * \frac{1}{2}\int_0^{\pi} [(4)^2 - (2-2\cos\theta)^2] \, d\theta = 10\pi$

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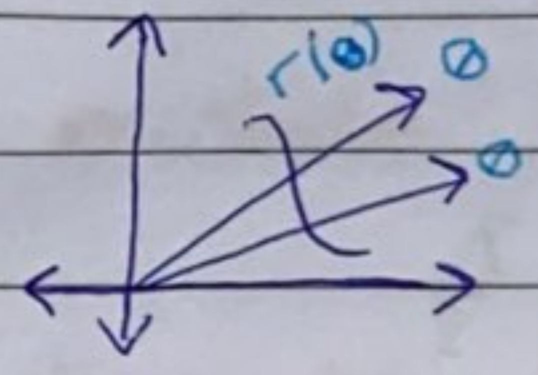
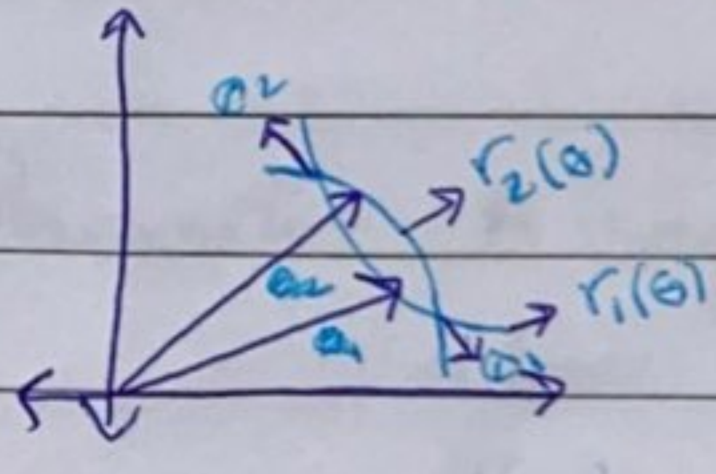
- Area in Polar curves

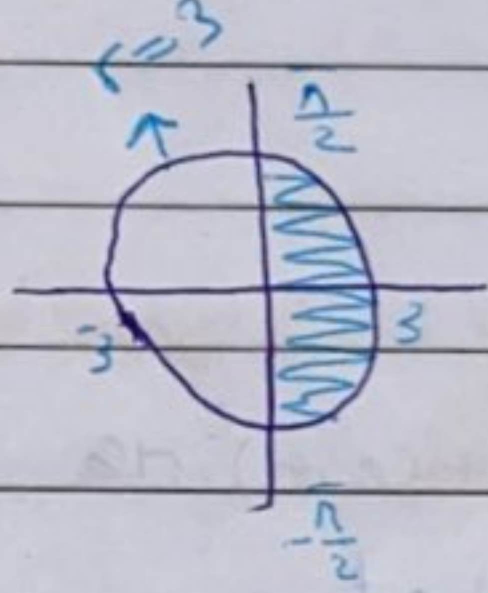


$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r(\theta))^2 \, d\theta$$



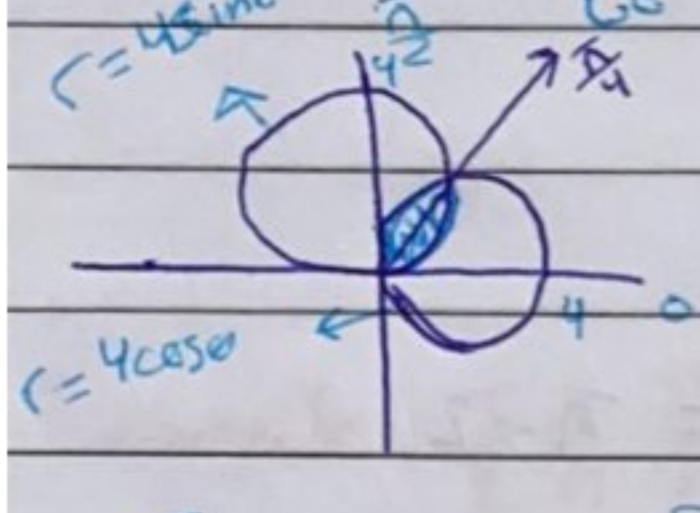
$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r_2(\theta))^2 - (r_1(\theta))^2$$

Example: find the area of the region in the right half plane and inside the circle $r=3$



$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3)^2 \, d\theta \ \ldots$$

Example: find the area of the region that is common to the circles $r = 4\cos\theta$, $r = 4\sin\theta$



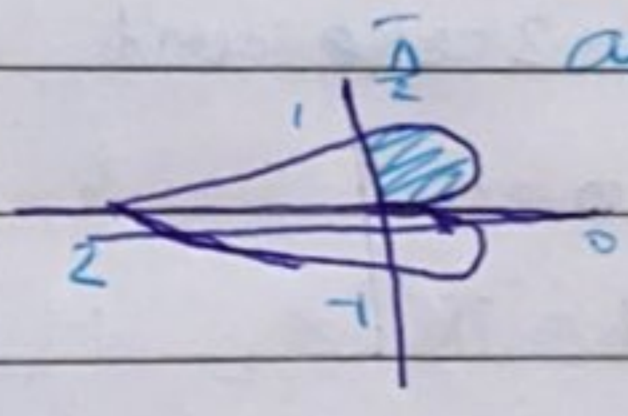
$4\sin\theta = 4\cos\theta$

$1 = \tan\theta$ $\quad A = \frac{1}{2}\left(\int_{0}^{\frac{\pi}{4}} (4\sin\theta)^2 + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (4\cos\theta)^2 \, d\theta\right) \ldots$

$\theta = \frac{\pi}{4}$

Example: find the area of the region inside $r=1-\cos\theta$ and in the first quadrant? ربع أول

$r = 1\times(1-\cos\theta)$



$$A = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} (1-\cos\theta)^2 \, d\theta$$

$$= \frac{3\pi}{8} - 1$$

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Identities

* $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

* $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

* $\sin(2x) = 2 \sin x \cos x$

* $\cos(2x) = \cos^2 x - \sin^2 x$
  $= 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x$

* $\sin^2 x + \cos^2 x = 1$

* $\tan^2 x + 1 = \sec^2 x$

* $1 + \cot^2 x = \csc^2 x$

* $\cos(-x) = \cos x$

* $\sin(-x) = -\sin x$

* $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

* $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

مصمم/ تغا حنل c

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1. $f(x) = c \Rightarrow f'(x) = 0$

2. $f(x) = ax \Rightarrow f'(x) = a$

3. $f(x) = x^n \Rightarrow f'(x) = n\,x^{n-1}$

4. $\frac{d}{dx}\left[a f(x) \pm b g(x)\right] = a f'(x) \pm b g'(x)$

5. $\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

6. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$

7. $\frac{d}{dx}\left[(f(x))^n\right] = n\,(f(x))^{n-1} * f'(x)$

8. $\frac{d}{dx}\left[\sqrt{f(x)}\right] = \frac{f'(x)}{2\sqrt{f(x)}}$

9. $\frac{d}{dx}\left[\sin(g(x))\right] = \cos(g(x)) \cdot g'(x)$

10. $\frac{d}{dx}\left[\cos(g(x))\right] = -\sin(g(x)) \cdot g'(x)$

11. $\frac{d}{dx}\left[\tan(g(x))\right] = \sec^2(g(x)) \cdot g'(x)$

12. $\frac{d}{dx}\left[\cot(g(x))\right] = -\csc^2(g(x)) \cdot g'(x)$

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13. $\frac{d}{dx}\left[\sec(g(x))\right] = \sec(g(x))\tan(g(x)) \cdot g'(x)$

14. $\frac{d}{dx}\left[\csc(g(x))\right] = -\csc(g(x))\cot(g(x)) \cdot g'(x)$

15. $\frac{d}{dx}\left[\ln(g(x))\right] = \frac{g'(x)}{g(x)}$

16. $\frac{d}{dx}\left[\log_a(g(x))\right] = \frac{g'(x)}{g(x)\cdot \ln a}$

17. $\frac{d}{dx}\left[a^{g(x)}\right] = a^{g(x)} \cdot g'(x) \cdot \ln a$

18. $\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)} \cdot g'(x)$

19. $\frac{d}{dx}\left[\sin^{-1}(g(x))\right] = \frac{g'(x)}{\sqrt{1-(g(x))^2}}$

20. $\frac{d}{dx}\left[\cos^{-1}(g(x))\right] = \frac{-g'(x)}{\sqrt{1-g^2(x)}}$

21. $\frac{d}{dx}\left[\tan^{-1}(g(x))\right] = \frac{g'(x)}{1+g^2(x)}$

22. $\frac{d}{dx}\left[\cot^{-1}(g(x))\right] = \frac{-g'(x)}{1+g^2(x)}$

23. $\frac{d}{dx}\left[\sec^{-1}(g(x))\right] = \frac{g'(x)}{|g(x)|\sqrt{g^2(x)-1}}$

24. $\frac{d}{dx}\left[\csc^{-1}(g(x))\right] = \frac{-g'(x)}{|g(x)|\sqrt{g^2(x)-1}}$

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25. $\frac{d}{dx}\left[\sinh(g(x))\right] = \cosh(g(x)) \cdot g'(x)$

26. $\frac{d}{dx}\left[\cosh(g(x))\right] = \sinh(g(x)) \cdot g'(x)$

27. $\frac{d}{dx}\left[\tanh(g(x))\right] = \operatorname{sech}^2(g(x)) \cdot g'(x)$

28. $\frac{d}{dx}\left[\coth(g(x))\right] = -\operatorname{csch}^2(g(x)) \cdot g'(x).$

29. $\frac{d}{dx}\left[\operatorname{sech}(g(x))\right] = -\operatorname{sech}(g(x))\tanh(g(x)) \cdot g'(x)$

30. $\frac{d}{dx}\left[\operatorname{csch}(g(x))\right] = -\operatorname{csch}(g(x))\coth(g(x)) \cdot g'(x)$

31. $\int a\, dx = ax + C$

32. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$

33. $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$

34. $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b) + C$

35. $\int \sin(ax+b)\, dx = \frac{-1}{a}\cos(ax+b) + C$

36. $\int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + C$

37. $\int \csc^2(ax+b)\, dx = \frac{-1}{a}\cot(ax+b) + C$

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(38) $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$

(39) $\int \csc(ax+b)\cot(ax+b)\,dx = \frac{-1}{a}\csc(ax+b) + C$

(40) $\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$

(41) $\int b^{ax+d}\,dx = \frac{b^{ax+d}}{a.\ln b} + C$

(42) $\int \frac{1}{x}\,dx = \ln|x| + C$

(43) $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

(44) $\int \frac{g'(x)}{1+g^2(x)}\,dx = \tan^{-1}(g(x)) + C$

(45) $\int \frac{g'(x)}{\sqrt{1-g^2(x)}}\,dx = \sin^{-1}(g(x)) + C$

(46) $\int \frac{g'(x)}{|g(x)|\sqrt{g^2(x)-1}}\,dx = \sec^{-1}(g(x)) + C$

(47) $\int \sinh(ax+b)\,dx = \frac{1}{a}\cosh(ax+b) + C$

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48. $\int \cosh(ax+b)\,dx = \frac{1}{a}\sinh(ax+b) + C$

49. $\int \operatorname{sech}^2(ax+b)\,dx = \frac{1}{a}\tanh(ax+b) + C$

50. $\int \operatorname{csch}^2(ax+b)\,dx = \frac{-1}{a}\coth(ax+b) + C$

51. $\int \operatorname{sech}(ax+b)\tanh(ax+b)\,dx = \frac{-1}{a}\operatorname{sech}(ax+b) + C$

52. $\int \operatorname{csch}(ax+b)\coth(ax+b)\,dx = \frac{-1}{a}\operatorname{csch}(ax+b) + C$

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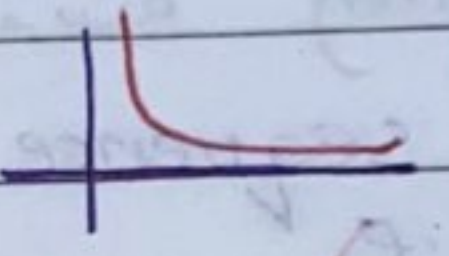
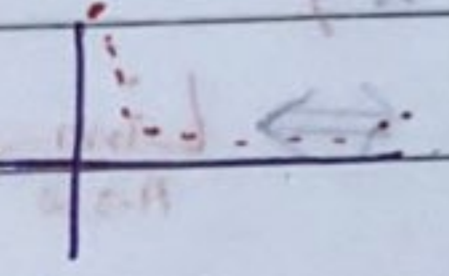
Subject Lec 13

- Sequences

ex. sketch $f(x) = \frac{1}{x}$, $x \in (0, \infty)$



Ex. sketch $f(x) = \frac{1}{x}$, $x \in \mathbb{N}$ natural number



Remark: A sequence of real numbers is just a function whose domain is "$\mathbb{N}$" denoted by $\{f(n)\}_{n=1}^{\infty}$

Range $\Rightarrow \mathbb{R}$, $f(n) : \mathbb{N} \rightarrow \mathbb{R}$

$(a_n)_{n=1}^{\infty}$, $\{a_n\}_{n=1}^{\infty}$, a_n, $n \geq 1$

Example given that $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$ Find the general form.

$a_1 = \frac{1}{2}$, $a_2 = \frac{2}{3}$, $a_3 = \frac{3}{4}$, $a_5 = \frac{5}{6}$

index

$\therefore a_n = \{\frac{n}{n+1}\}_{n=1}^{\infty}$

Example $\{\frac{1}{n}\}_{n=1}^{\infty}$ then $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = \frac{1}{4}$

$= \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

Example $\{1, 2, 4, \dots\} = \{2^{n-1}\}_{n=1}^{\infty}$

$a_1 \; a_2 \; a_3$ General Form

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Example: $\{-1, 1, -1, 1, -1, 1, \ldots\} = \{(-1)^n\}_{n=1}^{\infty}$
↳ alternating متذبذبة

iff

Theorem: • The sequence $\{a_n\}_{n=1}^{\infty}$ converge to L (if and only if)

$$\Leftrightarrow \lim_{n\to\infty} a_n = L$$

• if $\lim_{n\to\infty} a_n$ d.n.e $\Leftrightarrow$ $\{a_n\}$ diverge

Example: Determine whether the following sequence conv or div?

① $\{3^n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} 3^n = \infty$ $\therefore$ div

② $\{\cos\frac{\pi}{n}\}_{n=1}^{\infty}$, $\lim_{n\to\infty} (\frac{\pi}{n}) = \cos(0) = 1$ $\therefore$ conv to 1

③ $\left\{\frac{3n^2+5}{2n^2-1}\right\}_{n=1}^{\infty}$, $\lim_{n\to\infty} \frac{3n^2+5}{2n^2-1} = \frac{3}{2}$ $\therefore$ conv to $\frac{3}{2}$

④ $\left\{\frac{(1+n)^3}{3n^2+2}\right\}^{\infty}$, $\lim_{n\to\infty} \frac{(1+n)^3}{3n^2+2} = \infty$ $\therefore$ div

⑤ $\{\ln n\}_{n=1}^{\infty}$, $\lim_{n\to\infty} \ln n = \infty$ $\therefore$ div

⑥ $\left\{\frac{n^2+10n-5}{e^{2n}}\right\}_{n=1}^{\infty}$ $\lim_{n\to\infty} \frac{n^2+10n-5}{e^{2n}} = \frac{\infty}{\infty}$

L'H $\lim_{n\to\infty} \frac{2n+10}{2e^{2n}} = \frac{\infty}{\infty}$

L'H $\lim_{n\to\infty} \frac{2}{4e^{2n}} = 0$ $\therefore$ conv to 0

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Note that :-

① $\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a \quad , a \in \mathbb{R}$

② $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$

③ $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$

⑦ $\left\{\left(1+\frac{1}{n}\right)^{-3n}\right\}_{n=1}^{\infty}$ $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-3n} = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^{-3} = e^{-3}$

$\therefore$ Converge to e^{-3}

⑧ $\{n\sin(\frac{\pi}{n})\}_{n=1}^{\infty}$ $\lim_{n\to\infty} n \sin\frac{\pi}{n} \quad (\infty\cdot 0)$

L.H $\lim_{n\to\infty} \frac{\sin(\pi/n)}{n^{-1}} \quad \left(\frac{0}{0}\right)$

$\lim_{n\to\infty} \frac{\cos(\pi/n) * \frac{-\pi}{n^2}}{-(n)^{-2}}$

$\lim_{n\to\infty} \pi\cos(\pi/n) = \pi * 1 = \pi$

$\therefore$ converge to π

⑨ the limit of the seq $a_n = \ln(n^2+3n) - 2\ln(3n+1)$ is :-

$\Rightarrow \lim_{n\to\infty} \ln(n^2+3n) - \ln(3n+1)^2$

$= \lim_{n\to\infty} \ln\left(\frac{n^2+3n}{(3n+1)^2}\right) = \ln\left(\lim_{n\to\infty} \frac{n^2+3n}{(3n+1)^2}\right) = \ln\frac{1}{9} = -\ln 9$ (cont.)

⑩ limit of the seq $a_n = \sqrt{n^2+n+1} - n$ is :-

$\lim_{n\to\infty} \sqrt{n^2+n+1} - n \qquad \infty - \infty$

$= \lim_{n\to\infty} \left(\sqrt{n^2+n+1} - n\right)\left(\frac{\sqrt{n^2+n+1}+n}{\sqrt{n^2+n+1}+n}\right)$

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$$\lim_{n\to\infty} \frac{(n^2+n+1)-(n^2)}{\sqrt{n^2+n+1}+n}$$

$$\lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+n+1}+n}$$

$$\lim_{n\to\infty} \frac{n(1+\frac{1}{n})}{|n|\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+n}$$

$\hookrightarrow n$

$$\lim_{n\to\infty} \frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1}$$

$$\lim_{n\to\infty} \frac{(1+\frac{1}{n})}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1} = \frac{1+0}{\sqrt{1+0+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$ $\therefore$ converge to $\frac{1}{2}$

(11) $\left\{\left(\frac{n+1}{n-7}\right)^{3n}\right\}_{n=1}^{\infty}$ conv or div ?

$$\lim_{n\to\infty}\left(\frac{n+1}{n-7}\right)^{3n} = \lim_{n\to\infty}\left(\frac{\frac{n+1}{n}}{\frac{n-7}{n}}\right)^{3n}$$

$$\lim_{n\to\infty} \frac{(1+\frac{1}{n})^{3n}}{(1-\frac{7}{n})^{3n}} = \frac{(e^1)^3}{(e^{-7})^3} = e^3 \cdot e^{21} = e^{24}$$

$\therefore$ conv to e^{24}

(12) The limit of the seq $a_n = \sqrt[n]{n+1}$ is:

$$\lim_{n\to\infty} \sqrt[n]{n+1} = \lim_{n\to\infty} (n+1)^{\frac{1}{n}} \quad \infty^0$$

$y = (n+1)^{\frac{1}{n}}$

$\ln y = \frac{1}{n}\ln(n+1)$

$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln(n+1)}{n} \quad \frac{\infty}{\infty}$

L'H

$\ln y = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{1} = 0$

$\ln y = 0$

$y = e^0$

$y = 1$ $\therefore$ conv to 1

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(13) The limit of the seq $\{(4n+e^n)^{\frac{2}{n}}\}$ is:

$$\lim_{n\to\infty} (4n+e^n)^{\frac{2}{n}} \quad \infty^0$$

$$y = (4n+e^n)^{\frac{2}{n}}$$

$$\ln y = \frac{2}{n} \ln(4n+e^n)$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{2}{n} \ln(4n+e^n)$$

$$\ln y = \lim_{n\to\infty} 2 \cdot \frac{\frac{4+e^n}{4n+e^n}}{1}$$

$$\ln y = \lim_{n\to\infty} 2 \cdot \left(\frac{e^n}{4+e^n}\right)$$

$$\ln y = \lim_{n\to\infty} 2 \cdot \frac{e^n}{e^n} = 2$$

$\ln y = 2$

$y = e^2$

$\therefore$ conv to e^2

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Subject Lec 14 Date No.

• Segence Part II

Note that:-

• if $(a_n)_{n=1}^{\infty}$ conv to L then Both (a_{2n}) & (a_{2n+1}) conv to L

eventerms ↲ ↳ odd terms

• if a_{2n+1} "odd terms" conv to L
& a_{2n} "even terms" conv to L
then $\{a_n\}$ conv to L

• if a_{2n+1} "odd terms" conv to L
a_{2n} "even terms" conv to M
then $\{a_n\}$ div

Example: Let $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \begin{cases} 4+\frac{1}{n} & , n \text{ odd} \\ 2-\frac{1}{n^3} & , n \text{ even} \end{cases}$

is conv or div ?

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} 2-\frac{1}{n^3} = 2$$

a_n div

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 4+\frac{1}{n} = 4$$

Example: $\{\cos(n\pi)\}_{n=1}^{\infty}$ conv or div ?

$= \{-1, 1, -1, 1, -1, \ldots\}$ $a_1=-1, a_3=-1, a_5=-1 \ldots$

$\therefore a_{2n} = 1$, $\therefore a_{2n+1} = -1$ $a_2=1, a_4=1, a_6=1 \ldots$

a_{2n} conv to 1 a_{2n+1} conv to -1 $\therefore$ div

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Example: $\{1, 1, 1, 1, 1, 1 \ldots\}$

$\therefore |a_n| = 1 \quad \therefore$ conv to 1

Example: $\{1 + (-1)^n\}_{n=1}^{\infty}$ conv or div ?

$\{0, 2, 0, 2, 0, 2, 0, 2, \ldots\}$

$\Rightarrow a_{2n}$ conv to 2 , a_{2n+1} conv to 0 $\quad \therefore$ div

Example: $\{2, \overset{\to a_2}{\frac{1}{2}}, 4, \overset{\to a_4}{\frac{1}{4}}, 6, \frac{1}{6}, 8, \frac{1}{8} \ldots\}$

$a_1 = 1+1 \quad a_3 = 1+3 \quad a_5 = 5+1$

conv or div?

$$= \begin{cases} n+1 & , n \text{ odd} \to \lim_{n\to\infty} n+1 = \infty \\ \frac{1}{n} & , n \text{ even} \to \lim_{n\to\infty} \frac{1}{n} = 0 \end{cases} \quad \therefore \text{div}$$

- Alternating seq $((-1)^n f_n) \quad (((-1)^{n+1} f_n) \ldots$

Theorem: if $\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0$

Example: $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ conv or div ?

$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$, $\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \left|\frac{(-1)^n}{n}\right|$ conv to 0 then $\left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty}$ conv to zero

Example: $\left\{\frac{(-1)^{n+1}}{3^n}\right\}_{n=1}^{\infty}$ conv or div?

$\left|\frac{(-1)^{n+1}}{3^n}\right| = \frac{1}{3^n}$, $\lim_{n\to\infty} \frac{1}{3^n} = 0 \Rightarrow$ since $\left|\frac{(-1)^{n+1}}{3^n}\right|$ conv to 0 then $\left(\frac{(-1)^{n+1}}{3^n}\right)_{n=1}^{\infty}$ conv to zero

NOTEBOOK

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Example: $\left\{\frac{(-1)^n(n^2+1)}{2n^2+3n+5}\right\}_{n=1}^{\infty}$ conv or div ?

$|a_n| = \left|\frac{(-1)^n(n^2+1)}{2n^2+3n+5}\right| = \frac{n^2+1}{2n^2+3n+5}$ $\therefore$ conv to $\frac{1}{2}$

$a_{2n} = \frac{(-1)^{2n}((2n)^2+1)}{2(2n)^2+3(2n)+5} \Rightarrow \frac{4n^2+1}{8n^2+6n+5}$ $\therefore$ conv to $\frac{1}{2}$

$a_{2n+1} = \frac{(-1)^{2n+1}((2n+1)^2+1)}{2(2n+1)^2+3(2n+1)+5} = \frac{-4n^2-4n}{8n^2+14n+10}$ $\therefore$ conv to $\frac{-1}{2}$

$\rightarrow$ since odd term conv to $\frac{-1}{2}$

and even term conv to $\frac{1}{2}$ $\therefore$ div

Example: $\left\{\frac{(-1)^n}{n^2}\right\}$ conv or div

$|a_n| = \left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$ conv to zero

$\left\{\frac{(-1)^n}{n^2}\right\}_{n=1}^{\infty}$ conv to zero

Example: $\left\{\frac{(-1)^n n+3}{5n+2}\right\}_{n=1}^{\infty}$ conv or div ?

$|a_n| = \left|\frac{(-1)^n n+3}{5n+2}\right| = \frac{1}{5}$

$a_{2n} = \frac{(-1)^{2n}(2n)+3}{5(2n)+2} = \frac{2n+3}{10n+2}$ $\therefore$ conv to $\frac{1}{5}$

$a_{2n+1} = \frac{(-1)^{2n+1}(2n+1)+3}{5(2n+1)+2} = \frac{-2n+2}{10n+7}$ $\therefore$ conv to $\frac{-1}{5}$

$\therefore$ div

Subject Date No.

Example: $\left\{\frac{(-1)^n}{n^2+n}\right\}_{n=1}^{\infty}$ conv or div ?

$\left|\frac{(-1)^n}{n^2+n}\right| = \frac{1}{n^2+n}$ conv to zero

$\left\{\frac{(-1)^n}{n^2+n}\right\}_{n=1}^{\infty}$ conv to zero

Example: $\left\{\frac{(-1)^n n + n^2}{(3n+1)^2}\right\}$ conv or div ?

$\left|\frac{(-1)^n n + n^2}{(3n+1)^2}\right|$ conv to $\frac{1}{9}$

$a_{2n} = \frac{(-1)^{2n}(2n) + (2n)^2}{(3(2n)+1)^2} = \frac{2n + 4n^2}{(6n+1)^2}$ conv to $\frac{4}{36} = \frac{1}{9}$

$a_{2n+1} = \frac{(-1)^{2n+1}(2n+1) + (2n+1)^2}{(3(2n+1)+1)^2} = \frac{-2n-1+4n^2+1+8n}{(6n+3+1)^2}$ conv $\frac{4}{36} = \frac{1}{9}$

since even term conv to $\frac{1}{9}$ and odd terms to $\frac{1}{9}$

$\therefore \left\{\frac{(-1)^n n + n^2}{(3n+1)^2}\right\}$ conv to $\frac{1}{9}$

- Squeezing Theorem

if we ask $(C_n)_{n=1}^{\infty}$ conv or div,

and

$a_n \leq C_n \leq b_n$ then $\therefore (C_n)_{n=1}^{\infty}$ conv to L

a_n conv to L b_n conv to L

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Example: $\left\{\frac{\sin^2(n)}{n}\right\}_{n=1}^{\infty}$ conv or div?

$\therefore \lim_{n\to\infty} \frac{\sin^2 n}{n} = 0$, $0 \leq \sin^2 n \leq 1$

$0 \leftarrow 0 \leq \frac{\sin^2 n}{n} \leq \frac{1}{n} \to 0$

$\downarrow$ 0

Example: $\left\{\frac{n+\sin n}{n}\right\}_{n=1}^{\infty}$ conv or div?

$\frac{n+\sin n}{n} = 1 + \frac{\sin}{n}$ $\quad -1 \leq \sin n \leq 1$

$0 \leftarrow \frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \to 0$

$\downarrow$ 0

$\left\{\frac{n+\sin n}{n}\right\} = \lim_{n\to\infty} 1 + \frac{\sin}{n}$

$= 1+0 = 1$ $\therefore$ conv to 1

Example: $\left\{\frac{3\cos 2n}{n}\right\}_{n=1}^{\infty}$ conv or div?

$\lim_{n\to\infty} \frac{3\cos(2n)}{n}$ $\quad -1 \leq \cos(2n) \leq 1$

$0 \leftarrow \frac{-3}{n} \leq \frac{3\cos(2n)}{n} \leq \frac{3}{n} \to 0$

$\downarrow$ 0

$\therefore \left\{\frac{3\cos(2n)}{n}\right\}_{n=1}^{\infty}$ conv to 0

Example: $\{\sin(n)\}_{n=1}^{\infty}$ conv or div?

$\lim_{n\to\infty} \sin(n) =$ d.n.e. div

Example: $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n=1}^{\infty}$ conv or div?

$= \{1, 0, -1, 0, 1, 0, -1, 0, 1 \ldots\}$

$\Rightarrow$ even term conv to 0 , odd term div $\therefore$ div

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Example: $\{\sin(n\pi)\}_{n=1}^{\infty}$ conv or div ?

$= \{0, 0, 0, 0, 0, 0, 0 \ldots\}$

Theorem: $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$ conv to zero

$0 \leq \;\; = \frac{n}{n} \cdot \frac{(n-1)}{n} \cdot \frac{(n-2)}{n} \cdot \frac{(n-3)}{n} \cdots \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \leq \frac{1}{n}$

conv to zero — conv to zero

- Geometric seq $\{r^n\}_{n=1}^{\infty}$

$= \{r, r^2, r^3, r^4, \ldots\}$

$r^n = \begin{cases} \text{conv if } -1 < r \leq 1 \;: & r = 1 \rightarrow \text{conv to } 1 \\ & r = -1 \rightarrow (-1)^n \text{ div} \\ & -1 < r < 1 \rightarrow \text{conv to zero} \\ \text{div if } r \leq -1, \; r > 1 & \end{cases}$

Example: $\{(1)^n\}$ conv to 1

" : $\{(-1)^n\} = \{-1, 1, -1, 1 \ldots\}$ div

" : $\{(\frac{-1}{2})^n\}$ conv to zero

" : $\{(5)^n\}_{n=1}^{\infty}$ div

" : if $\left\{\frac{a^{n+1}}{2^n}\right\}_{n=1}^{\infty}$ conv find the values of a ?

$\frac{a^{n+1}}{2^n} = a \cdot (\frac{a}{2})^n$, conv if $-1 < \frac{a}{2} \leq 1$

$-2 < a \leq 2 \quad \therefore a \in (-2, 2]$

" : $\{(2^n + 3^n)^{\frac{1}{n}}\}$ ∞^0 L'H

or $\lim_{n\to\infty} (3^n((\frac{2}{3})^n + 1))^{\frac{1}{n}} = \lim_{n\to\infty} 3 \cdot ((\frac{2}{3})^n + 1)^{\frac{1}{n}} = 3 * (0+1)^0$

$= 3$

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- Sequence part III

- Sequence defined recursively

Example: The seq $\{a_n\}_{n=1}^{\infty}$ defined as:

$a_1 = 2$, $a_n = a_{n-1} + \frac{1}{a_{n-1}}$, $n \geq 2$

Find the 4-th term?

$a_1 = 2$

$a_2 = a_1 + \frac{1}{a_1} = 2 + \frac{1}{2} = \frac{5}{2}$

$a_3 = a_2 + \frac{1}{a_2} = \frac{5}{2} + \frac{2}{5} = \frac{29}{10}$

$a_4 = a_3 + \frac{1}{a_3} \Rightarrow \frac{29}{10} + \frac{10}{29} = \frac{941}{290}$

Example: find the first term of the seq in which

$a_n = 2a_{n-1}$ if $a_6 = 64$

$a_6 = 2a_5$ $\quad 64 = 2a_5$ $\quad a_5 = 32$

$a_5 = 2a_4$ $\quad 32 = 2a_4$ $\quad a_4 = 16$

$a_4 = 2a_3$ $\quad 16 = 2a_3$ $\quad a_3 = 8$

$a_3 = 2a_2$ $\quad 8 = 2a_2$ $\quad a_2 = 4$

$a_2 = 2a_1$ $\quad 4 = 2a_1$ $\quad a_1 = 2$

$\therefore a_1 = 2$

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Note that: Given that the seq $\{a_n\}_{n=1}^{\infty}$ Conv to 3

that is $\lim\limits_{n\to\infty} = L = 3$; So:

$\lim\limits_{n\to\infty} a_{n+5} = 3$, $\lim\limits_{n\to\infty} a_{n-4} = 3$

In General: $\lim\limits_{n\to\infty} a_{n+c} = 3$

Example: Giventhat $a_1 = \sqrt{6}$, $a_2 = \sqrt{6+\sqrt{6}}$, $a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}$

A. Find the recursive formula?

$a_1 = \sqrt{6}$

$a_2 = \sqrt{6+\sqrt{6}} = \sqrt{6+a_1}$ $\therefore a_2 = \sqrt{6+a_1}$

$a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}} = \sqrt{6+a_2}$ $\therefore a_3 = \sqrt{6+a_2}$

$\therefore a_{n+1} = \sqrt{6+a_n}$, $a_1 = \sqrt{6}$ $\forall n \geq 1$

B. Given the seq conv find it's Limit?

Let $\lim\limits_{n\to\infty} a_n = L \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = L$

Now $a_{n+1} = \sqrt{6+a_n}$

$\lim\limits_{n\to\infty} a_{n+1} = \sqrt{6+\lim\limits_{n\to\infty} a_n}$

$L = \sqrt{6+L}$

$L^2 = 6 + L$

$L^2 - L - 6 = 0$

$(L-3)(L+2) = 0$

✓ $L = 3$ or $L = -2$ ✗

$\therefore$ Conv to 3

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Example: Given that $a_1 = 1$, $a_n = \frac{1}{2}\left(a_{n-1} + \frac{3}{a_{n-1}}\right) \forall n \geq 2$

Find it's limit?

Let $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n-1} = L$

Now $\lim\limits_{n\to\infty} a_n = \frac{1}{2}\left(\lim\limits_{n\to\infty} a_{n-1} + \frac{3}{\lim\limits_{n\to\infty} a_{n-1}}\right)$

$L = \frac{1}{2}\left(L + \frac{3}{L}\right)$

$\left(2L = L + \frac{3}{L}\right) \times L$

$2L^2 = L^2 + 3$

$L^2 = 3 \implies L = \pm\sqrt{3}$

$\therefore \lim\limits_{n\to\infty} a_n = \sqrt{3}$ $\therefore$ conv to $\sqrt{3}$

- Montone sequence
  - ↳ increasing or decreasing

① increasing

Ex. $\{n\}_{n=1}^{\infty} = \{\underset{a_1}{1}, \underset{a_2}{2}, \underset{a_3}{3}, \underset{a_4}{4}, \underset{a_5}{5} \dots\}$

$\Rightarrow a_1 < a_2 < a_3 < a_4 < a_5 \dots a_n < a_{n+1}$

(i) $a_{n+1} - a_n > 0$ (ii) $\frac{a_{n+1}}{a_n} > 1$

(iii) $a_n = f(x)$, $\forall x \geq 1$ if $f'(x) > 0$

$\therefore$ increasing

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② decreasing

Ex. $\{\frac{1}{n}\}_{n=1}^{\infty}$, $\{\underset{a_1}{1}, \underset{a_2}{\frac{1}{2}}, \underset{a_3}{\frac{1}{3}}, \underset{a_4}{\frac{1}{4}}, \underset{a_5}{\frac{1}{5}} \cdots\}$

$\Rightarrow a_1 > a_2 > a_3 > a_4 > a_5 \cdots a_n > a_{n+1}$

(i) $a_{n+1} - a_n < 0$　　(ii) $\frac{a_{n+1}}{a_n} < 1$

(iii) $a_n = f(x)$ if $f'(x) < 0$

$\therefore$ decreasing

Example: Determine whether the following seq Montone or not.

① $\{\tan^{-1} n\}_{n=1}^{\infty}$　　$f(x) = \tan^{-1} x$, $\forall x \geq 1$

$f'(x) = \frac{1}{1+x^2} > 0$ $\forall x \geq 1$

$\therefore$ increasing $\therefore$ Montone

② $\{3 + \frac{1}{n}\}_{n=1}^{\infty}$　　$f(x) = 3 + \frac{1}{x}$, $\forall x \geq 1$

$f'(x) = \frac{-1}{x^2} < 0$, $\forall x \geq 1$

$\therefore$ decreasing $\forall n \geq 1$ $\therefore$ Montone

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(3) $\left\{\frac{(10)^n}{n!}\right\}_{n=1}^{\infty}$

$a_n = \frac{10^n}{n!}$, $a_{n+1} = \frac{(10)^{n+1}}{(n+1)!}$

$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(n+1)!} \div \frac{10^n}{n!}$

$= \frac{10^{n+1}}{(n+1)!} * \frac{n!}{10^n}$

$= \frac{10^n \cdot 10 \cdot n!}{(n+1) \cdot n! \cdot 10^n} = \frac{10}{n+1}$

$n=1 \rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{2} > 1$

$n=2 \rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{3} > 3$

$n=3 \rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{4} > 1$

$\vdots$

$n=9 \rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{10} = 1$

$n=10 \rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{11} < 1$

$n=11 \rightarrow \frac{a_{n+1}}{a_n} = \frac{10}{12} < 1$

$n=1$ >1 $n=9$ <1
$=1$

$\therefore$ decreasing $\forall n \geq 9$ "tail"

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④ $\{n^2+5n\}_{n=1}^{\infty}$ $f(x) = x^2+5x$, $f'(x) = 2x+5$

$\therefore$ increasing $\forall n \geq 1$ $\quad -\frac{5}{2} \quad 1 \quad +$

⑤ $\{\frac{5}{n+3}\}_{n=1}^{\infty}$ $f(x) = \frac{5}{x+3}$ $x \geq 1$, $f'(x) = \frac{-5}{(x+3)^2} < 0$

$\therefore$ decreasing $\forall n \geq 1$

⑥ $\{(-1)^n\}_{n=1}^{\infty}$ not Montone

⑦ $\{\frac{n}{n+2}\}_{n=1}^{\infty}$ $f(x) = \frac{x}{x+2}$ $\forall x \geq 1$

$$f'(x) = \frac{(x+2)-(x)}{(x+2)^2} = \frac{2}{(x+2)^2} > 0$$

$\therefore$ increasing $\forall n \geq 1$

⑧ $\{\ln n\}_{n=2}^{\infty}$ $f(x) = \ln x$, $x \geq 2$, $f'(x) = \frac{1}{x}$

$\therefore$ increasing $\forall n \geq 2$ $\quad 0 \quad 2 \quad +$

⑨ $\{\frac{n!}{4^n}\}_{n=1}^{\infty}$ $a_n = \frac{n!}{4^n}$, $a_{n+1} = \frac{(n+1)!}{4^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{4^{n+1}} * \frac{4^n}{n!} = \frac{n+1}{4}$$

$$n=1 \rightarrow \frac{a_{n+1}}{a_n} = \frac{2}{4} < 1$$
$$\vdots$$
$$n=3 \rightarrow \frac{a_{n+1}}{a_n} = \frac{4}{4} = 1$$
$$n=4 \rightarrow \frac{a_{n+1}}{a_n} = \frac{5}{4} > 1$$

$\therefore$ increasing $\forall n \geq 3$

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Subject Lec 16 Date No.

- Series

Let $\{a_n\}_{n=1}^{\infty}$ be any seq. then the corresponding series $a_1 + a_2 + a_3 \dots = \sum_{k=1}^{\infty} a_k$

Now, to find the sum of this series:

Let $S_1 = a_1$

$S_2 = a_1 + a_2 = S_1 + a_2$

$S_3 = a_1 + a_2 + a_3 = S_2 + a_3$

$\vdots$

$S_k = S_{k-1} + a_k$

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Now, $\{S_1, S_2, \dots, S_k\}$ $(S_k)_{k=1}^{\infty}$ seq of partial sums

- if $\lim_{k\to\infty} S_k = L$ then $\sum_{k=1}^{\infty} a_k$ conv and sum $= \lim_{k\to\infty} S_k$
- if S_k div then $\sum_{k=1}^{\infty} a_k$ div and has no sum

Example: determine whether the following series conv or div? if it's conv, find the sum.

[1] $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$,

$S_1 = \underbrace{1 - \frac{1}{2}}_{a_1}$

$S_2 = \underbrace{(1 - \frac{1}{2})}_{a_1} + \underbrace{(\frac{1}{2} - \frac{1}{3})}_{a_2} = 1 - \frac{1}{3}$

$S_3 = \underbrace{(1 - \frac{1}{3})}_{a_1 + a_2} + \underbrace{(\frac{1}{3} - \frac{1}{4})}_{a_3} = 1 - \frac{1}{4}$

$\vdots$

$S_k = 1 - \frac{1}{k+1}$

$\lim_{k\to\infty} S_k = 1$ $\therefore$ conv $\Sigma = 1$

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$\boxed{2}$ $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

$\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$\therefore \; 1 = A(n+1) + B(n)$ if $n=0 \rightarrow A=1$

if $n=-1 \rightarrow B=-1$

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum \frac{1}{n} - \frac{1}{n+1}$ $\boxed{1}$ نفس $\Longrightarrow \quad = 1$

$\boxed{3}$ $\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2+k}} = \sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{\sqrt{k(k+1)}} - \frac{\sqrt{k}}{\sqrt{k(k+1)}}$

$= \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}$

$S_1 = 1 - \frac{1}{\sqrt{2}}$

$S_2 = (\overset{a_1}{1 - \frac{1}{\sqrt{2}}}) + (\overset{a_2}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}}) = 1 - \frac{1}{\sqrt{3}}$

$S_3 = 1 - \frac{1}{\sqrt{4}}$

$\vdots$

$S_k = 1 - \frac{1}{\sqrt{k+1}}$, $\lim_{k\to\infty} S_k = \lim_{k\to\infty} 1 - \frac{1}{\sqrt{k+1}} = 1$

$\sum \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2+k}}$ conv and $\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2+k}} = 1$

$\boxed{4}$ $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \ln(n+1) - \ln(n)$

$S_1 = \ln 2 - \ln 1 = \ln 2$

$S_2 = \overset{a_1}{\ln 2} + (\overset{a_2}{\ln 3 - \ln 2}) = \ln 3$

$S_3 = \overset{a_1+a_2}{\ln 3} + (\overset{a_3}{\ln 4 - \ln 3}) = \ln 4$

$S_k = \ln(k+1)$, $\lim_{k\to\infty} S_k = \lim_{k\to\infty} \ln(k+1) = \infty$

$\therefore$ div and has no sum

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5 $\sum_{k=1}^{\infty} \tan^{-1}(k+1) - \tan^{-1}(k)$

$S_1 = \tan^{-1}(2) - \tan^{-1}(1)$

$S_2 = \tan^{-1}(2) - \tan^{-1}(1) + \tan^{-1}(3) - \tan^{-1}(2) = \tan^{-1}(3) - \tan^{-1}(1)$

$S_3 = \tan^{-1}(3) - \tan^{-1}(1) + \tan^{-1}(4) - \tan^{-1}(3) = \tan^{-1}(4) - \tan^{-1}(1)$

$\therefore S_k = \tan^{-1}(k+1) - \tan^{-1}(1)$

$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \tan^{-1}(k+1) - \tan^{-1}(1)$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

Example: if the series $\sum_{k=1}^{\infty} a_k$ has the nth partial sum $S_n = 2 + \frac{3n}{n+1}$ than $\sum_{k=1}^{\infty} a_k =$

$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 + \frac{3n}{n+1}$

$= 2 + 3 = 5$

<u>Hw.</u> conv or div ? if conv find it's sum.

① $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right) - \sin\left(\frac{1}{k+1}\right)$

② $\sum_{k=3}^{\infty} \frac{1}{k^2-1}$

③ $\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+2}$

④ $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right) - \cos\left(\frac{1}{(n+1)^2}\right)$

⑤ $\sum_{k=1}^{\infty} \frac{1}{2^k} - \frac{1}{2^{k+1}}$

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Subject Lec 17 Date No.

- Geometric series

$$\sum_{k=1}^{\infty} (r)^k = r + r^2 + r^3 + r^4 + \dots$$

Ex. $\sum_{k=1}^{\infty} 2^k$ Geometric

$\sum_{k=1}^{\infty} \frac{1}{4k}$ not Geometric

$\sum_{k=1}^{\infty} k^k$ not Geometric

$\sum_{k=1}^{\infty} k^3$ not Geometric

$\sum_{k=1}^{\infty} \frac{1}{3^k} = \sum \left(\frac{1}{3}\right)^k$ Geometric

Note that:

- $\sum_{k=1}^{\infty} r^k$ conv if $|r| < 1$ i.e $-1 < r < 1$ and has sum $\Rightarrow \sum_{k=m}^{\infty} r^k = \frac{r^m}{1-r}$

- $\sum_{k=1}^{\infty} r^k$ div if $|r| \geq 1$ and has no sum

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Example: Determine whether the following series conv or div, if it's conv, find it's sum.

$\boxed{1}$ $\sum_{k=0}^{\infty}\left(\frac{-1}{5}\right)^k$, Geometric series $\left|\frac{-1}{5}\right| = \frac{1}{5} < 1$ $\therefore$ conv

$\Rightarrow$ Sum $= \dfrac{\left(\frac{-1}{5}\right)^0}{1-\left(\frac{-1}{5}\right)} = \dfrac{1}{1+\frac{1}{5}} = \dfrac{5}{6}$

$\boxed{2}$ $\sum_{k=0}^{\infty}\left(\frac{2}{e}\right)^k$, Geometric series $\left|\frac{2}{e}\right| < 1$ $\therefore$ conv

$\Rightarrow$ Sum $= \dfrac{\left(\frac{2}{e}\right)^0}{1-\frac{2}{e}} = \dfrac{1}{\frac{e-2}{e}} = \dfrac{e}{e-2}$

$\boxed{3}$ $\sum_{k=1}^{\infty} 2^{\frac{-k}{2}} = \sum_{k=1}^{\infty}\left(\frac{1}{2^{\frac{1}{2}}}\right)^k = \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k$

, Geometric $\left|\frac{1}{\sqrt{2}}\right| < 1$ $\therefore$ conv

$\Rightarrow$ Sum $= \dfrac{(1/\sqrt{2})^1}{1-\frac{1}{\sqrt{2}}} = \dfrac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \dfrac{1}{\sqrt{2}-1}$

$\boxed{4}$ $\sum_{k=0}^{\infty} e^{-k} = \sum_{k=0}^{\infty}\left(\frac{1}{e}\right)^k$, $\left|\frac{1}{e}\right| < 1$ $\therefore$ conv

$\Rightarrow$ Sum $= \dfrac{(1/e)^0}{1-\frac{1}{e}} = \dfrac{1}{\frac{e-1}{e}} = \dfrac{e}{e-1}$

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$\boxed{5}$ $\sum_{n=2}^{\infty} \left(\frac{-1}{3}\right)^{n+1} = \sum_{n=2}^{\infty} \left(\frac{-1}{3}\right)^{n} \cdot \frac{-1}{3}$ $\quad \left|\frac{-1}{3}\right| < 1 \therefore$ conv

$= \frac{-1}{3} \cdot \frac{\left(\frac{-1}{3}\right)^2}{1-\left(\frac{-1}{3}\right)}$

$= \frac{-1}{3} \cdot \frac{1/9}{\frac{4}{3}} = \frac{-1}{36}$

Note that: if $\sum u_k$ conv then $\sum_{k=1}^{\infty} a u_k$ conv and $\sum_{k=1}^{\infty} a u_k = a \sum_{k=1}^{\infty} u_k$

$a u_1 + a u_2 + a u_3 + \dots$ $\quad \hookrightarrow a(u_1+u_2+u_3+\dots)$

$\boxed{6}$ $\sum_{k=1}^{\infty} (5)^{2k} \cdot (7)^{1-k} = \sum_{k=1}^{\infty} (5^2)^k \cdot \frac{7}{7^k}$

$= 7 \sum_{k=1}^{\infty} \left(\frac{25}{7}\right)^k$, $\left|\frac{25}{7}\right| > 1 \therefore$ div

$\Longrightarrow$ has no sum

$\boxed{7}$ $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n$, $\left|\frac{-1}{3}\right| < 1 \therefore$ conv

$\Longrightarrow$ Sum $= \frac{(-1/3)^0}{1-\left(\frac{-1}{3}\right)} = \frac{1}{4/3} = \frac{3}{4}$

$\boxed{8}$ $\sum_{n=1}^{\infty} e^n \pi^{1-n} = \sum_{n=1}^{\infty} \frac{e^n \cdot \pi}{\pi^n} = \pi \sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$, $\left|\frac{e}{\pi}\right| < 1 \therefore$ conv

$\Longrightarrow$ Sum $= \pi \cdot \frac{e/\pi}{1-\frac{e}{\pi}} = \pi \cdot \frac{e}{\pi - e} = \frac{\pi e}{\pi - e}$

$\boxed{9}$ $\sum_{n=1}^{\infty} 3^{2n} = \sum_{n=1}^{\infty} 9^n$, $|9| > 1$ div

$\boxed{10}$ $\sum_{k=2}^{\infty} 3^{-k+2} = \sum_{k=2}^{\infty} \frac{3^2}{3^k} = 9 \sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^k$ $\quad \left|\frac{1}{3}\right| < 1 \therefore$ Conv

$\Longrightarrow$ Sum $= \frac{9 \cdot \left(\frac{1}{3}\right)^2}{1-\frac{1}{3}} = \frac{3}{2}$

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11 $\sum_{k=1}^{\infty} \frac{4^{-k+1}}{3^{-2k+2}} = \sum_{k=1}^{\infty} \frac{4 \cdot 4^{-k}}{9 \cdot 3^{-2k}} = \frac{4}{9} \sum_{k=1}^{\infty} \left(\frac{3^2}{4}\right)^k$

$= \frac{4}{9} \sum \left(\frac{9}{4}\right)^k \quad , \left|\frac{9}{4}\right| > 1 \quad$ div

12 $\sum_{k=3}^{\infty} \frac{(-2)^{k+1}}{4^{\frac{1}{2}k+2}} = \sum_{k=3}^{\infty} \frac{-2 \cdot (-2)^k}{(4)^2 \cdot (4^{\frac{1}{2}})^k}$

$= \frac{-2}{16} \cdot \sum_{k=3}^{\infty} \left(\frac{-2}{2}\right)^k , \ |-1| = 1 \quad$ div

- Note that: if $\sum a_k$ conv, $\sum b_k$ conv
  then $\sum a_k + b_k$ conv
  and $\sum a_k + b_k = \sum a_k + \sum b_k$

13 $\sum_{k=1}^{\infty} \left(\frac{1}{\pi}\right)^k + \left(\frac{3}{\pi}\right)^k$

conv $\frac{1}{\pi} < 1$ conv $\frac{3}{\pi} < 1$

$\Longrightarrow$ Sum $= \sum_{k=1}^{\infty} \left(\frac{1}{\pi}\right)^k + \sum_{k=1}^{\infty} \left(\frac{3}{\pi}\right)^k = \frac{1/\pi}{1-\frac{1}{\pi}} + \frac{3/\pi}{1-3/\pi} \ldots$

14 $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k+2} = \left(\frac{2}{3}\right)^2 \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = \left(\frac{2}{3}\right)^2 \cdot \frac{\frac{2}{3}}{1-\frac{2}{3}} \ldots$

or $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k+2} = \sum_{k=1+2} \left(\frac{2}{3}\right)^{k+2-2} = \sum_{k=3}^{\infty} \left(\frac{2}{3}\right)^k$

$\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 \ldots$ $\quad \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 \ldots$

15 $\sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{7}{6^{k-1}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{6^k} \cdot 7$

$= 7 \cdot \frac{(-1/6)^0}{1+\frac{1}{6}} = 6$

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16 $\sum_{k=1}^{\infty} \frac{(4)^{k+2}}{(7)^{k-1}} = \frac{(4)^2}{(7)^{-1}} \cdot \sum_{k=1}^{\infty} \left(\frac{4}{7}\right)^k$

$= 16 \cdot 7 \cdot \sum_{k=1}^{\infty} \left(\frac{4}{7}\right)^k = 16 \cdot 7 \cdot \frac{4/7}{1-4/7}$

Example: if $\sum_{k=2}^{\infty} 2r^k = 1$ then $r = ?$

$2\sum_{k=2}^{\infty} r^k = 1 \rightarrow \sum_{k=2}^{\infty} r^k = \frac{1}{2}$

$\rightarrow \frac{r^2}{1-r} = \frac{1}{2}$

$\rightarrow 2r^2 = 1-r$

$\rightarrow 2r^2 + r - 1 = 0$

$\rightarrow (2r-1)(r+1) = 0$

$\checkmark\ r = \frac{1}{2}$ $\times\ r = -1 \rightarrow$ div $\therefore \boxed{r = \frac{1}{2}}$

Example: Find all values of r such that $\sum_{k=1}^{\infty} \frac{1}{r^k}$ conv.

$\sum \frac{1}{r^k}$ conv $\Rightarrow \sum \left(\frac{1}{r}\right)^k$

$\therefore \left|\frac{1}{r}\right| < 1 \longrightarrow -1 < \frac{1}{r} < 1$

$\longrightarrow -1 > r > 1$

$\therefore r < -1 \ ,\ r > 1$

$(-\infty, -1) \cup (1, \infty)$

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Subject Lec 18 Date No.

- Divergent test

Theorem: if $\sum_{k=1}^{\infty} a_k$ conv then $\lim_{k\to\infty} a_k = 0$

Div test: if $\lim_{k\to\infty} a_k \neq 0 \Rightarrow \sum a_k$ div

* conv or div?

Ex. ① $\sum_{k=1}^{\infty} (\frac{1}{3})^k$, Geometric series $|\frac{1}{3}| < 1$ $\therefore$ conv

$\lim_{k\to\infty} (\frac{1}{3})^k = 0$

② $\sum_{k=1}^{\infty} \frac{3-2k^2}{4+k+7k^2}$, $\lim_{k\to\infty} \frac{3-2k^2}{4+k+7k^2} = \frac{-2}{7} \neq 0$

$\therefore$ div by div test

③ $\sum_{k=1}^{\infty} (1-\frac{2}{k})^{3k}$, $\lim_{k\to\infty} (1-\frac{2}{k})^{3k} = (e^{-2})^3 \neq 0$

$\therefore$ div by div test

④ $\sum_{k=1}^{\infty} (\frac{3k}{3k+1})^k$, $\lim_{k\to\infty} (\frac{3k}{3k+1})^k = \lim_{k\to\infty} (\frac{3k+1}{3k})^{-k}$

$= \lim_{k\to\infty} (1+\frac{1}{3k})^{-k}$

$= (e^{\frac{1}{3}})^{-1} = \frac{1}{\sqrt[3]{e}} \neq 0$

$\therefore$ div by div test

⑤ $\sum_{k=1}^{\infty} \frac{1}{k^2+9}$, $\lim_{k\to\infty} \frac{1}{k^2+9} = 0$ $\therefore$ div test failed

⑥ ~~$\sum_{k=1}^{\infty}$~~ $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$, $\lim_{n\to\infty} \frac{n}{\ln(n)} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty \neq 0$

$\therefore$ div by div test

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Subject Date No.

⑦ $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2n^2+1}$, $\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2} \neq 0$ ∴ div by div test

A.S

Hw conv or div?

① $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{\sqrt{k}-1}$

② $\sum_{k=1}^{\infty} \frac{e^k+7}{k^2+3k+1}$

③ $\sum_{k=1}^{\infty} \frac{2^k+1}{3k}$

④ $\sum_{k=1}^{\infty} k^2 \sin^2\left(\frac{1}{k}\right)$

⑤ $\sum_{k=1}^{\infty} \frac{1}{2+3^{-k}}$

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Subject Date No.

- Integral Test

$\Rightarrow$ For $\sum_{k=1}^{\infty} a_n$, Let $a_n = f(x)$, $x \geq 1$

① $f(x) > 0$
② $f(x)$ decreasing
③ $f(x)$ continuous

$\Big] \ \forall\ x \geq 1$

$\Rightarrow$ then $\int_1^{\infty} f(x)\,dx \to$ conv $\to \sum a_n$ conv

$\to$ div $\to \sum a_n$ div

Example: $\sum_{k=3}^{\infty} \frac{\ln k}{k}$ conv or div?

1. $\lim_{k\to\infty} \frac{\ln k}{k} \overset{L'H}{=} \lim \frac{1/k}{1} = 0$ $\therefore$ div test failed

2. integral test: $f(x) = \frac{\ln x}{x}$, $\forall x \geq 3$

① $f(x) > 0$ $\forall \geq 3$

$e \quad 3$ ② $f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$ $\therefore$ dec $\forall x \geq 3$

③ $f(x)$, cont $(0, \infty)$ $\therefore$ cont $\forall x \geq 3$

Now,

$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Big|_1^t = \infty$ $\therefore$ div

$\therefore$ div by integral test

$\ln x = u$

$\frac{dx}{x} = du$

$\int \frac{\ln x}{x} = \int u\,du$

$= \frac{(\ln x)^2}{2}$

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Example: $\sum_{k=1}^{\infty} ke^{-k}$ conv or div?

① $\lim_{k\to\infty} ke^{-k} = \lim_{k\to\infty} \frac{k}{e^k} \overset{L'H}{=} \lim_{k\to\infty} \frac{1}{e^k} = 0$ ∴ div test failed

② integral test: $f(x) = xe^{-x}, \ \forall x \geq 1$

① $f(x) > 0 \quad \forall x \geq 1$

② $f'(x) = -xe^{-x} + e^{-x} = e^{-x}[-x+1]$ ∴ dec $\forall x \geq 1$

③ $f(x)$ cont $\forall x \geq 1$

Now,

$$\int_1^\infty xe^{-x}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x}$$

$$= \lim_{t\to\infty} \left[-xe^{-x} - e^{-x} \Big|_1^t\right]$$

$$= \lim_{t\to\infty} -te^{-t} - 0 - \left[-e^{-1} - e^{-1}\right]$$

$$= \underbrace{\lim_{t\to\infty} \frac{-t}{e^t}}_{0} + 2e^{-1} = 2e^{-1} \quad \therefore \text{conv}$$

u	dv
x	e^{-x}
1	$-e^{-x}$
0	e^{-x}

(+) (−)

∴ $\sum_{k=1}^{\infty} ke^{-k}$ conv by integral test

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Subject Date No.

Example: $\sum_{k=3}^{\infty} \frac{1}{k\ln k}$ conv or div?

1. $\lim_{k\to\infty} \frac{1}{k\ln k} = 0$ $\therefore$ div test failed

2. integral test: $f(x) = \frac{1}{x\ln x}$, $\forall x \geq 3$

① $f(x) > 0 \quad \forall x \geq 3$

$\therefore$ dec. $\forall x \geq 3$ ② $f'(x) = \frac{-(x\cdot\frac{1}{x} + \ln x)}{(x\ln x)^2} = \frac{-1-\ln x}{(x\ln x)^2}$

③ $f(x)$ cont $(1,\infty)$ $\therefore$ cont $\forall \geq 3$

Now,

$$\int_3^{\infty} \frac{1}{x\ln x} = \lim_{t\to\infty} \ln(\ln x)\Big|_3^t$$

$$= \lim_{t\to\infty} [\ln(\ln t) - \ln(\ln 3)] = \infty \therefore \text{div}$$

$\int \frac{1}{x\ln x}dx$

$\ln x = u$

$= \ln(\ln x)$

$\therefore \sum \frac{1}{k\ln k}$ div by integral test

Hw: ① $\sum_{k=3}^{\infty} \frac{\tan^{-1}k}{k^2+1}$ conv or div?

② $\sum_{k=1}^{\infty} k^2 e^{-k^3}$ conv or div?

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Subject Date No.

Example: $\sum_{k=3}^{\infty} \frac{1}{k^2+9}$ conv or div?

1. div test: $\lim_{k\to\infty} \frac{1}{k^2+9} = 0$ $\therefore$ failed

2. integral test: $f(x) = \frac{1}{x^2+9}$ $\forall x \geq 3$

① $f(x) > 0$ $\forall x \geq 3$

② $f'(x) = \frac{-2x}{x^2+9}$ $\therefore$ dec $\forall x \geq 3$

③ $f(x)$ cont $\forall x \geq 3$

Now,

$$\int_3^{\infty} \frac{1}{x^2+9}\,dx = \lim_{t\to\infty} \int_3^t \frac{dx}{x^2+9}$$
$$= \lim_{t\to\infty} \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right)\Big|_3^t$$
$$= \lim_{t\to\infty} \frac{1}{3} \left[\tan^{-1}\left(\frac{t}{3}\right) - \tan^{-1}\left(\frac{3}{3}\right)\right]$$
$$= \frac{1}{3}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$
$$= \frac{1}{3} \cdot \frac{\pi}{4} = \frac{\pi}{12} \quad \therefore \text{conv}$$

$\therefore \sum_{k=3}^{\infty} \frac{1}{k^2+9}$ conv by integral test

HW $\sum_{n=1}^{\infty} \frac{1}{n+1}$ conv or div?

N O T E B O O K

Subject Lec 19 Date No.

- P- series test

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \begin{cases} P>1 \longrightarrow \text{conv} \\ 0<P\leq 1 \longrightarrow \text{div} \end{cases}$$

* if $P=1$, $\sum \frac{1}{k}$ div and called Harmonic series

Ex. conv or div ?

① $\sum \frac{1}{k^3}$, $P=3>1$ $\therefore$ conv

② $\sum \frac{1}{\sqrt[3]{k}} = \sum \frac{1}{k^{\frac{1}{3}}}$, $P=\frac{1}{3}<1$ $\therefore$ div

③ $\sum k^{-5} = \sum \frac{1}{k^5}$, $P=5>1$ $\therefore$ conv

④ $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$, $P=\frac{1}{2}<1$ $\therefore$ div

⑤ $\sum_{k=1}^{\infty} \frac{k}{k^e} = \sum \frac{1}{k^{e-1}}$, $P=e-1>1$ $\therefore$ conv

⑥ $\sum_{k=1} \frac{3}{4k} = \frac{3}{4} \sum \frac{1}{k}$, $P=1$ $\therefore$ div

⑦ $\sum (\frac{2}{k})^4 = \sum \frac{2^4}{k^4} = 2^4 \sum \frac{1}{k^4}$, $P=4>1$ $\therefore$ conv

⑧ $\sum_{k=3}^{\infty} \frac{1}{(k-2)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$, $P=2>1$ $\therefore$ conv

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⑨ $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}} = \sum_{k=1+1} \frac{1}{\sqrt{k+1-1}} = \sum_{k=2}^{\infty} \frac{1}{\sqrt{k}}$, $p = \frac{1}{2}$ $\therefore$ div

$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \cdots$ $\qquad$ $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \cdots$ ✓

Example: determine whether the following series conv or div?

1. $\sum_{k=1}^{\infty} e^{-k}$: Geometric series
$= \sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$, $\left|\frac{1}{e}\right| < 1$ $\therefore$ conv by geometric series test

2. $\sum_{k=1}^{\infty} k^{-e} = \sum_{k=1}^{\infty} \frac{1}{k^e}$: $p = e > 1$ $\therefore$ conv by p-series test

3. $\sum_{k=1}^{\infty} e^k$: Geometric series, $|e| > 1$ $\therefore$ div

4. $\sum_{k=1}^{\infty} e^{\frac{1}{k}}$ not Geometric series
not P- series

$\Rightarrow$ div test: $\lim_{k \to \infty} e^{\frac{1}{k}} = 1 \neq 0$ $\therefore$ div by divergent test

- Limit comparision test $\frac{\text{Power}}{\text{Power}}$ (L.C.T)

* Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms
  - ⓐ choose $\sum_{k=1}^{\infty} b_k$
  - ⓑ Find $\rho = \lim_{k \to \infty} \frac{a_k}{b_k}$

$\Rightarrow$ if $\rho > 0$ then: if $\sum_{k=1}^{\infty} b_k$ conv $\Rightarrow \sum_{k=1}^{\infty} a_k$ conv
if $\sum_{k=1}^{\infty} b_k$ div $\Rightarrow \sum_{k=1}^{\infty} a_k$ div

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Subject Date No.

Example: $\sum_{k=1}^{\infty} \underbrace{\frac{1}{k^4+1}}_{a_k}$ conv or div ?

① $\sum b_k = \sum_{k=1}^{\infty} \frac{1}{k^4}$

② $\rho = \lim_{k \to \infty} \frac{1}{k^4+1} * k^4 = 1 > 0$

Now, $\sum b_k = \sum_{k=1}^{\infty} \frac{1}{k^4}$, $p = 4 > 1$ $\therefore$ conv

$\therefore \sum_{k=1}^{\infty} \frac{1}{k^4+1}$ conv by L.C.T

Example: $\sum \underbrace{\frac{2k^2+7}{10k^3+16k^2+9}}_{a_k}$ conv or div ?

① $\sum b_k = \sum \frac{2k^2}{10k^3} = \sum \frac{1}{5k}$

② $\rho = \lim_{k \to \infty} \frac{2k^2+7}{10k^3+16k^2+9} * \frac{5k}{1} = 1 > 0$

Now, $\sum b_k = \sum \frac{1}{5k}$ $\therefore$ div $p = 1$

$\therefore \sum a_k = \sum \frac{2k^2+7}{10k^3+16k^2+9}$ div by L.C.T

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Example: $\sum_{k=1}^{\infty} \frac{k^{4/3}}{k^2+3k+4}$ conv or div ?

① $\sum b_k = \sum \frac{k^{4/3}}{k^2} = \sum \frac{1}{k^{2/3}}$

② $\rho = \lim_{k\to\infty} \frac{k^{4/3}}{k^2+3k+4} \cdot \frac{k^{2/3}}{1} = 1 > 0$

Now, $\sum b_k = \sum \frac{1}{k^{2/3}}$ $p = 2/3 < 1$ $\therefore$ div

$\therefore \sum a_k = \sum \frac{k^{4/3}}{k^2+3k+4}$ div by L.C.T

Example: $\sum_{k=1}^{\infty} \frac{k}{(2k+1)^{15}}$ conv or div ?

① $\sum b_k = \sum \frac{k}{k^{15}} = \sum \frac{1}{k^{14}}$

② $\rho = \lim_{k\to\infty} \frac{k}{(2k+1)^{15}} \cdot k^{14} = \frac{1}{2^{15}} > 0$

Now $\sum b_k = \sum \frac{1}{k^{14}}$, $p = 14 > 1$ $\therefore$ conv

$\therefore \sum a_k = \sum_{k=1}^{\infty} \frac{k}{(2k+1)^{15}}$ conv by L.C.T

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Subject Date No.

Example: $\sum_{k=1}^{\infty} \frac{\sqrt{k}}{(2k+1)^2}$ conv or div ?

$\sum b_k = \sum \frac{k^{1/2}}{k^2} = \sum \frac{1}{k^{3/2}}$

$\rho = \lim_{k\to\infty} \frac{k^{1/2}}{(2k+2)^2} \cdot \frac{k^{3/2}}{1} = \frac{1}{4} > 0$

$\sum b_k = \sum \frac{1}{k^{3/2}}$, $P = 3/2 > 1$ conv

$\sum a_k = \sum \frac{\sqrt{k}}{(2k+1)^2}$ conv by L.C.T

Hw.

① $\sum_{k=1}^{\infty} \frac{3}{k^{5/2}+1}$

② $\sum_{k=1}^{\infty} \frac{n}{n^{3/2}-\frac{1}{2}}$

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- Alternating series Test (A.S.T)

$\sum (-1)^k a_k$, $\sum (-1)^{k+1} a_k$, $\sum \cos(\pi k)\, a_k$

* if ① $a_k > 0$ ② $\lim\limits_{k\to\infty} a_k = 0$ ③ a_k decreasing
then the series conv

* if $\lim\limits_{k\to\infty} a_k \neq 0 \Rightarrow$ div by div test

Example: $\sum\limits^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$ conv or div?

① $a_k = \frac{k+3}{k(k+1)} > 0$

② $\lim\limits_{k\to\infty} \frac{k+3}{k(k+1)} = 0$

③ $(a_k)' = \frac{k(k+1) - (k+3)(2k+1)}{(k^2+k)^2} = \frac{k^2+k-2k^2-7k-3}{(k^2+k)^2}$

$= \frac{-(k^2+6k+3)}{(k^2+k)^2} < 0 \therefore$ dec

$\therefore \sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$ conv by A.S.T

Example: $\sum (-1)^k \frac{k^2+1}{k+3}$, $\lim\limits_{k\to\infty} \frac{k^2+1}{k+3} = \infty \neq 0$

$\therefore$ div by div test

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Subject Date No.

Defn: ① $\sum a_k$ conv absolutely if $\sum |a_k|$ conv
② $\sum a_k$ div absolutely if $\sum |a_k|$ div

Theorem: ① if $\sum a_k$ conv abs $\Rightarrow$ $\sum a_k$ conv
② if $\sum a_k$ div $\Rightarrow$ $\sum a_k$ div abs

Note that: if $\sum a_k$ div abs + $\sum a_k$ conv by A.S.T. $\Rightarrow$
$\sum a_k$ conv conditionally

Examples:

[1] $\sum \frac{(-1)^k}{k^3}$, ① $\sum \left|\frac{(-1)^k}{k^3}\right| = \sum \frac{1}{k^3}$, $P = 3 > 1$ $\therefore$ conv abs
$\Rightarrow$ conv

[2] $\sum_{k=1} \frac{(-1)^k}{k}$, ① $\sum \left|\frac{(-1)^k}{k}\right| = \sum \frac{1}{k}$, $p = 1$ div abs
② $a_k = \frac{1}{k}$ by A.S.T
① $a_k > 0$ ② $\lim_{k \to \infty} \frac{1}{k} = 0$ ③ $(a_k)' = \frac{-1}{k^2} < 0 \Rightarrow$ dec
$\therefore$ conv by A.S.T.
+ div abs $\Rightarrow$ conv conditionally

[3] $\sum_{k=1}^{\infty} \frac{k\cos(\pi k)}{k^2+1}$ $\Rightarrow$

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3

$$\sum_{k=1}^{\infty} \frac{k\cos(\pi k)}{k^2+1}$$

① A.S: $\sum_{k=1}^{\infty} \left|\frac{k\cos(\pi k)}{k^2+1}\right| = \sum_{k=1}^{\infty} \frac{k}{k^2+1}$

(i) Now by L.C.T $b_k = \frac{k}{k^2} = \frac{1}{k}$

(ii) $\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k}{k^2+1} \cdot \frac{k}{1} = \lim \frac{k^2}{k^2+1} = 1 > 0$

$\sum \frac{1}{k}$ div $\Rightarrow \sum_{k=1}^{\infty} \frac{k}{k^2+1}$ div

$\therefore \sum \frac{k\cos(\pi k)}{k^2+1}$ div abs

Now A.S.T,

① $a_k = \frac{k}{k^2+1} > 0$

② $\lim_{k\to\infty} \frac{k}{k^2+1} = 0$

③ $(a_k)' = \frac{(k^2+1) - k(2k)}{(k^2+1)^2} = \frac{-k^2+1}{(k^2+1)^2} < 0$

$\therefore$ dec $\therefore$ conv by A.S.T

conv by A.S.T + div abs $\rightarrow \sum$ conv conditionally

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Subject Date No.

<u>Hw</u>

1. $$\sum_{k=4} \frac{(-1)^k}{\sqrt[3]{2k^2-3k}}$$

2. $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k+1)^{17}}$$

3. $$\sum_{k=3}^{\infty} \frac{(-1)^k \ln k}{k}$$

4. $$\sum_{n=1} \frac{(-1)^n \sqrt{n}}{n^3}$$

5. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)(n+1)}$$

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Subject Lec 21 Date No.

- Ratio Test $(! + ()^k)$

$\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$, if ① $\rho < 1 \to$ conv abs

② $\rho > 1 \to$ div

③ $\rho = 1 \to$ Test failed

- Root Test $()^k$

$\rho = \lim_{k\to\infty} (|a_k|)^{\frac{1}{k}}$ if ① $\rho < 1 \to$ conv abs

② $\rho > 1 \to$ div

③ $\rho = 1 \to$ Test failed

Examples:

[1] $\sum_{k=1}^{\infty} \frac{(-1)^k (2)^k}{k!}$

$\Rightarrow \rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k} = \lim_{k\to\infty} \frac{2}{k+1} = 0 < 1$

$\therefore$ conv abs by ratio test

[2] $\sum_{n=1}^{\infty} \left(\frac{2n-1}{5n+10}\right)^n$

$\Rightarrow \rho = \lim_{n\to\infty} \left(\left(\frac{2n-1}{5n+10}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{2n-1}{5n+10} = \frac{2}{5} < 1$

$\therefore$ conv by root test

[3] $\sum \frac{e^{2k}}{k^{2k}} = \sum \left(\frac{e^2}{k^2}\right)^k$

$\Rightarrow \rho = \lim \left(\left(\frac{e^2}{k^2}\right)^k\right)^{1/k} = \lim_{k\to\infty} \frac{e^2}{k^2} = 0 < 1$

$\therefore$ conv by root test

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$\boxed{4}$ $\sum \frac{1}{k!}$

$\Rightarrow \rho = \lim_{k\to\infty} \frac{1}{(k+1)!} \cdot k! = \lim_{k\to\infty} \frac{k!}{(k+1)k!} = \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1$

$\therefore$ conv by ratio test

$\boxed{5}$ $\sum_{n=1}^{\infty} \left(\frac{n+2}{n}\right)^{n^2}$

$\rho = \lim_{k\to\infty} \left(\left(\frac{n+2}{n}\right)^{n^2}\right)^{\frac{1}{n}} = \lim_{k\to\infty} \left(1+\frac{2}{n}\right)^n = e^2 > 1$

$\therefore$ div by root test

$\boxed{6}$ $\sum_{n=1}^{\infty} \left(1+\frac{3}{n}\right)^n$

$\Rightarrow \rho = \lim_{n\to\infty} \left(\left(1+\frac{3}{n}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} 1+\frac{3}{n} = 1$ $\therefore$ root test failed

div test: $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n = e^3 \neq 0$ $\therefore$ div by div test

$\boxed{7}$ $\sum k^{50} e^{-k} = \sum \frac{k^{50}}{e^k}$

$\Rightarrow \rho = \lim_{k\to\infty} \frac{(k+1)^{50}}{e^{k+1}} \cdot \frac{e^k}{k^{50}}$

$= \lim_{k\to\infty} \frac{1}{e} \cdot \left(\frac{k+1}{k}\right)^{50}$

$= \frac{1}{e} \left(\lim_{k\to\infty} \frac{k+1}{k}\right)^{50}$

$= \frac{1}{e} \cdot 1 = \frac{1}{e} < 1$

$\therefore$ conv by ratio test

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8 $\sum \left(\frac{n+1}{n+2}\right)^n$

① $\rho = \lim_{n\to\infty} \left(\left(\frac{n+1}{n+2}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n+1}{n+2} = 1 \therefore$ root test failed

② div test: $\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n = \lim_{n\to\infty} \frac{(n(1+\frac{1}{n}))^n}{(n(1+\frac{2}{n}))^n}$

$= \lim_{n\to\infty} \frac{(1+\frac{1}{n})^n}{(1+\frac{2}{n})^n} = \frac{e}{e^2} = \frac{1}{e} \neq 0 \therefore$ div by div test

9 $\sum \left(\frac{k+1}{3k+4}\right)^{2k}$

$\Rightarrow \rho = \lim_{k\to\infty} \left(\left(\frac{k+1}{3k+4}\right)^{2k}\right)^{\frac{1}{k}} = \lim_{k\to\infty} \left(\frac{k+1}{3k+4}\right)^2 = \frac{1}{9} < 0 \therefore$ conv by root test

10 $\sum \frac{(-1)^k k^k}{k!}$

$\Rightarrow \rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k}$

$= \lim_{k\to\infty} \frac{(k+1)^k \cdot k+1 \cdot k!}{(k+1)(k!) \cdot k^k}$

$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k$

$= \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e > 1$

$\therefore$ div by ratio test

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$\boxed{11}$ $\sum \frac{(k!)^2}{(2k)!}$

$\Rightarrow \rho = \lim \frac{((k+1)!)^2}{(2(k+1))!} \cdot \frac{(2k)!}{(k!)^2}$

$= \lim_{k\to\infty} \frac{((k+1)\, k!)^2 \cdot (2k)!}{(2k+2)! \cdot (k!)^2}$

$= \lim_{k\to\infty} \frac{(k+1)^2 \cdot (k!)^2 \cdot (2k)!}{(2k+2)(2k+1)\cdot(2k)!\,(k!)^2}$

$= \lim_{k\to\infty} \frac{(k+1)^2}{(2k+2)(2k+1)} = \frac{1}{4} < 1$ $\therefore$ conv by ratio test

Hw ① $\sum_{k=1}^{\infty} \frac{k!\, 10^k}{3^k}$

② $\sum_{n=1}^{\infty} \frac{(100)^n}{n!}$

③ $\sum \left(\frac{n}{n+2}\right)^{n^2}$

④ $\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$

⑤ $\sum_{n=2}^{\infty} \left(\frac{n}{\ln n}\right)^{2n}$

⑥ $\sum \left(\frac{k^2+1}{5k+1}\right)^{3k}$

⑦ $\sum \frac{\ln k}{e^k}$

⑧ $\sum \frac{2^k (k!)^2}{(2k+2)!}$

⑨ $\sum \frac{(-1)^k (2k-1)!}{3^k}$

NOTEBOOK

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Subject Lec 22 Date No.

- Power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

Examples:

[1] $\sum_{n=0}^{\infty} x^n$ $a_n = 1$, $x_0 = 0$

[2] $\sum_{n=0}^{\infty} n!\, x^n$ $a_n = n!$, $x_0 = 0$

[3] $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$ $a_n = \frac{1}{n!}$, $x_0 = 3$

Example: Find the value of x which make the series conv.

$\sum_{n=0}^{\infty} x^n$, $|x| < 1$ $x \in (-1, 1)$ "interval of convergence"

$\frac{1-(-1)}{2} = \frac{2}{2} = 1$ "radius of conv"

Theorem: for any power series ther exist radius of convergence and interval of convergence

- How to find the radius of conv?

radius $= \rho = \frac{1}{L}$

where $\rightarrow$ $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- How to find the interval of conv?

$|x - x_0| < \rho$

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Example: Find the radius and interval of conv of:

[1] $\sum_{k=0}^{\infty} \frac{(x+2)^k}{k \cdot 4^k}$, $a_k = \frac{1}{k \cdot 4^k}$, $x_0 = -2$

- To find the radius of conv:

① $L = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{1}{(k+1)\,4^{k+1}} \cdot k \cdot 4^k$

$= \lim_{k\to\infty} \frac{k}{4(k+1)} = \frac{1}{4}$

② $\rho = \frac{1}{L}$

$= \frac{1}{1/4} = 4$

- To find the interval of conv:

$|x+2| < 4 \Rightarrow -4 < x+2 < 4$

$-6 < x < 2$

① if $\boxed{x=-6}$, $\sum_{k=0}^{\infty} \frac{(-6+2)^k}{4^k \cdot k} = \sum \frac{(-4)^k}{4^k \cdot k} = \sum \frac{(-1)^k}{k}$

by A.S.T $\sum \frac{(-1)^k}{k}$ ∴ conv $[-6,$

② if $\boxed{x=2}$, $\sum \frac{(2+2)^k}{4^k k} = \sum \frac{4^k}{4^k k} = \sum \frac{1}{k}$ ∴ div $,2)$

∴ interval of conv $[-6, 2)$

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[2] $\sum_{k=0}^{\infty} k!\,(x+5)^k$, $a_k = k!$, $x_0 = 5$

① radius of conv

$\therefore L = \lim_{k\to\infty} \frac{(k+1)!}{k!} = \lim_{k\to\infty} k+1 = \infty$

$\therefore \rho = \frac{1}{\infty} = 0$

② interval of conv

$|x-5| < 0 \Rightarrow x = 5$ $\therefore$ conv only at 5

[3] $\sum_{k=0}^{\infty} \frac{(-1)^k\, 10^k\, (x-1)^k}{k!}$ $\quad a_k = \frac{(-1)^k\, 10^k}{k!}$

① radius of conv

$\therefore L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{10^{k+1}}{(k+1)!} \cdot \frac{k!}{10^k}$

$= \lim_{k\to\infty} \frac{10}{k+1} = 0$

$\therefore \rho = \frac{1}{0} = \infty$

② interval of conv $(-\infty, \infty)$

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Defn: Let $f(x)$ be diffble for all order at constant (x_0) then the taylor series for $f(x)$ is:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)\,(x-x_0)^k}{k!}$$

$$f(x) = f(x_0) + \frac{f'(x_0)\,(x-x_0)}{1!} + \frac{f^{(2)}(x_0)\,(x-x_0)^2}{2!} + \dots$$

Example: Let $f(x) = \frac{1}{x+3}$, Find the taylor series for $f(x)$ at $x_0 = 2$?

① $f(2) = \frac{1}{5}$

② $f'(2) = \frac{-1}{(x+3)^2}\Big|_{x=2} = \frac{-1}{25} = \frac{-1}{5^2}$

③ $f''(2) = \frac{2(x+3)}{(x+3)^4} = \frac{2}{(x+3)^3} = \frac{2}{125} = \frac{2}{5^3}$

④ $f^{(3)}(2) = 6(x+3)^{-4}\Big|_{x=2} = \frac{-6}{5^4}$

⑤ $f^{(4)}(2) = 24(x+3)^{-5}\Big|_{x=2} = \frac{24}{5^5}$

$\therefore f^{(k)}(2) = \frac{(-1)^k\, k!}{5^{k+1}}$

$\therefore$ taylor series for $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\, k!}{5^{k+1}} \cdot \frac{(x-2)^k}{k!}$

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Subject Lec 23 Date No.

- Maclaurin series Mac series

Recall that : - Power series = $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

- taylor series For $f(x)$ = $\sum_{n=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$

- Mac series For $f(x)$ = $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

Example: find the Mac series for $f(x) = e^x$

① $f^{(0)}(x) = e^x \longrightarrow f^{(0)}(0) = 1$

② $f^{(1)}(x) = e^x \longrightarrow f^{(1)}(0) = 1$

③ $f^{(2)}(x) = e^x \longrightarrow f^{(2)}(0) = 1$

⋮

④ $f^{(k)}(x) = e^x \longrightarrow f^{(k)}(0) = 1$

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \sum_{k=0} \frac{x^k}{k!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

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Subject Date No.

- Some important Maclaurin Series

① $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $x \in \mathbb{R}$

② $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!}$; $x \in \mathbb{R}$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

③ $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $x \in \mathbb{R}$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

④ $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$

$= 1 + x + x^2 + x^3 + \dots$

Eample: find the Maclaurin series for the following functions:

$\boxed{1}$ $e^{5x} = \sum_{n=0}^{\infty} \frac{(5x)^n}{n!} = \sum_{n=0}^{\infty} \frac{5^n x^n}{n!}$

$\boxed{2}$ $\frac{3}{e^x} = 3e^{-x} = 3\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3(-1)^n x^n}{n!}$

$\boxed{3}$ $xe^{-3x} = x\sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{x(-3)^n \cdot x^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^n \cdot x^{n+1}}{n!}$

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4 $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$

5 $\frac{1}{3-x} = \frac{1}{3(1-\frac{x}{3})} = \frac{1}{3} \cdot \frac{1}{1-\frac{x}{3}} = \frac{1}{3}\sum_{n=0}^{\infty}(\frac{x}{3})^n = \sum_{n=0}^{\infty}\frac{x^n}{3^{n+1}}$

6 $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty}(-x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k}$

تكامل $\tan^{-1}(x) = \sum_{k=0}^{\infty}(-1)^k \int x^{2k}dx = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$

$\therefore \tan^{-1}x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$

7 $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$

تكامل $\int \frac{1}{1+x} = \int \sum (-1)^k x^k$

$\therefore \ln(1+x) = \sum_{k=0}^{\infty}(-1)^k \cdot \frac{x^{k+1}}{k+1}$

8 $\cos(x^2) = \sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n \cdot x^{4n}}{(2n)!}$

9 $\sin(\frac{\pi}{2}+x) = \sin\frac{\pi}{2}\cos x + \cancel{\cos\frac{\pi}{2}} \sin x$

$= \cos x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$

10 $x\sin(2x^3) = x\sum_{k=0}^{\infty}\frac{(-1)^k (2x^3)^{2k+1}}{(2k+1)!}$

$= x\sum_{k=0}\frac{(-1)^k \cdot (2)^{2k+1} x^{3(2k+1)}}{(2k+1)!} = \sum_{k=0}\frac{(-1)^k (2)^{2k+1} \cdot x^{6k+3} \cdot x}{(2k+1)!}$

$= \sum_{k=0}^{\infty}\frac{(-1)^k \cdot 2^{(2k+1)} \cdot x^{6+4}}{(2k+1)!}$

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$\boxed{11}$ $2x^3 \cos(2x) = 2x^3 \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2 \cdot x^{2n+3}}{(2n)!}$

<u>HW</u> Find the Mac series for:

① $x \sin x$

② $x^2 e^{-4x}$

③ $\cos(\pi - x)$

④ $\ln\left(\frac{1+x}{1-x}\right)$

⑤ $\frac{x}{2+x}$

⑥ $\ln(1+4x)$

⑦ $\sinh x \rightarrow \frac{e^x - e^{-x}}{2}$

⑧ $\cosh x$

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Example: Find the sum of the following series:

[1] $\sum_{k=0}^{\infty} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{(1)^k}{k!} = e^1$

[2] $\sum_{k=2}^{\infty} \frac{2k}{k!} = e^2 - (\overset{k=0}{1} + \overset{k=1}{2}) = e^2 - 3$

[3] $\sum_{k=0}^{\infty} \frac{(-1)^k \cdot \pi^{2k+1}}{4^{2k+1} (2k+1)!} = \sum \frac{(-1)^k \cdot (\frac{\pi}{4})^{2k+1}}{(2k+1)!} = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

[4] $\sum_{k=1}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!} = \cos(\pi) - \overset{k=0}{1} = -1-1 = -2$

Hw. Find the sum of the following series.

① $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3^n}{n!}$

② $\sum_{n=2}^{\infty} \frac{(-1)^n \cdot 3^n}{n!}$

③ $\sum_{n=0}^{\infty} \frac{(-1)^n \cdot \pi^{2n}}{3^{2n-2} \cdot (2n)!}$

④ $\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n} (2n)!}$

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