تفاضل وتكامل 2

رانيا شقبوعة

للطالبة المبدعة

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إرادة - ثقة - تغيير

Subject : Lecture "1" 25 / Feb / 2024

Integration by Parts:-

$\int u . dv = u . v - \int v . du$

$$\begin{tikzcd} u \arrow[r] & du \\ dv \arrow[r] & v \arrow[ul] \arrow[u] \end{tikzcd}$$

حالات تكامل بالأجزاء باستخدام الجدول:

$\int (Poly)\, e^{ax+b}$

$\int (Poly)\, \sin(ax+b)$

$\int (Poly)\, \cos(ax+b)$

$\int (Poly)\, \sqrt{ax+b}$

Ex: $\int x \sin x . dx$

$u = x \qquad du = dx$

$dv = \sin x \qquad v = -\cos x$

$-\cos x\, (x) + \int \cos x . dx$

$-x \cos x + \sin x + C$

Ex: $\int \ln x . dx$

$u = \ln x \rightarrow du = \frac{1}{x}$

$dv = dx \rightarrow v = x$

$x \ln x - \int 1 . dx$

$x \ln x - x + C$

Ex: $\int x^2 e^x . dx$

$u = x^2 \qquad du = 2x$

$dv = e^x \qquad v = e^x$

$x^2 e^x - \int 2x e^x . dx$ other way →

$x^2 e^x - \underbrace{\int 2x e^x . dx}_{\text{نجزئ مرة أخرى}}$

$u = x \qquad du = 1$

$dv = e^x \qquad v = e^x$

$x e^x - \int e^x$

$x e^x - e^x$

$x^2 e^x - 2e^x x + 2e^x + C$

(Poly)

$$\begin{tikzcd} u & dv \\ x^2 \arrow[dr, "+"] & e^x \\ 2x \arrow[dr, "-"] & e^x \\ 2 \arrow[dr, "+"] & e^x \\ 0 & e^x \end{tikzcd}$$

$x^2 e^x - 2x e^x + 2e^x + C$

Subject: Lecture "1" 25 / Feb / 2024

Ex: $x^4 \sin 2x \, dx$

u		dv
x^4	$+$	$\sin 2x$
$4x^3$	$-$	$-\frac{\cos 2x}{2}$
$12x^2$	$+$	$-\frac{\sin 2x}{4}$
$24x$	$-$	$\frac{\cos 2x}{8}$
24	$+$	$\frac{\sin 2x}{16}$
0		$-\frac{\cos 2x}{32}$

$$-\frac{x^4}{2}\cos 2x + \frac{4x^3}{4}\sin 2x + \frac{12}{8}x^2\cos 2x - \frac{24}{16}x\sin 2x - \frac{24}{32}\cos 2x$$

Ex: $\int e^x \sin x \, dx$ (دوري)

$u = e^x \quad du = e^x$

$dv = \sin x \quad v = -\cos x$

$$-e^x \cos x + \underbrace{\int e^x \cos x \, dx}_{\text{by Parts}}$$

$\int e^x \cos x \, dx$

$u = e^x \quad du = e^x$

$dv = \cos x \quad v = \sin x$

$\sin x \, e^x - \int \sin x \, e^x$

$\int e^x \sin x \, dx = -e^x \cos x + \sin x \, e^x - \int \sin x \, e^x$

$2\int e^x \sin x \, dx = \sin x \, e^x - \cos x \, e^x$

$$\int e^x \sin x \, dx = \frac{\sin x \, e^x - \cos x \, e^x}{2}$$

Subject : Lecture (2) 27 / Feb / 2024

* Trigonometric integrals

1) $\int \sin x \, dx$

$= -\cos x + C$

other ways

2) $\int \sin^2 x \, dx$

$= \frac{-1}{2} \sin x \cos x + \frac{1}{2} \int 1 \, dx$

$= \frac{-1}{2} \sin x \cos x + \frac{1}{2} x + C$

$\longrightarrow \int \sin^2 x \, dx$

$= \frac{1}{2} \int 1 - \cos 2x \, dx$

$= \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) + C$

3) $\int \sin^3 x \, dx$

$= \frac{-1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx$

$= \frac{-1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C$

$\longrightarrow \int \sin x \sin^2 x \, dx$

$\int \sin x (1 - \cos^2 x) \, dx$

$y = \cos x \quad \frac{dy}{dx} = -\sin x$

$\int -(1 - y^2) \, dy = \int y^2 - 1$

$\frac{y^3}{3} - y + C$

$= \frac{\cos^3 x}{3} - \cos x + C$

4) $\int \sin^4 x \, dx$

$= \frac{-1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx$

$= \frac{-1}{4} \sin^3 x \cos x + \frac{3}{4} \left(\frac{-1}{2} \sin x \cos x + \frac{1}{2} x \right) + C$

* Reduction formulas :-

$\int \sin^n x \, dx = \frac{-1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$

$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$

Subject : Lecture (2) 27 / Feb / 2024

$\int \cos^6 x \, . dx$

$= \frac{1}{6} \cos^5 x \sin x + \frac{5}{6} \int \cos^4 x \, . dx$

$= \frac{1}{6} \cos^5 x \sin x + \frac{5}{6} \left(\frac{1}{4} \cos^3 x \sin x + \frac{3}{4} \int \cos^2 x \, . dx \right)$

$\frac{1}{2}(1 + \cos 2x)$

$= \frac{1}{6} \cos^5 x \sin x + \frac{5}{24} \cos^3 x \sin x + \frac{15}{48} \left(x + \frac{\sin 2x}{2} \right) + C$

$\int \sin^5 (3x+1) \, . dx$

Important

~~[illegible]~~

$y = 3x + 1$

$\frac{dy}{dx} = 3 \quad dy = 3 . dy$

$\int \frac{1}{3} \sin^5 y \, . dy$

$\frac{1}{3} \left(\frac{-1}{5} \sin^4 y \cos y + \frac{4}{5} \int \sin^3 y \, . dy \right)$

$\frac{1}{3} \left[\left(\frac{-1}{5} \sin^4 y \cos y + \frac{4}{5} \left(\frac{-1}{3} \sin^2 y \cos y + \frac{2}{3} \int \sin y \, . dy \right) \right] \right.$

$\frac{1}{3} \left[\frac{-1}{5} \sin^4 y \cos y + \frac{4}{5} \left(\frac{-1}{3} \sin^2 y \cos y + \frac{2}{3} \cos y + C \right] \right.$

$\frac{1}{3} \left[\frac{-1}{5} \sin^4 (3x+1) \cos (3x+1) + \frac{4}{5} \left(\frac{-1}{3} \sin^2 (3x+1) \cos (3x+1) - \frac{2}{3} \cos (3x+1) + c \right. \right.$

▸ Subject : Lecture (2) 27 / Feb /2024

$\int \sin^m x \cos^n x \,.\, dx$ $n = 2, 3 \ldots$ $m = 2, 3 \ldots$

even/odd → بنفك الـ odd

odd/odd → نفك الأصغر

even/even → أكثر من طريقة

Ex: $\int \sin^4 x \cos^5 x \,.\, dx$

$\int \sin^4 x \cos x \cos^4 x \,.\, dx$

$\int \sin^4 x \cos x \,(1 - \sin^2 x)^2 \,.\, dx$

$y = \sin x$

$\frac{dy}{dx} = \cos x$

$dx \,.\, \cancel{\cos} x = \frac{dy}{\cos x}$

$\int y^4 \cos x \,(1 - y^2)^2 \,.\, \frac{dy}{\cos x}$

$\int y^4 (1 - y^2)^2 \,.\, dy$

$\int y^4 - (1 - 2y^2 + y^4) \,.\, dy$

~~$\frac{y^5}{5} - \frac{y^7}{7} + C$~~ $\int y^4 - 2y^6 + y^8 \,.\, dy$

$\frac{y^5}{5} - \frac{2y^7}{7} + \frac{y^9}{9} + C$

~~$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$~~

$\frac{\sin^5 x}{5} - \frac{2}{7} \sin^7 x + \frac{\sin^9 x}{9} + C$

Ex: $\int \sin^3 x \cos^5 x \,.\, dy$

$\int \sin x \,(\sin^2 x) \cos^5 x \,.\, dx$

$\int \sin x \,(1 - \cos^2 x) \cos^5 x \,.\, dx$

$y = \cos x$

$\frac{dy}{dx} = -\sin x$

$dx = \frac{-dy}{\sin x}$

$\int \sin x \,(1 - y^2)\, y^5 \,.\, \frac{-dy}{\sin x}$

$-\int (1 - y^2)\, y^5 \,.\, dy$

$\int -y^5 + y^7 \,.\, dy$ → $-\frac{y^6}{6} + \frac{y^8}{8} + C$

$-\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C$

Subject : Lecture (2) 17 / feb / 2023

Ex : $\int \sin^4 x \cos^4 x \, dx$

① $\int (\sin x \cos x)^4 \, dx$

$\int \left(\frac{1}{2} \sin 2x\right)^4 \, dx$ نفس الفكرة

$= \frac{1}{16} \int \sin^4 2x \, dx$

$\vdots$

OR

② $\int \sin^4 x \, (1-\sin^2 x)^2 \, dx$

$\vdots$

OR

③ $\int (1-\cos 2x)^2 \cos^4 x \, dx$

$\int (1-2\cos^2 x+\cos^4 x) \cos^4 x \, dx$

$\vdots$

④ $\int (\sin^2 x \cos^2 x)^2 \, dx$

$\int \left(\frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)\right)^2$ الفكرة

$\frac{1}{16}\int (1-\cos^2 2x)^2 \, dx$

$\frac{1}{16} \int \sin^4 2x \, dx$

$\vdots$

Reduction formulas :

$\int \sin^n x \, dx = \frac{-1}{n} \sin^{n-1} \cos x + \frac{n-1}{n} \int \sin^{n-2} \, dx$

$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} \sin x + \frac{n-1}{n} \int \cos^{n-2} \, dx$

$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \qquad n \neq 1$

$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{x-2} x \, dx \qquad n \neq 1$

$\int \sin A \cos B dx = \frac{1}{2}\int \sin(A-B) + \sin(A+b) . dx$

$\int \sin A \sin B . dx = \frac{1}{2}\int \cos(A-B) - \cos(A+B) . dx$

$\int \cos A \cos B . dx = \frac{1}{2}\int \cos(A-B) + \cos(A+B) . dx$

$\sec^2 x = 1 + \tan^2 x$

$\tan^2 x = \sec^2 x - 1$

Ex: $\int \sin 4x \cos 5x . dx$

$= \frac{1}{2}\int \sin(-x) + \sin(9x) . dx$

$= \frac{1}{2}\int -\sin x + \sin 9x . dx$

$\frac{1}{2}\left(\cos x - \frac{\cos 9x}{9}\right) + C$

$\int \tan x . dx = -\ln|\cos x| = \ln|\sec x| + C$

$\int \tan^2 x . dx = \int \sec^2 x - 1 . dx$

$= \tan x + x + C$

$\int \sec x . dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) . dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} . dx$

$= \ln|\sec x + \tan x| + C$

$\sin^2 x = 1 - \cos 2x$

$\cos^2 x = 1 + \cos 2x$

~~Ex~~: $\int \tan^m x \sec^n x . dx \quad n, m = 2/3/4$

Ex: $\int \tan^4 x \sec^{\boxed{6}} x . dx \rightarrow$ even

$\int \tan^4 x \sec^2 x \sec^4 x . dx$

إذا كانت sec ← even ← نفكها

وإذا كانت tan ← odd ← نفكها

$y = \tan x$

$\frac{dy}{dx} = \sec^2 x$

$\int y^4 \sec^2 x \sec^4 x . \frac{dy}{\sec^2 x}$

$\int y^4 \sec^4 x . dx$

$\int y^4 (1 + \tan^2 x)^2 . dy$

$\int y^4 (1 + y^2)^2 . dy$

$\int y^4 + 2y^6 + y^5 . dy$

$$\int \tan^{\textcircled{5}} x \sec^7 x . dx$$ odd

$$y = \sec x \longrightarrow \frac{dy}{dx} = \sec x \tan x$$

$$\int \tan^5 x \, y^7 . \frac{dy}{\sec x \tan x}$$

$$\int y^6 . \tan^4 x . dy$$

$$\int y^6 (y^2-1)^2 . dy$$

$$\int y^{10} - 2y^8 + y^6 . dy$$

H.W:-

1) $\int \tan^4 x \sec^3 x . dx$

$$\int (\tan^2 x)^2 \sec^3 x . dx$$

$$\int (\sec^2 x - 1)^2 \sec^3 x . dx$$

$$\int \sec^7 x - 2\sec^5 x + \sec^3 x . dx$$

}

Red. For

2) $\int \tan^2 x \sec^3 x . dx$

$$\int (\sec^2 x - 1) \sec^3 x . dx$$

$$\int \sec^5 x - \sec^3 x . dx$$

}

Red for

2) $\int \tan^3 x \sec^4 x . dx$

$$y = \sec x$$

$$\frac{dy}{dx} = \sec x \tan x$$

$$dx = \frac{dy}{\sec x \tan x}$$

$$\int \tan^3 x . y^4 . \frac{dy}{\sec x \tan x}$$

$$\int \tan^2 x . y^3 . dy$$

$$\int (1 + y^2) y^3 . dy$$

$$\int y^3 + y^5 . dy$$

$$\frac{y^4}{4} + \frac{y^6}{6} + C$$

$$\frac{\sec^4 x}{4} + \frac{\sec^6 x}{6} + C$$

1) $\int \tan^3 x . dx$

$\int \tan^2 x \tan x . dx$

$\int (\sec^2 x - 1) \tan x . dx$

$\int \sec^2 x \tan x - \tan x . dx$

* $\int \sec^2 x \tan x . dx$

$y = \tan x$

$\frac{dy}{dx} = \sec^2 x$

$\int \sec^2 x . y . \frac{dy}{\sec^2 x}$

$\int y . dy$

$\frac{y^2}{2} - \ln|\cos x| + C$

$\frac{\tan^2 x}{2} - \ln|\cos x| + C$

2) $\int \tan^4 x . dx$

$\int \tan^2 x \tan^2 x . dx$

$\int (\sec^2 x - 1) \tan^2 x . dx$

$\int \sec^2 x \tan^2 x - \tan^2 x . dx$

* $\int \sec^2 x \tan^2 x . dx$ * $\int \tan^2 x . dx$

$y = \tan x$ $\int \sec^2 x - 1 . dx$

$\frac{dy}{dx} = \sec^2 x$ $\tan x - x + C$

$dx = \frac{dy}{\sec^2 x}$

$\int \sec^2 x . y^2 . \frac{dy}{\sec^2 x}$

$\int y^2 . dy$

$\frac{y^3}{3}$

$\frac{\tan^3 x}{3}$

$= \frac{\tan^3 x}{3} - \tan x + x + C$

Subject : / /

$\int \sec^3 x . dx$

$\int \sec x \sec^2 x . dx$

$u = \sec x \qquad du = \sec x \tan x$

$dv = \sec^2 x \qquad v = \tan x$

$= \sec x \tan x - \int \tan^2 \sec x . dx$

$= \sec x \tan x - \int (\sec^2 x - 1) \sec x . dx$

$= \sec x \tan x - \int \sec^3 x + \int \sec x . dx$

$\int \sec^3 x . dx = \dfrac{\sec x \tan x + \int \sec x}{2} + C$

$\sec^3 x$
$\csc^3 x$
[illegible]

9) $\int \sec^4 x . dx$

$\int \sec^2 x \sec^2 x . dx$

$\int (1 + \tan^2 x) \sec^2 x . dx$

$\int \sec^2 x + \sec^2 x \tan^2 x . dx \longrightarrow y = \tan x$!

$\tan x + \dfrac{\tan^3 x}{3} + C$

$\int \tan^2 x . \sec^3 x$

~~Lecture 4~~

*/ Trigomantic sub

$a^2 - x^2 \Longrightarrow x = a \sin\theta , \ \frac{-\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$

$a^2 + x^2 \Longrightarrow x = a \tan\theta , \ \frac{-\pi}{2} < \theta < \frac{\pi}{2}$

$x^2 - a^2 \Longrightarrow x = a \sec\theta , \ 0 < \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$.

~~4~~

Ex: $4 - x^2 \longrightarrow x = 2 \sin\theta$

$(10 + x^2)^{\frac{1}{2}} \longrightarrow x = \sqrt{10} \sin\theta$

$(x^2 - 7)^{\frac{3}{2}} \longrightarrow x = \sqrt{7} \sec\theta$

1) $\int \frac{\sqrt{9-x^2}}{x^2} \cdot dx$

$x = 3\sin\theta$

$\frac{dx}{d\theta} = 3\cos\theta$

$\int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} \cdot 3\cos\theta \cdot d\theta$

$\int \frac{\sqrt{9(1-\sin^2\theta)}}{9\sin^2\theta} \cdot 3\cos\theta \cdot d\theta$

$\int \frac{\cancel{3}|\cos\theta|}{\cancel{9}\sin^2\theta} \cdot \cancel{3}\cos\theta \cdot d\theta$ كوني بالربع الأول والرابع ال Cos فيه موجب

$\int \cot^2\theta \cdot d\theta$

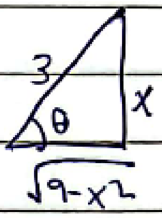
$\int \csc^2\theta - 1 \cdot d\theta$

$= -\cot\theta - \theta + C$

$x = 3\sin\theta$

$\sin\theta = \frac{x}{3}$

$\boxed{\sin^{-1}\frac{x}{3} = \theta}$



$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$

1) $9-(2x)^2 \longrightarrow 2x = 3\sin\theta$

$y = 2x$

$9 - y^2 \Longrightarrow y = 3\sin\theta$

$2x = 3\sin\theta$

2) $(x+3)^2-14 \Longrightarrow x+3=\sqrt{14}\ \sec\theta$

$y=x+3$

$y^2-14 \longrightarrow y=\sqrt{14}\ \sec\theta$

$x+3=\sqrt{14}\ \sec\theta$

3) $e^{4x}+5 \longrightarrow (e^{2x})^2+5$

$y=e^{2x}$

$y^2+5 \Longrightarrow y=\sqrt{5}\ \tan\theta$

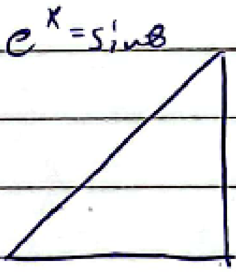
$e^{2x}=\sqrt{5}\ \tan\theta$

$e^{x}=\sin\theta$

~~[illegible]~~

~~$x \ln e = \ln \sin\theta$~~

~~$x = \ln \sin\theta$~~



▸ Subject : / /

2) $\int e^x \sqrt{1-e^{2x}}\,.dx$, $1-(e^x)^2$

$e^x = \sin\theta$ | $y = e^x$

$e^x . dx = \cos\theta . d\theta$ | $1-y^2$

$dx = \frac{\cos\theta}{e^x} . d\theta$ | $y = \sin\theta$

$e^x = \sin\theta$

$\int e^x \sqrt{1-\sin^2\theta}\ \frac{\cos\theta}{e^x} . d\theta$

$\int \sqrt{\cos^2\theta} . \cos\theta . d\theta$

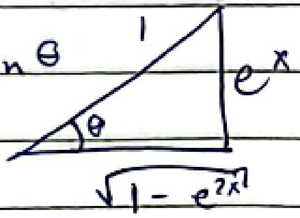
$\int \frac{1}{2} (1+\cos 2\theta . d\theta$

$\cancel{\int \frac{x}{\sqrt{4-(2\sin\theta)^2}}}$

$\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

$e^x = \sin\theta$

$\sin^{-1}(e^x) = \theta$



$\frac{1}{2}\left(\sin^{-1}(e^x) + \frac{2\sin\theta\cos\theta}{2}\right) + C$

$\frac{1}{2}\left(\sin^{-1}(e^x) + e^x\sqrt{1-e^{2x}}\right) + C$

3) $\int \frac{dx}{(x^4+2x^2+1}$

$\int \frac{dx}{(x^2+1)^2}$

$x = \tan\theta$

4) $\int \frac{x}{\sqrt{3-2x-x^2}} . dx$

[illegible]

$-(x^2+2x-3) \cancel{+ \left(\frac{b}{2}\right)^2}$

$-(x^2+2x+1-1-3) \cancel{+ \left(\frac{b}{2}\right)^2}$

$-((x+1)^2-4)$

$4-(x+1)^2$

$x+1 = 2\sin\theta$

$x = 2\sin\theta - 1$

$$\int \frac{x}{\sqrt{3-2x-x^2}} \cdot dx$$

إكمال المربع :-

$$\pm\left(\frac{b}{2}\right)^2$$

$$\int \frac{x}{\sqrt{-(x^2+2x-3)}} \cdot dx$$

$$\int \frac{x}{\sqrt{-(x^2+2x+1-1-3)}} \cdot dx$$

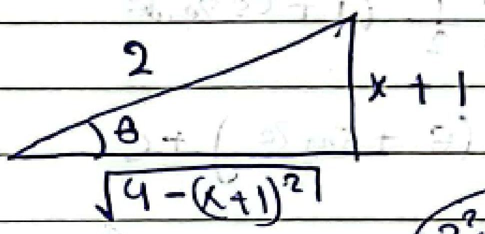
$$x = 2\sin\theta - 1$$

$$x+1 = 2\sin\theta$$

$$\sin\theta = \frac{x+1}{2}$$

$$\sin^{-1}\left(\frac{x+1}{2}\right) = \theta$$

$$\int \frac{x}{\sqrt{-(x^2+2x+1-4)}} \cdot dx$$



$$\int \frac{x}{\sqrt{-((x+1)^2-4)}} \cdot dx$$

$$2^2 = (x+1)^2 + y^2$$

$$y = \sqrt{4-(x+1)^2}$$

$$\int \frac{x}{\sqrt{4-(x+1)^2}}$$

$$y = x+1$$

$$x+1 = 2\sin\theta$$

$$x = 2\sin\theta - 1$$

$$dx = 2\cos\theta \cdot d\theta$$

$$\int \frac{x}{\sqrt{4-(2\sin\theta)^2}} \cdot 2\cos\theta \cdot d\theta$$

$$\int \frac{(2\sin\theta - 1) \cdot 2\cos\theta \cdot d\theta}{\sqrt{4-4\sin^2\theta}}$$

$$\int \frac{4\sin\theta\cos\theta - 2\cos\theta}{2\cos\theta} \cdot d\theta$$

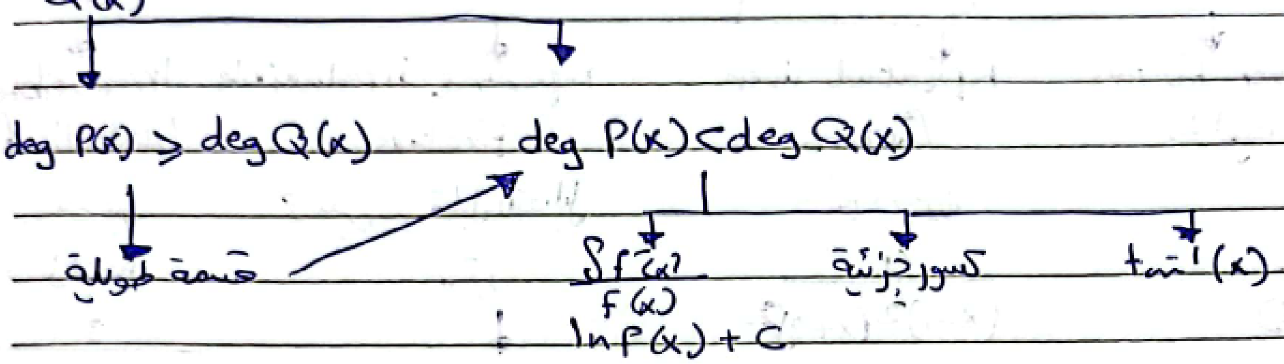
$$\int \frac{2\sin\theta\cos\theta - \cos\theta}{\cos\theta} \cdot d\theta \rightarrow$$

$$\int 2\sin\theta \cdot d\theta$$

$$-2\cos\theta + C$$

$$-2\left(\frac{\sqrt{4-(x+1)^2}}{2}\right) + C$$

$\frac{P(x)}{Q(x)}$



$* \int \frac{x^3+x}{x-1} \cdot dx$

$$\begin{array}{r} x^2+x+2 \\ x-1 \overline{\big) x^3+x} \\ \ominus x^3 \overset{\oplus}{-} x^2 \\ x^2+x \\ \ominus x^2 \overset{\oplus}{-} x \\ 2x \\ \ominus 2x \overset{\oplus}{-} 2 \\ 2 \end{array}$$

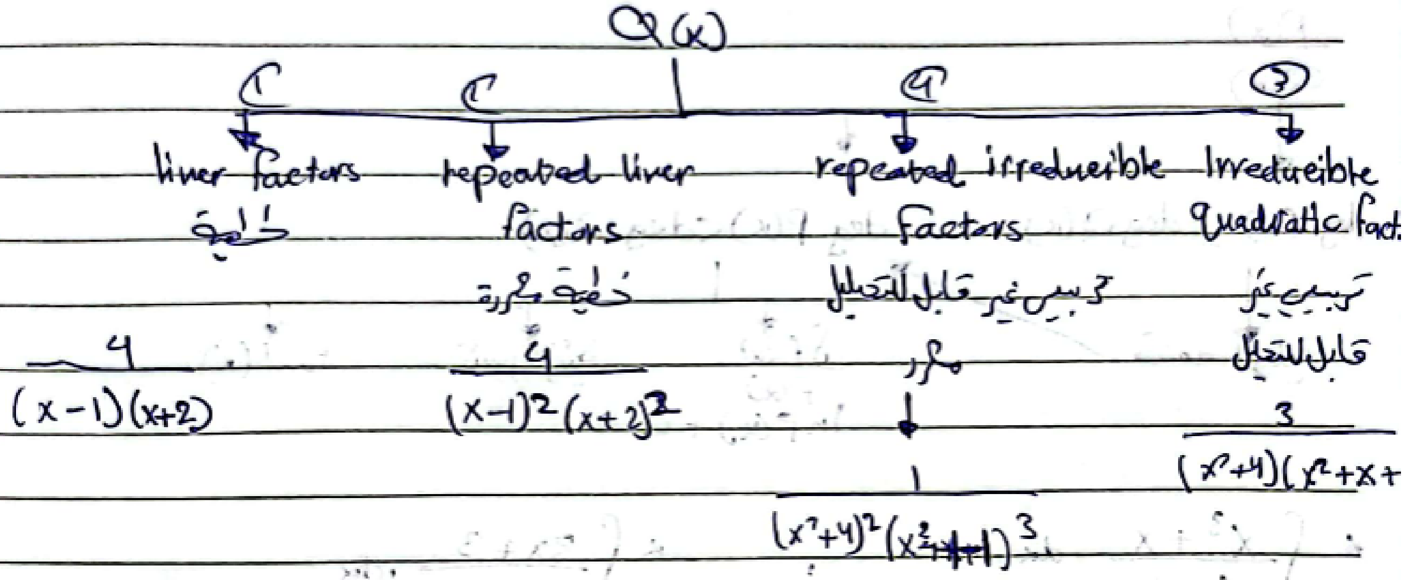
$\int x^2+x+2+\frac{2}{x-1} \cdot dx$

$\frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C$

$* \int \frac{3x+3}{x^2+x-2} \cdot dx$

$= \int \frac{2}{x-1} + \frac{1}{x+2}$

$2\ln|x-1| + \ln|x+2| + C$



1) $\frac{4}{(x-1)(x+2)}$

$= \frac{A}{(x-1)} + \frac{B}{(x+2)} = \frac{4}{(x-1)(x+2)}$

$A(x+2) + B(x-1) = 4$

$A(1+2) + 0 = 4$ $\boxed{x=1}$

$3A = 4$

$\boxed{A = \frac{4}{3}}$

$0 + B(-2-1) = 4$ $\boxed{x=-2}$

$-3B = 4$

$\boxed{B = \frac{-4}{3}}$

$\frac{4}{3(x-1)} + \frac{-4}{3(x+2)}$

2) $\frac{4}{(x-1)^2(x+2)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2} + \frac{F}{(x+2)^3}$

3) $\frac{3}{(x^2+4)^{\cancel{2}}(x^2+x+1)} = \frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+x+1)}$

4) $\frac{1}{(x^2+4)^2(x^2+x+1)^3} = \frac{A_1x+B_1}{x^2+4} + \frac{A_2x+B_2}{(x^2+4)^2} + \frac{A_3x+B_3}{(x^2+x+1)} + \cdots$

Ex:

* $\frac{x^2-1}{(x-2)(x+3)}$

$\frac{x^2-1}{x^2+x-6}$

$$\begin{array}{r} 1 \\ x^2+x-6 \,\big)\, x^2-1 \\ \ominus\; x^2 \overset{\ominus}{+} x \overset{\oplus}{-} 6 \\ -x-1+6 \\ -x+5 \end{array}$$

$\int 1 + \frac{-x+5}{x^2+x-6}$

$x \div$ [illegible]

$\int \frac{-x+5}{x^2+x-6} = \frac{A\cancel{x+B}}{(\cancel{x^2}+2-\cancel{6})} + \frac{B}{(x+3)} + 1$

~~$-x+5 = A(x+3) + B(x-2)$~~

* $\frac{1}{x^2(x-1)(x^2+x+1)(x^2+4)^2(x^2-1)}$

~~$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+F}{x^2+x+1} + \frac{Gx+H}{(x^2+4)}$~~

* $\frac{1}{x^2(x-1)^2(x+1)(x^2+x+1)(x^2+4)^2}$

$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} + \frac{E}{(x+1)} + \frac{A_1x+B_1}{x^2+x+1}$

$+ \frac{A_2x+B_2}{x^2+4} + \frac{A_4x+B_4}{(x^2+4)^2}$

Ex: $\int \frac{1}{x^2+x-2} \cdot dx$

$\int \frac{1}{(x-1)(x+2)} \cdot dx = \frac{A}{(X-1)} + \frac{B}{(x+2)}$

$1 = A(x+2) + B(x-1)$

$1 = B(-2-1)$ $\boxed{X=-2}$ حليل العبارة التكعيبية :

$1 = -3B$

$\boxed{B = \frac{-1}{3}}$

$1 = A(1+2) + B(0)$ $\boxed{X=1}$

$1 = 3A$

$\boxed{A = \frac{1}{3}}$

$\int \frac{1}{3(x-1)} + \frac{-1}{3(x+2)}$

$\frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|x+2) + C$

$\ln \sqrt[3]{\left|\frac{x-1}{x+2}\right|} + C$

③ $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} \cdot dx$

x^3-x^2-x+1

منطرق أخذ معاملات الحد المطلق $x=1/-1$

$x=1 \longrightarrow =0$

$(x-1)(x^2-1)$

$$\begin{array}{r|l} & x^2-1 \\ x-1 & x^3-x^2-x+1 \\ \ominus & x^3 \overset{\oplus}{-} x^2 \\ & -x+1 \\ \oplus & -x \overset{\ominus}{+} 1 \\ \hline & 0.0 \end{array}$$

$$\begin{array}{r|l} & x+1 \\ x^3-x^2-x+1 & x^4-2x^2+4x+1 \\ \ominus & x^4 \overset{\oplus}{-} x^3 \overset{\oplus}{-} x^2 \overset{\ominus}{+} x \\ & x^3-x^2+3x+1 \\ \ominus & x^3 \overset{\oplus}{-} x^2 \overset{\oplus}{-} x \overset{\ominus}{+} 1 \\ & 4x \end{array}$$

تكمل التكامل

$= \int x+1 \cdot dx + \int \frac{4x}{x^3-x^2-x+1} \cdot dx$

$\frac{x^2}{2} + x + \int \frac{4x}{(x-1)^2(x+1)}$

$\frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$

$= \frac{x^2}{2} - x + \frac{-2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C$

Ex: ① $\int \frac{3x^2+2}{x^3+2x-8} \cdot dx$

$\ln|x^3+2x-8| + C$

ليس كل كثير حدود على كثير حدود كسور جزئية

② $\int \frac{2x-1}{x^2+1} \cdot dx$

$= \int \frac{2x}{x^2} - \frac{1}{x^2+1}$

$\ln|x^2+1| - \tan^{-1}x + C$

* Strategy of integration

Ex :

1) $\int \frac{\tan^3 x}{\cos^3 x}\, dx$

$\int \sec^3 x \cdot \tan^3 x \cdot dx$

$y = \sec x$

$\frac{dy}{dx} = \sec x \tan x$

$\int y^3 \cdot \tan^3 x \cdot \frac{dy}{\sec x \tan x}$

$\int y^2 \cdot \tan^2 x \cdot dy$ → $\sec^2 x = 1 + \tan^2 x$ متطابقة

$\int y^2 (y^2 - 1) \cdot dy$

$\int y^4 - y^2 \cdot dy$

$\frac{y^5}{5} - \frac{y^3}{3} + C$

$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

2) $\int \frac{1}{1 - \cos x} \cdot dx \times \frac{1 + \cos x}{1 + \cos x}$

$\int \frac{1 + \cos}{(1 - \cos^2 x)} \cdot dx$

$\int \frac{1 + \cos x}{\sin^2 x} \cdot dx$

$\int \frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x} \cdot dx$

$\int \csc^2 x \cdot dx + \int \frac{\cos x}{\sin^2 x} \cdot dx$

$- \cot x + \cdot$

$- \cot x - \frac{1}{\sin x} + C$

$y = \sin x$

$\frac{dy}{dx} = \cos x$

$dx = \frac{dy}{\cos x}$

$\int \frac{\cos x}{y^2} \cdot \frac{dy}{\cos x}$

$\int \frac{1}{y^2} \cdot dy$

$\int y^{-2} \cdot dy$

$\frac{y^{-1}}{-1}$

$\frac{-1}{y} + C$

$\frac{-1}{\sin x} + C$

3) $\int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}} \cdot dx$ $2x = 3\tan\theta$ ممكن

لكن طويل

$y = 4x^2+9$

$\frac{dy}{dx} = 8x$

$dx = \frac{dy}{8x}$

$\int \frac{x^3}{y^{\frac{3}{2}}} \cdot \frac{dy}{8x}$

$\frac{1}{8} \int \frac{x^2}{y^{\frac{3}{2}}} \cdot dy$

$\frac{1}{8} \int \frac{\frac{y-9}{4}}{y^{\frac{3}{2}}} \cdot dy$

$\frac{1}{8} \int \frac{y-9}{4y^{\frac{3}{2}}} \cdot dy$

$\frac{1}{8} \int \frac{y^{-\frac{1}{5}}}{4} - \frac{9y^{-\frac{3}{2}}}{4} \cdot dy$

$\frac{1}{32} \int y^{-\frac{1}{5}} - 9y^{-\frac{3}{2}} \cdot dy$

4) $\int \sqrt{1-\sin x} \cdot dx$ ضرب بالمرافق

$\int \sqrt{1-\sin x} \times \frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}$

$\int \frac{\sqrt{\cos 2x}}{\sqrt{1+\sin x}} \cdot dx$

$\int \frac{\cos x}{\sqrt{1+\sin x}} \cdot dx$

$y = 1+\sin x$

Subject :

5) $\int e^{\sqrt{x}} \, dx$

$y = \sqrt{x}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$dy \, 2\sqrt{x} = dx$

$\int e^{y} \cdot 2\sqrt{x} \, dy$

$\int 2y e^{y} \, dy$

$2 \int y e^{y} \, dy$

$u = y$ $\quad du = 1$

$dv = e^{y}$ $\quad v = e^{y}$

$y e^{y} - \int e^{y}$

$y e^{y} - e^{y} + C$

$\sqrt{x} e^{\sqrt{x}} - e^{\sqrt{x}} + C$

6) $\int \sqrt{\frac{1-x}{1+x}} \, dx$

$\int \frac{\sqrt{1-x}}{\sqrt{1+x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}} \, dx$

$\int \frac{1-x}{\sqrt{1-x^2}} \, dx$

$\int \frac{1}{\sqrt{1-x^2}} \, dx + \frac{-x}{\sqrt{1-x^2}} \, dx$

$\sin^{-1} x \; +$

$x = \sin\theta$

7) $\int \frac{1}{\sqrt{x} - \sqrt[3]{x}} \, dx$

$\sqrt[2]{x}$ $\quad \sqrt[3]{x}$

$y^6 = x$

$dy \cdot 6y^5 = dx$

$\int \frac{1}{\sqrt{y^6} - \sqrt[3]{y^6}} \cdot 6y^5 \, dy$

$\int \frac{1}{y^3 - y^2} \cdot 6y^5 \, dy$

$6 \int \frac{y^5}{y^3 - y^2} \, dy$

$6 \int \frac{y^3}{y-1} \, dy$

$y^2 + y + 1$

$y - 1 \,\big)\, y^3$

$\ominus\, y^3 \overset{+}{-} y^2$

y^2

$\ominus\, y^2 \overset{+}{-} y$

y

$\ominus\, y \overset{+}{-} 1$

1

$\int y^2 + y + 1 + \int \frac{1}{y-1}$

$\frac{y^3}{3} + \frac{y^2}{2} + y + \ln|y-1| + C$

$6 \left[\frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| \right] + C$

Subject : / /

8) $\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$

$y = \sqrt{1+\sqrt{x}}$

$y^2 = 1+\sqrt{x}$

$2y\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$dy.\ 4y\sqrt{x} = dx$

$4y^3-4y\ .dy = dx$

$\int \frac{y}{x}(4y^3-4y)\,dy$

$4\int \frac{y^4-y^2}{(y^2-1)^2}\,dy$

$4\int \frac{y^2(y^2-1)}{(y^2-1)^2}\,dy$

$4\int \frac{y^2}{y^2-1}\,dy$

$y^2-1 \overline{)\ y^2\ }$ with quotient 1, remainder line $-y^2 \pm 1$

$\int 1 + \int \frac{1}{y^2-1}$

$4y + 4\int \frac{1}{(y-1)(y+1)}$

$\downarrow$

by Partial fractions

9) $\int \frac{xe^x}{\sqrt{1+e^x}}\,dx$

$y = \sqrt{1+e^x}$

$y^2 = 1+e^x$

$\cancel{2y}\cdot\frac{dy}{dx} = \frac{\cancel{1+}e^x}{2\sqrt{1+e^x}}$

$dx = \frac{2\sqrt{1+e^x}}{e^x}\,dy$

$dx = \frac{2y}{e^x}$

$\int \frac{xe^x}{y}\cdot\frac{2y}{e^x}\,dy$

$\int 2x\,dy$

$y = \sqrt{1+e^x}$

$y^2 = 1+e^x$

$y^2-1 = e^x$

$\ln|y^2-1| = x$

$2\int \ln|y^2-1|\,dy$

$u = \ln(y^2-1)$ $\quad du = \frac{2y}{y^2-1}$

$dv = dy$ $\quad v = y$

$2\left[y\ln|y^2-1| - \int \frac{2y^2}{y^2-1}\,dy\right]$

$\downarrow$

$\int$

$\downarrow$

Partial Frac

$$\int \frac{e^{2x}}{e^{2x}+3e^{x}+2} . dx$$

$u = e^{x}$

$\frac{du}{dx} = e^{x}$

$$\int \frac{u^2}{u^2+3u+2} . \frac{dx}{u}$$

$$\int \frac{u}{u^2+3u+2} . dx$$

$$\int \frac{u}{(u+2)(u+1)} . du$$

;

$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x2}}$

$\cos^{-1} x = \frac{-1}{\sqrt{1-x2}}$

$\tan^{-1} x = \frac{1}{1+x^2}$

$\cot^{-1} x = \frac{-1}{1+x^2}$

$\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

$\csc^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$

* Improper Integrals

Type 1 :- Infinite Intervals

1) $\int_{a}^{+\infty} f(x).dx = \lim_{t \to +\infty} \int_{a}^{t} f(x).dx$

2) $\int_{-\infty}^{a} f(x).dx = \lim_{x \to -\infty} \int_{t}^{a} f(x).dx$

Called convergent if limit exists & divergent if the limit d.n.e

3) $\int_{-\infty}^{+\infty} f(x).dx = \int_{-\infty}^{a} f(x).dx + \int_{a}^{+\infty} f(x).dx$

Subject : / /

Ex: $\int_{1}^{+\infty} \frac{1}{x}\,dx$

$= \lim_{t\to+\infty} \int_{1}^{t} \frac{1}{x}\,dx$

$= \lim_{t\to+\infty} \ln|x| \Big]_{1}^{t}$

$\lim_{t\to+\infty} \ln t - \ln 1$

$= \lim_{t\to 0} \ln t$

$= +\infty$

divergent

Ex: $\int_{1}^{+\infty} \frac{1}{x^2}\,dx$

$\lim_{t\to+\infty} \int_{1}^{t} \frac{1}{x^2}\,dx$

$\lim_{t\to+\infty} \frac{-1}{x} \Big]_{1}^{t}$

$\lim_{t\to+\infty} \left[\frac{-1}{t} - 1\right]$

$\lim_{t\to+\infty} \left[\overset{zero}{\cancel{\frac{-1}{\infty}}} - 1\right]$

$= 1$

Convergent

Theorem: $\int_{1}^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} p > 1 & \text{Convergent to } \frac{1}{p-1} \\ p \le 1 & \text{divergent} \end{cases}$

Ex = ① $\int_{1}^{+\infty} \frac{1}{x^3}\,dx$

Convergent = $\frac{1}{2}$

② $\int_{1}^{+\infty} \frac{1}{x^{10}}\,dx$ Convergent to = $\frac{1}{9}$

③ $\int_{1}^{+\infty} \frac{1}{\sqrt{x}}\,dx$ = divergent

④ $\int_{1}^{+\infty} x^{-\frac{2}{3}}\,dx = \int_{1}^{+\infty} \frac{1}{x^{\frac{2}{3}}}$ = divergent

Ex: $\int_{-\infty}^{0} x e^{x}\,dx$

$\lim_{t\to-\infty} \int_{t}^{0} x e^{x}\,dx$

$u = x \quad du = 1$

$dv = e^{x} \quad v = e^{x}$

$\lim_{t\to-\infty} x e^{x} - e^{x} \Big]_{t}^{0}$

$\lim_{t\to-\infty} \left[-1 - (te^{t} - e^{t})\right]$

التكملة بالصفحة التالية

Subject :

$$\lim_{t \to -\infty} e^{t} = 0$$

$$\lim_{t \to -\infty} te^{t} = -\infty . 0$$

$\downarrow$

l'Hospital rule

$$\lim_{t \to -\infty} \frac{t}{e^{t}}$$

$$\lim_{t \to -\infty} \frac{1}{-e^{t}} = 0$$

$$\lim_{t \to -\infty} \left(-1 - \cancel{\left(\overset{\text{Zero}}{te^{t}} - \overset{\text{Zero}}{e^{t}}\right)}\right)$$

$= -1$ Convergen

Ex: $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^2+1} \, dx$

$$\int_{-\infty}^{0} \frac{1}{1+x^2} \, dx + \int_{0}^{+\infty} \frac{1}{1+x^2} \, dx$$

$$\int_{-\infty}^{0} \frac{\cancel{\sin}}{1+x^2} \, dx$$

$$\lim_{t \to -\infty} \tan^{-1} x \Big]_{t}^{0}$$

$$\lim_{t \to 0} \left(0 - \tan^{-1} t\right)$$

$$= -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_{0}^{+\infty} \frac{1}{1+x^2} = \lim_{t \to +\infty} \int_{0}^{t} \frac{1}{1+x^2} \, dx$$

$$= \lim_{t \to +\infty} \tan^{-1} x \Big]_{0}^{t}$$

$$\lim_{t \to +\infty} \tan^{-1} t - \cancel{\tan^{-1} 0}^{\text{Zero}}$$

$$= \frac{\pi}{2}$$

Convergent

$$\frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

Convergent

Type 2: Discontinuous Integrands

1) If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^b f(x)\,dx$$

2) If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x).dx = \lim_{t\to a^+} \int_t^b f(x).dx$$

The improper integral is called convergent if limits exist & divergent if the limit does not exist.

3) If $f(x)$ is discontinuous at c, such that $a<c<b$, then

$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x).dx}_{①} + \underbrace{\int_c^b f(x).dx}_{②}$$

Ex: 1) $\int_2^5 \frac{1}{\sqrt{x-2}}\,.dx$ $\frac{1}{\sqrt{x-2}}$ discontinuous at $x=2$
↳ improper integral

$$\lim_{t\to 2^+} \int_t^5 (x-2)^{-\frac{1}{2}}$$

$$\lim_{t\to 2^+} 2(x-2)^{\frac{1}{2}} \Big]_t^5$$

$$\lim_{t\to 2^+} 2(3)^{\frac{1}{2}} - 2(t-2)^{\frac{1}{2}}$$

$$2\sqrt{3} - 2(2-2)^{\frac{1}{2}}$$

$$= 2\sqrt{3}$$ convergent

② $\int_0^{\frac{\pi}{2}} \sec x \, dx \qquad \sec\frac{\pi}{2} = \frac{1}{0} = \infty$

$= \lim_{t\to\frac{\pi}{2}^-} \int_0^t \sec x \, dx$

$= \lim_{t\to\frac{\pi}{2}^-} \ln|\sec x + \tan x| \Big]_0^t$

$= \lim_{t\to\frac{\pi}{2}^-} \ln|\sec t + \tan t| - \ln|\sec 0 + \tan 0|$

$\infty + \infty$

$= \infty$ divergab

③ $\int_0^3 \frac{dx}{x-1} \qquad \frac{1}{x-1}$ discontinuous at $x=1$

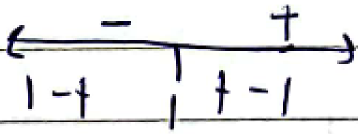
$\int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx$

$\lim_{t\to 1^-} \int_0^t \Big| \frac{1}{x-1}\,dx$

$= \lim_{t\to 1^-} \ln|x-1| \Big]_0^t$

$= \lim_{t\to 1^-} \ln|t-1| - \ln|0-1|$

$= \lim_{t\to 1^-} \ln|t-1|$



$\lim_{t\to 1^-} \ln|(1-t)|$

ما في داعي للتكملة مباشرة

$= -\infty$

$\int_0^3 \frac{dx}{x-1} \longrightarrow$ divergent

$\int_0^1 \frac{1}{x-1}\,dx$

divergab

▸ Subject : .... / /

Find the values of p for which the integral converges (limits exists)

1) $\int_{e}^{+\infty} \frac{1}{x(\ln x)^{2p}} \, dx$

$u = \ln x$

$dx = x \, du$

$x = e \rightarrow u = \ln e = 1$

$x \rightarrow \infty \rightarrow u \rightarrow +\infty$

$\int_{1}^{+\infty} \frac{1}{x \, u^{2p}} \cdot x \, du$

$\int_{1}^{+\infty} \frac{1}{u^{2p}} \, du \begin{cases} p \leq 1 \text{ div} \\ p > 1 \text{ con} \end{cases}$

$2p > 1 \Rightarrow p > \frac{1}{2} \quad (\frac{1}{2}, \infty)$

2) $\int_{a}^{b} \frac{1}{(x-a)^{p-1}} \, dx \quad \boxed{a<b}$

$p \neq 2$

$\lim_{t \to a^+} \int_{t}^{b} \frac{1}{(x-a)^{p-1}} \, dx$

$(+)^{2-p} \rightarrow \text{number}$

"والتعويضة" $\lim_{t \to a^+} (t-a)^{2-p}$

$2-p > 0$ ⊕ $\quad 2-p < 0$ ⊖

$\lim_{t \to a^+} \int_{t}^{b} (x-a)^{1-p} \, dx$

$\lim_{t \to a^+} \frac{(x-a)^{2-p}}{2-p} \Big]_{t}^{b}$

$0^+ \quad \bar{0}$

convergent $\quad \frac{1}{0} = \infty$

$= \lim_{t \to a^+} \frac{1}{2-p} \left((b-a)^{2-p} - (t-a)^{2-p} \right)$

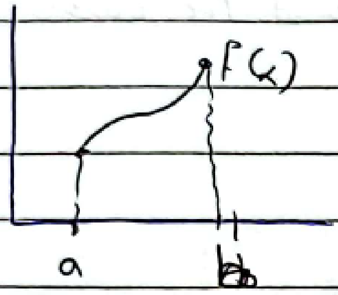
divergent

$2-p > 0$

Convergent $(-\infty, 2)$ $\quad \boxed{2 \geq p}$

if →

*Arc length



The Arc length formaulla

If f' is Continuous on $[a,b]$ then the length f the Curve $y=f(x)$

$a \leq x \leq b$ is $L = \int_a^b \sqrt{1+(f'(x))^2} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Ex: find the exact length of the curve:-

$f(x) = x^{\frac{3}{2}}$, $1 \leq x \leq 4$

$f'(x) = \frac{3}{2} x^{\frac{1}{2}} \rightarrow f'(x) = \frac{9}{4} x$

$L = \int_1^4 \sqrt{1+\frac{9}{4}x} \, dx$

$= \int_1^4 \left(1+\frac{9}{4}x\right)^{\frac{1}{2}} dx$

$= \frac{\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)\left(\frac{9}{4}\right)} \Big]_1^4$

$= \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$

$c \leq y \leq d$ Curve $x = g(y)$

$L = \int_c^d \sqrt{1+(g'(y))^2} \, dy$

$= \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2} \, dy$

Ex: set up the integral that represent the length of the curve:-

$x = y^2$ from (0,0) to (1,1)

$\frac{dx}{dy} = 2y \rightarrow \left(\frac{dx}{dy}\right)^2 = 4y^2$

$L = \int_0^1 \sqrt{1 + 4y^2}\,.dy$

$= \frac{(1+4y^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(4)} \Big]_0^1$

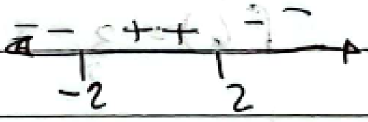
Ex: Find the arc length of the curve:-

$y = \sqrt{4 - x^2}$ $\xrightarrow[\text{the domain}]{}$ $4 - x^2 \geq 0$

domain $[-2, 2]$

$4 \geq x^2$

$\boxed{x = \pm 2}$



$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

$\frac{dy}{dx} = \frac{-2x}{2\sqrt{4-x^2}}$

$\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}} \longrightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4-x^2}$

$L = \int_{-2}^{2} \sqrt{1 + \frac{x^2}{4-x^2}}\,.dx$

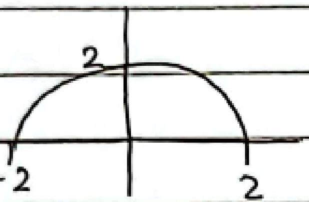
$= \int_{-2}^{2} \sqrt{\frac{4 - x^2 + x^2}{4 - x^2}}\,.dx$

$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\,.dx \longrightarrow 2\int_{-2}^{2} \sqrt{\frac{1}{4-x^2}}\,.dx$

$= 2\sin^{-1}\left(\frac{x}{2}\right)\Big]_{-2}^{2}$

$= 2\pi$

$f(x) = \sqrt{4-x^2}$ $[-2,2]$ other way $\sqrt{a^2-x^2}$, $-\sqrt{a} \le x \le \sqrt{a}$



$C = 2\pi r$

$\frac{1}{2} L = \pi r$

$= \boxed{2\pi}$

Area of surface of Revolution

f is Positive and has a continuous derivative we define the surface area obtained by rotating the curve $y = f(x)$, $a \le x \le b$ about x-axis

$$S = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\,.dx$$

Ex: Find the exact area of the surface ...

$f(x) = x^3$ $x \in [0,1]$ $f'(x) = 3x^2 \rightarrow (f'(x))^2 = 9x^4$

$$S = \int_0^1 2\pi(x^3)\sqrt{1+9x^4}\,.dx$$

$$= \int_0^1 2x^3\pi\sqrt{9x^4+1}\,.dx$$

$y = 9x^4 + 1$

$\frac{dy}{dx} = 36x^3$

$$= \int_0^1 2x^3\pi\sqrt{y}\,.\frac{dy}{36x^3}$$

$$= \frac{\pi}{18}\int_0^1 \sqrt{y}\,.dy$$

$\rightarrow \left(\frac{\pi}{18}\right)\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Big]_0^1$

$\frac{\pi}{27}\sqrt{(9x^4+1)^3}\Big]_0^1$

$= \frac{\pi}{27}\left[10\sqrt{10} - 1\right]$

Ex: $y = \sqrt{4-x^2}$, $-2 \le x \le 2$

$\frac{dy}{dx} = \frac{-2x}{2\sqrt{4-x^2}} \longrightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4-x^2}$

$= \int_{-2}^{2} 2\pi\sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx$

$= \int_{2}^{2} 2\pi\sqrt{4-x^2}\sqrt{\frac{4}{4-x^2}}\,dx$

$= \int_{-2}^{2} 4\pi\,dx$

$= 4\pi[2--2]$

$= 4\pi[4]$

$= 16\pi$

another way:- special case

$S = 4\pi r^2$

$S = 4\pi(2)^2$

$S = \boxed{16\pi}$

* Set up an Integral for the area of the Surface obtained by rotating the curve about x-axis:-

$y = \tan$, $0 \le x \le \frac{\pi}{3}$

$\frac{dy}{dx} = \sec^2 x \rightarrow \left(\frac{dy}{dx}\right)^2 = \sec^4 x$

$S = \int_{0}^{\frac{\pi}{3}} 2\pi \tan\sqrt{1+\sec^4 x}\,dx$

$S = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2} \, dx$, $y = f(x)$, $a \leq x \leq b$

$x = g(y)$, $c \leq y \leq d$ about x-axis

$S = \int_c^d 2\pi y \sqrt{1+\left(\frac{dx}{dy}\right)^2} \, dy$

Ex: find the exact area of the surface:-

$x = \frac{1}{3} (y^2+2)^{\frac{3}{2}}$, $1 \leq y \leq 2$

$\frac{dx}{dy} = \frac{1}{3} \left(\frac{3}{2}\right) (y^2+2)^{\frac{1}{2}} (2y)$

$\frac{dx}{dy} = y\sqrt{(y^2+2)} \longrightarrow \left(\frac{dx}{dy}\right)^2 = y^2 (y^2+2) = (y^4+2y^2)$

$S = \int_1^2 2\pi (y) \sqrt{1+(y^4+2y^2)} \, dy$

$= \int_1^2 2\pi y \sqrt{y^4+2y^2+1} \, dy$

$= \int_1^2 2\pi y \sqrt{(y^2+1)^2} \, dy$

$= 2\pi \int_1^2 y(y^2+1) \, dy$

$= 2\pi \int_1^2 y^3 + y \, dy$

$= 2\pi \left(\frac{y^4}{4} + \frac{y^2}{2} \right) \Big]_1^2$

$= \frac{21\pi}{4}$

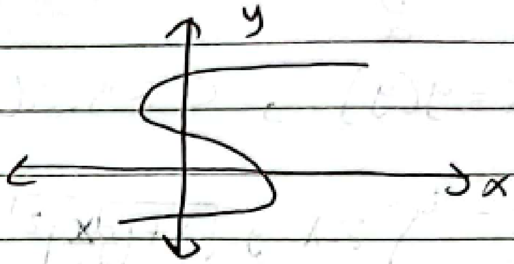
Subject : / /

Curves defined by Parametric equations:-

t = Parameter

$x = f(t)$

$y = g(t)$] → Parametric equations



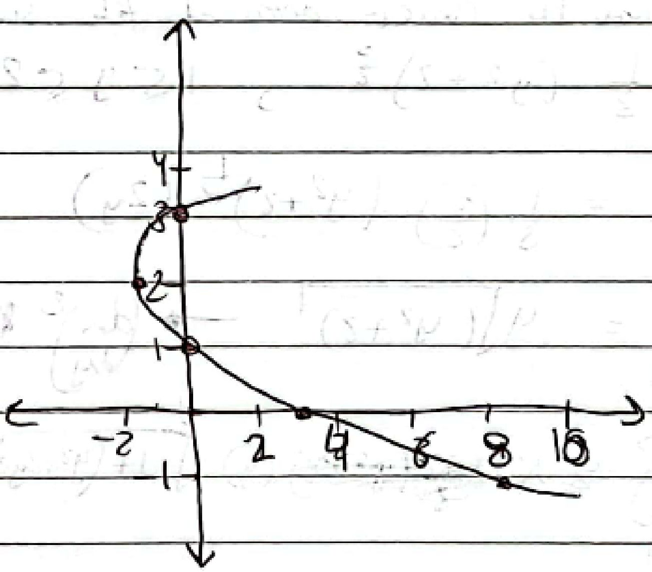
new method for describing curves

Ex: Sketch the curve defined by the Parametric equations:-

$x = t^2 - 2t$

$y = t + 1$

t	x	y	(x, y)
-2	8	-1	(8, -1)
-1	3	0	(3, 0)
0	0	1	(0, 1)
1	-1	2	(-1, 2)
2	0	3	(0, 3)



$t = y - 1$

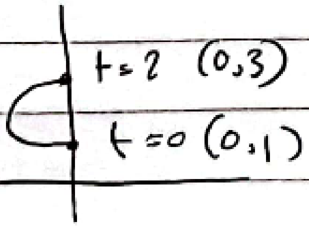
$x = t^2 - 2t$

$x = (y-1)^2 - 2(y-1)$

$\boxed{x = y^2 + 4y + 3}$

$x = t^2 - 2t$ $0 \leq t \leq 2$

$y = t + 1$



Ex: What curve represented by the following Parametric equations:-

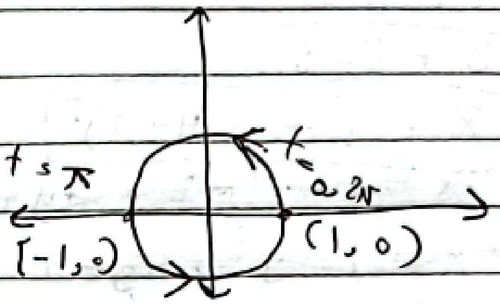
1) $X = \cos t$, $y = \sin t$ $\quad 0 \leq t \leq 2\pi$

$X^2 + y^2 = 1$

Circle ⇃

Centre $(0, 0)$

radius $= 1$



$t = 0 \rightarrow X = \cos 0 = 1 \quad (1, 0)$

$y = \sin 0 = 0$

$t = \pi \rightarrow X = \cos \pi = -1 \quad (-1, 0)$

$y = \sin \pi = 0$

$t = 2\pi \rightarrow X = \cos 2\pi = 1 \quad (1, 0)$

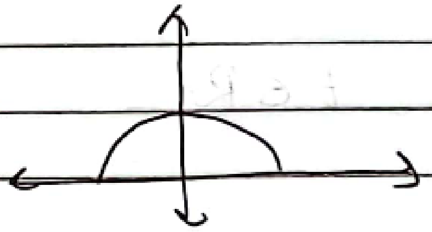
$y = \sin 2\pi = 0$

2) $X = \cos t$, $y = \sin t$ $\quad 0 \leq t \leq \pi$

Semi circle

Centre $(0, 0)$

radius $= (1)$



Ex: Eliminate the Parameter to find Cartesian equation of the curve:-

1) $X = 3 - 4t$

$y = 2 - 3t \qquad y = 2 - 3\left(\frac{3 - x}{4}\right)$ Line

$X - 3 = -4t$

$t = \frac{x - 3}{-4}$

$$\boxed{t = \frac{3 - X}{4}}$$

2) $x = 1 - t^2$ $\quad -2 \leq t \leq 2$

$y = t - 2$

$x = 1 - t^2$

$\boxed{y + 2 = t}$

$\boxed{x = 1 - (y+2)^2}$

~~$x = 1 - (y^2 + 4y + 4)$~~ $t = -2 \rightarrow y = -4$

~~$t = 2$ $x = 1 - y^2 - 4y$~~ $- 4$ $t = 2 \rightarrow y = 0$ $\quad -4 \leq y \leq 0$

~~$x = -y^2 - 4y - 3$~~

$t = -2$

3) $x = \sqrt{t}$ $\quad t \geq 0 \rightarrow x \geq 0$

$y = 1 - t$

$x^2 = t$

$\boxed{y = 1 - x^2}$, $x \geq 0$

4) $x = e^t$ $\quad , t \in R$

$y = e^{-t}$

$x = e^t$

$\ln x = t$

$y = e^{-\ln x}$

$y = e^{\ln \frac{1}{x}}$

$\boxed{y = \frac{1}{x}}$

5) $x = 3\cos t$ $\quad 0 \leq t \leq 2\pi$

$y = 4\sin t$ $\quad \cos^2 x \cdot 4\sin^2 x = 1$

$\cos t = \frac{x}{3}$ $\quad \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$

$\sin t = \frac{y}{4}$

Eillipse

قطع ناقص

5) $x = \cosh t \longrightarrow [1, \infty)$

$y = \sinh t$

$\cosh^2 t - \sinh^2 t = 1$

$x^2 - y^2 = 1$ Hypabolab

$\boxed{x \geq 1}$ قطع زائد

Para

Cart

t x, y

Ex: Find the Parametric equations:-

1) $x = y^2 - y$

$$\left.\begin{aligned} y &= t \\ x &= t^2 - t \end{aligned}\right\}$$

2) $y = 2x + 1$ $\quad 0 \leq x \leq 1$

$$\left.\begin{aligned} x &= t \\ y &= 2t + 1 \end{aligned}\right\} \quad 0 \leq t \leq 1$$

3) line segment from $(-2, 7)$ to $(3, -1)$

$P_1(x_1, x_2)$ $\quad x = x_1 + (x_2 - x_1)t$ $\quad 0 \leq t \leq 1$

$P_2(x_2\ y_2)$ $\quad y = y_1 + (y_2 - y_1)t$

$x = -2 + (3 - -2)t$

$$\left.\begin{aligned} &\boxed{x = -2 + 5t} \\ &y = 7 + (-1 - 7)t \\ &\boxed{y = 7 - 8t} \end{aligned}\right\}$$

Circle centre (h, k) radius R

$0 \leq t \leq 2\pi$

$x = h + r\cos t$

$y = k + r\sin t$

4) Circle centre $(5, 2)$, radius 2.

$(x-5)^2 + (y-2)^2 = 4$

$\left(\frac{x-5}{2}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 1$

$\cos^2 t + \sin^2 t = 1$

$$\left.\begin{aligned} \frac{x-5}{2} &= \cos t & 2\cos t + 5 &= x \\ \frac{y-2}{2} &= \sin t & 2\sin t + 2 &= y \end{aligned}\right\}$$

Calculus with Parametric Curves

$x = f(t)$

$y = g(t)$

$f(t)$, $g(t)$ are differentiable

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\boxed{y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}} \leftarrow \div \frac{dx}{dt} , \frac{dx}{dt} \neq 0$$

$$y'' = \frac{dy'}{dt} = \frac{dy'}{dx} \cdot \frac{dx}{dt} \quad \div \frac{dx}{dt} , \frac{dx}{dt} \neq 0$$

$$\boxed{y'' = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}}$$

① $x = t^2 + 1$

$y = t^2 + t$

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 2t + 1$$

$$\frac{dy}{dx} = \frac{2t+1}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{(2t)(2) - (2t+1)(2)}{4t^2}}{2t}$$

$$= \frac{4t - 4t - 2}{8t^3} = \frac{-1}{4t^3}$$

Subject :   /   /

① find $\frac{dy}{dx}\Big|_{t=1}$

$\frac{dy}{dx} = \frac{2t+1}{2t} \xrightarrow{t=1} \frac{dy}{dx}\Big|_{t=1} = \frac{2+1}{2} = \frac{3}{2}$

2) find $\frac{d^2y}{dx^2}\Big|_{t=1} \Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4t^3} \rightarrow \frac{d^2y}{dx^2} = \frac{-1}{4}$

$y = 2\sin t \rightarrow$ find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$

$x = 3\cos t$

$\frac{dy}{dt} = 2\cos t$

$\frac{dx}{dt} = -3\sin t$

$\frac{dy}{dx} = \frac{2\cos t}{-3\sin t}$

$$\frac{d^2y}{dx^2} = \frac{\frac{(-3\sin t)(-2\sin t) - (2\cos t)(-3\cos t)}{9\sin^2 t}}{-3\sin t}$$

$$= \frac{6\sin^2 t + 6\cos^2 t}{-27\sin^3 t}$$

$$\frac{d^2y}{dx^2} = \frac{6}{-27\sin^3 t} = \frac{-2}{9}\csc^3 t$$

Calculus with Parametric Curves:-

[1] Equ at tangent line

$$y - y_0 = m(x - x_0)$$

$$m = \text{slope} = \frac{dy}{dx}\Big|_{(x_0, y_0)}$$

[2] Curve has horizontal tangent

$$m = 0$$

$\frac{dy}{dx} = 0$ $\frac{dy}{dt} = 0$, $\frac{dx}{dt} \neq 0$

[3] vertical tangent

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ ~~$\frac{dx}{dt} \neq 0$~~

$\frac{dx}{dt} = 0$, $\frac{dy}{dt} \neq 0$

[4] $\frac{d^2y}{dx^2}$ Concave up $\frac{d^2y}{dx^2} > 0$

Concave down $\frac{d^2y}{dx^2} < 0$

Ex: A curve defined by the Parametric equations:-

$x = t^2$, $y = t^3 - 3t$

D show that C has two tangent at the Point (3,0) and find their Equations

$y = 0 \longrightarrow y = t^3 - 3t$

$0 = t^3 - 3t$

$\boxed{t = 0}$ مرفوض $\boxed{t = \pm\sqrt{3}}$ ✓

$x = 3$ $x = t^2$

$t = 0 \longrightarrow x = 0$

$t = \sqrt{3} \longrightarrow x = (\sqrt{3})^2 = 3$

$t = -\sqrt{3}$ $x = (-\sqrt{7})^2 = 3$

$y = t^3 - 3t$

$\frac{dy}{dt} = 3t^2 - 3$

$x = t^2$

$\frac{dx}{dt} = 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2-3}{2t}$$

t	slope $= m = \frac{dy}{dx} = \frac{3t^2-3}{2t}$	$y - y_0 = m(x - x_0)$ $(3,0)$
$\sqrt{3}$	$\sqrt{3}$	$y = \sqrt{3}(x-3)$
$-\sqrt{3}$	$\frac{-6}{2\sqrt{3}} = -\sqrt{3}$	$y = -\sqrt{3}(x-3)$

2) Find the Point on C, when the tangent is horizontal or vertical

$\frac{dy}{dx} = \frac{3t^2-3}{2t}$

Horizontal → $\frac{dy}{dt} = 0$ $3t^2 - 3 = 0$

$\boxed{t = \pm 1}$

$t = 1$ $t = -1$

$2t = 2 \neq 0$ $2t = -2 \neq 0$

✓ ✓

↓ ↓

$(1, -2)$ $(1, 2)$

$2t = 0 \rightarrow t = 0 \rightarrow 3t^2 - 3 \xrightarrow{t=0} -3 \neq 0$

Vertical tangent $x = t^2$ $\rightarrow (0,0)$

$y = t^3 - 3t$

Subject : / /

3) determinitae where the Curve is Concaveup & down.

$$\frac{dy}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{3t^2-3}{2t}$$

$$\frac{dy'}{dt} = \frac{(2t)(6t)-(3t^2-3)(2)}{4t^2}$$

$$= \frac{12t^2-6t^2+6}{4t^2}$$

$$= \frac{6t^2+6}{4t^2}$$

$$\frac{dy'}{dt} = \frac{3t^2+3}{2t^2}$$

$$\frac{dy}{dx^2} = \frac{\frac{3t^2+3}{2t^2}}{2t}$$

$$\frac{dy}{dx^2} = \frac{3t^2+3}{4t^3}$$

$\frac{dy}{dx^2} = 0 \rightarrow 3t^2+3=0 \quad 3t^2=-3 \quad X$

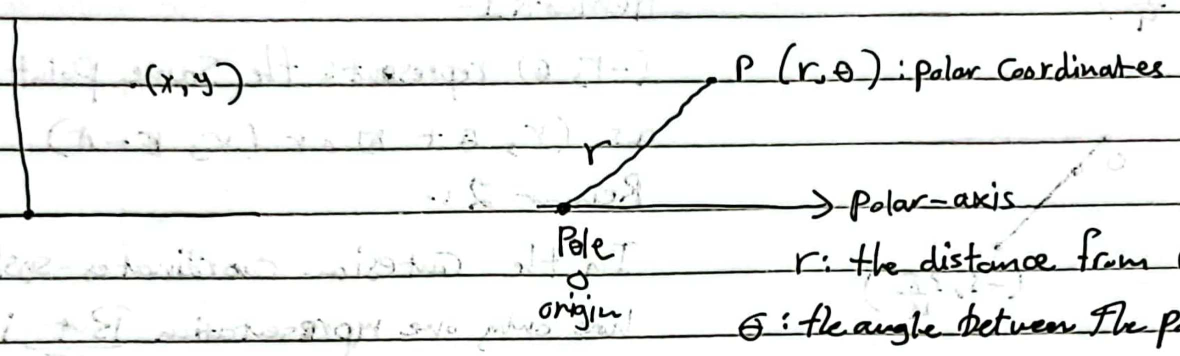
$4t^3=0 \quad \boxed{t=0}$

$\longleftarrow - - - \overset{}{\underset{0}{\circ}} + + + + \longrightarrow \quad y=$

Concave up $(0, \infty)$

Concave down $(-\infty, 0)$

* Polar cordinates

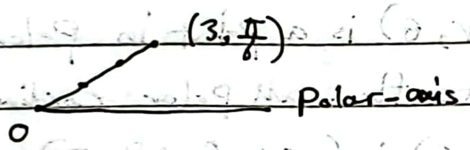


r: the distance from o to p

θ: the angle between the polar axis & the line op (radius)

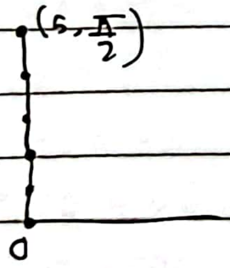
Example: plot the points whose Polar cordinates are given

1) $(3, \frac{\pi}{6})$

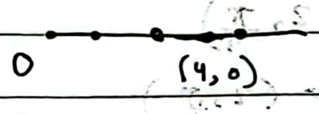


2) $(2, \frac{5\pi}{4})$

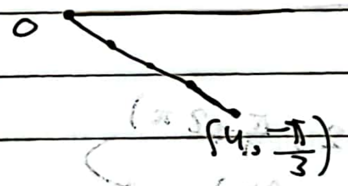
3) $(5, \frac{\pi}{2})$



4) $(4, 0)$

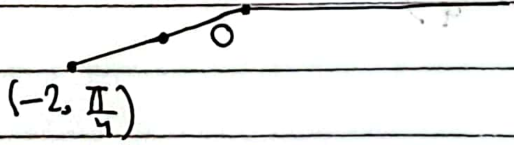


5) $(4, -\frac{\pi}{3})$

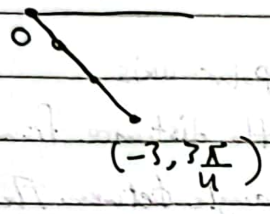


6) $(2, 3\pi)$

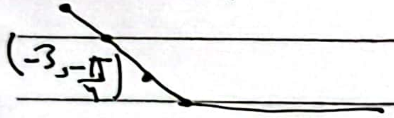
7) $(-2, \frac{\pi}{4})$



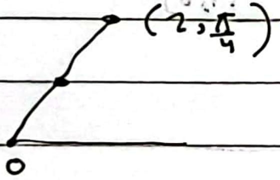
8) $(-3, \frac{3\pi}{4})$

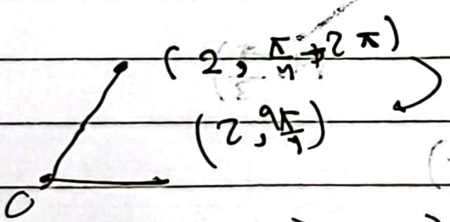


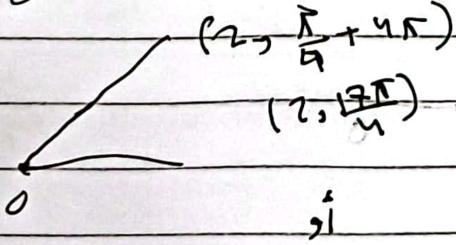
9) $(-3, \frac{-\pi}{4})$



Ex : $(p(2, \frac{\pi}{4})$







أو

$(2, \frac{-15\pi}{4})$ $(2, \frac{-7\pi}{4})$

$(-2, \frac{-3\pi}{4})$ $(-2, \frac{5\pi}{4})$

Remark 1-

$(-r, \theta)$ represents the Same point as $(r, \theta + \pi)$ or $(r, \theta - \pi)$

Remar 2 :-

In the Cartesian cordinates-system has only one representation <u>But</u> in the polarcoordinates-system each Point has many representation

If (r, θ) is a point in Polar Cordinates then all polar cordinates of (r, θ) is $(r + \theta + 2n\pi)$

$n : 0, \pm 1, \pm 2 \ldots)$ $(-r, \theta + \pi, 2n\pi)$

$n : 0, \pm 1, \pm 2 \ldots)$ $(-r, \theta + (2n+1)\pi)$

$(0, \theta) \rightarrow$ any value of θ

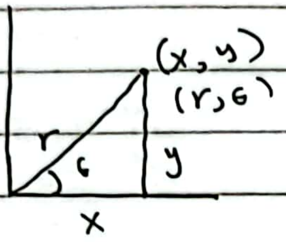
0
Pole

$(0, \frac{\pi}{2})$

$(0, \frac{\pi}{3})$

origin ~~(0,0)~~ $(0, 0)$

Relationship between Polar & Cartesian Coordinates:-



$\cos\theta = \frac{x}{y}$ $\sin\theta = \frac{y}{r}$

$\boxed{x = r\cos\theta}$ $\boxed{y = r\sin\theta}$

$\boxed{x^2 + y^2 = r^2}$ $\boxed{\tan\theta = \frac{y}{x}}$

Ex: Convert the following Points from Polar to Cartesian

1) $(2, \frac{\pi}{3})$

$x = r\cos\theta = 2\cos\frac{\pi}{3} = 1$

$y = r\sin\theta = 2\sin\frac{\pi}{3} = \sqrt{3}$

$(1, \sqrt{3})$

2) $(-6, \frac{2\pi}{3})$

$x = r\cos\theta = -6\cos\frac{2\pi}{3} = 3$

$y = r\sin\theta = -6\sin\frac{2\pi}{3} = -3\sqrt{3}$

$(3, -3\sqrt{3})$

Ex: Convert the following Points from Cartesian to Polar Coordinates

1) $(1, -1)$

$r^2 = x^2 + y^2$

$r = \sqrt{2}$

ع.إ $\tan\theta = \frac{-1}{1} = -1$

رابع $\theta = 2\pi - \frac{\pi}{4}$

$\theta = \frac{7\pi}{4}$

$(\sqrt{2}, \frac{7\pi}{4})$

2) $(-\sqrt{3}, 1)$

$r^2 = x^2 + y^2$

$r = 2$

ع.إ $\tan\theta = \frac{1}{-\sqrt{3}}$

ثاني

$\theta = \frac{5\pi}{6}$

$(2, \frac{5\pi}{6})$

Subject :

Ex: find the polar equation for the curve represented by given Cartesian equation

① $y = 2$

$r\sin\theta = 2$

$r = \frac{2}{\sin\theta}$

$r = 2\csc\theta$

2) $xy = 4$

$r\cos\theta \; r\sin\theta = 4$

$r^2 = \frac{4}{\sin\theta\cos\theta}$

$r^2 = \frac{4}{\frac{1}{2}\sin 2\theta}$

$r^2 = \frac{8}{\sin 2\theta}$

$r^2 = \sqrt{\frac{8}{\sin 2\theta}}$

$r^2 = 8\csc 2\theta$

Ex:

① $r = 2$

$r^2 = 4$

$x^2 + y^2 = r^2$

$\boxed{x^2 + y^2 = 4}$ → circle

2) $r = \sec\theta$

$r = \frac{1}{\cos\theta}$

$\boxed{r\cos\theta = 1}$ vertical line

$\boxed{x = 1}$

Subject : / /

3) $r = \sec\theta \tan\theta$

$r = \frac{1}{\cos\theta} \cdot \frac{y}{x} \times \cos\theta$

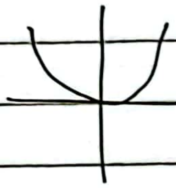
$r\cos\theta = \frac{y}{x}$

$x = \frac{y}{x}$

$y = x^2$

parabola

شكلها



4) $r = 3\cos\theta$

$r^2 = 3r\cos\theta \rightarrow 3x$

$x^2 + y^2 = r^2$

$x^2 + y^2 = 3x$

$x^2 - 3x + y^2 = 0$ (دائرة)

5) $\theta = \frac{\pi}{4}$

$\tan\theta = \tan\frac{\pi}{4} = 1$

$\frac{y}{x} = 1$

$\boxed{x = y}$ line

*Polar curves:-

- Lines

* horizontal line

$y = a$

$r\sin\theta = a$

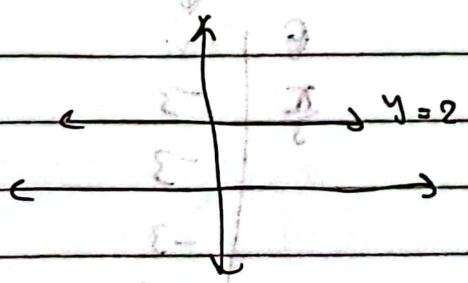
$r = \frac{a}{\sin\theta} = a\csc\theta$

$\boxed{r = a\csc\theta}$

Ex:-

$r = 2\csc\theta$

$y = 2$



* vertical line

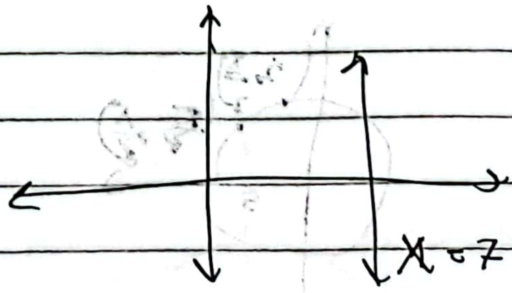
$x = a$

$r\cos\theta = a$

$\boxed{r = a\sec\theta}$

Ex:- $r = 7\sec\theta$

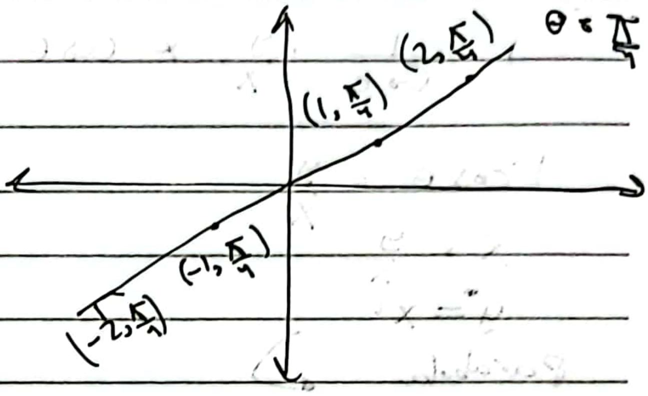
$\boxed{x = 7}$



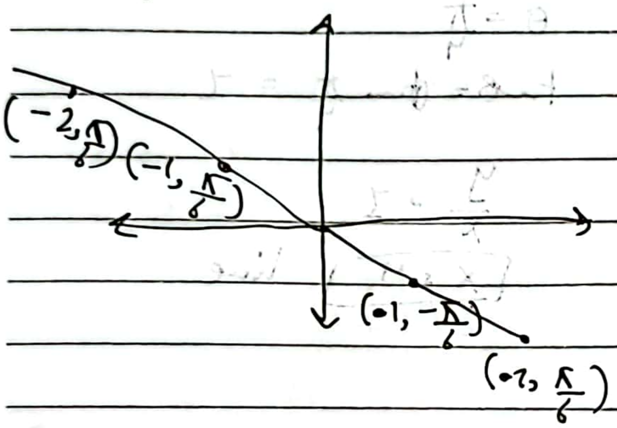
✯ The straight line Passes throughb Pole

$\boxed{\theta = \theta_0}$

Ex : $\theta = \frac{\pi}{4}$



$\theta = -\frac{\pi}{6}$



Polar Curves:-

Circle

1) $r = a$

Centre $(0,0)$

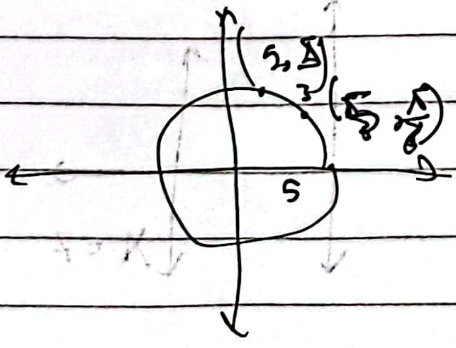
radius $(|a|)$

Ex: 1) $r = 5$

$(0,0)$

$r = 5$

$0 \leq \theta \leq 2\pi$

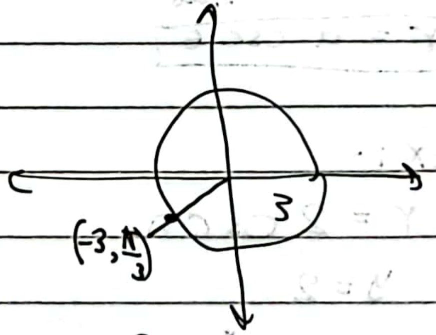


θ	r
0	5
$\frac{\pi}{3}$	5
$\frac{\pi}{6}$	5
⋮	⋮

2) $r = -3$

$(0,0)$

$r = 3$



θ	r
$\frac{\pi}{3}$	-3
	-3
	-3
	-3

② $r = 2a\cos\theta + 2a\sin\theta$

Centre (a, b) $\quad 0 \le \theta \le \pi$

radius $\sqrt{a^2 + b^2}$

Ex: $r = 4\cos\theta + 6\sin\theta$

$r = 2(2\cos\theta + 3\sin\theta)$

~~cut~~ Circle $(2, 3)$ ← Cartisaian not Polar

radius $\sqrt{4+9} = \sqrt{13}$

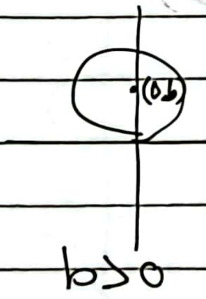
③ $r = 2a\cos\theta + 2b\sin\theta$ $\quad 0 \le \theta \le \pi$

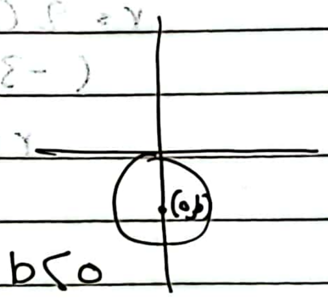
$\boxed{a = 0}$

$r = 2b\sin\theta$ $\quad 0 \le \theta \le \pi$

Circle centre $(0, b)$ ← Cartesian

radius r = $|b|$





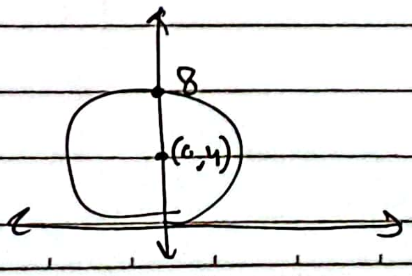
Ex:

1) $r = 8\sin\theta$

$r = 2(4\sin\theta)$

Circle centre $(0, 4)$

$r = 4$

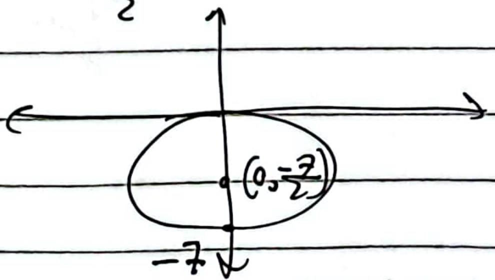


2) $r = -7\sin\theta$

$r = 2(\frac{-7}{2}\sin\theta)$

$(0, -\frac{7}{2})$

$r = \frac{7}{2}$



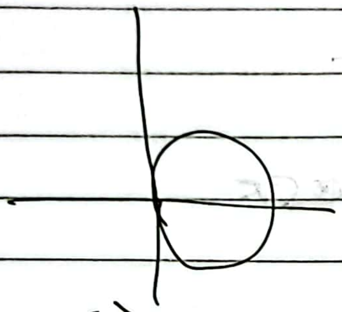
$\boxed{4}$ $r = 2a\cos\theta + 2b\sin\theta \quad 0 \leq \theta \leq \pi$

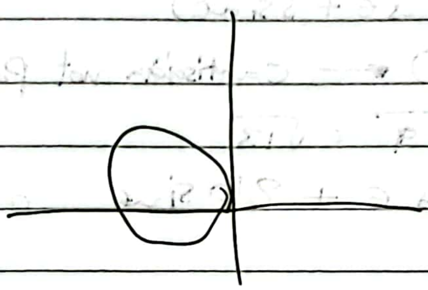
$b = 0$

$r = 2a\cos\theta \quad 0 \leq \theta \leq \pi$

Centre $(a, 0)$ ← Cartesian

$r = |a|$





Ex:

1) $r = 5\cos\theta$

$r = 2(\frac{5}{2}\cos\theta)$

$(\frac{5}{2}, 0)$

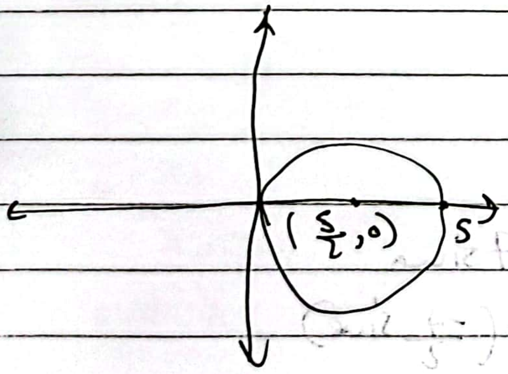
$r = \frac{5}{2}$

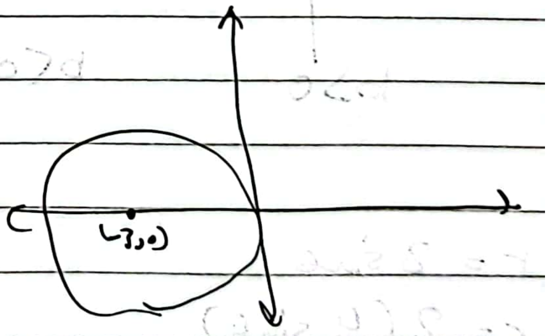
2) $r = -6\cos\theta$

$r = 2(-3\cos\theta)$

$(-3, 0)$

$r = 3$



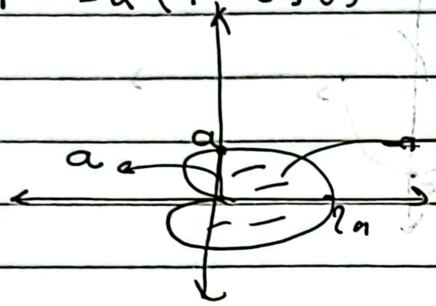


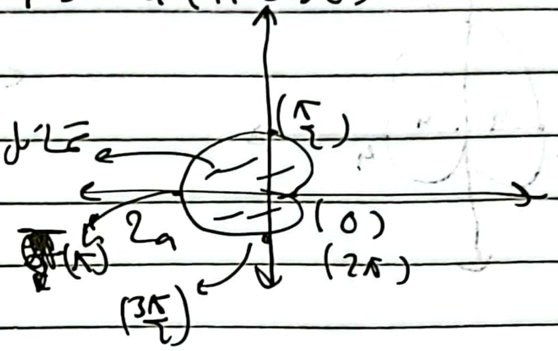
Polar curves $0 \leq \theta \leq 2\pi$

Cardioid $a > 0$

$r = a(1+\cos\theta)$ $r = a(1-\cos\theta)$

$r = -a(1-\cos\theta)$ $r = -a(1+\cos\theta)$

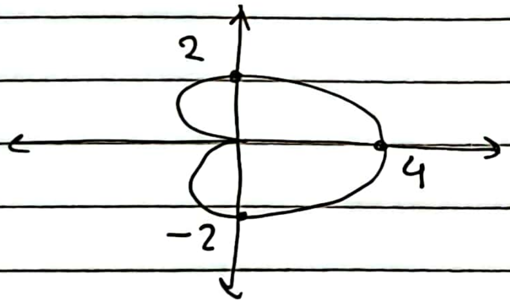


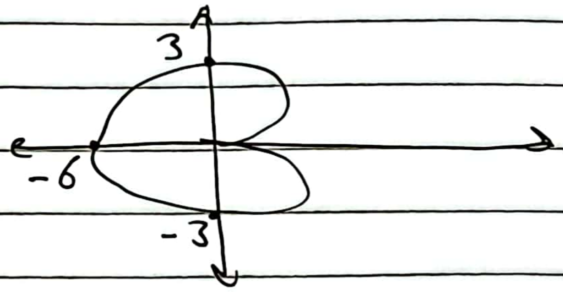


Ex:

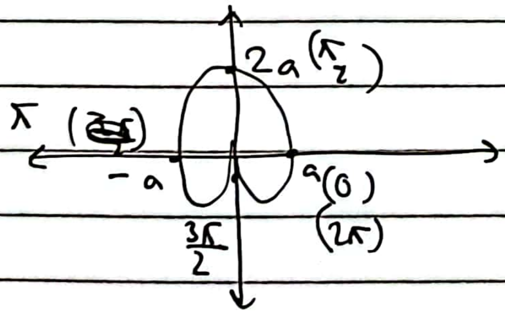
① $r = 2+2\cos\theta$ $r = 3-3\cos\theta$

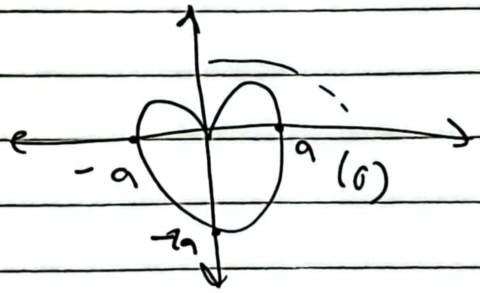
$r = 2(1+\cos\theta)$ $r = 3(1-\cos\theta)$





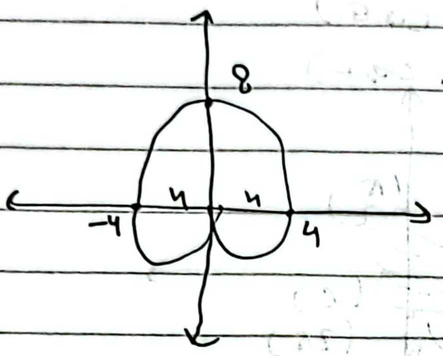
$r = a(1+\sin\theta)$ $r = a(1-\sin\theta)$





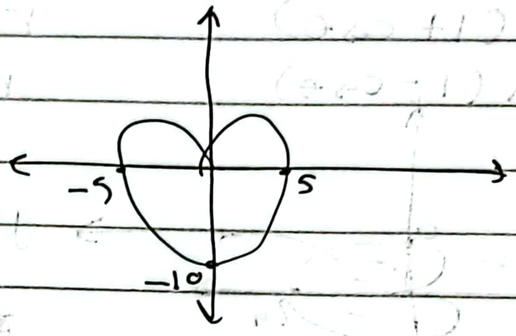
Ex : $r = 4 + 4\sin\theta$

$r = 4(1+\sin\theta)$



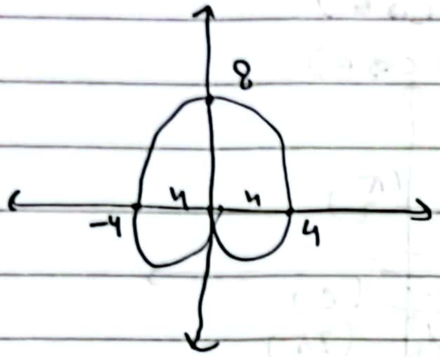
$r = 5 - 5\sin\theta$

$r = 5(1-\sin\theta)$



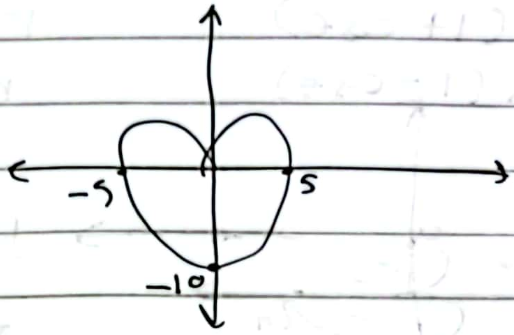
Ex : $r = 4 + 4\sin\theta$

$r = 4\,(1+\sin\theta)$

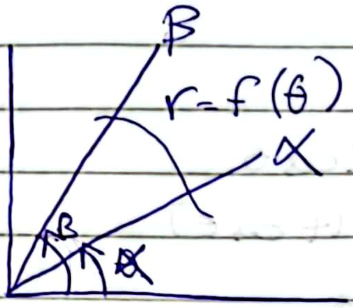


$r = 5 - 5\sin\theta$

$r = 5\,(1-\sin\theta)$

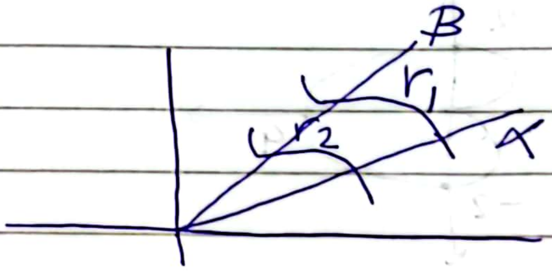


* Area in Polar Coordinates



$\boxed{1}$ $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$

$$= \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2\, d\theta$$

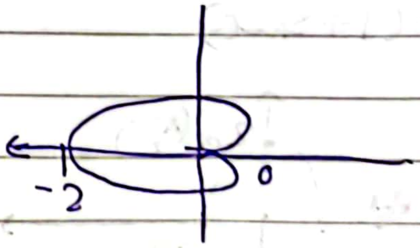


$r_1 > r_2$ بيأفقيه

$\boxed{2}$ $A = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 - (g(\theta))^2\, d\theta$

$$= \frac{1}{2}\int_{\alpha}^{\beta} r_1^2 - r_2^2\,.\,d\theta$$

Ex1 : Find the Area $r = 1 - \cos\theta$

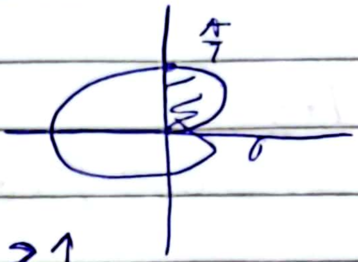


$$A = \frac{1}{2}\int_{0}^{2\pi} (1-\cos\theta)^2 . d\theta$$

or $2\left(\frac{1}{2}\int_{0}^{\pi}(1-\cos\theta)^2 . d\theta\right.$

Ex. 2

$$A = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} (1-\cos\theta)^2 . d\theta$$

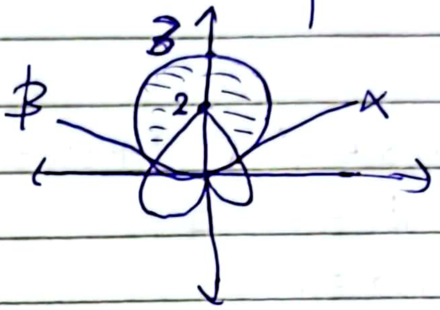


Ex (3)

$\int$ $3\sin\theta = 1+\sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6} / \frac{5\pi}{6}$



$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 - (1+\sin\theta)^2 . d\theta$$

or

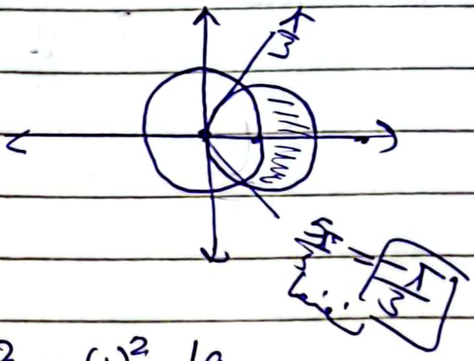
$$2 \times \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3\sin\theta)^2 - (1+\sin\theta)^2 . d\theta$$

$r = 2\cos\theta$

Ex 4 :- $1 = 2\cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$



~~$A = \frac{1}{2}\int_{0}(2)$~~ $A = 2\left[\frac{1}{2}\int_{0}^{\frac{\pi}{3}} (2\cos\theta)^2 - (1)^2 . d\theta\right.$

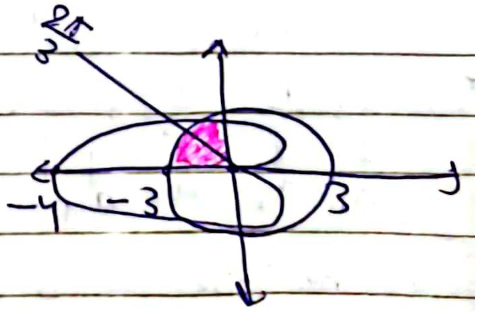
or

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos\theta)^2 - (1)^2 . d\theta$$

Ex 5: $r = 2 - 2\cos\theta$ / $r = 3$

$3 = 2 - 2\cos\theta$

$\theta = \frac{2\pi}{3}$



$A_{TOT} = A_1 + A_2$

$= \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (2 - 2\cos\theta)^2 \cdot d\theta + \frac{1}{2} \int_{\frac{2\pi}{3}}^{\pi} (3)^2 \cdot d\theta$

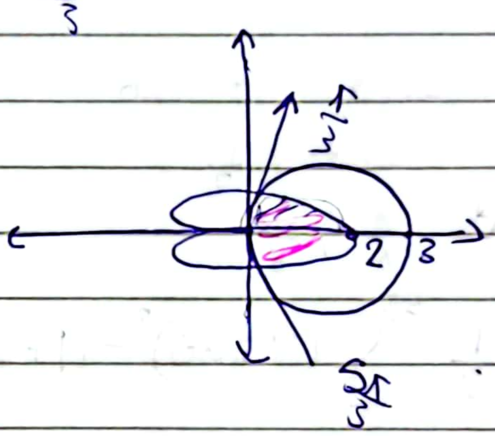
Ex 6 : $r = 3\cos\theta$ / $r = 1 + \cos\theta$

$3\cos\theta = 1 + \cos\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$ / $\frac{5\pi}{3}$



$A_{TOT} = (A_1 + A_2)(2)$

$= \frac{1}{2} \int_{0}^{\frac{\pi}{3}} (1 + \cos\theta)^2 \cdot d\theta + \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (3\cos\theta)^2 \cdot d\theta$

*Sequence :- list of numbers written in a definite order :-

$\underset{\text{first term}}{\overset{\downarrow}{a_1}}, \underset{\text{second term}}{\overset{\downarrow}{a_2}}, a_3 \dots \underset{\text{general term / nth term}}{\overset{\downarrow}{a_n}}, \dots$

Ex : ① $\underset{a_1}{2}, \underset{a_2}{4}, \underset{a_3}{6}, \underset{a_4}{8} \dots 2n$

$a_1 = n = 1 \rightarrow a_1 = 2 \times 1 = 2$

$a_2 = n = 2 \rightarrow a_2 = 2 \times 2 = 4$

② $\underset{a_1}{1}, \underset{a_2}{\frac{1}{2}}, \underset{a_3}{\frac{1}{3}}, \underset{a_4}{\frac{1}{4}}, \frac{1}{5} \dots \frac{1}{n}$

Def: An infinit seq is a functoin whose domain is the positive integers

$f(n) = a_n$, $n = 1, 2, 3 \dots$

notation: $\{a_1, a_2, a_3 \ldots\}$, $f(n) = a_n$

$\{a_n\}$ or $\{a_n\}_{n=1}^{n=\infty}$

Ex: find the first five terms -

① $a_n = \frac{n}{n+1}$

$a_1 = \frac{1}{2}$ $a_2 = \frac{2}{3}$ $a_3 = \frac{3}{4}$ $a_4 = \frac{4}{5}$ $a_5 = \frac{5}{6}$

② $a_n = \sqrt{n-3}$ $n \geq 3$

$a_3 = 0$ $a_4 = 1$ $a_5 = \sqrt{2}$ $a_6 = \sqrt{3}$ $a_7 = 2$

Ex: find the general term of the seq:

① $\underset{a_1}{\frac{1}{2}}, \underset{a_2}{\frac{1}{4}}, \underset{a_3}{\frac{1}{8}}, \underset{a_4}{\frac{1}{16}}, \ldots$

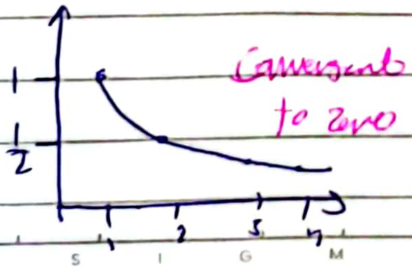
$a_n = \frac{1}{2^n}$

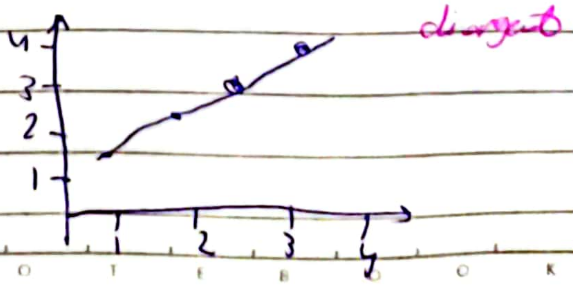
② $\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \ldots$

$a_n = \frac{(-1)^{n+1} n}{n+1}$

Graph of Sequences

$a_n = \frac{1}{n}$, $n = 1, 2, 3$ $a_n = n+1$





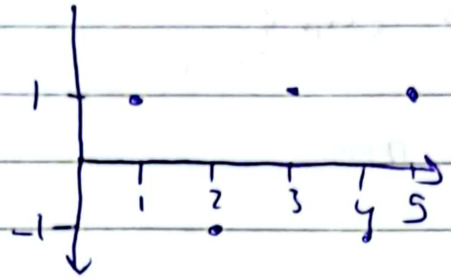
3) $a_n = (-1)^{n+1}$

a_1 $(-1)^2$ = (1,1)

(2,-1)

(3,1)

(4,-1)



divergent

Def. A seq $\{a_n\}$ is convergat to if $\lim_{n \to \infty} a_n = L$ otherwise its divergent

Theorem: If $\{a\}\{b_n\}$ $a_n \to L$ as $n \to +\infty$, Then $b_n \to M$

1) $\lim_{n \to +\infty} C = C$, $C \in R$

2) $\lim_{n \to +\infty} Ca_n = CL$

3) $\lim_{n \to +\infty} (a_n \pm b_n) = L \pm M$

4) $\lim_{n \to +\infty} (a_n b_n) = LM$

5) $\lim_{n \to +\infty} \left(\frac{a_n}{b_n}\right) = \frac{L}{M}$, $M \neq 0$

Theorem: If $\lim a_n = L$ & the function f is continuous at L then $\lim_{n \to +\infty} f(a_n) = f(L) \to f\left(\lim_{n \to +\infty} a_n\right) = f(L)$

▶ Subject : .... / /

Ex: Determine whether the seq conv & div?

① $\{n^2+2n+1\}$

$\lim_{n\to+\infty}(n^2+2n+1) \xrightarrow[\text{أكبر قوة}]{} \lim_{n\to+\infty} n^2 = +\infty$ div

2) $a_n = \frac{n^2}{n+1}$

$\lim_{n\to+\infty}\frac{n^2}{n+1} = \lim_{n\to+\infty}\frac{n^2}{n} = \lim_{n\to+\infty} n = +\infty$ div

أكبر قوة بالبسط على أكبر قوة
بالمقام.

3) $\{n e^{-n}\}$

$\lim_{n\to+\infty} n e^{-1} = +\infty(0)$ لوبيتال

$\lim_{n\to+\infty}\frac{n}{e^n} \xrightarrow{\text{لوبيتال}} \lim_{n\to+\infty} = \frac{1}{e^n} = \frac{1}{\infty} = 0$ Con to zero

④ $\{\ln(2n+1)-\ln n^2\}$

$\lim_{n\to+\infty}(2n+1)-\ln n = \infty-\infty$ لوبيتال

$\lim_{n\to\infty}\ln\left(\frac{2n+1}{n}\right) = \ln\left(\lim_{n\to+\infty}\frac{2n+1}{n}\right)$

أكبر قوة بالبسط مع معاملها على أكبر قوة
بالمقام مع معاملها.

$\ln\left(\lim_{n\to+\infty}\frac{2n}{n}\right)$

$= \ln 2$

5) $a_n = \sin n$

$\lim_{n \to +\infty} \sin n \longrightarrow$ div

6) $a_n = \left(1 + \frac{2}{n}\right)^{4n}$

$\lim_{n \to +\infty} \left(1 + \frac{2}{n}\right)^{4n}$

$= e^{(2)\cdot(4)}$

$= e^{8}$

$\lim_{x \to +\infty} \left(1 + \frac{a}{x}\right)^{bx}$

$= e^{ab}$

8) $a_n = \frac{(2n-1)!}{(2n+1)!}$ (ي، جيه)

$\lim_{n \to +\infty} \frac{(2n-1)!}{2n+1)!}$

$\lim_{n \to +\infty} \frac{\cancel{(2n-1)!}}{(2n+1)(2n)\cancel{(2n-1)!}}$

$= \lim_{n \to +\infty} = \frac{1}{\infty} = 0$

Con to zero

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 5 \cdot 4!$

$= 5 \cdot 4 \cdot 3!$

$n! = n \cdot (n-1)(n-2) \cdot 1$

9) $a_n = (3^n + 5^n)^{\frac{1}{n}}$

$\lim_{n \to \infty} (3^n + 5^n)^{\frac{1}{n}} = \infty^0$ لوبيتال

$\lim_{n \to \infty} \left(5^n \left(\left(\frac{3}{5}\right)^n + 1\right)\right)^{\frac{1}{n}}$ طريقة أخرى

$\lim_{n \to +\infty} 5 \left(\left(\frac{3}{5}\right)^n + 1\right)^{\frac{1}{n}}$

$= (5)(1)^0$

$= 5$ Con to 5

$\lim_{n \to +\infty} \left(\frac{3}{5}\right)^n = 0$

~~[illegible]~~ is

~~[illegible]~~

exponantial function

Theorem: (The squeezing Thm for Seq)

Let $\{a_n\} \{b_n\} \{C_n\}$ such that $a_n \leq b_n \leq C_n$

If $\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} C_n = L$, then $\lim_{n \to +\infty} b_n = L$

Ex: determain wheather the seq Con or div
$a_n = \frac{\sin n}{n}$

$\lim_{n \to +\infty} \frac{\sin n}{n}$ = zero by S.T Con to zero

$-1 \leq \sin n \leq 1 \quad \forall n$

$\frac{-1}{n} \leq \sin n \leq \frac{1}{n}$

$\lim_{n \to +\infty} \downarrow$ 0 بالتالي $\downarrow$ $\downarrow \lim_{n \to +\infty}$

$0 \leq 0 \leq 0$

Alternating sequences :- (divengent)

$+ / - / + / \ldots$

$- / + / - / \ldots$

ممكن تكون
$(-1)^{n+1}$
$(-1)^{n-1}$

$\{(-1)^a a_n\} = \{-a, a_2, -a_3, a_4, \ldots\}$

Ex: $a_n = (-1)^n = \{-1, 1, -1, 1, \ldots\}$ div

Ex: $(\cos n\pi)\, 2^n = a_n$

$\downarrow$

$\cos \pi (2)^1, \cos(2\pi) 2^2, (\cos 3\pi) 2^3,$

$-2, 2^2, -2^3$

Theorem: (1): If $\lim_{n \to +\infty} |a_n| = 0$ then $\lim_{n \to +\infty} a_n = 0$

Ex: Con or div

$\left\{\frac{(-1)^n}{n}\right\} \xrightarrow[\text{alternating}]{} \left\{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}\right\}$

$\lim_{n \to +\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n \to +\infty} \frac{1}{n} = 0$ Conv to 0

بتصير كلها موجبة

Subject : / /

Theorem: A seq converges to ℓ if the seq of even-numbered terms and odd-numbered terms converge to L.

$a_1, a_2, a_3, a_4, a_5, \ldots$

$$\left.\begin{matrix} a_1, a_3, a_5 \longrightarrow L \\ a_2, a_4, a_6 \longrightarrow L \\ n \to +\infty \end{matrix}\right\rangle a_1, a_2, a_3 \longrightarrow L$$

النظرية بالاتجاهين صحيحة

Ex:

$$a_n = (-1)^n \frac{n^2+1}{2n^2+3n+5}$$

$$\lim_{n \to +\infty} \left| \frac{(-1)^n n^2+1}{2n^2+3n+5} \right| = \lim_{n \to +\infty} \frac{n^2+1}{2n^2+3n+5} = \frac{1}{2} \neq 0$$

$$\left\{ \begin{matrix} \text{even} = a_{2n} = \frac{n^2+1}{2n^2+3n+5} \xrightarrow{n \to +\infty} \frac{1}{2} \\ \text{odd} = a_{2n+1} = \frac{(-1)n^2+1}{2n^2+3n+5} \xrightarrow{n \to +\infty} \frac{-1}{2} \end{matrix} \right\rangle \longrightarrow \text{div}$$

The sequence (r^n)

For what values of r is r^n convergent?

$r^n, r \in R$

$\{2^1, 2^2, 2^3, 2^4 \ldots\}$

$\{\frac{1}{5}, (\frac{1}{5})^2, (\frac{1}{5})^3 \ldots\}$

$$r^n \begin{cases} \text{con}, & -1 < r \leq 1 \\ \text{div}, & \text{otherwise} \end{cases}$$

div o con • div
-1 1

Ex:

$(\frac{1}{2})^n \longrightarrow$ con to 0

$(4)^n \longrightarrow$ div

$(-5)^n \longrightarrow$ div

$(\frac{e}{\pi})^n \longrightarrow$ con $= 0$

$r: -1 < r < 1$ con to 0

$r = 1$ con to 1

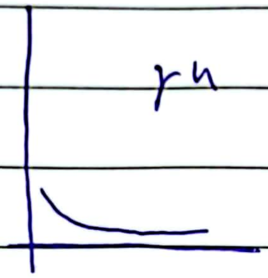
$-1 < r \leq 1$ Con ?

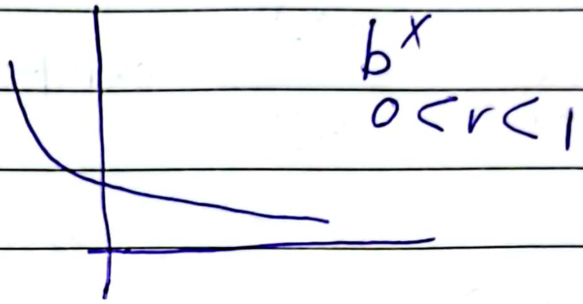
$r = 1 \longrightarrow r^n = \{1, 1, \}$ Con to 1

$r = 0 \longrightarrow \{0, 0 - \}$ Con to 0

$0 < r < 1$

$\lim_{n \to +\infty} r^n = 0$





$-1 < r < 0$

$0 < |r| < 1$

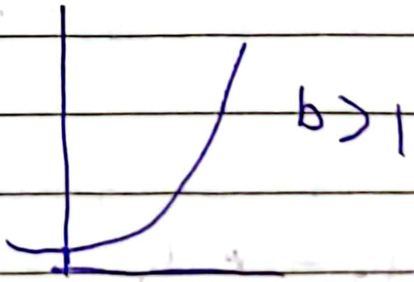
$\lim_{n \to +\infty} |a_n| = \lim_{n \to +\infty} a_n = 0$

$\lim_{r \to +\infty} |r^n| = \lim_{n \to +\infty} |r|^n = 0$

$\lim_{n \to +\infty} r^n = 0$ Con

$r > 1$

$\lim_{n \to +\infty} r^n = +\infty$ div



$r \leq -1 \longrightarrow (-1)^n = \{-1, 1, -1\}$ div

div $\longrightarrow \{-2^n, -3^n, -4^n\}$ — $(-2, 4, -8, -16)$

$+\infty$

$-\infty$

Ex: find the value of a, If the following seq is conv?

$\left\{\frac{a^{n+1}}{2^n}\right\}$ con $\quad a = ?$

$\downarrow$

$\lim_{n\to+\infty} \frac{a^{n+1}}{2^n}$ exists

$\lim_{n\to+\infty} \frac{a\,a^n}{2^n}$ exists

$a \lim_{n\to+\infty} \left(\frac{a}{2}\right)^n \longrightarrow r^n$? $\longrightarrow$ con $-1 \le r < 1$

$\frac{-2}{2} \le \frac{r}{2} \le 1$

$\boxed{-2 < a \le 2}$

$(-\infty, -2] \cup (2, \infty)$ $\longleftarrow$ div ولا

* Recursive Sequences

$a_n = 2n \longrightarrow \{2, 4, 6, 8\}$ div

$a_{10} \longrightarrow 2(10) = 20$

Ex:

$a_1 = 3 \quad , a_{n+1} = a_n + 2$

- $a_2 \longrightarrow a_{1+1} = a_1 + 2 = 3 + 2 = 5$
- $a_3 \longrightarrow a_{2+1} = a_2 + 2 = 5 + 2 = 7$
- $a_4 \longrightarrow a_{3+1} = a_3 + 2 = 7 + 2 = 9$

Ex: $a_1 = 1$, $a_{n+1} = \frac{1}{2}(a_n+3)$

1) find the first four terms

2) Assuming that the seq conv, find the limits.

1* $a_1 = 1$

$a_2 = a_{1+1} = \frac{1}{2}(a_1+3) = \frac{1}{2}(1+3) = 2$

$a_3 = \frac{1}{2}(a_2+3) = \frac{1}{2}(2+3) = \frac{5}{2}$

$a_4 = \frac{1}{2}(a_3+3) = \frac{1}{2}(\frac{5}{2}+3) = \frac{11}{4}$

$1, 2, \frac{5}{2}, \frac{11}{4}$

2* convergent → limits exists limit = L

$\lim\limits_{n\to+\infty} a_n = L$ as $n \to +\infty$

$a_n \to L$

$a_n = \frac{1}{2}(a_n+3)$

$\lim\limits_{n\to+\infty} (a_n+1) = L$

$\lim\limits_{n\to+\infty} (a_n+1) = \frac{1}{2} \lim\limits_{n\to+\infty} (a_n+3)$

$L = \frac{1}{2}(L+3)$

$2L = L+3$

$\boxed{L = 3}$

Ex: $a_1 = 1$ / $a_2 = 1$ / $a_{n+2} = a_n + a_{n+1}$

$a_1 = 1$

$a_2 = 1$ $\{1, 1, 2, 3, 5, 8, 13, 21\}$

$a_3 = a_{1+2} = a_1 + a_2 = 2$ Fibonaci seq

$a_4 = a_3 + a_2 = 1 + 2 = 3$

$a_5 = a_4 + a_3 = 2 + 3 = 5$

$a_6 = a_5 + a_4 = 5 + 3 = 8$

Ex: Determine the seq defined as follows is con or div

1) $a_1 = 1$ / $a_{n+1} = 4 - a_n$

$a_1 = 1$

$a_2 = 4 - a_1 = 4 - 1 = 3$

$a_3 = 4 - a_2 = 4 - 3 = 1$

$a_4 = 4 - a_3 = 4 - 1 = 3$

1/3/1/3 div

2) $a_1 = 2$, $a_{n+1} = 4 - a_n$

$a_1 = 2$

$a_2 = 4 - a_1 = 4 - 2 = 2$

$a_3 = 4 - a_2 = 4 - 2 = 2$

$a_4 = 4 - a_3 = 4 - 2 = 2$

2/2/2 con to 2

Ex: Consider the seq

$a_1 = \sqrt{6}$

$a_2 = \sqrt{6+\sqrt{6}}$

$a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}$

① Find a recursion formula for a_{n+1}

② Assuming that the seq converges, find the limits

* $a_1 = \sqrt{6}$ $\quad$ $a_{n+1} = ?$

$a_2 = \sqrt{6+\sqrt{6}}$

$a_2 = \sqrt{6+a_1}$

$a_{n+1} = \sqrt{6+a_n}$

* conv $\to$ limits exists

$\lim\limits_{n\to+\infty} a_n = L$

$\lim\limits_{n\to+\infty} a_{n+1} = L$

$a_{n+1} = \sqrt{6+a_n}$ $\quad$ limit

$\lim\limits_{n\to+\infty} a_{n+1} = \lim\limits_{n\to+\infty} \sqrt{6+a_n}$

$L = \sqrt{6+L}$

$L^2 = 6+L$

$L^2 - 6 - L = 0$

$\to L = -2$ X

$\to \boxed{L = 3} \to$ seq حدود ال موجب ✓

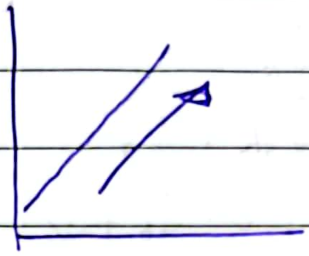
* Monotonic Sequences

1) Increasing seq

$a_1 < a_2 < a_3 < a_4$ —

Ex: 2/4/6/10 —

$2 < 4 < 6 < 8 <$ —

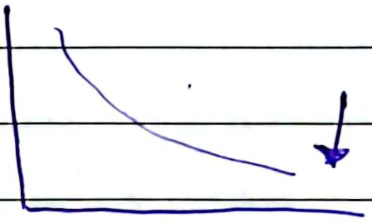


2) decreasing seq

$a_1 > a_2 > a_3 > a_4$ —

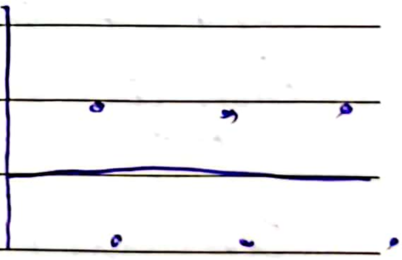
Ex = $1 / \frac{1}{2} / \frac{1}{3} / \frac{1}{4} / \frac{1}{5}$ —

$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4}$ —



3) Not inc / Not dec (Not Monotonic seq)

Ex: $(-1)^n$ $\{-1/1/-1/1 — \}$



	(−)	(÷)	
seq	Difference	Ratio (terms)	Derivative
Increasing	$a_{n+1} - a_n > 0$	$\frac{a_{n+1}}{a} > 1$	$f'(x) > 0$
decreasing	$a_{n+1} - a_n < 0$	$\frac{a_{n+1}}{a} < 1$	$f'(x) < 0$

Ex:

1) $a_n = \frac{3}{n+5}$

$a_{n+1} - a_n \rightarrow a(n+1) = \frac{3}{n+1+5}$

$a_{n+1} - a_n = \frac{3}{n+6} - \frac{3}{n+5}$

$= \frac{3(n+5) - 3(n+6)}{(n+6)(n+5)}$

$= \frac{-3}{(n+6)(n+5)}$

decreasing

2) $a_n = \frac{n}{n!}$

الأسهل مع المضرب
القسمة

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \div \frac{n^n}{n!}$

$= \frac{(n+1)^{n}(n+1)}{(n+1)(n)!} \times \frac{n!}{n^n}$

$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n > 1$

3) $a_n = \tan^{-1} x$

الأسهل معها المشتقة

$f(x) = \tan^{-1} x$

$f'(x) = \frac{1 ⊕}{1+x^2 ⊕} > 0$ increasing

4) $a_n = (-2)^{n+1}$

$4 / -8 / 16 / -32$

Not Mon

5) $a_n = \frac{n!}{6^n}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{6^{n+1}} \div \frac{n!}{6^n}$

$= \frac{(n+1)\cancel{n!}}{\cancel{6^n}\,6} \times \frac{\cancel{6^n}}{\cancel{n!}}$

$= \frac{n+1}{6}$? >1 or <1

$\frac{n+1}{6} > 1 \longrightarrow n+1 > 6$

$\longrightarrow n > 5 \longrightarrow n \geq 6$

increasing $n \geq 6$ $[6, \infty)$

6) $a_n = \frac{\ln n}{n}$

التكملة في
الصفحة التالية

$f'(x) = \frac{(x)(\frac{1}{\cancel{x}}) - (\ln x)}{x^2} = \frac{1-\ln x}{x^2}$

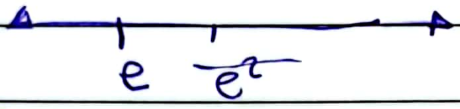
$f'(x) = \frac{1-\ln x}{x^2}$

$1-\ln x = 0$

$1 = \ln x$

$\boxed{x = e} \in D_f$

$x^2 = 0 \rightarrow x = 0 \in D_f$



$f'(e^2) = \frac{1-\ln e^2}{(e^2)^2}$

$= \frac{-1}{e^4}$ decreasing

$n \geq 3$

Limits

$$\boxed{1}\ \lim_{n\to\pm\infty}\frac{k}{n}=0$$

$$\boxed{2}\ \lim_{n\to\infty}(n)^k=\infty\quad k=1,2,3\ldots$$

$$\boxed{3}\ \lim_{n\to-\infty}(n^k)=\begin{cases}\to+\infty, & k=\text{even}\\ \to-\infty, & k=\text{odd}\end{cases}$$

$$\boxed{4}\ \lim_{n\to\pm\infty}(kn)=+\infty \text{ or } -\infty \rightarrow$$ حسب قيمة الثابت K

$$\boxed{5}\ \lim_{n\to\pm\infty}(f(n))^k=\left[\lim_{n\to\pm\infty}f(n)\right]^k$$

$$\boxed{6}\ \lim_{n\to\pm\infty}\frac{\text{Poly}}{\text{Poly}}=\frac{\text{أكبر قوة بالبسط مع معاملها}}{\text{أكبر قوة بالمقام مع معاملها}}$$

$$\boxed{7}\ \lim_{n\to\infty}k^n=\infty,\quad k>1$$

$$\boxed{8}\ \lim_{n\to-\infty}k^n=0,\quad k>1$$

$$\boxed{9}\ \lim_{n\to\infty}k^n=0,\quad -1<k\leq1$$

$$\boxed{10}\ \lim_{n\to-\infty}k^n=\infty\quad 0<k<1$$

$\boxed{11}$ $\lim_{n \to \infty} \ln(n) = +\infty$

$\boxed{12}$ $\lim_{n \to 0^+} \ln(n) = -\infty$

* The sequeeze theorem

$k(x) \leq f(x) \leq g(x)$

if $\lim_{x \to c} k(x) = \lim_{x \to c} g(x) = L$

then $\lim_{x \to c} f(x) = L$

* L'Hopital's Rule:-

$$\lim_{x \to k} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}, \frac{0}{0}, 0.\infty, \infty - \infty, 0^0, 1^\infty, \infty^0$$

$$\longrightarrow \lim_{x \to k} \frac{f'(x)}{g'(x)}$$

لو ضل الناتج بعد الاشتقاق نفس الأشي بنقدر نعمل
لوبيتال كمان مرة.

$$\lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^{bx} = e^{kb}$$

* Infinite Series :-

$a_1, a_2, a_3, \ldots$ Seq

$a_1 + a_2 + a_3 + a_4 + \ldots$ Series $\sum_{n=1}^{+\infty} a_n = \sum a_n$

Ex :

① $1+2+3+4 \ldots \sum_{n=1}^{+\infty} n$

② $1+\frac{1}{2}+\frac{1}{3}+\ldots \sum_{n=1}^{+\infty} \frac{1}{n}$

③ Definitoin :- $\sum_{n=1}^{\infty} a_n$, the seq of Partial sum $\{S_n\}_{n=1}^{+\infty}$

$S_1 = a_1$

$S_2 = a_1 + a_2$ $\Longrightarrow S_1 = a_1$

$S_3 = a_1 + a_2 + a_3$ $S_2 = S_2 + a_2$

$\vdots$ $S_3 = S_2 + a_3$

$S_n = a_1 + a_2 + \ldots a_{n-1} + a_n$ $S_n = S_{n-1} + S_n$

$\{S_n\}_{n=1}^{+\infty}$ Con $\longrightarrow \lim_{n \to +\infty} S_n = L$

$\sum_{n=1}^{+\infty} a_n$ Con $\Longrightarrow \sum_{n=1} a_n = L$

$\{S_n\}_{n=1}^{+\infty}$ div $\Longleftrightarrow \sum_{n=1}^{+\infty} a_n$ div (has no sum)

* Infinite Series :-

$a_1 , a_2 , a_3 , \ldots$ Seq

$a_1 + a_2 + a_3 + a_4 + \ldots$ Series $\sum_{n=1}^{+\infty} a_n = \sum a_n$

Ex :

① $1+2+3+4 \ldots \sum_{n=1}^{+\infty} n$

② $1+\frac{1}{2}+\frac{1}{3}+\ldots \sum_{n=1}^{+\infty} \frac{1}{n}$

Definitoin :- $\sum_{n=1}^{\infty} a_n$, the seq of Partial sum $\{S_n\}_{n=1}^{+\infty}$

$S_1 = a_1$

$S_2 = a_1 + a_2$ $\longrightarrow S_1 = a_1$

$S_3 = a_1 + a_2 + a_3$ $\quad S_2 = S_2 + a_2$

$\vdots \quad S_3 = S_2 + a_3$

$S_n = a_1 + a_2 + \ldots a_{n-1} + a_n \quad S_n = S_{n-1} + S_n$

$\{S_n\}_{n=1}^{+\infty}$ Con $\longrightarrow \lim_{n \to +\infty} S_n = L$

$\sum_{n=1}^{+\infty} a_n$ Con $\Longrightarrow \sum_{n=1}^{+\infty} a_n$ ~~S_n~~ $= L$

$\{S_n\}_{n=1}^{+\infty}$ div $\Longleftrightarrow \sum_{n=1}^{+\infty} a_n$ div (has no sum)

Ex: If the series $\sum_{n=1}^{\infty} a_n$ has the nth Partial sum

$S_n = 2 + \frac{3n}{n+1}$, find the sum of $\sum_{n=1}^{\infty} a_n$

$\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} 2 + \frac{3n}{n+1}$, $2 + 3 = 5$ Con

$\sum_{n=1}^{\infty} a_n$ Conv, $\sum_{n=1}^{+\infty} a_n = 5$

Ex: Con or div

① $\sum_{n=1}^{+\infty} (-1)^n = -1 + 1 + -1 + 1$

$S_1 = a_1 - 1$ $\{-1, 0, -1, 0 \dots\}$

$S_2 = 0$ $\lim_{n \to +\infty} S_n =$ dim.e div

$S_3 = -1$

$S_4 = 0$ $\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} n = +\infty$

Ex: $\sum_{n=1}^{+\infty} 1 = 1 + 1 + 1$

div

(has no Sum)

$S_1 = 1$

$S_2 = S_1 + 1 = 2$

$S_3 = S_2 + 1 = 3$

$\vdots$

$S_n = n$

Infinite series:-

Ex: Con or div & find sum:-

1) $\sum_{n=1}^{+\infty} \frac{1}{n} - \frac{1}{n+1}$

$S_1 = 1 - \frac{1}{2}$

$S_2 = S_1 + a_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$ telescoping sum

$S_3 = S_2 + a_3 = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$

$S_4 = S_3 + a_4 = 1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{3} = 1 - \frac{1}{3}$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n \to +\infty} \left(1 - \cancel{\frac{1}{n+1}}^{\text{zero}}\right) = 1 - 0 = 1 \text{ Con}$$

2) $\sum_{n=1}^{+\infty} \ln\left(\frac{n+1}{n}\right)$

$\sum_{n=1}^{+\infty} \ln(n+1) - \ln(n)$

$S_1 = \ln 2 - \ln 1 = \ln 2$

$S_2 = S_1 + a_2 = \ln 2 + \ln 3 - \ln 2 = \ln 3$

$S_3 = S_2 + a_3 = \ln 3 + \ln(4) - \ln 3 = \ln 4$

$S_n = \ln(n+1)$

$\lim_{n \to +\infty} (\ln(n+1)) = +\infty$ div

has no sum

③ $\sum_{n=1}^{+\infty} \frac{1}{9n^2+3n-2}$

$\sum \frac{1}{(3n-1)(3n-2)} = \frac{A}{3n-1} + \frac{B}{3n+2}$

$= \frac{A(3n+2)+B(3n-1)}{(3n-1)(3n+2)}$

$1 = A(3n+2)+B(3n-1)$

$1 = 0 + B(-2-1)$ $\quad n = \frac{-2}{3}$

$1 = -3B$

$\boxed{B = -\frac{1}{3}}$

$1 = A3$ $\quad n = \frac{1}{3}$

$\boxed{A = \frac{1}{3}}$

$\sum_{n=1}^{\infty} \frac{1}{3(3n-1)} - \frac{1}{3(3n+2)}$

$S_1 = \frac{1}{6} - \frac{1}{15}$

$S_2 = S_1 + a_2 = \frac{1}{6} - \frac{1}{15} + \frac{1}{15} - \frac{1}{24} = \frac{1}{6} - \frac{1}{24}$

$S_n = \frac{1}{6} - \frac{1}{3(3n+2)}$

$\lim_{n\to+\infty}\left(\frac{1}{6} - \frac{1}{3(3n+2)}\right) = \frac{1}{6}$ Con

Geometric Series:-

Ex: $5^1 + 5^2 + 5^3 + 5^4 + 5^5 + \dots$

$\frac{5^2}{5} = 5$, $\frac{5^3}{5^2} = 5$, $\dots \rightarrow$ ratio

$\sum_{n=m}^{+\infty}$ r: ration

Ex: Which of this is Geo seies:-

1) $\sum_{n=5}^{\infty} 4^n = 4^5 + 4^6 + 4^7 + \dots$ G.S

2) $\sum_{k=1}^{\infty} \frac{1}{3^{k+1}} = \frac{1}{3^2} + \frac{1}{3^3} \dots$ G.S

3) $\sum_{k=1}^{\infty} k^3 = 1^3, 2^3 \dots$ Not G.S

4) $\sum_{k=1}^{+\infty} (2k)^k = 2^1 + 4^2 + 6^3$ Not G.S

Theorem:-

$$\sum_{n=m}^{+\infty} r^n = \begin{cases} \text{con if } |r| < 1 \ (-1 < r < 1) \\ \text{div if } |r| \geq 1 \ (r \geq 1 \text{ or } r \leq -1) \end{cases}$$

if $\sum_{n=m}^{+\infty} r^n$ con the $\sum_{n=m}^{\infty} r^m = \frac{r^m}{1-r}$ first term ratio

Ex: determine whether the series Con or div and find sum.

① $\sum_{n=0}^{+\infty} \left(\frac{-1}{5}\right)^n$ Con $\longrightarrow$ $-1 < \frac{4}{5} < 1$

$$\sum_{n=0}^{+\infty} \left(\frac{-1}{5}\right)^n = \frac{\left(\frac{-1}{5}\right)^0}{1--\frac{1}{5}} = \frac{1}{1+\frac{1}{5}} = \frac{5}{6}$$

2) $\sum_{k=1}^{+\infty} \left(\frac{\pi}{e}\right)^k \longrightarrow$ div

③ $\sum_{n=1}^{\infty} 5^n \, 7^{-n+2} =$

$$= \sum_{n=1}^{\infty} 5^n \, 7^{-n} \, 7^2$$

$7^2 \sum_{n=1}^{\infty} \left(\frac{5}{7}\right)^n \longrightarrow$ Con

$$7^2 \left(\frac{\frac{5}{7}}{1-\frac{5}{7}}\right) = \frac{245}{2}$$

4) Ex: if $\sum_{n=2}^{+\infty} 2r^n = 1$ find n

$\sum_{n=2}^{+\infty} 2r^n$ Con

Theorem
$\sum a_n$ Con
then
$\sum C a_n = C \sum a_n$

$2\sum_{n=2}^{+\infty} r^n = 1$

$\sum_{n=2}^{+\infty} r^n = \frac{1}{2}$

$\frac{r^2}{1-r} = \frac{1}{2}$

$2r^2 = 1-r$

$2r^2 + r - 1 = 0$

$\boxed{r=\frac{1}{2}}$ $\cancel{\boxed{r=-1}}$

$\leftarrow \boxed{|r|<1}$

* The divergent test

* $\sum a_n$, if $\lim_{n\to+\infty} a_n \neq 0$ then $\sum a_n$ divi

* if $\lim_{n\to+\infty} a_n = 0$ (Test fail)

Ex: Con or div

1) $\sum_{n=1}^{\infty} \frac{2n-1}{3n+4}$

$\lim_{n \to +\infty} \frac{2n-1}{3n+4} = \frac{2}{3} \neq 0$

2) $\sum_{n=1}^{+\infty} \left(1 - \frac{3}{n}\right)^{n}$

$\lim_{n \to +\infty} \left(1 - \frac{3}{n}\right)^{n} = e^{-3}$ div

3) $\sum_{n=1}^{+\infty} \cos n\pi$

$\lim_{n \to \infty} \cos n\pi$ = div

4) $\sum_{n=1}^{\infty} \ln n$

$\lim_{n \to +\infty} \ln n = +\infty$ div

5) $\sum_{n=1}^{\infty} \left(\frac{1}{e^{n}}\right)$

$\lim_{n \to \infty} \left(\frac{1}{e^{\infty}}\right) = 0$ (Test fail)

$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ Geometric series Con $-1 < \frac{1}{e} < 1$

8) $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n \to \infty} \frac{1}{n} = 0$ test fail

$$\begin{tikzcd} & \text{Cov} \\ \lim_{n \to +\infty} a_n = 0 \arrow[ur] \arrow[dr] & \\ & \text{div} \end{tikzcd}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ div (p-series)

* The Integral test

$\sum_{n=m}^{+\infty} a_n$, $a_n = f(n)$ $[m, \infty)$

1) f Positive
2) f Continuous
3) f decreasing

$\Bigg\}$ $[m, \infty)$

If $\int_{m}^{+\infty} f(x).dx$ Cov $\rightarrow$ $\sum_{m}^{+\infty} a_n$ Cov

$\int_{m}^{+\infty} f(x).dx$ div $\longrightarrow$ $\sum_{m}^{+\infty} a_n$ div

Ex: Con or div

$$\sum_{n=3}^{+\infty} \frac{\ln n}{n}$$

$$\lim_{n \to +\infty} \frac{\ln n}{n} = 0$$

1) $f(x) = \frac{\ln x}{x}$

2) $f(x) = \frac{\ln x}{x}$ ⊕ ⊕ $x > 3$

2) $f(x) = \frac{\ln x}{x}$ $x = 0$ $[3, \infty)$ متصل لا X بس اطراف المجال

3) $f(x) = \frac{\ln x}{x}$

$$f'(x) = \frac{1 - \ln x}{x^2}$$

$1 - \ln x = 0$ ////|- - - - - - - - $[f^-]$ (e, 3)

$$\boxed{x = e}$$

تحقق الشروط الثلاث

$$\int_3^{+\infty} \frac{\ln x}{x} \cdot dx = \lim_{t \to +\infty} \int_3^{t} \frac{\ln x}{x} \cdot dx$$

بالتعويض

$$= \lim_{t \to +\infty} \frac{(\ln x)^2}{2} \Big]_3^{t}$$

$$= \lim_{t \to +\infty} \frac{(\ln t)^2}{2} - \frac{(\ln 3)^2}{2} = +\infty \quad \text{div}$$

2) $\sum_{n=1}^{\infty} n e^{-n^2}$, $f(x) = x e^{-x^2}$

1) $x e^{-x^2} \to \oplus \; \oplus > 0 \; [1, \infty)$

2) $f(x) = \frac{x}{e^{x^2}}$ Con $[1, \infty)$

3) $f'(x) = e^{-x^2} (-2x^2 + 1)$

$f'(x) = 0$

$e^{-x^2} \neq 0 \quad -2x^2 + 1 = 0$

$$\boxed{x = \pm \frac{1}{\sqrt{2}}}$$

$\int_1^{\infty} x e^{-x^2} \, dx$

$= \lim_{t \to +\infty} \int_1^t x e^{-x^2} \, dx$ [illegible]

$= \lim_{t \to +\infty} \frac{-1}{2} e^{-x^2} \Big]_1^t$

$\lim_{t \to +\infty} \underbrace{\cancel{\frac{-1}{2} e^{-t^2}}}_{\text{zero}} + \frac{1}{2} e^{-1} = \frac{1}{2} e^{-1}$ Con

* P-series

1) $\sum_{n=1}^{\infty} \frac{1}{n}$ Con or div

1) $f(x) = \frac{1}{x} > 0 \quad [1, \infty)$

2) $f(x) = \frac{1}{x} \quad x = 0 \notin [1, \infty)$

3) $f'(x) = \frac{-1}{x^2} < 0$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to+\infty} \ln|x|\Big]_1^t$$

$$= \lim_{t\to+\infty} \ln t - \ln t = +\infty$$

div

* $\sum_{n=1}^{\infty} \frac{1}{n}$ div [harmonic series]

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{con}, & p > 1 \\ \text{div}, & p \leq 1 \end{cases}$$

p-series

Ex: Con or div

1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\downarrow$

con - Pseries

$2 > 1$

2) $\sum_{n=1}^{\infty} \frac{1}{n^\pi}$

$\downarrow$

Con p-series

$\pi > 1$

3) $\sum_{n=3}^{\infty} \frac{1}{n^{\frac{2}{5}}}$

$\downarrow$

div p-series

$\frac{2}{5} < 1$

4) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n+3}} = \sum_{n=5}^{+\infty} \frac{1}{\sqrt{n}}$ div P-series

ازالة ال 3 زدناها

لل 2

لتغير الحد

The limit Comparison Test.

$\sum a_n$, $\sum b_n$ are series with Positive terms & $C = \lim_{n \to +\infty} \frac{a_n}{b_n}$
if $C > 0$ & $C \neq 0$, Then Both series are Con
or div

Ex: Con or div

1) $\sum_{1}^{\infty} \frac{3n^3 - 2n^2 + 4}{n^7 - n^3 + 2}$

$\sum b_n = \sum_{1}^{\infty} \frac{3n^3}{n^7} = \sum_{1}^{\infty} \frac{3}{n^4}$ $\curvearrowright$ (P-series $\to$ Con)

$C = \lim_{n \to +\infty} \frac{a_n}{b_n} = \lim_{n \to +\infty} \frac{3n^3 - 2n^2 + 4}{n^7 - n^3 + 2} \times \frac{n^4}{3}$ نحول القسمة ضرب

$= \lim_{n \to +\infty} \frac{3n^7 - 2n^6 + 4n^4}{3n^7 - 3n^3 + 6}$

$= \lim_{n \to +\infty} \frac{3n^7}{3n^7} = \boxed{1} > 0$

$\sum a_n$, $\sum b_n$ are Con by L.C.T $\sum \frac{3n^3 - 2n^2 + 4}{n^7 - n^3 + 2}$ Con

② $\sum_{1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$

$\sum b_n = \sum \frac{2n^2}{n^{\frac{5}{2}}} = \sum \frac{2}{n^{\frac{1}{2}}}$ (P-series div)

$C = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \times \frac{\sqrt{n}}{2}$

$= \frac{1}{2} \lim_{n\to 0} \frac{\sqrt{n}\, n^2(2+\frac{3}{n})}{\sqrt{n^5(\frac{5}{n^5}+1)}}$ 5

$= \frac{1}{2} \lim_{n\to+\infty} \frac{n^{\frac{5}{2}}(2+\frac{3}{n})}{n^{\frac{5}{2}}\sqrt{\frac{5}{n^5}+1}}$

$= \frac{1}{2} \lim_{n\to+\infty} \frac{2+\cancelto{zero}{\frac{3}{n}}}{\sqrt{\cancelto{zero}{\frac{5}{n^5}}+1}} = \frac{1}{2} \times \frac{2}{1} = \boxed{1}$

$1 > 0 \neq \infty$

$\sum \frac{2n^2+3n}{\sqrt{5+n^5}}$ div L.C.T $\sum a_n$, $\sum b_n \to$ div

③ $\sum_{n=1}^{\infty} \frac{3+9^n}{5+10^n}$

$\sum b_n = \left(\frac{9}{10}\right)^n \longrightarrow$ G.S Con

$C = \lim_{n\to\infty} \left(\frac{3+9^n}{5+10^n}\right) \times \left(\frac{10}{9}\right)^n$

$$\lim_{n\to+\infty} \frac{9^n\left(\frac{3}{9^n}+1\right)10^n}{10^n\left(\frac{5}{10^n}+1\right)9^n}$$

$$= \lim_{n\to+\infty} \frac{\overset{Zero}{\cancel{\frac{3}{9^n}}}+1}{\underset{Zero}{\cancel{\frac{5}{10^n}}}+1} = 1 > 0 \neq \infty$$

$\boxed{Con}$

آخر خيار

- **The Comparison Test**

suppose that $\sum a_n$ and $\sum b_n$ are series with Positive terms, Such that

$\sum a_n \leq \sum b_n$

- If $\sum b_n$ is convergent then $\sum a_n$ is also Convergent
- If $\sum a_n$ is divergent then $\sum b_n$ is also div

Ex: $\sum_{n=1}^{+\infty} \frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$

$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ ⟶ P-Series Con

$\sum \frac{5}{2n^2+4n+3} < \underset{Con}{\sum \frac{5}{2n^2}}$

↳ Con by C.T

Ex: $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

$-1 \leq \sin n \leq 1$

$\underbrace{0 \leq \sin^2 n \leq 1}$

$\sin^2 n \leq 1$

$\frac{\sin^2 n}{n} \leq \frac{1}{n^2} \leftarrow$ P-series Con

$\sum \frac{\sin^2 n}{n} < \sum \frac{1}{n^2}$

$\hookrightarrow$ Con too

Ex: $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^3}$

$-\frac{\pi}{2} < \tan^{-1} n < \frac{\pi}{2}$

$\frac{\tan^{-1} n}{n^3} < \frac{\pi}{2n^3} \rightarrow$ P-series Con

$\sum \frac{\tan^{-1} n}{n^3}$ Con

Ex: $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ $\qquad$ $\sum_{n=1}^{\infty} \frac{1}{n}$ P-series div

$\ln n < n$ $\qquad$ $\sum \frac{1}{n} < \sum \frac{1}{\ln n}$

$\frac{1}{\ln n} > \frac{1}{n}$ $\qquad$ $\downarrow$ div $\qquad$ $\downarrow$ div by C T

Ex: $\sum_{n=1}^{\infty} \frac{5^n+1}{2^n-1}$

$2^n - 1 < 2^n$

$\frac{1}{2^n-1} > \frac{1}{2^n} \times 5^n+1$

$\frac{5^n+1}{2^n-1} > \frac{5^n+1}{2^n}$

$\frac{5^n+1}{2^n} > \frac{5^n}{2^n}$ ↲

$\left(\frac{5}{2}\right)^n \to$ div

So $\frac{5^n+1}{2^n} \to$ div

So $\frac{5^n+1}{2^n-1} \to$ div

* Alternating Series Test

$\sum_{n=1}^{\infty} (-1)^n a_n$, $a_n > 0$

Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

شرط :-

$$\left.\begin{array}{l} ① \lim\limits_{n \to +\infty} a_n = 0 \\ ② \ a_n \text{ decreasing} \end{array}\right\} \Longrightarrow \sum (-1)^n a_n \text{ is Convergent}$$

Ex: Con or div

① $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ Alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

$$① \lim_{n \to +\infty} \frac{1}{n} = 0$$

$$② f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2} < 0 \text{ decreasing}$$

$\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ Con by A.S.T.

$$② \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\ln(n+4)}$$

$$① \lim_{n \to +\infty} \frac{1}{\ln(n+4)} = \frac{1}{\ln\infty} = \frac{1}{\infty} = 0$$

$$② f(x) = \frac{1}{\ln(x+4)} \rightarrow f'(x) = \frac{(-)\left(\frac{1}{x+4}\right)}{\ln(x+4)^2} = \frac{\ominus}{\oplus} < 0$$

$\sum \frac{(-1)^{n-1}}{n(n+4)}$ Con by A.S.T

3) $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n} \longrightarrow \lim_{n\to+\infty} \cancel{|(-1)^n|} \frac{1}{e^n}$

① $\lim_{n\to\infty} \frac{1}{e^n} = \frac{1}{e^{+\infty}} , \frac{1}{\infty} = 0$ ✓

2) $f(x) = \frac{1}{e^x} = e^{-x} = -e^{-x} < 0$ Con

$\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{2n+1}$

1) $\lim_{n\to+\infty} \frac{n+1}{2n+1} = \lim_{n\to+\infty} \frac{n}{2n} = \frac{1}{2}$

D.T ⟵ A.S.T إذا فشل

$$\lim_{n\to+\infty} (-1)^n \frac{n+1}{2n+1} \begin{cases} \xrightarrow{\text{even}} \lim_{n\to+\infty} \frac{n+1}{2n+1} = \frac{1}{2} \\ \xrightarrow{\text{odd}} \lim_{n\to+\infty} \frac{-(n+1)}{2n+1} = \frac{-1}{2} \end{cases} \boxed{\nexists}$$

div

* Ratio Test

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = C$$

① $C<1$ $\sum a_n$ Con (Absolutely Con)

② $C>1$ or $C=\infty$ $\sum a_n$ div

③ $C=1$ Test fail (no conclusion)

Ex: Con or div

① $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{3^{n+1}}{(n+1)!} * \frac{n!}{3^n}$$

$$= \lim_{n\to+\infty} \frac{3^n\,3}{(n+1)(n)!} \cdot \frac{\cancel{n!}}{\cancel{3^n}}$$

$$= \lim_{n\to+\infty} \frac{3}{(n+1)} = 0 \quad \text{Con}$$

* Root test

$$\lim_{n\to+\infty} |a_n|^{\frac{1}{n}} = C \qquad \sqrt[n]{|a_n|}$$

① $C<1$ $\sum a_n$ convergent (Abs Con)

② $C>1$ or $C=\infty$ $\sum a_n$ (div)

③ $C=1$ Test fail (no conclusion)

Ex: Con or div

① $\sum_{n=2}^{\infty} \left(\frac{2n+1}{4n-5}\right)^{n}$

$\lim_{n \to +\infty} \left(\left|\left(\frac{2n+1}{4n-5}\right)^{n}\right|\right)^{\frac{1}{n}}$ $\oplus$ $\oplus$

$\lim_{n \to +\infty} \left(\left(\frac{2n+1}{4n+9}\right)^{n}\right)^{\frac{1}{n}}$

$\lim_{n \to +\infty} \frac{2n}{4n} = \frac{1}{2}$ Con

② $\sum_{n=1}^{\infty} \left(\frac{n}{100}\right)^{n}$

$\lim_{n \to +\infty} \left(\left(\frac{n}{100}\right)^{n}\right)^{\frac{1}{n}} = +\infty$ div by R.T

③ $\sum_{n=2}^{+\infty} \left(\frac{-1}{\ln n}\right)^{n}$

$\sum_{n=2}^{+\infty} \left(\left(\frac{-1}{\ln n}\right)^{n}\right)$ *

$\sum_{n=2}^{+\infty} (-1)^{n} \left(\frac{1}{\ln n}\right)^{n} \longrightarrow \lim_{n \to +\infty} \left(\left(\frac{1}{\ln n}\right)^{n}\right)^{\frac{1}{n}}$

$\lim_{n \to +\infty} \frac{1}{\ln n} = \frac{1}{\infty}$, so $0 < 1$

Con by R.T

Algebraic Properties of infinite Series

* if $\sum a_n$ & $\sum b_n$ are convergent, then $\sum a_n + \sum b_n$ & $\sum a_n - b_n$ are convergent

- $\sum a_n + \sum b_n = \sum a_n + b_n$
- $\sum a_n - b_n = \sum a_n - \sum b_n$

* If c is a nonzero Constant, then

- If $\sum a_n$ convergent $\longleftrightarrow$ $\sum c a_n$ convergent, $\sum c a_n = c \sum a_n$
- If $\sum a_n$ divergent $\longleftrightarrow$ $\sum c a_n$ div

Ex: $\sum_{n=1}^{\infty} \frac{3}{4^n} - \frac{2}{6^{n-1}}$

$$= \sum_{n=1}^{\infty} \underset{\underset{G.S}{\downarrow}}{3\left(\frac{1}{4}\right)^n} - \underset{\underset{G.S}{\downarrow}}{2\left(\frac{1}{5}\right)^{n-1}}$$

$$= 3 \underset{Con}{\sum \left(\frac{1}{4}\right)^n} - 2 \underset{Con}{\sum \left(\frac{1}{5}\right)^{n-1}}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \longrightarrow \frac{\left(\frac{1}{4}\right)}{1 - \frac{1}{4}} = \frac{1}{3}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n-1} = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4} \longrightarrow 3\left(\frac{1}{3}\right) - 2\left(\frac{5}{4}\right)$$

$$= 1 - \frac{5}{2}$$

$$= \boxed{\frac{-3}{2}}$$

2) $\sum_{n=1}^{\infty} \frac{1}{e^n} + \frac{1}{n(n+1)}$

$\sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$\downarrow$ $\qquad\qquad$ $\downarrow$

Con $\qquad$ by T.S. Con

$\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{1-\frac{1}{e}} = \frac{1}{e-1}$

$\sum_{n=1}^{\infty} \frac{1}{e^n} + \frac{1}{n(n+1)}$

$= \frac{1}{e-1} - 1$

$= \frac{e}{e-1}$

T.S

$\sum \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$A(n+1) + B(n) = 1$

$n = 0 \quad A = 1$

$n = -1 \quad B = -1$

$\sum \frac{1}{n(n+1)} = \sum \frac{1}{n} - \frac{1}{n+1}$

$S_1 = 1 - \frac{1}{2}$

$S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3}$

$S_n = 1 - \frac{1}{n+1}$ $\quad$ zero

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \cancel{\frac{1}{n+1}}$

$= \boxed{1}$

$\boxed{2}$ if $\sum a_n$ Con & $\sum b_n$ div, then $\sum a_n \mp \sum b_n$ div

Ex: $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \frac{1}{n}$

$\downarrow$ $\qquad\qquad$ $\downarrow$

Con $\qquad$ div (harmonic series)

$\downarrow$

div

③ if $\sum a_n$ & $\sum b_n$ divs Then $\sum a_n \pm b_n$ may either Con or div

Ex: $\sum_{n=1}^{\infty} -1$ div

$\sum_{n=1}^{\infty} 1$ div

$$\left.\right\} \sum_{n=1}^{\infty} -1+1 = \sum_{n=1}^{\infty} 0 = 0 \text{ Con}$$

Ex: $\sum_{n=1}^{\infty} \frac{1}{n}$ div

$\sum_{n=1}^{\infty} \frac{5}{n}$ div

$$\left.\right\} \sum_{n=1}^{\infty} \frac{1}{n} + \frac{5}{n} = \sum_{n=1}^{\infty} \frac{6}{n} \text{ div}$$

*Strategy of test series

① $\sum_{n=2}^{+\infty} \frac{1}{n\sqrt{\ln n}}$

Integral Test

② $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

Root test

③ $\sum_{n=1}^{\infty} \frac{1}{n^3}$ P-Series

④ $\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}$ G.S

⑤ $\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$ Ration Test

⑥ $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

Limit Comparision test

⑦ $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

Limit Comparison Test

⑧ $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

Divergent test

⑨ $\sum_{n=1}^{\infty} \left(\sqrt[n]{n} - 1\right)^n$

Root Test

⑩ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$

A.S.T

⑪ $\sum_{n=1}^{\infty} e^{\frac{1}{n}} - e^{\frac{1}{n+1}}$

$S = e - e^{\frac{1}{2}}$

$S_2 = e - e^{\frac{1}{2}} + e^{\frac{1}{2}} - e^{\frac{1}{3}}$

T.S

⑫ $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ Root test

⑬ $\sum_{n=1}^{\infty} \frac{(-5)^n}{n!}$ Ration Test

⑭ $\sum_{n=1}^{\infty} \frac{n+1}{2n+1}$ divergent Series

⑮ $\sum_{n=1}^{\infty} \frac{2^{n-1} \, 3^{n+1}}{n^n}$ Root Test

⑯ $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ G.S

⑰ $\sum_{n=1}^{\infty} \frac{n^2 \, 2^{n-1}}{(-5)^n}$

- جذري
- Root Test
- Ration Test
- A.S.T

Absolute Convergence, Conditional Convergence

$\sum a_n = a_1 + a_2 + a_3 + \cdots$

? conv & div

$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$ Con

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = |-1| + \left|\frac{1}{2}\right| + \left|\frac{-1}{3}\right| + \left|\frac{1}{4}\right| + \cdots$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ div

$\boxed{1}$ If $\sum |a_n|$ is conv $\Rightarrow$ $\sum a_n$ Absolutely conv

$\boxed{2}$ If $\sum |a_n|$ is div & $\sum a_n$ conv $\Rightarrow$ $\sum a_n$ Conditionaly conv

$\boxed{3}$ if $\sum |a_n|$ is div & $\sum a_n$ div

$\sum a_n$ div

1) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n$

$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n =$ ~~Con G.S~~ $\sum \left(\frac{1}{2}\right)^n$ Con G.S

than $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n$ ~~Con~~ Abs. Con $\rightarrow$ Con

② $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$

$\sum \left|\frac{(-1) n^3}{3^n}\right| = \sum_{n=1}^{\infty} \frac{n^3}{3^n}$? Con or div

$\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$

$= \lim_{n\to+\infty} \frac{1}{3} \cdot \frac{(n+1)^3}{n^3}$

$= \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3$

$= \frac{1}{3}$ Con

So $\sum \frac{(-1)^n (n^3)}{3^n} \longrightarrow$ Abs. Con

③ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ div P-Series

$\sum \frac{(-1)^n}{n}$ A.S.T

1. $\lim_{n\to+\infty} \frac{1}{n} = 0$

2. $\frac{1}{n}$ decreasing

$\sum \frac{(-1)^n}{n}$ Con

$\sum \left|\frac{(-1)^n}{n}\right|$ div

$\sum \frac{(-1)^n}{n}$ con. Convergent

④ $\sum \frac{(-1)^n(2n-1)!}{3^n}$

$\sum \left|\frac{(-1)^n\,(2n-1)!}{3^n}\right| = \sum \frac{(2n-1)!}{3^n}$ Con or div ?

$\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{(2(n+1)-1)!}{3^{n+1}} \times \frac{3^n}{(2n-1)!}$

$= \lim_{n\to+\infty} \frac{(2n+1)!\,3^n}{3^n\,3(2n-1)!}$

$= \lim_{n\to+\infty} \frac{1}{3}(2n+1)(2n) = +\infty$ div by Ratio Test

$\sum \frac{(2n-1)!}{3^n}$ div by Ratio Test

⑤ $a_1 = 2$, $a_{n+1} = \frac{5n+1}{4n+3}\,a_n$

$\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{5(n)+1}{4n+5}\,a_n \times \frac{1}{a_n}$

$= \lim_{n\to+\infty} \frac{5n+1}{4n+5}$

$\frac{5}{4}$ by Ratio Test div

$\sum a_n$	$\sum \|a_n\|$	
Con	Con	abs Con
Con	div	Con Con
div	div	div
div	Con	X

$\sum a_n$	$\sum \lvert a_n \rvert$	
Con	Con	abs Con
Con	div	Con Con
div	div	div
div	Con	X

*Power series

$$* \sum_{n=0}^{\infty} C_n X^n = C_0 + C_1 X + C_2 X^2 + C_3 X^3 \cdots$$

X: Variable

C_n = Constant (coffficients)

$* \sum_{n=0}^{\infty} C_n (X-a)^n$ Power series centered at a

$$= C_0 + C_1 (X-a) + C_2 (X-a)^2 + C_3 (X-a)^3 \cdots$$

Ex: 1) $\sum_{n=0}^{\infty} \frac{x^n}{5^n}$ centered at 0.

$$1 + \frac{x}{5} + \frac{x^2}{5^2} + \frac{x^3}{5^3}$$

$\downarrow \quad \downarrow \quad \downarrow$

$C_1 = 1 \quad C_2 = \frac{1}{5} \quad C_3 = \frac{1}{25}$

② $\sum_{n=1}^{\infty} nx^n$ centered at 0

$x+2x^2+3x^3+4x^4+\dots$

$C_1=1$ $C_2=2$ $C_3=3$

③ $\sum_{n=1} \frac{(x-3)^n}{n+1}$ Centered at 3

$\frac{x-3}{2}+\frac{(x-3)^2}{3}+\frac{(x-3)^3}{4}+\dots$

④ $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt{n}}$ Centered at $\frac{1}{2}$

$2x-1+\frac{(2x-1)^2}{\sqrt{2}}+\frac{(2x-1)^3}{\sqrt{3}}+\dots$

$f(x)=\sum_{n=1}^{\infty} C_n x^n$, Domain: The set of all x for which the series converges.

Ex: $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$

$x=1 \longrightarrow \sum \frac{1}{2^n}$ con G.S $-1<\frac{1}{2}<1$

$x=6 \longrightarrow \sum \frac{6^n}{2^n} = \sum 3^n$ div G.S

$\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$ $-1<\frac{x}{2}<1$ $-2<x<2$ Domain

Conv.

Thm: $\sum_{n=0}^{\infty} C_n (x-a)^n$

1) The series converges only when $x = a$
2) The series Converges for all x
3) There is a positive number R such then series Converges if $|x-a| < R$ & div $|x-a| > R$

	Radius of Convergence	interval of Convergence
1)	$R = 0$	$\{a\}$
2)	$R = +\infty$	$(-\infty, \infty)$
3)	R	$[a-R, a+R]$
		[illegible] [illegible]

Ratio test / Root test

Ex: find the radius of Convergence & the interval of Convergence

1) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)}$

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\,x^{n+1}}{3^{n+1}\,(n+2)}\cdot\frac{3^n\,(n+1)}{(-1)^n\,n\,(x^n)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x(n+1)}{3(n+2)}\right|$$

$$= \lim_{n\to\infty}\frac{1}{3}\,|x|\,\frac{n+1}{n+2}$$

$$= \frac{1}{3}x < 1$$

$$x < 3$$

Radius of Convergence 3

يجب التأكد منهم $\leftarrow$ $-3 < x < 3$

$-3 < x \leq 3$

$\boxed{x=-3}$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n}(3)^n}{3^n(n+1)}$

$\displaystyle\sum_{n=0}^{\infty}\frac{1}{n+1}$ $\rightarrow$ div / P. series $\rightarrow$ $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$

$\boxed{x=3}$ $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ Con by A.S.T

② $\sum_{n=1}^{\infty} \frac{(X-2)^n}{n^2+1}$

$\lim_{n \to +\infty} \left| \frac{(X-2)^{n+1}}{(n+1)^2+1} \times \frac{(n^2+1)}{(X-2)^n} \right|$

$\lim_{n \to +\infty} \left| \frac{(X-2)(n^2+1)}{(n+1)^2+1} \right|$

$\lim_{n \to +\infty} |X-2| \frac{n^2+1}{n^2+2n+2}$

$= |X-2|$

$|X-2| < 1$

$X < 3$ $\boxed{X=3}$ radius of Convergence

$-1 < X-2 < 1$

$1 < X < 3$

التحقق منهم $\boxed{X=1} \rightarrow \frac{(-1)^n}{n^2+1}$ Con by A.S.T

$\boxed{X=3} \rightarrow \frac{(1)^n}{n^2+1}$ Con by L.C.T

③ $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

$\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \left|\frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n}\right|$

$= \lim_{n\to+\infty} \frac{|x|}{(n+1)}$

$=$ Zero   ذلك يعني أنه $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

$\downarrow$

Converges for all Values of X

$R = +\infty$

④ $\sum_{n=1}^{\infty} n^n (x-3)^n$

Root test $\lim_{n\to+\infty} \left|n^n (x-3)^n\right|^{\frac{1}{n}}$

Ratio صعب

$\lim_{n\to+\infty} |n(x-3)| = \lim_{n\to+\infty} n|x-3| = +\infty$

$\downarrow$

div

لو عوضنا $X=3$

$\lim_{n\to+\infty} 0 = 0$ $\boxed{\text{Con}}$

$\downarrow$

بس قيمة وحدة

Rad of Con $= 0$

Interval Of Con $\{3\}$

Taylor & Maclaurin Series

Taylor series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^n$

$$f(x) = f(a) + \frac{f'(a)}{1!}\underset{(x-a)}{\downarrow} + \frac{f''(a)}{2!}\underset{(x-a)^2}{\downarrow} + \frac{f'''(a)}{3!}\underset{(x-a)^3}{\downarrow} + \dots$$

$\sum_{n=0}^{\infty} C_n(x-a)^n$, $C_n = \frac{f^{n}(a)}{n!}$

If $a=0$, we call it maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!}(x-0)^n = \frac{f^{n}(0)}{n!}x^n$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Ex: find taylor series for the function $f(x) = 2^x$ centered at $x=1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(1)}{n!}(x-1)^n$$

$f(x) = 2^x \quad f(1) = 2$

$f'(x) = 2^x \ln 2 \longrightarrow f'(1) = 2\ln 2$

$f''(x) = 2^x(\ln 2)^2 \longrightarrow f''(1) = 2(\ln 2)^2.$

$f'''(x) = 2^x(\ln 2)^2 \longrightarrow f'''(1) = 2(\ln 2)^3$

$\vdots$

$f^n(x) = 2^x(\ln 2)^n \longrightarrow f^{(n)}(1) = 2(\ln 2)^n$

$$f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^n(1)}{n!}(x-1)^n +$$

$$= 2 + 2\ln 2(x-1) + \frac{2(\ln 2)^2}{2!}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + \frac{2(\ln 2)^n(x-1)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{2(\ln 2)^n}{n!}(x-1)^n$$

Ex: find the Maclaurin Series for the $f(x) = e^x$

$$\sum_{n=0}^{\infty} \frac{f^n(0)\, x^n}{n!}$$

$f(x) = e^x \longrightarrow f(0) = e^0 = 1$

$f'(x) = e^x \longrightarrow f'(0) = 1$

$\vdots$

$f^n(x) = e^x \longrightarrow f^n(0) = 1$

$$f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!} + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex: find Maclaurin Series for the function $f(x) = \sin x$

$$\left.\begin{aligned} f(x) &= \sin x \longrightarrow f(0) = 0 \\ f'(x) &= \cos x \longrightarrow f'(0) = 1 \\ f''(x) &= -\sin x \longrightarrow f''(0) = 0 \\ f'''(x) &= -\cos x \longrightarrow f'''(0) = -1 \end{aligned}\right\}$$

$$\vdots \qquad \vdots$$

$$f^{2n}(x) = \sin x \longrightarrow f^{2n}(0) = (-1)^n \sin 0 = 0$$

$$f^{(2n+1)}(x) = (-1)^n (\cos x) \longrightarrow f^{(2n+1)}(0) = (-1)^n \cos 0 = (-1)^n$$

$$f(0) + \frac{f'(0)x}{1} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \dots \frac{f^{2n}(0)x^{2n}}{(2n)!} + \frac{f^{(2n+1)}(0)x^{2n+1}}{(2n+1)!}$$

$$0 + x + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \frac{(-1)^n (x^{2n+1})}{(2n+1)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{2n})}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{2n+1})}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3$$

$$R = 1$$

Ex:

1) $f(x) = \sin 3x$

$$\sin 3x = \sum_{0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

2) $f(x) = \frac{1}{1+x^2}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$= \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x^{2n}) = \frac{1}{1+x^2}$$

3) $f(x) = \frac{1}{x+2}$ $\quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$= \frac{1}{2\left(\frac{x}{2}+1\right)}$$

$$= \frac{1}{2\left(1-\left(\frac{-x}{2}\right)\right)}$$

$$= \frac{1}{2}\left[\frac{1}{1-\left(\frac{-x}{2}\right)}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n =$$

4) $f(x) = x e^{-x}$ $\qquad e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{-x^n}{n!}$$

$$x e^{-x} = x \sum_{n=0}^{\infty} \frac{-x^n}{n!}$$

$$= \sum_{n=0}^{\infty} x \frac{(-1)^n x^n}{n!}$$

$$x e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{n+1})}{n!}$$

5) $F(x) = \sin^2 x$ $\qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$= \frac{1}{2}\left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}\right)$$

$$= \frac{1}{2}\left(1 - 1 - \sum_{n=1}^{\infty} \frac{(2x)^{2n}(-1)^n}{(2n)!}\right.$$

$$= -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2^{2n-1}\, x^{2n}}{(2n)!}$$

6) $f(x) = \cosh x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} + \dots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^3}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \frac{x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

* Find the sum of the series

1) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} = \boxed{e^2}$ $\qquad e^x = \sum_{n=0} \frac{x^n}{n!}$

2) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x^{4n})}{n!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}$$

③ $\sum_{n=0}^{\infty} \frac{(-1)^{n-1} 3^n}{n 5^n} = \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \left(\frac{3}{5}\right)^n}{n}$

$= \ln\left(1+\frac{3}{5}\right) = \ln \frac{8}{5}$

④ $\sum_{n=0}^{\infty} \frac{(-1)^n (\pi)^{2n+2}}{4^{2n+2} (2n+1)!}$ $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{2n+1})}{(2n+1)!}$ ✓

$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+2}}{(2n+1)!}$ $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{2n})}{(2n)!}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right) \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}$

$= \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}$

$= \frac{\pi}{4} \sin\frac{\pi}{4} = \frac{\pi}{4\sqrt{2}}$

5) $\frac{-\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots$ $\sin x = \sum \frac{(-1)^n (x^{2n+1})}{(2n+1)!}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n (\pi^{2n+1})}{(2n+1)!}$ $\sin \pi = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}$

$= \pi + \dots$

$= \sin \pi - \pi$

$0 - \pi$

$= \boxed{-\pi}$