ElCoM
ELECTRICAL COMPUTER MECHATRONICS

تفاضل و تكامل 2

الاء القدومي

للطالـب المبدع
عمر النصلة

إرادة - ثقـة - تغيير

{Calculus 1} Review

① $f(x) = \sin x$
Domain : R
Rang : $[-1,1]$

② $f(x) = \cos(x)$
Domain : R
Rang : $[-1,1]$

③ $f(x) = \tan(x)$
Domain : R
Rang: $R - [\pm\frac{\pi}{2}, \pm\frac{\pi 3}{2}]$

④ $f(x) = \sin^{-1}(x)$
Domain : $[-1,1]$
Rang :- $[-\frac{\pi}{2}, \frac{\pi}{2}]$

⑤ $f(x) = \cos^{-1}(x)$
Domain : $[-1,1]$
Rang : $[0,\pi]$

⑥ $f(x) = \tan^{-1}(x)$
Domain : R
Rang : $(\frac{-\pi}{2}, \frac{\pi}{2})$

⑦ $f(x) = \csc^{-1}(x)$
Domain: $(-\infty,-1] \cup [1,\infty)$
Rang : $(0,\frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

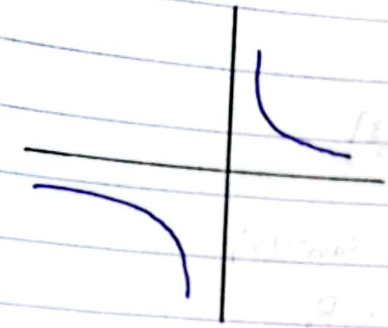
⑧ $f(x) = \sec^{-1}(x)$
Domain : $(-\infty,1] \cup [1,\infty)$
Range : $[0,\frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

⑨ $f(x) = \cot^{-1}(x)$
Domain : R
Rang : $(0,\pi)$

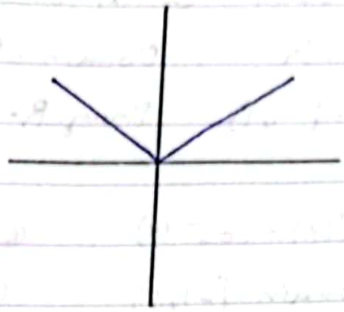
* المتطابقات

1. $\sin^2 a + \cos^2 a = 1$
2. $\sin(2a) = 2\sin(a)\cos(a)$
3. $\cos(2a) = 2\cos^2 a - 1$
4. $\cos(2a) = 1 - 2\sin^2(a)$
5. $\cos(2a) = \cos^2(a) - \sin^2(a)$
6. $\sec^2(a) = 1 + \tan^2(a)$
7. $\csc^2(a) = 1 + \cot^2(a)$
8. $\sin^2(a) = \frac{1}{2}(1 - \cos(2a))$
9. $\cos^2(a) = \frac{1}{2}(1 + \cos(2a))$
10. $\sin(a)\cos(b) = \frac{1}{2}(\sin(a-b) + \sin(a+b))$
11. $\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$
12. $\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$
13. $\sin(a+2\pi) = \sin(a)$
14. $\cos(a+2\pi) = \cos(a)$

① $f(x) = \frac{1}{x}$

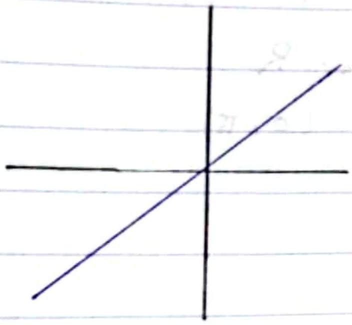


② $f(x) = |x|$

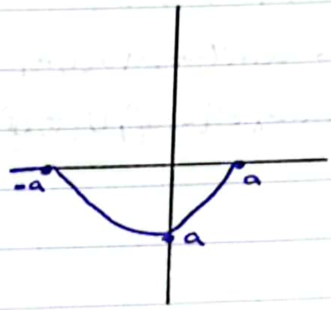


③ $f(x) = x^2$

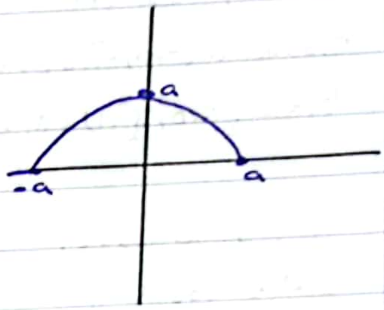
④ $f(x) = x$



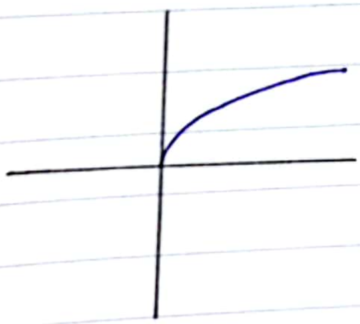
⑤ $f(x) = -\sqrt{a^2 - x^2}$



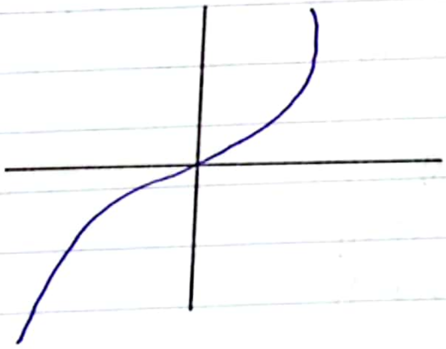
⑥ $f(x) = \sqrt{a^2 - x^2}$



⑦ $f(x) = \sqrt{x}$



⑧ $f(x) = x^3$



* $y = f(x) + c$ shift the graph $y = f(x)$ a distance c units upward
* $y = f(x) - c$ 〃 〃 〃 〃 〃 〃 〃 downward
* $y = f(x+c)$ 〃 〃 〃 〃 〃 〃 〃 to the left
* $y = f(x-c)$ 〃 〃 〃 〃 〃 〃 〃 〃 〃 right
* $y = -f(x)$ reflect the graph $y = f(x)$ about the x axis
* $y = f(-x)$ 〃 〃 〃 〃 〃 〃 y axis

* reflect : انعكاس

* Domain $f \circ g = \{ x \in D_{\text{داخلي}}$ and $g(x) \in D_{\text{خارجي}} \}$ * Domain $f \circ g = D_{\text{داخلي}} \cap D$ الناتج عن التركيب بعد التبسيط

*Inverse function (f^{-1})

Domain : Rang f(x)

Rang : Domain f(x)

* كيفية ايجادها :-

① نساوي $f(x) = y$

② x موضع القانون

③ نبدل ال x ب f^{-1} وكل y ب x

* Vertical line test

to know if it a function or no

* Horizontal line test

to Know if it one to one function or no

* قوانين الـ log و الـ ln

① $b^{\log_b x} = x$ ② $\log_b a = \frac{\ln a}{\ln b}$ ③ $\log_b ac = \log_b a + \log_b c$

④ $\log_b b^x = x$ ⑤ $\log_b \frac{a}{c} = \log_b a - \log_b c$ ⑥ $e^{\ln x} = x$ ⑦ $\ln e^x = x$

⑧ $(f \circ f^{-1})(x) = x$ ⑨ $\log_b a^r = r \log_b a$

* Limits

* $\frac{f}{g} \rightarrow \lim_{x \to a} \frac{f}{g}$

① عدد

② $\frac{0}{0}$ → اعادة تعريف / قسمة طويلة / ضرب مرافق / تحليل

③ $\frac{\text{عدد}}{\text{صفر}}$ → d.n.e / $-\infty$ / $+\infty$

* عدد على صفر إذا كان الجواب طرف $+\infty$ وطرف $-\infty$ يكون الجواب d.n.e

*vertical asymptote

طريقة :-

① اصفار المقام

② limit لكل من اصفار المقام

③ الناتج $+\infty$ او $-\infty$ يكون هو الجواب او الـ 2 مع بعض

*squeeze theorem:-

$3x \le f(x) \le x^2+2$ $\lim_{x \to 1} f(x)$?

$\lim_{x \to 1} 3x = 3$

$\lim_{x \to 1} x^2+2 = 3$ so $\lim_{x \to 1} f(x) = 3$

* continuity

f(a) defind

$\lim_{x \to a} f(x) = f(a)$

الاتصال على فترة

① الفترات

② نقاط التشعب

③ الاطراف

Theorem :-

① $\lim_{x \to \pm\infty} (f(x))^n = \left(\lim_{x \to \pm\infty} f(x)\right)^n$

② $\lim_{x \to \pm\infty} k\, f(x) = k \lim_{x \to \pm\infty} f(x)$

③ $\lim_{x \to \pm\infty} k = k$

④ $\lim_{x \to +\infty} x^n = +\infty \,,\; n = 1, 2, 3, 4, \dots$

⑤ $\lim_{x \to -\infty} x^n = \begin{cases} -\infty \,, & n = 1, 3, 5, 7 \\ +\infty \,, & n = 2, 4, 6, 8 \end{cases}$

⑥ $\lim_{x \to \pm\infty} (a_1 x + a_2 x^2 + a_3 x^3 + a_n x^n) = \lim_{x \to \pm\infty} a_n x^n$

⑦ if $r > 0$ is arational number them $\lim_{x \to \pm\infty} \frac{1}{x^r} = 0$

⑧ $\lim_{x \to \infty} \frac{e^{5x} - e^x}{e^{4x} - e^{3x}} = \lim_{x \to \infty} \frac{e^{5x}}{e^{4x}}$

⑨ $\lim_{x \to -\infty} \frac{e^{5x} - e^x}{e^{4x} - e^{3x}} = \lim_{x \to -\infty} \frac{e^x}{e^{3x}}$

⑩ إذا كان جذر , مناخذ اكبر أس مع اكبر أس داخل الجذر ,

$\lim_{x \to -\infty} \frac{2 - x^2}{\sqrt{4x^4 + x}} = \lim_{x \to -\infty} \frac{-x^2}{\sqrt{4x^4}} = \lim_{x \to -\infty} \frac{-x^2}{2x^2} = \frac{-1}{2}$

✗ Derivation

* قانون التعريف العام للمشتقة

$f'(x) = \frac{f(x+h) - f(x)}{h}$

* $\sin^{-1}(g(x))$

$\frac{g'(x)}{\sqrt{1 - g(x)^2}}$

* $\cos^{-1}(g(x))$

$\frac{-g'(x)}{\sqrt{1 - g(x)^2}}$

* $\tan^{-1}(g(x))$

$\frac{g'(x)}{1 + g(x)^2}$

* $\csc^{-1}(g(x))$

$\frac{-g'(x)}{g(x)\sqrt{g(x)^2 - 1}}$

$\sec^{-1}(g(x))$

$\frac{g'(x)}{g(x)\sqrt{g(x)^2 - 1}}$

$\cot^{-1}(g(x))$

$\frac{-g'(x)}{1 + g^2(x)^2}$

* differentiable:-

1- $f(x)$ و $\lim_{x \to a} f(x)$ متساويات

2- متصلة

3- المشتقة = $f'(x) = f'_+(x)$, $\lim f'_-(x)$

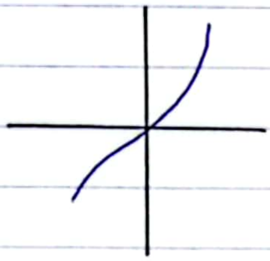
* Hyperbolic function:-

*** sinh(x)**

Domain:- R

Rang:- ~~[1,∞)~~ R

$$= \frac{e^x - e^{-x}}{2}$$

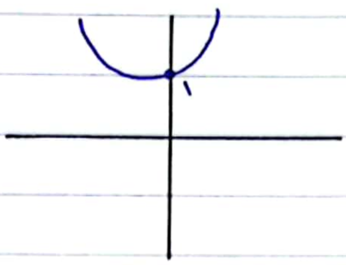


*** cosh(x)**

Domain: R

Rang: $[1,\infty)$

$$= \frac{e^x + e^{-x}}{2}$$

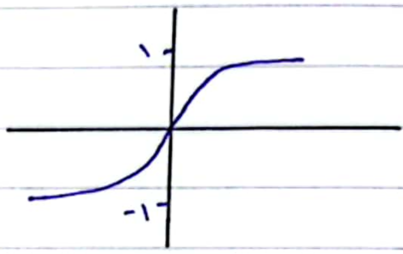


*** tanh(x)**

Domain:- R

Rang:- $(-1,1)$

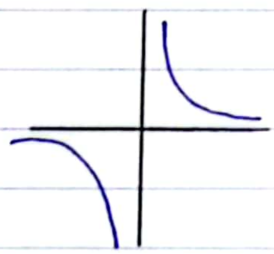
$$= \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



*** csch(x)**

Domain:- $R \setminus \{0\}$

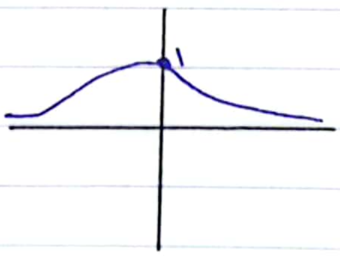
Rang:- $R \setminus \{0\}$



*** sech(x)**

Domain: R

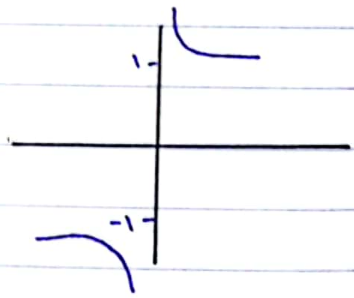
Rang: $(0,1]$



*** coth(x)**

Domain:- $R \setminus \{0\}$

Rang: $(-\infty,-1) \cup (1,\infty)$



* Theorems:-

1. $\cosh + \sinh = e^x$
2. $\cosh(x) - \sinh(x) = e^{-x}$
3. $\cosh^2(x) - \sinh^2(x) = 1$
4. $1 - \tanh^2(x) = \operatorname{sech}^2(x)$
5. $\coth^2(x) - 1 = \operatorname{csch}^2(x)$
6. $\sinh^2(x) = \frac{1}{2}(\cancel{\cosh 2}\ \cosh(2x) - 1)$
7. $\cosh^2(x) = \frac{1}{2}(1 + \cosh(2x))$
8. $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$
9. $\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$
10. $\sinh(x-y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$
11. $\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$
12. $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$
13. $\cosh(2x) = 2\cosh^2(x) - 1$
14. $\cosh(2x) = 2\sinh^2(x) + 1$

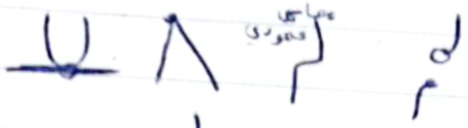
⑥ Applications of Differntiation

$f'(x)$

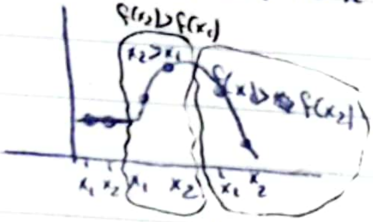
a critical number ($a \in D_f$) قيمة حرجة

- $f'(a) = 0$ has a horizontal tangent line
- $f'(a)$ d.n.e

ا صفار البسط وا صفار المقام و
النقاط التشعب ونقاط عدم الاتصال



* f increasing :-
  ① if $x_2 > x_1 \to f(x_2) > f(x_1)$ $\underline{+ + + + +}$ $f' > 0$
* f Decresing
  if $x_2 > x_1 \longrightarrow f(x_2) < f(x_1)$ $\underline{- - - -}$ $f' < 0$
* f constant
  $f(x_1) = f(x_2)$ $\forall x$ $f' = 0$



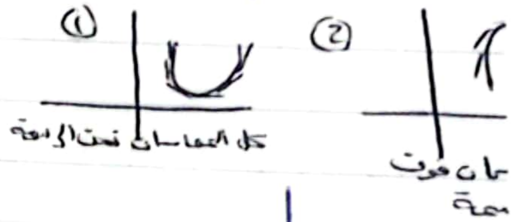
عظمى محلية a local max if $f(a) \geq f(x)$ when x is near a
صغرى محلية a local min if $f(a) \leq f(x)$ " " " " "

عظمى مطلقة a absolute max if $f(a) \geq f(x)$ $\forall x \in D_f$
صغرى مطلقة a absolute min if $f(a) < f(x)$ $\forall x \in D_f$

$f''(x)$

التقعر Concavit[y]

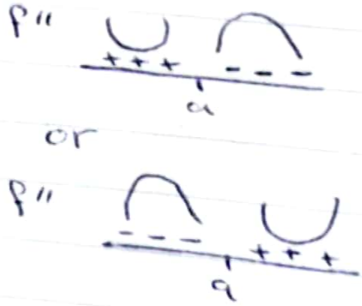
* f concave upward مقعر للأعلى $\underline{+ +}$
① (The graph of f lies abo[ve] of its tangent} on an
* f concave downward مقعر للأسفل $\underline{-}$
② (The graph of f lies bel[ow] all of its tangents or interval)



a inflection point نقطة

① $a \in D_f$
② f cont at $x = a$

* the curve changes fro[m] concave upward or concave downward



* The mean Value Theorem : (M.V.T)

① f cont on $[a, b]$
② f differentiable on (a, b)

Then $\exists c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a} = \dfrac{\Delta y}{\Delta x}$

Inderterminate forms and L'Hospitals Rule:-

① $\frac{0}{0}, \frac{+\infty}{+\infty}, \frac{-\infty}{-\infty}, \frac{+\infty}{-\infty}, \frac{-\infty}{+\infty}$

نشتق البسط ونشتق المقام حتى يبسط الحل

② $(0 \cdot \pm\infty), (+\infty \cdot -\infty), (-\infty \cdot +\infty)$

نغير الشكل حيث ما يصيرو بسط على مقام
(يعني حالة ١ ثم نشتق)

③ $1^{\pm\infty}, \pm\infty^{0}, 0^{0}$

١- نساوي لـ y ما في داخل الـ lim
2- نأخذ ln للطرفين
3- نأخذ المقدار ونجد الـ lim
(حيث ما يصير زي الحالة ①)
٦- الجواب = $e^{\text{جواب نقطة ③}}$

* Remark:-

① $\lim_{x \to \pm\infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$ ② $\lim_{x \to 0} (1 + ax)^{bx} = e^{ab}$

* Intergration

* $\int_{-a}^{a}$ odd function $= 0$

* $\int_{-a}^{a}$ even function $= 2\int_{0}^{a}$ even function

* $\int \frac{1}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$
* $\int \frac{g'(x)}{\sqrt{1 - (g(x))^2}} = \sin^{-1}(g(x)) + c$

* $\int \frac{1}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$
* $\int \frac{g'(x)}{g(x)\sqrt{g(x)^2 - 1}} = \sec^{-1}(g(x)) + c$

* $\int \frac{1}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
* $\int \frac{g'(x)}{a + g(x)^2} = \tan^{-1}(g(x)) + c$

* $\frac{d}{dx}\int_{a}^{b} f(x)\,dx = 0$
* $\frac{d}{dx}\int f(x) = f(x)$

* The fundamental Theorem of calculas:-

$$\frac{d}{dx}\int_{f(x)}^{g(x)} h(t)\,dt = h(g(t)) \cdot g'(t) - h(f(x)) \cdot f'(x)$$

* Area between Two curves:-

حالة ① :-
رسمه وإلى اعلى بتكون قبل الثانية وبينهم (-)

حالة ②
* يوجد رسمة بساويهم ببعض وبطلع نقاط الإلتقاء وبعوض ~~النقاط~~ بين النقاط في كل مرحلة عشان اعرف مين احط اول

Calculas 2

Week 1

* lecture 1

Integration by parts: التكامل بالأجزاء

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\int_a^b u \cdot dv = u \cdot v \Big]_a^b - \int_a^b v \cdot du$$

u	dv
نشتق	نكامل
du	v

$du \xrightarrow{-\int} v$

Ex:- ① $\int \overset{u}{\ln x} \, \overset{dv}{dx}$

$u = \ln x \qquad dv = dx \qquad = x \ln x - \int \frac{1}{x} \cdot x \, dx$

$du = \frac{1}{x} dx \xrightarrow{-\int} v = x$

$$\boxed{= x \ln x - x + C}$$

* Make it easier:-

① $\int$ اقتران $* \ln(\ )$ — dv , u

② $\int$ اقتران $* \begin{matrix} \sin^{-1}(\ ) \\ \cos^{-1}(\ ) \\ \tan^{-1}(x) \end{matrix}$ — dv , u

③ $\int$ كثير حدود $* e^{\text{خطي}}$ — u , dv

④ $\int$ كثير حدود $* \begin{matrix} \cos(\text{خطي}) \\ \sin(\text{خطي}) \end{matrix}$ — u , dv

⑤ $\int e^{\text{خطي}} * \begin{matrix} \sin(\text{خطي}) \\ \cos(\text{خطي}) \end{matrix}$ — dv , u

ويمكن العكس

⑥ $\int \text{poly} * (\text{خطي})^n$ — u , dv

Ex:- ① $\int_0^1 \ln(x^2+1) \, dx$

$u = \ln(x^2+1) \qquad dv = dx \qquad = x \ln(x^2+1) - \int \frac{2x}{x^2+1} * x \, dx$

$du = \frac{2x}{x^2+1} \qquad v = x \qquad = x \ln(x^2+1) - \int \frac{2x^2}{x^2+1} dx$

طرق الحل:-
① قسمة طويلة
② اضافة وطرح

$$= x \ln(x^2+1) - 2 \int \frac{x^2+1-1}{x^2+1} dx$$

*الحل عن الإضافة والطرح

$$= x \ln(x^2+1) - 2 \int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1} dx$$

$$= x \ln(x^2+1) - 2(x - \tan^{-1}(x)) \Big]_0^1$$

$$= 1 \ln 2 - 2(1 - \tan^{-1}(1)) - (0 \ln 1 - 2(0 - \tan^{-1}(0)))$$

$$= \ln 2 - 2 + 2\frac{\pi}{4} - 0 - 0 + 0$$

$$\boxed{= \ln 2 - 2 + \frac{\pi}{2}}$$

Lecture 2

Ex:- ① $\int \cos(x) * \ln(\cos x)\, dx$

$u = \ln(\cos(x)) \qquad dv = \cos(x)$

$du = \frac{-\sin x}{\cos x} \qquad v = \sin x$

$= \sin(x)\ln(\cos x) - \int \frac{-\sin^2 x}{\cos x}\, dx$

$= \sin(x)\ln(\cos x) + \int \frac{1-\cos^2 x}{\cos x}\, dx$

$= \sin(x)\ln(\cos x) + \int \frac{1}{\cos x} - \cos(x)\, dx$

$= \sin(x)\ln(\cos x) + \int \sec x - \cos x\, dx$

$= \sin(x)\ln(\cos x) - \sin x + \int \sec(x)$

$\int \sec(x) \left(\frac{\sec x + \tan x}{\sec x + \tan(x)} \right)$

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$

$= \ln|\sec x + \tan x|$

$= \sin(x)\ln(\cos x) - \sin x + \ln|\sec x + \tan x| + c$

---

② $\int_0^{\frac{1}{3}} \overset{u}{\tan^{-1}(3x)}\, \overset{dv}{dx}$

$u = \tan^{-1}(3x) \qquad dv = dx$

$du = \frac{3}{1+9x^2}\, dx \qquad v = x$

~~$= x\tan^{-1}(3x) - \int x \tan^{-1}(3x)\, dx$~~

$= x\tan^{-1}(3x) - \int \frac{3x}{1+9x^2}\, dx$

$= x\tan^{-1}(3x) - \frac{1}{6}\int \frac{18x}{1+9x^2}\, dx$

$= x\tan^{-1}(3x) - \frac{1}{6}\ln|1+9x^2| \Big]_0^{\frac{1}{3}}$

$= \frac{1}{3}\tan^{-1}(3 * \frac{1}{3}) - \frac{1}{6}\ln|1+9(\frac{1}{3})^2| - \left(0\tan^{-1}(0) - \frac{1}{6}\ln|1+9(0)^2|\right)$

$= \frac{1}{3} * \frac{\pi}{4} - \frac{1}{6}\ln 2$

$= \frac{\pi}{12} - \frac{1}{6}\ln 2$

Lecture 3

تكامل بالأجزاء باستخدام الجدول :-

Ex:- ① $\int x^2 e^{-x} dx$

u		dv
x^2	$+$	e^{-x}
$2x$	$-$	$-e^{-x}$
2	$+$	e^{-x}
0		$-e^{-x}$
$-\int$		

$$\boxed{= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C}$$

مش دايما منتجاهلها

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* حالات استخدام الجدول

① $\int$ كثير حدود * $e^{\text{خطي}}$ ② $\int$ كثير حدود * $\cos(\text{خطي})$ ③ $\int e^{\text{خطي}} \cos(\text{خطي})$

عادي يكون $(\sin)$ او $(\cos)$

---

② $\int x^2 \sin(2x) dx$

u		dv
x^2	$+$	$\sin(2x)$
$2x$	$-$	$-\frac{1}{2}\cos(2x)$
2	$+$	$-\frac{1}{4}\sin(2x)$
0		$\frac{1}{8}\cos(2x)$
$-\int$		

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) - \int 0 * \frac{1}{8}\cos(2x) + C$$

$$\boxed{= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C}$$

---

③ $\int e^{2x} \sin(x) dx$

u		dv
e^{2x}	$+$	$\sin(x)$
$2e^{2x}$	$-$	$-\cos(x)$
$4e^{2x}$		$-\sin(x)$
$+\int$		

$$\int e^{2x}\sin(x) = -e^{2x}\cos(x) + 2e^{2x}\sin(x) + \int -4e^{2x}\sin(x)$$

$$\int e^{2x}\sin(x) = -e^{2x}\cos(x) + 2e^{2x}\sin(x) - 4\int e^{2x}\sin(x)$$

$$5\int e^{2x}\sin(x) = -e^{2x}\cos(x) + 2e^{2x}\sin(x)$$

$$\int e^{2x}\sin(x) = \boxed{\frac{-e^{2x}\cos(x) + 2e^{2x}\sin(x)}{5} + C}$$

نستعمل الجدول حيث ما يرجع للحد الأصلي بغض النظر عن الإشارة

H.W

① $\int_0^{\frac{1}{2}} x \cos(\pi x)\,dx$

$u = x \qquad dv = \cos(\pi x)$

$du = 1 \qquad v = \frac{\sin(\pi x)}{\pi}$

$= \frac{x}{\pi}\sin(\pi x) - \int \frac{\sin(\pi x)}{\pi}\,dx$

$= \frac{x}{\pi}\sin(\pi x) + \frac{\cos(\pi x)}{\pi^2}\Big]_0^{\frac{1}{2}}$

$= \frac{1}{2\pi}\sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2}) \cdot \frac{1}{\pi^2} - (0 + \frac{1}{\pi^2})$

$= \frac{1}{2\pi} - \frac{1}{\pi^2}$

② $\int e^{-2\theta}\sin(3\theta)\,d\theta$

u		dv
$e^{-2\theta}$	+	$\sin(3\theta)$
$-2e^{-2\theta}$	−	$-\frac{\cos(3\theta)}{3}$
$4e^{-2\theta}$	$+\int$	$-\frac{\sin(3\theta)}{9}$

$\int e^{-2\theta}\sin(3\theta)\,d\theta = \frac{-e^{-2\theta}\cos(3\theta)}{3} - \frac{2e^{-2\theta}\sin(3\theta)}{9} - \int \frac{4e^{-2\theta}\sin(3\theta)}{9}$

$\int e^{-2\theta}\sin(3\theta)\,d\theta = \frac{-1}{3}\cos(3\theta)e^{-2\theta} - \frac{2}{9}e^{-2\theta}\sin(3\theta) - \frac{4}{9}\int e^{-2\theta}\sin(3\theta)$

$\frac{13}{9}\int e^{-2\theta}\sin(3\theta)\,d\theta = \frac{-1}{3}\cos(3\theta)e^{-2\theta} - \frac{2}{9}e^{-2\theta}\sin(3\theta)$

$\int e^{-2\theta}\sin(3\theta)\,d\theta = \frac{9}{13}\left(\frac{-1}{3}\cos(3\theta)e^{-2\theta} - \frac{2}{9}e^{-2\theta}\sin(3\theta)\right)$

③ $\int (x^2+2x+1)e^{-x}\,dx$

u		dv
x^2+2x+1	+	e^{-x}
$2x+2$	−	$-e^{-x}$
2	+	e^{-x}
0		$-e^{-x}$

$= -(x^2+2x+1)e^{-x} - (2x+2)e^{-x} - 2e^{-x} + c$

⑤ $\int (x^2+2x)(x+1)^4\,dx$

u		dv
x^2+2x	+	$(x+1)^4$
$2x+2$	−	$\frac{(x+1)^5}{5}$
2	+	$\frac{(x+1)^6}{30}$
0		$\frac{(x+1)^7}{210}$

$= \frac{1}{5}(x^2+2x)(x+1)^5 - \frac{1}{30}(2x+2)(x+1)^6 + \frac{1}{105}(x+1)^7 + c$

4. $\int_1^3 \sqrt{x}\ \tan^{-1}(\sqrt{x})\,dx$

$u = \sqrt{x}$ $\quad\quad$ $\int \sqrt{x}\tan^{-1}(u) * 2\sqrt{x}\,du$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ $\quad\quad$ $\int u \tan^{-1}(u) * 2u\,du$

$\int 2u^2 \tan^{-1}(u)\,du$

$dx = 2\sqrt{x}\,du$ $\quad\quad$ $u = \tan^{-1}(u)$ $\quad dv = 2u^2$

$du = \frac{1}{1+u^2}$ $\quad v = \frac{2u^3}{3}$

$\frac{2}{3}u^3\tan^{-1}(u) - \frac{2}{3}\int \frac{u^3}{1+u^2}$ $\longrightarrow$ $u^2+1 \overline{)\,u^3}$ quotient u; u^3+u; $-u$

$\frac{2}{3}u^3\tan^{-1}(u) - \frac{2}{3}\int u - \frac{u}{u^2+1}$

$\frac{2}{3}u^3\tan^{-1}(u) - \frac{2}{6}u^2 - \frac{2}{3} * \frac{1}{2}\ln|u^2+1|$

$\frac{2}{3}u^3\tan^{-1}(u) - \frac{1}{3}u^2 - \frac{1}{3}\ln|u^2+1|$

$\frac{2}{3}\sqrt{x}^3\tan^{-1}(\sqrt{x}) - \frac{1}{3}\sqrt{x}^2 - \frac{1}{3}\ln|\sqrt{x}^2+1|\Big]_1^3$

$2\sqrt{3}\tan^{-1}(\sqrt{3}) - 1 - \frac{1}{3}\ln 4 - \left(\frac{2}{3} * \frac{\pi}{4} - \frac{1}{3} - \frac{1}{3}\ln 2\right)$

$2\sqrt{3} * \frac{\pi}{3} - 1 - \frac{1}{3}\ln 4 - \frac{\pi}{6} + \frac{1}{3} + \frac{1}{3}\ln 2$

$\frac{2}{\sqrt{3}}\pi - \frac{2}{3} - \frac{\pi}{6} - \frac{1}{3}\ln 4 + \frac{1}{3}\ln 2$

$0.988\pi - \frac{2}{3} + \ln\sqrt[3]{\frac{2}{4}}$

$0.988\pi - \frac{2}{3} - \frac{1}{3}\ln 2$

$2.437 - \frac{1}{3}\ln 2$

Lecture 4

Ex:- ① $\int x^3 \sin(x^2)\,dx$

$y = x^2$ $\qquad \int x^3 \sin(y) \frac{dy}{2x}$

$\frac{dy}{dx} = 2x$

$dx = \frac{dy}{2x}$ $\qquad \frac{1}{2}\int y \sin(y)\,dy$

~~u = y dv = sin(y)~~

~~du = dy v = -cos(y)~~

u	dv
y	$\sin(y)$
1 (+)	$-\cos(y)$
0 (−)	$-\sin(y)$

$= \frac{1}{2}(-y\cos(y) + \sin(y)) + c$

$\boxed{\frac{1}{2}(-x^2\cos(x^2) + \sin(x^2)) + C}$

② $\int \cos(\ln x)\,dx$

$y = \ln x$

$dy = \frac{1}{x}dx$ $\qquad \int \cos(y)\, x\, dy$

$dx = x\,dy$ $\qquad \int \cos(y)\, e^y\, dy$

$u = e^y \qquad dv = \cos(y)$

e^y	$\cos(y)$
e^y (+)	$\sin(y)$
e^y (−)	$-\cos(y)$
$+\int$	

$= e^y \sin y + e^y \cos(y) + \int -\cos(y) e^y\,dy$

$\int \cos(y) e^y = e^y \sin(y) + e^y \cos(y) - \int \cos(y) e^y$

$2\int \cos(y) e^y = e^y \sin(y) + e^y \cos(y)$

$\int \cos(y) e^y = \frac{1}{2}(e^y \sin(y) + e^y \cos(y)) + C$

$= \frac{1}{2}(e^{\ln x} \sin(\ln x) + e^{\ln x}\cos(\ln x)) + C$

$\boxed{= \frac{1}{2}(x \sin(\ln x) + x\cos(\ln x)) + C}$

③ suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$ $f'(4) = 3$ and f'' is cont find the Value of $\int_1^4 x\, f''(x)\,dx$?

u	dv
x	$f''(x)$
	$f'(x) = v$
$du = 1$	

$\int_1^4 x\, f''(x) = x\, f'(x) - \int f'(x) * 1\,dx$

$= x\, f'(x) - f(x)\Big]_1^4$

$= 4 f'(4) - f(4) - (f'(1) - f(1))$

$= 4(3) - 7 - (5 - 2)$

$= 12 - 7 - 3$

$= 2$

H.W

④ $\int (\sin^{-1})^2 dx$

$u = (\sin^{-1}(x))^2 \quad dv = dx$

$du = 2\sin^{-1} * \frac{1}{\sqrt{1-x^2}} \quad v = x$

$= x\sin^{-1}(x)^2 - \int 2x \sin^{-1} \frac{1}{\sqrt{1-x^2}}$

$= x\sin^{-1}(x)^2 - 2\int \sin^{-1}(x) \frac{x}{\sqrt{1-x^2}}$

$u = \sin^{-1}(x) \quad dv = \frac{x}{\sqrt{1-x^2}} \cdots ①$

$du = \frac{1}{\sqrt{1-x^2}} \quad v = -\sqrt{1-x^2}$

$= x\sin^{-1}(x) - 2\left(-\sin^{-1}(x)\sqrt{1-x^2} - \int -\sqrt{1-x^2} * \frac{1}{\sqrt{1-x^2}} dx\right.$

$= x\sin^{-1^2}(x) - 2\left(-\sin^{-1}(x)\sqrt{1-x^2} + x\right)$

$= x(\sin^{-1}(x))^2 + 2\sin^{-1}(x)\sqrt{1-x^2} - 2x + C$

① $\int \frac{x}{\sqrt{1-x^2}} = \int \frac{x}{\sqrt{u}} \cdot x \cdot \frac{du}{-2x}$

$u = 1-x^2$

$du = -2x\,dx$

$dx = \frac{du}{-2x}$

$= \frac{-1}{2}\int \frac{1}{\sqrt{u}}$

$= \frac{-1}{2}\int u^{\frac{-1}{2}} du$

$= \frac{-1}{2} 2\sqrt{u}$

$= \frac{-1}{2} 2\sqrt{1-x^2}$

$= -\sqrt{1-x^2}$

Lecture 5

Trigonometric Integrats

* $\int \sin^n x\, dx = \frac{-1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

* $\int \cos^n x\, dx = \frac{+1}{n}\cos^{n-1} \sin x + \frac{n-1}{n}\int \cos^{n-2} dx$

Ex:- prove that $\int \cos^n x = \frac{1}{n}\cos^{n-1}\sin x + \frac{n-1}{n}\int \cos^{n-2} dx$?

$\int \cos^n(x)\, dx = \int \underbrace{\cos^{n-1}(x)}_{u}\ \underbrace{\cos(x)\, dx}_{dv}$

$u = \cos^{n-1}(x) \quad dv = \cos(x) dx$

$du = (n-1)\cos^{n-2}(x) * -\sin x \quad v = \sin(x)$

$\sin x \cos^{n-1}(x) - \int \sin(x)(n-1)\cos^{n-2}(x) * -\sin x$

$\sin x \cos^{n-1}(x) + \int (n-1)\cos^{n-2}(x)\sin^2 x\, dx$

$\cos^{n-1}(x)\sin x + (n-1)\int \cos^{n-2}(x)(1-\cos^2 x)dx$

$\cos^{n-1}(x)\sin x + (n-1)\int \cos^{n-2} - \cos(x)^{n-2+2} dx$

$\int \cos^n(x) = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x) - (n-1)\int \cos^n(x)$

~~(1+n-1)∫ cos^n(x) = cos^{n-1}~~ sin

$(1+n-1)\int \cos^n(x) = \cos^{n-1}(x)\sin x + (n-1)\int \cos^{n-2}(x)$

$n\int \cos^n(x) = \cos^{n-1}(x)\sin x + (n-1)\int \cos^{n-2}(x)$

$\int \cos^n(x) = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)$

Ex:-

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① $\int \sin^4 x\, dx = \frac{-1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\, dx$

$= \frac{-1}{4}\sin^3 x\cos x + \frac{3}{4}\int \frac{1}{2}(1-\cos(2x))$

$= \frac{-1}{4}\sin^3 x\cos x + \frac{3}{4} * \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)$

$= \frac{-1}{4}\sin^3 x\cos x + \frac{3}{8}x - \frac{3}{16}\sin(2x) + C$

② $\int \cos^3(3x)\, dx$

$y = 3x$ $\qquad \int \cos^3(y)\frac{dy}{3}$

$\frac{y}{3} = x$

$dy = 3dx$ $\qquad \frac{1}{3}\int \cos^3(y)\,dy$

$dx = \frac{dy}{3}$ $\qquad = \frac{1}{3}\left(\frac{1}{3}\cos^2(y)\sin(y) + \frac{2}{3}\int \cos(y)\right)$

$= \frac{1}{3}\left(\frac{1}{3}\cos^2(3x)\sin(3x) + \frac{2}{3}\sin(3x)\right) + C$

H.W

① $\int \sin^2(5x)\, dx$

$y = 5x$ $\qquad \frac{1}{5}\int \sin^2(y)\,dy$

$dy = 5dx$

$dx = \frac{dy}{5}$ $\qquad \frac{1}{5}\left(-\frac{1}{2}\sin(y)\cos(y) + \frac{1}{2}\int \sin^0(y)\,dy\right)$

$\frac{1}{5}\left(-\frac{1}{2}\sin(y)\cos(y) + \frac{1}{2}y + C\right.$

$\frac{1}{5}\left(\frac{-1}{2}\sin(5x)\cos(5x) + \frac{1}{2}(5x)\right) + C$

② $\int \sin^6(x)\, dx$

$= \frac{-1}{6}\sin^5(x)\cos(x) + \frac{5}{6}\int \sin^4(x)\, dx$

$= \frac{-1}{6}\sin^5(x)\cos(x) + \frac{5}{6}\left(\frac{-1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx\right)$

$= \frac{-1}{6}\sin^5(x)\cos(x) + \frac{5}{6}\left(\frac{-1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \frac{1}{2}(1-\cos(2x))\,dx\right)$

$= \frac{-1}{6}\sin^5(x)\cos(x) + \frac{5}{6}\left(\frac{-1}{4}\sin^3(x)\cos(x) + \frac{3}{8}x - \frac{3}{16}\sin(2x)\right) + C$

$= \frac{-1}{6}\sin^5(x)\cos(x) - \frac{5}{24}\sin^3(x)\cos(x) + \frac{15}{48} - \frac{15}{96}\sin(2x) + C$

③ $\int \cos^4(7x)\,dx$

$y = 7x \qquad \frac{1}{7}\int \cos^4(y)\,dy$

$dy = 7dx$

$dx = \frac{dy}{7}$

$= \frac{1}{7}\left(\frac{1}{4}\cos^3(y)\sin(y) + \frac{3}{4}\int(\cos^2(y))\,dy\right.$

$= \frac{1}{7}\left(\frac{1}{4}\cos^3(y)\sin(y) + \frac{3}{4}\int \frac{1}{2}(1+\cos(2y))\,dy\right)\Big)$

$= \frac{1}{7}\left(\frac{1}{4}\cos^3(y)\sin(y) + \frac{3}{8}y + \frac{3}{16}\sin(2y)\right) + c$

$= \frac{1}{28}\cos^3(7x)\sin(7x) + \frac{3}{8}x + \frac{3}{112}\sin(14x) + c$

④ $\int \sin^3(x)\,dx$

$= -\frac{1}{3}\sin^2(x)\cos(x) + \frac{2}{3}\int \sin(x)\,dx$

$= -\frac{1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x) + C$

Lecture 6

$\int \sin^m(x)\cos^n(x)$

* If (n) odd save one cos
  use $\cos^2(x) = 1 - \sin^2 x$
  let $u = \sin(x)$

* If (m) odd save one sin
  use $\sin^2(x) = 1 - \cos^2(x)$
  let $u = \cos(x)$

* If (n) and (m) even :-
  ① $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$
  ② $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
  ③ $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$

Ex:- ① $\int \sin^3(x)\cos^2(x)\,dx$

$\int \sin^2(x)\cos^2(x)\sin(x)\,dx$

$\int (1-\cos^2(x))\cos^2(x)\sin(x)\,dx$

$u = \cos(x)$

$du = -\sin x\,dx$

$dx = \frac{du}{-\sin x}$

$\int (1-u^2)u^2 \sin(x)\frac{du}{-\sin(x)}$

$-\int u^2 - u^4\,du$

$-\frac{u^3}{3} + \frac{u^5}{5} + c$

$\left\{-\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C\right\}$

② $\int \sin^6(5x)\cos^3(5x)\,dx \qquad y = 5x, \quad \frac{dy}{5} = dx$

$\frac{1}{5}\int \sin^6(y)\cos^3(y)\,dy$

$\frac{1}{5}\int \sin^6(y)\cos^2(y)\cos(y)\,dy$

$u = \sin y$

$du = \cos(y)dx$

$dx = \frac{dy}{\cos(y)}$

$\frac{1}{5}\int \sin^6 y\,(1-\sin^2 y)\cos(y) * \frac{dy}{\cos y}$

$\frac{1}{5}\int u^6(1-u^2)\,dy$

$\boxed{\frac{1}{35}\sin^7(5x) - \frac{1}{45}\sin(5x) + c}$

③ $\int \sin^2(x)\cos^4(x)\,dx$

$= \int \frac{1}{2}(1-\cos(2x))\left(\frac{1}{2}(1+\cos(2x))\right)^2 dx$

$= \int \frac{1}{2}(1-\cos(2x)) * \frac{1}{4}(1+2\cos(2x)+\cos^2(2x))$

$= \frac{1}{8}\int 1+2\cos(2x)+\cos^2(2x)-\cos(2x)-2\cos^2(2x)-\cos^3(2x)\,dx$

$= \frac{1}{8}\int 1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\,dx$

$= \frac{1}{8}\left(x+\frac{\sin(2x)}{2}-\int\frac{1}{2}(1+\cos(4x))-\int\cos^3(2x)\,dy\right.$ $\qquad y=2x$, $dy=2dx$, $dx=\frac{dy}{2}$

$= \frac{1}{8}\left(x+\frac{1}{2}\sin(2x)-\int\frac{1}{2}(1+\cos(4x))-\frac{1}{2}\int\cos^3(y)\,dy\right.$

$= \frac{1}{8}\left(x+\frac{1}{2}\sin(2x)-\frac{1}{2}x-\frac{1}{8}\sin(4x)-\frac{1}{2}\left(\frac{1}{3}\cos^2 y\sin y+\frac{2}{3}\int\cos(y)\,dy\right.\right.$

$= \frac{1}{8}\left(x+\frac{1}{2}\sin(2x)-\frac{1}{2}x-\frac{1}{8}\sin(4x)-\frac{1}{6}\cos^2(2x)\sin(2x)-\frac{1}{3}\sin(2x)\right)+C$

$*\ \sin A\cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$ (دائما ال sin A ويكون اول)

$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$

$\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

Ex:- ① $\int\cos(4x)\sin(3x)\,dx$

$\int\frac{1}{2}(\sin(3x-4x)+\sin(3x+4x))$

$\frac{1}{2}\int -\sin x+\sin(7x)$

$\frac{1}{2}\cos(x) - \frac{1}{14}\cos x + C$

② $\int\cos(2x)\cos(4x)\,dx$

$\int\frac{1}{2}(\cos(2x-4x)+\cos(2x+4x)\,dx$

$\frac{1}{2}\int\cos(2x)+\cos(6x)\,dx$

$\frac{1}{2}\left(\frac{\sin 2x}{2}+\frac{\sin(6x)}{6}\right)+C$

① $\int \cos^5(2x)\sin^2(x)\,dx$

$y = 2x$

$dy = 2dx$

$\int \cos^5(2x) * \frac{1}{2}(1-\cos(2x))$

$\frac{1}{2}\int \cos^5(2x) - \cos^6(2x)\,dx$

$\frac{1}{4}\int \cos^5(y) - \cos^6(y)\,dy$

① $\int \cos^5(y) = \frac{1}{5}\cos^4 x \sin x + \frac{4}{5}\int \cos^3(x)$

$\frac{1}{5}\cos^4(x)\sin x + \frac{4}{5}\left(\frac{1}{3}\cos^2(x)\sin x + \frac{2}{3}\int \cos(x)\right)dx$

$\frac{1}{5}\cos^4(x)\sin x + \frac{4}{15}\cos^2(x)\sin x + \frac{8}{15}\sin(x)$

② $\int \cos^6(y) = \frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{6}\int \cos^4(x)$

$\frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{6}\left(\frac{1}{4}\cos^3(x)\sin(x) + \frac{3}{4}\int \cos^2(x)\right)$

$\frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{24}\cos^3(x)\sin(x) + \frac{10}{48}x + \frac{10}{96}\sin x + C$

$\frac{1}{4}\left(\frac{1}{5}\cos^4(x)\sin(x) + \frac{4}{15}\cos^2(x)\sin(x) + \frac{8}{15}\sin(x) + \frac{1}{6}\cos^5(x)\sin(x) + \frac{5}{24}\cos^3(x)\sin(x) + \frac{10}{48}x + \frac{10}{96}\sin x\right) + C$

② $\int x\sin(2x)\sin(3x)$

$\int x\left(\frac{1}{2}(\cos(-x) - \cos(5x)\right)$

$\frac{1}{2}\int x(\cos(x) - \cos(5x))$

$\frac{1}{2}\left(\int x\cos(x) - x\cos(5x)\right)$

x	$\cos(x)$
1	$\sin(x)$
0	$-\cos(x)$

x	$\cos(5x)$
1	$\frac{\sin(5x)}{5}$
0	$\frac{-\cos(5x)}{25}$

$\frac{1}{2}\left(x\sin(x) + \cos(x) + \frac{1}{5}x\sin(5x) + \frac{1}{25}\cos(5x)\right)$

③ $\int \cos^4(x)\sin^3(x)\,dx$

$\int \cos^4(x)\sin^2(x)\sin(x)\,dx$

$u = \cos(x)$

$du = -\sin(x)$

$dx = \frac{du}{-\sin(x)}$

$\int \cos^4(x)(1-\cos^2(x))\sin(x)$

$\int u^4(1-u^2)\sin(x)\frac{du}{-\sin(x)}$

$-\int u^4 - u^6\,du$

$-\frac{u^5}{5} + \frac{u^7}{7}\,du$

$-\frac{\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C$

4) $\int \sin^4(x)\cos^4(x)\,dx$

$$\int \sin^4(x)\,(1-\sin^2(x))^2\,dx$$

$$\int \sin^4(x)\,(1-2\sin^2(x)+\sin^4(x))\,dx$$

$$\int \sin^4(x)-2\sin^6(x)+\sin^8(x)\,dx$$

$$\overset{①}{\int \sin^4(x)} - \overset{②}{\int 2\sin^6(x)} + \overset{③}{\int \sin^8(x)}\,dx$$

① $\int \sin^4(x)\,dx$

$$=\frac{-1}{4}\sin^3(x)\cos(x)+\frac{3}{4}\int \sin^2(x)\,dx$$

$$=\frac{-1}{4}\sin^3(x)\cos(x)+\frac{3}{4}\int \frac{1}{2}(1-\cos(2x))$$

$$-\frac{1}{4}\sin^3(x)\cos(x)+\frac{3}{8}x-\frac{3}{16}\sin(2x)$$

② $2\int \sin^6(x)$

$$\frac{-1}{6}\sin^5(x)\cos(x)+\frac{5}{6}\int \sin^4(x) = \frac{-1}{6}\sin^5(x)\cos(x)+\frac{5}{6}\left(\frac{-1}{4}\sin^3(x)\cos(x)+\frac{3}{4}\int \sin^2(x)\right)dx$$

$\&$ $\frac{1}{2}(1-\cos(2x))$

$$=\frac{-1}{6}\sin^5(x)\cos(x)+\frac{-5}{24}\sin^3(x)\cos(x)+\frac{15}{48}x-\frac{15}{96}\sin(2x)$$

③ $\int \sin^8(x)$ حل على ورقة خارجية

$$\int \sin^8(x) = \frac{-1}{8}\sin^7(x)\cos(x)+\frac{7}{8}\left(\frac{-1}{6}\cos(x)\sin^5(x)+\frac{5}{6}\left(\frac{-1}{4}\sin^3(x)\cos(x)+\frac{3}{8}\left(x-\frac{1}{2}\sin(2x)\right)\right)\right)$$

$$=\frac{-1}{4}\sin^3(x)\cos(x)+\frac{3}{8}\left(x-\frac{1}{2}\sin(2x)\right)-2\left(\frac{-1}{6}\sin^5(x)\cos(x)+\frac{5}{6}\left(\frac{-1}{4}\sin^3(x)\cos(x)+\frac{3}{8}\left(x-\frac{1}{2}\sin(2x)\right)\right)\right)$$

$$+$$

$$\left(\frac{-1}{8}\sin^7(x)\cos(x)+\frac{7}{8}\left(-\frac{\cos(x)\sin^5(x)}{6}+\frac{5}{6}\left(\frac{-1}{4}\sin^3(x)\cos(x)+\frac{3}{8}\left(x-\frac{1}{2}\sin(2x)\right)\right)\right)\right)$$

$$=\frac{-1}{4}\sin^3(x)\cos(x)+\frac{3}{8}\left(x-\frac{1}{2}\sin(2x)\right)-2\left(\frac{-1}{6}\sin^5(x)\cos(x)+\frac{5}{6}\left(\frac{-1}{4}\sin^3(x)\cos(x)+\frac{3}{8}\left(x-\frac{1}{2}\sin(2x)\right)\right)\right)$$

$$-$$

$$\frac{1}{8}\cos(x)\sin^7(x)+\frac{7}{8}\left(\frac{-1}{6}\sin^5(x)\cos(x)+\frac{5}{6}\left(\frac{-1}{4}\sin^3(x)\cos(x)+\frac{3}{8}\left(x-\frac{1}{2}\sin(2x)\right)\right)\right)$$

Week 2

* Lecture 7:-

* $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

* $\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} dx$

* $\int \tan x\,dx = \int \frac{\sin x}{\cos x} = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| = \ln|\sec x| + C$

* $\int \sec x\,dx = \int \sec(x) \frac{(\sec x + \tan(x))}{\sec x + \tan x} = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan(x)} = \ln|\sec x + \tan x| + C$

* Ex:-

① $\int \tan^3(2x)\,dx = \frac{1}{2}\int \tan^3(y)\,dy = \frac{1}{2}\left(\frac{\tan^2(x)}{2} - \int \tan(x)\,dx\right)$

$y = 2x$

$dx = \frac{dy}{2}$ $= \frac{1}{2}\left(\frac{1}{2}\tan^2(x) + \ln|\cos x|\right) + C$

② $\int \sec^4(x)\,dx = \frac{\sec^2(x)\tan(x)}{3} + \frac{2}{3}\int \sec^2(x)\,dx$

$= \frac{1}{3}\sec^2(x)\tan(x) + \frac{2}{3}\tan(x) + C$

*:-

$\int \sec^m(x)\tan^n(x)$

* if $\sec(x)$ is even save $\sec^2(x)$ and But $\sec^2(x) = 1 + \tan^2 x$

Then substitute $u = \tan(x)$

* if power of tan is odd save $\sec x \tan x$ use $\tan^2(x) = \sec^2(x) - 1$

〃 〃 $u = \sec(x)$

* Ex:- ① $\int \tan^5(x)\sec(x)\,dx$

$= \int \tan^4(x)\sec x \tan x = \int (\tan^2(x))^2 \sec x \tan x\,dx$

$= \int (\sec^2 x - 1)^2 \sec x \tan x$ $u = \sec x$

$du = \sec x \tan x\,dx$

$= \int (\sec^2(x) - 1)^2 \sec x \tan x \frac{du}{\sec x \tan x}$ $dx = \frac{du}{\sec x \tan x}$

$= \int \cancel{\sec x}(u^2 - 1)^2\,du$

$= \int u^4 - 2u^2 + 1\,du = \frac{u^5}{5} - \frac{2u^3}{3} + u = \frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3}\ldots$

② $\int \tan^3(x)\sec^4(x)\,dx$:-

$= \int \tan^3(x)\sec^2(x)\sec^2(x)\,dx$

$= \int \tan^3(x)(1+\tan^2(x))\sec^2(x)$

$u = \tan x$

$du = \sec^2(x)\,dx$

$dx = \frac{du}{\sec^2(x)}$

$= \int u^3(1+u^2)\sec^2(x)\frac{du}{\sec^2 x}$

$= \int u^3 + u^5\,du$

$= \frac{u^4}{4} + \frac{u^6}{6} + c = \frac{\tan^4(x)}{4} + \frac{\tan^6(x)}{6} + c$

③ $\int \tan^4(x)\sec x\,dx$

$= \int (\tan^2 x)^2 \sec x\,dx$

$= \int (\sec^2(x)-1)^2 \sec x\,dx$

$= \int (\sec^4(x) - 2\sec^2(x) + 1)\sec x\,dx$

$= \int \overset{①}{\sec^5(x)} - 2\overset{②}{\sec^3(x)} + \overset{③}{\sec x}\,dx$

① $\int \sec^5(x)$

$= \frac{\sec^3(x)\tan(x)}{4} + \frac{3}{4}\int \sec^3(x)\,dx$

$= \frac{\sec^3(x)\tan(x)}{4} + \frac{3}{4}\left(\frac{\sec(x)\tan x}{2} + \frac{1}{2}\int \sec(x)\,dx\right.$

$= \frac{\sec^3(x)\tan(x)}{4} + \frac{3}{8}\sec(x)\tan(x) + \frac{3}{8}\ln|\sec x + \tan x|$

② $\int \sec^3(x)$

$= \frac{\sec(x)\tan x}{2} + \frac{1}{2}\int \sec(x)\,dx$

$= \frac{\sec(x)\tan(x)}{2} + \frac{1}{2}\ln|\sec x + \tan x|$

③ $\int \sec(x) = \ln|\sec x + \tan(x)$

$= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{8}\sec(x)\tan(x) + \frac{3}{8}\ln|\sec x + \tan x| - \sec(x)\tan x - \ln|\sec x + \tan x| + \ln|\sec x + \tan x| + c$

$= \frac{1}{4}\sec^3(x)\tan(x) - \frac{5}{8}\sec(x)\tan(x) + \frac{3}{8}\ln|\sec x + \tan x| + c$

* ① $\int \cot(x) = \ln|\sin x| + c$

② $\int \csc x\,dx = -\ln|\csc x + \cot x| + c$

$= \ln|\csc x - \cot x| + c$

③ $\cot^2 x = \csc^2(x) - 1$

* $\int \cot^n dx = \frac{-\cot^{n-1} x}{n-1} - \int \cot^{(n-2)} x\,dx + c$

* $\int \csc^n(x) = \frac{-\cot x \csc^{n-2}(x)}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx$

Lecture 8 :- Trigonometric substitution

$*\ \sqrt{a^2-x^2} \rightarrow x = a\sin\theta \quad -\frac{\pi}{2} \le \theta < \frac{\pi}{2}$

$\sqrt{a^2+x^2} \rightarrow x = a\tan\theta \quad -\frac{\pi}{2} \le \theta < \frac{\pi}{2}$

$\sqrt{x^2-a^2} \rightarrow x = a\sec\theta \quad \pi \le \theta < \frac{3\pi}{2}$

$\cos^2(\theta) = 1-\sin^2\theta$

$\sec^2(\theta) = 1+\tan^2\theta$

$\tan^2\theta = \sec^2(\theta)-1$

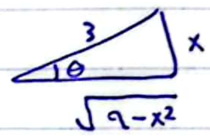
$\sin^2(\theta) = 1-\cos^2\theta$

***Ex :-** ① $\int \frac{\sqrt{9-x^2}}{x^2} = \int \frac{\sqrt{3^2-x^2}}{x^2}dx$

$x = a\sin\theta \rightarrow x = 3\sin\theta$

$dx = a\cos\theta \rightarrow dx = 3\cos\theta$

$\sin\theta = \frac{x}{3} \rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$



$= \int \frac{\sqrt{9-(3\sin\theta)^2}}{(3\sin\theta)^2} * 3\cos\theta\, d\theta$

$= \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta} 3\cos\theta\, d\theta$

$= \int \frac{\sqrt{9(1-\sin^2\theta)}}{9\sin^2\theta} 3\cos\theta\, d\theta$

$= \int \frac{\cancel{3}\sqrt{1-\sin^2\theta} * \cancel{3}\cos\theta}{\cancel{9}\sin^2\theta} = \int \frac{\sqrt{\cos^2\theta}\cos\theta}{\sin^2\theta} = \int \frac{|\cos\theta|\cos\theta}{\sin^2\theta} = \int \frac{\cos^2\theta}{\sin^2\theta} = \int \cot^2(\theta)$

$\int \cot^2\theta = \int \csc^2\theta - 1\, d\theta = -\cot\theta - \theta + C = -\cot\left(\sin^{-1}\left(\frac{x}{3}\right)\right) - \sin^{-1}\left(\frac{x}{3}\right) + C$

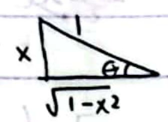
$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$

② $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} dx = \int \frac{1}{(\sqrt{1-x^2})^3} dx$

$x = 1\sin\theta$

$dx = \cos\theta\, d\theta \rightarrow \cancel{d\theta = d}$

$\theta = \sin^{-1}(x)$



$= \int \frac{1}{(\sqrt{1-\sin^2\theta})^3} \cos\theta\, d\theta$

$= \int \frac{1}{(\sqrt{\cos^2\theta})^3} \cos\theta\, d\theta$

$= \int \frac{1}{\cos^3\theta} \cos\theta\, d\theta$

$= \int \frac{1}{\cos^2\theta}\, d\theta$

$= \int \sec^2\theta\, d\theta$

$= \tan\theta + C$

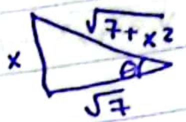
$= \tan(\sin^{-1}(x)) + C = \frac{x}{\sqrt{1-x^2}} + C$

③ $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{7+x^2}}\,dx = \int_0^{\sqrt{x}} \frac{x^2}{\sqrt{(\sqrt{7})^2+x^2}}\,dx$

$x = a\tan\theta \rightarrow x = \sqrt{7}\tan\theta$

$dx = \sqrt{7}\sec^2\theta$

$\theta = \tan^{-1}\left(\frac{x}{\sqrt{7}}\right)$



$\int \frac{(\sqrt{7}\tan^2\theta)^2}{\sqrt{\sqrt{7}^2+(\sqrt{7}\tan\theta)^2}}\ \sqrt{7}\sec^2\theta\,d\theta$

$= \int \frac{7\tan^2\theta\,\sqrt{7}\sec^2\theta}{\sqrt{7+7\tan^2\theta}}$

$= \int \frac{7\sqrt{7}\tan^2\theta\sec^2\theta}{\sqrt{7}\sqrt{1+\tan^2\theta}}\,d\theta$

$= \int 7\,\frac{\tan^2\theta\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta = \int \frac{7\tan^2\theta\sec^2\theta}{\sec\theta}\,d\theta = \int 7\tan^2\theta\sec\theta\,d\theta$

$= \int 7(\sec^2\theta - 1)(\sec\theta)$

$= \int \overset{①}{7\sec^3\theta} - \overset{②}{7\sec\theta}\,d\theta$

① $\int 7\sec^3\theta = 7\int \sec^3\theta\,d\theta$

② $7\int \sec\theta$

$7\left(\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)$

$7\ln|\sec\theta+\tan\theta|$

$\frac{7}{2}\sec\theta\tan\theta + \frac{7}{2}\ln|\sec\theta+\tan\theta| - 7\ln|\sec\theta+\tan\theta|$

$\frac{7}{2}\sec\theta\tan\theta - \frac{7}{2}\ln|\sec\theta+\tan\theta|$

$\cancel{\frac{7}{2}\sec}\ \frac{7}{2}\,\frac{\sqrt{7+x^2}}{\sqrt{7}} * \frac{x}{\sqrt{7}} - \frac{7}{2}\ln\left|\frac{\sqrt{7+x^2}}{\sqrt{7}} + \frac{x}{\sqrt{7}}\right|\Bigg]_0^{\sqrt{2}}$

$\frac{7}{2}\left(\frac{\sqrt{7+x^2}}{\sqrt{7}} * \frac{x}{\sqrt{7}} - \ln\left|\frac{\sqrt{7+x^2}}{\sqrt{7}} * \frac{x}{\sqrt{7}}\right|\right)\Bigg]_0^{\sqrt{2}}$

$\frac{7}{2}\left(\left(\frac{\sqrt{7+2}}{\sqrt{7}} * \frac{\sqrt{2}}{\sqrt{7}} - \ln\left|\frac{\sqrt{7+2}}{\sqrt{7}} + \frac{\sqrt{2}}{\sqrt{7}}\right|\right) - \left(\frac{\sqrt{7}}{\sqrt{7}} * 0 - \ln\left|\frac{\sqrt{7}}{\sqrt{7}} + 0\right|\right)\right)$

$\frac{7}{2}\left(\frac{\sqrt{9}}{\sqrt{7}} * \frac{\sqrt{2}}{\sqrt{7}} - \ln\left|\frac{\sqrt{9}}{\sqrt{7}} + \frac{\sqrt{2}}{\sqrt{7}}\right| - 0\right)$

$\frac{7}{2}\left(\frac{3\sqrt{2}}{7} - \ln\left|\frac{3+\sqrt{2}}{\sqrt{7}}\right|\right)$

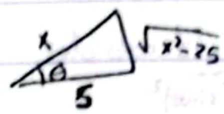
Lecture 9:-

Ex:- ① $\int \frac{\sqrt{x^2-25}}{x}\,dx$:

$x = a\sec\theta \rightarrow x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$\theta = \sec^{-1}\left(\frac{x}{5}\right)$



$\int \frac{\sqrt{(5\sec\theta)^2-5^2}}{5\sec\theta}\,5\sec\theta\tan\theta\,d\theta$

$\int \frac{\sqrt{25\sec^2\theta-25}}{1}\tan\theta\,d\theta$

$\int 5\sqrt{\sec^2\theta-1}\,\tan\theta\,d\theta$

$\int 5\sqrt{\tan^2\theta}\,\tan\theta\,d\theta$

$\int 5\tan^2\theta\,d\theta$

$5\int(\sec^2\theta-1)\,d\theta$

$5(\tan\theta-\theta)+C = 5\left(\frac{\sqrt{x^2-25}}{5} - \sec^{-1}\left(\frac{x}{5}\right)\right)+C = \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right)+C$

② $\int e^x\sqrt{1-e^{2x}}\,dx = \int e^x\sqrt{1-(e^x)^2}$

$y = e^x$

$dy = e^x dx$

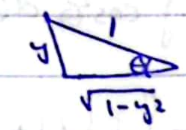
$dx = \frac{dy}{e^x} = \frac{dy}{y}$

$y = a\sin\theta$

$= \sin\theta$

$dy = \cos\theta\,d\theta$

$\theta = \sin^{-1}(y)$



$= \int y\sqrt{1-y^2}\,\frac{dy}{y} = \int\sqrt{1-y^2}\,dy$

$= \int\sqrt{1-\sin^2\theta}\,d\theta\cos\theta$

$= \int\sqrt{\cos^2\theta}\,\cos\theta\,d\theta$

$= \int\cos^2\theta\,d\theta$

$= \int\frac{1}{2}(1+\cos(2\theta))\,d\theta$

$= \frac{1}{2}\left(\theta+\frac{\sin(2\theta)}{2}\right)+C$

$= \frac{1}{2}\left(\theta+\frac{1}{2}\,2\sin\theta\cos\theta\right)+C$

$= \frac{1}{2}(\theta+\sin\theta\cos\theta)$

$= \frac{1}{2}\left(\sin^{-1}(y)+y\sqrt{1-y^2}\right)+C$

$= \frac{1}{2}\sin^{-1}(y)+\frac{1}{2}y\sqrt{1-y^2}+C = \frac{1}{2}\sin^{-1}(e^x)+\frac{1}{2}e^x\sqrt{1-e^{x^2}}+C$

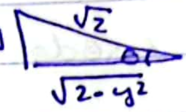
3 $\int \frac{\cos t}{\sqrt{2-\sin^2 t}}\,dt =$

$\cancel{\int \frac{\cos t}{2 - \sin}}$

$y = \sin t$
$dy = \cos t\, dt$
$dt = \frac{dy}{\cos t}$

$\int \frac{\cos t}{\sqrt{2-y^2}} \frac{dy}{\cos t} = \int \frac{dy}{\sqrt{2-y^2}}$

$y = \sqrt{2}\sin\theta$
$dy = \sqrt{2}\cos\theta\, d\theta$
$\sin\theta = \frac{y}{\sqrt{2}} \rightarrow \theta = \sin^{-1}\left(\frac{y}{\sqrt{2}}\right)$



$\int \frac{1}{\sqrt{2-(\sqrt{2}\sin\theta)^2}} \sqrt{2}\cos\theta\, d\theta$

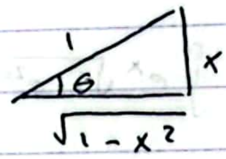
$\int \frac{1}{\sqrt{2-2\sin^2\theta}} \sqrt{2}\cos\theta\, d\theta$

$\int \frac{1}{\sqrt{2}\sqrt{1-\sin^2\theta}} \sqrt{2}\cos\theta\, d\theta = \int \frac{1}{\sqrt{2}\cos\theta}\sqrt{2}\cos\theta\, d\theta = \frac{1}{\cancel{\sqrt{2}}}\theta + C$

$= \sin^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$ $\quad$ $= \sin^{-1}\left(\frac{\sin t}{\sqrt{2}}\right) + C$

* Lecture 10 :-

Ex:- ① $\int \cos(\sin^{-1}(x))\,dx$



$= \int \sqrt{1-x^2}\,dx$

$x = \sin\theta$
$dx = \cos\theta\, d\theta$
$\theta = \sin^{-1}(x)$

$= \int \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$

$= \int \cos^2\theta\, d\theta$

$= \int \left(\frac{1}{2}(1+\cos(2\theta)\right)d\theta$

$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$

$= \frac{1}{2}\sin^{-1}(x) + \frac{1}{4}\, 2\sin\theta\cos\theta + C$

$= \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C$

② $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-2x)}} = \int \frac{dx}{\sqrt{-(x^2-2x+1-1)}}$

نحتاج اكمال التربيع

$(\frac{1}{2}\times\text{المعامل})^2 = (\frac{1}{2}(2))^2$

$= 1$

$= \int \frac{dx}{\sqrt{-(x-1)(x-1)-1}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$ $\quad y = x-1$, $dy = dx$

$= \int \frac{1}{\sqrt{1-y^2}}\,dy$ $\quad y = \sin\theta$, $dy = \cos\theta\,d\theta$

$= \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$ $\quad \theta = \sin^{-1}(y)$

$= \int 1\,d\theta = \theta + c = \sin^{-1}(y) + c = \sin^{-1}(x-1) + c$

③ The most proper trigonometric substitution to solve

$\int \frac{x}{x^2+2x+2}\,dx$ is :

a) $x = \tan\theta$ $\quad$ c) $x = -1+\tan\theta$

b) $x = \tan\theta + 1$ $\quad$ d) $x = 2+\tan\theta$

$\int \frac{x}{x^2+2x+1-1+2}\,dx$ → اكمال مربع

$\int \frac{x}{(x+1)^2+1}\,dx$ $\quad y = x+1$, $dy = dx$

$\int \frac{y-1}{y^2+1}\,dy$

→ $y = a\tan\theta$

$= \tan\theta$

$y = x+1 = \tan\theta$

$x = \tan\theta - 1$

Lecture 11 :- Integration of rational function by partial fractions

تكامل الإقترانات النسبية

$\int \frac{f}{g}$ where $\frac{f}{g}$ rational function

[1] degree f < degree g

[2] degree f ≥ degree g

[1] degree f < degree g

Ex:- ① $\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x)+c$

② $\int \frac{3}{x+1}\,dx = 3\ln|x+1|+c$

H.W :-

① $\int \frac{1}{x^2-4x+4}\,dx = \int \frac{1}{(x-2)^2}$ $y = x-2$, $dy = dx$

$= \int \frac{1}{y^2}\,dy = \frac{-y^{-1}}{1}+c$

$= \frac{-1}{(x-2)}+c$

② $\int \frac{4x-2}{x^2-x-2}dx = \frac{1}{2}\ln|x^2-2-2|+c$

③ $\int \frac{3x+1}{3x^2+2x-1}\,dx = \frac{1}{2}\ln|3x^2+2x-1|$

partial fractions :-

① $\frac{f}{g} = \frac{f}{(ax+b)(cx+d)(ex+f)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{c}{ex+f}$

[2] $\frac{f}{(ax+b)^2(cx+d)} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(cx+d)}$

[3] $\frac{f}{(a_1x^2+b_1x+C_1)(a_2x^2+b_2x+C_2)} = \frac{Ax+B}{a_1x^2+b_1x+C_1} + \frac{Cx+D}{a_2x^2+b_2x+C_2}$

تربيعية لا تحلل

[4] $\frac{f}{(a_1x^2+b_1x+C_1)^2(a_2x^2+b_2x+C_2)}$

$= \frac{Ax+B}{(a_1x^2+b_1x+C_1)} + \frac{Cx+D}{(a_1x^2+b_1x+C_1)^2} + \frac{ex+F}{a_2x^2+b_2x+C_2}$

Ex:- ① write out the form of the partial fraction decomposition

① $\frac{7x+1}{x^3-x^2} = \frac{7x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

② $\frac{x^3-3x}{(x^2-2)^2} = \frac{x^3-3x}{(x-\sqrt{2})^2(x+\sqrt{2})^2} = \frac{A}{x-\sqrt{2}} + \frac{B}{(x-\sqrt{2})^2} + \frac{C}{x+\sqrt{2}} + \frac{D}{(x+\sqrt{2})^2}$

③ $\frac{5}{(x^2+5)^3(x-1)^3x^2} = \frac{Ax+B}{x^2+5} + \frac{Cx+D}{(x^2+5)^2} + \frac{Ex+F}{(x^2+5)^3} + \frac{G}{x-1} + \frac{H}{(x-1)^2} + \frac{J}{(x-1)^3} + \frac{I}{x} + \frac{K}{x^2}$

④ $\dfrac{3x^2+x+4}{x^4+3x^2+2} = \dfrac{3x^2+x+4}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{(x^2+1)} + \dfrac{Cx+D}{x^2+2}$

⑤ $\dfrac{2x}{(x^3-1)(x^2-1)} = \dfrac{2x}{(x-1)(x^2+x+1)(x-1)(x+1)} = \dfrac{2x}{(x-1)^2(x+1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1} + \dfrac{Dx+E}{x^2+x+1}$

لو ٥ تحلل

H.w :-

* $\dfrac{4x+1}{x^3-x^2-x+1} = \dfrac{4x+1}{x^2(x-1)-(x-1)} = \dfrac{4x+1}{(x-1)(x^2-1)} = \dfrac{4x+1}{(x-1)(x-1)(x+1)} = \dfrac{4x+1}{(x-1)^2(x+1)}$

$= \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$

* $\dfrac{1-x^2}{x^3(x^2+2)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{Dx+E}{x^2+2} + \dfrac{Fx+G}{(x^2+2)^2}$

Lecture 12 :-

① $\displaystyle\int \frac{3x+8}{x^3+5x^2+6x}\,dx = \int \frac{3x+8}{x(x^2+5x+6)}\,dx = \int \frac{3x+8}{x(x+2)(x+3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+3}$

$3x+8 = A(x+2)(x+3) + B(x)(x+3) + C(x)(x+2)$

$x=0$ $\quad 8 = 6A + 0 + 0 \rightarrow A = \frac{4}{3}$

$x=-2$ $\quad 2 = 0 - 2B + 0 \rightarrow B = -1$

$x=-3$ $\quad -1 = 0 + 0 + 3C \rightarrow C = \frac{-1}{3}$

$\displaystyle\int \frac{3x+8}{x^3+5x^2+6x}\,dx = \int \frac{\frac{4}{3}}{x} + \frac{-1}{x+2} + \frac{\frac{-1}{3}}{x+3} = \frac{4}{3}\ln|x| - \ln|x+2| - \frac{1}{3}\ln|x+3| + C$

② $\displaystyle\int_2^4 \frac{dx}{x^2(x^2-1)} = \int_2^4 \frac{1}{x^2(x-1)(x+1)} = \int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}\,dx$

$1 = A(x)(x-1)(x+1) + B(x-1)(x+1) + C(x^2)(x+1) + D(x-1)(x^2)$

$x=0$ $\quad 1 = 0 + -B + 0 + 0 \rightarrow B = -1$

$x=1$ $\quad 1 = 0 + 0 + 2C + 0 \rightarrow C = \frac{1}{2}$

$x=-1$ $\quad 1 = 0 + 0 + 0 + -2D \rightarrow D = \frac{-1}{2}$

$x=2$ $\quad 1 = 6A + 3B + 12C + 4D$

$1 = 6A + -3 + 6 + -2$

$1 = 6A + 1 \rightarrow 6A = 0 \rightarrow \cancel{A = \frac{0}{3}}\ A = 0$

$\displaystyle\int \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}\,dx$

$\displaystyle= \int_2^4 \frac{0}{x} + \frac{-1}{x^2} + \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\,dx$

$\displaystyle= \int_2^4 -x^{-2} + \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}\,dx$

$= \frac{1}{x} + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| \Big]_2^4$

$= \frac{1}{4} + \frac{1}{2}\ln 3 - \frac{1}{2}\ln 5 - \left(\frac{1}{2} + \frac{1}{2}\ln 1 - \frac{1}{2}\ln 3\right)$

$= \frac{-1}{2}\ln 3 - \ln\sqrt{5}$

Home works week 2

Lecture 7

① $\int \sec^3(x)\,dx$

$\frac{\sec^1(x)\tan(x)}{2} + \frac{1}{2}\int \sec(x)\,dx$

$\boxed{\frac{\sec(x)\tan(x)}{2} + \frac{1}{2}\ln|\sec(x)+\tan(x)| + C}$

② $\int \sec 7(x)\,dx$

$y = 7x$

$dy = 7dx$

$dx = \frac{dy}{7}$

$7\int \sec(y)\,dy$

$7\ln|\sec(y)+\tan(y)| + C$

$\boxed{7\ln|\sec(7x)+\tan(7x)| + C}$

③ $\int \tan^3 x \sec^3(x)\,dx$

$\int \tan^2(x)\sec^2(x)\sec x\tan x\,dx$

$\int (\sec^2 - 1)\sec^2(x)\sec(x)\tan(x)$

$u = \sec x$

$du = \sec x\tan x\,dx$

$\int (u^2-1)u^2 \sec x\tan x \frac{dx}{\sec x\tan x}$

$\int u^3 - u^2\,du$

$\frac{u^4}{4} - \frac{u^3}{3} + C = \boxed{\frac{\sec^4(x)}{4} - \frac{\sec^3(x)}{3} + C}$

④ $\int \tan^2(x)\sec^4(x)$

$\int \tan^2(x)\sec^2(x)\sec^2(x)\,dx$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$\int u(1+u^2)\sec^2(x)\frac{dx}{\sec^2 x}$

$\int u + u^3$

$\frac{u^2}{2} + \frac{u^4}{4} + C$

$\boxed{\frac{\tan^2(x)}{2} + \frac{\tan^4(x)}{4} + C}$

⑤ $\int \cot^n x\,dx$

$\frac{-\cot^{n-1}(x)}{n-1} - \int \cot^{n-2}(x)\,dx + C$

⑥ $\int \csc^n x\,dx$

$\frac{-\cot(x)\csc(x)^{n-2}}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2}(x)\,dx$

⑦ $\int \csc^4(x)\cot^6(x)\,dx$

$\int \csc^2(x)\csc^2(x)\cot^6(x)\,dx$

$\int \csc^2(x)(1+\cot^2(x))\cot^6(x)$

$u = \cot(x)$ $\qquad dx = \frac{du}{-\csc^2(x)}$

$du = -\csc^2(x)dx$

$\int \csc^2(x)(1+u^2)u^6 \frac{du}{-\csc^2(x)}$

$-\frac{u^7}{7} - \frac{u^9}{9} + C$

$\boxed{-\frac{\cot^7(x)}{7} - \frac{\cot^9(x)}{9} + C}$

⑧ $\int_{\pi/4}^{\pi/2} \cot^3 x\,dx$

$\frac{-\cot^2(x)}{2} - \int \cot(x)\,dx$

$\left(\frac{-\cot^2(x)}{2} - \ln|\sin x|\right]_{\pi/4}^{\pi/2}$

$\frac{-}{2}0 - \ln|1| - \left(-\frac{1}{2}1 - \ln|\right) = \frac{1}{2}$

⑨ $\int_{\pi/6}^{\pi/3} \csc^3(x)\,dx$

$\frac{-\cot(x)\csc(x)}{2} + \frac{1}{2}\int \csc(x)\,dx$

$\frac{-\cot(x)\csc(x)}{2} + \frac{1}{2}\ln|\csc(x)+\cot(x)|\Big]_{\pi/6}^{\pi/3}$

$-\frac{1}{2}\frac{\sqrt{3}}{3}\frac{2\sqrt{3}}{3} - \frac{1}{2}\ln\left|\frac{\sqrt{3}\,2}{3} + \frac{\sqrt{3}}{3}\right| - \left(-\frac{1}{2}\sqrt{3}\,2 - \frac{1}{2}\ln|2+\sqrt{3}\right.$

$\boxed{\frac{-1}{3} - \frac{1}{2}\ln|\sqrt{3}| + \sqrt{3} + \frac{1}{2}\ln|2+\sqrt{3}|}$

Lecture 10

① $\int \sqrt{1-4x^2}\,dx = \int \sqrt{4(\frac{1}{4}-x^2)}\,dx$

$= \int 2\sqrt{(\frac{1}{2})^2 - x^2}\,dx$ $\quad x = \frac{1}{2}\sin\theta$

$= \int 2\sqrt{\frac{1}{4} - (\frac{1}{2}\sin\theta)^2}\,\frac{1}{2}\cos\theta\,d\theta$ $\quad dx = \frac{1}{2}\cos\theta\,d\theta$

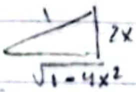
$\theta = \sin^{-1}(2x)$

$= \int 2\sqrt{\frac{1}{4}(1-\sin^2\theta)} * \frac{1}{2}\cos\theta\,d\theta$

$= \int 2\,\frac{1}{2}\cos\theta * \frac{1}{2}\cos\theta\,d\theta$

$= \frac{1}{2}\int \cos^2\theta = \frac{1}{2}\int \frac{1}{2}(1+\cos(2\theta))\,d\theta$

$= \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + c$



$\frac{1}{4}\theta + \frac{1}{4}\sin\theta\cos\theta + c$

$\frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2}$

② $\int \sqrt{3-(2x+5)^2}\,dx$ $\quad y = 2x+5$, $dy = 2dx$

$\int \sqrt{3-y^2}\,\frac{dy}{2}$ $\quad y = \sqrt{3}\sin\theta$

$\frac{1}{2}\int \sqrt{3-y^2}$ $\quad dy = \sqrt{3}\cos\theta\,d\theta$

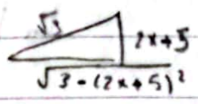
$\theta = \sin^{-1}(\frac{y}{\sqrt{3}})$

$\frac{1}{2}\int \sqrt{3-(\sqrt{3}\sin\theta)^2}\,\sqrt{3}\cos\theta\,d\theta$

$\frac{1}{2}\int \sqrt{3(1-\sin^2\theta)}\,\sqrt{3}\cos\theta\,d\theta$

$\frac{3}{2}\int \cos^2\theta\,d\theta = \frac{3}{2}\int \frac{1}{2}(1+\cos(2\theta))\,d\theta$

$\frac{3}{4}\theta + \frac{3}{8}\sin(2\theta) + c$



$\frac{3}{4}\sin^{-1}(\frac{2x+5}{\sqrt{3}}) + \frac{3}{4}\sin\theta\cos\theta$

$\frac{3}{4}\sin^{-1}\left(\frac{2x+5}{\sqrt{3}}\right) + \frac{3}{4}\left(\frac{2x+5}{\sqrt{3}}\right)\left(\sqrt{3-(2x+5)^2}\right) + c$

$\frac{\tan^2(\theta)}{2} - \int \tan\theta\,d\theta$

$\frac{\tan^2\theta}{2} + \ln|\cos\theta|$

$\left(\frac{4x^2}{18}\right) + \ln\left|\frac{3}{\sqrt{4x^2+9}}\right|$

③ $\int \sec^{-1}(x)\,dx$

$u = \sec^{-1}(x)$ $\quad dv = dx$

$du = \frac{1}{x\sqrt{x^2-1}}$ $\quad v = x$

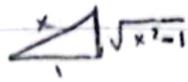
$-\int$

$x\sec^{-1}(x) - \int x\,\frac{1}{x\sqrt{x^2-1}}\,dx$

$\sec^{-1}(x) - \int \frac{1}{\sqrt{x^2-1}}\,dx$

$\int \frac{1}{\sqrt{x^2-1}}\,dx$ $\quad x = \sec(\theta)$, $dx = \sec\theta\tan\theta\,d\theta$, $\theta = \sec^{-1}(x)$

$\int \frac{1}{\sqrt{\sec^2 x - 1}}\sec\theta\tan\theta\,d\theta$



$\int \frac{1}{\tan\theta}\sec\theta\tan\theta\,d\theta$

$\int \sec\theta\,d\theta$

$\ln|\sec\theta + \tan\theta|$

$\sec^{-1}(x) - \ln|x + \sqrt{x^2-1}| + c$

④ $\int \frac{dx}{x^2-4x+5}$ $\quad$ اكمال مربع

$\left(\frac{-4}{2}\right)^2 = 4$

$\int \frac{dx}{x^2-4x+4-4+5}\,dx$

$\int \frac{dx}{(x-2)^2+1}$ $\quad y = x-2$, $dy = dx$

$\int \frac{dx}{y^2+1} = \tan^{-1}(y) + c$

$= \tan^{-1}(x-2) + c$

⑤ $\int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\,dx = \frac{3}{32}$ show that:-

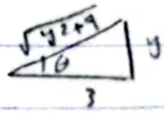
$\int \frac{x^3}{\sqrt{9+(2x)^2}}\,dx$ $\quad y = 2x \rightarrow x = \frac{y}{2}$, $dy = 2dx$, $dx = \frac{dy}{2}$

$\frac{1}{16}\int \frac{y^3}{(\sqrt{9+y^2})^3}\,dy$

$y = 3\tan\theta$, $dy = 3\sec^2\theta\,d\theta$

$\frac{1}{16}\int \frac{(3\tan\theta)^3}{(\sqrt{9+9\tan^2\theta})^3}\,3\sec^2\theta\,d\theta$ $\quad \theta = \tan^{-1}(\frac{y}{3})$

$\frac{1}{16}\int \frac{27\tan^3\theta}{\cancel{3}\sqrt{1+\tan^2\theta}}\,\cancel{3}\sec^2\theta$



$\frac{27}{16}\int \frac{\tan^3}{\sec^2\theta}\sec^2\theta\,d\theta$

$\frac{27}{16}\int \tan^3\theta$

⑥ $\int \sqrt{x(4-x)}$

$\int \sqrt{4x-x^2} = \sqrt{-(-4x+x^2)} = \sqrt{-(x^2-4x+4-4)}$

$= \sqrt{-((x-2)^2-4)} = \sqrt{4-(x-2)^2}$ $\quad y = x-2$

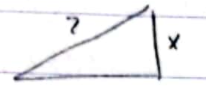
$= \int \sqrt{4-y^2}$ $\quad dy = dx$

$x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$= \int \sqrt{4-4\sin^2\theta}\, d\theta$ $\quad \theta = \sin^{-1}(\frac{x}{2})$

$= \int 2\sqrt{1-\sin^2\theta}\, d\theta$



$= \int 2\cos\theta \, d\theta$

$= 2\sin\theta + c = 2\sin(\frac{x}{2}) + c$

⑦ show that $\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$

$\int \frac{1}{a^2+x^2}$ $\quad x = a\tan\theta$

$dx = a\sec^2\theta \, d\theta$

$\int \frac{1}{a^2+(a\tan\theta)^2} \sec^2\theta \, d\theta$ $\quad \theta = \tan^{-1}(\frac{x}{a})$

$\int \frac{1}{a(1+\tan^2\theta)} \sec^2\theta \, d\theta$

$\int \frac{1}{a} \frac{\sec^2\theta}{\sec^2\theta} = \int \frac{1}{a}\, d\theta$

$= \frac{1}{a}\theta + c$

$= \frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$

⑧ show that $\int \frac{1}{\sqrt{a^2+x^2}} dx = \ln(x+\sqrt{a^2+x^2}) + c$

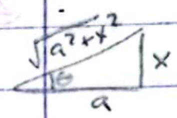
$\int \frac{1}{\sqrt{a^2+x^2}} dx$ $\quad x = a\tan\theta.$

$dx = a\sec^2\theta \, d\theta$

$\theta = \tan^{-1}(\frac{x}{a})$

$\int \frac{1}{\sqrt{a^2+a^2\tan^2\theta}} a\sec^2\theta \, d\theta$

$\int \frac{1}{a\sqrt{1+\tan^2\theta}} a\sec^2\theta$



$\int \frac{1}{\sec\theta} \sec^2\theta$

$\int \sec\theta \, d\theta$

$\ln|\sec\theta + \tan\theta| + c$

$\ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + c$

$\ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right| + c$

$\ln\left|\sqrt{a^2+x^2}+x\right| \underbrace{-\ln|a| + c}_{C_1}$

Lecture 13

Week 3

① $\int \frac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)}\,dx$

$= \int \frac{Ax+b}{x^2+1} + \frac{cx+D}{x^2+3}\,dx$

$x^3+3x^2+x+9 = (Ax+b)(x^2+3)+(cx+D)(x^2+1)$

$x^3+3x^2+x+9 = Ax^3+3Ax+bx^2+3b+cx^3+cx+Dx^2+D$

$x^3+3x^2+x+9 = (A+C)x^3+(B+D)x^2+(3A+C)x+(3B+D)$

$A+C=1$ --- ① ①-③ $\rightarrow -2A=0 \rightarrow A=0$

$B+D=3$ --- ② ① $\rightarrow A+C=1 \rightarrow C=1$

$3A+C=1$ --- ③ ②-④ $\rightarrow -2B=-6 \rightarrow B=3$

$3B+D=9$ --- ④ ② $\rightarrow B+D=3 \rightarrow D=0$

$\int \frac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)} = \int \frac{3}{x^2+1} + \int \frac{x}{x^2+3}$

$3\tan^{-1}(x) + \frac{1}{2}\ln|x^2+3| + c$

② $\int \frac{e^x}{(e^x-2)(e^{2x}+1)}\,dx$ $y=e^x$

$dy = e^x dx$

$dx = \frac{dy}{e^x}$

$\int \frac{e^x}{(y-2)(y^2+1)}\frac{dy}{e^x}$

$\int \frac{1}{(y-2)(y^2+1)}dy = \left(\int \frac{A}{y-2} + \int \frac{By+c}{y^2+1}\right)dy$

$1 = A(y^2+1)+(By+c)(y-2)$

$1 = Ay^2+A+By^2-2By+cy-2c$

$1 = (A+B)y^2+(c-2B)y+(A-2c)$

① $A+B=0$ ---> $A=-B$

② $c-2B=0$

③ $A-2c=1$ ---> $-B-2c=1$

$2*$② $-$ ③

$2c-4B=0$

$-B-2c=1$

$-5B=+1$ $B=\frac{-1}{5}$

$A=\frac{1}{5}$

$c+\frac{2}{5}=0$ $c=\frac{-2}{5}$

$\int \frac{1}{(y-2)(y^2+1)} = \int \frac{\frac{1}{5}}{y-2} + \frac{\frac{-1}{5}y-\frac{2}{5}}{y^2+1}\,dy$

$= \int \frac{\frac{1}{5}}{y-2} - \frac{1}{5}\int \frac{y}{y^2+1} + \frac{-2}{5}\int \frac{1}{y^2+1}$

$= \frac{1}{5}\ln|y-2| - \frac{1}{10}\ln|y^2+1| - \frac{2}{5}\tan^{-1}(x) + c$

$= \frac{1}{5}\ln|e^x-2| - \frac{1}{10}\ln|e^{2x}+1| - \frac{2}{5}\tan^{-1}(e^x) + c$

Lecture 14

[2] $\int \frac{f}{g}$, f degree $\geq$ degree g (قسمة طويلة)

$$g \overline{\smash{)}\,f} \quad \text{quotient } q,\ \text{remainder } r \quad \rightarrow \quad \boxed{\frac{f}{g} = q + \frac{r}{g}}$$

① $\int \frac{x^2-7}{x+3}\,dx$

$$\begin{array}{r} x-3 \\ x+3 \overline{\smash{)}\,x^2-7} \\ \underline{x^2+3x} \\ -3x-7 \\ \underline{-3x-9} \\ 2 \end{array}$$

$$\int x-3+\frac{2}{x+3}$$

$$\boxed{\frac{x^2}{2}-3x+2\ln|x+3|+c}$$

② $\int \frac{x^4+7x^3+12x^2+11x}{x^2+2x+2}\,dx$

$$\begin{array}{r} x^2+5x \\ x^2+2x+2 \overline{\smash{)}\,x^4+7x^3+12x^2+11x} \\ \underline{x^4+2x^3+2x^2} \\ 5x^3+10x^2+11x \\ \underline{5x^3+10x^2+10x} \\ x \end{array}$$

$$\int x^2+5x+\frac{x}{x^2+2x+2}\,dx$$

$$\int x^2+\int 5x+\int\frac{x}{x^2+2x+2}$$

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ليس كسور جزيئية

$$\int x^2+5x+\int\frac{x}{x^2+2x+1-1+2}$$

$$\int x^2+5x+\int\frac{x}{(x+1)^2+1}$$

$y = x+1$
$dy = dx$
$x = y-1$

$$\int x^2+5x+\int\frac{y-1}{y^2+1}$$

$$\int x^2+5x+\int\frac{y}{y^2+1}-\int\frac{1}{y^2+1}$$

$$\frac{x^3}{3}+5x^2+\frac{1}{2}\ln|y^2+1|-\tan^{-1}(y)+c$$

$$\boxed{\frac{x^3}{3}+5x^2+\frac{1}{2}\ln|(x+1)^2+1|-\tan^{-1}(x+1)+c}$$

① $\int \frac{7\cos\theta}{\sin^2\theta + 5\sin\theta - 6}\, d\theta$

$y = \sin\theta$

$dy = \cos\theta\, dx \rightarrow dx = \frac{dy}{\cos\theta}$

$\int \frac{7\cos\theta}{y^2+5y-6} \frac{dy}{\cos\theta}$

$\int \frac{7}{y^2+5y-6}\, dy$

$\int \frac{7}{(y+6)(y-1)}\, dy = \frac{A}{y+6} + \frac{B}{y-1}$

$7 = A(y-1) + B(y+6)$

$y=1 \quad 7 = 7B \rightarrow B=1$

$y=-6 \quad 7 = -7A + 0 \rightarrow A=-1$

$\int \frac{-1}{y+6} + \frac{1}{y-1}\, dy$

$-\ln|y+6| + \ln|y-1| + C$

$-\ln|\sin\theta+6| + \ln|\sin\theta - 1| + C$

② $\int \frac{5+2\ln x}{x(1+\ln x)^2}$

$y = \ln x$

$dy = \frac{1}{x}dx \rightarrow dx = x\,dy$

$\int \frac{5+2y}{x(1+y)^2}\, x\,dy$

$\int \frac{5+2y}{(1+y)^2}\, dy = \int \frac{5+2y}{y^2+2y+1}$

$= \int \frac{A}{y+1} + \frac{B}{(y+1)^2}\, dy$

$5+2y = A(y+1) + B$

$y=-1 \quad 5-2 = B \quad B=3$

$y=0 \quad 5 = A+3 \rightarrow A=2$

$\int \frac{2}{y+1} + \frac{3}{(y+1)^2}$

$2\ln|y+1| + -3\frac{1}{y+1} + C$

$2\ln|\ln x+1| - \frac{3}{\ln x+1} + C$

③ $\int (\sin^{-1})^2 dx$

$u = (\sin^{-1}(x))^2 \quad dv = dx$

$du = 2(\sin^{-1}(x)) * \frac{1}{\sqrt{1-x^2}}dx \quad v = x$

$= x\sin^{-1}(x) - 2\int \frac{x}{\sqrt{1-x^2}} \sin^{-1}(x)\, dx$

$u = \sin^{-1}(x) \quad dv = \frac{x}{\sqrt{1-x^2}}$

$du = \frac{1}{\sqrt{1-x^2}} \quad v = -\sqrt{1-x^2}$ غلط

$= x\sin^{-1}(x) - 2\left(\sin^{-1}(x)\sqrt{1-x^2} + \int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\, dx\right)$

$= x\sin^{-1}(x) - 2(\sin^{-1}(x)\sqrt{1-x^2} + x) + C$

④ $\int \frac{1}{\sqrt{\sqrt{x}+1}}\, dx$

Let $y = \sqrt{\sqrt{x}+1} \quad y^2 = \sqrt{x}+1$

$y^2 - 1 = \sqrt{x} \quad 2y\,dy = \frac{1}{2\sqrt{x}}dx$

$dx = 4\sqrt{x}\,y\,dy$

$\int \frac{4\sqrt{x}\,y\,dy}{y}$

$\int \frac{4(y^2-1)y\,dy}{y} = \int 4(y^2-1)\,dy$

$4\left(\frac{y^3}{3} - y\right) + C$

$4\left(\frac{(\sqrt{\sqrt{x}+1})^3}{3} - \sqrt{\sqrt{x}+1}\right) + C$

5 $\int \frac{\tan^3 x}{\cos^3(x)}\,dx$

$= \int \frac{1}{\cos^3(x)} \tan^3(x)\,dx = \int \sec^3(x)\tan^3(x)\,dx$

$\cancel{u = \tan x}$

$\cancel{du = \sec^2 x\,dx}$

$\cancel{dx = \frac{du}{\sec^2(x)}}$

$\cancel{\int \sec^3(x)\, u^3 \frac{du}{\sec^2(x)}}$

$u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$dx = \frac{du}{\sec(x)\tan(x)}$

$\int \frac{\cancel{u^3}\ \sec^3(x)\tan^3(x)}{\sec(x)\tan(x)}\,du$

$\int \sec^2(x)\tan^2(x)\,du$

$= \int u^2(\sec^2(x)-1)\,du$

$\int u^2(u^2-1)\,du$

$\int u^4 - u^2\,du$

$\frac{u^5}{5} - \frac{u^3}{3} + C$

$\frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C$

Lecture 16

① $\int \frac{xe^x}{(1+x)^2}\,dx$

$u = xe^x \qquad dv = \frac{1}{(1+x)^2}$

$du = xe^x + e^x \qquad v = -\frac{1}{1+x}$

$= -\frac{xe^x}{1+x} + \int \frac{xe^x + e^x}{1+x}$

$= \frac{-xe^x}{1+x} + \int e^x\,dx$

$\boxed{= \frac{-xe^x}{1+x} + e^x + c}$

② $\int \ln(x+\sqrt{x^2+1})\,dx$

$u = \ln(x+\sqrt{x^2+1}) \qquad dv = dx$

$du = \frac{1 + \frac{2x}{2\sqrt{x^2+1}}}{(x+\sqrt{x^2+1})} \qquad v = x$

$= \frac{\sqrt{x^2+1} + x}{(x+\sqrt{x^2+1})(\sqrt{x^2+1})}$

$= \frac{1}{\sqrt{x^2+1}}$

$= x\ln(x+\sqrt{x^2+1}) - \int \frac{x}{\sqrt{x^2+1}}\,dx$

$\int \frac{x}{\sqrt{x^2+1}} \qquad x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\int \frac{\tan\theta}{\sqrt{\tan^2\theta+1}}\,d\theta\,\sec^2\theta$

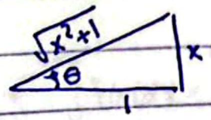
$\int \frac{\tan\theta}{\sec\theta}\sec^2\theta\,d\theta$

$\int \tan\theta\sec\theta\,d\theta \qquad u = \sec\theta$

$du = \sec\theta\tan\theta\,dx$

$\int du \qquad dx = \frac{du}{\sec\theta\tan\theta}$

$= u = \sec\theta = \sqrt{x^2+1}$



$\boxed{= \left(x\ln(x+\sqrt{x^2+1}) - \sqrt{x^2+1}\right) + C}$

③ $\int e^{\sqrt{x}}\,dx$

$y = \sqrt{x}$

$dy = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2\sqrt{x}\,dy$

$\int e^y\,2y\,dy$

u	dv
$2y$	e^y
2	e^y
0	e^y

$2e^y y - 2e^y + c$

$\boxed{2e^{\sqrt{x}}\sqrt{x} - 2e^{\sqrt{x}} + c}$

④ $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$

نفرض x = y قوة
قوة حاصل ضربهم

$x = y^6 \to y = \sqrt[6]{x}$

$dx = 6y^5\,dy$

$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,6y^5\,dy$

$\int \frac{\sqrt{y^6}}{1+\sqrt[3]{y^6}}\,6y^5\,dy$

$\int \frac{y^3}{1+y^2}\,6y^5\,dy$

$6\int \frac{y^8}{1+y^2} = \int \frac{6y^8}{y^2+1}$

$$\begin{array}{r} 6y^6 - 6y^4 + 6y^2 - 6 \\ y^2+1 \,\big)\, 6y^8 \qquad\qquad \\ 6y^8 + 6y^6 \\ \hline -6y^6 \\ -6y^6 - 6y^4 \\ \hline 6y^4 \\ 6y^4 + 6y^2 \\ \hline -6y^2 \\ -6y^2 - 6 \\ \hline 6 \end{array}$$

$\int 6y^6 - 6y^4 + 6y^2 - 6 + \frac{6}{y^2+1}$

$\frac{6y^7}{7} - \frac{6y^5}{5} + \frac{6y^3}{3} - \frac{6y^2}{2} + 6\,\cancel{\tfrac{1}{1}}\,6\tan^{-1}(y) + c$

$\boxed{\frac{6(\sqrt[6]{x})^7}{7} - \frac{6(\sqrt[6]{x})^5}{5} + 2\sqrt[6]{x}^3 - 3\sqrt[6]{x}^2 - 6\tan^{-1}(\sqrt[6]{x}) + C}$

Lecture 16 — نيسان 16

⑤ $\int \frac{1}{1+e^x}\,dx$

$y = e^x$

$dy = e^x dx \rightarrow dx = \frac{dy}{e^x}$

$\int \frac{1}{1+y}\,\frac{dy}{y}$

$\int \frac{1}{y+y^2}\,dy = \int \frac{A}{y} + \frac{B}{y+1}$

$1 = A(y+1) + B(y)$

$y=0$ $1 = A$

$y=-1$ $1 = -B \rightarrow B = -1$

$\int \frac{1}{y} + \frac{-1}{y+1}$

$\ln|y| - \ln|y+1| + c$

$\ln|e^x| - \ln|e^x+1| + c$

$x - \ln|e^x+1| + c$

⑥ $\int \frac{\sqrt{x+4}}{x}\,dx$

Let $y = \sqrt{x+4}$

$y^2 = x+4$

$2y\,dy = 1\,dx$

$\int \frac{y}{y^2-4}\,2y\,dy$

$\int \frac{2y^2}{y^2-4}$

$2y^2 \div (y^2-4) = 2$, $2y^2 - 8$, remainder 8

$\int 2 + \frac{8}{y^2-4}$

$2y + 8\int \frac{1}{(y-2)(y+2)}$ $\frac{A}{y-2} + \frac{B}{y+2}$

$8 = A(y+2) + B(y-2)$

$y=-2$ $8 = -4B \rightarrow B = -2$

$y=2$ $8 = 4A \rightarrow A = 2$

$2y + \int \frac{2}{y-2} + \frac{-2}{y+2}\,dy$ $\int \frac{2}{y-2} + \frac{-2}{y+2}$

$2y + 2\ln|y-2| - 2\ln|y+2| + c$

$2\sqrt{x+4} + 2\ln|\sqrt{x+4}-2| - 2\ln|\sqrt{x+4}+2| + c$

⑦ $\int \frac{1}{1+\cos^2 x}\,dx$

$\int \frac{1}{\sin^2(x) + \cos^2(x) + \cos^2(x)}\,dx$

$\int \frac{1}{\sin^2(x) + 2\cos^2(x)}\,dx$

$\int \frac{1}{\cos^2(x)\,(\tan^2(x)+2)}\,dx$

$\int \frac{\sec^2(x)}{\tan^2(x)+2}\,dx$

$y = \tan x$

$dy = \sec^2(x)\,dx$

$dx = \frac{dy}{\sec^2(x)}$

$\int \frac{\sec^2(x)}{y^2+2}\,\frac{dy}{\sec^2(x)}$

$\int \frac{1}{y^2+2}\,dy$

$y = \sqrt{2}\tan\theta$

$dy = \sqrt{2}\sec^2\theta\,d\theta$

$\int \frac{1}{(\sqrt{2}\tan\theta)^2+2}\,\sqrt{2}\sec^2(\theta)\,d\theta$

$\int \frac{1}{2(\tan^2\theta+1)}\,\sqrt{2}\sec^2\theta\,d\theta$

$\int \frac{1}{\sqrt{2}}\,d\theta$

$\frac{1}{\sqrt{2}}\theta + c$

$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + c$

$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan(x)}{\sqrt{2}}\right) + c$

Lecture 17 ~~Nawasee~~ Week 4 (39)

Improper Integrals التكاملات المعتلة

Type 1
Infinite intervals

$\int_a^{\infty} f(x)dx$, $\int_{-\infty}^{b} f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx$

Type 2
Discontinuous Integrals
(f has a vertical asymptote)

$\int_a^b f(x)\,dx$

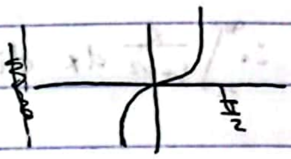
ممكن ما يكون متصل عند a او b او نقطة بينهم

Ex:- ① $\int_1^{\infty} e^{-x}dx$ improper integral (∞)

② $\int_2^3 \frac{1}{x^2}$ not Improper غير متصل فقط عند (0)

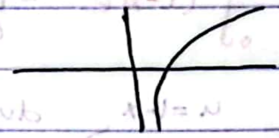
③ $\int_0^{\frac{\pi}{2}} \tan x\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x}dx$ Improper integral

↳ $\frac{\pi}{2}$ تحول المقام (0)



④ $\int_{-\infty}^{5} \frac{1}{x-2}dx$ Improper $(-\infty$, discont at $x=2)$

⑤ $\int_0^1 \ln x\,dx$ Improper (dis cont at $x=0$)



⑥ $\int_1^5 \frac{dx}{x+2}$ not Improper Integral

* If $\int_a^t f(x)$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

* If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

* $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx$$

if the Limits exists we call it (convergent)

If the Limit does not exists we call it (divergent)

Ex :- ①

$$\int_{-1}^{\infty} \frac{x}{1+x^2} \to \lim_{t\to+\infty} \int_{-1}^{t} \frac{x}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln|1+x^2|\Big]_{-1}^{t}$$

$$= \lim_{t\to\infty}\left(\frac{1}{2}\ln|1+t^2| - \frac{1}{2}\ln|2|\right)$$

$$= \frac{1}{2}\lim_{t\to\infty}\ln(1+t^2) - \frac{1}{2}\lim_{t\to\infty}\ln|2|$$

$$= \frac{1}{2}\ln\left(\overset{\infty}{\lim_{t\to\infty}(1+t^2)}\right) - \left(\frac{1}{2}\ln 2\right)$$

$= \infty -$ number $= \infty$

$\therefore \int_{-1}^{\infty}\frac{x}{1+x^2}\,dx$ divergent

lecture 18

Ex :- ①

$$\int_0^{\infty} (1-x)e^{-x} = \lim_{t\to\infty}\int_0^t (1-x)e^{-x}\,dx$$

$u = 1-x$ $\quad dv = e^{-x}$

-1 $\quad +$ $\quad -e^{-x}$

0 $\quad -$ $\quad e^{-x}$

$+\int$

$$= \lim_{t\to\infty}\left(-(1-x)e^{-x}+e^{-x}\right)\Big]_0^t$$

$$= \lim_{t\to\infty}\left(-(1-t)e^{-t}+e^{-t} - \left(-(1-0)e^{-0}+e^{0}\right)\right)$$

$$= \lim_{t\to\infty} e^{-t}\left(-(1-t)+1\right)$$

$$= \lim_{t\to\infty} e^{-t}(t) = 0\cdot\infty$$

$$= \lim_{t\to\infty}\frac{t}{e^t} = \frac{\infty}{\infty} = \frac{1}{e^t}$$

$\lim \frac{1}{e^t} = 0$ $\qquad 0 = \frac{1}{\infty}$

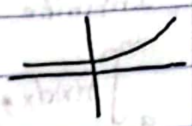
$\therefore \int_0^{\infty}(1-x)e^{-x}\,dx$ converge to 0

②

$$\int_{-\infty}^{0} e^{4x}\,dx \to \lim_{t\to-\infty}\int_t^0 e^{4x}\,dx$$

$$= \lim_{t\to-\infty}\frac{e^{4x}}{4}\Big]_t^0$$

$$= \lim_{t\to-\infty}\left(\frac{1}{4} - \frac{e^{4t}}{4}\right)$$



$$= \frac{1}{4} - 0 = \frac{1}{4}$$

$\therefore \int_{-\infty}^{0} e^{4x}\,dx$ converge to $\frac{1}{4}$

② $\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx \to \int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_0^{\infty}\frac{1}{1+x^2}\,dx$

بحل واحد فيهم وإذا طلع dive بكون الجواب dive

$$= \lim_{t\to-\infty}\int_t^0\frac{1}{1+x^2} + \lim_{t\to\infty}\int_0^t\frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty}\tan^{-1}x\Big]_t^0 + \lim_{t\to\infty}\tan^{-1}(x)\Big]_0^t$$

$$= \lim_{t\to-\infty} 0 - \tan^{-1}t + \lim_{t\to\infty}(\tan^{-1}t - 0)$$

$$= --\frac{\pi}{2} + \frac{\pi}{2}$$

$$= 2\frac{\pi}{2} = \pi$$

$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx =$ converge to π

Theorem :-

$$\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \to \text{converge to } \frac{1}{p-1} & \text{if } p>1 \\ \to \text{Diverge} & \text{if } p\le 1 \end{cases}$$

Ex:- $\int_1^\infty x^{-3}\,dx = \int_1^\infty \frac{1}{x^3}\,dx \to p=3>1$ (converge) to $\frac{1}{3-1}=\frac{1}{2}$

Ex:- Find the value of p for which the integral converge and evalute the integral for those value of p

① $\int_1^\infty \frac{1}{x^{p-2}}\,dx \begin{cases} \to \text{converg if } p-2>1 \to p>3 & (3,\infty) \\ \to \text{diverge if } p-2\le 1 \to p\le 3 & (-\infty,3] \end{cases}$

$\int_1^\infty \frac{1}{x^{p-2}}$ converge to $\frac{1}{p-2-1} = \frac{1}{p-3}$

② $\int_e^\infty \frac{1}{x(\ln x)^{2p}}\,dx$

$y=\ln x \qquad x=e \to y=1$

$dy=\frac{1}{x}dx \qquad x=\infty \to \infty=y$

$x\,dy=dx$

$\int_1^\infty \frac{x}{x\,y^{2p}}\,dy$

$\int_1^\infty \frac{1}{y^{2p}}\,dy \begin{cases} \to \text{converge to } \frac{1}{2p-1} & \text{if } 2p>1 \to p>\frac{1}{2} \\ \to \text{diverge if } 2p\le 1 \to p\le\frac{1}{2} \end{cases}$

Lecture 19

* If f is cont on [a,b) and is discont at b then:-

$$\int_a^b f(x)dx = \lim_{x\to b^-} \int_a^t f(x)dx$$

* if f is cont on (a,b] and discont at a then

$$\int_a^b f(x)dx = \lim_{x\to a^+} \int_t^b f(x)dx$$

limit exist (convergent)

Limit d.n.e (divergent)

* $\int_a^b f(x)dx$ (a,b) discont

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Ex:- $\int_0^{\frac{\pi}{2}} \cot x \, dx \to \frac{\cos x}{\sin x} \to \sin 0 = 0$

$$= \lim_{t\to 0^+} \int_t^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx$$

$$= \lim_{t\to 0^+} \ln|\sin x| \Big]_t^{\frac{\pi}{2}}$$

$$= \lim_{t\to 0^+} \left(\ln|\sin \tfrac{\pi}{2}| - \ln|\sin t| \right)$$

$$= \lim_{t\to 0^+} \ln|\sin t|$$

$$= +\infty$$

$\therefore \int_0^{\frac{\pi}{2}} \cot x \, dx$ is divergent

② $\int_0^1 \frac{1}{\sqrt{1-x^2}}$

$$\lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x^2}} dx$$

$$\lim_{t\to 1^-} \sin^{-1}(x) \Big]_0^t$$

$$\lim_{t\to 1^-} \sin^{-1}(t) - \sin^{-1}(0)$$

$$\lim_{t\to 1^-} \sin^{-1}(t)$$

$$= \frac{\pi}{2}$$

$\therefore \int_0^1 \frac{1}{\sqrt{1-x^2}} dx$ is convergent to $\frac{\pi}{2}$

③ $\int_{-2}^{2} \frac{1}{x^2} dx = \underbrace{\int_{-2}^{0} \frac{1}{x^2} dx}_{①} + \underbrace{\int_0^2 x^{-2} dx}_{②}$

① $\lim_{t\to 0^-} \int_{-2}^{t} x^{-2} dx$

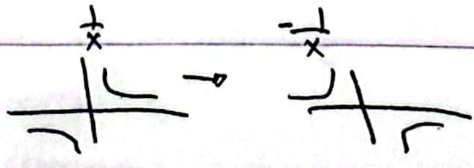
$$\lim_{t\to 0^-} \frac{-1}{x} \Big]_{-2}^{t}$$

$$= \lim_{t\to 0^-} \frac{-1}{t} + \frac{1}{2}$$

$$= +\infty \to \text{d.n.e}$$

$\therefore \int_{-2}^{2} \frac{1}{x}$ is divergent

عشان واحد وطلع divergent مش ضروري اكمل



(4) $\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$ يوجد مشكلتين

$\infty, 0$

$= \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

$= \lim_{t \to 0^+} \int_t^1 \frac{1}{\sqrt{x}(x+1)} dx + \lim_{t \to \infty} \int_1^t \frac{dx}{\sqrt{x}(x+1)}$

$y = \sqrt{x}$ ~~dy~~

$y^2 = x$

$2y\,dy = dx$

$= \lim_{t \to 0^+} \int_t^1 \frac{1}{y(y^2+1)} dy\, 2y + \lim_{t \to \infty} \int \frac{2y}{y(y^2+1)} dy$

$= \lim_{t \to 0^+} \int \frac{2}{y^2+1} + \lim_{t \to \infty} \int \frac{2}{y^2+1}$

$= \lim_{t \to 0^+} 2\tan^{-1}(y) + \lim_{t \to \infty} 2\tan^{-1} y$

$= \lim_{t \to 0^+} 2\tan^{-1}(\sqrt{x}) \Big]_t^1 + \lim_{t \to \infty} 2\tan^{-1}(\sqrt{x}) \Big]_1^t$

$= \lim_{t \to 0^+} \left(2\tan^{-1}(1) - 2\tan^{-1}(\sqrt{t})\right) + \lim_{t \to \infty} \left(2\tan^{-1}(\sqrt{t}) - 2\tan^{-1}(1)\right)$

$= 2\frac{\pi}{4} - 2(0) + \frac{2\pi}{2} - 2\frac{\pi}{4}$

$= \frac{2\pi}{4} + \frac{2\pi}{2} - \frac{2\pi}{4}$

$= \frac{\pi}{2} + \pi - \frac{\pi}{2} = \pi$

$\therefore \int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$ convergent at $x = \pi$

Ex: Find the values of p for which the integrent is ~~prope~~ improper

$\int_0^9 \frac{1}{\sqrt{x} - p} dx$

$0 \le x \le 9$

$\int_0^9 \sqrt{x} - 0 \quad \sqrt{x} = 0$

$\sqrt{x} - p \ge 0$

$0 \le \sqrt{x} \le 3$

$\int_0^9 \sqrt{x} - 1 \to x = 1$

$p = \sqrt{x}$

$0 \le p \le 3$

$\int_0^9 \sqrt{x} - 2 \quad x = 4$

$\vdots$

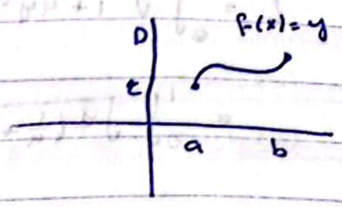
Week 5

~~Lecture~~ Lecture 20,22,24 مش مطلوب

Lecture 21 Arc length طول قوس المنحنى

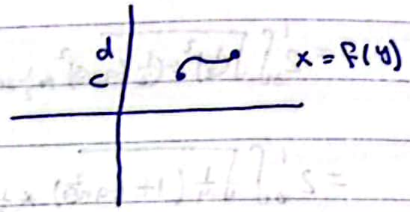
⊛ If f' is cont on $[a,b]$ then the length of the curve $y=f(x)$ on $[a,b]$ is :-

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$



⊛ If f' is cont on $[c,d]$ then the length of the curve $x=f(y)$ on $[c,d]$ is :-

$$L = \int_c^d \sqrt{1+(f'(y))^2}\, dy = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$



Ex:- find the arc length of the curve

① $y=x^{\frac{3}{2}}$ from $(1,1)$ to $(2,2\sqrt{2})$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} \longrightarrow$ cont from (1) to (2)

$$L = \int_1^2 \sqrt{1+\left(\tfrac{3}{2}x^{\frac{1}{2}}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1+\tfrac{9}{4}x}\, dx$$

$$= \int_1^2 \left(1+\tfrac{9}{4}x\right)^{\frac{1}{2}} dx$$

$$= \frac{2\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}\, 4}{3*9}\Bigg]_1^2$$

$$= \frac{8\left(1+\frac{9}{4}x\right)^{\frac{3}{2}}}{27}\Bigg]_1^2$$

$$= \frac{8}{27}\left(\left(\tfrac{11}{2}\right)^{\frac{3}{2}} - \left(\tfrac{13}{4}\right)^{\frac{3}{2}}\right)$$

② $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $1 \le x \le 2$

$\frac{dy}{dx} = \frac{1}{2}x - \frac{1}{2x} = \frac{2x^2-2}{4x}$ cont $[1,2]$

$$L = \int_1^2 \sqrt{1+\left(\frac{2x^2-2}{4x}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1+\frac{4x^4-8x^2+4}{16x^2}}\, dx$$

$$= \int_1^2 \sqrt{\frac{16x^2+4x^4-8x^2+4}{16x^2}}$$

$$= \int_1^2 \sqrt{\frac{x^4+2x^2+1}{4x2}}$$

$$= \int_1^2 \frac{\sqrt{(x^2+1)^2}}{2x}$$

$$= \int_1^2 \frac{x^2+1}{2x} = \int_1^2 \frac{1}{2}x + \frac{1}{2x}$$

$$= \frac{1}{4}x^2 + \frac{1}{2}\ln|x|\Big]_1^2$$

$$= \frac{3}{4} + \frac{1}{2}\ln 2$$

③ $x=y^2$ from $\overset{a\ c}{(0,0)}$ to $\overset{b\ d}{(1,1)}$

$\frac{dx}{dy}=2y$ cont from $[c,d]$

$$L=\int_c^d \sqrt{1+(2y)^2}\,dy$$

$$L=\int_0^1 \sqrt{1+4y^2}\,dy$$

$$=\int_0^1 \sqrt{4\left(\tfrac14+y^2\right)}\,dy$$

$$=2\int_0^1 \sqrt{\left(\tfrac12\right)^2+y^2}\,dy$$

$y=\frac12\tan\theta$

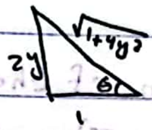
$2y=\tan\theta$

$dy=\frac12\sec^2\theta$

$\theta=\tan^{-1}(2y)$

$$=2\int_0^1 \sqrt{\left(\tfrac12\right)^2+\left(\tfrac12\tan\theta\right)^2}*\tfrac12\sec^2\theta$$

$$=2\int_0^1 \sqrt{\tfrac14\left(1+\tan^2\theta\right)}*\tfrac12\sec^2\theta$$



$$=\frac12\int \sec^3\theta\,d\theta$$

$$=\frac12\left(\frac{\sec\theta\tan\theta}{2}+\frac12\int\sec\theta\,d\theta\right)$$

$$=\frac14\sec\theta\tan\theta+\frac14\ln|\sec\theta+\tan\theta|$$

$$=\frac14\sqrt{1+4y^2}\,2y+\frac14\ln\left|\sqrt{1+4y^2}+2y\right|\Big]_0^1$$

$$=\frac12\sqrt{1+4}+\frac14\ln|\sqrt5+2|-\left(0+\frac14\ln 1\right)$$

$$=\frac{\sqrt5}{2}+\frac14\ln(\sqrt5+2)$$

Lecture 23

Area of surface of revolution

مشتقة

⊛ If f is positive and has a cont derivative then the surface area obtained by rotating the curve about x-axis :-

$$S = \int 2\pi y \, ds$$

$y = f(x)$, $a \le x \le b$

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

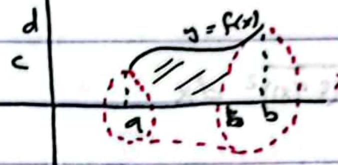
إذا كانت x بدلالة الـ t و y بدلالة t

بتكون فيك :-

$a \le t \le b$ $$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$x = f(y)$, $c \le y \le d$

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dy$$



دوران

Ex :- find the area of the surface generated by revolving the given curve about x-axis

① $y = \sqrt{x}$, $x \in [1, 4]$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$$S = \int_1^4 2\pi y \, ds$$

$$S = \int_1^4 2\pi y \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} \, dx$$

$$= \int_1^4 2\pi \sqrt{x} \sqrt{\frac{4x+1}{4x}}$$

$$= \int_1^4 2\pi \sqrt{x} \, \frac{\sqrt{4x+1}}{2\sqrt{x}}$$

$$= \int_1^4 \pi \sqrt{4x+1}$$

$$= \frac{\pi (4x+1)^{\frac{3}{2}}}{6} \Big]_1^4$$

$$= \frac{\pi (17)^{\frac{3}{2}}}{6} - \frac{\pi (5)^{\frac{3}{2}}}{6}$$

② $x = 1 + 2y^2$ $1 \le y \le 2$

$$S = \int_1^2 2\pi y \sqrt{1 + (4y)^2} \, dy$$

$$= \int_1^2 2\pi y \sqrt{1 + 16y^2} \, dy$$

Let $z = 1 + 16y^2$

$dz = 32y \, dy$

$dy = \frac{dz}{32y}$

$$= \int_{\ast}^{\ast} 2\pi y \, \frac{\sqrt{z}}{32y} \, dz$$

$$\int_{\ast}^{\ast} \frac{\pi}{16} \sqrt{z} \, dz$$

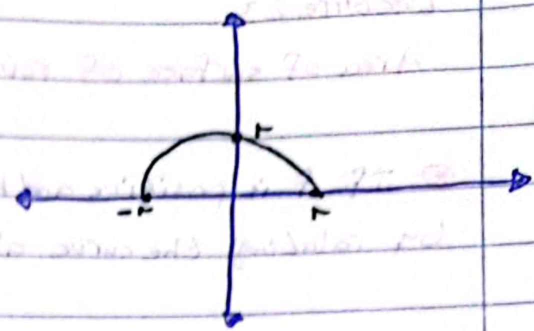
$$\frac{\pi}{16} z^{\frac{3}{2}} \ast \frac{2}{3} = \frac{\pi}{24} (1 + 16y^2)^{\frac{3}{2}} \Big]_1^2$$

$$\frac{\pi}{24} (1 + 16(4))^{\frac{3}{2}} - \frac{\pi}{24} (1 + 16)^{\frac{3}{2}}$$

Lecture 25

Remark:-

$f(x)=\sqrt{r^2-x^2}$, $-r \le x \le r$



- Area of circle $= \pi r^2$
- circumference $= 2\pi r$.
- surface area of ball (sphere) $= 4\pi r^2$

Ex:- Use the arc length formula to drive the formula of circumference circle of radias r

$f(x)=\sqrt{r^2-x^2}$ $\quad -r \le x \le r$

نجد محيط نصف الدائرة ثم نضرب بـ 2

$L=\int_a^b \sqrt{1+(f'(x))^2}\,dx$

$L=\int_{-r}^{r}\sqrt{1+\left(\frac{-2x}{2\sqrt{r^2-x^2}}\right)^2}$

$=\int_{-r}^{r}\sqrt{1+\frac{x^2}{r^2-x^2}}$

$=\int_{-r}^{r}\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}} = \int_{-r}^{r}\frac{r}{\sqrt{r^2-x^2}}dx = r\sin^{-1}\left(\frac{x}{r}\right)\Big]_{-r}^{r}$

$r\sin^{-1}(1)-r\sin^{-1}\left(\frac{-r}{r}\right)$

$r\sin^{-1}(1)+r\sin^{-1}(1)$

$= r\pi \rightarrow \frac{1}{2}$ circunferce of circle

*circllmfence of circle $= 2\pi r$

(2) Use the surface area formula to drive the formula for the surface of the ball (sphere)

$$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{-r}^{r} 2\pi \sqrt{r^2-x^2} \sqrt{1+\left(\frac{-2x}{2\sqrt{r^2-x^2}}\right)^2}\, dx$$

$$= \int_{-r}^{r} 2\pi \sqrt{r^2-x^2}\, \frac{r}{\sqrt{r^2-x^2}}$$

$$= \int_{-r}^{r} 2\pi r \, dx = \cancel{2\pi r\, dx}\ 2\pi r x \Big]_{-r}^{r}$$

$$2\pi r r - 2\pi r(-r) = 4\pi r^2$$

(3) for $f(x)=\sqrt{4-x^2}$, $-2 \le x \le 2$ find

(a) Area enclosed by $f(x)$ and x axis

$A = \frac{1}{2}$ Area of circle

$= \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(4) = 2\pi$

or

$\int_{-2}^{2} \sqrt{4-x^2}\, dx \ldots\ldots$

(b) Arc length of cure on $[-2,2]$

Arc length $= \frac{1}{2}$ ciraunference of circle

$L = \frac{1}{2}(2\pi r) = 2\pi$

$\int_{-2}^{2} \sqrt{} \ldots$

(c) surface area of the solid generated by revolving (دوران) the curve about x-axis

$S = 4\pi r^2$
$= 16\pi$

or $S = \int_a^b 2\pi y \, ds$

Lecture 26

50

Parametric equation

Remark :-

* circle with center a, b, radius r

$$(x-a)^2+(y-b)^2=r^2$$

motion : counter clockwise

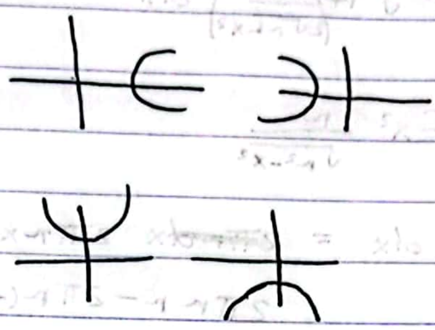
* Line

$$y=ax+b$$

* Parabola القطع مكافئ

$$(y-k)^2 = 4c(x-h)$$

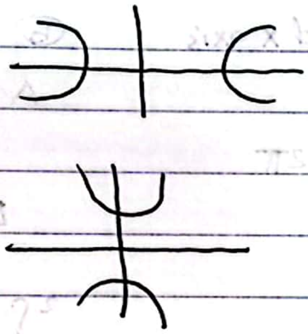
$$(x-h)^2 = 4c(y-k)$$



* Hyper bola قطع الزائد

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

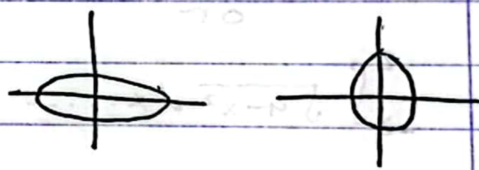
$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$



* Ellipse قطع الناقص

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2}=1$$

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2}=1$$

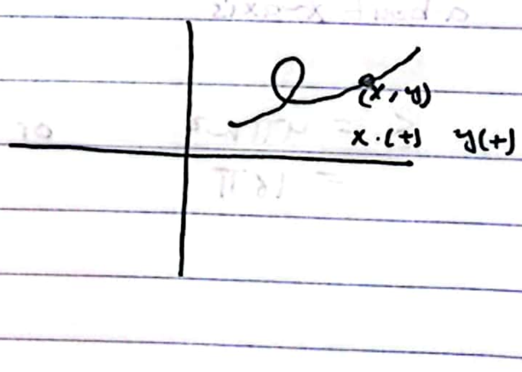


* The parametric equation of $y=f(x)$ is :

$$C: \left.\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}\right\}$$

t : parameter

C : Graph (Parametric curve)



استخدام ال t

Ex:- Find the graph of parametric equation(eliminate the ~~par~~ parameter)

① C : $x = t - 1$ $\quad 0 \le t \le 5$
$y = t + 1$

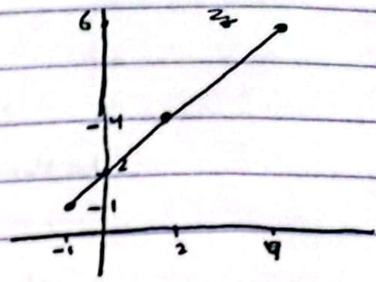
$x + 1 = t$

$y = t + 1$

$y = x + 1 + 1$

$y = x + 2$

t	x	y	(x, y)
0	-1	1	(-1,1)
1	0	2	(0,2)
3	2	4	(2,4)
5	4	6	(4,6)



$y = x + 2$
from (-1,1) to (4,6) Line

② $x = t + 2$
$y = t^2$ $\quad -2 \le t \le 2$

$(x - 2) = t$

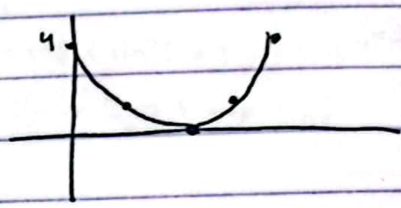
$y = (x - 2)^2$

$y = x^2 - 2x + 4$

parabola

from (0,4) to (4,4)

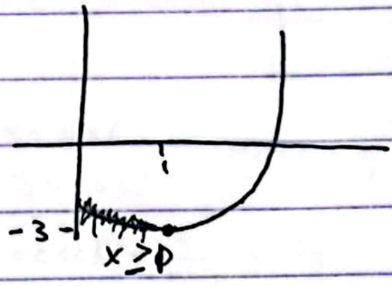
t	x	y	(x, y)
-2	0	4	(0,4)
-1	+1	1	(1,1)
0	2	0	(2,0)
1	3	1	(3,1)
2	4	4	(4,4)



③ $x = t^2 + 1$ $\quad x - 1 = t^2$ $\quad x \ge \phi$
$y = t^4 - 3$

$y = (x - 1)^2 - 3$
$= x^2 - 2x + 1 - 3$
$= x^2 - 2x - 2$
parabolal

t	x	y	(x, y)

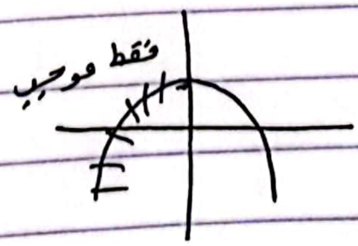


④ $x = \sqrt{t}$ $\quad x^2 = t$

$y = 1 - t$

$y = 1 - x^2$

$x \ge 0$



Week 6

Lecture 27:-

Ex:-

* Find the graph of parametric equation (eliminate the parameter)

① $x = 3\cos t$

$y = 3\sin t \qquad 0 \le t \le 2\pi$

لما يكون زوايا مختلفة نستخدم متطابقات

$\frac{x}{3} = \cos t$

$\frac{y}{3} = \sin t$

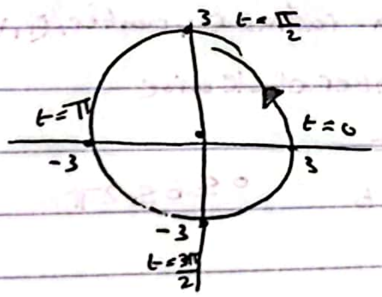
$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$

$\frac{y^2}{9} + \frac{x^2}{9} = 1$

$y^2 + x^2 = 9$

circle with center (0,0) radias 3

t	x	y	(x,y)
0	3	0	(3,0)
$\frac{\pi}{2}$	0	3	(0,3)
π	-3	0	(-3,0)
$\frac{3\pi}{2}$	0	-3	(0,-3)



oriented counter clockwise

② $x = 3\sec t \qquad 0 \le t \le 2\pi$

$y = 4\tan t$

$\frac{x}{3} = \sec t$

$\frac{y}{4} = \tan t$

$\sec^2 t - \tan^2 t = 1$

$\frac{x^2}{9} - \frac{y^2}{4} = 1$

Hyperbola

③ $x = 5\cos^2 t$

$y = 3\sin^2 t$

$\frac{x}{5} = \cos^2 t \qquad x \ge 0$

$\frac{y}{3} = \sin^2 t \qquad y \ge 0$

لأنه x, y موجبين بالأصل

$\cos^2 t + \sin^2 t = 1$

$\frac{x}{5} + \frac{y}{3} = 1 \to$ Line

④ $x = \cosh t$

$y = \sinh t$

$\cosh^2 t - \sinh^2 t = 1$

$x^2 - y^2 = 1 \dashrightarrow$ Hyperbola

$x \ge 1 \to \cosh$

⑤ $x = \sin t \qquad 0 < t < \frac{\pi}{2}$

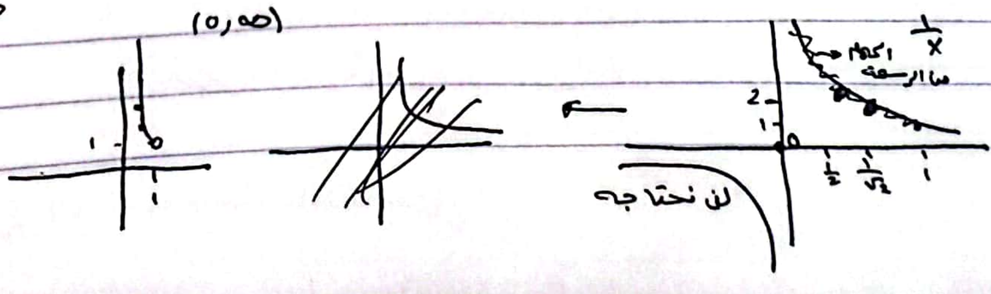
$y = \csc t$

$y = \csc t = \frac{1}{\sin t} = \frac{1}{x}$

t	x	y	(x,y)
0	0	غ.م	$(0,\infty)$
$\frac{\pi}{6}$	$\frac{1}{2}$	2	$(\frac{1}{2},2)$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$(\frac{1}{\sqrt{2}},\sqrt{2})$
$\frac{\pi}{2}$	1	1	(1,1)

$0 < x < 1$

$1 < y < \infty$



$x = 5 + 2\cos t$

$y = 3 + 2\sin t \qquad 0 \le t \le 2\pi$

$\frac{x-5}{2} = \cos t$

$\frac{y-3}{2} = \sin t$

$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x-5}{2}\right)^2 + \left(\frac{y-3}{2}\right)^2 = 1$

$\frac{(x-5)^2}{4} + \frac{(y-3)^2}{4} = 1$

$(x-5)^2 + (y-3)^2 = 4$

circle with center $(+5,3)$
radias 2

Lecture 28 :-

parametric equation :-

* The line pass through $(x_0, y_0), (x_1, y_1)$ المستقيم الذي يمر بالنقطتين

$x(t) = x_0 + (x_1 - x_0)t$

$y(t) = y_0 + (y_1 - y_0)t$

* The line $y = ax + b$

$x(t) = t$

$y(t) = at + b \qquad t \in \mathbb{R}$

* The line $x = a$

$x(t) = a$

$y(t) = t \qquad t \in \mathbb{R}$

• The line $y = b$

$x(t) = t$

$y(t) = b \qquad t \in \mathbb{R}$

* The circle with radias r, center (a, b) and counter clockwise

$x(t) = a + r\cos t$

$y(t) = b + r\sin t \qquad 0 \le t \le 2\pi$

Ex :- find parametric equation

① The line $y = 2x+1$ $\quad 0 \le x < 1$

$x(t) = t \rightarrow x = t$

$y(t) = at+b \rightarrow y = 2t+1$

$0 \le x \le 1$ if $0 \le t \le 1$

② The line pass through $\overset{x_0\, y_0}{(1,5)}, \overset{x_1\, y_1}{(2,8)}$

$x = x_0 + (x_1 - x_0)t$

$y = y_0 + (y_1 - y_0)t \quad 0 \le t \le 1$

$x = 1 + (2-1)t \rightarrow x = 1+t$

$y = (5) + (8-5)t \quad y = 5+3t \quad 0 \le t \le 1$

$y = ax+b \leftarrow$ او ممكن نحلها كالتالي

$x = t$

$y = at+b$

③ The circle with radias 4, centerd at $(1,-3)$ oriented counterclockwise

$x(t) = a + r\cos t$

$y(t) = b + r\sin t \quad 0 \le t \le 2\pi$

$x = 1 + 4\cos t$

$y = -3 + 4\sin t \quad 0 \le t \le 2\pi$

④ $(x+1)^2 + (y-2)^2 = 25$

Circle with center $(-1,2)$ radius 5

$x = -1 + 5\cos t$

$y = 2 + 5\sin t \quad 0 \le t \le 2\pi$

⑤ semicircle with radias 5, centered at the origen oriented counterclockwise

$x = a + r\cos t \rightarrow x = 0 + 5\cos t$

$y = b + r\sin t \rightarrow y = 0 + 5\sin t$

$0 \le t \le \pi \rightarrow$ عشان نصير semicircle

وممكن فترات أخرى

Lecture 29 :- Calculus with parametric curves :-

* Tangents :- $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

* slop of tangent line $= \left.\dfrac{dy}{dx}\right|_t$

* Horizontal tangent line $\dfrac{dy}{dt} = 0, \dfrac{dx}{dt} \ne 0$

* vertical tangent line $\dfrac{dy}{dt} \ne 0, \dfrac{dx}{dt} = 0$

* $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

* If $\dfrac{d^2y}{dx^2} > 0$ (concave upward)

* If $\dfrac{d^2y}{dx^2} < 0$ (concave downward)

Ex:- Concider $\boxed{\begin{array}{l} x = t^2 \\ y = t^3 - 3t \end{array}} \rightarrow C$

① show that C has 2 tangents at (3,0) and find their equations :-

when $x = 3 \rightarrow 3 = t^2 \rightarrow t = \pm\sqrt{3}$

$y = 0 \rightarrow 0 = t^3 - 3t \rightarrow t = 0 \quad t = \pm\sqrt{3}$

$t = \pm\sqrt{3}$

~~when~~ $t = \pm\sqrt{3}$ at $(3,0)$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

Slop : $\left.\frac{dy}{dt}\right|_{t=\sqrt{3}} = \frac{3\sqrt{3}^2 - 3}{2\sqrt{3}} = \sqrt{3}$

equation of tangent $y - y_1 = m(x - x_1)$

$y - 0 = \sqrt{3}(x - 3)$ ---①

when $\left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = \frac{3 \cdot \sqrt{3}^2 - 3}{-2\sqrt{3}} = -\sqrt{3}$

$y - 0 = -\sqrt{3}(x - 3)$ ---②

lecture 30 غير مطلوب

② find the point on C when the tangent is horizontal or vertical

① Horizontal $\rightarrow \frac{dy}{dt} = 0$

$\frac{dy}{dt} = 3t^2 - 3 = 0 \rightarrow t = \pm 1$

at $t = 1 \rightarrow x = 1 \quad y = -2 \quad (1,-2)$

at $t = -1 \quad x = 1 \quad y = 2 \quad (1,2)$

② vertical $\frac{dx}{dt} = 0 \ , \frac{dy}{dt} \neq 0$

$2t = 0 \rightarrow t = 0$

at $t = 0 \rightarrow x = 0 \quad y = 0 \quad (0,0)$

③ Determind where the curve is concave upward or downward

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

$$\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3}{2}\left(t - \frac{1}{t}\right)\right)}{2t}$$

$$= \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t} \rightarrow \frac{3(t^2+1)}{4t^3}$$

$3(t^2+1) = 0 \rightarrow \neq 0$

$4t^3 = 0 \rightarrow t = 0$

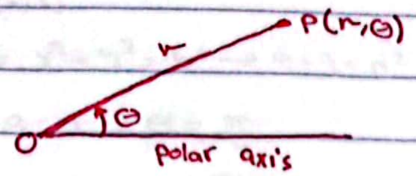
$- - - \quad | \quad + + +$

0

concave up when $t > 0$

concave down when $t < 0$

Polar coordinates :-



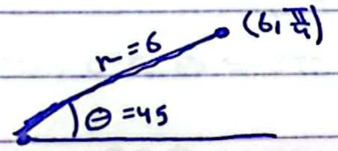
(r,θ) polar coordinates

r distance from p to O

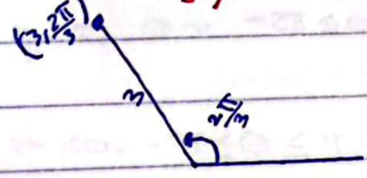
θ polar angle

Ex:- plot the points whose polar coordinates are given

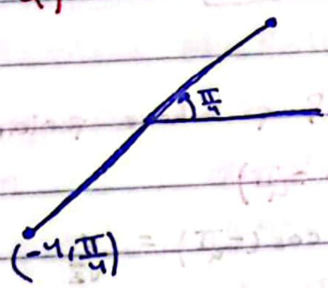
① $(\overset{r}{6}, \overset{\theta}{\frac{\pi}{4}})$



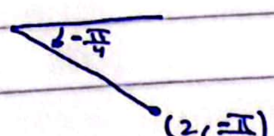
② $(3, \frac{2\pi}{3})$



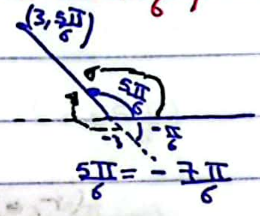
③ $(-4, \frac{\pi}{4})$



④ $(2, \frac{-\pi}{4})$



⑤ $(3, \frac{5\pi}{6})$



$(-3, \frac{-\pi}{6}) = (3, \frac{5\pi}{6})$

* In fact the point represented by polar coordinates (r,θ) is also represented by:-

① $(r, \theta + 2n\pi)$

② $(r, \theta - 2\pi n)$

③ $(-r, \theta + \pi + 2n\pi) = (-r, \theta + (2n+1)\pi)$

④ $(-r, \theta - \pi + 2n\pi) = (-r, \theta(2n-1)\pi)$

Ex:- ① $(3, \frac{5\pi}{6} + 2\pi) = (3, \frac{17\pi}{6})$

② $(3, \frac{5\pi}{6} - 2\pi) = (3, \frac{7\pi}{6})$

③ $(-3, \frac{5\pi}{6} + \pi) = (-3, \frac{11\pi}{6})$

④ $(-3, \frac{5\pi}{6} - \pi) = (-3, \frac{-\pi}{6})$

Ex:- plot the point

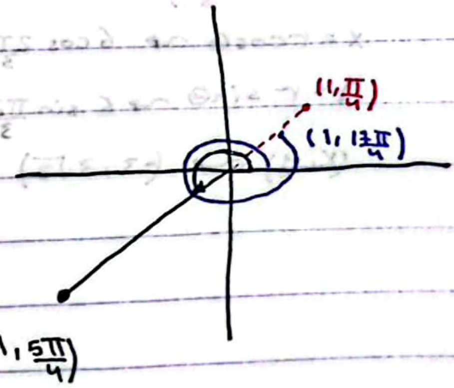
① $(-1, \frac{9\pi}{4})$

$(-1, \frac{9\pi}{4} + 2\pi) = (-1, \frac{17\pi}{4})$

$(-1, \frac{9\pi}{4} - 2\pi) = (-1, \frac{\pi}{4})$

$(1, \frac{9\pi}{4} + \pi) = (1, \frac{13\pi}{4})$

$(1, \frac{9\pi}{4} - \pi) = (1, \frac{5\pi}{4})$



$(1, \frac{5\pi}{4})$

$(-1, \frac{\pi}{4})$

$(-1, \frac{17\pi}{4})$

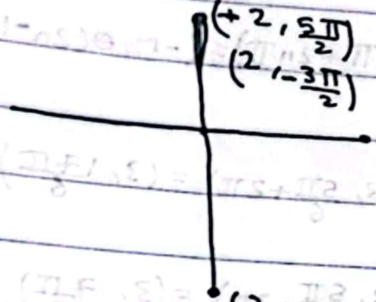
② $(-2, \frac{\pi}{2})$

$(-2, \frac{\pi}{2}+2\pi) = (-2, \frac{5\pi}{2})$

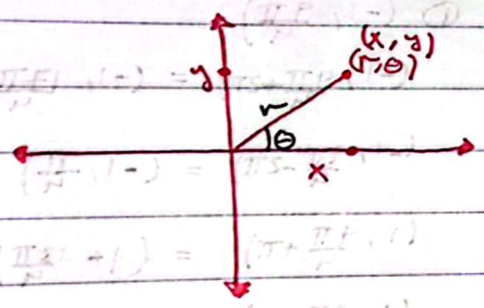
$(-2, \frac{\pi}{2}-2\pi) = (-2, -\frac{3\pi}{2})$

$(2, \frac{\pi}{2}+\pi) = (2, \frac{3\pi}{2})$

$(2, \frac{\pi}{2}-\pi) = (2, -\frac{\pi}{2})$



Lecture 32 — Relation between polar (r, θ) and rectangular (cartesion) (x, y) coordinates



$$\boxed{\begin{array}{l} x^2+y^2=r^2 \\ \tan\theta = \frac{y}{x} \end{array}}$$

$\cos\theta = \frac{x}{r} \rightsquigarrow \boxed{x = r\cos\theta}$

$\sin\theta = \frac{y}{r} \rightsquigarrow \boxed{y = r\sin\theta}$

Ex:- find the rectangular coordinates (x,y) of p whose polar coordinates

① $(6, \frac{2\pi}{3})$ (r, θ)

$x = r\cos\theta \rightsquigarrow 6\cos\frac{2\pi}{3} = -3$

$y = r\sin\theta \rightsquigarrow 6\sin\frac{\pi 2}{3} = 3\sqrt{3}$

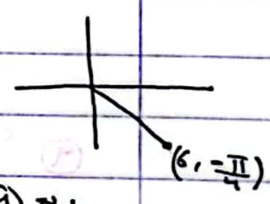
$(x, y) \rightsquigarrow (-3, 3\sqrt{3})$ ②

② $(6, \frac{-\pi}{4})$

$x = 6\cos(\frac{-\pi}{4}) = \frac{6}{\sqrt{2}}$

$y = 6\sin(-\frac{\pi}{4}) = \frac{-6}{\sqrt{2}}$

$(x, y) \rightsquigarrow (\frac{6}{\sqrt{2}}, \frac{-6}{\sqrt{2}})$ ④



Convert the points from cartesion to polar coordinates

① $(3,3)$

$r \to x^2+y^2=r^2 \to 9+9=r^2 \to r=\sqrt{18}$

$\tan\theta = 1 \quad \theta = \frac{\pi}{4}$

$(r,\theta) \sim (\sqrt{18}, \frac{\pi}{4})$

② $(-2,-2\sqrt{3})$

$r = \sqrt{x^2+y^2} = \sqrt{4+12} = 4$

$\tan\theta = \frac{-2\sqrt{3}}{-2} = \sqrt{3} \quad \theta = \frac{\pi}{3}$

$(r,\theta) \sim (4, \frac{\pi}{3}) + \pi$

$(4, \frac{4\pi}{3})$ عشان ربع ③

③ $(-\sqrt{3},1)$

$r = \sqrt{3+1} = \sqrt{4} = 2$

$\tan\theta = -\frac{1}{\sqrt{3}} \to \theta = \frac{\pi}{6}$ ② ربع

$\theta = \frac{5\pi}{6}$ ② ربع

$(r,\theta) \sim (2, \frac{5\pi}{6})$

$r>0,\ 0 \le \theta \le 2\pi \to (2, \frac{5\pi}{6})$

$r<0,\ 0 \le \theta \le 2\pi \to (-2, \frac{5\pi}{6}+\pi)$

$(-2, \frac{5\pi}{6}-\pi)$

$r>0,\ \theta \sim -2\pi \le \theta < 0 \to (2, \frac{5\pi}{6}-2\pi)$

$r<0,\ -\pi \le \theta \le \pi \sim (-2, \frac{-\pi}{6})$

Lecture 33 :-

Ex:- Replace the following cartesion equation by equivelant polar equation

① $2xy=5$

$2r\cos\theta\, r\sin\theta = 5$

$2r^2\cos\theta\sin\theta = 5$

$r^2\sin(2\theta) = 5$

② $(x^2+y^2)^2 = x^2-y^2$

$(r^2)^2 = (r\cos\theta)^2-(r\sin\theta)^2$

$r^4 = r^2\cos^2\theta - r^2\sin^2\theta$

$r^2 = \cos^2\theta - \sin^2\theta$

$r^2 = \cos(2\theta)$

Ex:-

Replace the following polar equation by equivelant rectangular equation

(1) $r\cos\theta = -4$

$x = -4$

vertical line

(2) $r = \cos\theta$

$rr = r\cos\theta$

$r^2 = x$

$x^2+y^2 = x$

$x^2 - x + y^2 = 0$

$x^2 - x + \frac{1}{4} - \frac{1}{4} + y^2 = 0$

$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$

circlue center $(\frac{1}{2}, 0)$

radias $(\frac{1}{2})$

(3) $r = \sec\theta \tan\theta$

$r = \frac{1}{\cos\theta}\tan\theta$

$r\cos\theta = \tan\theta$

$r\cos\theta = \tan\theta$

$x = \frac{y}{x}$

$y = x^2$

(4) $\theta = \frac{\pi}{3}$

$\tan\theta = \tan\frac{\pi}{3}$

$\tan\theta = \sqrt{3}$

$\frac{y}{x} = \sqrt{3}$

$y = \sqrt{3}x$ Line

Week 7

lecture 34

polar curves:-

The graph of a polar equation:-

$r = f(\theta)$

$f(r, \theta) = 0$

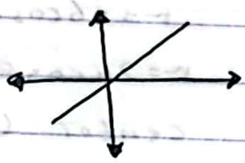
Ex:- Consist of all points p that have at least one polar ~~curves~~ representation whose coordinates satisfy the equation.

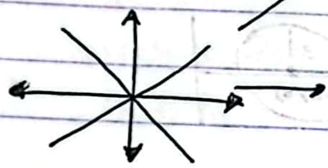
① $\theta = a$ (straight line)

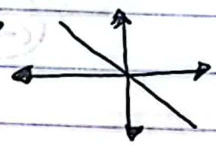
$\tan\theta = \tan a$

$\frac{y}{x} = \tan a$

$y = \tan a \, x$





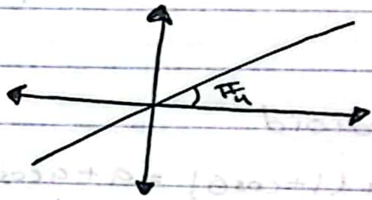


sketch the graph $\theta = \frac{\pi}{4}$

$\tan\theta = \tan\frac{\pi}{4}$

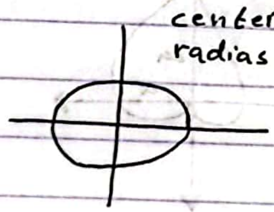
$\frac{y}{x} = 1$

$\boxed{y = x}$



② $r = a$ $\quad 0 \le \theta \le 2\pi$

center (0,0)
radias $|a|$



$r^2 = a^2$

$x^2 + y^2 = a^2$

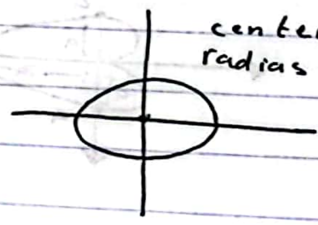
sketch the graph of $r = 3$

$r = 3$

$r^2 = 9$

$x^2 + y^2 = 9$

center (0,0)
radias = 3



③ $r = 2a\cos\theta + 2b\sin\theta$

circlue with radias $\sqrt{a^2+b^2}$ and center (a, b)

$r^2 = 2ar\cos\theta + 2br\sin\theta$

$x^2 + y^2 = 2ax + 2by$

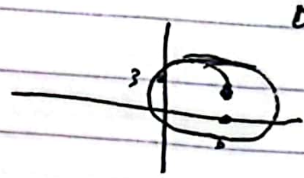
$x^2 - 2ax + a^2 - a^2 + y^2 - 2by + b^2 - b^2 = 0$

$(x-a)^2 + (y-b)^2 = a^2 + b^2$

sketch the graph

$r = 10\cos\theta + 6\sin\theta$

$r = \sqrt{100+36}$



$r = 2a\cos\theta + 2b\sin\theta$

$a \to 5$

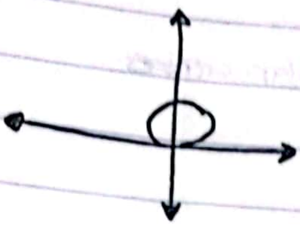
$b \to 3$ $\quad r = \sqrt{34}$

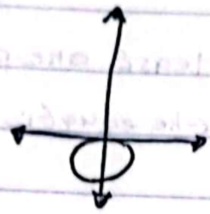
④ if $a=0$, $r=2b\sin\theta$

circle with center $(0,b)$

radias $|b|$

* if $b>0$ if $b<0$



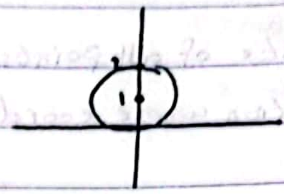


$r=2\sin\theta$

$r=2b\sin\theta \rightarrow b=1$

circle $(0,1)$

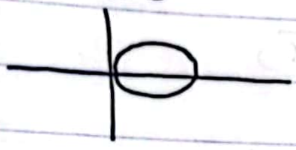
radias $|1|$

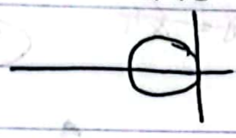


⑤ If $b=0$, $r=2a\cos\theta$

center $(a,0)$ radias $|a|$

$a>0$ $a<0$

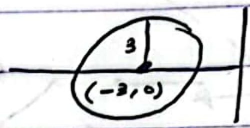




$r=-6\cos\theta$

$r=2a\cos\theta \rightarrow a=-3$

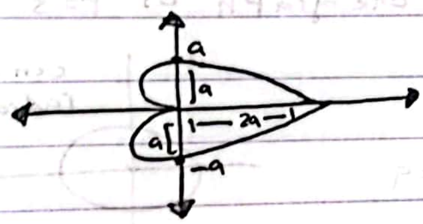
center $(-3,0)$ radias $|-3|=3$



⑥ cardioid

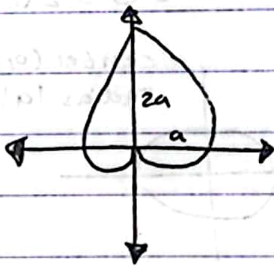
$r=a(1+\cos\theta)=a+a\cos\theta$

or $r=-a(1-\cos\theta)$



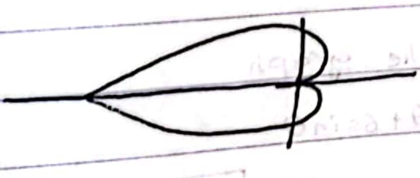
$r=a(1+\sin\theta)=a+a\sin\theta$

or $r=-a(1-\sin\theta)$



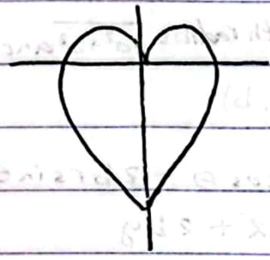
$r=a(1-\cos\theta)=a-a\cos\theta$

or $r=-a(1+\cos\theta)$



$r=a(1-\sin\theta)=a-\sin\theta$

or $r=-a(1+\sin\theta)$

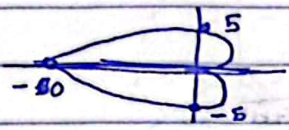


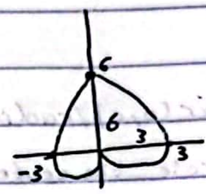
Ex:-

① $r = 5 - 5\cos\theta$

$= 5(1-\cos\theta)$

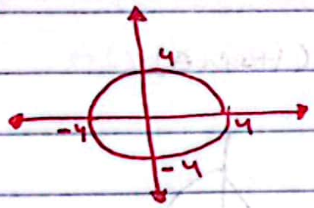
② $r = 3(1+\sin\theta)$





Ex: find on equation (In polar) for

①



$x^2+y^2 = 4^2$

circle center $(0,0)$

radias $= 4$

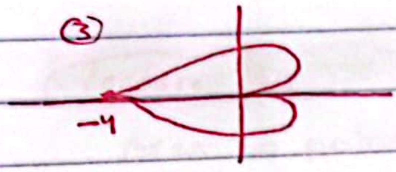
②

circle with center $(0,2)$

radia $= 2$

$r = 4\sin\theta$

③

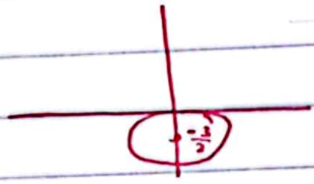


$r = a(1+\cos\theta)$

$a = 2$

$r = 2(1-\cos\theta)$

④



$r = 2b\sin\theta$

$r = 2(\frac{3}{2})\sin\theta$

$r = 3\sin\theta$

$\boxed{b = \frac{3}{2}}$

lecture 39

من 35 لـ 38 غير مطلوب

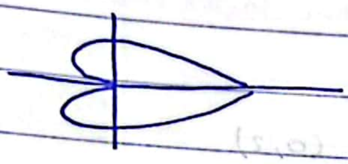
64

Area in polar

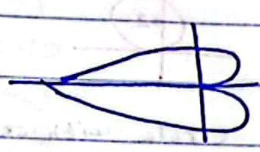
Recall:-

- $\theta = a \rightarrow$ line
- $a = r \rightarrow$ circle / center $(0,0)$ / radias $|a|$
- $r = 2a\cos\theta + 2b\sin\theta \rightarrow$ circle the center (a,b) radias $\sqrt{a^2+b^2}$
- $r = 2b\sin\theta \rightarrow$ center $(0,b)$ radias $|b|$
- $r = 2a\cos\theta \rightarrow$ " $(a,0)$ = $|a|$

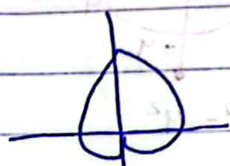
- $r = a(1+\cos\theta)$



- $r = a(1-\cos\theta)$



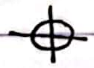
- $r = a(1+\sin\theta)$



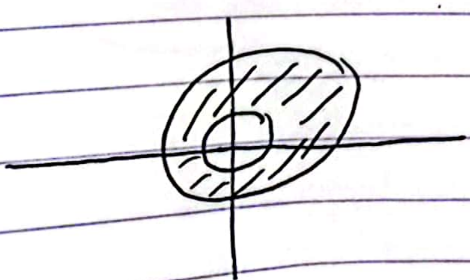
- $r = a(1-\sin\theta)$

Ex:-- sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions

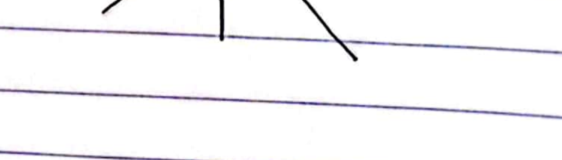
① $1 \le r \le 2$

$r=1$ 

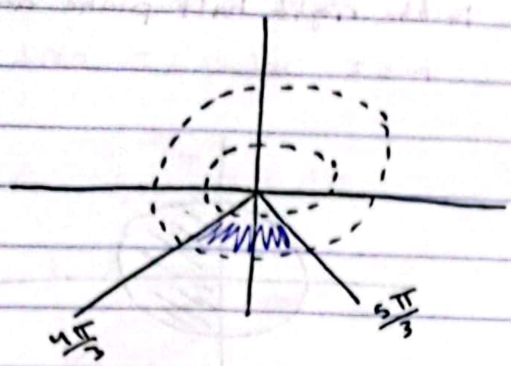
$r=2$ 



② $r \ge 0$, $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$



③ $2 < r < 3$, $\frac{4\pi}{3} \le \theta \le 5\frac{\pi}{3}$



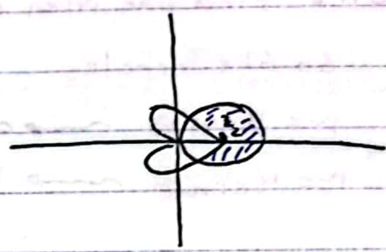
Ex: ① sketch the region inside $r=(1-\sin\theta)$

or " " " bounded by $r=(1-\sin\theta)$

② sketch the region inside $r=3\cos\theta$ outside $r=1+\cos\theta$

$r=3\cos\theta$

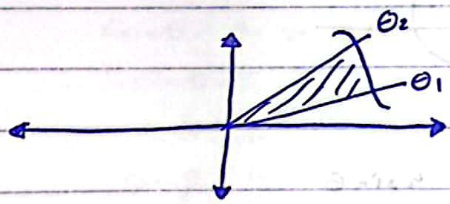
$2a=3 \rightarrow a=\frac{3}{2}$



Lecture 40

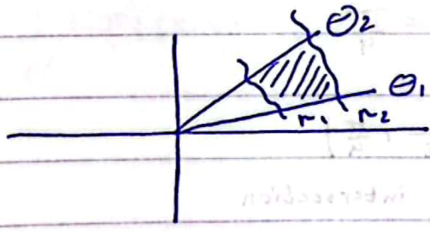
Area in polar :-

- $\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$.

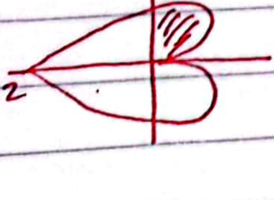


الأبعد — الأقرب

- $\frac{1}{2}\int_{\theta_1}^{\theta_2} (r_2)^2-(r_1)^2 \, d\theta$



Ex:- Find the area of shaded region



$r=a(1-\cos\theta)$

$r=1-\cos\theta$

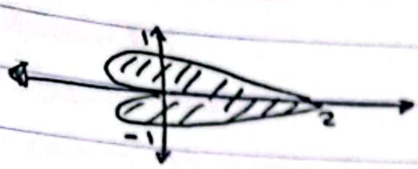
$\theta=0$ $\theta=\frac{\pi}{2}$

$A=\frac{1}{2}\int_{0}^{\frac{\pi}{2}} r^2 \, d\theta$

$=\frac{1}{2}\int_{0}^{\frac{\pi}{2}} (1-\cos\theta)^2 \, d\theta$

وبلا مني الجواب

Ex:- find the area of the regio. Inside $r = 1+\cos\theta$



$r = 1+\cos\theta$

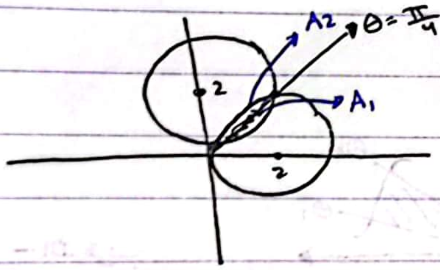
$0 \le \theta \le 2\pi$

$$\frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2$$ ومنطلع الجواب

Ex:- find the area that is common to the circles.

$r = 4\cos\theta \rightsquigarrow a = 2$

$r = 4\sin\theta \rightsquigarrow b = 2$



$4\cos\theta = 4\sin\theta$

$\theta = \frac{\pi}{4}$

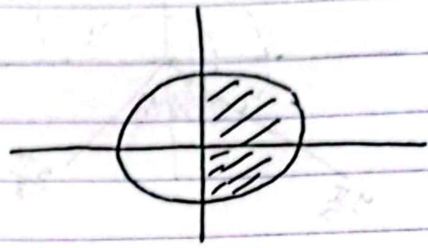
$\left(\frac{4}{\sqrt{2}}, \frac{\pi}{4}\right)$

point of intersection

$A = A_1 + A_2$

$$A = \frac{1}{2}\int_0^{\frac{\pi}{4}} (4\sin\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos\theta\, d\theta$$

Ex:- find the area of the region in the right half plane and Insid $r = 3$



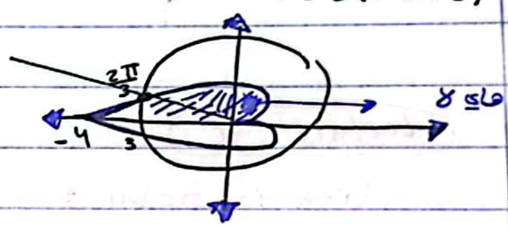
$$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3^2\, d\theta$$

Ex: find the area in the secound quardrant that is common

ربع ثاني

$r = 3$, $r = 2 - 2\cos\theta$

$r = 2(1-\cos\theta)$



To find point of Intersection

$3 = 2 - 2\cos\theta$

$\cos\theta = \frac{-1}{2}$ $\quad \theta = \frac{2\pi}{3}$

$\left(3, \frac{2\pi}{3}\right)$

$A = A_1 + A_2$

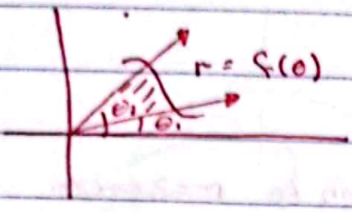
$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (2-2\cos\theta)^2 d\theta + \frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi} 3^2\, d\theta$$

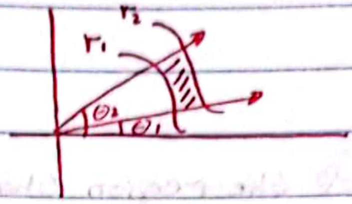
Week 8

Lecture 41

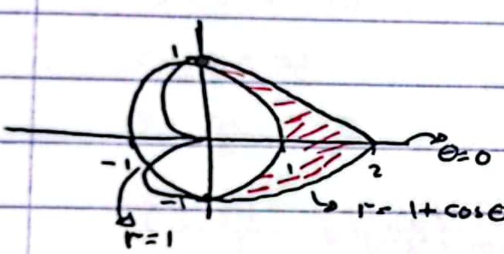
Area In polar

* $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \, d\theta$ * $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} (r_2)^2 - (r_1)^2 \, d\theta$





Ex:- find the area out side the circle $r=1$ and Inside $r = 1+\cos\theta$



$1+\cos\theta = 1$

$\cos\theta = 0$

$\theta = \frac{\pi}{2}, \frac{-\pi}{2}$

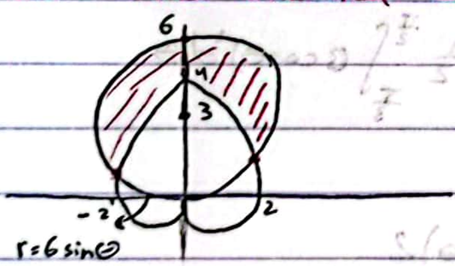
$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos\theta)^2 - 1^2 \, d\theta$

or

$A = \frac{1}{2} * 2 \int_{0}^{\frac{\pi}{2}} (1+\cos\theta)^2 - 1^2 d\theta$

لأنهم متماثلات

Ex:- find the area Inside $r = 6\sin\theta$ out side $r = 2+2\sin\theta$



$r = 2b\sin\theta \rightarrow b = 3 \quad (0,3), |b| = 3 = r$

$6\sin\theta = 2 + 2\sin\theta$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$(3, \frac{\pi}{6})\ (3, \frac{5\pi}{6})$

Points of Intersection

$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (6\sin\theta)^2 - (2+2\sin\theta)^2$

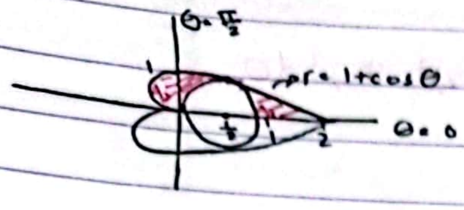
or

$A = \frac{1}{2} * 2 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (6\sin\theta)^2 - (2+2\sin\theta)^2$

Lecture 42 (68)

Ex:- Find the area of the region In upper half plane outside نصف العلوي $r=\cos\theta$ Inside $r=1+\cos\theta$

$2a=1 \rightarrow a=\frac{1}{2}$

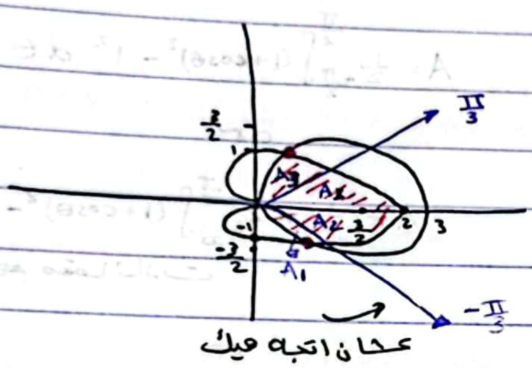


$A = A_1 + A_2$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos\theta)^2 - (\cos\theta)^2 + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (1+\cos\theta)^2 d\theta$

Ex:- Find the area of the region that is common to $r=3\cos\theta$ $r=1+\cos\theta$

$2a=3$
$a=\frac{3}{2}$
$(\frac{3}{2},0)$



$3\cos\theta = 1+\cos\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{7\pi}{3}$ or $-\frac{\pi}{3}$

$A = A_1 + A_2 + A_3$

$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{-\frac{\pi}{3}} (3\cos\theta)^2 d\theta + \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (1+\cos\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (3\cos\theta)^2 d\theta$

or

$A = \left(\frac{1}{2}\int_0^{\frac{\pi}{3}} (1+\cos\theta)^2 + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (3\cos\theta)^2 d\theta\right)2$

Ex:- Find the area enclosed by $r = 4\cos\theta + 2\sin\theta$

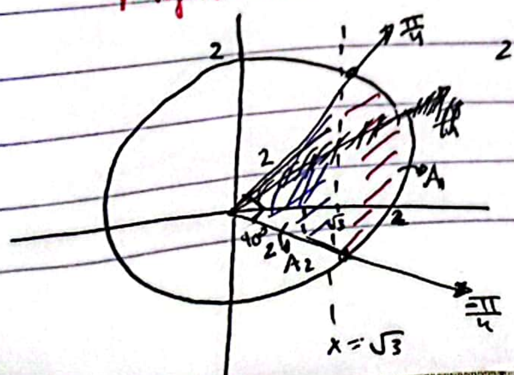
$2a=4$, $a=2$ $2b=2$, $b=1$

$(2,1)$ $r=\sqrt{5}$

Area $= \pi r^2$

$= \pi 5$

Ex:- Find the area In the region that is Inside $r=2$ and to the right of $r=\sqrt{3}\sec\theta \rightarrow r=\frac{\sqrt{3}}{\cos\theta} \rightarrow r\cos\theta=\sqrt{3}$ $x=\sqrt{3}$ vertical line



$2 = \sqrt{2}\sec\theta$

$\sec\theta = \sqrt{2}$

$\theta = \frac{\pi}{4}, -\frac{\pi}{4}$

$(2,\frac{\pi}{4})$ $(-2,\frac{\pi}{4})$

$A_1 = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2^2 d\theta = \pi$

$A_2 = \frac{1}{2}(2)(2) = 2$

$A = A_1 - A_2$

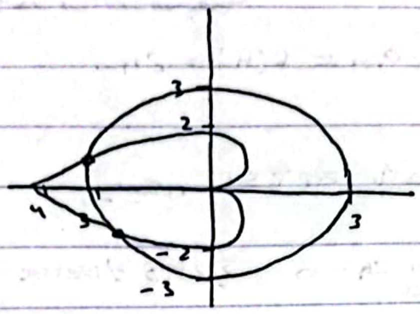
$= \pi - 2$

Lecture 43

Ex:- find all points of Intersection   نرسم ونجد النقاط

① $r = 3$

$r = 2 - 2\cos\theta$



$3 = 2 - 2\cos\theta$

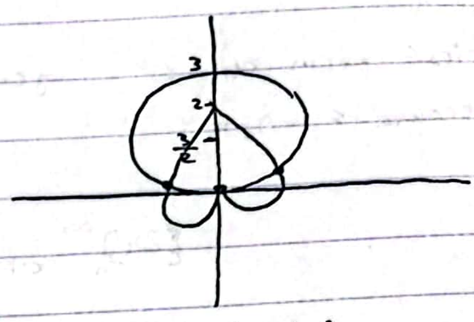
$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$(3, \frac{2\pi}{3})$ $(3, \frac{4\pi}{3})$

② $r = 3\sin\theta$ $2b = 3 \rightarrow b = \frac{3}{2}$

$r = 1 + \sin\theta$



$3\sin\theta = 1 + \sin\theta$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$(\frac{3}{2}, \frac{\pi}{6})$ $(\frac{3}{2}, \frac{5\pi}{6})$ $(0, \theta)$ → pole

$(0, 0)$

$(0, \pi)$

$(0, \frac{\pi}{2})$

$(0, \frac{3\pi}{2})$

نجرب بهم وأي معطي نفس الرقم بعد التعويض في ② بكون الجواب

الجواب $(0, \pi)$ لـ $r = 3\sin\theta$

$(0, \frac{3\pi}{2})$ لـ $r = 1 + \sin\theta$

ومنكتبها بدل نقاط pole

Lecture 44+45 غير مطلوبين

Lecture 46

70

Sequences المتتاليات

$$f(x) = a_n \quad \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5, \dots \\ [2, & 4, & 6, & 8, & 10, 12, \dots] \end{matrix}$$

$n = x$

تعريفها : اقتران مجاله مجموعة الأعداد الطبيعية الموجبة ومداه مجموعة جزئية من الأعداد الحقيقية

first term $a_1 = 2$ general term $= a_n = f(n) = 2n$

seccound $= a_2 = 4$

ويمكن الكتابة بأشكال اخرى

$\{2n\}_{n_1}^{\infty}$ or $\{2n\}$ $\lim_{n\to\infty} 2n = \infty$ $\{2n\}$ diverge

Ex:- Find general term of :-

① $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

$a_1 = 1$

$a_2 = \frac{1}{2}$ $a_n = \frac{1}{n}$

$a_3 = \frac{1}{3}$

$\{\frac{1}{n}\}_{n=1}^{\infty}$

$\left.\begin{matrix} n \to \infty \\ \frac{1}{n} \to 0 \end{matrix}\right\} \to \lim_{n\to\infty} \frac{1}{n} = 0 \to$ converge to 0

② $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$

$a_1 = \frac{1}{2}$

$a_2 = \frac{2}{3}$ $a_n = \frac{n}{n+1}$

$a_3 = \frac{3}{4}$

$\{\frac{n}{n+1}\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \frac{n}{n+1} = 1 \to$ converge to 1

③ $\{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \dots\}$

$a_1 = -1$

$a_2 = \frac{1}{2}$ $a_n \frac{(-1)^n}{n}$

$a_3 = \frac{-1}{3}$

$\{\frac{(-1)^n}{n}\}_{n=1}^{\infty}$ or $\left\{\begin{matrix} \frac{-1}{n}, n \text{ (odd)} \\ \frac{1}{n}, n \text{ (even)} \end{matrix}\right\}$

* Definition :- A sequence $\{a_n\}$ has the limit L and we write:-

$\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$

* If $\lim_{n\to\infty} a_n$ exist we say the sequence converges (or is convergent)

otherwise we say the sequence diverges (or is divergent)

* If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is constant then

1) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

2) $\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n$

3) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$, If $\lim_{n\to\infty} b_n \neq 0$

5) $\lim_{n\to\infty} (a_n)^p = (\lim_{n\to\infty} a_n)^p$, If $p > 0$ and $a_n > 0$

Lecture 47

Ex:- Find the first four terms and find the limit

① $\left\{ \frac{n-1}{2n+1} \right\}_{n=1}^{\infty}$

$a_1 = 0$

$a_2 = \frac{1}{5}$

$a_3 = \frac{2}{7}$

$a_4 = \frac{3}{9} = \frac{1}{3}$

$\lim_{n \to \infty} \frac{n-1}{2n+1} = \frac{\infty}{\infty}$

$\lim_{n \to \infty} \frac{1}{2} = \frac{1}{2}$

$\left\{ \frac{n-1}{2n+1} \right\}_{n=1}^{\infty} \longrightarrow \frac{1}{2}$ as $n \to \infty$

or

$\left\{ \frac{n-1}{2n+1} \right\}_{n=1}^{\infty}$ converge to $\frac{1}{2}$

② $\left\{ \frac{2n-1}{3n^2-1} \right\}_{n=1}^{\infty}$

$a_1 = \frac{1}{2}$

$a_2 = \frac{3}{11}$

$a_3 = \frac{5}{26}$

$a_4 = \frac{7}{47}$

$\lim_{n \to \infty} \frac{2n-1}{3n^2-1} = \lim_{n \to \infty} \frac{\infty}{\infty}$

$\lim_{n \to \infty} \frac{2}{6n} = 0$

$\therefore \left\{ \frac{2n-1}{3n^2-1} \right\}$ converge to 0

Ex: Determine whether the sequence converge or diverge

① $\{8-2n\}_{n=1}^{\infty}$

$\lim_{n \to \infty} 8-2n = \lim_{n \to \infty} -2n = -\infty$

diverge

② $\left\{ \frac{n^2+3n-1}{e^{2n}} \right\}_{n=1}^{\infty}$

$\lim_{n \to \infty} \frac{n^2+3n-1}{e^{2n}} = \frac{\infty}{\infty}$

$\lim_{n \to \infty} \frac{2n+3}{2e^{2n}} = \frac{\infty}{\infty}$

$\lim_{n \to \infty} \frac{2}{4e^{2n}} = 0$ converge

③ $\{\tan^{-1}(\sqrt[n]{n})\}_{n=1}^{\infty}$

$\lim_{n \to \infty} \tan^{-1}(\sqrt[n]{n}) = \tan^{-1}\left(\lim_{n \to \infty} n^{\frac{1}{n}}\right)$ ∞^0

$\lim_{n \to \infty} n^{\frac{1}{n}}$

$y = n^{\frac{1}{n}}$

$\ln y = \frac{1}{n} \ln n$

$\lim_{n \to \infty} \frac{1}{n} \ln n = 0 \cdot \infty$

$\lim_{n \to \infty} \frac{\ln n}{n} = \frac{\infty}{\infty}$

$\lim_{n \to \infty} \frac{\frac{1}{n}}{1} = 0$

$e^0 = 1$

$\tan^{-1}(1) = \frac{\pi}{4}$ converge to $\frac{\pi}{4}$

④ $\left\{ \left(1+\frac{1}{n}\right)^{-n} \right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$

Recall :-

$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^{bn} = e^{ab}$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = e^{1(-1)} = e^{-1} = \frac{1}{e}$

Converge to $\frac{1}{e}$

⑤ $\left\{ \left(\frac{n+3}{n+1}\right)^{n} \right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \left(\frac{n+3}{n+1}\right)^{n} \to 1^{\infty}$

$\lim_{n\to\infty} \left(\frac{\frac{n+3}{n}}{\frac{n+1}{n}}\right)^{n} = \frac{\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^{n}}{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}}$

$= \frac{e^{3}}{e^{1}} = e^{2}$

converge to e^2

⑥ $\left\{ \sqrt{n^2+n+1} - n \right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \sqrt{n^2+n+1} - n * \frac{\sqrt{n^2+n+1}+n}{\sqrt{n^2+n+1}+n}$

$\lim_{n\to\infty} \frac{n^2+n+1-n^2}{\sqrt{n^2+n+1}+n} = \lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+n+1}+n}$

$\lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+n}$

$\lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1\right)}$

$\lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1}\right) = \frac{1+0}{\sqrt{1+0+0}+1} = \frac{1}{2}$

converge to $\frac{1}{2}$

Lecture 48

Theorem:- A sequence converge to L Iff even terms converge to L and odd terms converge to L

Ex:- Determine whether the sequence converge

① $\left\{\frac{(-1)^{n+1}}{n}\right\}_{n=1}^{\infty} = \left\{\frac{1}{1}, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \dots\right\}$

$$= \begin{cases} \frac{-1}{n}, & n \text{ even} \\ \frac{1}{n}, & n \text{ odd} \end{cases}$$

$$\lim \frac{(-1)^{n+1}}{n} = \begin{cases} \lim\limits_{n\to\infty} \frac{-1}{n}, & n \text{ even} & = 0 \\ \lim\limits_{n\to\infty} \frac{1}{n}, & n \text{ odd} & = 0 \end{cases}$$ converge to 0

② $\left\{\frac{(-1)^{n+1} n}{2n+1}\right\}_{n=1}^{\infty}$

$$= \begin{cases} \frac{n}{2n+1}, & n \text{ odd} \\ \frac{-n}{2n+1}, & n \text{ even} \end{cases}$$

$$\lim \frac{(-1)^{n+1} n}{2n+1} = \begin{cases} \lim\limits_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \\ \lim\limits_{n\to\infty} \frac{-n}{2n+1} = \frac{-1}{2} \end{cases}$$ diverge

*Theorem: If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$

Ex:- Determine whether the sequence converge

① $\left\{\frac{(-1)^{n+1}}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \dots\right\}$

$$\begin{cases} \frac{1}{n}, & n \text{ odd} \\ \frac{-1}{n}, & n \text{ even} \end{cases}$$

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

By Theorem $\lim\limits_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$

converge to 0

Week 9

75

lecture 49

* The sequence $\{r^n\}_{n=1}^{\infty}$

[1] converge to zero If $-1<r<1$

Ex:- $\{(\frac{-1}{2})^n\}_{n=1}^{\infty} = \begin{cases} \frac{-1}{2^n}, & \text{odd} \\ \frac{1}{2^n}, & \text{even} \end{cases}$

[2] converge to 1 if $r=1$

Ex:- $\{(1)^n\}_{n=1}^{\infty} = \{1,1,1,1,\dots\}$

→ converge If $-1<r\leq 1$

[3] Diverge If $r>1$, $r\leq -1$

Ex:- $\{(-3)^n\}_{n=1}^{\infty} = \begin{cases} \infty \\ -\infty \end{cases}$ d.n.E

$\{(3^n)\}_{n=1}^{\infty} \rightarrow \{\infty$

Ex:- Determine whether the sequence converge or diverge

① $\{(\frac{-1}{2})^n\}_{n=1}^{\infty}$

$\lim_{n\to\infty}(\frac{-1}{2})^n = \begin{cases} \lim_{n\to\infty} \frac{-1}{2^n} = 0 & \text{odd} \\ \lim_{n\to\infty} \frac{1}{(2)^n} = 0 & \text{even} \end{cases}$

Converge to Zero

② $\{\frac{1}{(3)^n}\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \frac{1}{(3)^n} = 0$

converge to Zero

③ $\{\frac{\pi^n}{5^n}\}_{n=1}^{\infty} = \{(\frac{\pi}{5})^n\}_{n=1}^{\infty}$

$r = \frac{\pi}{5} < 1$ converge to Zero

④ $\{\frac{3^{n+2}}{5^n}\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \frac{3^{n+2}}{5^n} = \lim_{n\to\infty} \frac{3^2 \cdot 3^n}{5^n}$

$= \lim_{n\to\infty} 9 \frac{3^n}{5^n} = 9 \lim_{n\to\infty} (\frac{3}{5})^n$

$r = \frac{3}{5} < 1$ $= 9(0) = 0$

⑤ $\{(\frac{1}{2})^{-n}\}_{n=1}^{\infty} = \{2^n\}_{n=1}^{\infty}$

$r = 2 > 1$ diverge

⑥ $\{\frac{1}{2+3^{-n}}\}_{n=1}^{\infty}$

$\lim_{n\to\infty} \frac{1}{2+3^{-n}} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{(3)^n}}$

$= \frac{1}{2+0} = \frac{1}{2}$

converge to $\frac{1}{2}$

Ex:- find The value of (a) such that $\{(\frac{a}{3})^n\}_{n=1}^{\infty}$ converge

$r = \frac{a}{3}$ converg If :-

$-1 < \frac{a}{3} \leq 1$

$-3 < a \leq 3$

$a \in (-3, 3]$

Lecture 50

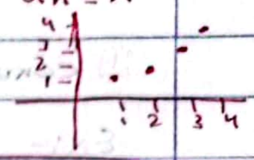
Secqueezing Theorem

نقاط فقط

* Let a_n, c_n, b_n sequences such that $a_n \le c_n \le b_n$
If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$ then $\lim_{n\to\infty} c_n = L$

رسمة sequence

$a_n = n$



Ex:- Determine whether the sequence converge

① $\left\{\frac{\sin^2 n}{n}\right\}_{n=1}^{\infty}$ $\to c_n$

$-1 \le \sin n \le 1$

$\frac{0}{n} \le \frac{\sin^2 n}{n} \le \frac{1}{n}$

$0 \le \frac{\sin^2 n}{n} \le \frac{1}{n}$

$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$

Then by Theorem

$\lim_{n\to\infty} \frac{\sin^2 n}{n} = 0$

converge to 0

② $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$ $\to c_n$

$n! = n(n-1)(n-2), \dots$

$(n+1)! = (n+1)\cdot n(n-1)(n-2)$

$= (n+1)n!$

$= (n+1)n(n-1)!$

$(2n)! = 2n(2n-1)(2n-2)!$

$n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1$

$$0 < \frac{n(n-1)(n-2)\dots 3*2*1}{\underbrace{n\cdot n\cdot n\cdot n\cdot \dots \cdot n}_{n \text{ terms}}} <$$

$$0 < \frac{1}{n}\left(\frac{2*3*4\dots(n-2)(n-1)n}{n\cdot n\cdot n\dots n\cdot n\cdot n}\right) \le \frac{1}{n}(1)$$

البسط رح يكون اقل من او يساوي 1

$0 < \frac{n!}{n^n} \le \frac{1}{n}$

$\lim_{n\to\infty} 0 = 0$

$\lim_{n\to\infty} \frac{1}{n} = 0$

by Theorem

$\lim_{n\to\infty} \frac{n!}{n^n} = 0$

Sequences defined recursvely :-

$a_1 = c$

$a_1, a_2, a_3, a_4, \dots, a_n, a_{n+1}, a_{n+2}, \dots$

بيعطيك اول حد وبيطلب الباقي

Ex:- $a_1 = 1$

$a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right)$ for all $n \geq 1$

لو طلب a_{10}

$a_{10} = \frac{1}{2}\left(a_9 + \frac{3}{a_9}\right)$

وبدي اعرف a_9

① find a_2, a_3, a_4

$$a_2 = \frac{1}{2}\left(a_1 + \frac{3}{a_1}\right) = \frac{1}{2}(1+3) = 2$$

$$a_3 = \frac{1}{2}\left(a_2 + \frac{3}{a_2}\right) = \frac{1}{2}\left(2 + \frac{3}{2}\right) = \frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4}$$

$$a_4 = \frac{1}{2}\left(a_3 + \frac{3}{a_3}\right) = \frac{1}{2}\left(\frac{7}{4} + \frac{3}{\frac{7}{4}}\right) = \frac{1}{2}\left(\frac{7}{4} + \frac{12}{7}\right) = \frac{51}{28} * \frac{1}{2} = \frac{51}{56}$$

② Limit for a_n

Let $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$

$a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right)$

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{3}{a_n}\right)$

$L = \frac{1}{2}\left(L + \frac{3}{L}\right)$

$2L = L + \frac{3}{L}$

$L = \frac{3}{L}$

حسب سلوك Lim انه مثل موجب

$L^2 = 3$

$L = \pm\sqrt{3}$

$\therefore L = +\sqrt{3}$

$a_1, a_2, a_3, \dots, a_n, a_{n+1}, a_{n+2}$

$a_n = \frac{1}{n}$

a_n $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_{n+2} = \dots$

$\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} \frac{1}{n+2} = \dots$

Lecture 51

Definition :- A sequence $\{a_n\}$ Is called

Increasing If $a_1 < a_2 < a_3 \ldots\ldots a_n < a_{n+1} < a_{n+2} < \ldots$ ($a_n < a_{n+1}$ for all $n \geq 1$)

Decreasing If $a_1 > a_2 > a_3 > \ldots\ a_n > a_{n+1} > a_{n+2} \ldots$ ($a_n > a_{n+1}$ for all $n \geq 1$)

A sequence Is monotonic if it is increasing or decreasing

Testing for monotonicity

	Increasing ($a_{n+1} > a_n$)		Decreasing ($a_{n+1} < a_n$)
① Differences	$a_{n+1} - a_n > 0$		$a_{n+1} - a_n < 0$
② Ratio	$\frac{a_{n+1}}{a_n} > 1$	يفضل $\frac{n!}{10^n}$	$\frac{a_{n+1}}{a_n} < 1$
③ Differntiation	$a_n = f(x)$ $f'(x) > 0$	يفضل مثل ا $\tan^{-1} n$ $\ln n$	$a_n = f(x)$ $f'(x) < 0$

حدود موجبة

Ex:- Determine whether the sequence monotonic

① $\left\{ 1 - \frac{1}{n} \right\}_{n=1}^{\infty}$

$a_n = 1 - \frac{1}{n} \quad a_{n+1} = 1 - \frac{1}{n+1}$

① $a_{n+1} - a_n$

$1 - \frac{1}{n+1} - 1 + \frac{1}{n}$

$\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)}$ → موجب

Increasing

② $\frac{a_{n+1}}{a_n} = \frac{1 - \frac{1}{n+1}}{1 - \frac{1}{n}} = \frac{\frac{n+1-1}{n+1}}{\frac{n-1}{n}}$

$\frac{\frac{n}{n+1}}{\frac{n-1}{n}} = \frac{n^2}{(n^2-1)} > 1$

Increasing

③ $f(x) = 1 - \frac{1}{x}$

$f'(x) = \frac{1}{x^2}$

$\therefore f'(x) > 0$

Increasing

~~$f(x) = 1 - \frac{1}{x+1}$~~

~~$f'(x) =$~~

② $\left\{\frac{10^n}{n!}\right\}_{n=1}^{\infty}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} = \frac{10^{n+1} * n!}{(n+1)!\ 10^n}$$

$$= \frac{10^n\ 10^1\ n!}{(n+1)n!\ 10^n} = \frac{10}{n+1} > \text{or} <$$

$n=1 \rightarrow 5>1$

$n=2 \rightarrow \frac{10}{3}>1$

$n=3 \rightarrow \frac{10}{4}>1$

$n=9 \rightarrow \frac{10}{10} \cancel{>} 1$

$n=10 \rightarrow \frac{10}{11}<1$

$n=12 \rightarrow \frac{10}{13}<1$

$n=100 \rightarrow \frac{10}{101}<1$

رح تضل اقل من 1

Decreacing

$\frac{10}{n+1} \leq 1 \quad \forall n \geq 10$

or

$\forall n > 9$

وهاد الجواب

$\frac{10}{n+1} \leq \ \forall n \geq 9$

③ $\{n^5 e^{-n}\}_{n=1}^{\infty}$

$a_n = f(x)$

$= x^5 e^{-x}$

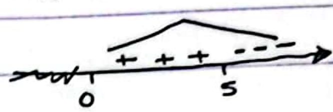
$f'(x) = x^5(-e^{-x}) + e^{-x} \cdot 5x^4$

$= x^4 e^{-x}(5-x) = 0$

$x^4 = 0 \rightarrow x = 0$

$e^{-x} \neq 0$

$5 - x = 0 \rightarrow x = 5$



Decreasing $(5, \infty)$

Increasing $(0, 5)$

$a_n = \{n^5 e^{-n}\}_{n=1}^{\infty}$

Decreasing

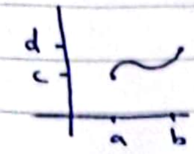
لأنه ~~...~~ اخر حدود بوصفها

ملخص من فيديو 21 لـ 52

① Arc length (L)

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

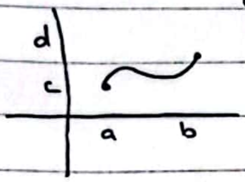
$$L = \int_c^d \sqrt{1+(f'(y))^2}\,dy$$



② Area of surface (S)

$$S = \int 2\pi y\,ds$$

$$S = \int_a^b 2\pi y\sqrt{1+(f'(x))^2}\,dx \qquad \int_c^d 2\pi y\sqrt{1+(f'(y))^2}\,dy$$



③ + ④ + ⑤ $f(x) = \sqrt{r^2 - x^2}$

Area of circle $= \pi r^2$

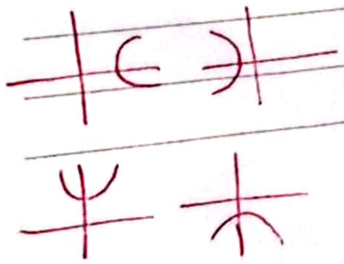
circumference $= 2\pi r$

surface area of ball (sphere) $= 4\pi r^2$

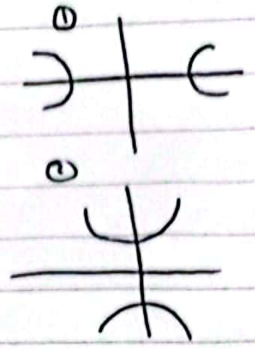
⑥ circle with center a,b radius r

$$(x-a)^2 + (y-b)^2 = r^2$$

⑦ parabola



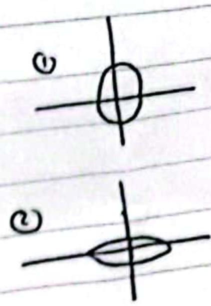
⑧ Hyperbola



① $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

② $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

⑨ Ellipse



① $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

② $\dfrac{(y-k)^2}{a^2} + \dfrac{(x-h)^2}{b^2} = 1$

⑩ parametric equation

ⓐ The line pass through (x_0, y_0), (x_1, y_1)

$x(t) = x_0 + (x_1 - x_0)t$

$y(t) = y_0 + (y_1 - y_0)t$

ⓑ The line $y = ax + b$

$x(t) = t$

$y(t) = at + b$

ⓒ The line $a = x$

$x(t) = a$

$y(t) = t$

ⓓ The line $b = y$

$x(t) = t$

$y(t) = b$

ⓔ The circle with radias r, center (a,b) and counter clock wise

$x(t) = a + r\cos t \qquad 0 \le t \le 2\pi$

$y(t) = b + r\sin t$

⑪ calculas with parametric curves

① Tangents :- $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

② slop of tangent line :- $\left.\dfrac{dy}{dx}\right|_{t=}$

③ Horizontal tangent line :- $\dfrac{dy}{dt} = 0$, $\dfrac{dx}{dt} \neq 0$

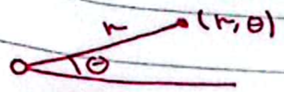
④ vertical 〃 〃 :- $\dfrac{dy}{dt} \neq 0$, $\dfrac{dx}{dt} = 0$

⑤ $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

⑥ If $\dfrac{d^2y}{dx^2} > 0$ (concave upward)

⑦ 〃 〃 < 0 (concave downward)

Polar coordinate



* we can write (r,θ) with a differente ways :-

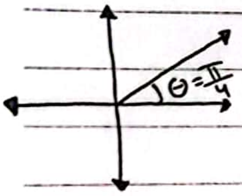
① $(r, \theta+2\pi)$
② $(r, \theta-2\pi)$
③ $(-r, \theta+\pi)$
④ $(-r, \theta-\pi)$

* Relation between polar coordinat and rectangular (x,y)

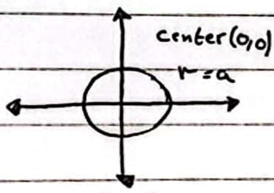
① $x^2+y^2=r^2$
② $\tan\theta=\frac{y}{x}$
③ $\cos\theta=\frac{x}{r} \rightarrow x=r\cos\theta$
④ $\sin\theta=\frac{y}{r} \rightarrow y=r\sin\theta$

* sketch The graph

1) $\theta=\frac{\pi}{4}$



2) $r=a \quad 0\le\theta\le 2\pi$



$r^2=a^2$
$x^2+y^2=a^2$

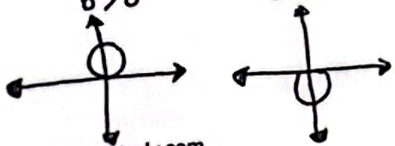
3) $r=2a\cos\theta+2b\sin\theta$

circle with radias $\sqrt{a^2+b^2}$ and center (a,b)

4) If $a=0$, $r=2b\sin\theta$

center $(0,b)$, radias $|b|$

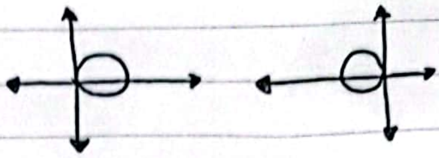
$b>0$ $\qquad$ $b<0$



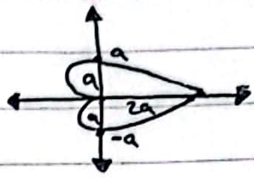
⑤ If $b=0$, $r=2a\cos\theta$

center $(a,0)$ radias $|a|$

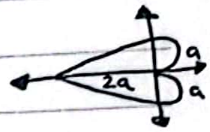
$a>0$ $\qquad$ $a<0$



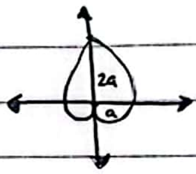
6) ⓐ $r=a(1+\cos\theta)$
$r=-a(1-\cos\theta)$



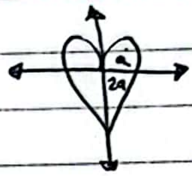
ⓑ $r=a(1-\cos\theta)$
$r=-a(1+\cos\theta)$



ⓒ $r=a(1+\sin\theta)$
$r=-a(1-\sin\theta)$

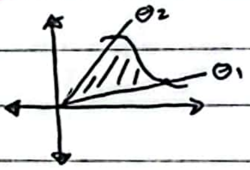


$r=a(1-\sin\theta)$
$r=-a(1+\sin\theta)$

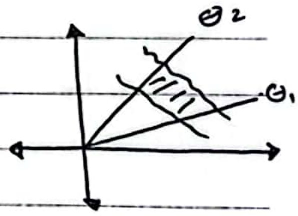


* Area In polar

$$A=\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta$$



$$A=\frac{1}{2}\int_{\theta_1}^{\theta_2} (r_2)^2-(r_1)^2$$



Sequences المتتاليات

*① general term $= a_n = f(n) =$ ___

② If $\lim_{n\to\infty} a_n$ exist we say converge

" " not " " " diverge

③ If $\{a_n\}$ and $\{b_n\}$ ~~exist~~ are convergent sequences and C is constant then :-

1) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

2) $\lim_{n\to\infty} C a_n = C \lim_{n\to\infty} a_n$

3) $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, If $\lim_{n\to\infty} b_n \neq 0$

5) $\lim_{n\to\infty} (a_n)^p = (\lim_{n\to\infty} a_n)^p$, If $p>0$ and $a_n>0$

* Increasing and Decreasing for sequnces

↳ If $a_1 < a_2 < a_3 < \dots a_n < a_{n+1} < a_{n+2}$

Decreasing :- $a_1 > a_2 > a_3 > a_4 > \dots a_n > a_{n+1}$

Testing for monotonicity

↳ (If increasing or Decreasing)

	Increasing	Decreasing
① Difference	$a_{n+1} - a_n > 0$	$a_{n+1} - a_n < 0$
② Ratio	$\frac{a_{n+1}}{a_n} > 1$	$\frac{a_{n+1}}{a_n} < 1$
③ Differentiation	$a_n = f(x)$ $f'(x) > 0$	$a_n = f(x)$ $f'(x) < 0$

* رقم 3 يفضل لـ $\tan^{-1}$, $\sin^{-1}$, ... وكمان ln

* رقم 2 يفضل لـ $\frac{n!}{10^n}$

* بس اعمل وحدة منهم بجوز منها وبشوف إذا Increasing او Decreasing

* Theorem :-

A sequence converge to L Iff even terms converge to L and odd terms converge to L

تستخدم اكثر اشي لما يكون الأس (n+1) ونفس الشك ومكن عادي بس قوة (n)

* The sequence $\{r^n\}_{n=1}^{\infty}$

$\left.\begin{array}{l} \boxed{1} \text{ converge to Zero if } -1<r<1 \\ \boxed{2} \quad " \quad " \ 1 \text{ if } r=1 \end{array}\right\} \to \text{converge}$

$\boxed{3}$ Diverge if $r>1$ or $r \leq -1$

* secqueezing Theorem :- $a_n < C_n < b_n$

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$ Then $\lim_{n\to\infty} C_n = L$

lecture 52 غير مطلوب

Week 10

lecture 53

Series المتسلسلات

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + u_4 + \dots$$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$$

Ex:-

① $\sum_{n=1}^{\infty} \frac{3}{10^n}$ → a_n

$$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \dots$$

② $-1 + 1 + -1 + 1 + -1 + \dots$

$\sum_{n=1}^{\infty} (-1)^n$ or $\sum_{n=0}^{\infty} (-1)^{n+1}$

Defination

If $\sum u_k = u_1 + u_2 + \dots$

$S_1 = u_1$

$S_2 = u_1 + u_2 = S_1 + u_2$

$S_3 = u_1 + u_2 + u_3 = S_2 + u_3$

$S_n = u_1 + u_2 + u_3 + \dots + u_n = S_{n-1} + u_n$

Then $\{S_1, S_2, S_3, \dots\} = \{S_n\}_{n=1}^{\infty}$

is called sequence of partial sums

- If $\lim_{n\to\infty} S_n = S$ then $\sum u_k = S$ (converge to S)

- If $\lim_{n\to\infty} S_n$ = d.n.e then $\sum u_k$ has no sum (diverge)

Ex:- Determine whather the series converge:-

① $1 + -1 + 1 + -1 + 1 + -1 + \dots = \sum_{n=1}^{\infty} (-1)^{n+1}$

$\cancel{\lim_{n\to\infty} (-1)^{n+1}}$ $\lim_{n\to\infty} S_n = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

$S_1 = u_1 = 1$

$S_2 = S_1 + u_2 = 0$

$S_3 = S_2 + u_3 = 1$

$S_4 = S_3 + u_4 = 0$

→ $\{1, 0, 1, 0, \dots\}$

$S_n = \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$

seq of partial sums is Diverge

$\therefore 1 + -1 + 1 + -1 + \dots = \sum_{n=1}^{\infty} (-1)^{n+1}$ has no sum

② $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

$S_1 = u_1 = 1 - \frac{1}{2} = \frac{1}{2}$

$S_2 = S_1 + u_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$

$S_3 = S_2 + u_3 = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$

$S_4 = S_3 + u_4 = 1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} = 1 - \frac{1}{5}$

$\vdots$

$S_n = 1 - \frac{1}{n+1} \quad \therefore \left\{1 - \frac{1}{n+1}\right\}_{n=1}^{\infty}$

$\lim_{n\to\infty} S_n \rightarrow \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$

converge to 1

$\therefore \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$ converge to 1

$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1$

③ $\sum_{k=1}^{\infty} \frac{1}{k^2+5k+6} = \frac{1}{(k+3)(k+2)}$

$\frac{1}{(k+3)(k+2)} = \frac{A}{k+3} + \frac{B}{k+2}$

$1 = A(k+2) + B(k+3)$

$k=-2 \quad 1 = B$

$k=-3 \quad -1 = A$

$\sum_{k=1}^{\infty} \frac{-1}{k+3} + \frac{1}{k+2} = \sum_{k=1}^{\infty} \frac{1}{k+2} - \frac{1}{k+3}$

$S_1 = \frac{1}{3} - \frac{1}{4}$

$S_2 = S_1 + u_2 = \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5}$

$= \frac{1}{3} - \frac{1}{5}$

$S_3 = S_2 + u_3 = \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6} = \frac{1}{3} - \frac{1}{6}$

$\vdots$

$S_n = \frac{1}{3} - \frac{1}{n+3}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3} - \frac{1}{n+3} = \frac{1}{3} - 0 = \frac{1}{3}$

converge to $\frac{1}{3}$

Lecture 54

Ex:- Determine whether the series converge

① $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \ln(n+1) - \ln(n)$

$S_1 = \ln 2 - \ln 1 = \ln 2$

$S_2 = S_1 + u_2 = \ln 2 + \ln 3 - \ln 2 = \ln 3$

$S_3 = S_2 + u_3 = \ln 3 + \ln 4 - \ln 3 = \ln 4$

$\vdots$

$S_n = \ln(n+1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1) = +\infty$

$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ has no sum (diverge)

② $\sum_{k=2}^{\infty} \ln\left(1 - \frac{1}{k^2}\right) = \sum_{k=2} \ln\left(1-\frac{1}{k}\right)\left(1+\frac{1}{k}\right)$

$= \sum_{k=2}^{\infty} \ln\left(1-\frac{1}{k}\right) + \ln\left(1+\frac{1}{k}\right)$

$S_2 = \ln\left(1-\frac{1}{2}\right) + \ln\left(1+\frac{1}{2}\right) = \ln\frac{1}{2} + \ln\frac{3}{2}$

$S_3 = S_2 + u_3 = \ln\frac{1}{2} + \underbrace{\ln\frac{3}{2} + \ln\frac{2}{3}}_{=0} + \ln\frac{4}{3}$

$S_3 = \ln\frac{1}{2} + \ln\frac{4}{3}$

$S_4 = S_3 + u_4 = \ln\frac{1}{2} + \underbrace{\ln\frac{4}{3} + \ln\frac{3}{4}}_{=0} + \ln\frac{5}{4}$

$S_4 = \ln\frac{1}{2} + \ln\frac{5}{4}$

$\vdots$

$S_n = \ln\frac{1}{2} + \ln\frac{n+1}{n}$

$\lim_{n\to\infty} \ln\frac{1}{2} + \ln\frac{n+1}{n} = \ln\frac{1}{2} + \ln\left(\lim_{n\to\infty}\left(\frac{n+1}{n}\right)\right)$

$= \ln\frac{1}{2} + 0 = \ln\frac{1}{2}$

converge to $\frac{1}{2}$ 　 sum $= \frac{1}{2}$

③ $\sum_{K=1}^{\infty} \frac{\sqrt{K+1}-\sqrt{K}}{\sqrt{K(K+1)}} = \sum_{K=1}^{\infty} \frac{\sqrt{K+1}}{\sqrt{K}\sqrt{1+K}} - \frac{\sqrt{K}}{\sqrt{K}(\sqrt{K+1})}$

$= \sum_{K=1}^{\infty} \left(\frac{1}{\sqrt{K}} - \frac{1}{\sqrt{K+1}}\right)$

$S_1 = 1 - \frac{1}{\sqrt{2}}$

$S_2 = S_1 + U_2 = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}$

$S_3 = 1 - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} = 1 - \frac{1}{\sqrt{4}}$

⋮

$S_n = 1 - \frac{1}{\sqrt{n+1}}$

$\lim_{n\to\infty} 1 - \frac{1}{\sqrt{n+1}} = 1 - 0 = 1$

converge to 1

Ex:- calculate the sum of $\sum a_n$ whose partial sums are given

① $S_n = 2 - 3(0.8)^n$

$\sum a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 - 3(0.8)^n$

$= 2 - 3\lim (0.8)^n$

$= 2 + 0 = 2$

② $S_n = 2 + \frac{3n}{n+1}$

$\lim_{n\to\infty} 2 + \frac{3n}{n+1} = 2 + 3 = 5$

Ex:- suppose we know the sum of the first n terms of the series is $S_n = a_1 + a_2 + \dots + a_n = 2 + \frac{3n}{n+1}$ find the sum of the series

$\sum a_n$ = sum of series $= \lim_{n\to\infty} S_n$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 + \frac{3n}{n+1} = 2 + 3 = 5$

* Theorem:-

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum c a_n$ (where c constant), $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$

① $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$

② $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

Lecture 55

Geometric series:-

((constant)^k)

$$\sum a_k = \sum_{k=1}^{\infty} a r^{k-1}$$

a حد أول ، r نسبة ثابتة

Ex:- ① $3+1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots$

$\frac{a_2}{a_1}=\frac{1}{3}$

$\frac{a_3}{a_2}=\frac{\frac{1}{3}}{1}=\frac{1}{3}$

$\frac{a_4}{a_3}=\frac{\frac{1}{9}}{\frac{1}{3}}=\frac{1}{3}$ Geometric series

$$\sum_{k=1}^{\infty} 3\left(\frac{1}{3}\right)^{k-1} = \sum_{k=0}^{\infty} 3\left(\frac{1}{3}\right)^{k}.$$

② $18+0.18+0.0018+\dots$

$\frac{a_2}{a_1}=0.01$

$\frac{a_3}{a_2}=0.01 \rightarrow r=0.01$

Geometric series

$$\sum_{k=1}^{\infty} 18(0.01)^{k-1}$$

③ $1+2+3+4+5+6\dots$

$\frac{a_2}{a_1}=2$

$\frac{a_3}{a_2}=\frac{3}{2}$ not geometric series

Theorem:-

$$\sum_{k=n}^{\infty} r^k = \begin{cases} \text{converge} = \frac{r^n}{1-r} & \text{if } |r|<1 \rightarrow -1<r<1 \\ \text{diverge} & \text{if } |r|\ge 1 \rightarrow r\ge 1,\ r\le -1 \end{cases}$$

Ex: Determine whether the series converge or diverge

① $\sum_{k=2}^{\infty}\left(\frac{1}{2}\right)^{k-1} = \sum_{k=2}^{\infty}\left(\frac{1}{2}\right)^{k}\left(\frac{1}{2}\right)^{-1} = 2\sum_{k=2}\left(\frac{1}{2}\right)^{k} \rightarrow r=\frac{1}{2},\ |r|=\frac{1}{2}<1$ converge

The sum $2\left(\frac{\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}}\right) = 2\left(\frac{\frac{1}{4}}{\frac{1}{2}}\right)=1$

طريقة اخرى في الكتابة

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k-1+1} = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k}$$

$$\sum_{k=6}^{\infty}\frac{1}{k+6} = \sum_{k=7}\frac{1}{(k-1)+6} = \sum_{k=0}\frac{1}{k+6+6}$$

② $\sum_{k=1}^{\infty}\frac{5}{4^k} = 5\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k,\ r=\frac{1}{4}\ \ |r|=\frac{1}{4}<1$ converge to

$5 * \frac{\left(\frac{1}{4}\right)}{1-\frac{1}{4}} = \frac{5}{3}$

③ $\sum_{k=1}^{\infty} 3^{2k}\,5^{1-k}$

$$\sum_{k=1}^{\infty} ((3)^k)^2\, 5^1 (5)^{-k}$$

$$5\sum_{k=1}^{\infty}\left(\frac{9}{5}\right)^k \quad r=\frac{9}{5},\ |r|=\frac{9}{5}>1 \text{ diverge}$$

④ $\sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{7}{9^{k-1}}$

$7 \sum_{k=1}^{\infty} \left(\frac{-1}{9}\right)^{k-1}$

$7 \sum_{k=0}^{\infty} \left(\frac{-1}{9}\right)^{k}$ $r = \frac{-1}{9}$, $|r| = \frac{1}{9} < 1$

$\frac{7 * \left(\frac{-1}{9}\right)^0}{1 - -\frac{1}{9}} = \frac{63}{10}$ converge

⑤ $\sum_{k=5}^{\infty} \left(\frac{e}{\pi}\right)^{k-1}$

$\sum_{k=4}^{\infty} \left(\frac{e}{\pi}\right)^{k}$

$r = \frac{e}{\pi}$, $|r| = \frac{e}{\pi} \approx \frac{2.7}{3.14} < 1$

converge to $\frac{\left(\frac{e}{\pi}\right)^4}{1 - \frac{e}{\pi}} = \frac{e^4}{e^3(\pi - e)}$

⑥ $\sum_{k=1}^{\infty} e^{-k}$

$\sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$ $\frac{1}{e} = r$, $|r| = \frac{1}{e} < 1$

$\frac{\frac{1}{e}}{1 - \frac{1}{e}}$ converge to

⑦ $\sum_{k=1}^{\infty} \cos(\pi k)$

$= \cos(\pi) + \cos(2\pi) + \cos(3\pi) + \dots$

$-1 + 1 + -1 + 1 + \dots$

$\sum_{k=1}^{\infty} (-1)^k = \sum_{k=0}^{\infty} (-1)^{k+1}$

نحتاج هاي

$r = -1$, $|r| = 1 = 1$ diverge

⑧ $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27}$

$\frac{a_2}{a_1} = \frac{-\frac{10}{3}}{5} = -\frac{2}{3}$

$\frac{a_3}{a_2} = \frac{\frac{20}{9}}{\frac{-10}{3}} = -\frac{2}{3}$

or

$5\left(1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \dots\right)$

$\frac{a_2}{a_1} = \frac{-2}{3}$

$\frac{a_3}{a_2} = \frac{-2}{3}$

$5 \sum_{k=1}^{\infty} 1 \left(\frac{-2}{3}\right)^{k-1} = 5 \sum_{k=0} \left(\frac{-2}{3}\right)^k$

$r = \frac{-2}{3}$, $|r| = \frac{2}{3} < 1$

$\frac{5 * \left(\frac{-2}{3}\right)^0}{1 - \frac{-2}{3}} = \cancel{15}$ converge to 3

Lecture 56

Ex:- Determine whether the series converge

$$\sum_{k=1}^{\infty}\left(\left(\frac{1}{\pi}\right)^k+\left(\frac{3}{\pi}\right)^k\right)$$

$\left|\frac{1}{\pi}\right|<1$ converg $\left|\frac{3}{\pi}\right|<1$ converg

$$\sum_{k=1}^{\infty}\left(\frac{1}{\pi}\right)^k+\sum_{k=1}^{\infty}\left(\frac{3}{\pi}\right)^k$$

$$\frac{\frac{1}{\pi}}{1-\frac{1}{\pi}}+\frac{\frac{3}{\pi}}{1-\frac{3}{\pi}}=\frac{1}{\pi-1}+\frac{3}{\pi-3}$$

Ex:- If $\sum_{k=2}^{\infty} 2r^k=1$ find r

$$2\sum_{k=2}^{\infty} r^k$$

$$2\frac{r^2}{1-r}=1$$

$$2r^2=1-r$$

$$2r^2+r-1=0$$

$$(2r-1)(r+1)$$

$2r=1$ $r=-1$ غير مقبولة لأنه تخليه diverge

$r=\frac{1}{2}$

Ex: find the set of all values of r for which series $\sum_{k=1}^{\infty}\frac{1}{r^k}$ converg

$$\sum_{k=1}^{\infty}\left(\frac{1}{r}\right)^k \qquad \left|\frac{1}{r}\right|<1$$

$$\frac{\frac{1}{r}}{1-\frac{1}{r}} \qquad \frac{1}{|r|}<1$$

$$|r|>1$$

$$r>1,\ r<-1$$

The set of all values of r for which the series converge

$$(1,\infty)\cup(-\infty,-1)$$

Ex: white The number $0.\overline{3}$ as a ratio of Integers (In the form $\frac{a}{b}$)

$$0.\overline{3}=0.333333$$

$$=0.3+0.03+0.003+0.0003+\cdots$$

$$\frac{0.03}{0.3}=0.1$$

$$\frac{0.003}{0.03}=0.1$$

$$0.\overline{3}=\sum_{k=1}^{\infty}\frac{3}{10}\left(\frac{1}{10}\right)^{k-1}$$

$$=\frac{3}{10}\sum_{k=0}^{\infty}\left(\frac{1}{10}\right)^k$$

$$\frac{3}{10}\ \frac{\left(\frac{1}{10}\right)^0}{1-\frac{1}{10}}=\boxed{\frac{1}{3}}$$

Ex: find all values of x that make the series converge and find the sum of series

① $x-x^3+x^5-x^7+x^9-\ldots$

$r=\frac{-x^3}{x}=-x^2 \qquad a=x$

$$\sum_{k=1}^{\infty}x(-x^2)^{k-1}$$

$r=-x^2$

$|r|=|-x^2|<1$

$x^2<1$

$\sqrt{x^2}<1$

$x<1$

$-1<x<1$

$$x\in(-1,1)$$

$$x\sum_{k=0}(-x^2)^k$$

$$\frac{x\ (-x^2)^0}{1-(-x^2)}=\frac{x}{1+x^2}$$

Lecture 57

Theorem:- (The divergent test)

If $\sum a_k$ converge then $\lim_{k\to\infty} a_k = 0$

If $\lim_{k\to\infty} a_k \neq 0$ then $\sum a_k$ Diverge (has no sum)

Note that:- If $\lim_{k\to\infty} a_k = 0$ then the series maybe converge or diverge

Ex:- Determine whether the series converge

① $\sum_{k=1}^{\infty} \frac{e^k}{e^k+1}$

$\lim_{k\to\infty} \frac{e^k}{e^k+1} = 1 \neq 0$ series diverge

② $\sum_{k=1}^{\infty} \frac{k^2}{2k^2+1}$

$\lim_{k\to\infty} \frac{k^2}{2k^2} = \frac{1}{2} \neq 0$ series diverge

③ $\sum_{k=1}^{\infty} \frac{1}{2+3^{-k}}$

$\lim \frac{1}{2+\frac{1}{3^k}} = \frac{1}{2} \neq 0$ series diverge

④ $\sum_{k=1}^{\infty} \frac{3k-1}{k^3+5}$

$\lim_{k\to\infty} \frac{3k-1}{k^3+5} = \lim \frac{3}{k^2} = 0$

test faild (maybe converge or diverge)

⑤ $\sum_{k=1}^{\infty} \left(1+\frac{1}{k}\right)^{-k}$

$\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^{-k} = e^{-1} = \frac{1}{e} \neq 0$

diverge

⑥ $\sum_{k=1}^{\infty} k^2 \sin^2\left(\frac{1}{k}\right)$

$\lim_{k\to\infty} k^2 \sin^2\left(\frac{1}{k}\right) = \infty . 0$

$u = \frac{1}{k}$ $k = \frac{1}{u}$

$k\to\infty$ $u\to 0$

$\lim_{u\to 0} \frac{1}{u^2} \sin^2(u)$

$\left(\lim_{u\to 0} \frac{\sin(u)}{u}\right)^2 = 1^2 = 1 \neq 0$

diverge

⑦ $\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{e}}$

$\lim_{k\to\infty} \frac{1}{e^{\frac{1}{k}}}$

$= \frac{1}{e^0} = \frac{1}{1} = 1 \neq 0$

diverge

⑧ $\sum_{k=1}^{\infty} \cos k$

$\lim_{k\to\infty} \cos k$ d.n.e

diverge

lecture 58 — The Integral test

Theorem :- consider $\sum_{k=a}^{\infty} a_k$, let $f(x) = a_k$, $k \geq a$

1. If $f(x) > 0$ $\forall x \geq a$ (f is positive on $[a,\infty)$)
2. $f(x)$ is continuous on $[a,\infty)$
3. $f(x)$ is decreasing on $[a,\infty)$ ($f'(x) < 0$ $\forall x \geq a$)

then $\int_a^{\infty} f(x)\,dx$ and $\sum a_k$ are both convergent

Ex :- Determine whether the series are converge or diverge

① $\sum_{k=1}^{\infty} \frac{k}{1+k^2} \rightarrow f(x) = \frac{x}{1+x^2}$

① $f(x) > 0$ ✓

② f cont on $[1,\infty)$ ✓

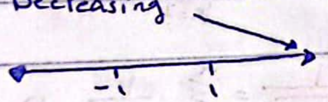
③ $f'(x) = \frac{(1+x^2) - 2x^2}{(1+x^2)^2}$

$= \frac{1-x^2}{(1+x^2)^2}$

$1 - x^2 = 0$

$x = \pm 1$

Decreasing



so $\int_1^{\infty} \frac{k}{1+k^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{x}{1+x^2}\,dx$

$\lim_{t\to\infty} \frac{1}{2} \ln|1+x^2| \Big]_1^t$

$\lim_{t\to\infty} \left(\frac{1}{2} \ln|1+t^2| - \frac{1}{2} \ln 2\right)$

$= \infty - \frac{1}{2}\ln|2| = \infty$

so its Diverge

By Integral test:

$\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ Diverge

② $\sum_{n=3}^{\infty} \frac{\ln n}{n} \rightarrow f(x) = \frac{\ln x}{x}$

① $f(x) > 0$ ✓

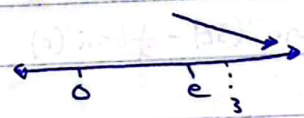
② f cont on $[3,\infty)$ ✓ (متصل الاقتران من $(0,\infty)$)

③ $f'(x) = \frac{\frac{1}{x}x - \ln x}{x^2}$

$= \frac{1 - \ln x}{x^2}$

$1 - \ln x = 0 \rightarrow x = e$

$x^2 = 0 \rightarrow x = 0$



f decreasing $[3,\infty)$

$\int_3^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_3^{t} \frac{\ln x}{x}$

$u = \ln x$

$du = \frac{1}{x} \rightarrow dx = x\,du$

$\lim_{t\to\infty} \int_3^{t} \frac{u}{x}\,x\,du = \lim_{t\to\infty} \int_3^{t} u\,du$

$= \lim_{t\to\infty} \frac{(\ln x)^2}{2} \Big]_3^t$

$= \lim_{t\to\infty} \left(\frac{(\ln t)^2}{2} - \frac{(\ln 3)^2}{2}\right)$

$= \infty \rightarrow$ Diverge

$\sum_{n=3}^{\infty} \frac{\ln n}{n}$ Diverge

③ $\sum_{k=0}^{\infty} \frac{1}{1+9k^2}$ $\quad f(x)=\frac{1}{1+9x^2}$

① $f(x)>0$ ✓

② $f(x)$ cont on $[0,\infty)$

③ $f'(x) = \frac{-1 * 18x}{(1+9x^2)^2} = \frac{-18x}{(1+9x^2)^2} \neq 0$

$-18x = 0 \rightarrow x = 0$

Decreasing

0

$\int_0^{\infty} \frac{1}{1+9x^2} = \lim_{t\to\infty} \int_0^t \frac{1}{1+9x^2}$

$\int_0^t \frac{1}{1+9x^2} = \int_0^t \frac{1}{9\left((\frac{1}{3})^2+x^2\right)}dx$

~~$x = \frac{1}{3}\tan$~~

$= \lim_{t\to\infty}\left(\frac{1}{9}\tan^{-1}(3x)\right]_0^t \rightarrow) * \left(\frac{1}{3}\right)$ يرجع

$= \lim_{t\to\infty} \frac{1}{3}\tan^{-1}(3t) - \frac{1}{3}\tan^{-1}(0)$

$= \frac{1}{3} * \frac{\pi}{2} = \cancel{} = \frac{\pi}{6}$

converge to $\frac{\pi}{6}$

By Integral test

$\sum_{k=0}^{\infty} \frac{1}{1+9k^2}$ converge

فـا التكامل converge وإله جواب (رقم)

الـ series (converge) بس بدون رقم

Lecture 59

Theorem : converge of p-series

$\sum \frac{1}{k^p} \begin{cases} \text{converge If } p>1 \\ \text{Diverge If } 0<p\leq 1 \end{cases}$

special case $\sum \frac{1}{k}$ harmonic series (Diverge)

Ex: Determine whether the following series converge

① $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$

$p=\frac{1}{3} \in (0,1]$ By series Its Diverge

② $\sum_{k=1}^{\infty} \frac{1}{k^4}$ $\quad p=4>1$ converge

③ $\sum_{k=1}^{\infty} k^{-5/3} = \sum_{k=1}^{\infty} \frac{1}{k^{5/3}}$ $\quad p=\frac{5}{3}>1$ converge

④ $\sum_{k=1}^{\infty} \frac{1}{k+6} = \sum_{k=7}^{\infty} \frac{1}{k}$ $\quad p=1$ (Diverge)

⑤ $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k+3}} = \sum_{k=5}^{\infty} \frac{1}{k^{1/2}}$ $\quad p=\frac{1}{2}<1$ Diverge

⑥ $\sum_{k=1}^{\infty} \frac{3}{5k} = \frac{3}{5}\sum_{k=1}^{\infty} \frac{1}{k}$ $\quad p=1$ Diverge

⑦ $\sum_{k=1}^{\infty} \left(\frac{4}{k}\right)^2 = \sum_{k=1}^{\infty} \frac{16}{k^2}$ $\quad p=2>1$ converge

Lecture 60

The comparison test :-

Theorem :-

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and $\sum a_k \leq \sum b_k$

- If the bigger series $\sum b_k$ converge then the smaller $\sum a_k$ converge
- " " smaller " $\sum a_k$ Diverge " " bigger $\sum b_k$ Diverge

Ex :- Determine whether the series converge

① $\sum_{k=1}^{\infty} \frac{1}{3+k^4}$

$k^4+3 > k^4$

$\sum \frac{1}{k^4+3} < \sum \frac{1}{k^4}$ → $P>1$ converge

* then by comparison test

$\sum \frac{1}{3+k^4}$ converge

② $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

$\sin k \leq 1$

$\frac{\sin^2 k}{k^2} \leq \frac{1}{k^2}$

$\sum \frac{\sin^2 k}{k^2} \leq \sum \frac{1}{k^2}$

$P=2>1$ converge

then $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$ converge

③ $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^3}$

$\tan^{-1}(k) < \frac{\pi}{2}$

$\sum \frac{\tan^{-1}(k)}{k^3} < \sum \frac{\frac{\pi}{2}}{k^3}$

$\sum \frac{\tan^{-1}(k)}{k^3} < \frac{\pi}{2} \sum \frac{1}{k^3}$

$3>1$ converge → $\sum_{k=1}^{\infty} \frac{\tan^{-1}(k)}{k^3}$ converge

④ $\sum_{k=1}^{\infty} \frac{5^k+1}{2^k-1}$

$5^k+1 > 5^k$ → $5^k+1 > 5^k$

$2^k-1 < 2^k$ → $\frac{1}{2^k-1} > \frac{1}{2^k}$

$\sum \frac{5^k+1}{2^k-1} > \sum \frac{5^k}{2^k}$ → Geometric series

$|r| = \frac{5}{2} > 1$

Diverge

∴ then $\sum_{k=1}^{\infty} \frac{5^k+1}{2^k-1}$ Diverge

Lecture 61

Limit comparison test

Theorem:-

let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose $c = \lim_{k\to\infty} \frac{a_k}{b_k}$

* If $0 < c < \infty$ then the series both converge or both Diverge

Ex:- Determine whether the following series converge

① $\sum_{k=1}^{\infty} \frac{5}{k^3+2} \to a_k$

$b_k :- \sum \frac{5}{k^3} \to$ converge

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{5}{k^3+2} * \frac{k^3}{5}$$

$$= \lim_{k\to\infty} \frac{k^3}{k^3+2} = 1 \in (0,\infty)$$

by Theorem $\sum_{k=1}^{\infty} \frac{5}{k^3+2}$ converge

③ $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ $\quad b_n = \sum \frac{1}{2^n}$ Geometric series

$r = \frac{1}{2} < 1$ converge

$$= \lim_{n\to\infty} \frac{1}{2^n-1} * \frac{2^n}{1}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^n-1} = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2^n \ln 2}{2^n \ln 2} = 1 \in (0,\infty)$$ both converge

④ $\sum_{k=1}^{\infty} \frac{k}{k^{\frac{3}{2}} - \frac{1}{2}}$ $\quad b_k = \sum \frac{k}{k^{\frac{3}{2}}} = \sum \frac{1}{k^{\frac{1}{2}}}$

$\therefore P = \frac{1}{2} < 1$ Diverge

$$= \lim_{k\to\infty} \frac{k}{k^{\frac{3}{2}} - \frac{1}{2}} * \frac{k^{\frac{1}{2}}}{1}$$

$$= \lim_{k\to\infty} \frac{k^{\frac{3}{2}}}{k^{\frac{3}{2}} - \frac{1}{2}} = \frac{\infty}{\infty}$$

$$= \lim_{k\to\infty} \frac{\frac{3}{2}\sqrt{k}}{\frac{3}{2}\sqrt{k}} = 1 \in (0,\infty)$$ both Diverge

② $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \to a_n$

$$b_n :- \sum_{n=1}^{\infty} \frac{2n^2}{\sqrt{n^5}} = 2\sum_{n=1}^{\infty} \frac{n^2}{n^{\frac{5}{2}}} = 2\sum_{n=1}^{\infty} n^{-\frac{1}{2}}$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

$\frac{1}{2} < 1$ Diverge

$$\lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} * \frac{\sqrt{n^5}}{2n^2}$$

$$= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \frac{n^{\frac{1}{2}}}{2}$$

$$= \lim_{n\to\infty} \frac{2n^{\frac{5}{2}} + 3n^{\frac{3}{2}}}{2\sqrt{5+n^5}}$$

$$= \lim_{n\to\infty} \frac{2n^{\frac{5}{2}} + 3n^{\frac{3}{2}}}{n^{\frac{5}{2}}\sqrt{\frac{5}{n^5}+1}} * \frac{1}{2}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{\frac{2n^{\frac{5}{2}}}{n^{\frac{5}{2}}} + \frac{3n^{\frac{3}{2}}}{n^{\frac{5}{2}}}}{\sqrt{\frac{5}{n^5}+1}}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{2+\frac{3}{n}}{\sqrt{\frac{5}{n^5}+1}} = \frac{1}{2} \times 2 = 1$$

both Diverge

Lecture 62 Alternating series :-

* An Alternating series has one of the following two forms :-

- $\sum (-1)^k a_k \; , a_k > 0$
- $\sum (-1)^{k+1} a_k \; , a_k > 0$

Ex:- $-1, 1, -1, 1, -1, 1$ alternating series

$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ not alternating series

Theorem :- Alternating series test :-

An alternating series converge If :-

① $a_1 > a_2 > a_3 > \dots$ (Decreasing)

② $\lim_{n \to \infty} a_n = 0$

Ex:- Determine whether the series converge

① $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ $\quad a_k = \frac{1}{k}$

① a_k decreasing ? ✓

$f'(x) = \frac{-1}{x^2}$ (always negative)

② $\lim_{k \to \infty} \frac{1}{k} = 0$ ✓

By The test

$\sum_{k=1} \frac{(-1)^{k+1}}{k}$ converge

② $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{3^k}$ $\quad a_k = \frac{k}{3^k}$

$a_{k+1} = \frac{k+1}{3^{k+1}}$

$f(x) = \frac{k}{3^k}$

① $\frac{k+1}{3^{k+1}} * \frac{3^k}{k} = \frac{1}{3}\left(\frac{k+1}{k}\right)$

(Decreasing) أصغر من ١

② $\lim_{k \to \infty} \frac{k}{3^k} = \frac{\infty}{\infty}$

$\lim_{k \to \infty} \frac{1}{3^k 3} = 0$

converge

③ $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$ $\quad a_k = \frac{k+3}{k(k+1)}$

$a_{k+1} = \frac{k+4}{(k+1)(k+2)}$

① $\frac{k+4}{(k+1)(k+2)} * \frac{k(k+1)}{k+3}$

$\frac{k^2+4k}{k^2+5k+6} < 1$ Decreasing

② $\lim_{k \to \infty} \frac{k+3}{k(k+1)} = 0$ converge

Lecture 63

Definition:- A series $\sum a_k$ is said to be:-

① Absolutly convergent If
$$\sum_{k=1}^{\infty} |a_k| = |a_1| + |a_2| + |a_3| + \dots \quad \text{converge}$$

② Absolutly Diverge If :-
$$\sum_{k=1}^{\infty} |a_k| = |a_1| + |a_2| + |a_3| + \dots \quad \text{Diverge}$$

Theorem:-

If $\sum |a_k|$ converge ($\sum a_k$ absolutly converge) then $\sum a_k$ converge

Theorem

If $\sum a_k$ converg and absolutly Diverge then $\sum a_k$ called conditionally convergent

Ex:- classify the series absolutly converge, conditionally convergent or absolutly Diverge

① $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \dots$

take $|\;| \Rightarrow 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$|r| = \frac{1}{2} < 1$ converge

so the series is absolutily converge

then By Theorem the series converge

② $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

take $|\;| \to 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

$\sum_{k=1}^{\infty} \frac{1}{k}$ Diverge

$\sum (-1)^{k+1} \frac{1}{k}$ absolutly Diverge

ما بقدر احكي إذا converge (بدي احلها حل عادي)

$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ alternating series

$a_k = \frac{1}{k}$ Decreasing

$f'(x) = \frac{-1}{x^2}$

② $\lim \frac{1}{k} = 0$

By alternating series test converge

By Theorem (absolutly Div + con)

then the series is conditionally convergent

③ $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{k(k+3)}$

take $|a_k|$

$\frac{k+2}{k(k+3)}$ a_k $b_k = \sum \frac{k}{k^2} = \frac{1}{k}$ Diverge

$c = \lim_{k \to \infty} \frac{a_k}{b_k} = \frac{k+2}{k(k+3)} * \frac{k}{1}$

$= \lim_{k \to \infty} \frac{k^2+2k}{k^2+3k} = 1 \in (0, \infty)$

By test $\sum \frac{k+2}{k(k+3)}$ Diverge

so $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{k(k+3)}$ absolutly Diverge

Now $\sum (-1)^{k+1} \frac{k+2}{k^2+3k}$

① $a_k = \frac{k+2}{k^2+3k}$ Decreasing

② $\lim_{k \to \infty} \frac{k+2}{k^2+3k} = 0$

By The test $\sum (-1)^{k+1} \frac{k+2}{k^2+3k}$ converge

so $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+2}{k^2+3k}$ conditionally convergent

④ $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$

take $|\ |$

$\sum_{k=1}^{\infty} \frac{|\cos k|}{k^2}$

$\frac{|\cos k|}{k^2} \leq \frac{1}{k^2}$

$\sum \frac{|\cos k|}{k^2} < \sum \frac{1}{k^2}$

$p = 2 > 1$ converge

By comparison test $\sum_{k=1}^{\infty} \frac{|\cos k|}{k^2}$ converge

By Theorem $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ absolutly converge

By Theorem $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converge

Lecture 64

The Ratio test :-

let $\sum a_k$ be series with non zero terms and

$$L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \begin{cases} \to L<1 \text{ absolutily } \cancel{\text{Diverge}} \text{ converge (converge)} \\ \to L>1,\ L=+\infty \quad \text{Diverge} \\ \to L=1 \text{ the test faild} \end{cases}$$

Ex:- Determine whether the series converge

① $\sum_{k=1}^{\infty} \frac{1}{k!}$

$a_k = \frac{1}{k!} \quad a_{k+1} = \frac{1}{(k+1)!}$

$\lim_{k\to\infty} \left|\frac{k!}{(k+1)!}\right| = \lim_{k\to\infty} \frac{k!}{(k+1)\,k!}$

$= \lim_{k\to\infty} \frac{1}{1+k} = 0$

series $\sum_{k=1}^{\infty} \frac{1}{k!}$ absolutly converge

so converge

② $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k} \qquad a_{k+1} = \frac{(2(k+1))!}{4^{k+1}}$

$L = \lim_{k\to\infty} \left|\frac{(2k+2)!}{4^{k+1}} * \frac{4^k}{(2k)!}\right|$

$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)\cancel{(2k)!}}{\cancel{4^k}\,4} * \frac{\cancel{4^k}}{\cancel{2k!}}$

$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{4} = \infty$

By Ratio test the series

$\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$ Diverge

③ $\sum_{n=1}^{\infty} \frac{n^n}{n!} \qquad a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

$L = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}}{(n+1)!} * \frac{n!}{n^n}\right|$

$= \lim_{n\to\infty} \left|\frac{(n+1)^n\,(n+1)\,n!}{(n+1)\,n!\,n^n}\right|$

$= \lim_{n\to\infty} \left|\frac{(n+1)^n}{n^n}\right|$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n =$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e^1$ Diverge

④ $\sum_{k=1}^{\infty} \frac{(-1)^k k^3}{3^k} \qquad a_{k+1} = \frac{(-1)^{k+1}(k+1)^3}{3^{k+1}}$

$L = \lim_{k\to\infty} \left|\frac{(-1)^{k+1}(k+1)^3}{3^{k+1}} * \frac{3^k}{(-1)^k k^3}\right|$ $\boxed{(-1)^k \text{ or } {}^{k+1} \text{ رح تروح بالمطلق}}$

$= \lim_{k\to\infty} \left|\frac{(k+1)^3}{3k^3}\right|$

$= \frac{1}{3} \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^3 = \frac{1}{3} < 1$

absolutly converge so converge

Week 12

Lecture 68, 69, 73
غير مطلوب

lecture 65

Theorem: The Root test $(\ )^k$

Let $\sum a_k$ be series:

$$L = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} (|a_k|)^{\frac{1}{k}}$$

- If $L<1$ absolutely converge (converge)
- If $L>1$, $L=+\infty$ Diverge
- If $L=1$ the test faild

Ex:- Determine whether the series converge

① $\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$

$$L = \lim_{k\to\infty} |a_k|^{\frac{1}{k}} = \lim_{k\to\infty} \left(\left|\left(\frac{4k-5}{2k+1}\right)\right|^{\frac{1}{k}}\right)^k$$

$$= \lim_{k\to\infty} \left(\frac{4k-5}{2k+1}\right)$$

$$= 2 > 1$$

By Root test

$\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$ Diverge

② $\sum_{k=1}^{\infty} (-1)^k \frac{k}{5^k}$

$$L = \lim_{k\to\infty} \left(\left|(-1)^k \frac{k}{5^k}\right|\right)^{\frac{1}{k}}$$

$$= \lim_{k\to\infty} \frac{k^{\frac{1}{k}}}{5}$$

$$= \frac{1}{5} \lim_{k\to\infty} k^{\frac{1}{k}} \to \infty^0$$

$y = k^{\frac{1}{k}}$

$\ln y = \ln k^{\frac{1}{k}}$

$\ln y = \frac{1}{k} \ln k$

$\lim_{k\to\infty} \frac{\ln k}{k} = \frac{\infty}{\infty}$

$\lim_{k\to\infty} \frac{1}{k} = 0$

$= \frac{1}{5} e^0 = \frac{1}{5}$

converge

③ $\sum_{k=1}^{\infty} \left(1+\frac{2}{k}\right)^{k^2}$

$$L = \lim_{k\to\infty} \left(\left|1+\frac{2}{k}\right|^{k^2}\right)^{\frac{1}{k}}$$

$$= \lim_{k\to\infty} \left(1+\frac{2}{k}\right)^k$$

$$= e^2 > 1 \quad \text{Diverge}$$

④ $\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^n = (-1)^n \left(\frac{2n}{n+1}\right)^n$

$$L = \lim_{k\to\infty} \left(\left|(-1)^n \left(\frac{2n}{n+1}\right)^n\right|\right)^{\frac{1}{n}}$$

$$= \lim_{k\to\infty} \frac{2n}{n+1} = 2 > 1$$

Diverge

Lecture 66

Power series :-

A power series is the series of the form :-

$\boxed{1}$ $$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x^1 + c_2 x^2 + \dots,$$

is called power series in x, canterd at zero, or obout zero

where x is avariable and c_n are constant called coefficients of the series

$\boxed{2}$ $$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a)^1 + c_2 (x-a)^2 + \dots$$

is called power series of $(x-a)$ or power series centard at (a) or power series about (a)

- The sum of series is a function (has ~~Inbai~~ Infinity many terms)

$f(x) = c_0 + c_1 x + c_2 x^2 + \dots$

Ex:- $\sum_{n=0}^{\infty} x^n$ which converge If $-1 < x < 1$ Diverge $x \geq 1$, $x \leq -1$

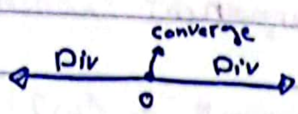
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

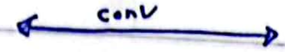
② $$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x^1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

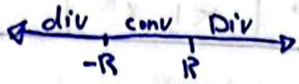
③ $$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 + \frac{-x^2}{2!} + \frac{x^4}{4!} + \dots$$

Theorem:-

For any power series $\sum_{n=0}^{\infty} c_n x^n$ one of the following true

① 

② 

③ 

Remark:- for ③

① at $x=R$ or $x=-R$ depending on the particular series حسب العدد بشوف

② R = Radias of convergence اذا conv

③ $(-R, R)$ = Interval of convergence

Ex:- Find the interval (الفترة) and radias of convergence of the series

① $\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^k}{3^k (k+1)}$

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{(-1)^{k+1} \cdot x^{k+1}}{3^{k+1} \cdot (k+2)} * \frac{3^k (k+1)}{(-1)^k \cdot x^k}$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{x^k} x^1 \cdot \cancel{3^k} (k+1)}{\cancel{3^k} 3 \, (k+2) \cancel{x^k}} \right|$$

$$= \lim \frac{k+1}{3(k+2)} |x|$$

$$= \frac{1|x|}{3} \lim_{k\to\infty} \frac{k+1}{k+2}$$

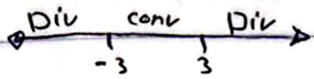
$$= \frac{|x|}{3}$$

① $L = \frac{|x|}{3} < 1$ absolutily conv

$= |x| < 3 \rightarrow -3 < x < 3 \; (-3, 3)$

② $L = \frac{|x|}{3} > 1$ Div

$x \geq 3$, $x \leq -3$ ⟵



at $x=3$

$$\sum_{k=0}^{\infty} \frac{(-1)^k \cancel{3^k}}{\cancel{3^k}(k+1)} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}$$

By altennating series test

① $a_k = \frac{1}{k+1}$ Dec ✓

② $\lim \frac{1}{k+1} = 0$ ✓

conv at $x=3$

at $x=-3$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (-3)^k}{3^k (k+1)} = \sum_{k=0}^{\infty} \frac{\overbrace{(-1)^k (-1)^k}^{(-1)^{2k}} \cancel{3^k}}{(k+1)\cancel{3^k}} = \sum_{k=0}^{\infty} \frac{(-1)^{2k}}{k+1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k+1} = \sum_{k=1}^{\infty} \frac{1}{k} \text{ Div at } x=-3$$

الخلاصة

The Interval $(-3, 3]$

Radias $\rightarrow 3 - 0 = 3$

نهاية الفترة - المركز

(98)

Lecture 67

Ex:- Find the interval and radias of convergence

① $\sum_{K=0}^{\infty} \frac{x^K}{K!}$

$L = \lim_{K\to\infty} \left| \frac{x^{K+1}}{(K+1)!} * \frac{K!}{x^K} \right|$

$= \lim_{K\to\infty} \frac{x}{K+1}$

$= x \lim_{K\to\infty} \frac{1}{K+1}$

$= x \cdot 0 = 0 < 1$ converge

converge on all $\mathbb{R}$

interval of convergent $= (-\infty, \infty)$

Radias $= +\infty$

② $\sum_{K=0}^{\infty} K!\, x^K$

$L = \lim_{K\to\infty} \left| \frac{(K+1)!\, x^{K+1}}{K!\, x^K} \right|$

$L = \lim_{K\to\infty} (K+1)(x)$

$= |x| \lim_{K\to\infty} (K+1)$

$= +\infty \quad \forall x \neq 0$ Diverge

If $x = 0$, $\Sigma = 0$ converge

Div — conv (at 0) — Div

Interval of converge $= \{0\}$

Radias $= 0$

③ $\sum_{K=0}^{\infty} x^K$

$L = \lim_{K\to\infty} \left| \frac{x^{K+1}}{x^K} \right|$

$= \lim_{K\to\infty} |x|$

$= |x|$ →
- $|x| < 1$ → $-1 < x < 1$
- $|x| > 1$ → $x > 1,\ x < -1$
- $|x| = 1$ → $x = 1$, $x = -1$

at $x = 1$ → $\sum_{K=1}^{\infty} (+1)^K$ Div.

at $x = -1$ → $\sum_{K=1}^{\infty} (-1)^K$ Div

Div — (-1: Div) — conv — (1: Div) — Div

Interval of convergence $= (-1, 1)$

Radias $=$ 〃 $= 1$

Lecture 70 Maclaurin series

Definition: Its taylor series about $x=0$ is:-

$$\sum_{k=0}^{\infty} f(0)^{k} \frac{x^k}{k!} = f(0) + f'(0)x + f''(0) * \frac{x^2}{2!} + f'''(0) * \frac{x^3}{3!} + \dots$$

Ex: Find Maclaurin series for

① $f(x) = e^x$ $x=0$

$f(x) = e^x \rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \rightarrow f'(0) = 1$

$f''(x) = e^x \rightarrow f''(0) = 1$

$f^{(3)}(x) = e^x \rightarrow f'''(0) = 1$

$$\sum f^{k}(0) \frac{x^k}{k!} = f(0) + f'(0)x^1 + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

② $f(x) = \sin x$

$f(x) = \sin x \rightarrow f(0) = 0$

$f'(x) = \cos x \rightarrow f'(0) = 1$

$f''(x) = -\sin x \rightarrow f''(0) = 0$

$f'''(x) = -\cos x \rightarrow f'''(0) = -1$

$$\sum f^{k}(0) \frac{x^k}{k!} = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots$$

$$= 0 + x + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + 0$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$f(x)$	Maclaurin series	Interval of convergence
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	$(-\infty, \infty)$
$\sin x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	$(-\infty, \infty)$
$\cos x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	$(-\infty, \infty)$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	$(-1, 1)$
$\ln(1+x)$	$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$	$(-1; 1]$
$\tan^{-1}(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$	$(-1, 1)$

Lecture 71

Ex:- find the Maclarin series for (find a power series representation of) :-

① e^{2x}

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^{k}}{k!} \rightarrow e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^{k}}{k!}$$

② $\frac{1}{1+x} = \frac{1}{1--x} = \sum_{k=0}^{\infty} (-x)^{k}$

$$= \sum_{k=0}^{\infty} (-1)^{k} x^{k}$$

③ $\frac{1}{1+x^2} = \frac{1}{1--x^2} = \sum_{k=0}^{\infty} (-x^2)^{k}$

$$= \sum_{k=0}^{\infty} (-1)^{k} (x)^{2k}$$

④ $\ln x$

$$\ln(x+x-1) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^{k}}{k}$$

⑤ $\cos(\pi x)$

$$\sum_{k=0}^{\infty} (-1)^{k} \frac{(\pi x)^{2k}}{2k!}$$

⑥ $\cosh x = \frac{e^{x}+e^{-x}}{2}$

$$= \frac{\sum_{k=0}^{\infty} \frac{x^{k}}{k!} + \sum_{k=0}^{\infty} \frac{(-1)^{k}(x)^{k}}{k!}}{2}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1+(-1)^{k}) x^{k}}{k!}$$

⑦ $x e^{x}$

Maclarin

لكثير الحدود هو كثير الحدود نفسه

$$x \sum_{k=0}^{\infty} \frac{x^{k}}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(x)^{k+1}}{k!}$$

⑧ $\tan^{-1}(x^3)$

$$\sum_{k=0}^{\infty} \frac{(-1)^{k} (x^3)^{2k+1}}{2k+1}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^{k} x^{6k+3}}{2k+1}$$

⑨ $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

$$\frac{1}{2}\left(1 - \sum \frac{(-1)^{k} (2x)^{2k}}{2k!}\right)$$

$$\frac{1}{2} - \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^{k} (2x)^{2k}}{2k!}$$

مفوض $k=0$

$$\frac{1}{2} - \frac{1}{2}(1) - \frac{1}{2} \sum_{k=1}^{\infty} \frac{2^{2n} (-1)^{n} x^{2n}}{(2n)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)(-1)^{n} 2^{2n} x^{2n}}{2(2n)!}$$

السؤال بدء $k=1$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{n+1} 2^{2n} x^{2n}}{2(2n)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}$$

Ex: Express ① $\frac{1}{x+2}$ as the sum of power series and find the Interval of convergence

$$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2(1-\frac{-x}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{2^{n+1}}$$

The series converge when $\frac{x}{2}<1 \rightarrow x<2 \rightarrow -2<x<2$

Interval $(-2,2)$

② $\frac{x^3}{x+2}$

$$\frac{x^3}{x+2} = x^3\frac{1}{x+2} = x^3\sum\frac{(-1)^n x^n}{2^{n+1}} = \sum_{n=0}^{\infty}\frac{x^{3+n}(-1)^n}{2^{n+1}}$$

$$\left(\frac{-x}{2}\right)<1 \rightarrow x\in(-2,2)$$

Lecture 72

Determine whether the series converge or diverge and find the sum :-

① $\sum_{k=0}^{\infty} \frac{2^k}{k!} = e^2$

converge to e^2

② $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} = \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})^k}{k!} = e^{-\frac{1}{2}}$

③ $\pi + \frac{-\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots$

$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sin \pi = 0$

④ $\sum_{k=0}^{\infty} \frac{(-1)^k (\pi^{2k+1})}{4^{2k+1}(2k+1)}$

$\sum_{k=0}^{\infty} \frac{(-1)^k (\frac{\pi}{4})^k}{(2k+1)} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

⑤ $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k k}$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} (\frac{1}{2})^k}{k}$

$= \ln(1 + \frac{1}{2}) = \ln \frac{3}{2}$

⑥ $\frac{-\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots$

لو كان في π قبل $\frac{-\pi^3}{3!}$ كا نكون sin

فكانه نقلناه على الطرف الثاني

$\frac{-\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} = \sin \pi - \pi$

$= 0 - \pi$

$= -\pi$

⑦ $\frac{1}{1(2)} + \frac{-1}{(2)(2)^2} + \frac{1}{(3)(2)^3} + \frac{-1}{(4)(2)^4} + \dots$

$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k 2^k} = \frac{(-1)^k (\frac{1}{2})^k}{k}$

$x \in (-1, 1)$ to be conv

$= \ln(1 + x)$

$= \ln(1 + \frac{1}{2})$

$= \ln \frac{3}{2}$

converge to $\ln \frac{3}{2}$

⑧ $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+2}}{3^{2n+2} (2n+1)!}$

$\sum_{n=0}^{\infty} \frac{(-1)^n (\pi) \pi^{2n+1}}{3 \, 3^{2n+1} (2n+1)!}$

$\frac{\pi}{3} \sin \frac{\pi}{3} = \frac{\pi}{3} \frac{\sqrt{3}}{2}$

ملخص من فيديو 53 ل 74

Series :-

① sequence of partial

نكتب ال series على شكل sequence

ثم ناخذ ال Lim :-

If $\lim S_n = s$ (converge to (s))

" " $S_n = \pm\infty$ (Diverge)

② Geometric series :-

$a_n = ar^{n-1}$

بخليها r^n بدون اي اشي مع القوة ثم :-

$$\sum_{n=k}^{\infty} r^n = \begin{cases} \text{converge If } |r|<1,\ -1<r<1 \\ \text{Diverge If } |r|\ge 1,\ r\ge 1,\ r\le 1 \end{cases}$$

If converge its equal $\left(\frac{r^n}{1-r}\right)$

③ The Divergent test :-

If $\lim_{k\to\infty} a_k = 0$ Then $\sum a_k$ converge

If $\lim_{k\to\infty} a_k \neq 0$ " $\sum a_k$ (Diverge)

④ The Integral test :-

$\sum_{k=a}^{\infty} a_k$, $f(x) = a_k$

① If $f(x) > 0$

② F is continuous on $[a, \infty)$

③ $f(x)$ decreasing

Then $\int_a^{\infty} f(x)\,dx$ and $\sum a_k$ are both converge

⑤ converge of p-series

$$\sum \frac{1}{k^p} \begin{cases} \text{If } p>1 \text{ converge} \\ \text{If } 0<p\le 1 \text{ Diverge} \end{cases}$$

⑥ The comparison Test :-

- $\sum a_k$, $\sum b_k$ are series with positive terms. ($\sum b_k \ge \sum a_k$)
- If the bigger ($\sum b_k$) converge then the smaller ($\sum a_k$) converge
- If the smaller ($\sum a_k$) Diverge then the bigger ($\sum b_k$) Diverge

⑦ The Limit comparison test :-

$\sum a_k, \sum b_k$ series with positive terms and suppose $c = \lim_{k\to\infty} \frac{a_k}{b_k}$

If $0 < c < \infty$ the both are converge or both are Diverge

- مبجيب a_k او b_k وبشوف وحدة منهم إذا conv او Div وبحل على Test والجواب إذا بطلع إذا بين $(0,\infty)$ بكون ال 2 نفس إذا جربت عليها اول اشي

⑧ Altelenating test :-

has one of the following forms:-

$\sum (-1)^k a_k$, $a_k > 0$

$\sum (-1)^{k+1} a_k$, $a_k > 0$

It is converge If :-

① $a_1 > a_2 > a_3$ (Decreasing)

② $\lim_{n\to\infty} a_n = 0$

وعشان اعرف إذا Decreasing

رح ارجع ل 3 قوانين

① $a_{n+1} - a_n < 0$

② $\frac{a_{n+1}}{a_n} < 1$

③ $f'(x) < 0$

→ Decreasing

* Absolutly (conv or Div)

Absoulktly converge If

$\sum_{k=1}^{\infty} a_k = |a_1| + |a_2| + |a_3| \cdots$ converge

Absolutly Diverge If

$\sum_{k=1}^{\infty} a_k = |a_1| + |a_2| + |a_3| \cdots$ Diverge

* If $\sum |a_k|$ converge ($\sum a_k$ absoulutly conv)
  Then $\sum a_k$ converge
* If $\sum a_k$ converge and absolutly Diverge then $\sum a_k$ called conditionally convergent

* باخذ القيمة المطلقة لـ $\sum a_k$ وبطلع الحدود بكتبهم على شكل series وبشوف إذا converge او Diverge

(9) Ratio test :-

$\sum a_k$ be series with non zero terms so :-

$$L = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \begin{cases} L<1 \text{ (absoulutly conv) and conv} \\ L>1,\ L=\infty \text{ Diverge} \\ L=1, \text{ The test faild} \end{cases}$$

نستعملها اكثر اشي عشان $\frac{1}{k!}$ او n^n

(10) The Root Test :- $(\ )^k$

$$L = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} (|a_k|)^{\frac{1}{k}} \begin{cases} L<1 \text{ absoulutly conv} \\ L>1, \text{ Div or } L=\infty \\ L=1 \text{ The Test faild} \end{cases}$$

بس انه برفعه للقوة $(\frac{1}{k})$ وبعديها برابطه وبشوف إذا conv او Div

(11) power series :- $\sum_{n=0}^{\infty} x^n$

for any power series :-

(1) ← Div, conv (at 0), Div →

(2) ← conv →

(3) ← Div | conv | Div → (at $-R$ and R)

لنقطة (3) :- عند R و $-R$ بعرف من التعويض

$(-R, R)$ Interval (الفترة)

$R \to$ Radias of convergence

لما اعوض مكان x قيمة R او $-R$ واشوف الجواب إذا conv او Div

(12) Maclaurin series

f(x)	Maclaurin series	Interval
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	$(-\infty, \infty)$
$\sin x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	$(-\infty, \infty)$
$\cos x$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2k!}$	$(-\infty, \infty)$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	$(-1, 1)$
$\ln\|1+x\|$	$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$	$(-1, 1]$
$\tan^{-1}(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)}$	$(-1, 1)$

الملخص :(1) عشان اطلع إذا con او Div رح احل عليهم

(2) " " ايضا conv واطلع قيمتها بحل على :-

(A) Geometric series (r^n)

(B) Tayler series (Maclaurin)

(C) Telescoping seiles (seq of partial)

(3) سؤال عام بدون ما يحدد طريقة :- يفضل

اول اشي p - series

(2) Geometric

(3) Telescoping

(4) Alternating

Series Convergence/Divergence Flow Chart