

Reference Frames and rotations

DESIGN OF MECHATRONICS SYSTEMS

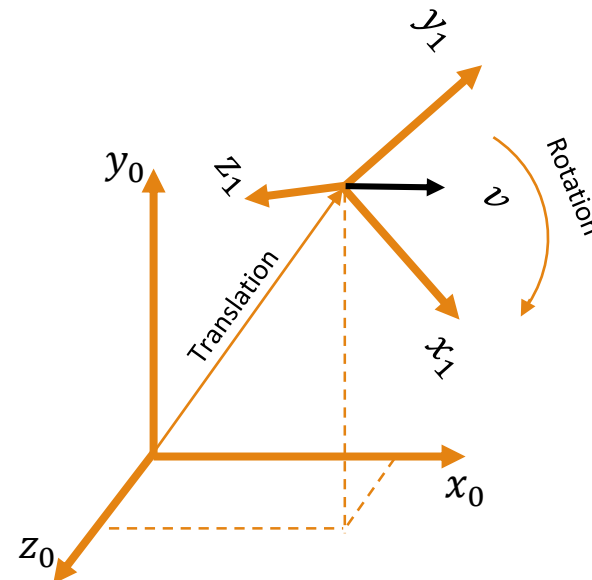
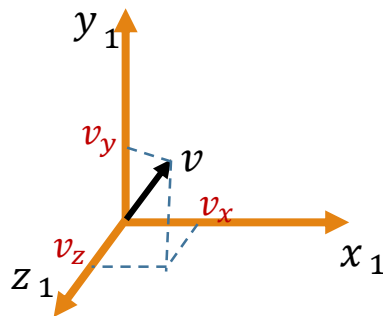
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Vector and Reference frames

Any vector in the Cartesian coordinate system can be given in three components $v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$

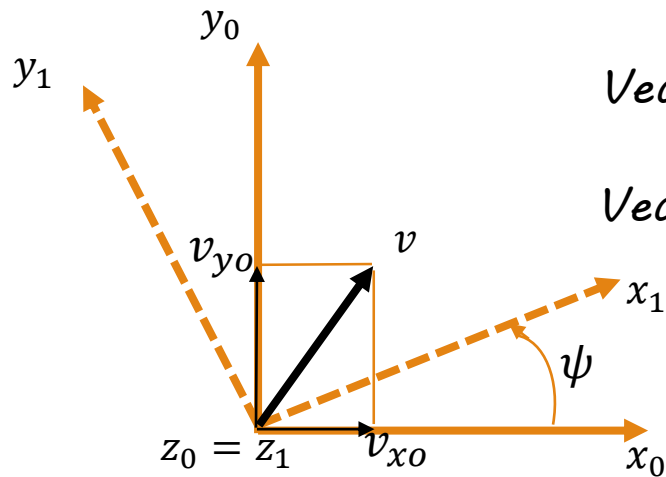
One coordinate frame is transformed into another through two basic operations:

- Rotations
- Translations



Rotation Matrices

- Vector v can be expressed in both frames \mathcal{F}^0 and \mathcal{F}^1 .
- The rotation occurs around z axis and the angle of rotation is ψ .



Vector v in \mathcal{F}^0 $v^0 = \begin{bmatrix} v_{x0} \\ v_{y0} \end{bmatrix}$

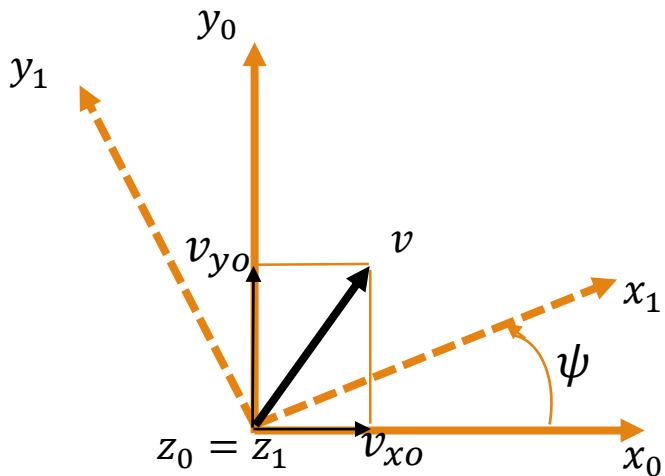
Vector v in \mathcal{F}^1 $v^1 = \begin{bmatrix} v_{x1} \\ v_{y1} \end{bmatrix} = \begin{bmatrix} v_{x0} \cos \psi + v_{y0} \sin \psi \\ -v_{x0} \sin \psi + v_{y0} \cos \psi \end{bmatrix}$

$$= \begin{bmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{bmatrix} \begin{bmatrix} v_{x0} \\ v_{y0} \end{bmatrix}$$

Rotation Matrices

$$v^1 = \begin{bmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{bmatrix} \begin{bmatrix} v_{x0} \\ v_{y0} \end{bmatrix}$$

$$v^1 = R_0^1 v_0 \quad \text{rotation matrix } R$$



- The relationship between the vector in \mathcal{F}^0 and the vector in \mathcal{F}^1 is the rotation matrix R .
- The notation R_0^1 is used to denote a rotation matrix from coordinate frame \mathcal{F}^0 to coordinate frame \mathcal{F}^1 .
- By representing the vector in 3 axes we will have a 3x3 rotation matrix as follow

$$R_0^1 = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$v^0 = R_1^0 v^1 = R_0^{1^{-1}} v_1 = R_0^{1T} v_1$$

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Rotation Matrices

$$R_0^1 = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For example if the readings of the vector v in \mathcal{F}^0 are $v^0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and the rotation angle is $\psi = 45^\circ$, then the vector can be represented in \mathcal{F}^1 by:

$$v^1 = \begin{bmatrix} \cos 45 & \sin 45 & 0 \\ -\sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \cos 45 + 2 \sin 45 \\ -\sin 45 + 2 \cos 45 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.121 \\ 0.7071 \\ 3 \end{bmatrix}$$

$$v^0 = \begin{bmatrix} \cos 45 & -\sin 45 & 0 \\ \sin 45 & \cos 45 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2.121 \\ 0.7071 \\ 3 \end{bmatrix} = \begin{bmatrix} 2.121 \cos 45 - 0.7071 \sin 45 \\ 2.121 \sin 45 + 0.7071 \cos 45 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$R_0^{1^{-1}} = R_0^{1^T} = R_1^0$$

Rotation Matrices

If the rotation (right handed) happens about y axis with an angle θ . Then the rotation matrix can be given by:

$$R_0^1 = R(\theta) = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

If the rotation (right handed) happens about x axis with and angle ϕ . Then the rotation matrix can be given by:

$$R_0^1 = R(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix}$$

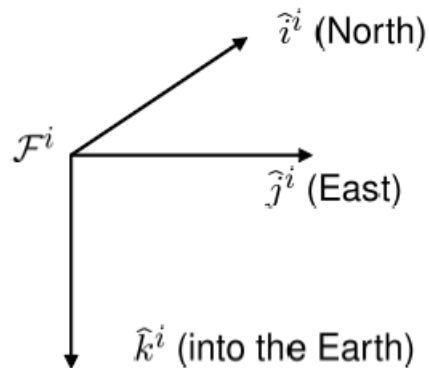
Rotation matrices have the following properties:

$$\begin{aligned} R_a^{b^{-1}} &= R_a^{b^T} = R_b^a \\ R_b^c R_a^b &= R_a^c \\ |R_a^b| &= 1 \end{aligned}$$

Frames in robotics application

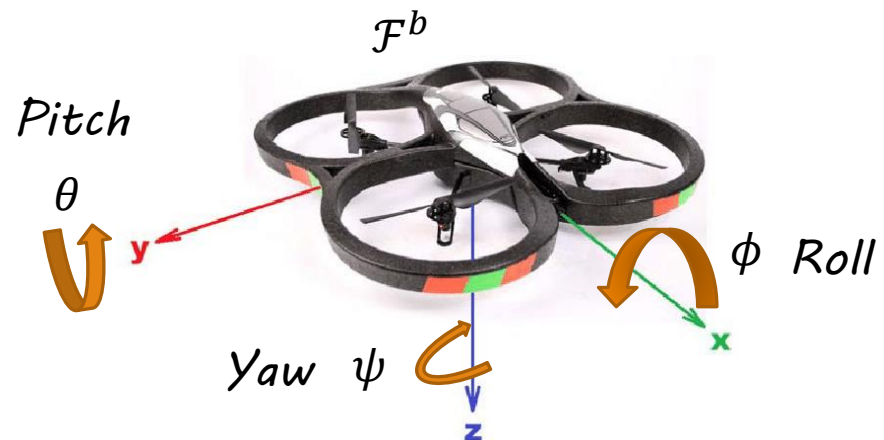
The inertial frame \mathcal{F}^i

The inertial coordinate system is an earth fixed coordinate system with origin at the defined home location.



The body frame \mathcal{F}^b

The body frame is fixed on a rigid body in the space and is obtained by three rotations from the inertial frame



Frames in robotics application

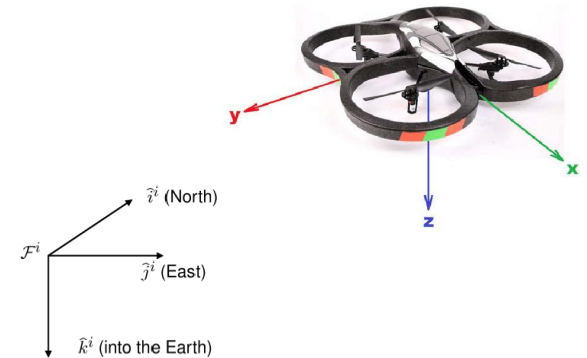
The transformation from the inertial frame to the body from is produced by a sequential rotations around 3 axes:

$$R_i^b = R(\phi)R(\theta)R(\psi)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ s\phi s\theta c\psi - c\phi s\psi & s\phi s\theta s\psi + c\phi c\psi & s\phi c\theta \\ c\phi s\theta c\psi + s\phi s\psi & c\phi s\theta s\psi - s\phi c\psi & c\phi c\theta \end{pmatrix}$$

where $c\phi = \cos \phi$ and $s\phi = \sin \phi$.



Frames in robotics application

Consider the gravity vector is represented in inertial frame by

the vector $g = \begin{bmatrix} 0 \\ 0 \\ 9.82 \end{bmatrix} m/s^2$. How this vector is represented in quadcopter frame (body frame), when

A) $\phi = 10^\circ, \theta = 15^\circ, \psi = 0^\circ$

B) $\phi = 10^\circ, \theta = 15^\circ, \psi = 100^\circ$

