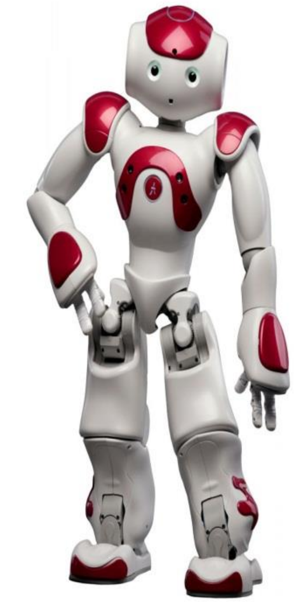

Robotics
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Office Hour: to be posted on office door

Textbook: John Craig "Introduction to Robotics mechanics and control," 3ed Ed., Pearson Education Inc.

Topics covered :

1. Introduction
2. Spatial descriptions and transformations
3. Manipulator kinematics
4. Inverse manipulator kinematics
5. Jacobians: Velocities and static forces
6. Manipulator dynamics
7. Trajectory generation

What is a robot?

- Many different definitions for robots exist.

- A robot is a reprogrammable, multifunctional manipulator designed to move material, parts, tools, or specialized devices through variable programmed motions for the performance of a variety of tasks." (Robot Institute of America).

Automation vs. robots

Automation: Machinery designed to carry out a specific task

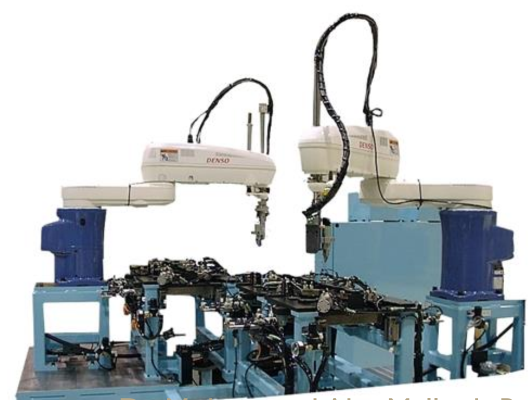
-Bottling machine

-Dishwasher

Robots: machinery designed to carry out a variety of tasks

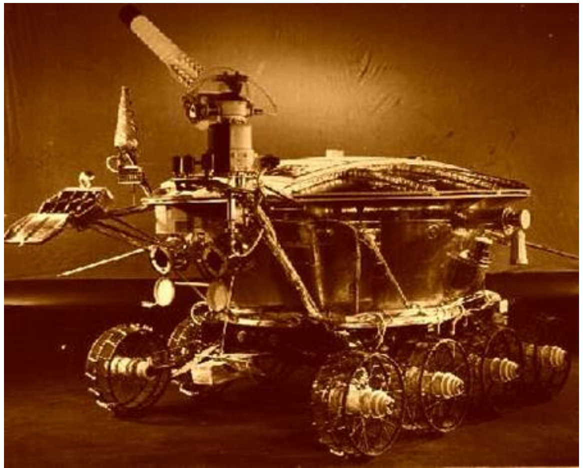
-Pick and place arms

-Mobile robots



Robots Classification

- Manipulators: robotic arms. These are most commonly found in industrial settings.
- Mobile Robots: unmanned vehicles
- Hybrid Robots: mobile robots with manipulators
- Humanoid robot



Applications

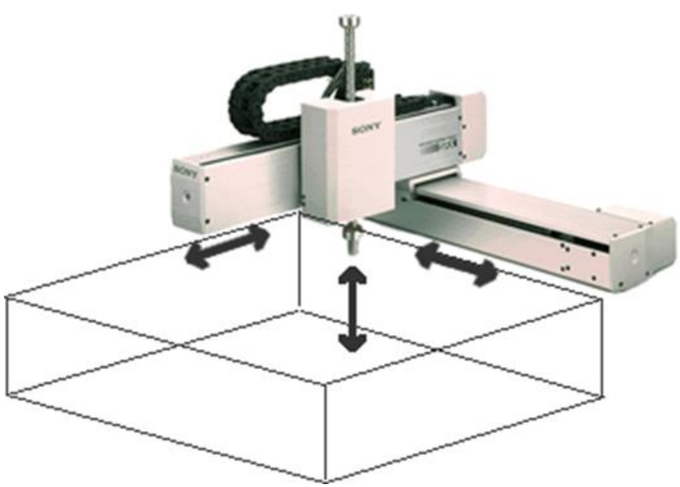
Dangerous:
-Space exploration
-chemical spill cleanup
-disarming bombs
-disaster cleanup

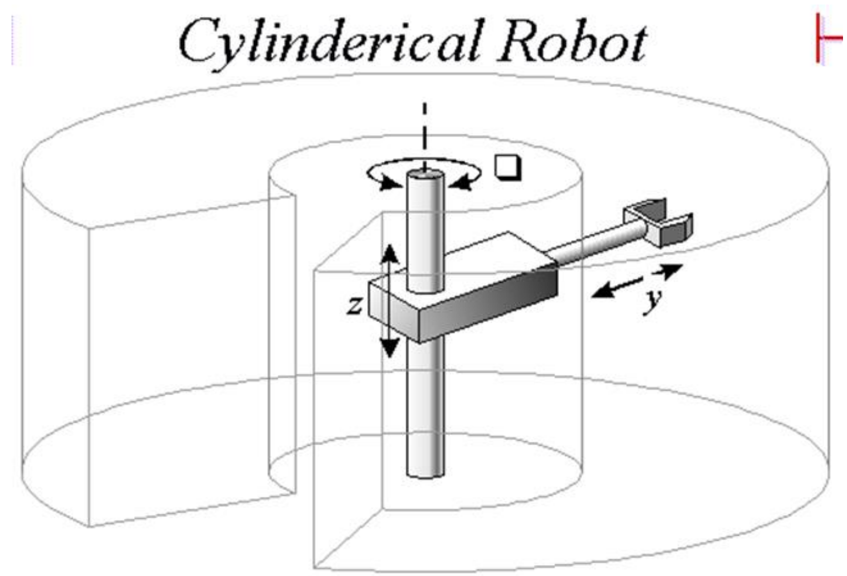
Repetitive
-Welding car frames
-part pick and place
-manufacturing parts.

High precision or high speed
-Electronics chips
-Surgery
-precision machining

Measures of performance

- Work space
  - The space within which the robot operates.
  - Larger volume costs more but can increase the capabilities of a robot





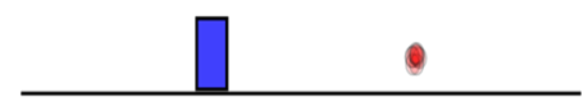
Measures of performance

- Speed and acceleration
  - Faster speed often reduces resolution or increases cost
  - Varies depending on position, load.
  - Speed can be limited by the task the robot performs (welding, cutting)

Measures of performance

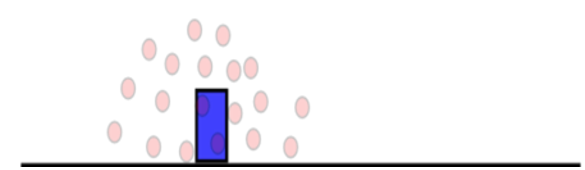
- Accuracy

➢ The difference between the actual position of the robot and the programmed position

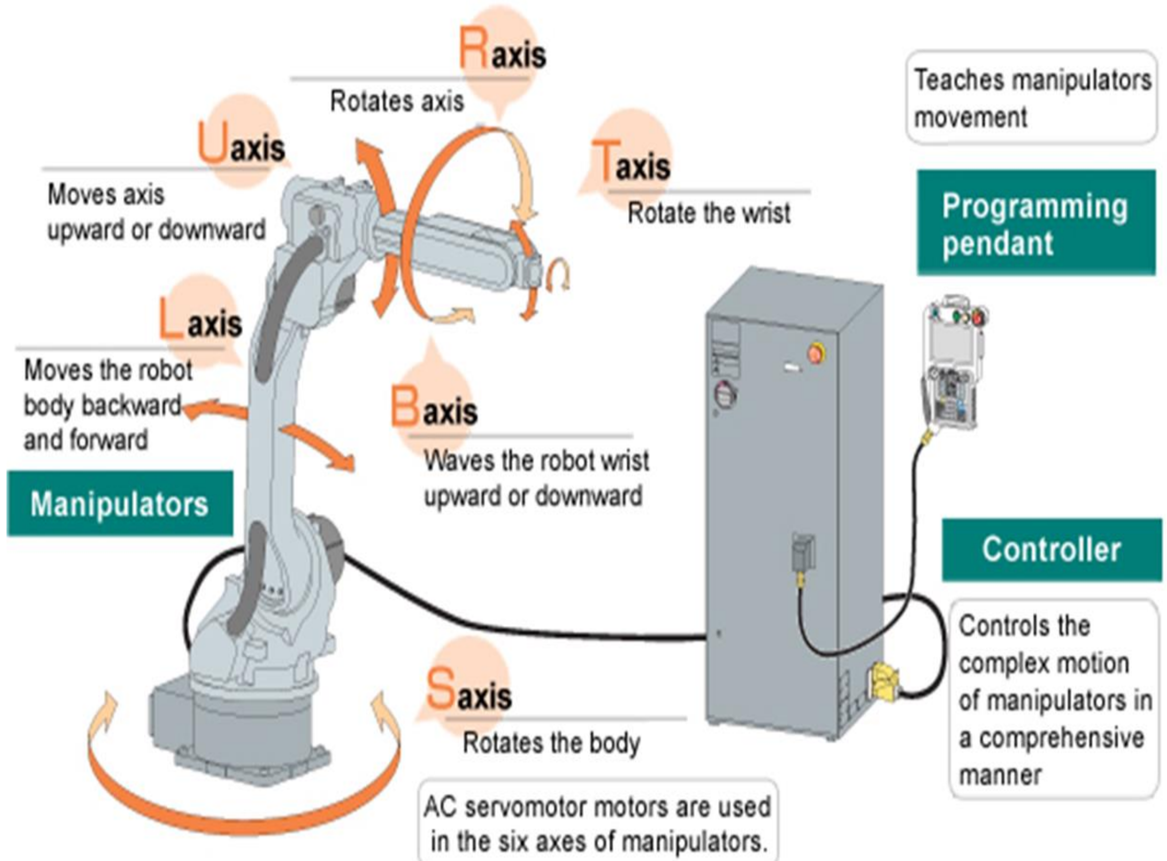


- Repeatability

➢ Will the robot always return to the same point under the same conditions?



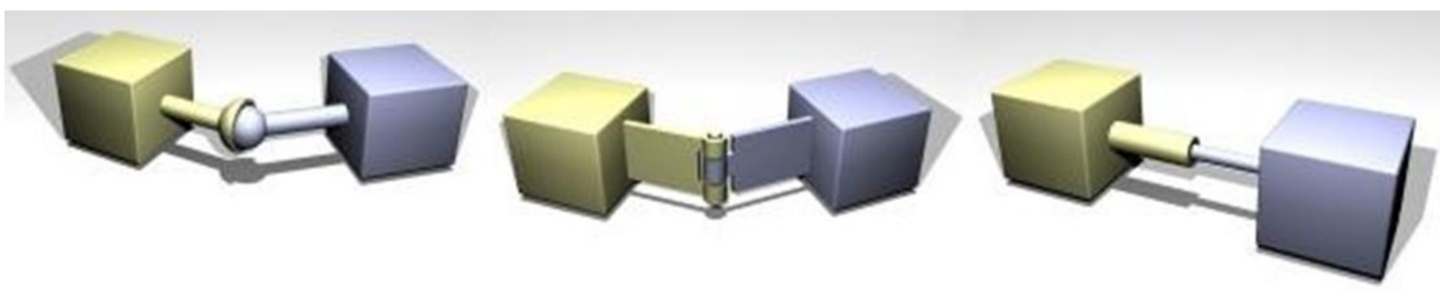
Robot Components



- Body
- End Effectors
- Actuators
- Sensors
- Controller
- Software

Robot: Body

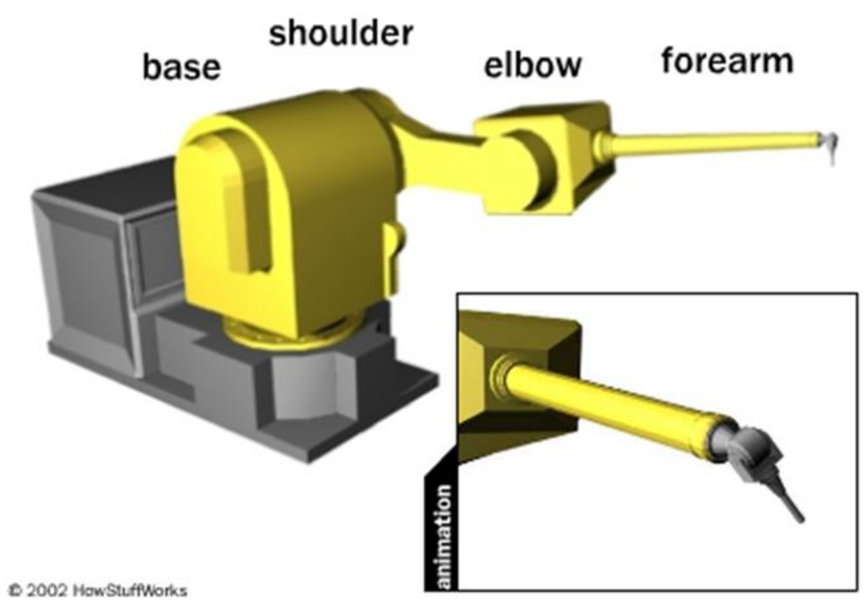
- Consists of links and joints
- A link is a part, a shape with physical properties.
- A joint is a constraint on the spatial relations of two or more links.
- These are just a few examples…



Ball joint Revolute (hinge) joint Prismatic (slider) joint

Degrees of Freedom

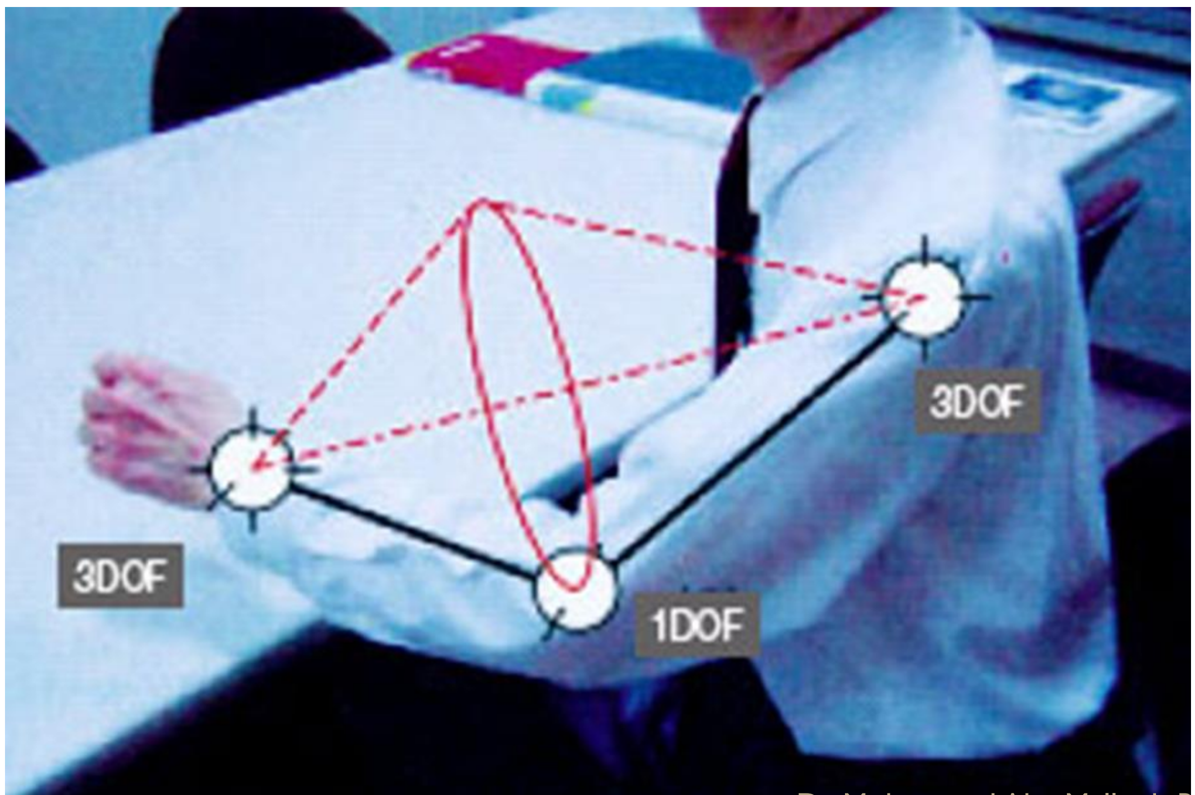
- Joints constraint free movement, measured in "Degrees of Freedom" (DOFs).
- Joints reduce the number of DOFs by constraining some translations or rotations.
- Robots classified by total number of DOFs



How many DOFs can you identify in your arm?

Degrees of Freedom

How many DOFs can you identify in your arm?



Robot: End Effectors

- Component to accomplish some desired physical function
- Examples:
  - ✓ Hands
  - ✓ Torch
  - ✓ Wheels
  - ✓ Legs

Robot: Actuators

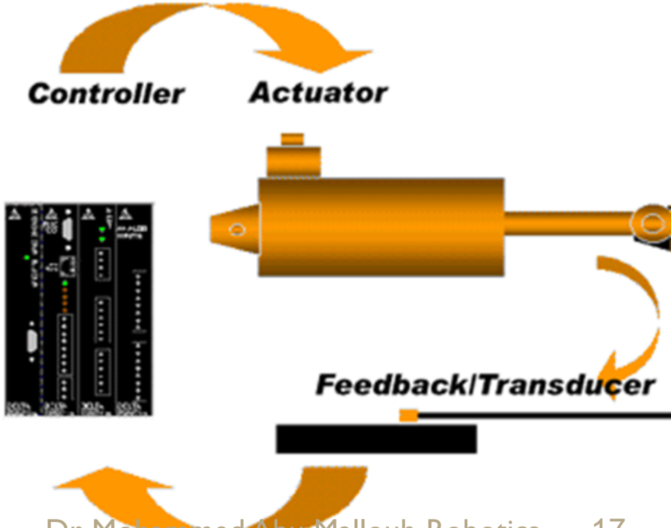
- Actuators are the “muscles” of the robot.
- These can be electric motors, hydraulic systems, pneumatic systems, or any other system that can apply forces to the system.

Robot: Sensors

- Rotation encoders
- Cameras
- Pressure sensors
- Limit switches
- Optical sensors
- Sonar

Robot: Controller

- Controllers direct a robot how to move.
- There are two controller paradigms
- ➢ Open-loop controllers execute robot movement without feedback.
- ➢ Closed-loop controllers execute robot movement and judge progress with sensors. They can thus compensate for errors.

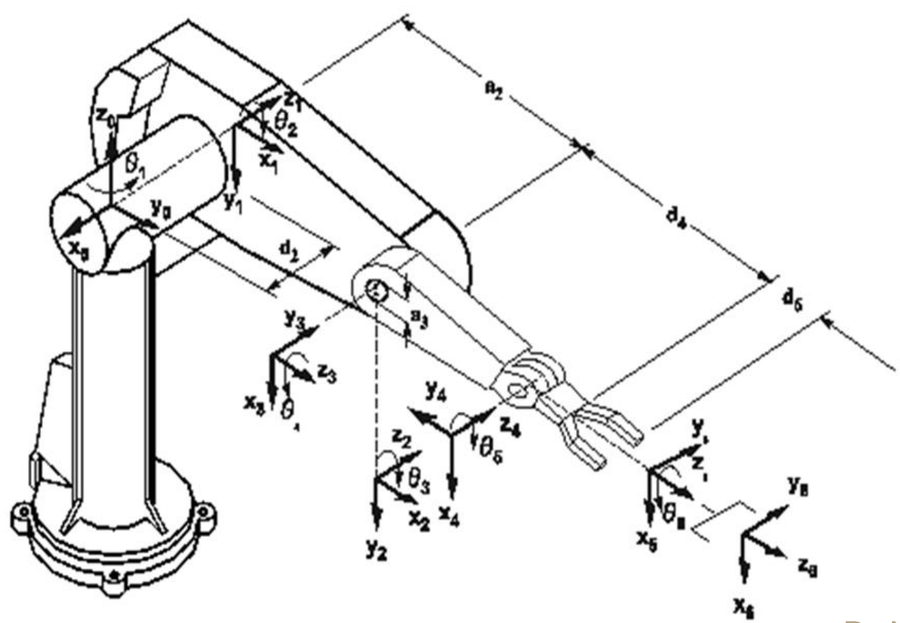


Kinematics

- Kinematics is the study of motion without regard for the forces that cause it.
- It refers to all time-based and geometrical properties of motion.
- It ignores concepts such as torque, force, mass, energy, and inertia.

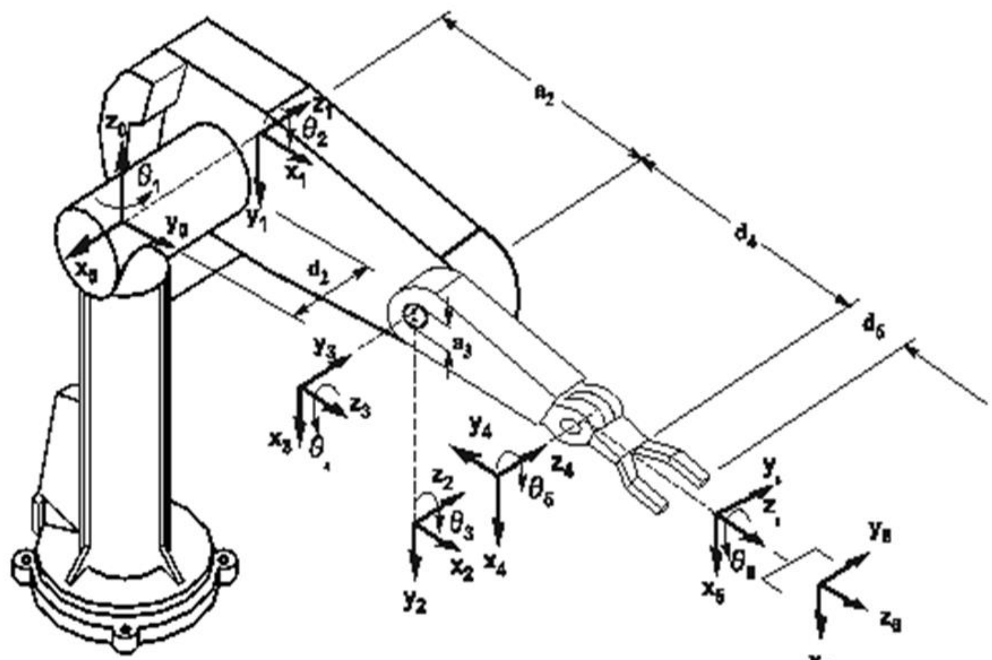
Forward Kinematics

- For a robotic arm, this would mean calculating the position and orientation of the end effector given all the joint variables.

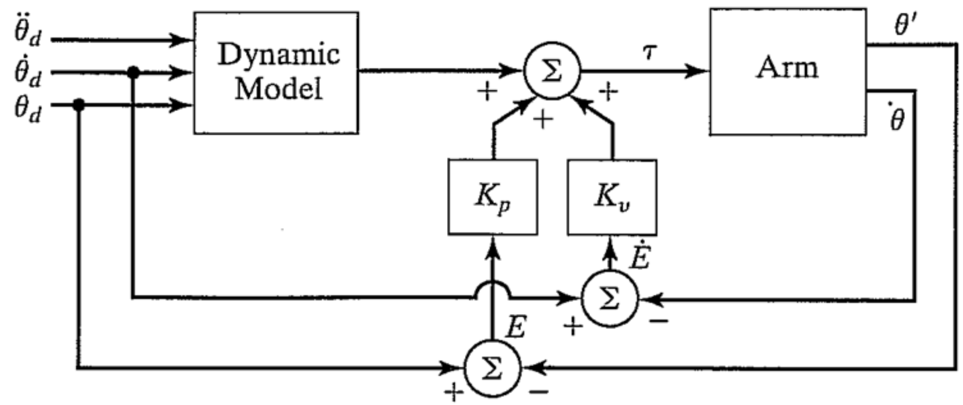


Inverse Kinematics

- Inverse Kinematics is the reverse of Forward Kinematics.
- It is the calculation of joint values given the positions, orientations, and geometries of mechanism's parts.
- It is useful for planning how to move a robot in a certain way.



Dynamics



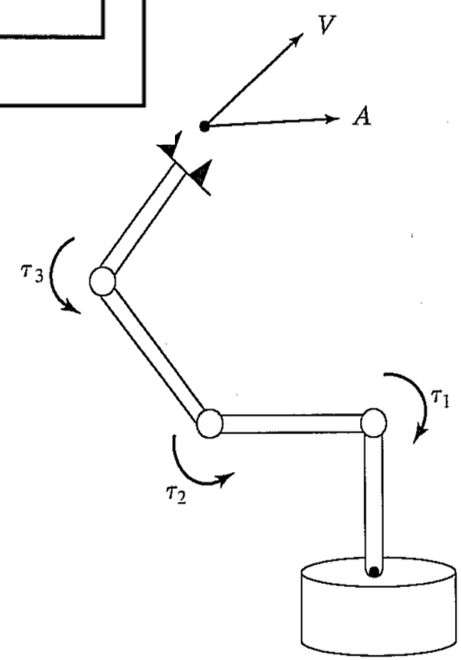


FIGURE 1.10:The relationship between the torques applied by the actuators and the resulting motion of the manipulator is embodied in the dynamic equations.

Trajectory generating

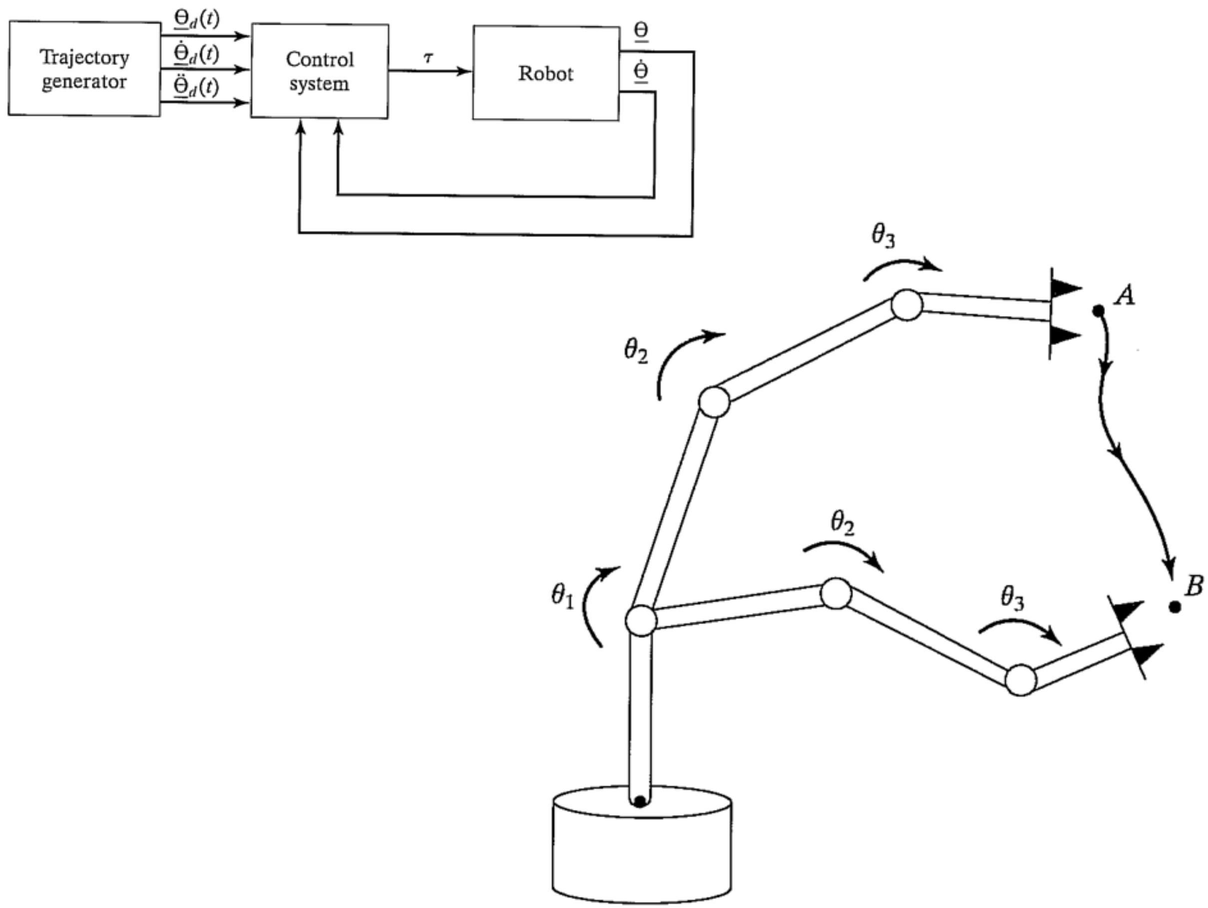


FIGURE 1.11:In order to move the end-effector through space from point A to point B , we must compute a trajectory for each joint to follow.

Position Control

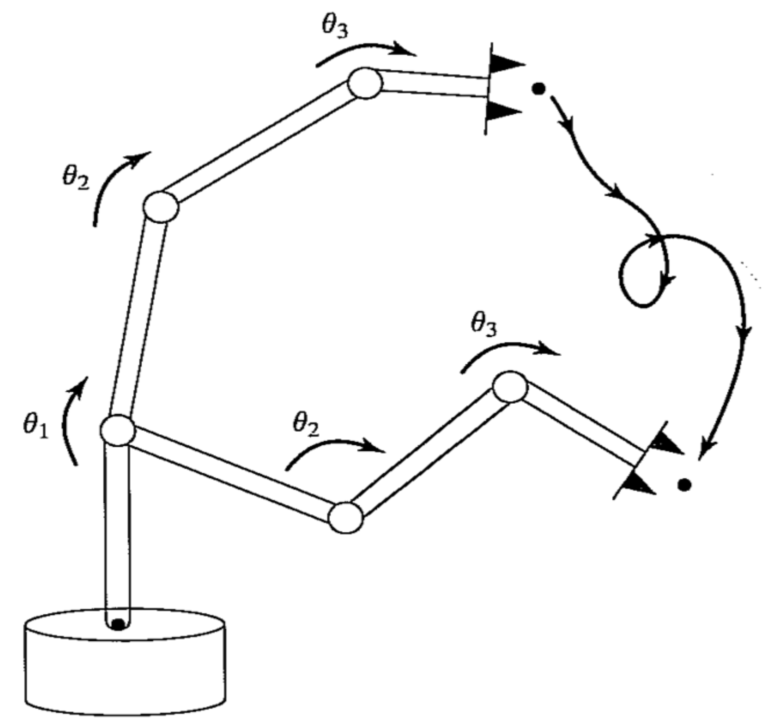


FIGURE 1.13: In order to cause the manipulator to follow the desired trajectory , a position-control system must be implemented. Such a system uses feedback from joint sensors to keep the manipulator on course.

Force Control

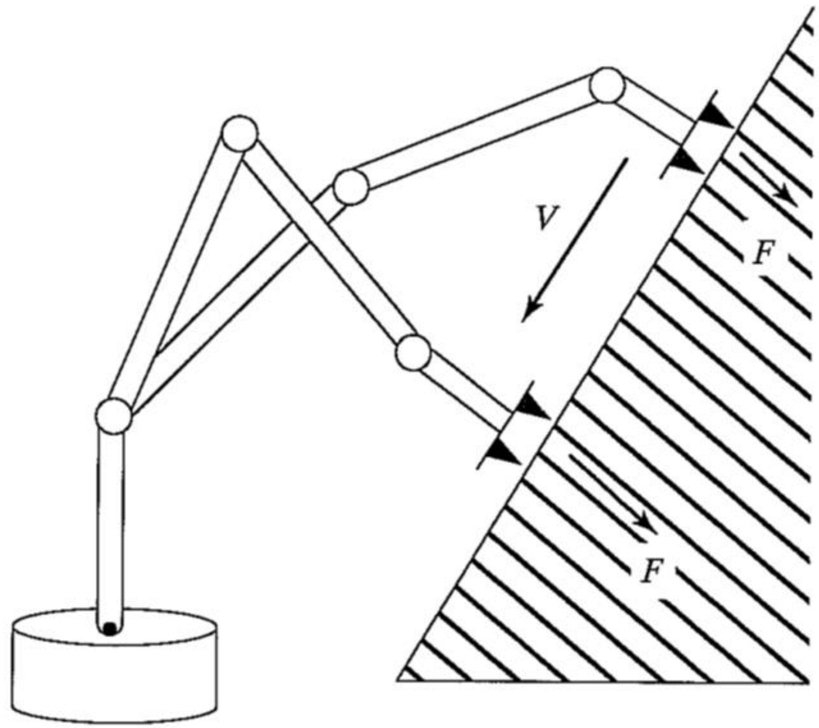
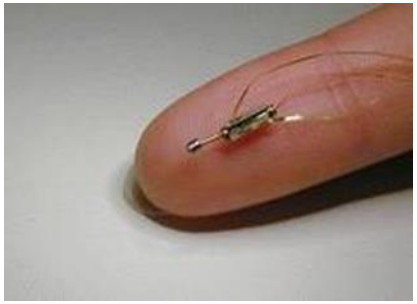
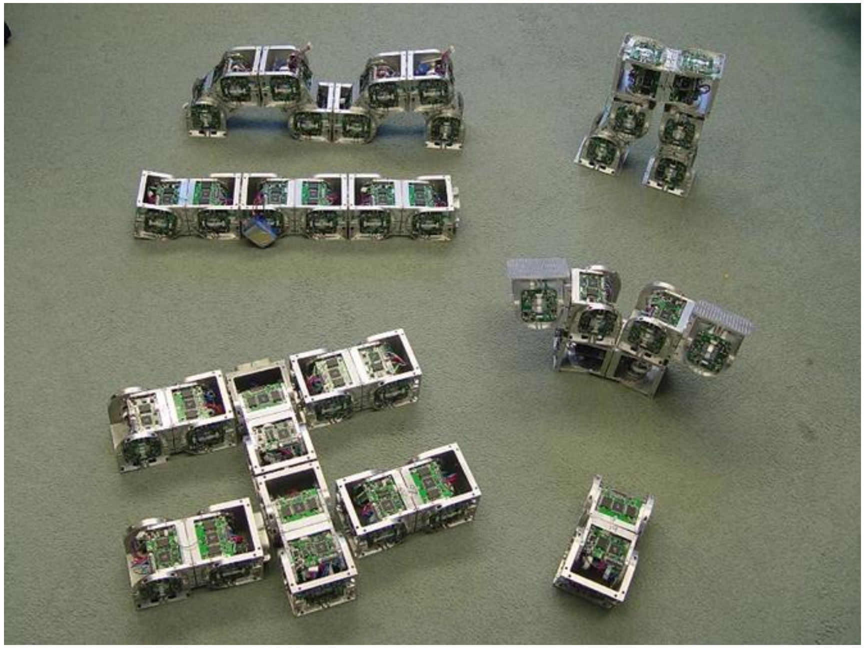


FIGURE 1.14: In order for a manipulator to slide across a surface while applying a constant force , a hybrid position-force control system must be used.

New direction

- Nanobots





- Reconfigurable Robot

Chapter 2
Spatial descriptions

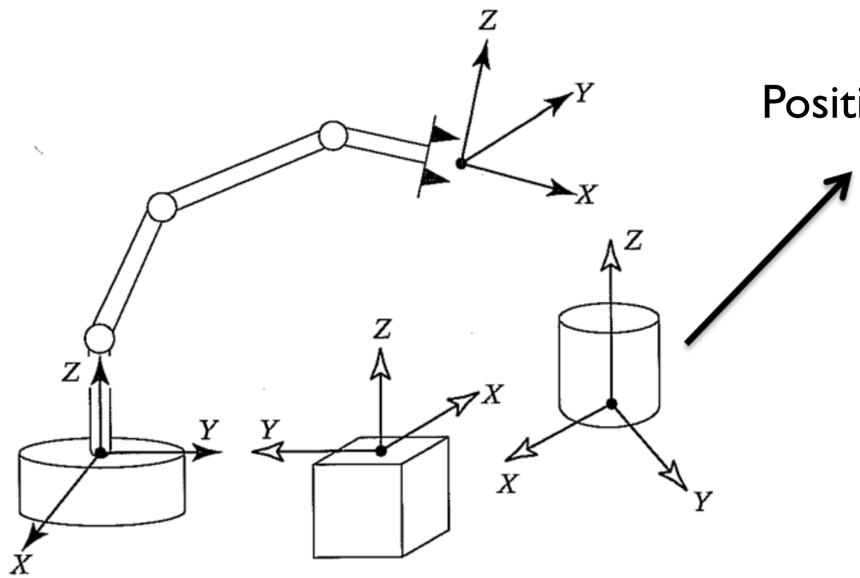
2.2 DESCRIPTIONS :
POSITIONS,ORIENTATIONS,AND FRAMES

2.3 MAPPINGS :
CHANGING DESCRIPTION FROM FRAME TO FRAME

2.4 OPERATORS :
TRANSLATIONS ,ROTATIONS, AND TRANSFORMATIONS

Introduction:

In the study of robotics, we are constantly concerned with the location of objects in three-dimensional space. These objects are the links of the manipulator, the parts and tools with which it deals, and other objects in the manipulator's environment.



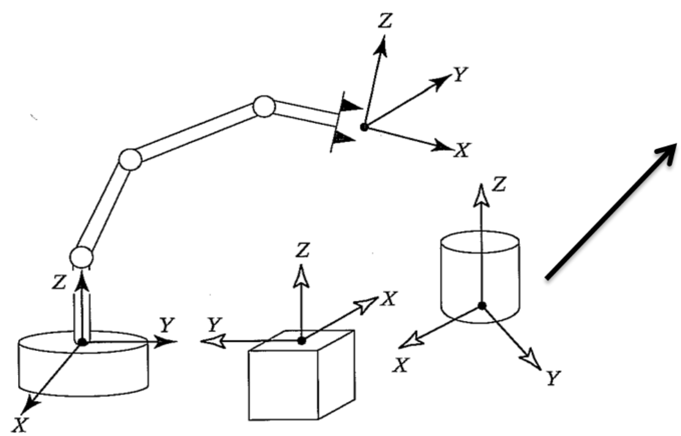
Position and orientation

FIGURE 1.5: Coordinate systems or "frames" are attached to the manipulator and to objects in the environment.

Introduction: cont.

In order to describe the position and orientation of a body in space, we will always attach a coordinate system, or **frame**, rigidly to the object. We then proceed to describe the position and orientation of this frame with respect to some reference coordinate system. (See Fig. 1.5.)

A **description** is used to specify attributes of various objects with which a manipulation system deals. These objects are parts, tools, and the manipulator it self. In this section, we discuss the description of positions, of orientations, and of an entity that contains both of these descriptions: the frame.



Position and orientation

FIGURE 1.5: Coordinate systems or "frames" are attached to the manipulator and to objects in the environment.

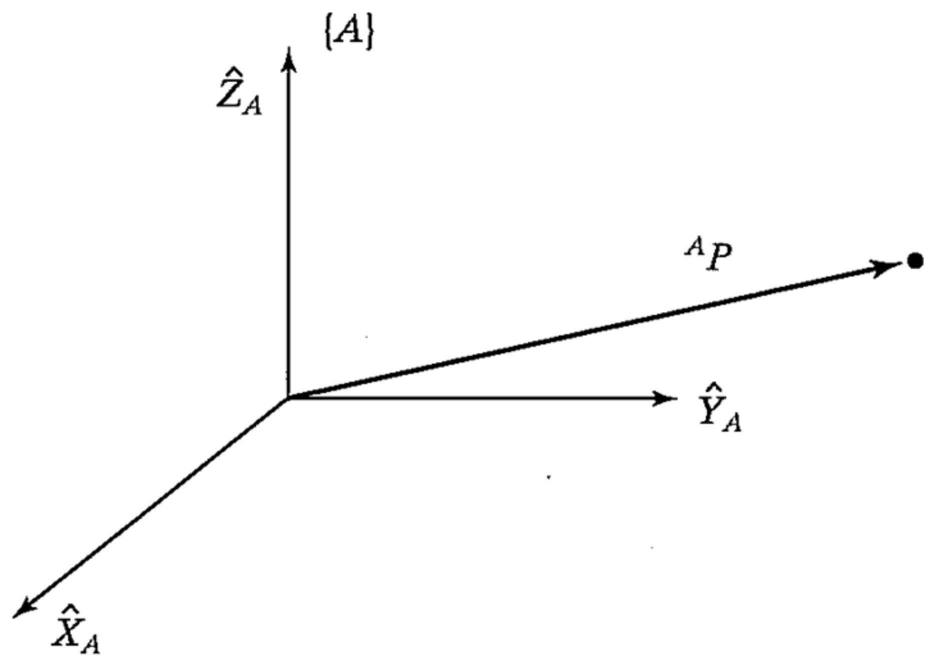
Description of a position

Once a coordinate system is established, we can locate any point in the universe with a 3 × 1 position vector. Because we will often define many coordinate systems.

$$|x| = 1,$$

$$|y| = 1,$$

$$|z| = 1,$$

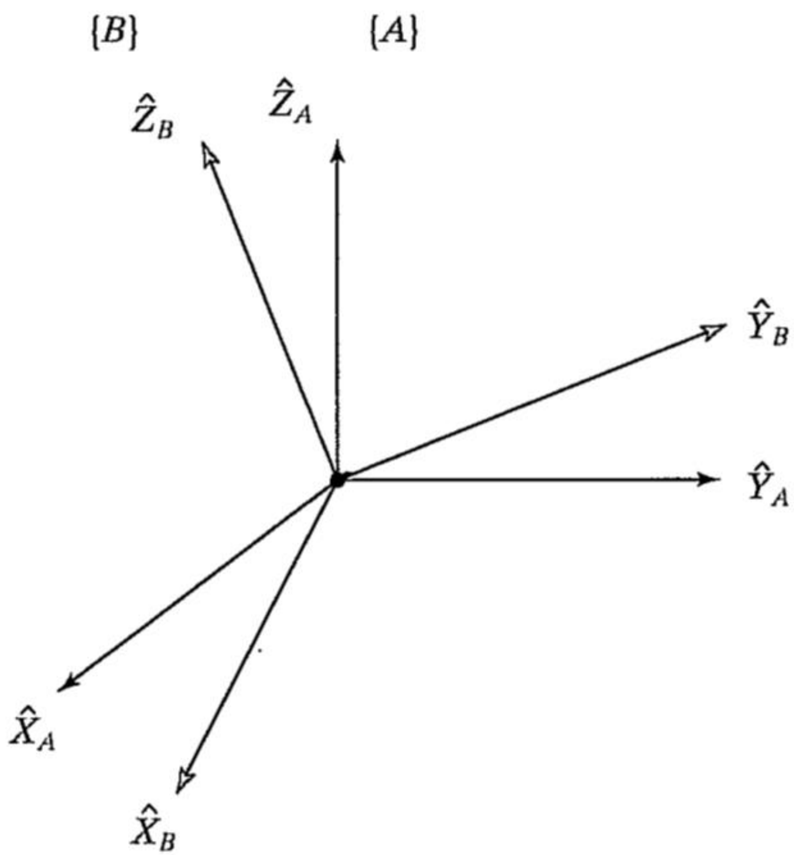


$$A_P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

FIGURE 2.1: Vector relative to frame (example).

Description of an orientation

Often, we will find it necessary not only to represent a point in space but also to describe the **orientation** of a body in space .

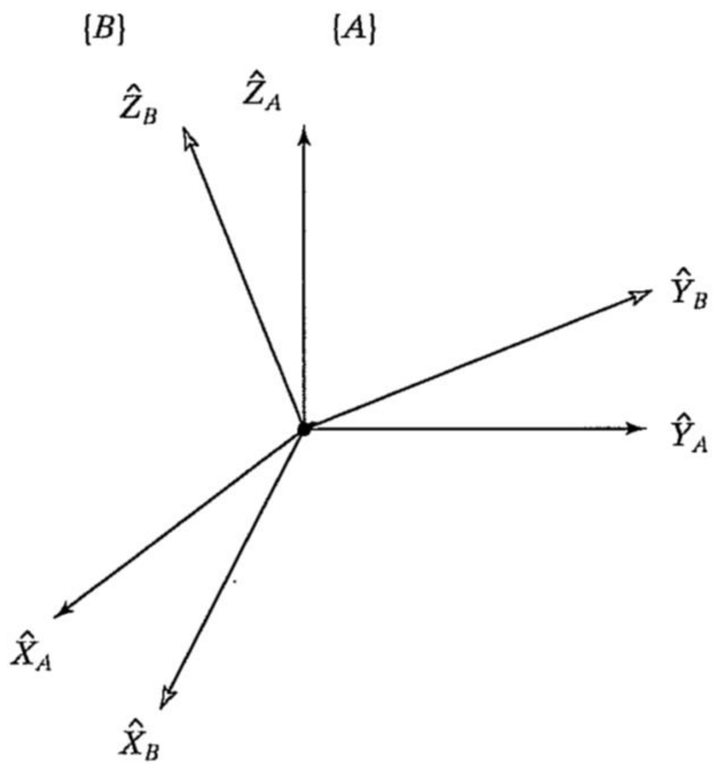


Rotation matrix : {B} relative to {A}

$$ {}^{A}_{B}R = \left[{}^{A}\hat{X}_{B} \ {}^{A}\hat{Y}_{B} \ {}^{A}\hat{Z}_{B} \right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

We can give expressions for the scalars r_{ij} in (2.2) by nothing that the components of any vector are simply the projections of that vector on to the unit directions of its reference frame. Hence , each component of ${}^{A}_{B}R$ in (2.2) can be written as the dot product of a pair of unit vectors :

$$ {}^{A}_{B}R = \left[{}^{A}\hat{X}_{B} \; {}^{A}\hat{Y}_{B} \; {}^{A}\hat{Z}_{B} \right] = \begin{bmatrix} \hat{X}_{B} \cdot \hat{X}_{A} & \hat{Y}_{B} \cdot \hat{X}_{A} & \hat{Z}_{B} \cdot \hat{X}_{A} \\ \hat{X}_{B} \cdot \hat{Y}_{A} & \hat{Y}_{B} \cdot \hat{Y}_{A} & \hat{Z}_{B} \cdot \hat{Y}_{A} \\ \hat{X}_{B} \cdot \hat{Z}_{A} & \hat{Y}_{B} \cdot \hat{Z}_{A} & \hat{Z}_{B} \cdot \hat{Z}_{A} \end{bmatrix} \quad (2.3) $$



<u>Note: dot product for vector</u>

Further inspection of (2.3) shows that the rows of the matrix are the unit vectors of {A} expressed in {B} ;that is,

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}_{A}^{T} \\ {}^{B}\hat{Y}_{A}^{T} \\ {}^{B}\hat{Z}_{A}^{T} \end{bmatrix} \qquad (2.4) $$

Hence, ${}^{A}_{B}R$*, the description of frame {A} relative to {B}, is given by the transpose of (2.3); that is,*

$$ {}^{B}_{A}R = {}^{A}_{B}R^{T}. \qquad (2.5) $$

This suggests that the inverse of a rotation matrix is equal to its transpose , a fact that can be easily verified as

$$ {}^{A}_{B}R^{T}\ {}^{A}_{B}R = \begin{bmatrix} {}^{B}\hat{X}_{A}^{T} \\ {}^{B}\hat{Y}_{A}^{T} \\ {}^{B}\hat{Z}_{A}^{T} \end{bmatrix} \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = I_{3}, \qquad (2.6) $$

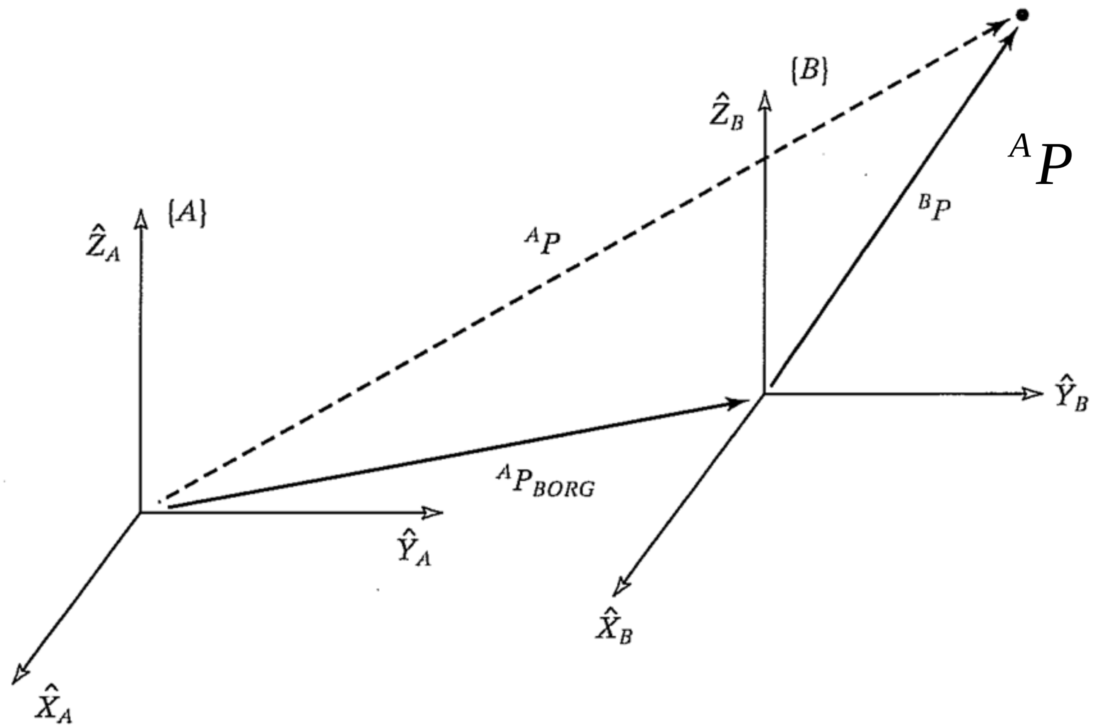
Where I_3 *is the 3×3 identity matrix. Hence,*

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T}, \qquad (2.7) $$

Note: example rotation matrix

2.3 MAPPING: CHANGING DESCRIPTIONS FROM FRAME TO FRAME

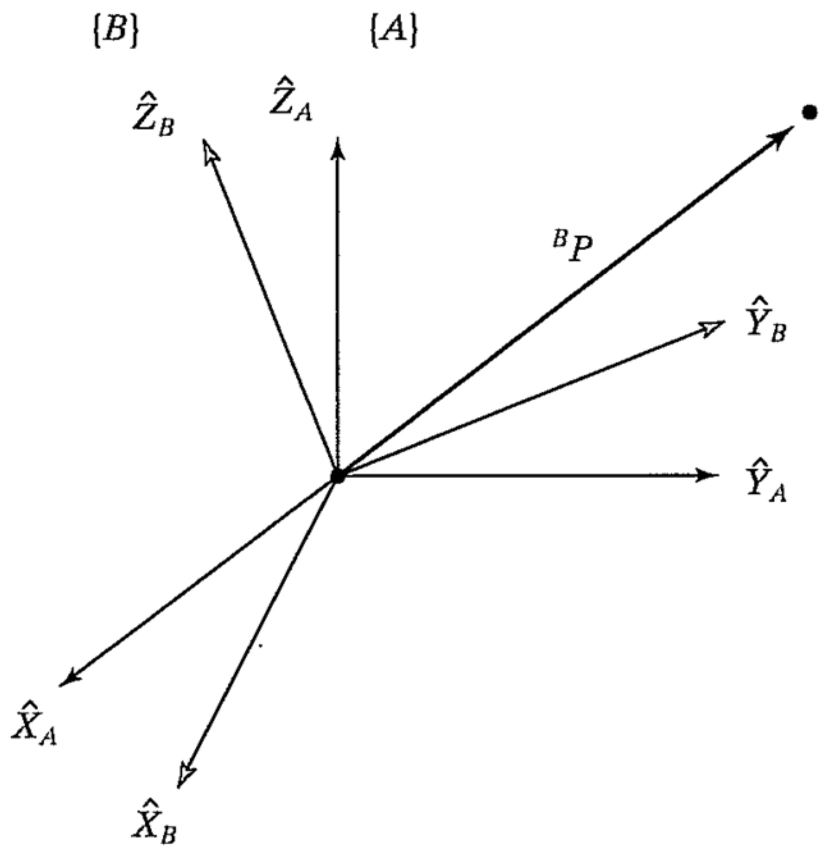
In a great many of the problems in robotics , we are concerned with expressing the same quantity in terms of various reference coordinate systems. The previous section introduced descriptions of positions, orientations, and frames; we now consider the mathematics of **mapping** in order to change descriptions from frame to frame.



$$^{A}P = {}^{B}P + {}^{A}P_{BORG}$$

FIGURE 2.4: Translational mapping

Mappings involving rotated frames



$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P. $$

FIGURE 2.5: Rotating the description of a vector.

Mappings involving rotated frames

EXAMPLE 2.1

Figure 2.6 shows a frame {B} that is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees. Here , $\hat{Z}$ is pointing out of the page .

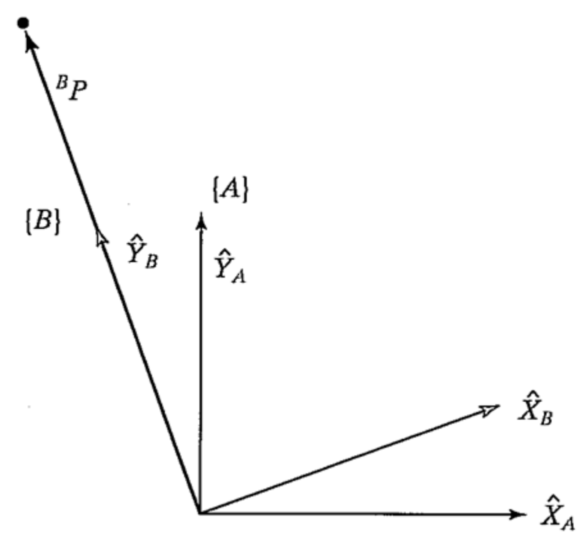


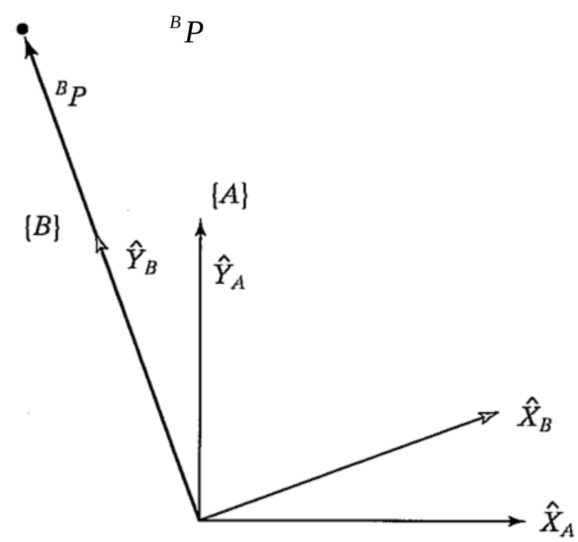
FIGURE 2.6: {B} rotated 30 degrees about $\hat{Z}$.

Mappings involving rotated frames

EXAMPLE 2.1

Figure 2.6 shows a frame {B} that is rotated relative to frame {A} about $\hat{Z}$ by 30 degrees. Here , $\hat{Z}$ is pointing out of the page .

$$ {}^{B}P = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix} $$



$$ {}^{A}P = {}^{A}_{B}R \, {}^{B}P = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix} $$

$$ R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

FIGURE 2.6: {B} rotated 30 degrees about $\hat{Z}$

APPENDIX A

Formulas for rotation about the principle axes by :

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad \text{(A.1)}$$

$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad \text{(A.2)}$$

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(A.3)}$$

Mappings involving general frames

Very often, we know the description of a vector with respect to some frame {B},and we would like to know its description with respect to another frame,{A}. We now consider the general case of mapping. Here, the origin of frame {B} is not coincident with that of frame {A} but has a general vector offset. The vector that locates {B}'s origin is called ${}^{A}P_{BORG}$. Also {B} is rotated with respect to {A}, as described by .
Given ${}^{B}P$, we wish to compute ${}^{A}P$, as in Fig. 2.7.

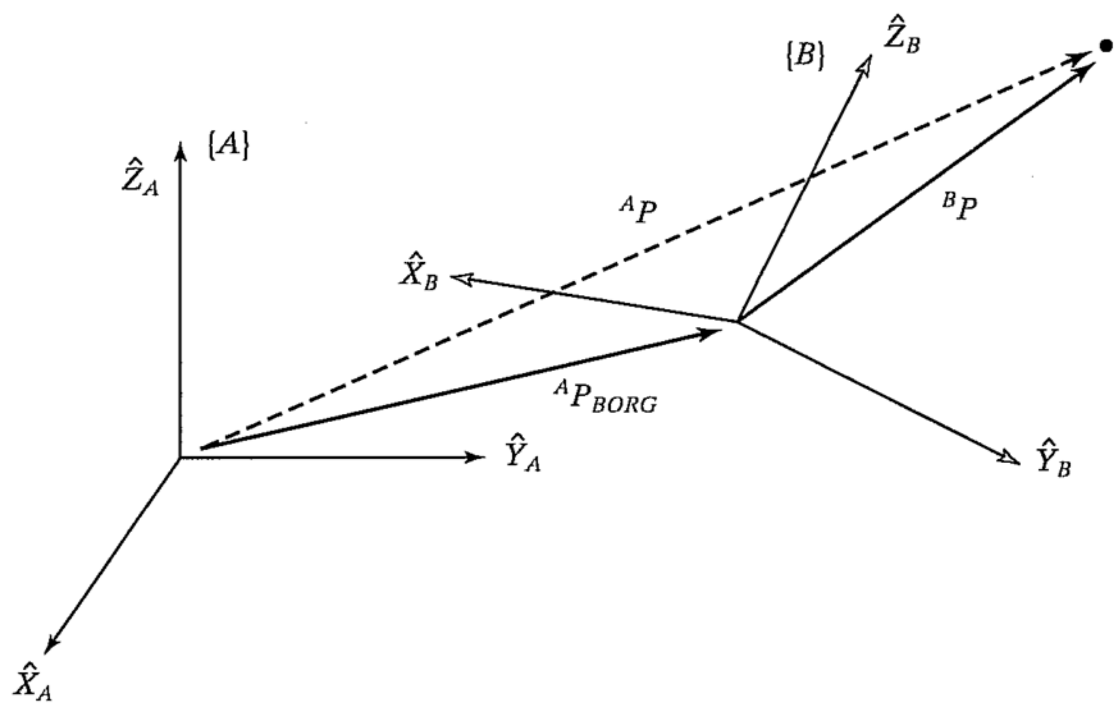
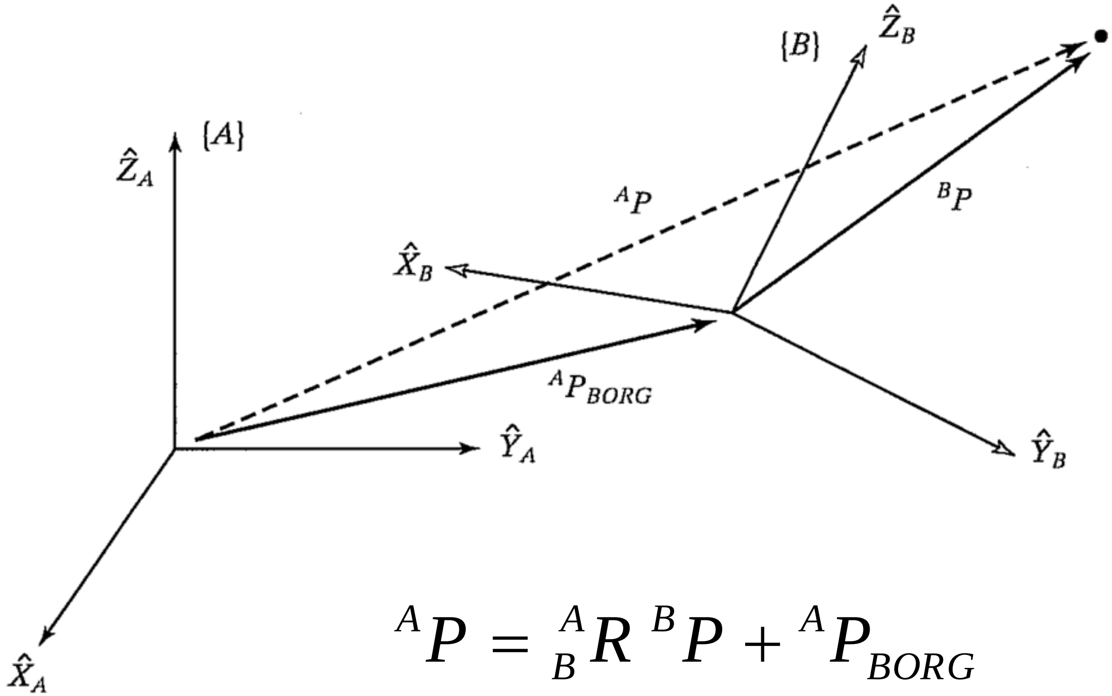


FIGURE 2.7 : General transform of a vector .



$$^{A}P = {}^{A}_{B}R\, {}^{B}P + {}^{A}P_{BORG}$$

Homogeneous transform

$$^{A}P = {}^{A}_{B}T\, {}^{B}P$$

$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} ^{A}_{B}R & ^{A}P_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix}$$

EXAMPLE 2.2

Figure 2.8 shows a frame {B}, which is rotated relative to frame {A} about $\hat{Z}$ by 30 degree, translated 10 units in $\hat{X}_A$, and translated 5 units in $\hat{Y}_A$. Find ${}^{A}P$, where

$${}^{B}P = [3.0 \; 7.0 \; 0.0]^T .$$

The definition of frame {B} is

$$ {}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2.21)$$

Given

$${}^{B}P = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix}, \qquad (2.22)$$

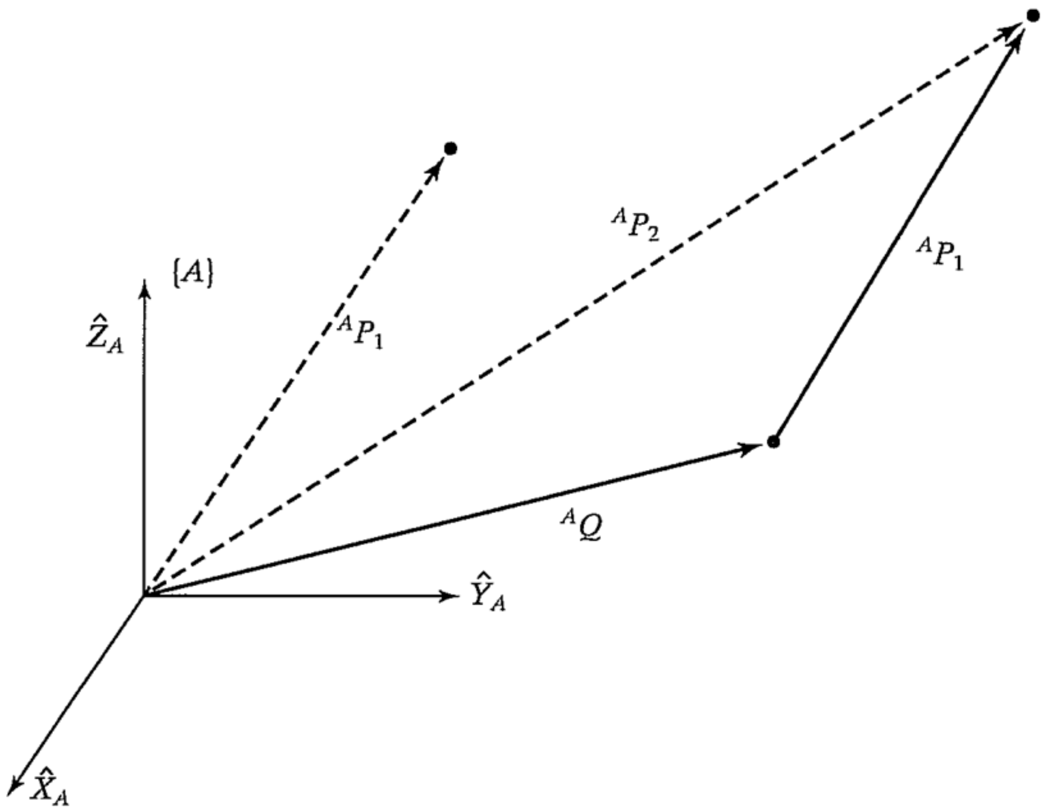
We use the definition of {B} just given as a transformation:

$${}^{A}P = {}^{A}_{B}T \; {}^{B}P = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}, \qquad (2.23)$$

2.4 OPERATORS: TRANSLATIONS, ROTATIONS, AND TRANSFORMATIONS

The same mathematical forms used to map points between frames can also be interpreted as Operators that translate points, rotate vectors, or do both .This section illustrates this interpretation of the mathematics we have already developed.

Translational operators



$${}^A P_2 = {}^A P_1 + {}^A Q$$

$${}^A P_2 = D_Q(q)\,{}^A P_1.$$

Rotational operators

$$^{A}P_2 = R\,^{A}P_1$$

$$^{A}P_2 = R_K(\theta)\,^{A}P_1$$

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

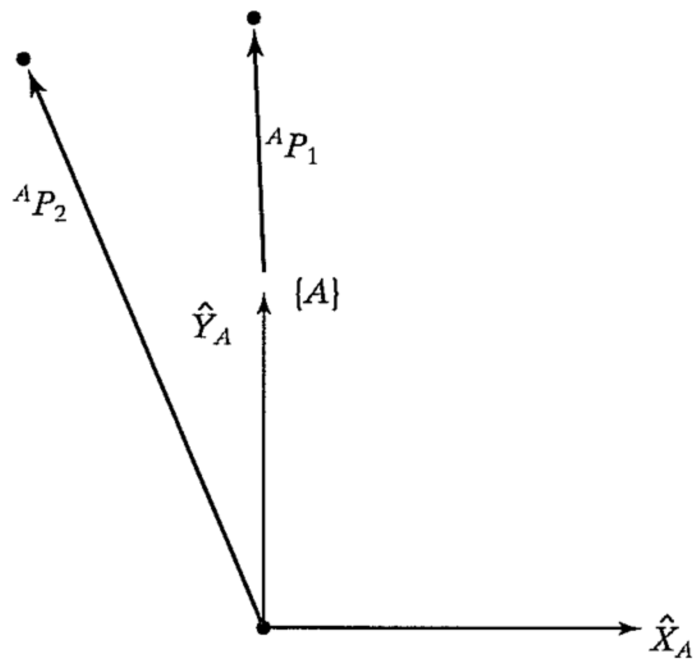


FIGURE 2.10: The vector $^{A}P_1$ rotated 30 degrees about $\hat{Z}$.

Rotational operators

Figure 2.10 shows a vector ${}^{A}P_{1}$. We wish to compute the vector obtained by rotating this vector about $\hat{Z}$ by 30 degrees. Call the new vector ${}^{A}P_{2}$.
The rotation matrix that rotates vectors by 30 degrees about $\hat{Z}$ is the same as the rotation matrix that describes a frame rotated 30 degrees about $\hat{Z}$ relative to the reference frame . Thus , the correct rotational operator is

$$R_Z(30.0) = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \qquad (2.30)$$

Given

$${}^{A}P_{1} = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix}, \qquad (2.31)$$

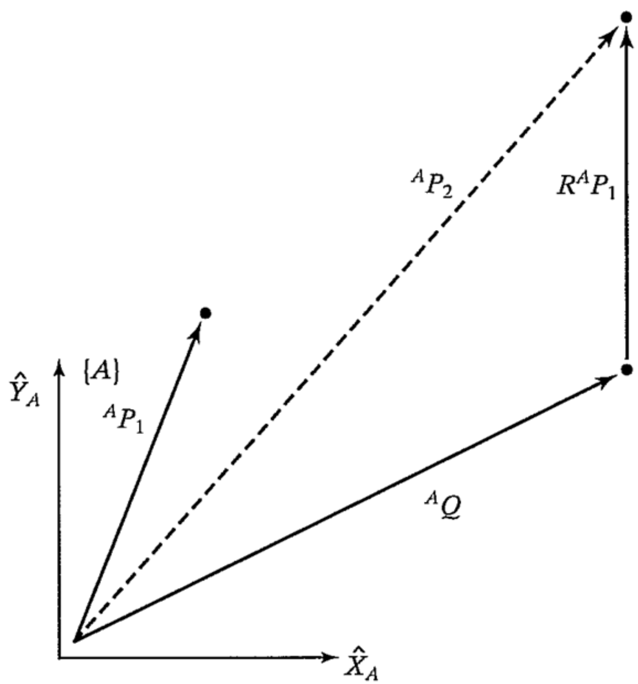
We calculate ${}^{A}P_{2}$ as

$${}^{A}P_{2} = R_Z(30.0)\,{}^{A}P_{1} = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}, \qquad (2.32)$$

Transformation operators

EXAMPLE 2.4

Figure 2.11 shows a vector ${}^{A}P_1$. We wish to rotate it about $\hat{Z}$ by 30 degrees and translate it 10 units in $\hat{X}_A$ and5 units in $\hat{Y}_A$. Find ${}^{A}P_2$, where ${}^{A}P_1 = [3.0\ 7.0\ 0.0]^T$.



$$T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{A}P_2 = T\ {}^{A}P_1$$

$${}^{A}P_2 = T\ {}^{A}P_1 = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}$$

FIGURE 2.11:The vector ${}^{A}P_1$ rotated and translated to form ${}^{A}P_2$.

2.5 SUMMARY OF INTERPRETATIONS

We have introduced three interpretations of this homogeneous transform:

1. It is a descriptions of a frame. ${}^{A}_{B}T$ describes the frame {B} relative to the frame {A}. Specifically, the columns of ${}^{A}_{B}R$ are unit vectors defining the directions of the principal axes of {B}, and ${}^{A}P_{BORG}$ locates the position of the origin of {B} .
2. It is a transform mapping. ${}^{A}_{B}T$ maps ${}^{B}P \rightarrow {}^{A}P$.
3. It is a transform operator. T operates on ${}^{A}P_1$ to create ${}^{A}P_2$.

2.6 TRANSFORMATION ARITHMETIC
Compound transformations

we have ${}^{C}P$ and wish to find ${}^{A}P$.

$$ {}^{B}P = {}^{B}_{C}T\,{}^{C}P $$

$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}P $$

$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P $$

$$ {}^{A}_{C}T = {}^{A}_{B}T\,{}^{B}_{C}T. $$

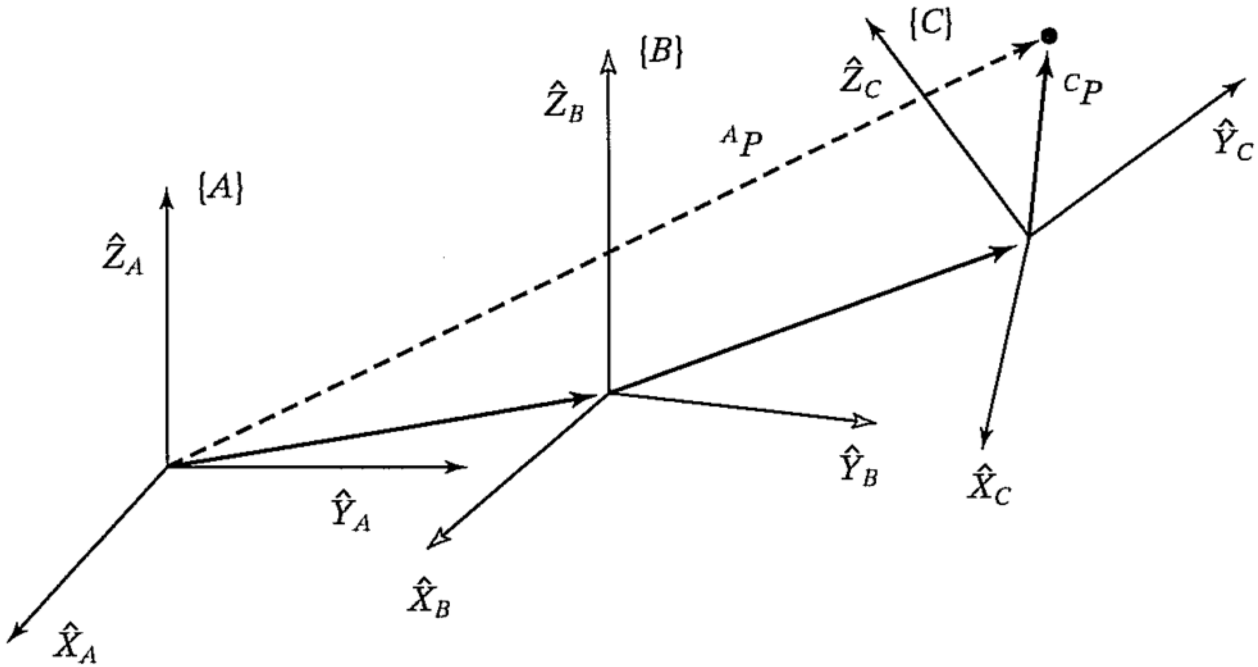


FIGURE 2.12 : Compound frames: Each is known relative to previous one .

EXAMPLE 2.5

Figure 2.13 shows a frame {B} that is rotated relative to frame {A} about 30 degrees and translated four units in $\hat{X}_A$ and three units in $\hat{Y}_A$ .Thus, we have a description of ${}^A_BT$ .Find ${}^B_AT$.

The frame defining {B} is

$$ {}^A_BT = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 4.0 \\ 0.500 & 0.866 & 0.000 & 3.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2.68) $$

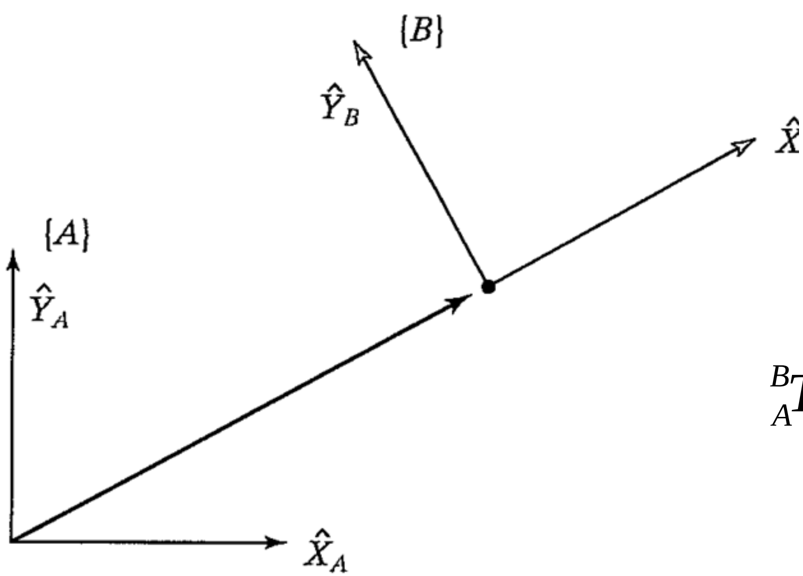
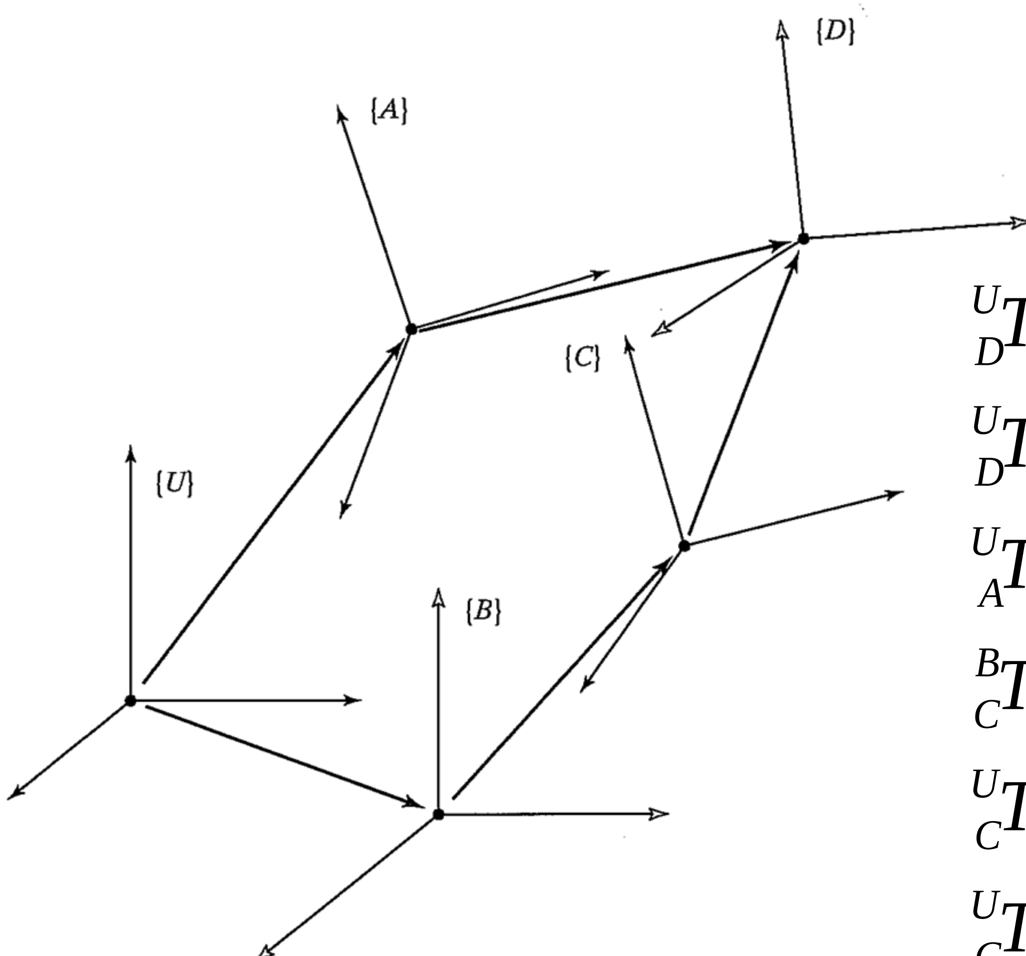


FIGURE 2.13 : {B} relative to {A} .

$$ {}^B_AT = \begin{bmatrix} 0.866 & 0.500 & 0.000 & -4.964 \\ -0.500 & 0.866 & 0.000 & -0.598 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

2.7 TRANSFORM EQUATIONS



$$ {}^{U}_{D}T = {}^{U}_{A}T \, {}^{A}_{D}T ; $$

$$ {}^{U}_{D}T = {}^{U}_{B}T \, {}^{B}_{C}T \, {}^{C}_{D}T . $$

$$ {}^{U}_{A}T \, {}^{A}_{D}T = {}^{U}_{B}T \, {}^{B}_{C}T \, {}^{C}_{D}T . $$

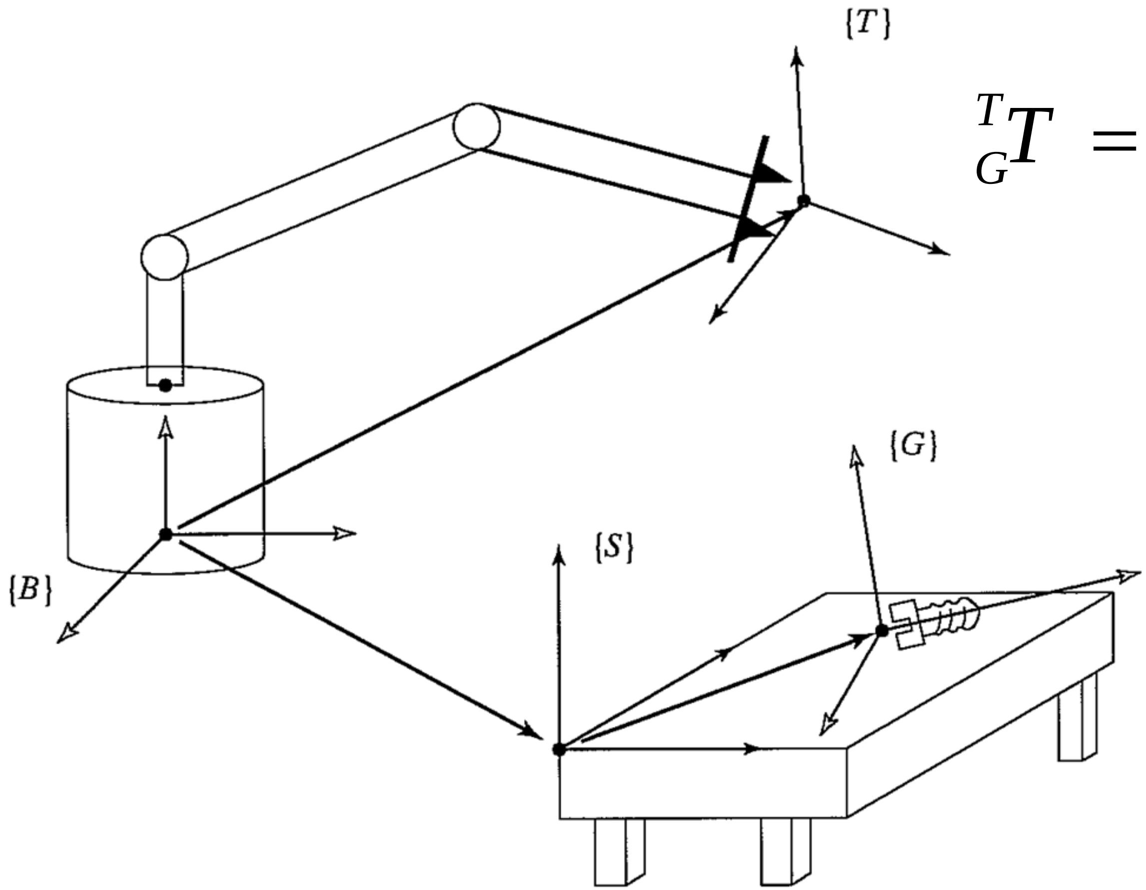
$$ {}^{B}_{C}T = {}^{U}_{B}T^{-1} \, {}^{U}_{A}T \, {}^{A}_{D}T \, {}^{C}_{D}T^{-1} . $$

$$ {}^{U}_{C}T = {}^{U}_{A}T \, {}^{D}_{A}T^{-1} \, {}^{D}_{C}T $$

$$ {}^{U}_{C}T = {}^{U}_{B}T \, {}^{B}_{C}T . $$

FIGURE 2.14: Set of transforms forming a loop .

2.7 TRANSFORM EQUATIONS



$$ {}^{T}_{G}T = {}^{B}_{T}T^{-1}\, {}^{B}_{S}T\, {}^{S}_{G}T . $$

FIGURE 2.16 : Manipulator reaching for a bolt .

2.8 MORE ON REPRESENTATION OF ORIANTATION

Rotation matrix determinant is +1

$$R = \begin{bmatrix} \hat{X} & \hat{Y} & \hat{Z} \end{bmatrix}.$$

$$\left|\hat{X}\right| = 1,$$
$$\left|\hat{Y}\right| = 1,$$
$$\left|\hat{Z}\right| = 1,$$
$$\hat{X} \cdot \hat{Y} = 0,$$
$$\hat{X} \cdot \hat{Z} = 0,$$
$$\hat{Y} \cdot \hat{Z} = 0.$$

Clearly, the nine elements of a rotation matrix are not all independent . In fact, given a rotation matrix, R , it is easy to write down the six dependencies between the elements. Therefore, rotation matrix can be specified by just three parameters.

2.8 MORE ON REPRESENTATION OF ORIANTATION

- Rotation matrices are useful as operators. Their matrix from is such that, when multiplied by a vector , they perform the rotation operation.
- Human operator at a computer terminal who wishes to type in the specification of the desired orientation of a robot's hand would have a hard time inputting a nine-element matrix with orthonormal columns. A representation that requires only three numbers would be simpler .

X-Y-Z fixed angles

One method of describing the orientation of frame {B} is as follows:

Start with the frame coincident with a known reference frame {A}.

Rotate {B}first about $\hat{X}_A$ by an angle γ,then about $\hat{Y}_A$ by angle β ,and ,finally ,about $\hat{Z}_A$ by an angle α

Each of the three rotations takes place about an axis in the fixed reference frame {A}. We will call this convention for specifying an orientation **X-Y-Z fixed angles** .the word "fixed" refers to the fact that the rotations are specified about the fixed (i.e. nonmoving) reference frame (Fig 2.17).sometimes this convention is referred to as roll, pitch, yaw angles, but care must be used ,as this name is often given to other related but different conventions.

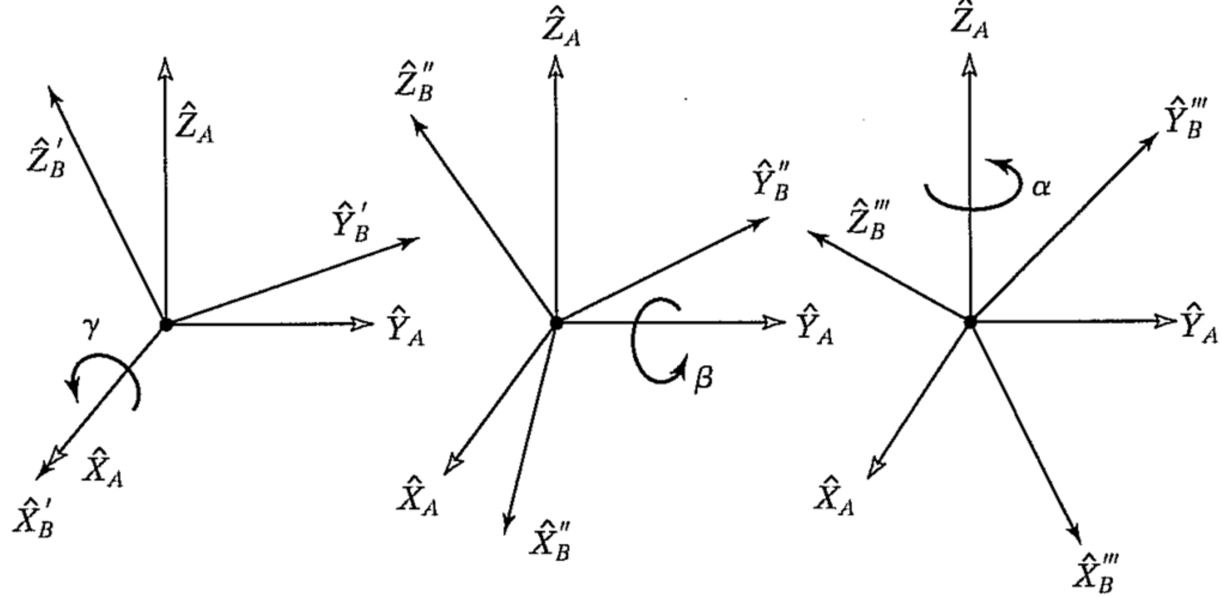


FIGURE 2.17:X-Y-Z fixed angles. Rotations are performed in the order $R_X\,(\gamma), R_Y\,(\beta), R_Z\,(\alpha)$.

$$
{}^{A}_{B}R_{XYZ}\left(\gamma,\beta,\alpha\right) = R_Z\,(\alpha), R_Y\,(\beta) R_X\,(\gamma)
$$

$$
= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}
$$

$$ {}_{B}^{A}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma & -s\alpha c\gamma & c\alpha s\beta c\gamma & +s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma & -c\alpha c\gamma & s\alpha s\beta c\gamma & +c\alpha s\gamma \\ -s\beta & c\beta s\gamma & & \mathrm{c}\beta\mathrm{c}\gamma & \end{bmatrix}. $$

$$ {}_{B}^{A}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. $$

$$ \beta = A\tan 2(-r_{31},\sqrt{r_{11}^{2}+r_{21}^{2}}), $$

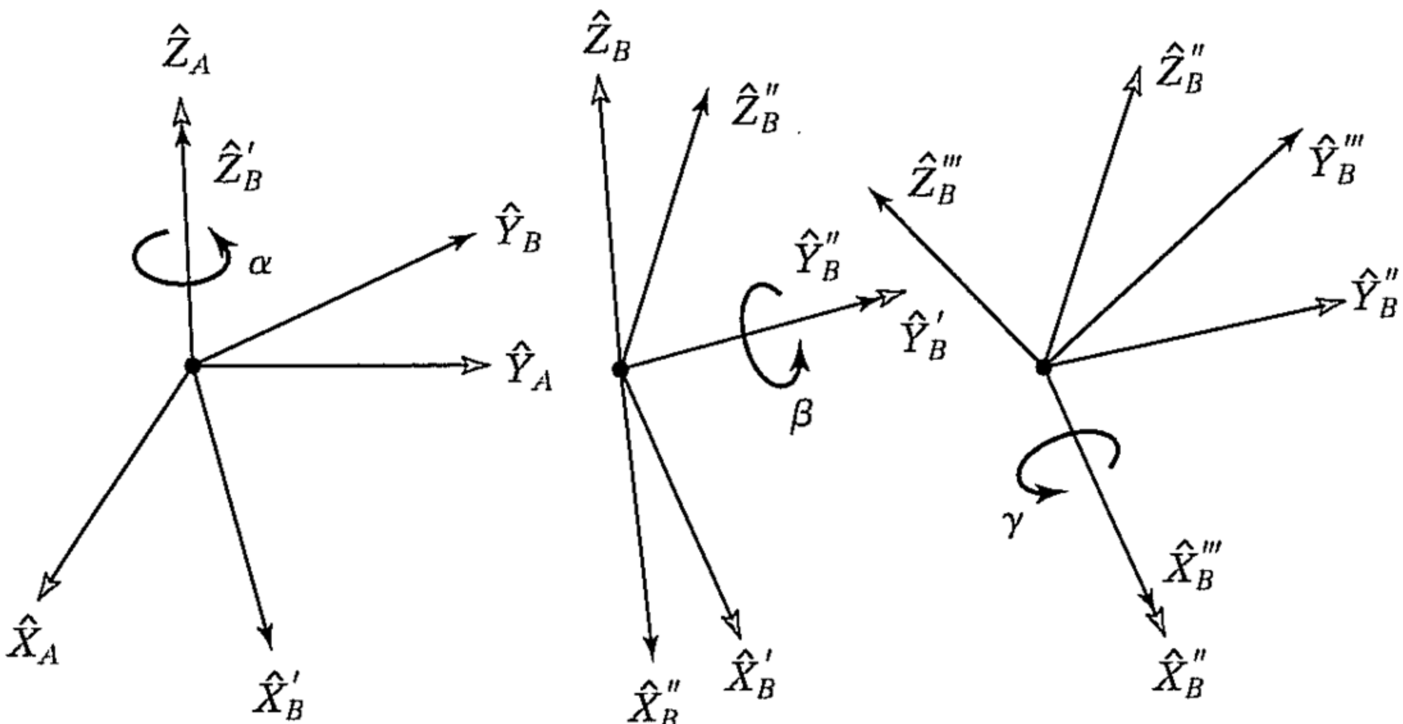
$$ \alpha = A\tan 2(r_{21},/c\beta, r_{11}/c\beta), $$

$$ \gamma = A\tan 2(r_{31},/c\beta, r_{33}/c\beta). $$

Note: use handout

Z-Y-X Euler angles

Another possible description of a frame {B} is as follows:
Start with the frame coincident with a known frame {A}. Rotate {B} first about $\hat{Z}_B$ by an angle α then about $\hat{Y}_B$ by an angle β, and , finally , about $\hat{X}_B$ by an angle γ .
In this representation ,each rotation is performed about an axis of the moving system {B} rather than one of
The fixed reference {A}.Such sets of three rotations



$$ {}^{A}_{B}R_{ZYX} = R_Z(\alpha), R_Y(\beta) R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}, $$

$$ {}^{A}_{B}R_{ZYX}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma \quad -s\alpha c\gamma & c\alpha s\beta c\gamma \quad +s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma \quad -c\alpha c\gamma & s\alpha s\beta c\gamma \quad +c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}. $$

Note: use handout

EXAMPLE 2.7

Consider two rotations , one about $\hat{Z}$ by 30 degrees and one about $\hat{X}$ by 30 degrees:

$$R_Z(30) = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \qquad (2.60)$$

$$R_X(30) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.866 & -0.500 \\ 0.000 & 0.500 & 0.866 \end{bmatrix} \qquad (2.61)$$

$$R_Z(30)R_X(30) = \begin{bmatrix} 0.87 & -0.43 & 0.25 \\ 0.50 & 0.75 & -0.43 \\ 0.00 & 0.50 & 0.87 \end{bmatrix}$$

$$\neq R_X(30)R_Z(30) = \begin{bmatrix} 0.87 & -0.50 & 0.00 \\ 0.43 & 0.75 & -0.50 \\ 0.25 & 0.43 & 0.87 \end{bmatrix} \qquad (2.62)$$

Z-Y-Z Euler angles

Another possible description of a frame {B} is

Start with the frame coincident with a known frame {A}.rotate {B} first about $\hat{Z}_B$ by an angle α, then about $\hat{Y}_B$ by an angle β, and , finally , about Z_b by an angle γ.

$$
{}^{A}_{B}R_{ZYZ}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & \mathrm{c}\beta \end{bmatrix}.
$$

$$
{}^{A}_{B}R_{ZYZ}(\alpha,\beta,\gamma)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.
$$

$$
\beta = A\tan 2(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}),
$$

$$
\alpha = A\tan 2(r_{23}, / s\beta, r_{13} / s\beta),
$$

$$
\gamma = A\tan 2(r_{32}, / s\beta, -r_{31} / s\beta).
$$

APPENDIX B
The 24 angle-set conventions

The 12 Euler angle sets

The 12 fixed angle sets

Equivalent angle-axis representation

Start with the frame coincident with a known frame {A}; then rotate {B} about the vector ${}^{A}\hat{K}$ by an angle θ according to the right-hand rule.

$$R_K(\theta) = \begin{bmatrix} k_x k_x \upsilon\theta + c\theta & k_x k_y \upsilon\theta - k_z s\theta & k_x k_z \upsilon\theta + k_y s\theta \\ k_x k_y \upsilon\theta + k_z s\theta & k_y k_y \upsilon\theta + c\theta & k_y k_z \upsilon\theta - k_x s\theta \\ k_x k_z \upsilon\theta - k_y s\theta & k_y k_z \upsilon\theta + k_x s\theta & k_z k_z \upsilon\theta + c\theta \end{bmatrix}$$

where $c\theta = \cos\theta, s\theta = \sin\theta, \upsilon\theta = 1 - \cos\theta, and\ {}^{A}\hat{K} = \begin{bmatrix} k_x k_y k_z \end{bmatrix}^T$

$${}^{A}_{B}R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

$$\theta = A\cos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} & -r_{23} \\ r_{13} & -r_{31} \\ r_{21} & -r_{12} \end{bmatrix}$$

EXAMPLE 2.8

A frame {B} is descried as initially coincident with {A}.we then rotate {B} about the vector ${}^{A}\hat{K} = [0.7070\ 7070\ 0]^T$ (passing through the origin)by an amount $\theta = 30$ degrees . Given the frame description of {B}.
Substituting in to (2.80) yields the rotation-matrix part of the frame description.
There was no translation of the origin ,so the position vector is $[0,\ 0,\ 0]^T$.
Hence,

$$ {}^{A}_{B}T = \begin{bmatrix} 0.933 & 0.067 & 0.354 & 0.0 \\ 0.067 & 0.933 & -0.354 & 0.0 \\ 0.354 & 0.354 & 0.866 & 0.0 \end{bmatrix} \qquad (2.83) $$

2.27 [15] Referring to Fig. 2.25,give the value of ${}^{A}_{B}T$
2.28 [15] Referring to Fig. 2.25,give the value of ${}^{A}_{C}T$
2.29 [15] Referring to Fig. 2.25,give the value of ${}^{B}_{C}T$

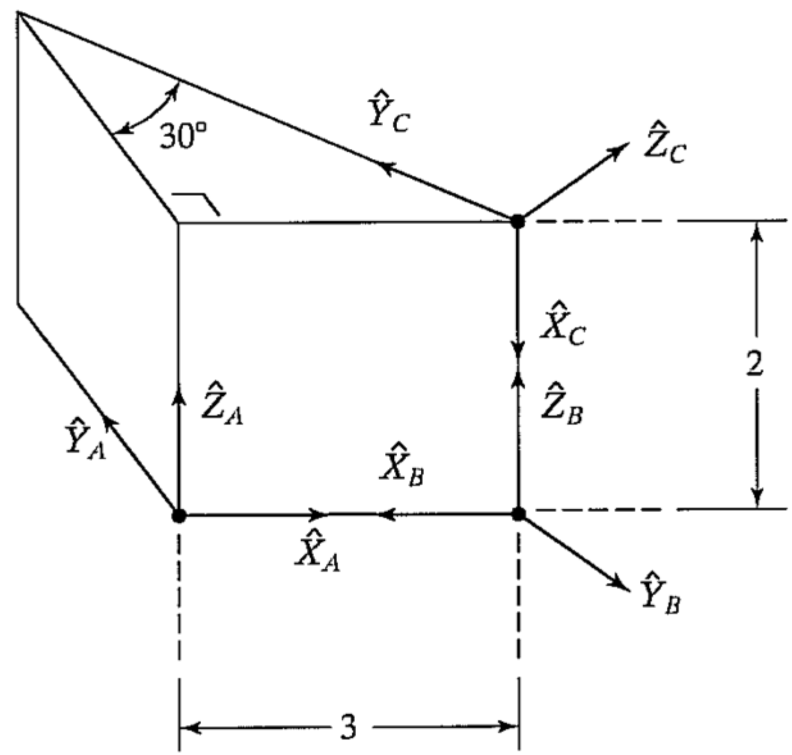


FIGURE 2.25: Frames at the corners of a wedge.

Note: use handout

Chapter 3
Manipulator Kinematics

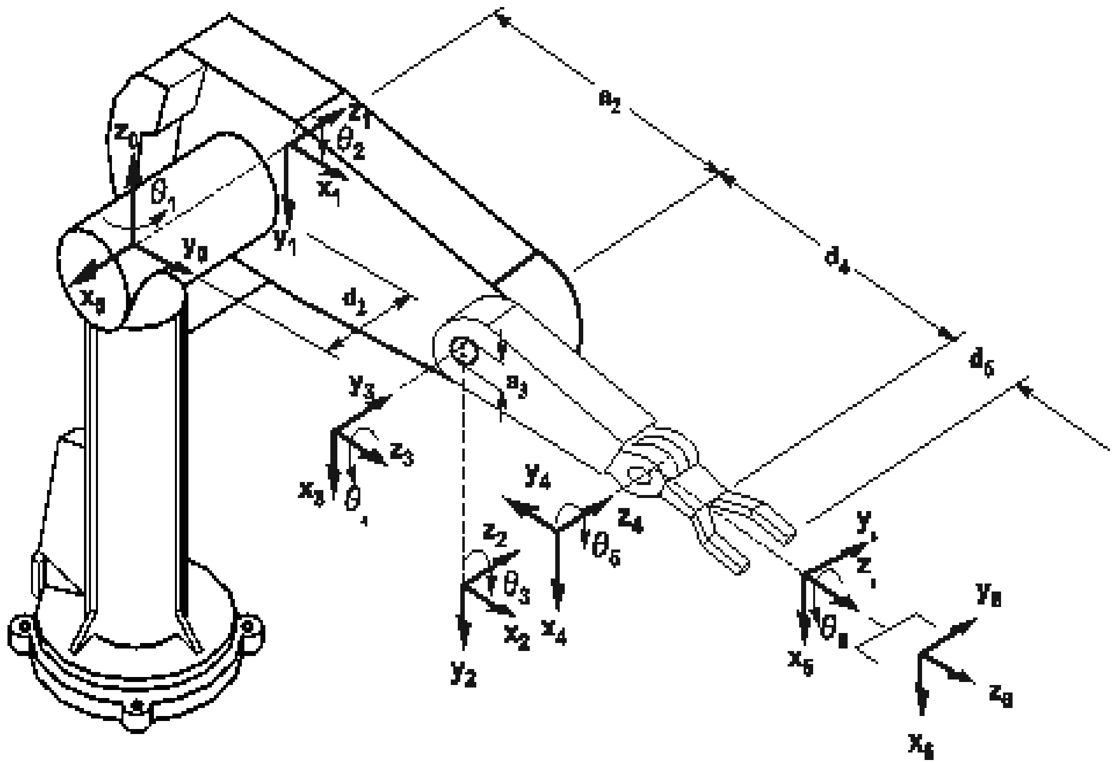
Kinematics

- Kinematics is the study of motion without regard for the forces that cause it.
- It refers to time-based and geometrical properties of motion.
- It ignores concepts such as torque, force, mass, energy, and inertia.

Teach pendent

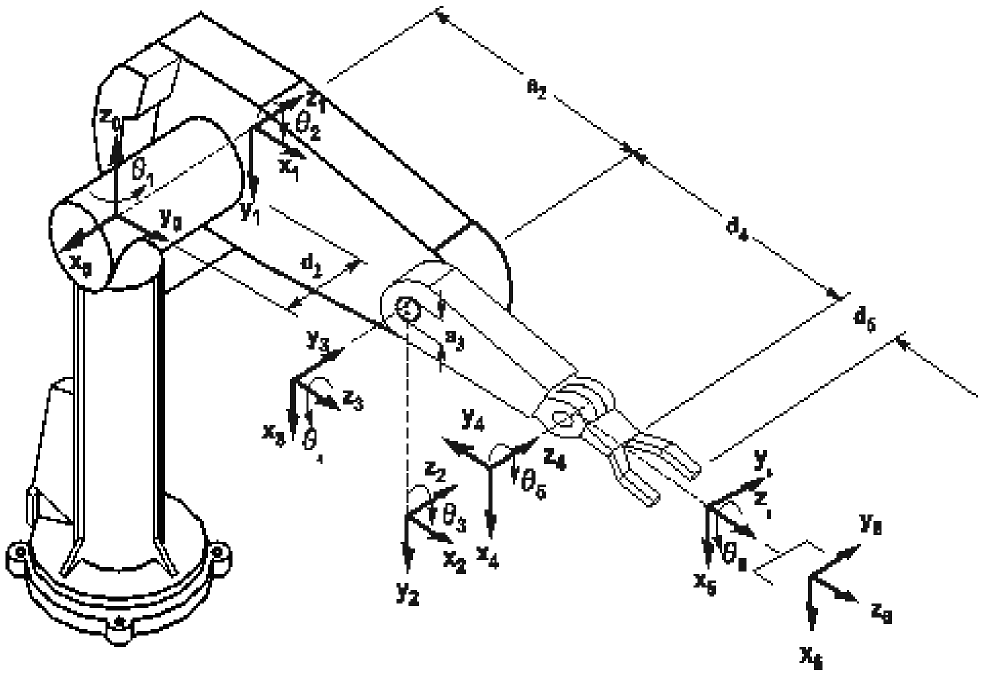
Forward Kinematics

- For a robotic arm, this would mean calculating the position and orientation of the end effector given all the joint variables.



Inverse Kinematics

- Inverse Kinematics is the reverse of Forward Kinematics. (!)
- It is the calculation of joint values given the positions, orientations, and geometries of mechanism’s parts.
- It is useful for planning how to move a robot in a certain way.

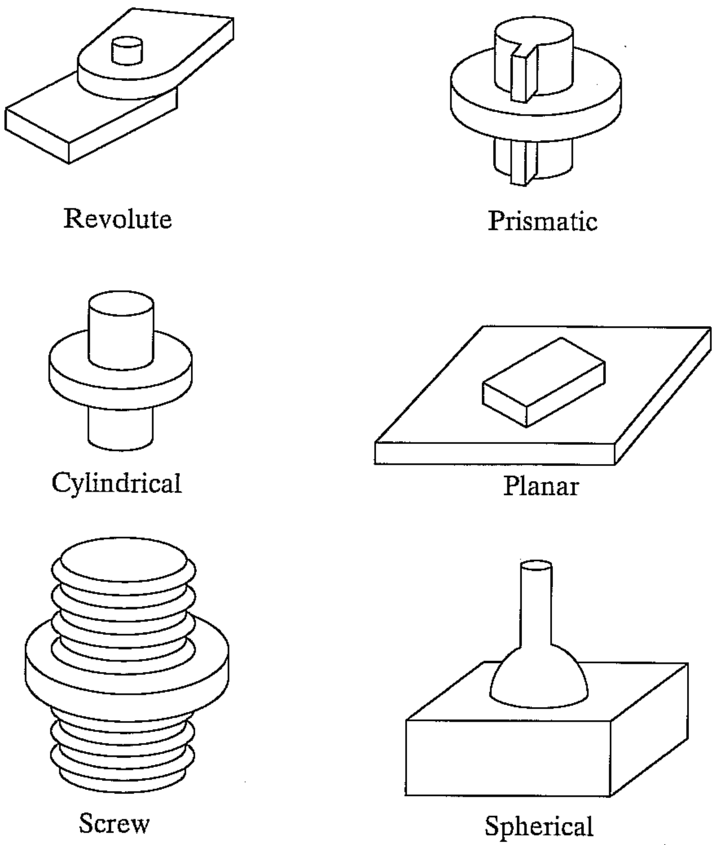


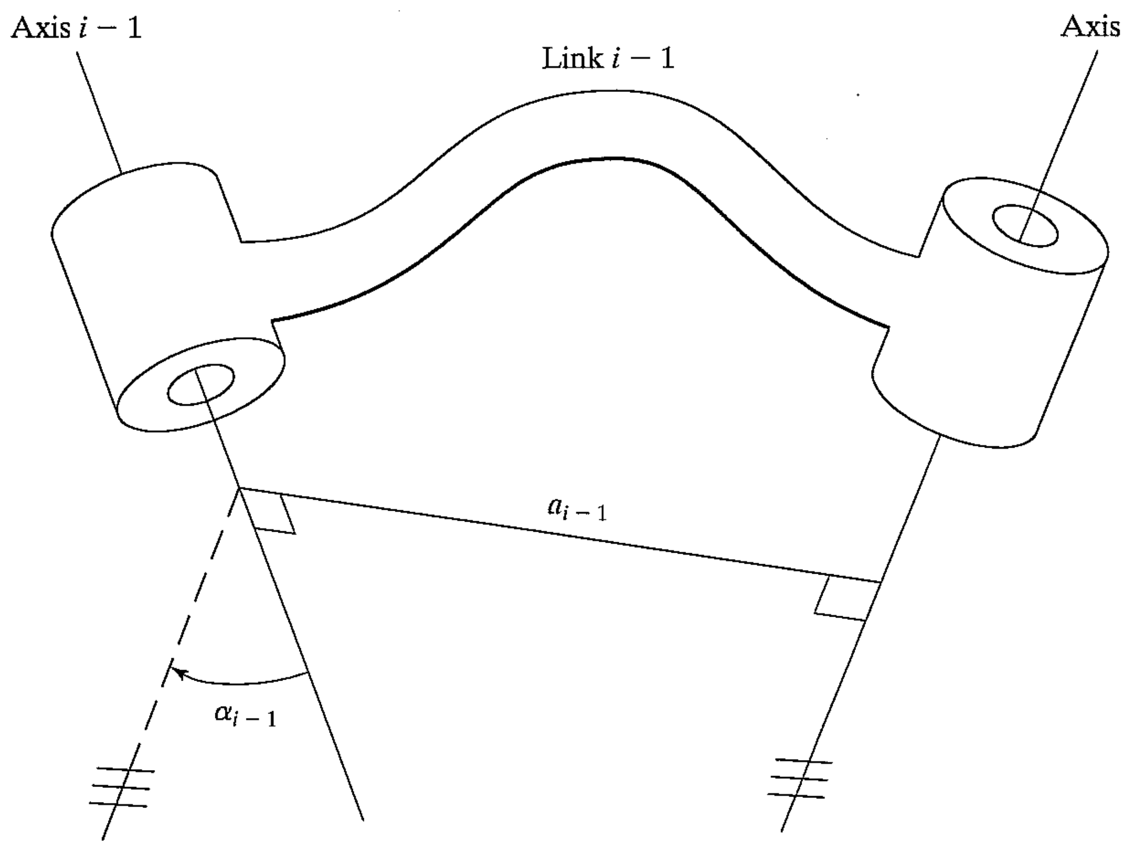
Mobility (Degree of freedom)

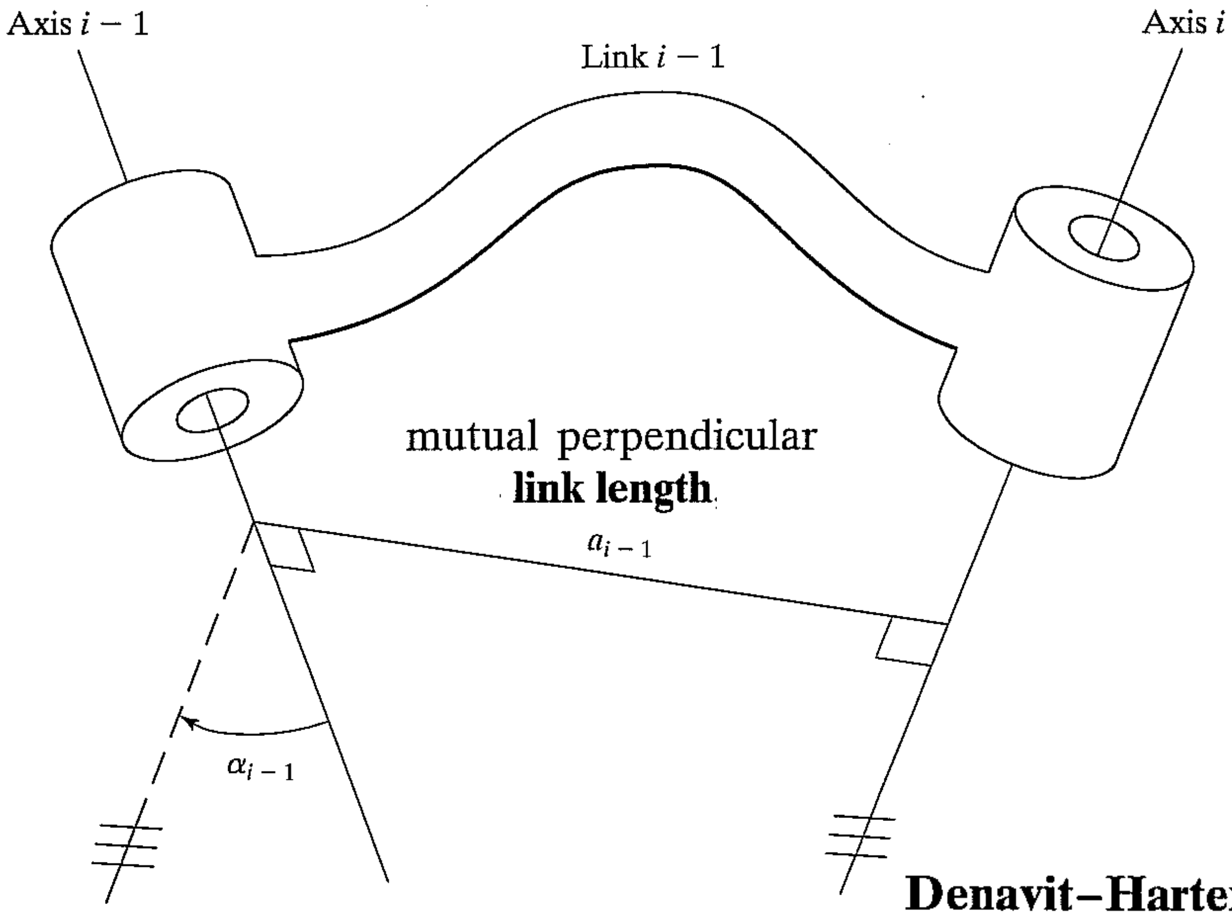
- See notes

3.2 LINK DESCRIPTION

A manipulator may be thought of as a set of bodies connected in a chain by joints. These bodies are called links. Joints form a connection between a neighboring pair of links.







Denavit–Hartenberg notation

link twist.

This angle is measured from axis $i-1$ to axis i in the right-hand sense about a_{i-1}.

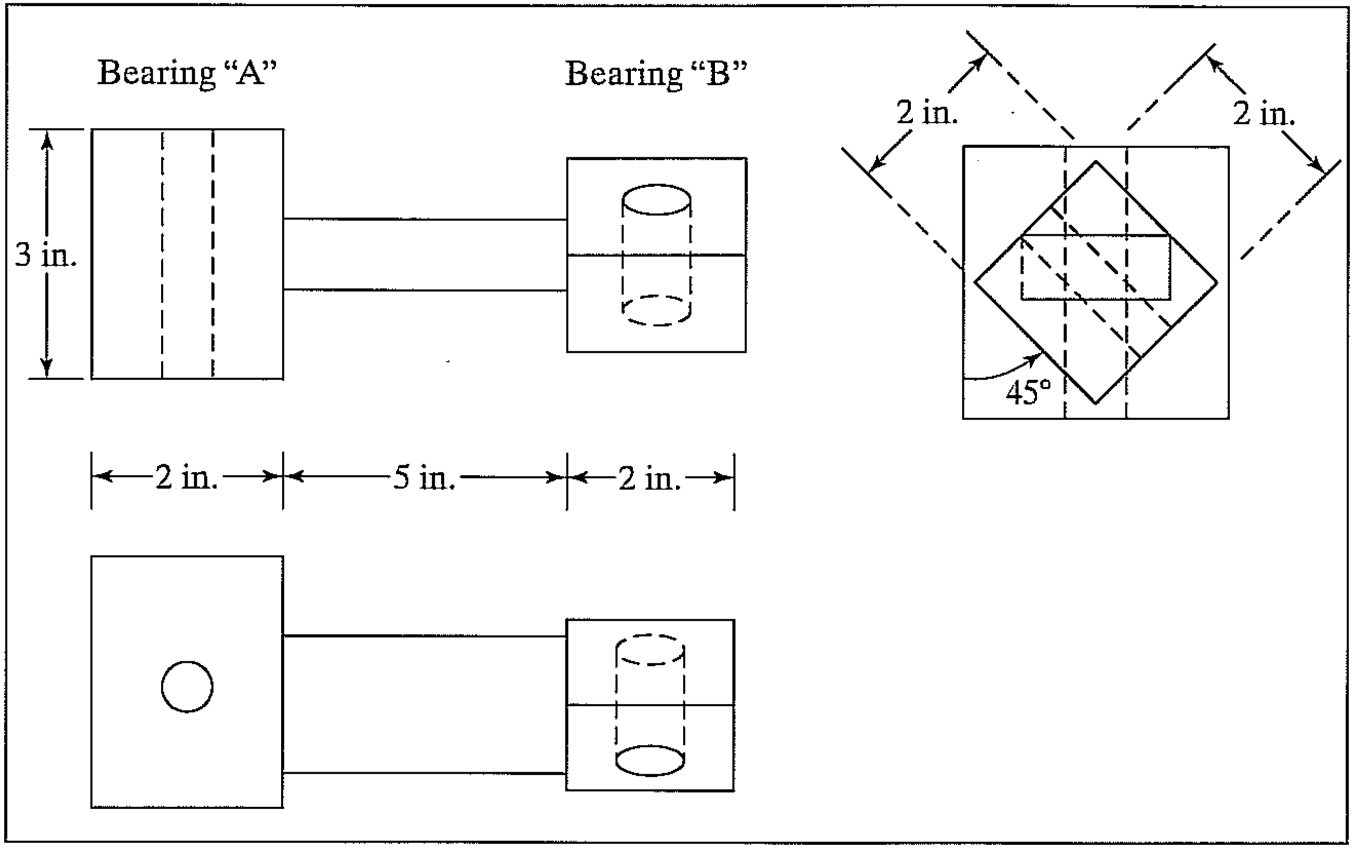


FIGURE 3.3: A simple link that supports two revolute axes.

link length is 7 inches.

Link twist is +45 degrees.

3.3 LINK-CONNECTION DESCRIPTION

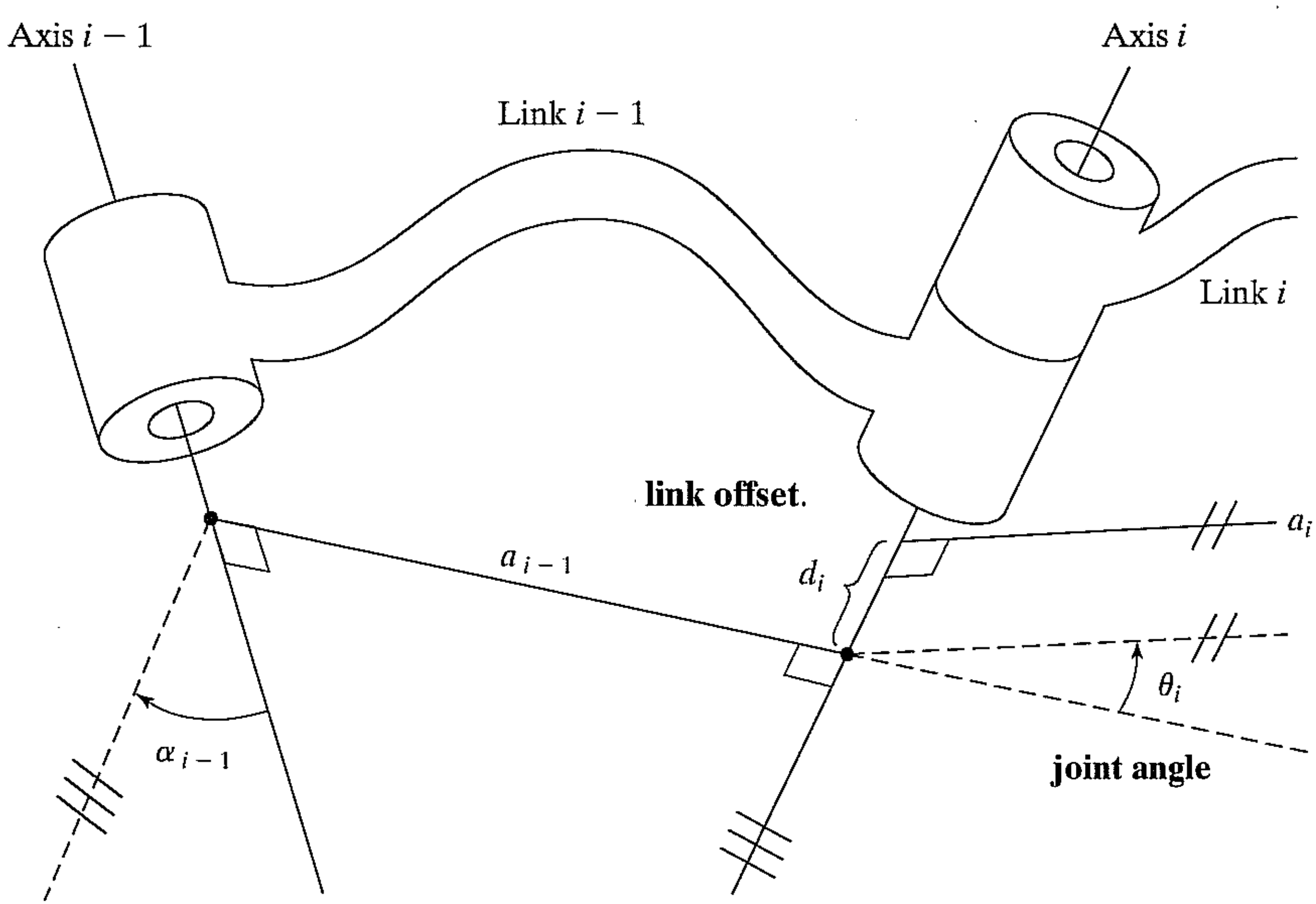
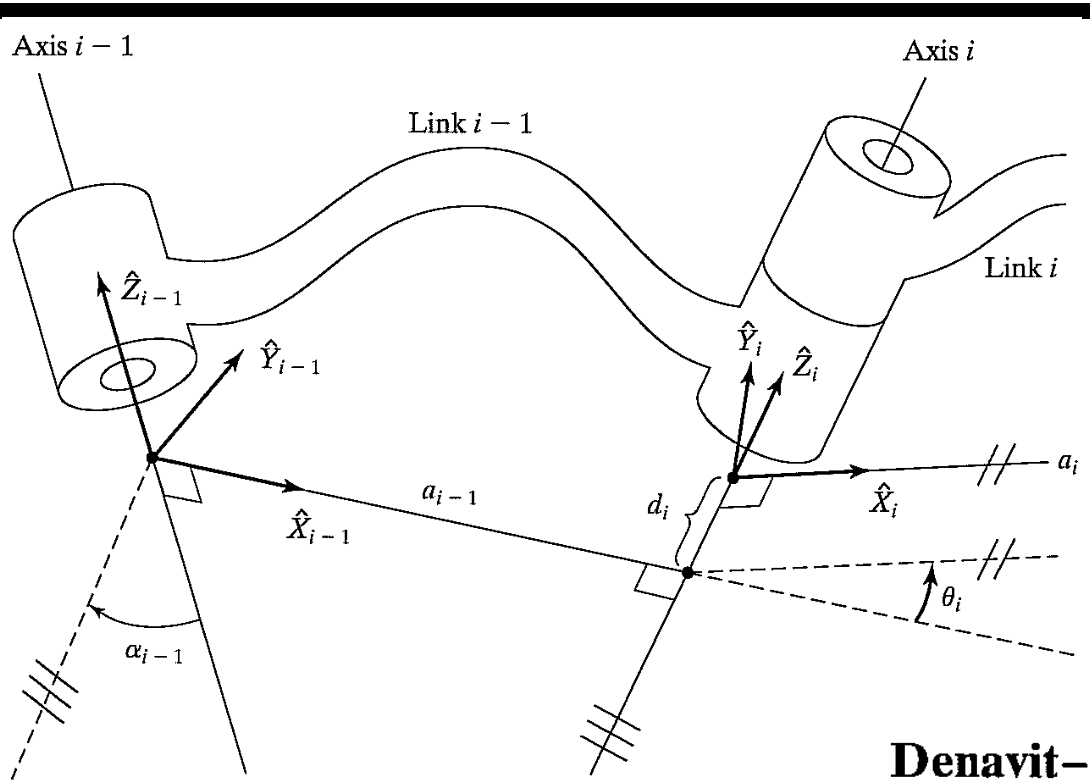


FIGURE 3.4: The link offset, d, and the joint angle, θ, are two parameters that may be used to describe the nature of the connection between neighboring links.



Denavit–Hartenberg notation

a_{i-1} = *the distance from* $\hat{Z}_{i-1}$ *to* $\hat{Z}_i$ *measured along* $\hat{X}_{i-1}$

α_{i-1} = *the angle from* $\hat{Z}_{i-1}$ *to* $\hat{Z}_i$ *measured about* $\hat{X}_{i-1}$

d_i = *the distance from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured along* $\hat{Z}_i$; and

θ_i = *the angle from* $\hat{X}_{i-1}$ *to* $\hat{X}_i$ *measured about* $\hat{Z}_i$.

3.4 CONVENTION FOR AFFIXING FRAMES TO LINKS

Intermediate links in the chain

The convention we will use to locate frames on the links is as follows: The $\hat{Z}$-axis of frame $\{i\}$, called $\hat{Z}_i$, is coincident with the joint axis i. The origin of frame $\{i\}$ is located where the a_i perpendicular intersects the joint i axis. $\hat{X}_i$ points along a_i in the direction from joint i to joint $i+1$.

In the case of $a_i = 0$, $\hat{X}_i$ is normal to the plane of $\hat{Z}_i$ and $\hat{Z}_{i+1}$. We define α_i as being measured in the right-hand sense about $\hat{X}_i$, and so we see that the freedom of choosing the sign of α_i in this case corresponds to two choices for the direction of $\hat{X}_i$. $\hat{Y}_i$ is formed by the right-hand rule to complete the ith frame. Figure 3.5 shows the location of frames $\{i-1\}$ and $\{i\}$ for a general manipulator.

3.4 CONVENTION FOR AFFIXING FRAMES TO LINKS

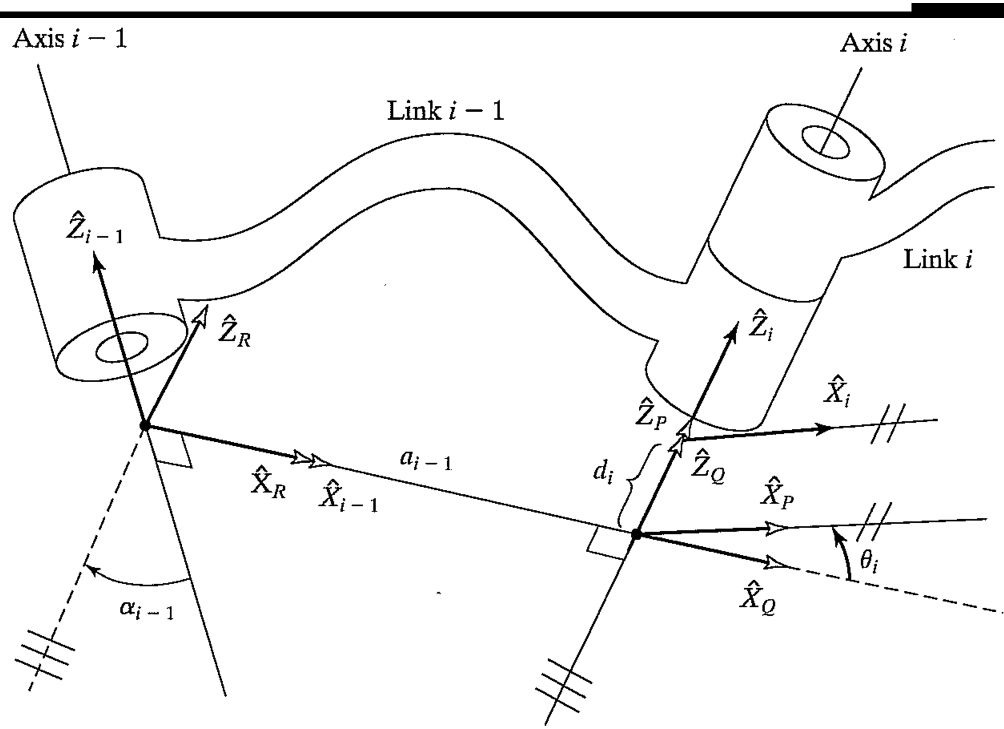
First and last links in the chain

We attach a frame to the base of the robot, or link 0, called frame {0}. This frame does not move; for the problem of arm kinematics, it can be considered the reference frame. We may describe the position of all other link frames in terms of this frame.

Frame {0} is arbitrary, so it always simplifies matters to choose $\hat{Z}_0$ along axis 1 and to locate frame {0} so that it coincides with frame {1} when joint variable 1 is zero. Using this convention, we will always have $a_0 = 0.0$, $\alpha_0 = 0.0$. Additionally, this ensures that $d_1 = 0.0$ if joint 1 is revolute, or $\theta_1 = 0.0$ if joint 1 is prismatic.

For joint n revolute, the direction of $\hat{X}_N$ is chosen so that it aligns with $\hat{X}_{N-1}$ when $\theta_n = 0.0$, and the origin of frame $\{N\}$ is chosen so that $d_n = 0.0$. For joint n prismatic, the direction of $\hat{X}_N$ is chosen so that $\theta_n = 0.0$, and the origin of frame $\{N\}$ is chosen at the intersection of $\hat{X}_{N-1}$ and joint axis n when $d_n = 0.0$.

Derivation of link transformations



$$ {}^{i-1}_{i}T = {}^{i-1}_{R}T \; {}^{R}_{Q}T \; {}^{Q}_{P}T \; {}^{P}_{i}T. $$

$$ {}^{i-1}_{i}T = R_X(\alpha_{i-1}) D_X(a_{i-1}) R_Z(\theta_i) D_Z(d_i), $$

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

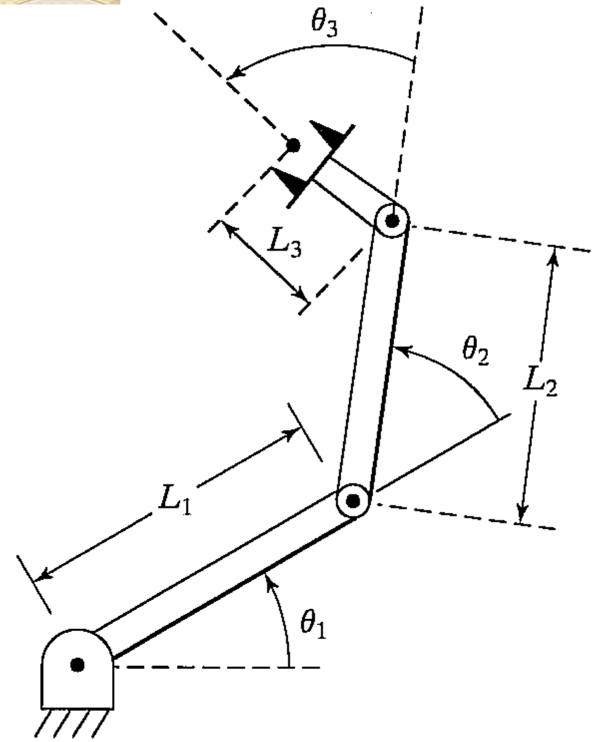
$$ {}^{0}_{N}T = {}^{0}_{1}T \; {}^{1}_{2}T \; {}^{2}_{3}T \ldots {}^{N-1}_{N}T. $$

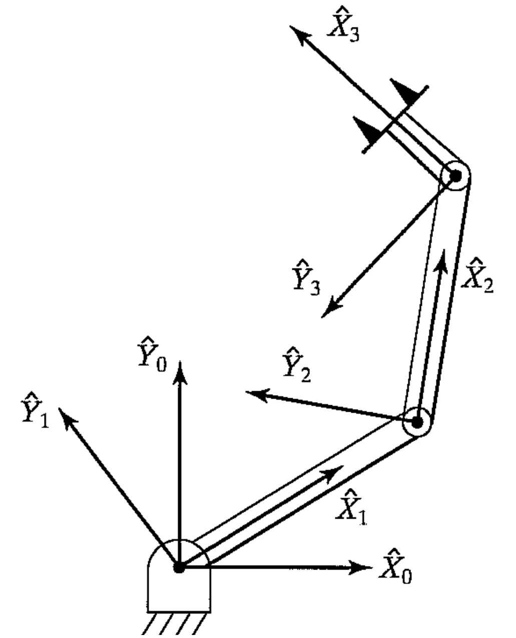
Summary of link-frame attachment procedure

The following is a summary of the procedure to follow when faced with a new mechanism, in order to properly attach the link frames:

1. Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two of these neighboring lines (at axes i and $i+1$).
2. Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the ith axis, assign the link-frame origin.
3. Assign the $\hat{Z}_i$ axis pointing along the ith joint axis.
4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.
5. Assign the $\hat{Y}_i$ axis to complete a right-hand coordinate system.
6. Assign $\{0\}$ to match $\{1\}$ when the first joint variable is zero. For $\{N\}$, choose an origin location and $\hat{X}_N$ direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

RRR (or $3R$) mechanism.

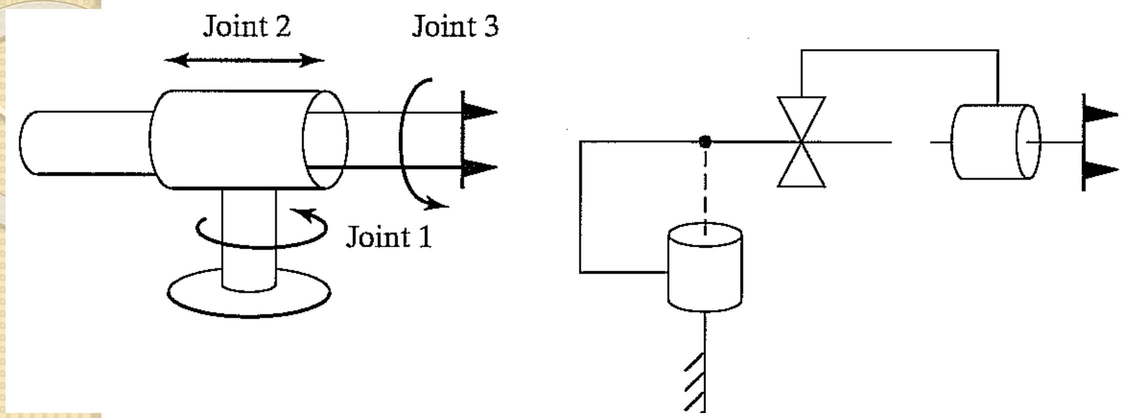




i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

See notes

RPR mechanism



See notes

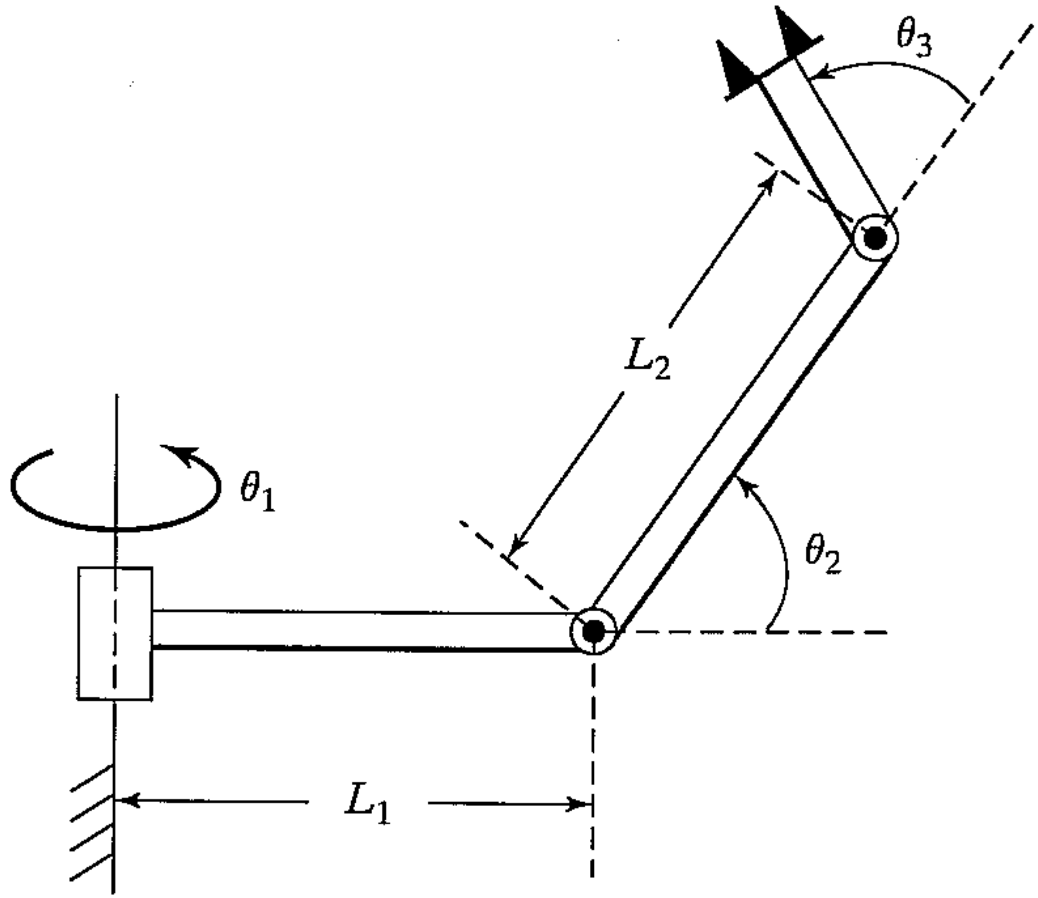


FIGURE 3.29: The $3R$ nonplanar arm (Exercise 3.3).

See notes

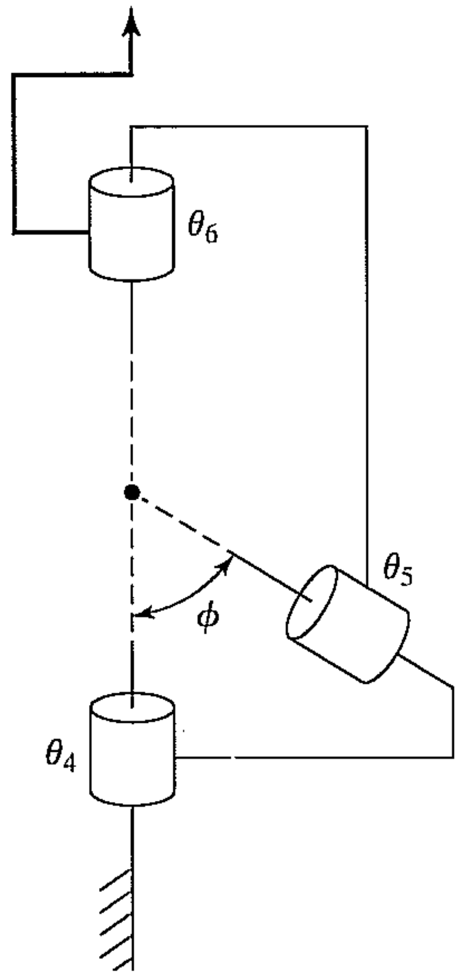
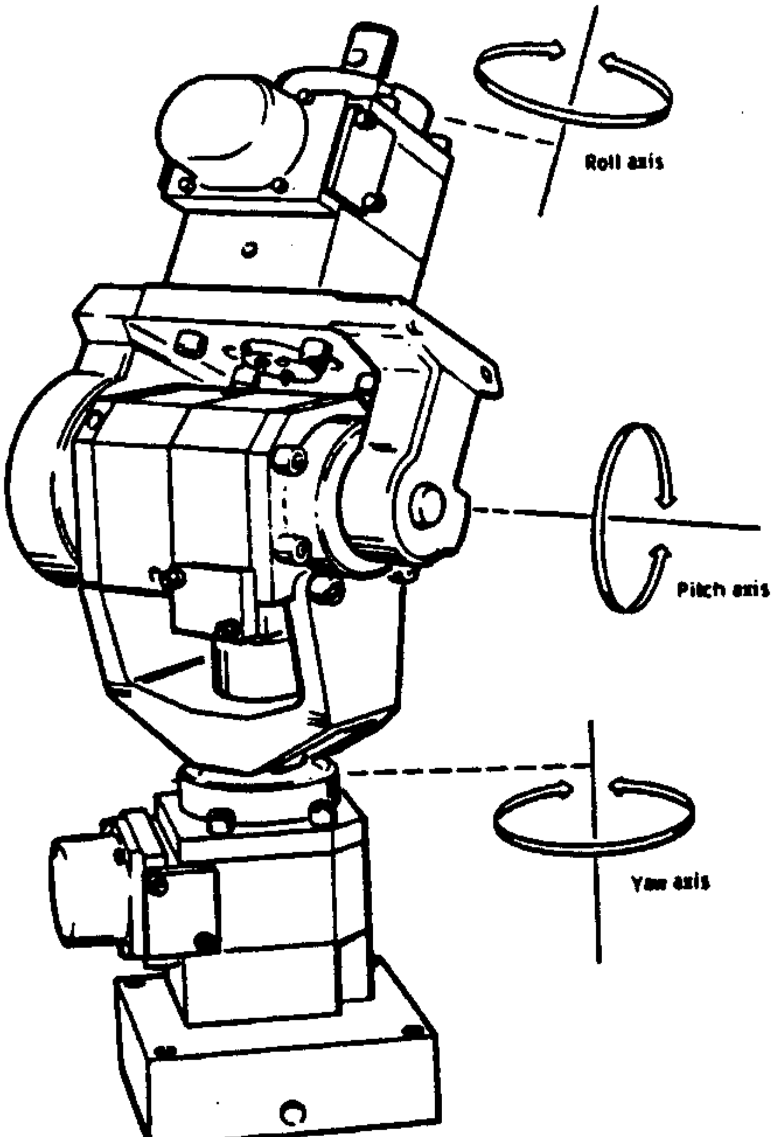


FIGURE 3.33: $3R$ nonorthogonal-axis robot (Exercise 3.11).

See notes


Roll axis
Pitch axis
Yaw axis

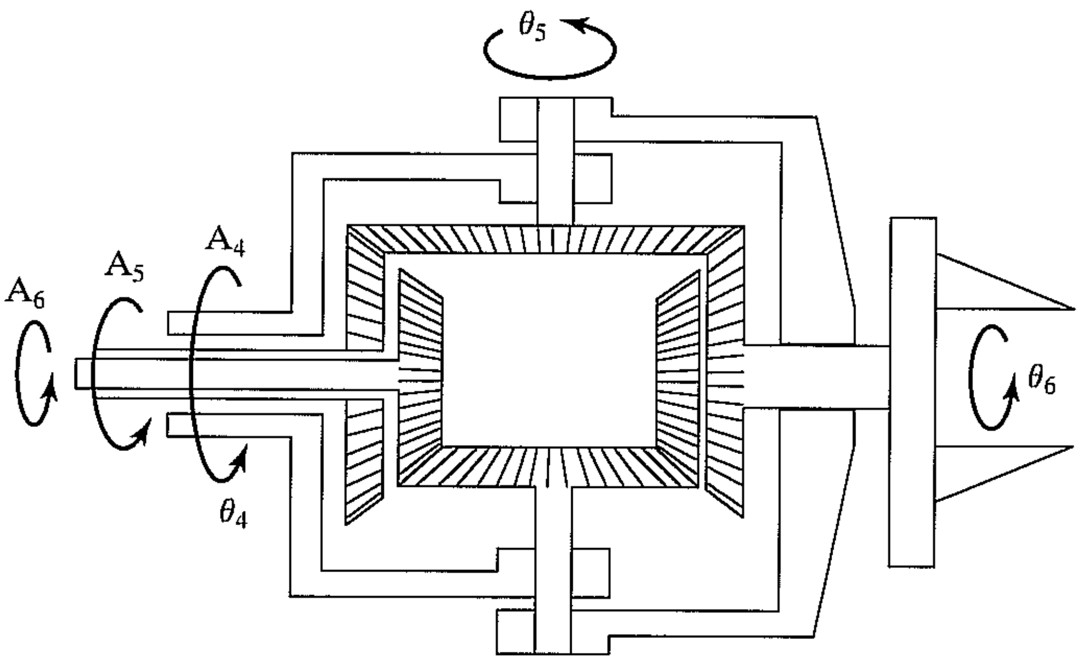
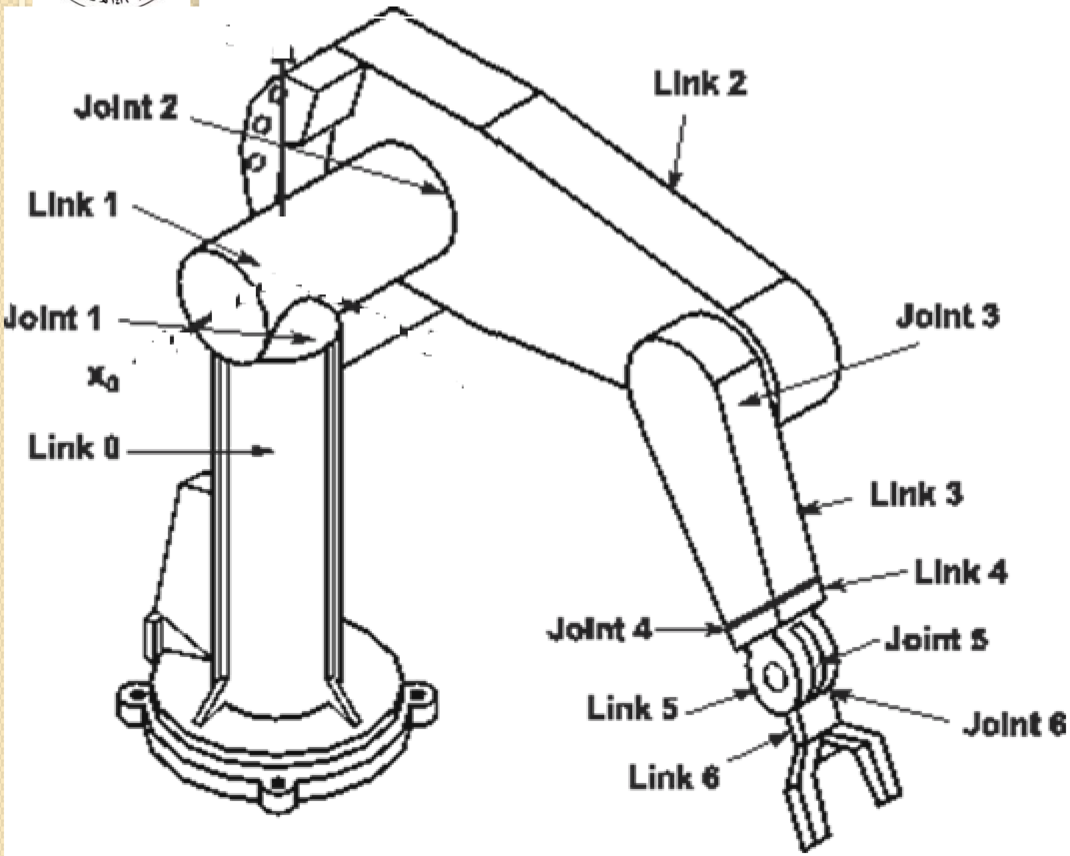
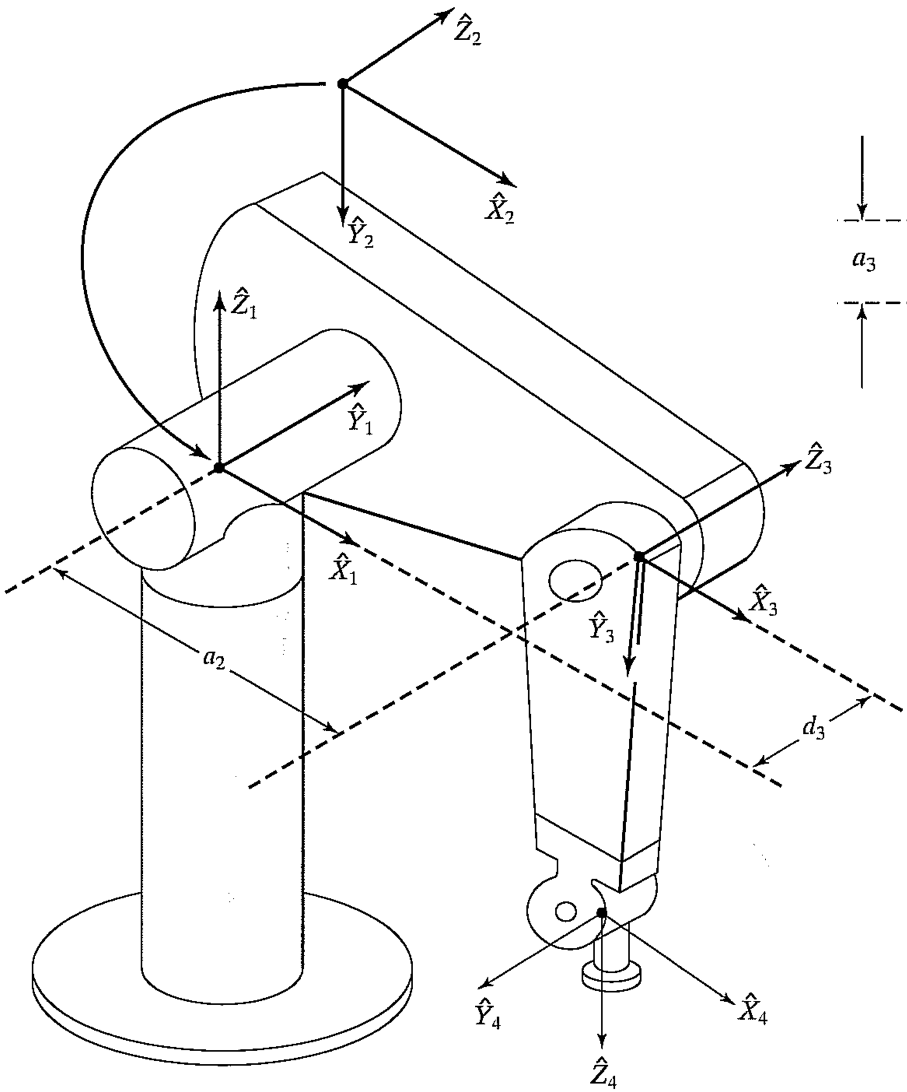
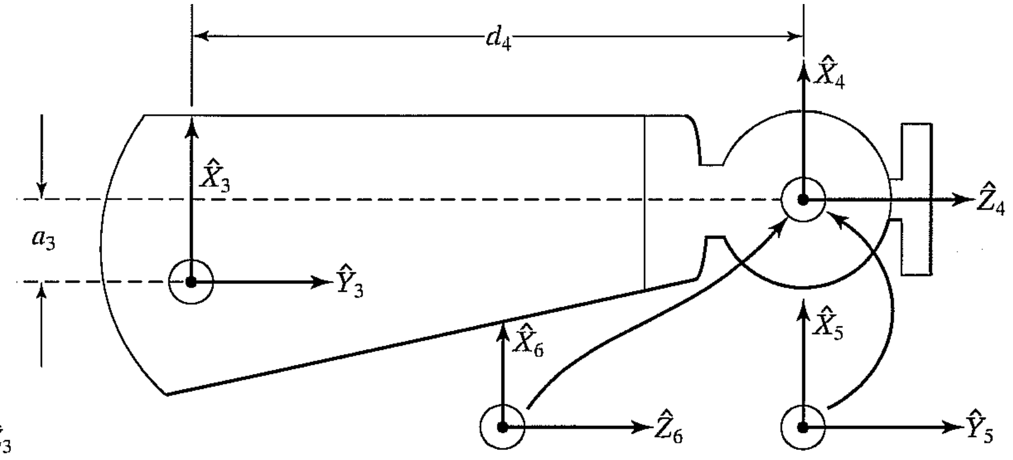


FIGURE 8.8: An orthogonal-axis wrist driven by remotely located actuators via three concentric shafts.



PUMA robot 6DOF


$\hat{Z}_2$
$\hat{X}_2$
$\hat{Y}_2$
$\hat{Z}_1$
$\hat{Y}_1$
$\hat{X}_1$
$\hat{Z}_3$
$\hat{X}_3$
$\hat{Y}_3$
a_2
d_3
$\hat{Y}_4$
$\hat{X}_4$
$\hat{Z}_4$


d_4
$\hat{X}_4$
$\hat{X}_3$
$\hat{Z}_4$
a_3
$\hat{Y}_3$
$\hat{X}_6$
$\hat{X}_5$
$\hat{Z}_6$
$\hat{Y}_5$

See notes

$$r_{11} = c_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6),$$

$$r_{21} = s_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6 - c_1(s_4c_5c_6 + c_4s_6),$$

$$r_{31} = -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6,$$

$$r_{12} = c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6),$$

$$r_{22} = s_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6),$$

$$r_{32} = -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6,$$

$$r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5,$$

$$r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5,$$

$$r_{33} = s_{23}c_4s_5 - c_{23}c_5,$$

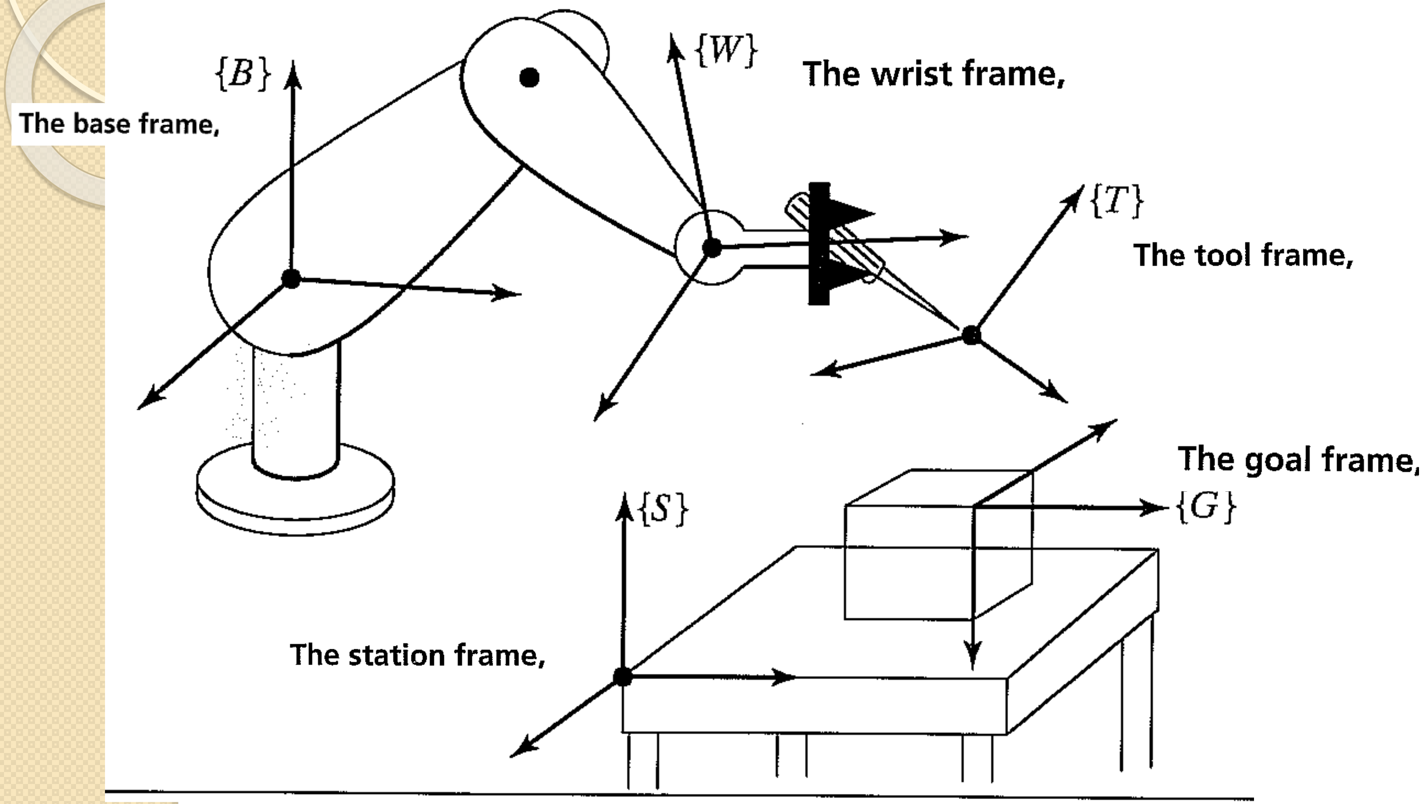
$$p_x = c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1,$$

$$p_y = s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1,$$

$$p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}.$$

$$ {}^0_6T = {}^0_1T\, {}^1_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3.8 FRAMES WITH STANDARD NAMES



3.6 ACTUATOR SPACE, JOINT SPACE, AND CARTESIAN SPACE

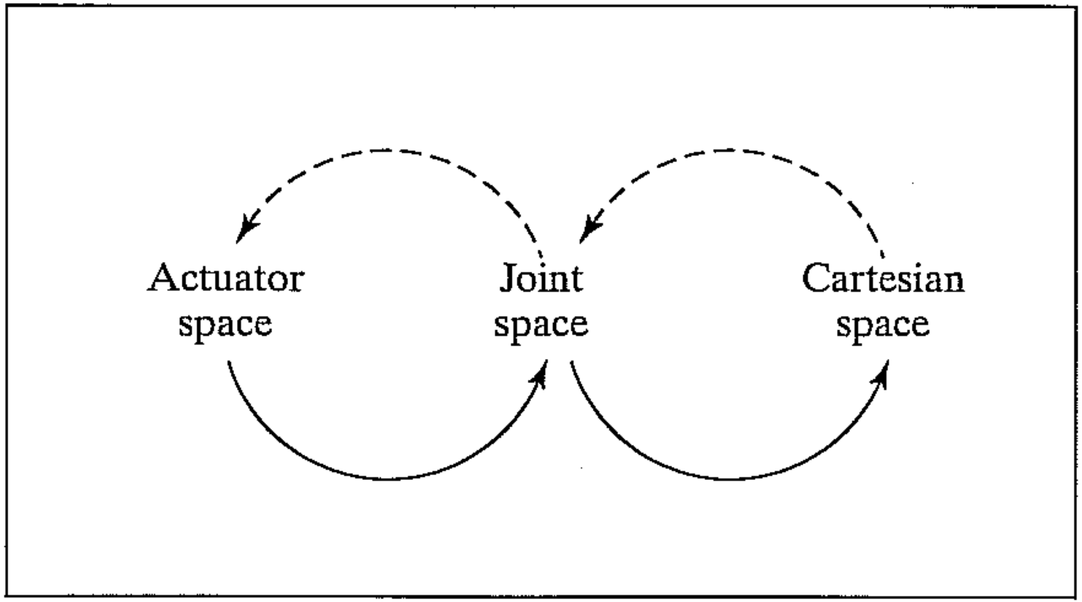


FIGURE 3.16: Mappings between kinematic description

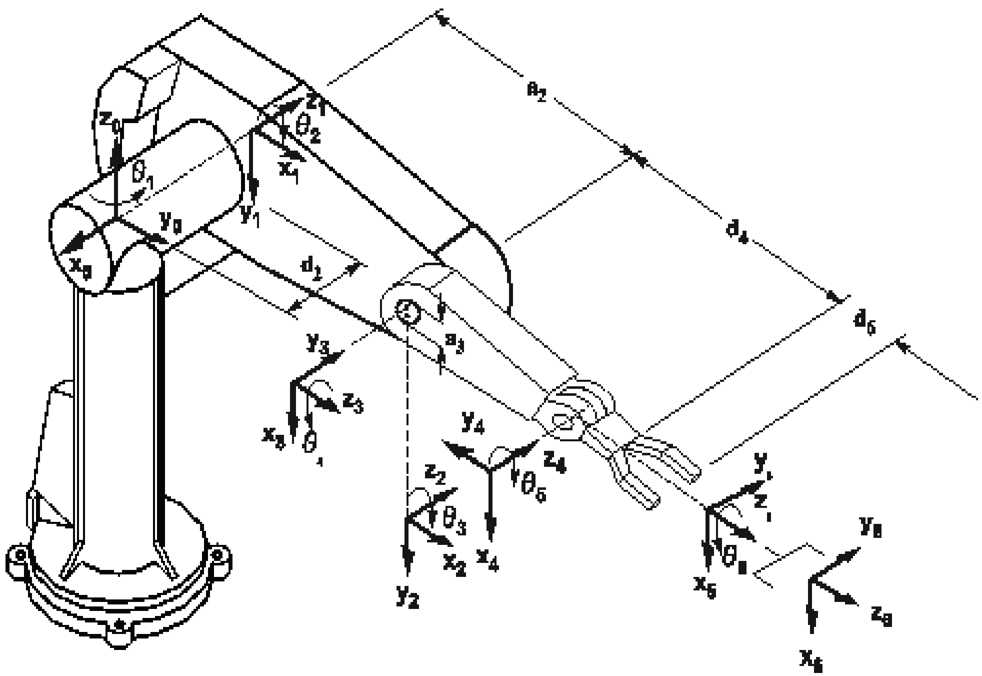
Robotics

Chapter 4
Inverse Manipulator Kinematics

4.1 INTRODUCTION

- Inverse Kinematics is the reverse of Forward Kinematics. (!)
- It is the calculation of joint values given the positions, orientations, and geometries of mechanism's parts

Given the numerical value of ${}^{0}_{N}T$, we attempt to find values of $\theta_1, \theta_2, \ldots, \theta_n$.



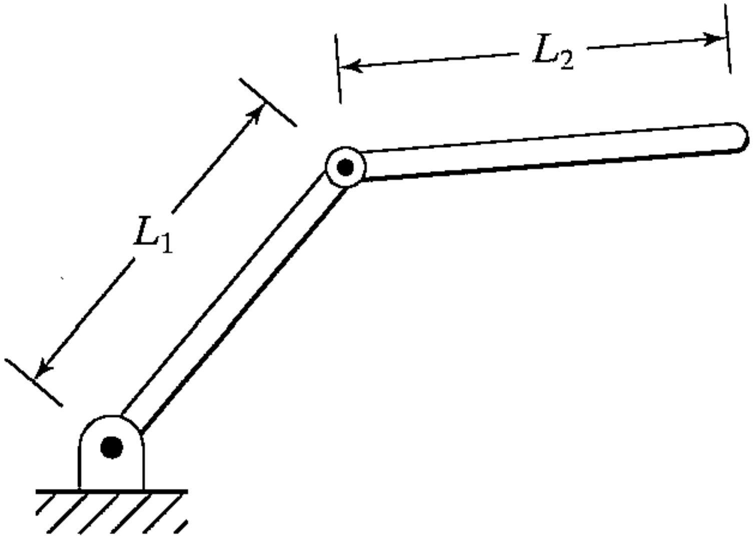
4.2 SOLVABILITY

Existence of solutions

The question of whether any solution exists at all raises the question of the manipulator's **workspace**. Roughly speaking, workspace is that volume of space that the end-effector of the manipulator can reach. For a solution to exist, the specified goal point must lie within the workspace. Sometimes, it is useful to consider two definitions of workspace: **Dextrous workspace** is that volume of space that the robot end-effector can reach with all orientations. That is, at each point in the dextrous workspace, the end-effector can be arbitrarily oriented. The **reachable workspace** is that volume of space that the robot can reach in at least one orientation. Clearly, the dextrous workspace is a subset of the reachable workspace.

If $l_1 = l_2$:   If $l_1 \neq l_2$,

Consider the workspace of the two-link manipulator in Fig. 4.1. If $l_1 = l_2$, then the reachable workspace consists of a disc of radius $2l_1$. The dextrous workspace consists of only a single point, the origin. If $l_1 \neq l_2$, then there is no dextrous workspace, and the reachable workspace becomes a ring of outer radius $l_1 + l_2$ and inner radius $|l_1 - l_2|$. Inside the reachable workspace there are two possible orientations of the end-effector. On the boundaries of the workspace there is only one possible orientation.

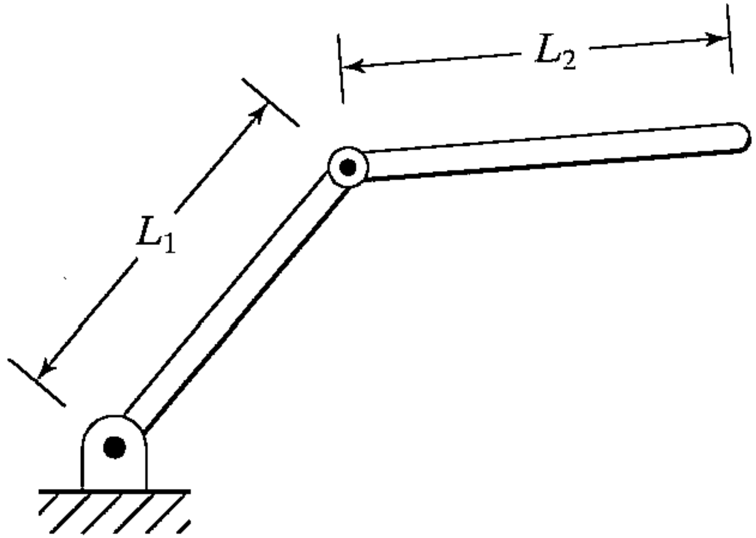


- See notes

If $L_1 > L_2$

- Reachable Workspace
- Dextrous workspace
- No of solutions (inner and boundary)

$$0 \le \theta_1 \le 360, \quad 0 \le \theta_2 \le 360$$

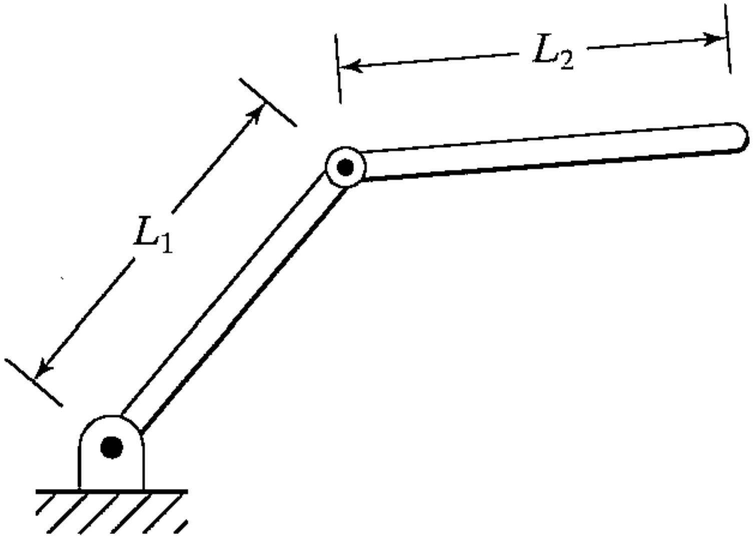


- See notes

If $L_1 > L_2$

- Reachable Workspace
- Dextrous workspace
- No of solutions (inner and boundary)

$$0 \le \theta_1 \le 360, \;\; 0 \le \theta_2 \le 180$$

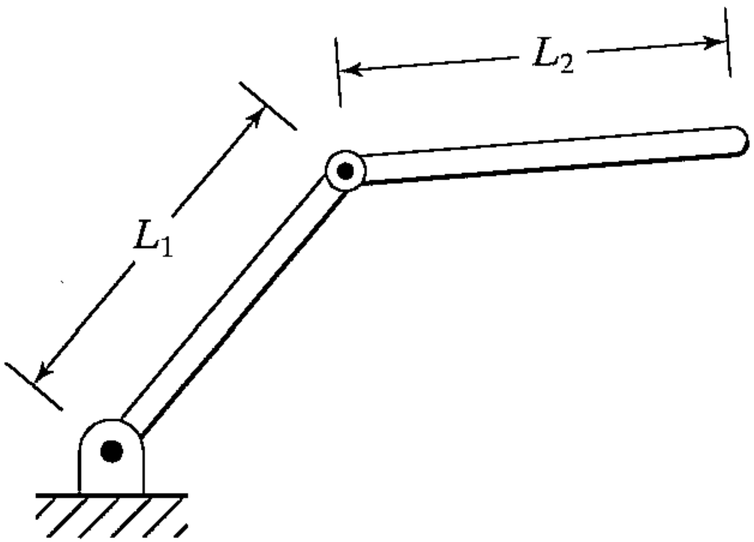


- See notes

If $L_2 > L_1$

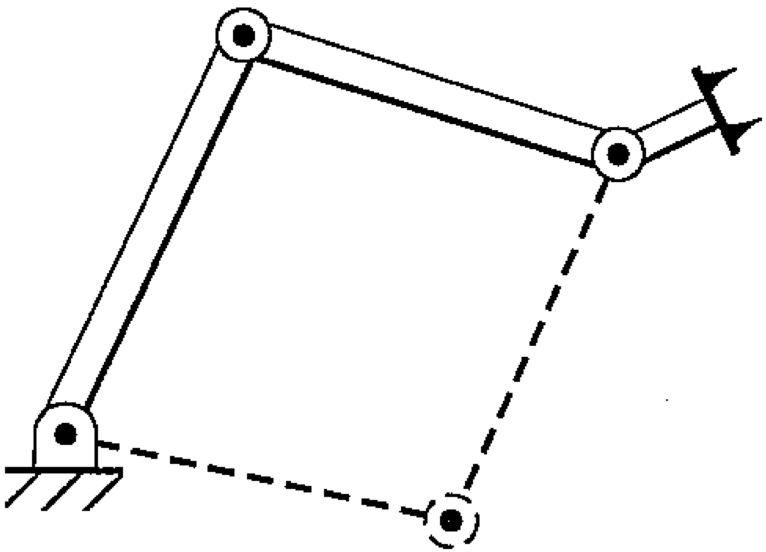
- Reachable Workspace
- Dextrous workspace
- No of solutions (inner and boundary)

$$0 \le \theta_1 \le 360 , \quad 0 \le \theta_2 \le 360$$

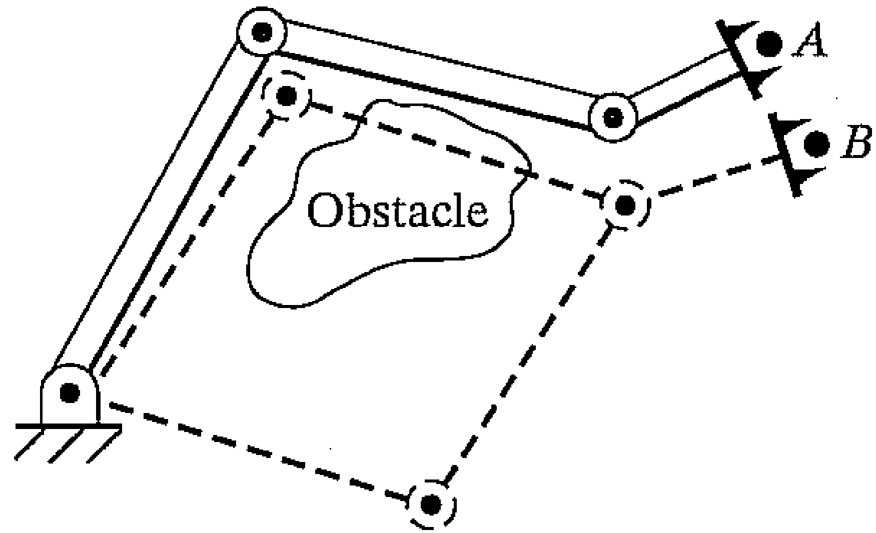


- See notes

Multiple solutions



Dashed lines indicate a second solution.



closest solution.

. The presence of obstacles

moving smaller joints

PUMA 560 can reach certain goals with eight different solutions.

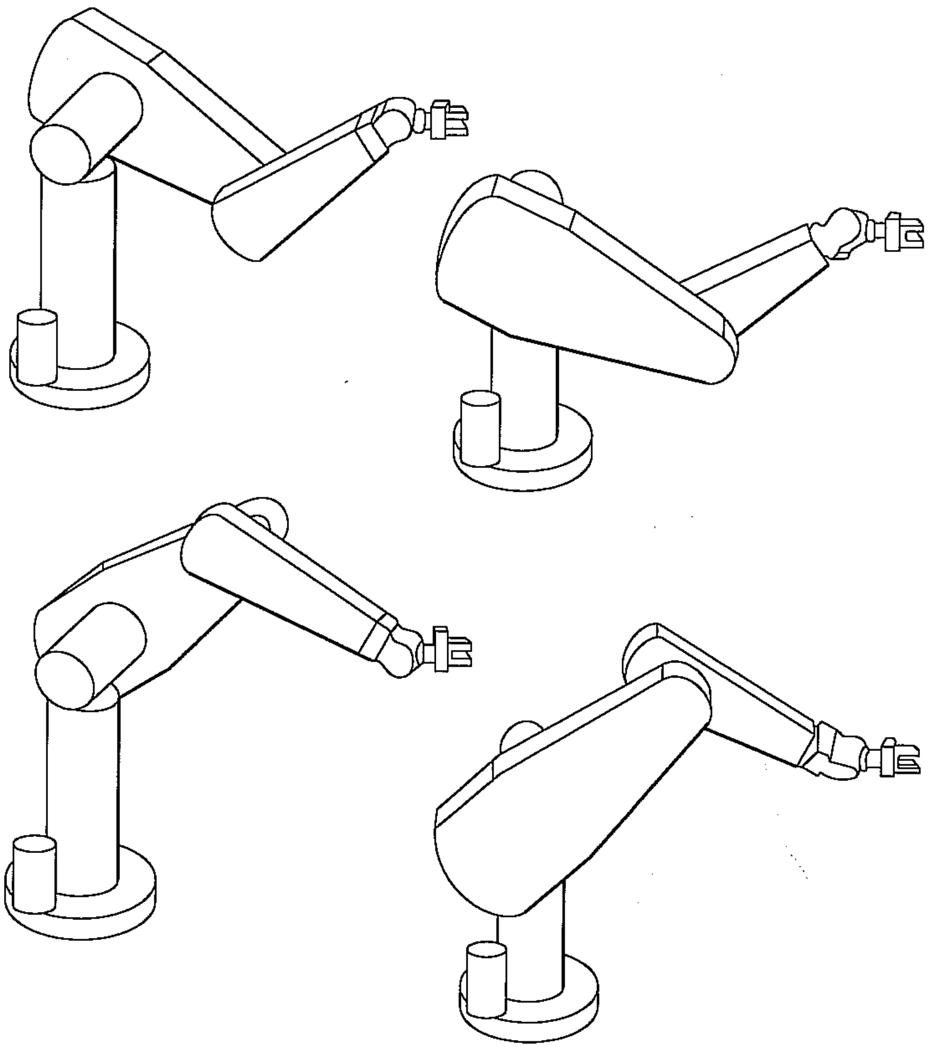


FIGURE 4.4: Four solutions of the PUMA 560.

$$\theta_4' = \theta_4 + 180°,$$

$$\theta_5' = -\theta_5,$$

$$\theta_6' = \theta_6 + 180°.$$

a_i	Number of solutions
$a_1 = a_3 = a_5 = 0$	$\leqslant 4$
$a_3 = a_5 = 0$	$\leqslant 8$
$a_3 = 0$	$\leqslant 16$
All $a_i \neq 0$	$\leqslant 16$

FIGURE 4.5: Number of solutions vs. nonzero a_i.

Method of solution

closed-form solutions and **numerical solutions**.

We will restrict our attention to closed-form solution methods.

"closed form" means a solution method based on analytic expressions

Within the class of closed-form solutions, we distinguish two methods of obtaining the solution: **algebraic** and **geometric**. These distinctions are somewhat hazy: Any geometric methods brought to bear are applied by means of algebraic expressions, so the two methods are similar. The methods differ perhaps in approach only.

A major recent result in kinematics is that, according to our definition of solvability, *all systems with revolute and prismatic joints having a total of six degrees of freedom in a single series chain are solvable*. However, this general solution is a numerical one. Only in special cases can robots with six degrees of freedom be solved analytically. These robots for which an analytic (or closed-form) solution exists are characterized either by having several intersecting joint axes or by having many α_i equal to 0 or ± 90 degrees. Calculating numerical solutions is generally time consuming relative to evaluating analytic expressions; hence, it is considered very important to design a manipulator so that a closed-form solution exists. Manipulator designers discovered this very soon, and now virtually all industrial manipulators are designed sufficiently simply that a closed-form solution can be developed.

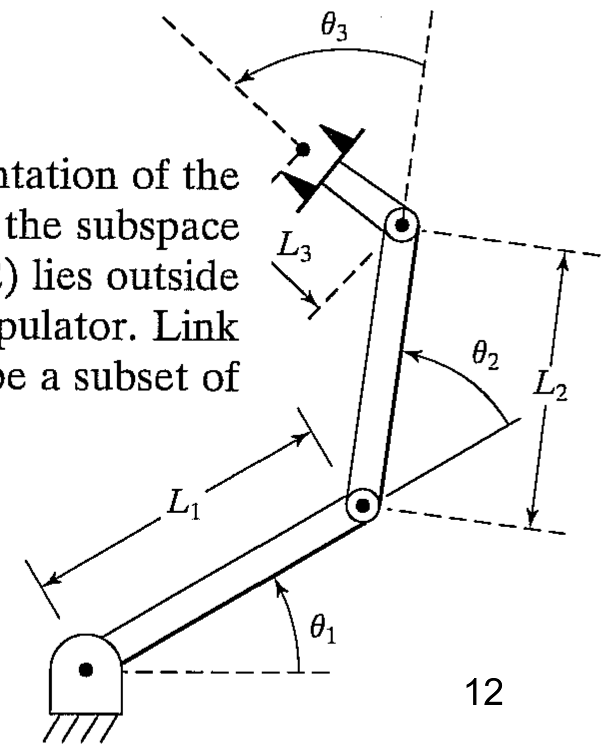
4.3 THE NOTION OF MANIPULATOR SUBSPACE WHEN $n < 6$

Give a description of the subspace of ${}^{B}_{W}T$ for the three-link manipulator

The subspace of ${}^{B}_{W}T$ is given by

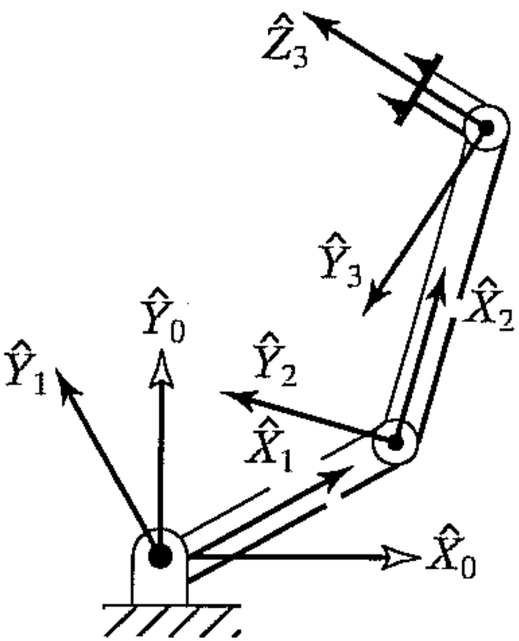
$$
{}^{B}_{W}T = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

where x and y give the position of the wrist and ϕ describes the orientation of the terminal link. As x, y, and ϕ are allowed to take on arbitrary values, the subspace is generated. Any wrist frame that does not have the structure of (4.2) lies outside the subspace (and therefore lies outside the workspace) of this manipulator. Link lengths and joint limits restrict the workspace of the manipulator to be a subset of this subspace.



Algebraic solution

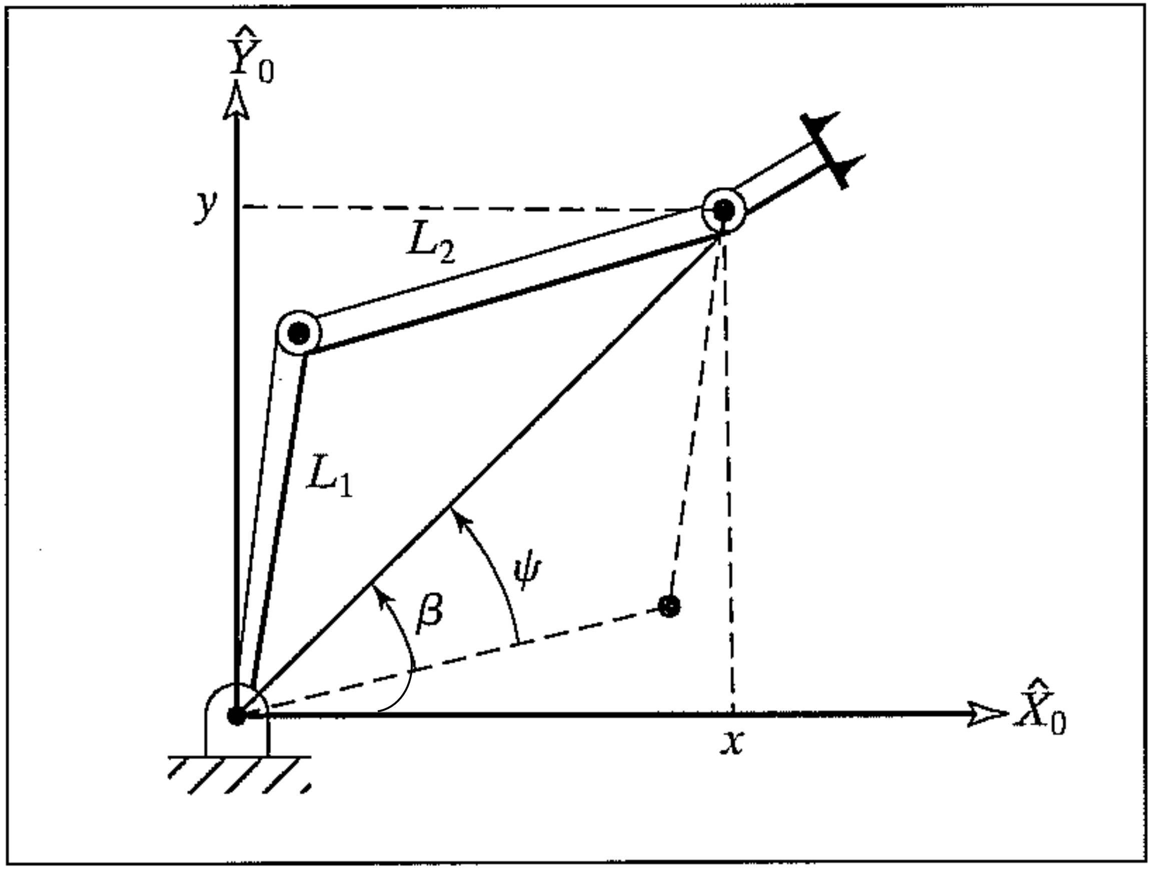
$$ {}^{0}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0.0 & l_1s_1 + l_2s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



given

$$ {}^{0}_{3}T = \begin{bmatrix} c_{\phi} & -s_{\phi} & 0.0 & x \\ s_{\phi} & c_{\phi} & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Geometric solution



ALGEBRAIC SOLUTION BY REDUCTION TO POLYNOMIAL

Transcendental equations are often difficult to solve because, even when there is only one variable (say, θ), it generally appears as $\sin\theta$ and $\cos\theta$. Making the following substitutions, however, yields an expression in terms of a single variable, u:

$$u = \tan\frac{\theta}{2},$$

$$\cos\theta = \frac{1-u^2}{1+u^2}, \tag{4.35}$$

$$\sin\theta = \frac{2u}{1+u^2}.$$

See notes

EXAMPLE 4.3

Convert the transcendental equation

$$a \cos\theta + b \sin\theta = c$$

into a polynomial in the tangent of the half angle, and solve for θ.

Substituting from (4.35) and multiplying through by $1 + u^2$, we have

$$a(1 - u^2) + 2bu = c(1 + u^2).$$

Collecting powers of u yields

$$(a + c)u^2 - 2bu + (c - a) = 0,$$

which is solved by the quadratic formula:

$$u = \frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a + c}.$$

Hence,

$$\theta = 2\tan^{-1}\left(\frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a + c}\right).$$

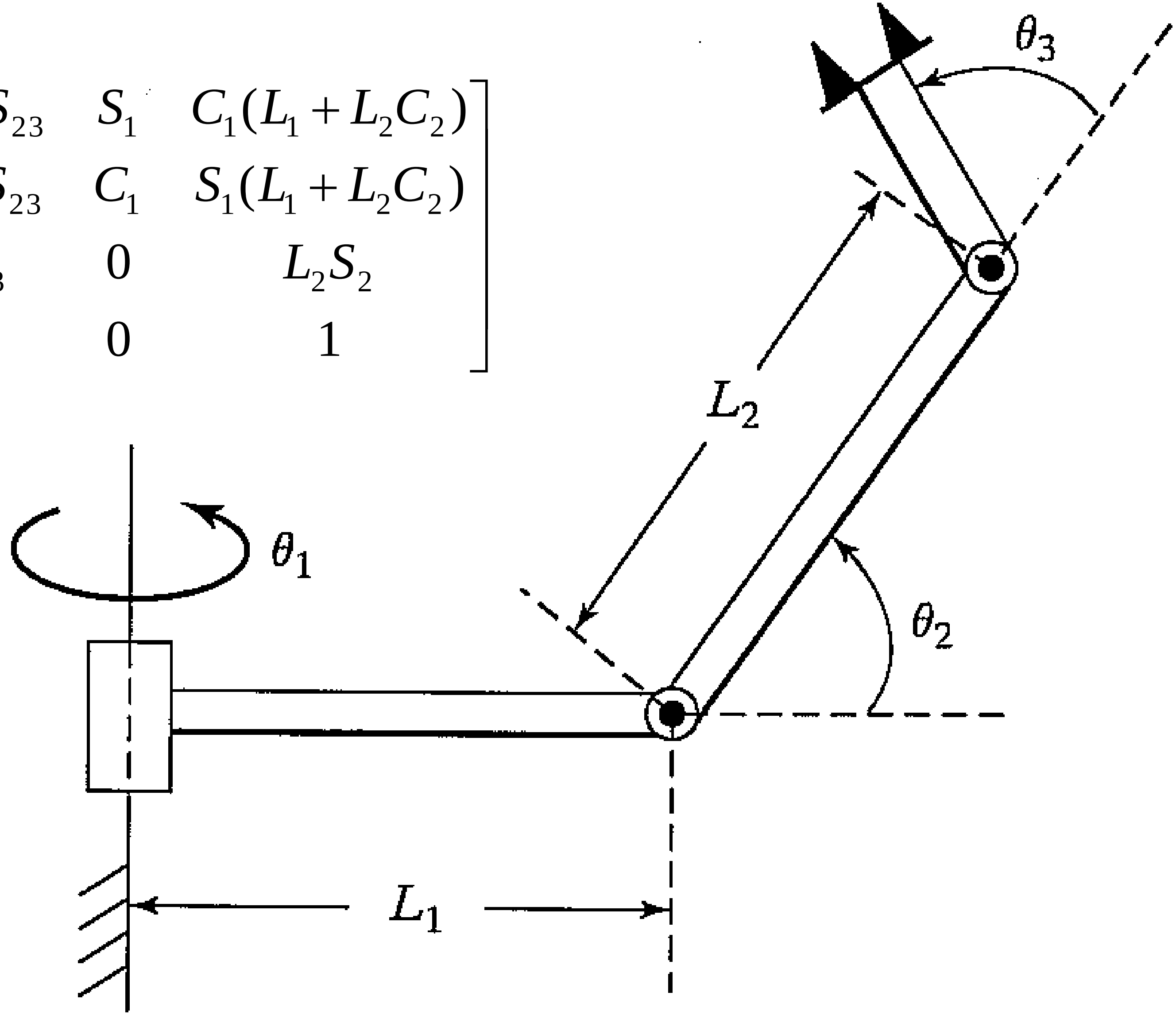
Appendix C another solution

4.6 PIEPER'S SOLUTION WHEN THREE AXES INTERSECT

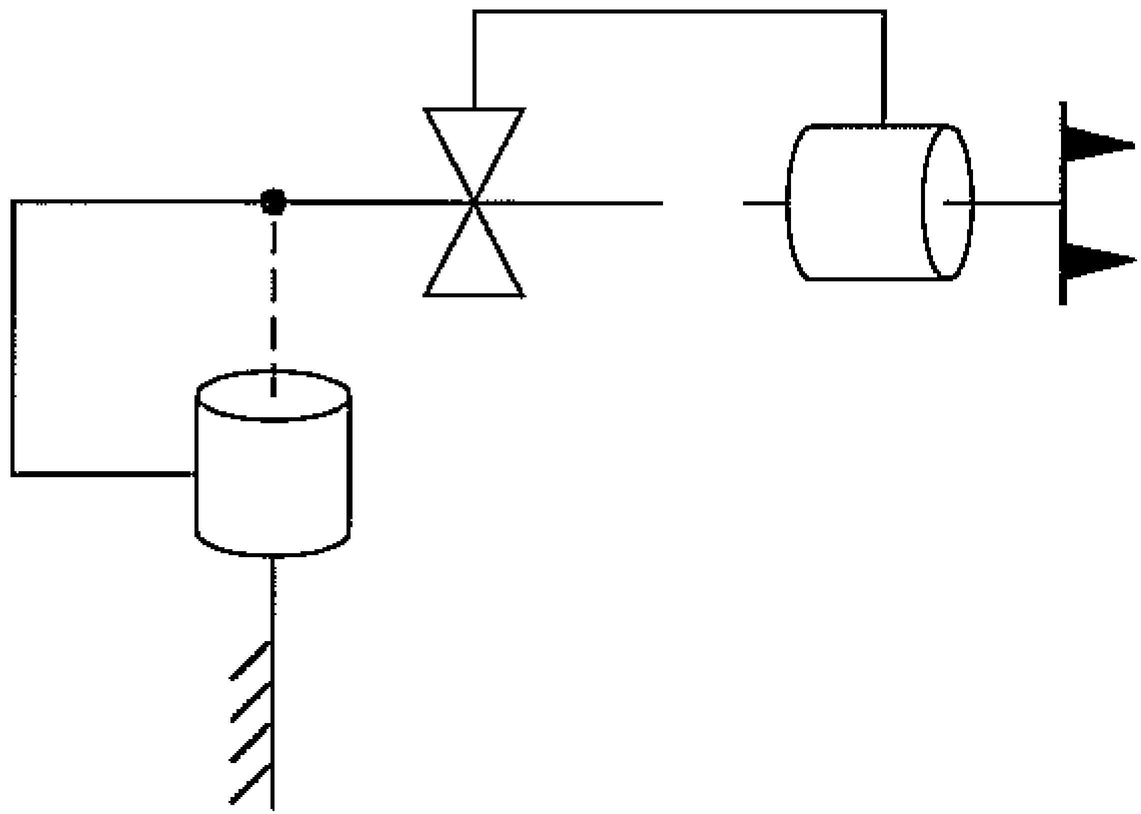
As mentioned earlier, although a completely general robot with six degrees of freedom does not have a closed-form solution, certain important special cases can be solved. Pieper [3, 4] studied manipulators with six degrees of freedom in which three consecutive axes intersect at a point.[2] In this section, we outline the method he developed for the case of all six joints revolute, with the last three axes intersecting. His method applies to other configurations, which include prismatic

- Given the below transformation matrix solve inverse kinematic problem
- Sketch the workspace

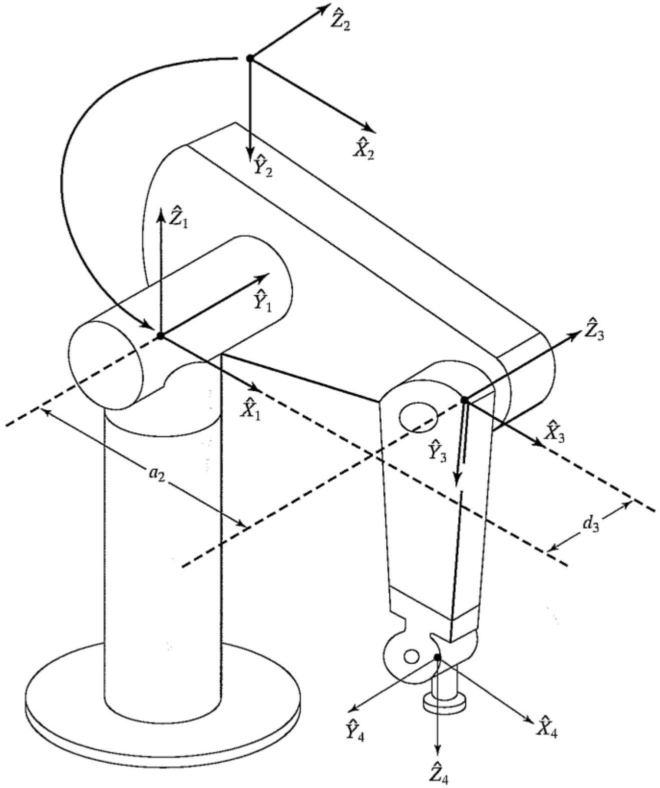
$$
{}^{0}_{3}T = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & C_1(L_1 + L_2C_2) \\ S_1S_{23} & -S_1S_{23} & C_1 & S_1(L_1 + L_2C_2) \\ S_{23} & C_{23} & 0 & L_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



•solve inverse kinematic problem
•Sketch the workspace



(b)



$$[{}^0_1T(\theta_1)]^{-1}\,{}^0_6T = {}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6). \tag{4.55}$$

Inverting ${}^0_1T$, we write (4.55) as

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T, \tag{4.56}$$

$$ {}^1_6T = {}^1_3T\, {}^3_6T = \begin{bmatrix} {}^1r_{11} & {}^1r_{12} & {}^1r_{13} & {}^1p_x \\ {}^1r_{21} & {}^1r_{22} & {}^1r_{23} & {}^1p_y \\ {}^1r_{31} & {}^1r_{32} & {}^1r_{33} & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T, $$

$$ -s_1 p_x + c_1 p_y = d_3. $$

$$ \begin{aligned} {}^1r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6, \\ {}^1r_{21} &= -s_4c_5c_6 - c_4s_6, \\ {}^1r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6, \\ {}^1r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6, \\ {}^1r_{22} &= s_4c_5s_6 - c_4c_6, \\ {}^1r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6, \\ {}^1r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\ {}^1r_{23} &= s_4s_5, \\ {}^1r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\ {}^1p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\ {}^1p_y &= d_3, \\ {}^1p_x &= -a_3s_{23} - a_2s_2 - d_4c_{23}. \end{aligned} $$

Pz

$$ \begin{aligned} p_x &= \rho\cos\phi, \\ p_y &= \rho\sin\phi, \\ \rho &= \sqrt{p_x^2 + p_y^2}, \\ \phi &= \operatorname{Atan2}(p_y, p_x). \end{aligned} $$

$$ c_1 s_\phi - s_1 c_\phi = \frac{d_3}{\rho}. $$

$$ \sin(\phi - \theta_1) = \frac{d_3}{\rho}. $$

$$ \cos(\phi - \theta_1) = \pm\sqrt{1 - \frac{d_3^2}{\rho^2}}, $$

$$ \phi - \theta_1 = \operatorname{Atan2}\left(\frac{d_3}{\rho}, \pm\sqrt{1 - \frac{d_3^2}{\rho^2}}\right). $$

$$ \theta_1 = \operatorname{Atan2}(p_y, p_x) - \operatorname{Atan2}\left(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}\right). $$

Note that we have found two possible solutions for θ_1, corresponding to the plus-or-minus sign in (4.64). Now that θ_1 is known, the left-hand side of (4.56) is known. If we equate both the (1,4) elements and the (3,4) elements from the two sides of (4.56), we obtain

Pz

$$\begin{aligned} c_1 p_x + s_1 p_y &= a_3 c_{23} - d_4 s_{23} + a_2 c_2, \\ -p_x &= a_3 s_{23} + d_4 c_{23} + a_2 s_2. \end{aligned} \tag{4.65}$$

If we square equations (4.65) and (4.57) and add the resulting equations, we obtain

$$a_3 c_3 - d_4 s_3 = K, \tag{4.66}$$

where

$$K = \frac{p_x^2 + p_y^2 + p_x^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2}. \tag{4.67}$$

Same method as before

$$\theta_3 = \text{Atan2}(a_3, d_4) - \text{Atan2}(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}).$$

$$[{}^0_3T(\theta_2)]^{-1}{}^0_6T = {}^3_4T(\theta_4){}^4_5T(\theta_5){}^5_6T(\theta_6), \tag{4.69}$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^3_6T, \tag{4.70}$$

$${}^3_6T = {}^3_4T\,{}^4_6T = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

where ${}^3_6T$ is given by equation (3.11) developed in Chapter 3. Equating both the (1,4) elements and the (2,4) elements from the two sides of (4.70), we get

$$\begin{aligned} c_1c_{23}p_x + s_1c_{23}p_y - s_{23}p_z - a_2c_3 &= a_3, \\ -c_1s_{23}p_x - s_1s_{23}p_y - c_{23}p_z + a_2s_3 &= d_4. \end{aligned} \tag{4.71}$$

These equations can be solved simultaneously for s_{23} and c_{23}, resulting in

$$s_{23} = \frac{(-a_3 - a_2 c_3) p_z + (c_1 p_x + s_1 p_y)(a_2 s_3 - d_4)}{p_z^2 + (c_1 p_x + s_1 p_y)^2},$$

$$c_{23} = \frac{(a_2 s_3 - d_4) p_z - (a_3 + a_2 c_3)(c_1 p_x + s_1 p_y)}{p_z^2 + (c_1 p_x + s_1 p_y)^2}. \tag{4.72}$$

The denominators are equal and positive, so we solve for the sum of θ_2 and θ_3 as

$$\begin{aligned}\theta_{23} = \mathrm{Atan2}[&(-a_3 - a_2 c_3) p_z - (c_1 p_x + s_1 p_y)(d_4 - a_2 s_3),\\ &(a_2 s_3 - d_4) p_z - (a_3 + a_2 c_3)(c_1 p_x + s_1 p_y)].\end{aligned} \tag{4.73}$$

Equation (4.73) computes four values of θ_{23}, according to the four possible combinations of solutions for θ_1 and θ_3; then, four possible solutions for θ_2 are computed as

$$\theta_2 = \theta_{23} - \theta_3, \tag{4.74}$$

Now the entire left side of (4.70) is known. Equating both the (1,3) elements and the (3,3) elements from the two sides of (4.70), we get

$$\begin{aligned} r_{13}c_1c_{23} + r_{23}s_1c_{23} - r_{33}s_{23} &= -c_4s_5, \\ -r_{13}s_1 + r_{23}c_1 &= s_4s_5. \end{aligned} \tag{4.75}$$

$$\theta_4 = \mathrm{Atan2}(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + r_{33}s_{23}). \tag{4.76}$$

$$[{}^0_4T(\theta_4)]^{-1}\, {}^0_6T = {}^4_5T(\theta_5){}^5_6T(\theta_6), \tag{4.77}$$

where $[{}^0_4T(\theta 4)]^{-1}$ is given by

$$\begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & s_1c_{23}c_4 - c_1s_4 & -s_{23}c_4 & -a_2c_3c_4 + d_3s_4 - a_3c_4 \\ -c_1c_{23}s_4 + s_1c_4 & -s_1c_{23}s_4 - c_1c_4 & s_{23}s_4 & a_2c_3s_4 + d_3c_4 + a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.78}$$

$${}^4_6T = {}^4_5T\, {}^5_6T = \begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.10}$$

$$[{}^0_4T(\theta_4)]^{-1}\,{}^0_6T = {}^4_5T(\theta_5){}^5_6T(\theta_6), \tag{4.77}$$

where $[{}^0_4T(\theta 4)]^{-1}$ is given by

$$\begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & s_1c_{23}c_4 - c_1s_4 & -s_{23}c_4 & -a_2c_3c_4 + d_3s_4 - a_3c_4 \\ -c_1c_{23}s_4 + s_1c_4 & -s_1c_{23}s_4 - c_1c_4 & s_{23}s_4 & a_2c_3s_4 + d_3c_4 + a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.78}$$

$${}^4_6T = {}^4_5T\,{}^5_6T = \begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.10}$$

$$\begin{aligned} r_{13}(c_1c_{23}c_4 + s_1s_4) + r_{23}(s_1c_{23}c_4 - c_1s_4) - r_{33}(s_{23}c_4) &= -s_5, \\ r_{13}(-c_1s_{23}) + r_{23}(-s_1s_{23}) + r_{33}(-c_{23}) &= c_5. \end{aligned} \tag{4.79}$$

Hence, we can solve for θ_5 as

$$\theta_5 = \mathrm{Atan2}(s_5, c_5), \tag{4.80}$$

Applying the same method one more time, we compute $({}^0_5T)^{-1}$ and write (4.54) in the form

$$({}^0_5T)^{-1}\,{}^0_6T = {}^5_6T(\theta_6). \tag{4.81}$$

Equating both the (3,1) elements and the (1,1) elements from the two sides of (4.77) as we have done before, we get

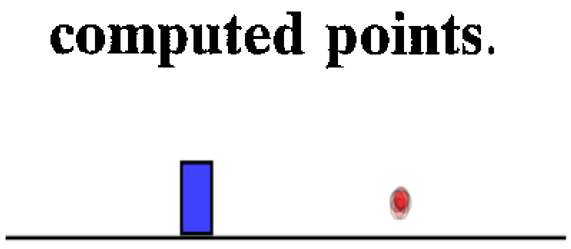
$$\theta_6 = \mathrm{Atan2}(s_6, c_6), \tag{4.82}$$

where

$$\begin{aligned}
s_6 &= -r_{11}(c_1c_{23}s_4 - s_1c_4) - r_{21}(s_1c_{23}s_4 + c_1c_4) + r_{31}(s_{23}s_4),\\
c_6 &= r_{11}[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5] + r_{21}[(s_1c_{23}c_4 - c_1s_4)c_5 - s_1s_{23}s_5]\\
&\quad - r_{31}(s_{23}c_4c_5 + c_{23}s_5).
\end{aligned}$$

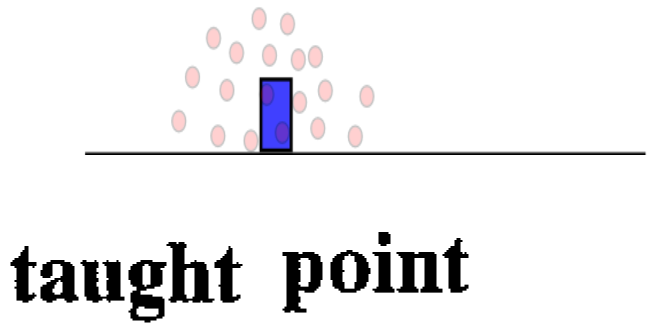
4.10 REPEATABILITY AND ACCURACY

- Accuracy
  - –The difference between the actual position of the robot and the programmed position



- Repeatability

  Will the robot always return to the same point under the same control conditions?



4.10 REPEATABILITY AND ACCURACY

Many industrial robots today move to goal points that have been taught. A **taught point** is one that the manipulator is moved to physically, and then the joint position sensors are read and the joint angles stored. When the robot is commanded to return to that point in space, each joint is moved to the stored value. In simple "teach and playback" manipulators such as these, the inverse kinematic problem never arises, because goal points are never specified in Cartesian coordinates. When a manufacturer specifies how precisely a manipulator can return to a taught point, he is specifying the **repeatability** of the manipulator.

4.10 REPEATABILITY AND ACCURACY

Any time a goal position and orientation are specified in Cartesian terms, the inverse kinematics of the device must be computed in order to solve for the required joint angles. Systems that allow goals to be described in Cartesian terms are capable of moving the manipulator to points that were never taught—points in its workspace to which it has perhaps never gone before. We will call such points **computed points**. Such a capability is necessary for many manipulation tasks. For example, if a computer vision system is used to locate a part that the robot must grasp, the robot must be able to move to the Cartesian coordinates supplied by the vision sensor. The precision with which a computed point can be attained is called the **accuracy** of the manipulator.

4.10 REPEATABILITY AND ACCURACY

The accuracy of a manipulator is bounded by the repeatability. Clearly, accuracy is affected by the precision of parameters appearing in the kinematic equations of the robot. Errors in knowledge of the Denavit–Hartenberg parameters will cause the inverse kinematic equations to calculate joint angle values that are in error. Hence, although the repeatability of most industrial manipulators is quite good, the accuracy is usually much worse and varies quite a bit from manipulator to manipulator. Calibration techniques can be devised that allow the accuracy of a manipulator to be improved through estimation of that particular manipulator's kinematic parameters [10].

Robotics

Chapter 5
Jacobians: velocities and static forces

INTRODUCTION

In this chapter, we expand our consideration of robot manipulators beyond static-positioning problems. We examine the notions of linear and angular velocity of a rigid body and use these concepts to analyze the motion of a manipulator. We also

Differentiation of position vectors

$$^{B}V_{Q} = \frac{d}{dt}\,^{B}Q = \lim_{\Delta t \to 0} \frac{^{B}Q(t+\Delta t) - ^{B}Q(t)}{\Delta t}.$$

$$^{A}(^{B}V_{Q}) = \frac{^{A}d}{dt}\,^{B}Q.$$

$$^{B}(^{B}V_{Q}) = ^{B}V_{Q}.$$

$$^{A}(^{B}V_{Q}) = ^{A}_{B}R\,^{B}V_{Q}.$$

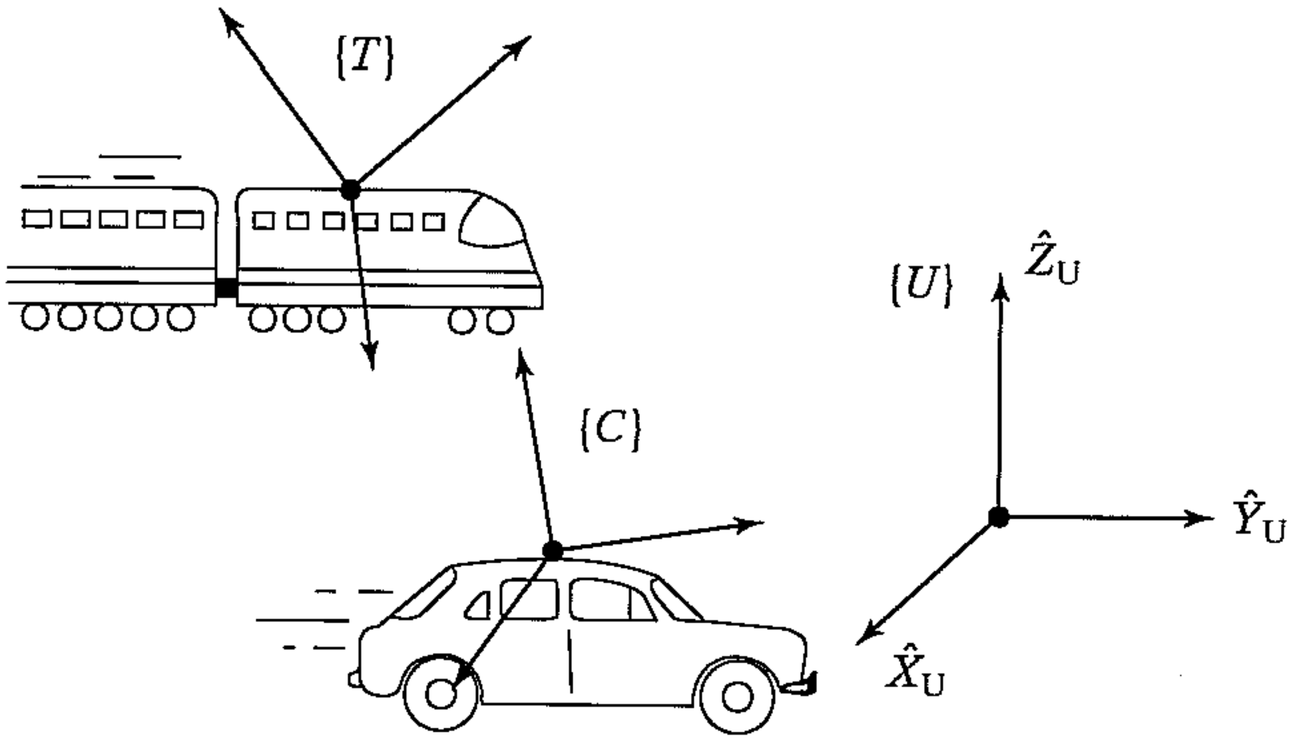
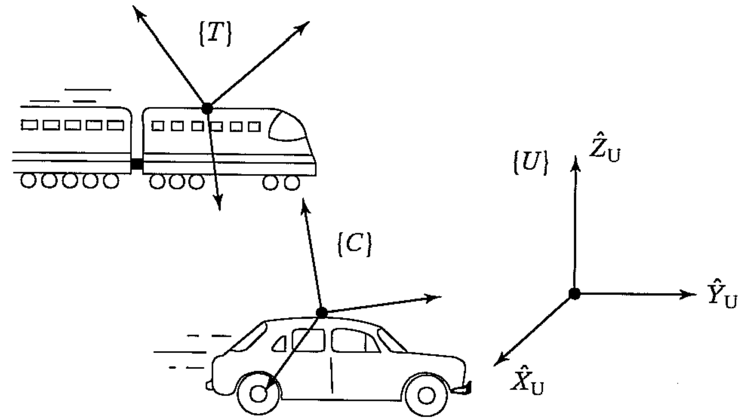


FIGURE 5.1: Example of some frames in linear motion.

Figure 5.1 shows a fixed universe frame, $\{U\}$, a frame attached to a train traveling at 100 mph, $\{T\}$, and a frame attached to a car traveling at 30 mph, $\{C\}$. Both vehicles are heading in the $\hat{X}$ direction of $\{U\}$. The rotation matrices, ${}^U_T R$ and ${}^U_C R$, are known and constant.



ion.

What is $\frac{{}^{U}d}{dt}\,{}^{U}P_{CORG}$?

$$\frac{{}^{U}d}{dt}\,{}^{U}P_{CORG} = {}^{U}V_{CORG} = \upsilon_C = 30\hat{X}.$$

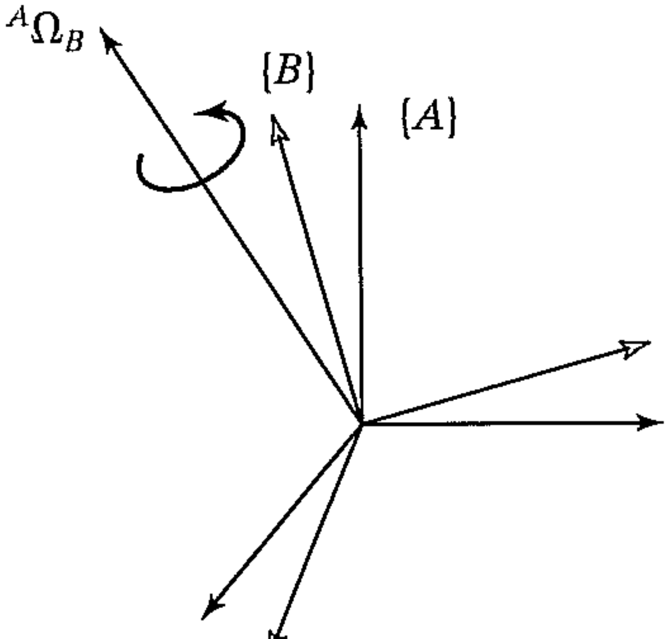
What is ${}^{C}({}^{U}V_{TORG})$?

$${}^{C}({}^{U}V_{TORG}) = {}^{C}\upsilon_T = {}^{C}_{U}R\upsilon_T = {}^{C}_{U}R(100\hat{X}) = {}^{U}_{C}R^{-1}\,100\hat{X}.$$

What is ${}^{C}({}^{T}V_{CORG})$?

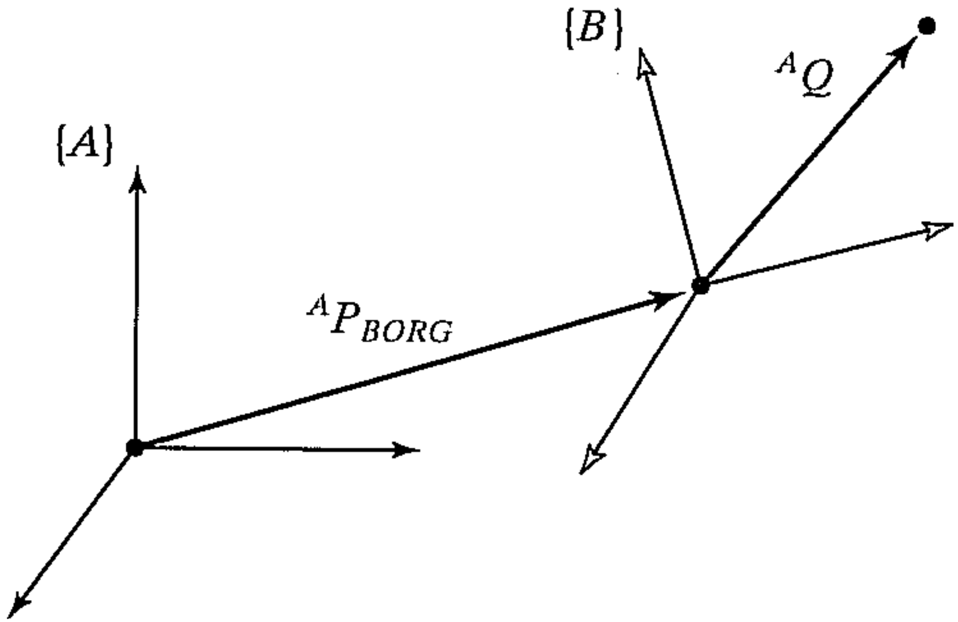
$${}^{C}({}^{T}V_{CORG}) = {}^{C}_{T}R\,{}^{T}V_{CORG} = -{}^{U}_{C}R^{-1}\,{}^{U}_{T}R\,70\hat{X}.$$

The angular velocity vector



In Fig. 5.2, $^A\Omega_B$ describes the rotation of frame $\{B\}$ relative to $\{A\}$. Physically, at any instant, the direction of $^A\Omega_B$ indicates the instantaneous axis of rotation of $\{B\}$ relative to $\{A\}$, and the magnitude of $^A\Omega_B$ indicates the speed of rotation. Again, like any vector, an angular velocity vector may be expressed in any coordinate system, and so another leading superscript may be added; for example, $^C(^A\Omega_B)$ is the angular velocity of frame $\{B\}$ relative to $\{A\}$ expressed in terms of frame $\{C\}$.

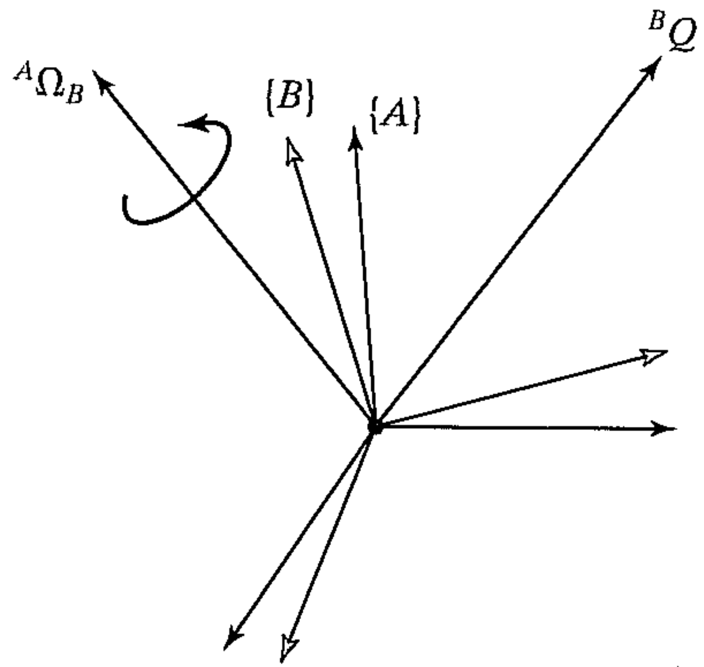
5.3 LINEAR AND ROTATIONAL VELOCITY OF RIGID BODIES



No rotation

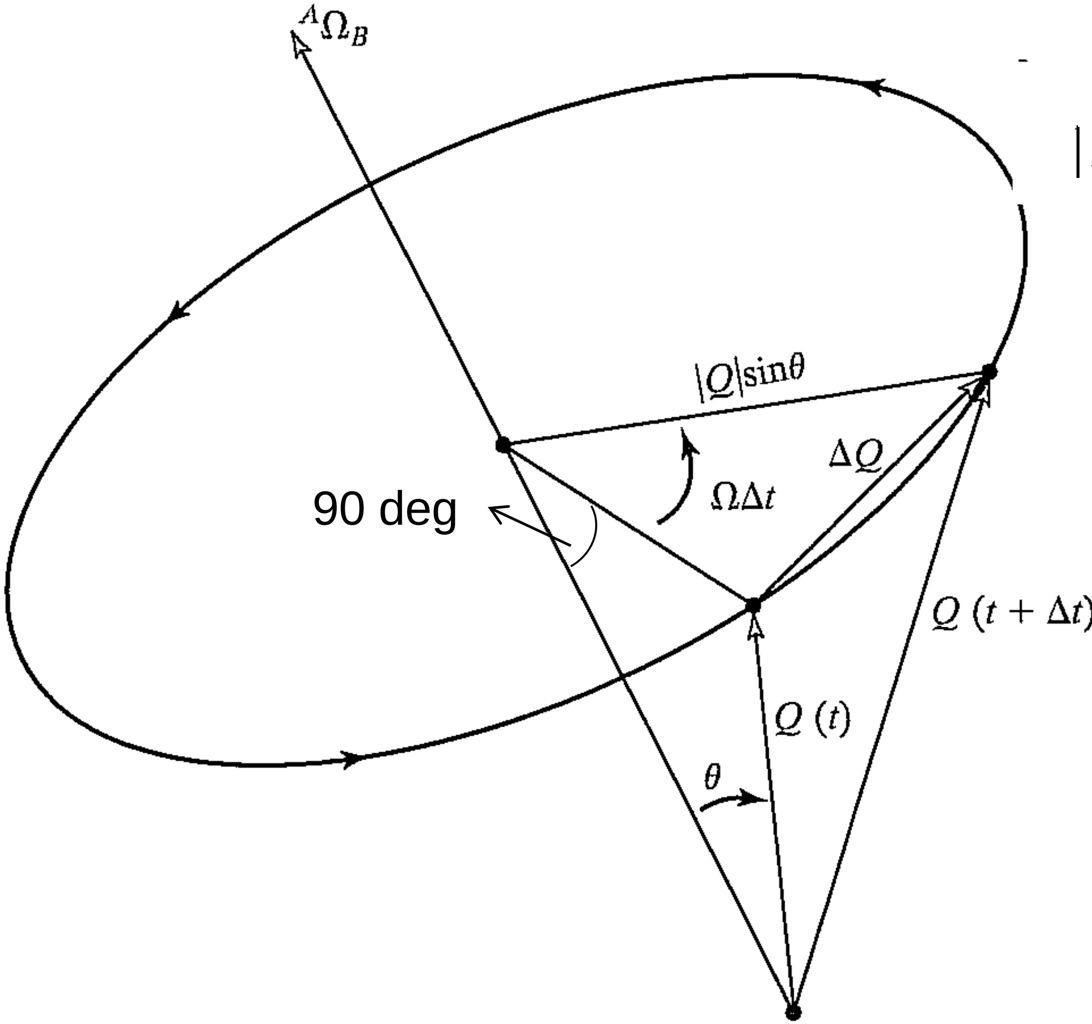
$$^AV_Q = {^AV_{BORG}} + {^A_BR}\,{^BV_Q}.$$

Equation (5.7) is for only that case in which relative orientation of $\{B\}$ and $\{A\}$ remains constant.



No translation

FIGURE 5.4: Vector ^BQ, fixed in frame $\{B\}$, is rotating with respect to frame $\{A\}$ with angular velocity $^A\Omega_B$.



$$|\Delta Q| = (|{}^A Q| \sin\theta)(|{}^A\Omega_B|\Delta t).$$

$${}^A V_Q = {}^A\Omega_B \times {}^A Q.$$

the vector Q could also be changing with respect to frame $\{B\}$,

$${}^A V_Q = {}^A({}^B V_Q) + {}^A\Omega_B \times {}^A Q. \qquad {}^A V_Q = {}^A_B R\, {}^B V_Q + {}^A\Omega_B \times {}^A_B R\, {}^B Q.$$

Simultaneous linear and rotational velocity

We can very simply expand (5.12) to the case where origins are not coincident by adding on the linear velocity of the origin to (5.12) to derive the general formula for velocity of a vector fixed in frame $\{B\}$ as seen from frame $\{A\}$:

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}Q \tag{5.13}$$

5.6 VELOCITY "PROPAGATION" FROM LINK TO LINK

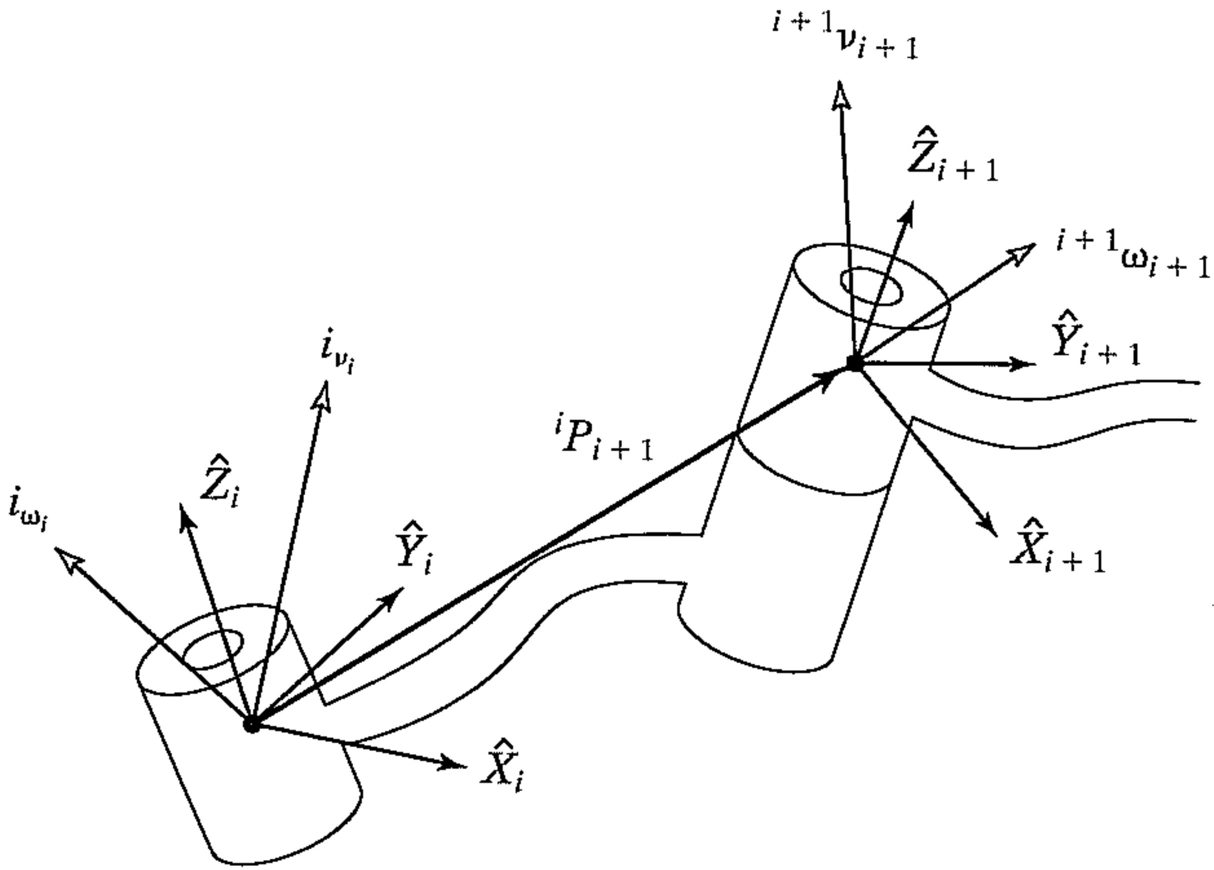


FIGURE 5.7: Velocity vectors of neighboring links.

Rotational velocities can be added when both ω vectors are written with respect to the same frame. Therefore, the angular velocity of link $i+1$ is the same as that of link i plus a new component caused by rotational velocity at joint $i+1$. This can be written in terms of frame $\{i\}$ as

$$^{i}\omega_{i+1} = {}^{i}\omega_i + {}^{i}_{i+1}R\, \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}. \tag{5.43}$$

Note that

$$\dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}. \tag{5.44}$$

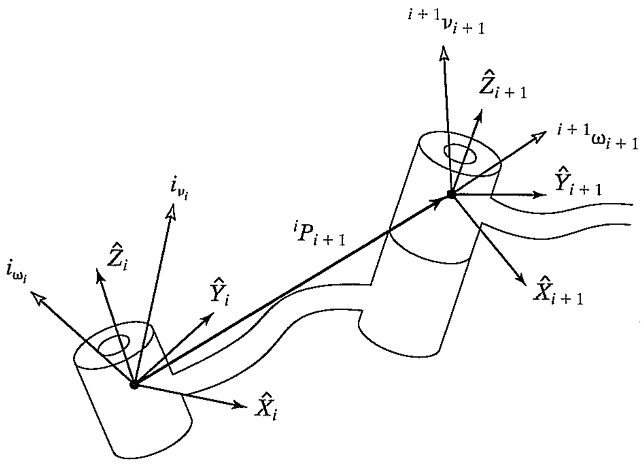


FIGURE 5.7: Velocity vectors of neighboring links.

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}.$$

Forward propagation

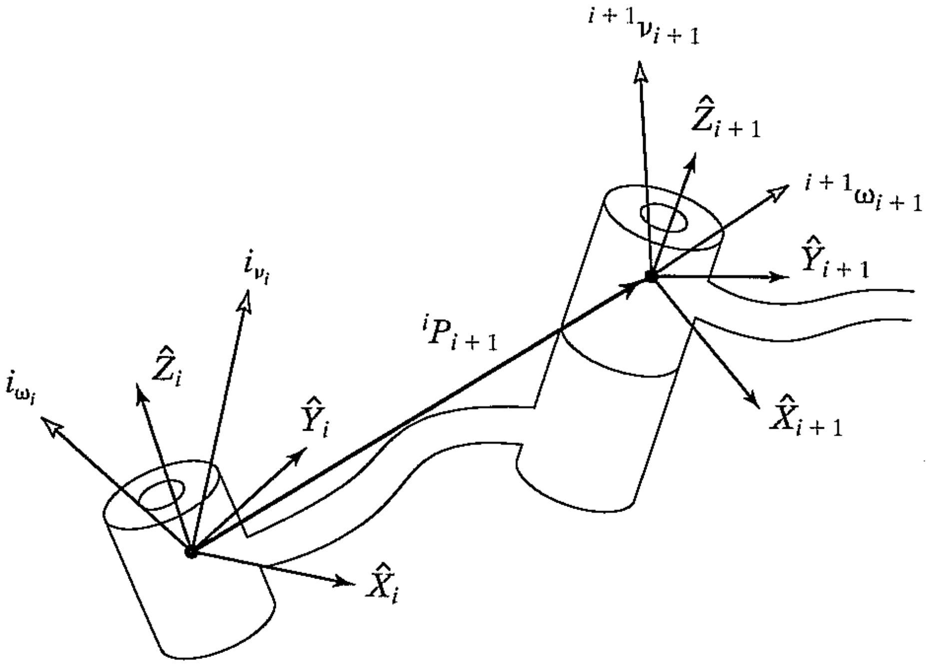


FIGURE 5.7: Velocity vectors of neighboring links.

For revolute

$$ {}^{i}\upsilon_{i+1} = {}^{i}\upsilon_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}. $$

$$ {}^{i+1}\upsilon_{i+1} = {}^{i+1}_{i}R({}^{i}\upsilon_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}). $$

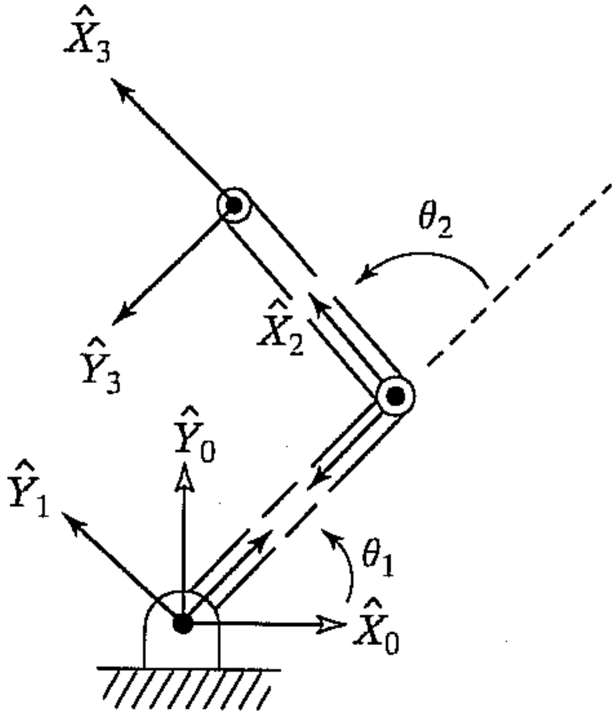
The corresponding relationships for the case that joint $i+1$ is prismatic are

$$
\begin{aligned}
{}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\, {}^{i}\omega_i, \\
{}^{i+1}\upsilon_{i+1} &= {}^{i+1}_{i}R({}^{i}\upsilon_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}.
\end{aligned} \tag{5.48}
$$

Applying these equations successively from link to link, we can compute ${}^{N}\omega_N$ and ${}^{N}\upsilon_N$, the rotational and linear velocities of the last link. Note that the resulting velocities are expressed in terms of frame $\{N\}$. This turns out to be useful, as we will see later. If the velocities are desired in terms of the base coordinate system, they can be rotated into base coordinates by multiplication with ${}^{0}_{N}R$.

EXAMPLE 5.3

A two-link manipulator with rotational joints is shown in Fig. 5.8. Calculate the velocity of the tip of the arm as a function of joint rates. Give the answer in two forms—in terms of frame {3} and also in terms of frame {0}.



$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

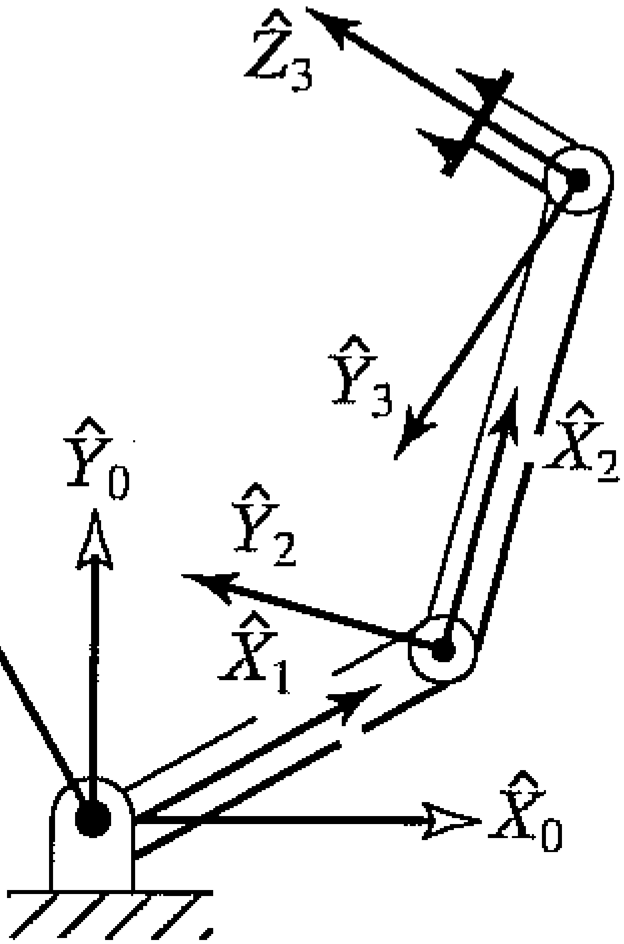
$$
{}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

- See notes

5.7 JACOBIANS

- See notes

- See notes



$$\begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix}_{\text{6x1}} = {}^{B}\nu = {}^{B}J(\Theta)_{\text{6xn}}\,\dot{\Theta}_{\text{nx1}},$$

$$\begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}_{B}R & 0 \\ \hline 0 & {}^{A}_{B}R \end{array}\right] \begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix}.$$

- See notes

$$\begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}_{B}R & 0 \\ \hline 0 & {}^{A}_{B}R \end{array}\right] {}^{B}J(\Theta)\dot{\Theta}.$$

$${}^{A}J(\Theta)_{\text{6xn}} = \left[\begin{array}{c|c} {}^{A}_{B}R_{\text{3x3}} & 0_{\text{3x3}} \\ \hline 0_{\text{3x3}} & {}^{A}_{B}R_{\text{3x3}} \end{array}\right]_{\text{6x6}} {}^{B}J(\Theta)_{\text{6xn}}.$$

5.8 SINGULARITIES

Most manipulators have values of Θ where the Jacobian becomes singular. Such locations are called **singularities of the mechanism** or **singularities** for short. All manipulators have singularities at the boundary of their workspace, and most have loci of singularities inside their workspace. An in-depth study of the classification of

- See notes

$$\mathbf{B} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

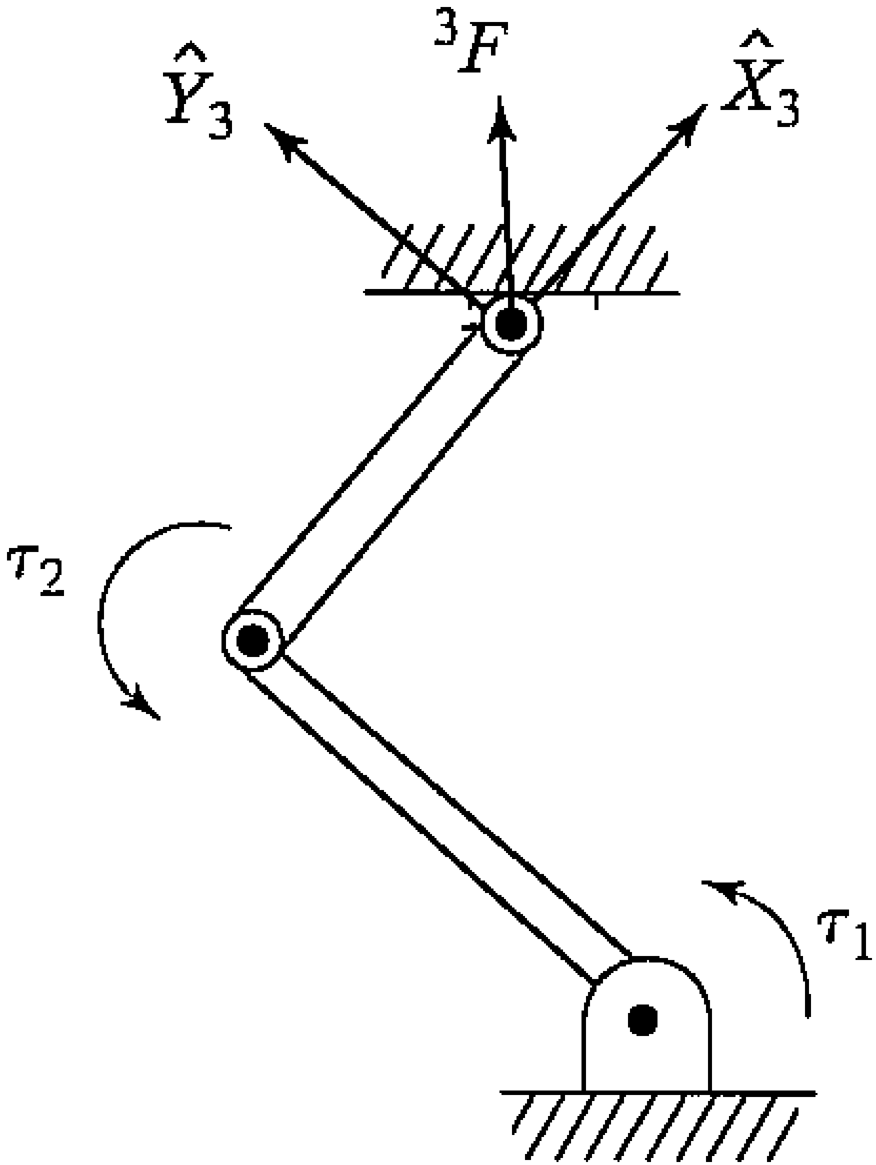
If $AD - BC \neq 0$, then B has an inverse, denoted B^{-1}

$$\mathbf{B}^{-1} = \frac{1}{AD-BC}\begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

5.9 STATIC FORCES IN MANIPULATORS

The chainlike nature of a manipulator leads us quite naturally to consider how forces and moments "propagate" from one link to the next. Typically, the robot is pushing on something in the environment with the chain's free end (the end-effector) or is perhaps supporting a load at the hand. We wish to solve for the joint torques that must be acting to keep the system in static equilibrium.

- See notes



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5.10 JACOBIANS IN THE FORCE DOMAIN

- See notes

Robotics

Chapter 6
Manipulator dynamics

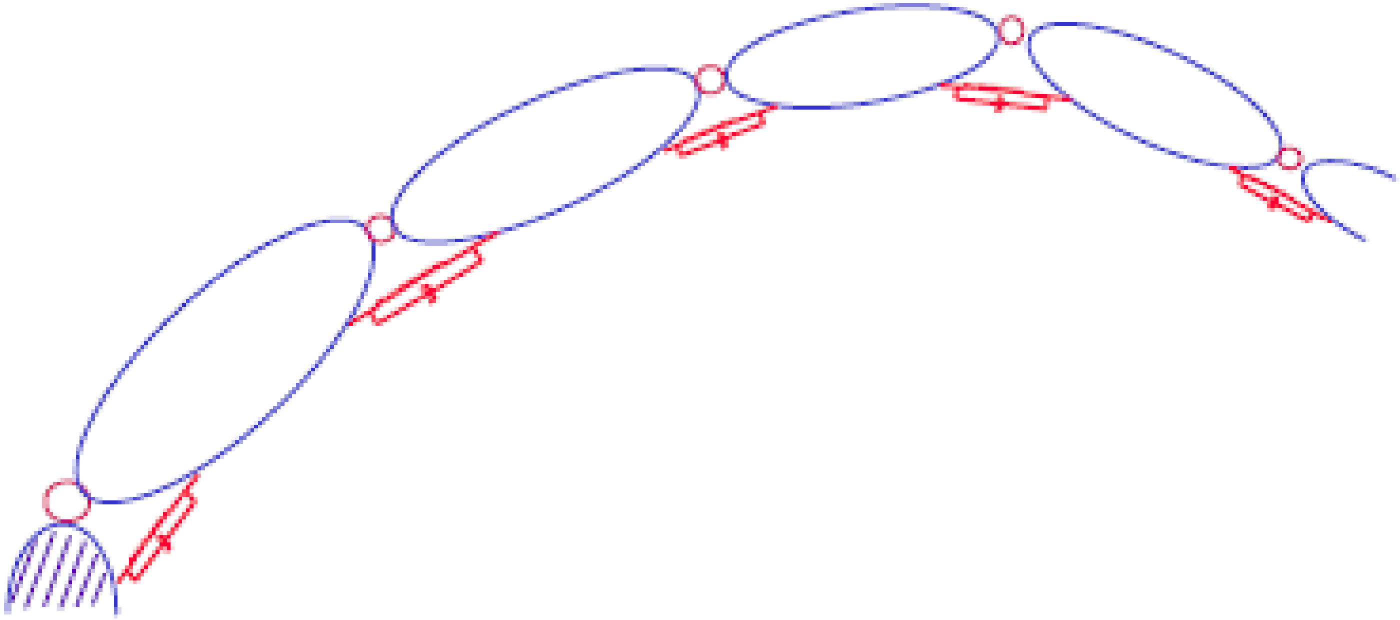
6.1 INTRODUCTION

Our study of manipulators so far has focused on kinematic considerations only. We have studied static positions, static forces, and velocities; but we have never considered *the forces required to cause motion*. In this chapter, we consider the equations of motion for a manipulator—the way in which motion of the manipulator arises from torques applied by the actuators or from external forces applied to the manipulator.

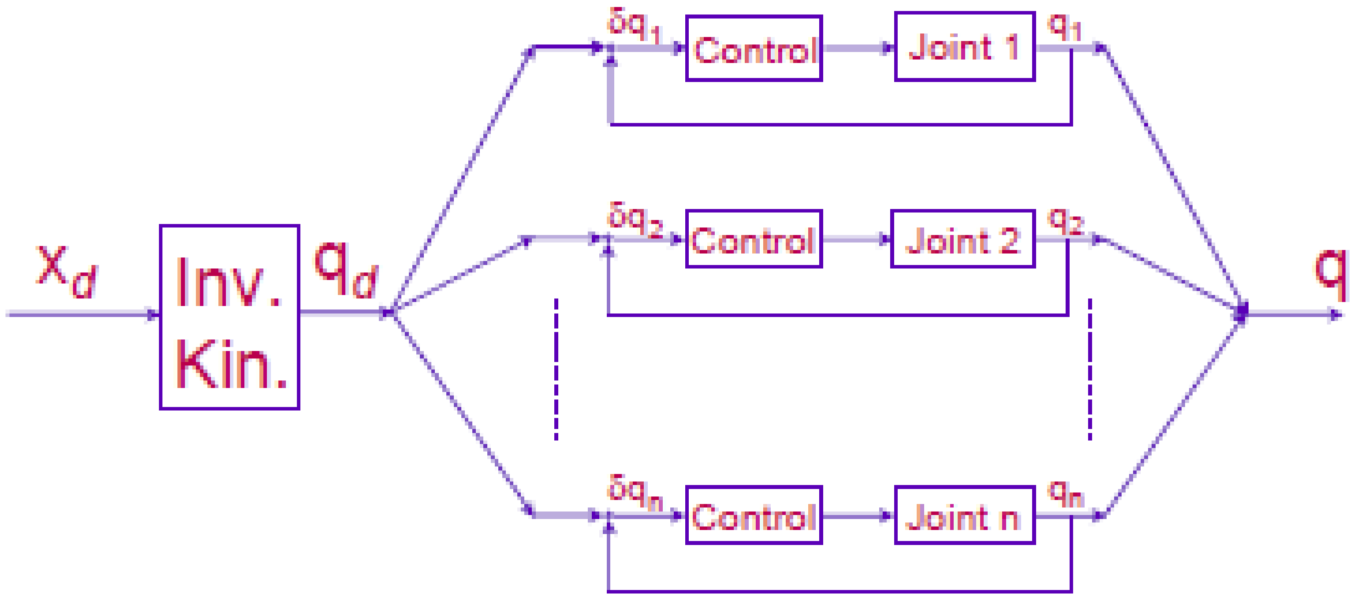
There are two problems related to the dynamics of a manipulator that we wish to solve. In the first problem, we are given a trajectory point, Θ, $\dot{\Theta}$, and $\ddot{\Theta}$, and we wish to find the required vector of joint torques, τ. This formulation of dynamics is useful for the problem of controlling the manipulator (Chapter 10). The second problem is to calculate how the mechanism will move under application of a set of joint torques. That is, given a torque vector, τ, calculate the resulting motion of the manipulator, Θ, $\dot{\Theta}$, and $\ddot{\Theta}$. This is useful for simulating the manipulator.

See notes

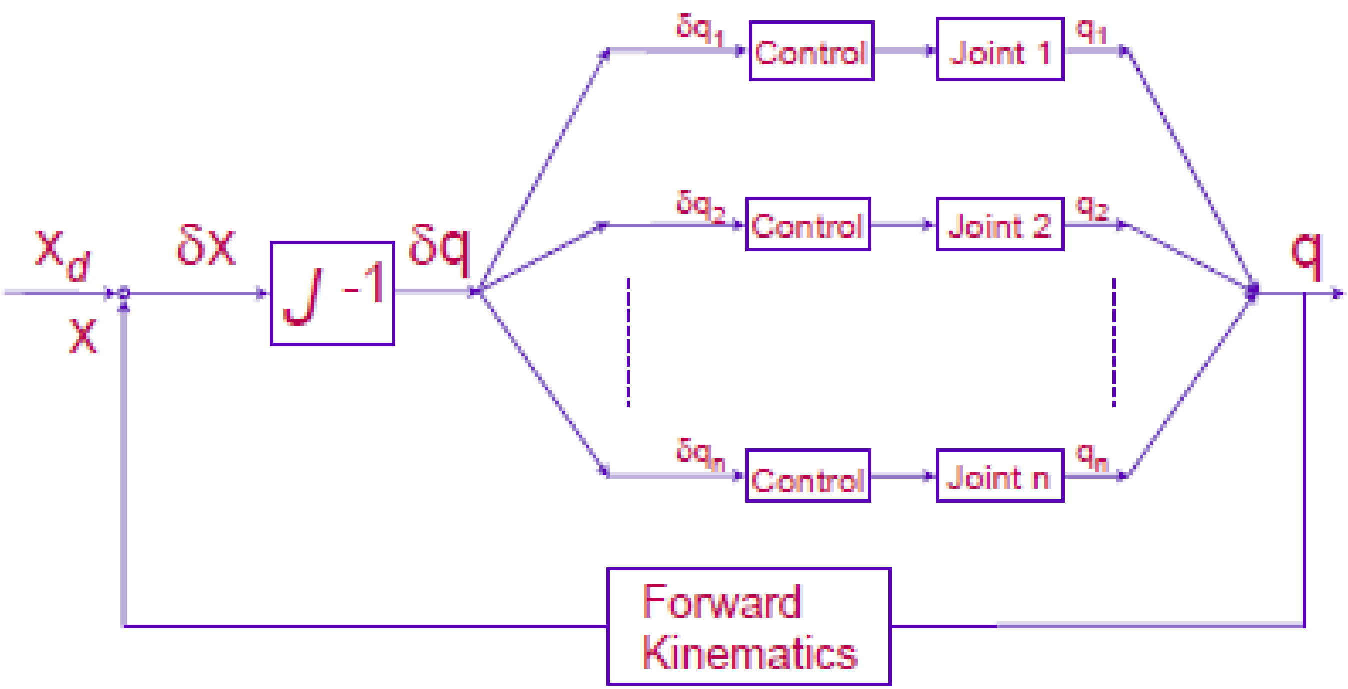
Joint-Space Control



Joint Space Control

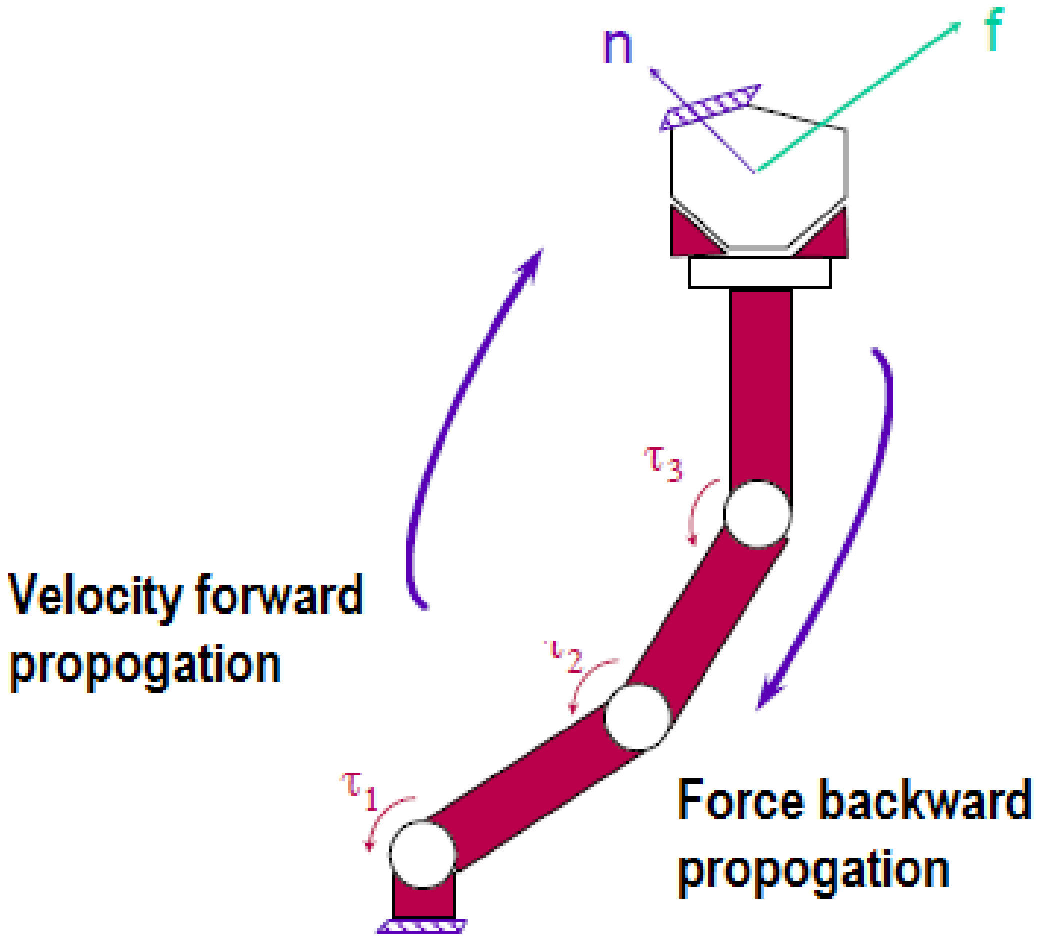


Resolved Motion Rate Control



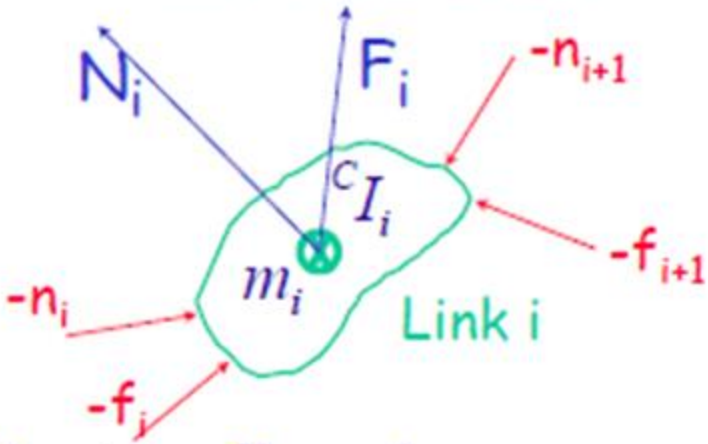
Dynamics review (2D)

See notes


n
f
τ_3
τ_2
τ_1
Velocity forward propogation
Force backward propogation

Formulations

Newton-Euler



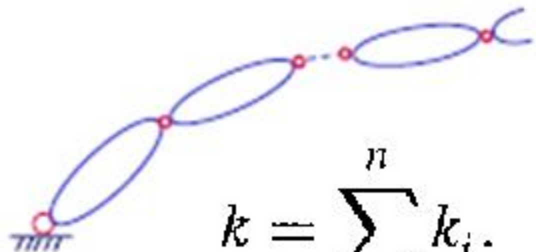
Newton: $F_i = m_i \dot{\mathbf{v}}_{C_i}$

Euler: $N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i$

Eliminate Internal Forces and Moments

$$\Gamma_i = \begin{cases} n_i^T . Z_i & \text{revolute} \\ f_i^T . Z_i & \text{prismatic} \end{cases}$$

Lagrange



$$k = \sum_{i=1}^{n} k_i .$$

$$u = \sum_{i=1}^{n} u_i .$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Theta}} - \frac{\partial \mathcal{L}}{\partial \Theta} = \tau,$$

$$\mathcal{L}(\Theta, \dot{\Theta}) = k(\Theta, \dot{\Theta}) - u(\Theta).$$

See notes

The force and torque acting on a link

Having computed the linear and angular accelerations of the mass center of each link, we can apply the Newton–Euler equations (Section 6.4) to compute the inertial force and torque acting at the center of mass of each link. Thus we have

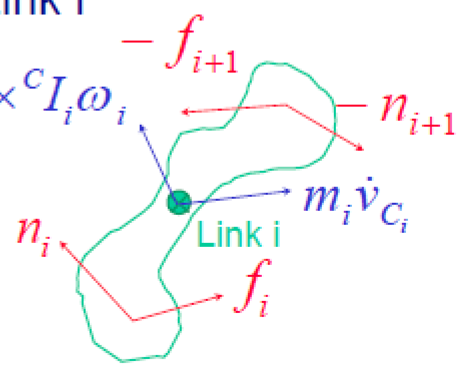
$$F_i = m\dot{v}_{C_i},$$

$$N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i, \tag{6.37}$$

where $\{C_i\}$ has its origin at the center of mass of the link and has the same orientation as the link frame, $\{i\}$.

Dynamic forces on Link i

$${}^{c}I_i\dot{\omega}_i + \omega_i \times {}^{c}I_i\omega_i$$



$$m_i\dot{\mathbf{v}}_{C_i} = \sum forces$$

$${}^{c}I_i\dot{\omega}_i + \omega_i \times {}^{c}I_i\omega_i = \sum moments / c_i$$

Inertial forces/moments

$$F_i = m_i\dot{\mathbf{v}}_{C_i}$$

$$N_i = {}^{C}I_i\dot{\omega}_i + \omega_i \times {}^{C}I_i\omega_i$$

Inward iterations to compute forces and torques

Having computed the forces and torques acting on each link, we now need to calculate the joint torques that will result in these net forces and torques being applied to each link.

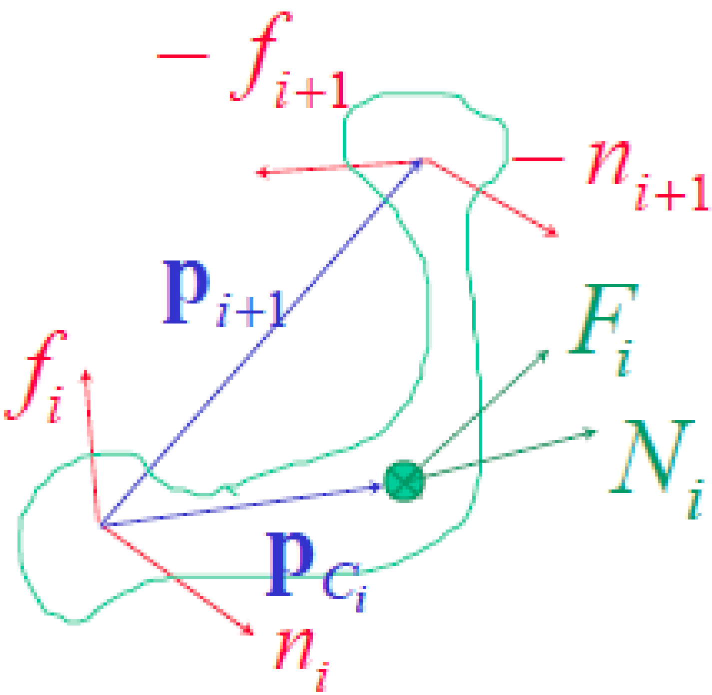
We can do this by writing a force-balance and moment-balance equation based on a free-body diagram of a typical link. (See Fig. 6.5.) Each link has forces and torques exerted on it by its neighbors and in addition experiences an inertial force and torque. In Chapter 5, we defined special symbols for the force and torque exerted by a neighbor link, which we repeat here:

f_i = force exerted on link i by link $i-1$,
n_i = torque exerted on link i by link $i-1$.

By summing the forces acting on link i, we arrive at the force-balance relationship:

$$ {}^{i}F_i = {}^{i}f_i - {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}. \tag{6.38}$$

By summing torques about the center of mass and setting them equal to zero, we arrive at the torque-balance equation:



$$F_i = f_i - f_{i+1}$$

$$N_i = n_i - n_{i+1} + (-\mathbf{p}_{C_i}) \times f_i + (\mathbf{p}_{i+1} - \mathbf{p}_{C_i}) \times (-f_{i+1})$$

Finally, we can rearrange the force and torque equations so that they appear as iterative relationships from higher numbered neighbor to lower numbered neighbor:

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i, \tag{6.41} $$

$$ {}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}. \tag{6.42} $$

These equations are evaluated link by link, starting from link n and working inward toward the base of the robot. These *inward force iterations* are analogous to the static force iterations introduced in Chapter 5, except that inertial forces and torques are now considered at each link.

As in the static case, the required joint torques are found by taking the $\hat{Z}$ component of the torque applied by one link on its neighbor:

$$ \tau_i = {}^{i}n_i^T\,{}^{i}\hat{Z}_i. \tag{6.43} $$

For joint i prismatic, we use

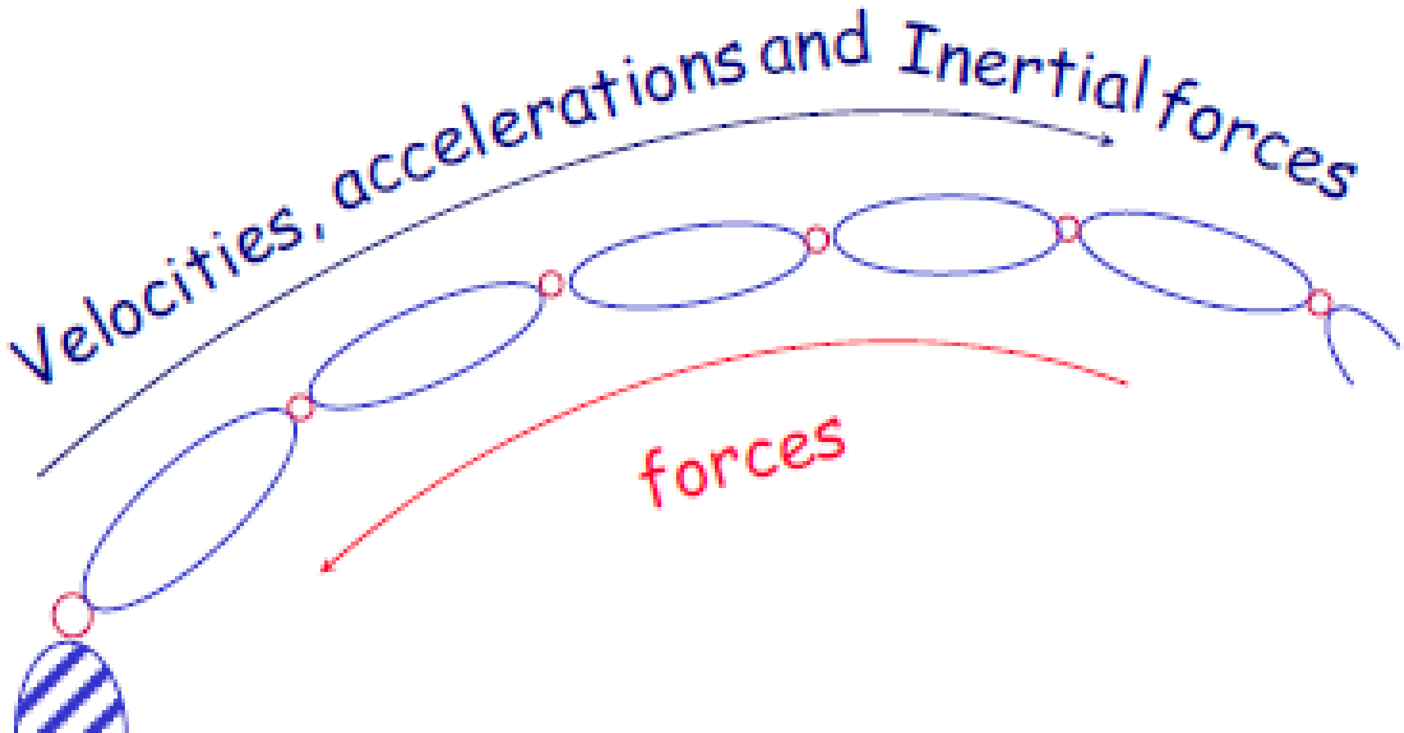
$$ \tau_i = {}^{i}f_i^T\,{}^{i}\hat{Z}_i, \tag{6.44} $$

where we have used the symbol τ for a linear actuator force.

See notes for gravity

The iterative Newton–Euler dynamics algorithm

The complete algorithm for computing joint torques from the motion of the joints is composed of two parts. First, link velocities and accelerations are iteratively computed from link 1 out to link n and the Newton–Euler equations are applied to each link. Second, forces and torques of interaction and joint actuator torques are computed recursively from link n back to link 1. The equations are summarized next for the case of all joints rotational:



Outward iterations: $i : 0 \to 5$

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}, \tag{6.45} $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}, \tag{6.46} $$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i), \tag{6.47} $$

$$ \begin{aligned} {}^{i+1}\dot{v}_{C_{i+1}} = {}&{}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} \\ &+ {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1}, \end{aligned} \tag{6.48} $$

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}, \tag{6.49} $$

$$ {}^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}. \tag{6.50} $$

Inward iterations: $i : 6 \to 1$

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i, \tag{6.51} $$

$$ \begin{aligned} {}^{i}n_i = {}&{}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i \\ &+ {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}, \end{aligned} \tag{6.52} $$

$$ \tau_i = {}^{i}n_i^T\,{}^{i}\hat{Z}_i. \tag{6.53} $$

6.3 MASS DISTRIBUTION

In systems with a single degree of freedom, we often talk about the mass of a rigid body. In the case of rotational motion about a single axis, the notion of the *moment of inertia* is a familiar one. For a rigid body that is free to move in three dimensions, there are infinitely many possible rotation axes. In the case of rotation about an arbitrary axis, we need a complete way of characterizing the mass distribution of a rigid body. Here, we introduce the **inertia tensor**

We shall now define a set of quantities that give information about the distribution of mass of a rigid body relative to a reference frame. Figure 6.1 shows a rigid body with an attached frame. Inertia tensors can be defined relative to any frame, but we will always consider the case of an inertia tensor defined for a frame attached to the rigid body. Where it is important, we will indicate, with a leading superscript, the frame of reference of a given inertia tensor. The inertia tensor relative to frame $\{A\}$ is expressed in the matrix form as the 3×3 matrix

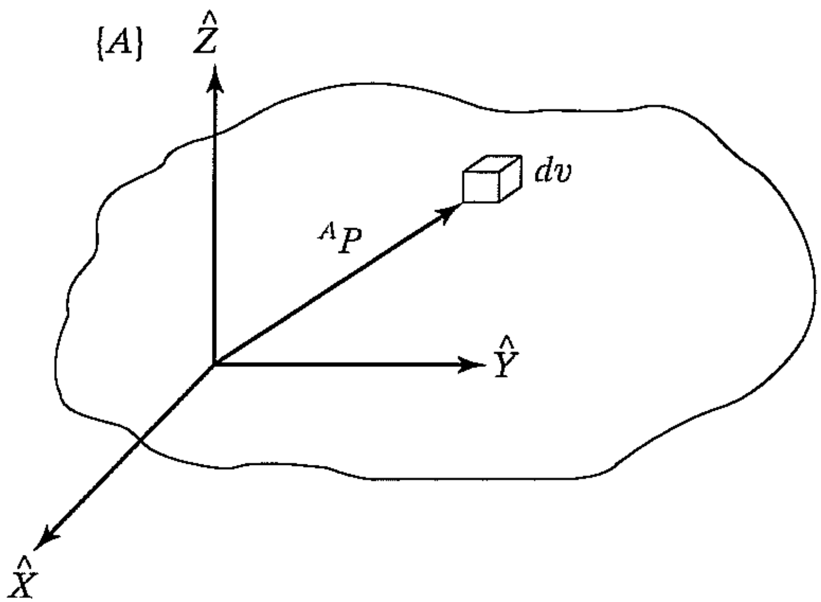
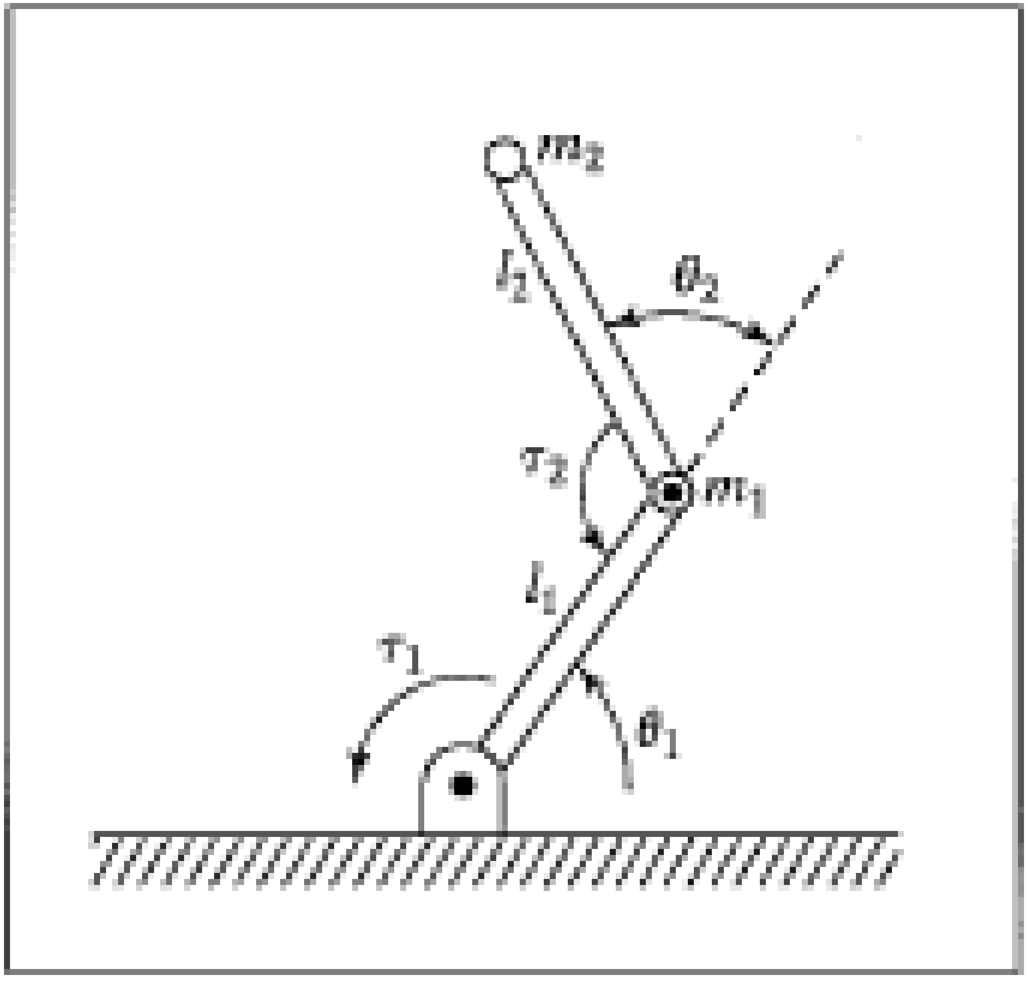


FIGURE 6.1: The inertia tensor of an object describes the object's mass distribution. Here, the vector ${}^{A}P$ locates the differential volume element, dv.

6.7 AN EXAMPLE OF CLOSED-FORM DYNAMIC EQUATIONS

Here we compute the closed-form dynamic equations for the two-link planar manipulator shown in Fig. 6.6. For simplicity, we assume that the mass distribution is extremely simple: All mass exists as a point mass at the distal end of each link. These masses are m_1 and m_2.

See notes



First, we determine the values of the various quantities that will appear in the recursive Newton–Euler equations. The vectors that locate the center of mass for each link are

$$^{1}P_{C_1} = l_1 \hat{X}_1,$$
$$^{2}P_{C_2} = l_2 \hat{X}_2.$$

Because of the point-mass assumption, the inertia tensor written at the center of mass for each link is the zero matrix:

$$^{C_1}I_1 = 0,$$
$$^{C_2}I_2 = 0.$$

There are no forces acting on the end-effector, so we have

$$f_3 = 0,$$
$$n_3 = 0.$$

The base of the robot is not rotating; hence, we have

$$\omega_0 = 0,$$
$$\dot{\omega}_0 = 0.$$

To include gravity forces, we will use

$$ {}^{0}\dot{v}_0 = g\hat{Y}_0. $$

The rotation between successive link frames is given by

$$ {}^{i}_{i+1}R = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0.0 \\ s_{i+1} & c_{i+1} & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, $$

$$ {}^{i+1}_{i}R = \begin{bmatrix} c_{i+1} & s_{i+1} & 0.0 \\ -s_{i+1} & c_{i+1} & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}. $$

We now apply equations (6.46) through (6.53).

The outward iterations for link 1 are as follows:

$$^{1}\omega_{1} = \dot{\theta}_{1}\, {}^{1}\hat{Z}_{1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{bmatrix},$$

$$^{1}\dot{\omega}_{1} = \ddot{\theta}_{1}\, {}^{1}\hat{Z}_{1} = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{1} \end{bmatrix},$$

$$^{1}\dot{v}_{1} = \begin{bmatrix} c_{1} & s_{1} & 0 \\ -s_{1} & c_{1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} gs_{1} \\ gc_{1} \\ 0 \end{bmatrix},$$

$$^{1}\dot{v}_{C_{1}} = \begin{bmatrix} 0 \\ l_{1}\ddot{\theta}_{1} \\ 0 \end{bmatrix} + \begin{bmatrix} -l_{1}\dot{\theta}_{1}^{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} gs_{1} \\ gc_{1} \\ 0 \end{bmatrix} = \begin{bmatrix} -l_{1}\dot{\theta}_{1}^{2} + gs_{1} \\ l_{1}\ddot{\theta}_{1} + gc_{1} \\ 0 \end{bmatrix},$$

$$^{1}F_{1} = \begin{bmatrix} -m_{1}l_{1}\dot{\theta}_{1}^{2} + m_{1}gs_{1} \\ m_{1}l_{1}\ddot{\theta}_{1} + m_{1}gc_{1} \\ 0 \end{bmatrix},$$

$$^{1}N_{1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6.54}$$

The outward iterations for link 2 are as follows:

$$^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix},$$

$$^2\dot\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_1 + \ddot\theta_2 \end{bmatrix},$$

$$^2\dot v_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_1\dot\theta_1^2 + gs_1 \\ l_1\ddot\theta_1 + gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\ddot\theta_1 s_2 - l_1\dot\theta_1^2 c_2 + gs_{12} \\ l_1\ddot\theta_1 c_2 + l_1\dot\theta_1^2 s_2 + gc_{12} \\ 0 \end{bmatrix},$$

$$^2\dot v_{C_2} = \begin{bmatrix} 0 \\ l_2(\ddot\theta_1 + \ddot\theta_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -l_2(\dot\theta_1 + \dot\theta_2)^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} l_1\ddot\theta_1 s_2 - l_1\dot\theta_1^2 c_2 + gs_{12} \\ l_1\ddot\theta_1 c_2 + l_1\dot\theta_1^2 s_2 + gc_{12} \\ 0 \end{bmatrix}, \tag{6.55}$$

$$^2F_2 = \begin{bmatrix} m_2 l_1\ddot\theta_1 s_2 - m_2 l_1\dot\theta_1^2 c_2 + m_2 g s_{12} - m_2 l_2(\dot\theta_1 + \dot\theta_2)^2 \\ m_2 l_1\ddot\theta_1 c_2 + m_2 l_1\dot\theta_1^2 s_2 + m_2 g c_{12} + m_2 l_2(\ddot\theta_1 + \ddot\theta_2) \\ 0 \end{bmatrix},$$

$$^2N_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The inward iterations for link 2 are as follows:

$$^2f_2 = {}^2F_2,$$

$$^2n_2 = \begin{bmatrix} 0 \\ 0 \\ m_2l_1l_2c_2\ddot{\theta}_1 + m_2l_1l_2s_2\dot{\theta}_1^2 + m_2l_2gc_{12} + m_2l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}. \qquad (6.56)$$

The inward iterations for link 1 are as follows:

$$^1f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2l_1s_2\ddot{\theta}_1 - m_2l_1c_2\dot{\theta}_1^2 + m_2gs_{12} - m_2l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2l_1c_2\ddot{\theta}_1 + m_2l_1s_2\dot{\theta}_1^2 + m_2gc_{12} + m_2l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} -m_1l_1\dot{\theta}_1^2 + m_1gs_1 \\ m_1l_1\ddot{\theta}_1 + m_1gc_1 \\ 0 \end{bmatrix},$$

$$^1n_1 = \begin{bmatrix} 0 \\ 0 \\ m_2l_1l_2c_2\ddot{\theta}_1 + m_2l_1l_2s_2\dot{\theta}_1^2 + m_2l_2gc_{12} + m_2l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ m_1l_1^2\ddot{\theta}_1 + m_1l_1gc_1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ m_2l_1^2\ddot{\theta}_1 - m_2l_1l_2s_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2l_1gs_2s_{12} \\ +m_2l_1l_2c_2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2l_1gc_2c_{12} \end{bmatrix}. \qquad (6.57)$$

Extracting the $\hat{Z}$ components of the ${}^{i}n_{i}$, we find the joint torques:

$$\tau_1 = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2(2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 \dot{\theta}_2^2$$
$$-2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1,$$
$$\tau_2 = m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2). \quad (6.58)$$

Equations (6.58) give expressions for the torque at the actuators as a function of joint position, velocity, and acceleration. Note that these rather complex functions arose from one of the simplest manipulators imaginable. Obviously, the closed-form equations for a manipulator with six degrees of freedom will be quite complex.

THE STRUCTURE OF A MANIPULATOR'S DYNAMIC EQUATIONS

It is often convenient to express the dynamic equations of a manipulator in a single equation that hides some of the details, but shows some of the structure of the equations.

The state-space equation

When the Newton–Euler equations are evaluated symbolically for any manipulator, they yield a dynamic equation that can be written in the form

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta), \tag{6.59}$$

where $M(\Theta)$ is the $n \times n$ **mass matrix** of the manipulator, $V(\Theta, \dot{\Theta})$ is an $n \times 1$ vector of centrifugal and Coriolis terms, and $G(\Theta)$ is an $n \times 1$ vector of gravity terms. We use the term **state-space equation** because the term $V(\Theta, \dot{\Theta})$, appearing in (6.59), has both position and velocity dependence [3].

Each element of $M(\Theta)$ and $G(\Theta)$ is a complex function that depends on Θ, the position of all the joints of the manipulator. Each element of $V(\Theta, \dot{\Theta})$ is a complex function of both Θ and $\dot{\Theta}$.

We may separate the various types of terms appearing in the dynamic equations and form the mass matrix of the manipulator, the centrifugal and Coriolis vector, and the gravity vector.

$$M(\Theta) = \begin{bmatrix} l_2^2 m_2 + 2 l_1 l_2 m_2 c_2 + l_1^2 (m_1 + m_2) & l_2^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2^2 m_2 + l_1 l_2 m_2 c_2 & l_2^2 m_2 \end{bmatrix}. \qquad (6.60)$$

Any manipulator mass matrix is symmetric and positive definite, and is, therefore, always invertible.

The velocity term, $V(\Theta, \dot{\Theta})$, contains all those terms that have any dependence on joint velocity. Thus, we obtain

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}. \qquad (6.61)$$

A term like $-m_2 l_1 l_2 s_2 \dot{\theta}_2^2$ is caused by a **centrifugal force**, and is recognized as such because it depends on the square of a joint velocity. A term such as $-2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2$ is caused by a **Coriolis force** and will always contain the product of two different joint velocities.

The gravity term, $G(\Theta)$, contains all those terms in which the gravitational constant, g, appears. Therefore, we have

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}. \qquad (6.62)$$

Note that the gravity term depends only on Θ and not on its derivatives.