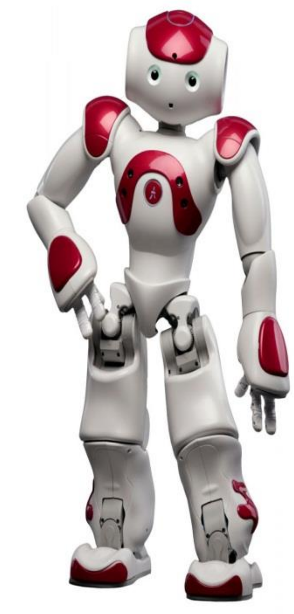
Robotics
Dr. Mohammed Abu malloul



E-mail: mmallouh@hu.edu.jo
Office: E3045, send e-mail to set appointment
Office Hour: to be posted on office door

Textbook: John Craig "Introduction to Robotics mechanics and control," 3ed Ed., Pearson Education Inc.

Topics covered :

1. Introduction
2. Spatial descriptions and transformations
3. Manipulator kinematics
4. Inverse manipulator kinematics
5. Jacobians: Velocities and static forces
6. Manipulator dynamics

What is a robot?

- Many different definitions for robots exist.

- A robot is a reprogrammable, multifunctional manipulator designed to move material, parts, tools, or specialized devices through variable programmed motions for the performance of a variety of tasks." (Robot Institute of America).

Automation vs. robots

Automation: Machinery designed to carry out a specific task

-Bottling machine

-Dishwasher

Robots: machinery designed to carry out a variety of tasks

-Pick and place arms

-Mobile robots

Robots Classification

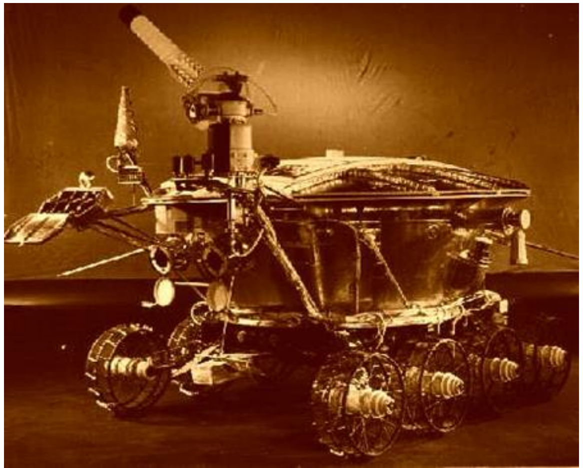
- Manipulators: robotic arms. These are most commonly found in industrial settings. https://www.jabil.com/blog/ten-popular-industrial-robot-applications.html

- Mobile Robots: unmanned vehicles

- Hybrid Robots: mobile robots with manipulators

- Humanoid robot

https://www.bostondynamics.com/atlas



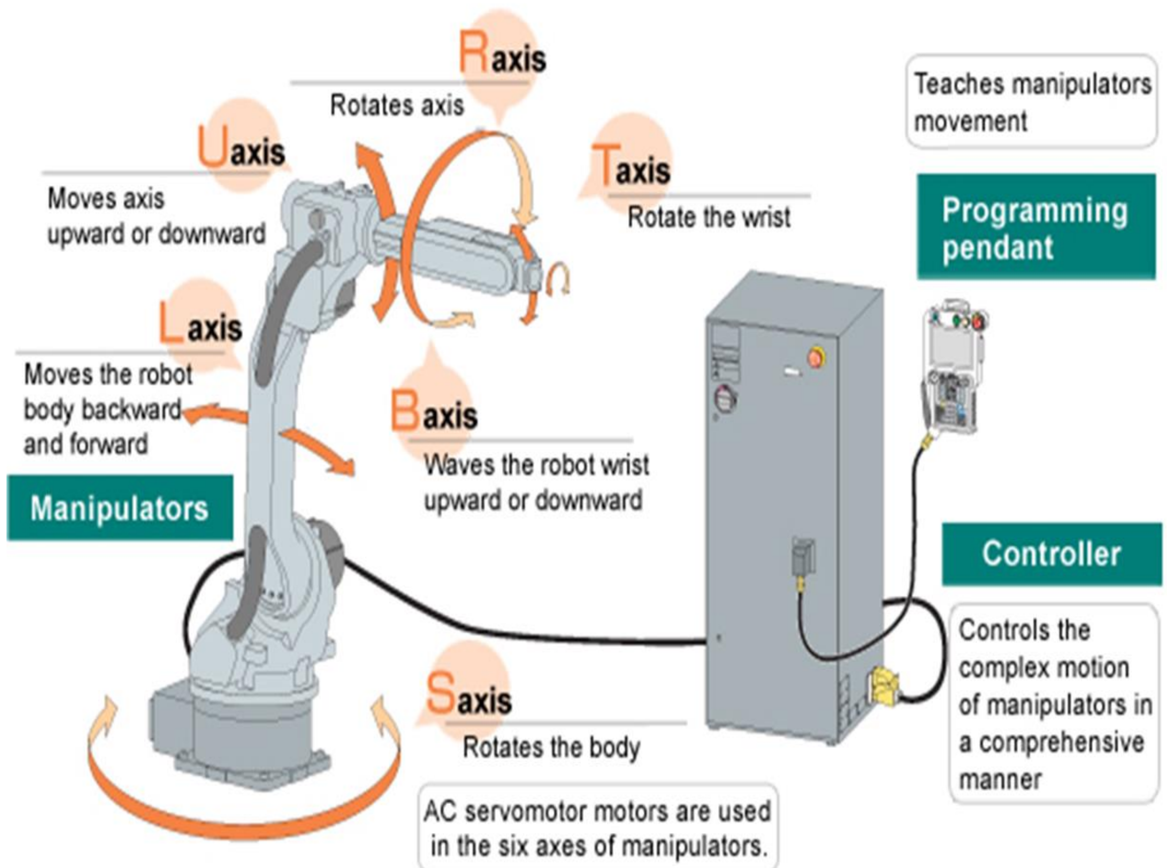
Applications

Dangerous:
-Space exploration
-chemical spill cleanup
-disarming bombs
-disaster cleanup

Repetitive
-Welding car frames
-part pick and place
-manufacturing parts.

High precision or high speed
-Electronics chips
-Surgery
-precision machining

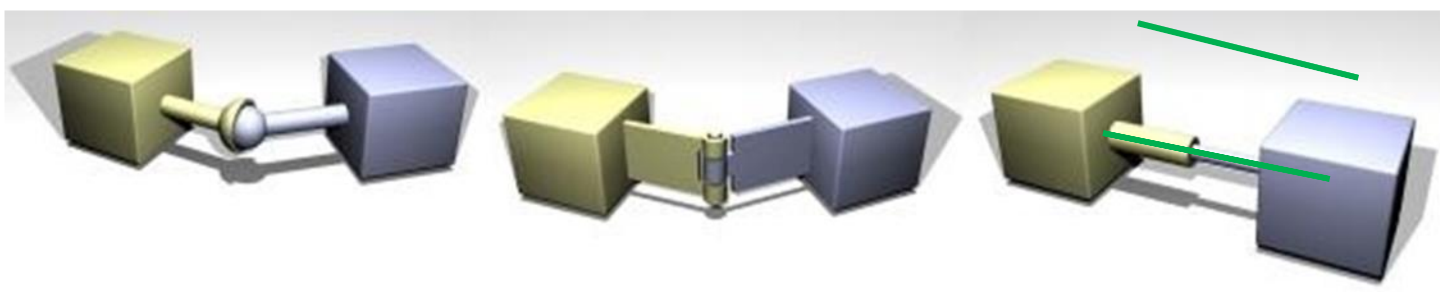
Robot Components



- Body
- End Effectors
- Actuators
- Sensors
- Controller
- Software

Robot: Body

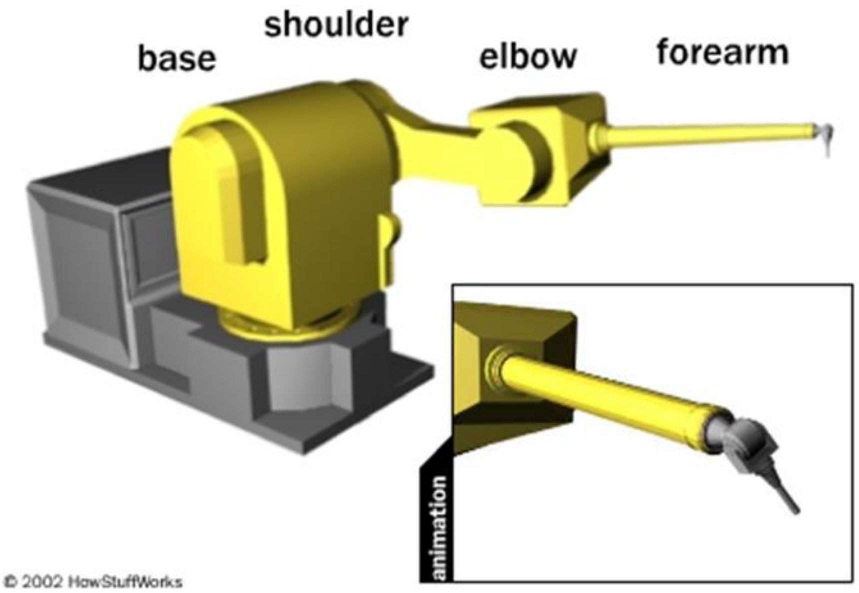
- Consists of links and joints
- A link is a part, a shape with physical properties.
- A joint is a constraint on the spatial relations of two or more links.
- These are just a few examples…



Ball joint Revolute (hinge) joint Prismatic (slider) joint

Degrees of Freedom

- Joints constraint free movement, measured in “Degrees of Freedom” (DOFs).
- Joints reduce the number of DOFs by constraining some translations or rotations.
- Robots classified by total number of DOFs



How many DOFs can you identify in your arm?

Robot: End Effectors

- Component to accomplish some desired physical function
- Examples:
  - ✓ Hands
  - ✓ Torch
  - ✓ Wheels
  - ✓ Legs

Robot: Actuators

- Actuators are the “muscles” of the robot.

- These can be electric motors, hydraulic systems, pneumatic systems, or any other system that can apply forces to the system.

Robot: Sensors

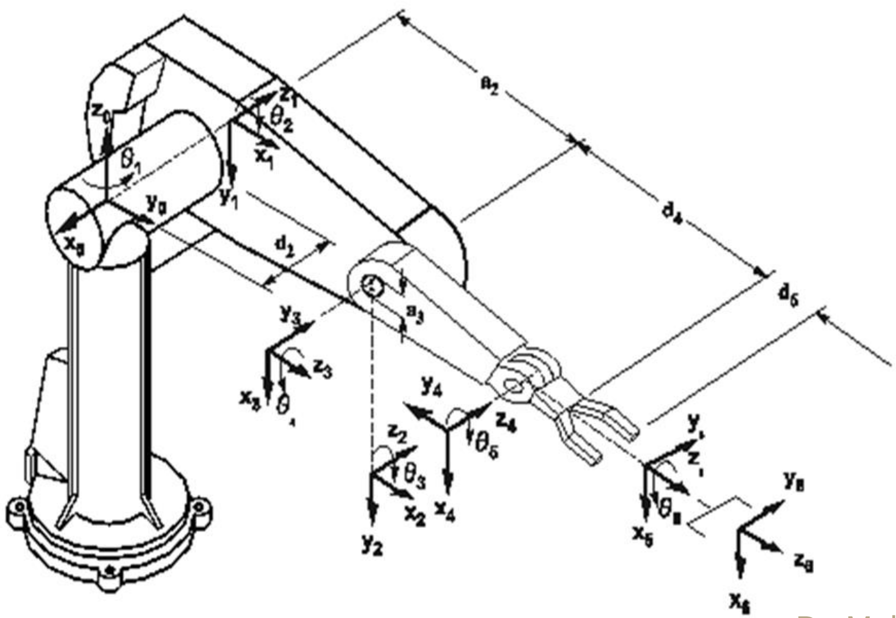
- Rotation encoders
- Cameras
- Pressure sensors
- Limit switches
- Optical sensors
- Sonar

Kinematics

- Kinematics is the study of motion without regard for the forces that cause it.
- It refers to all time-based and geometrical properties of motion.
- It ignores concepts such as torque, force, mass, energy, and inertia.

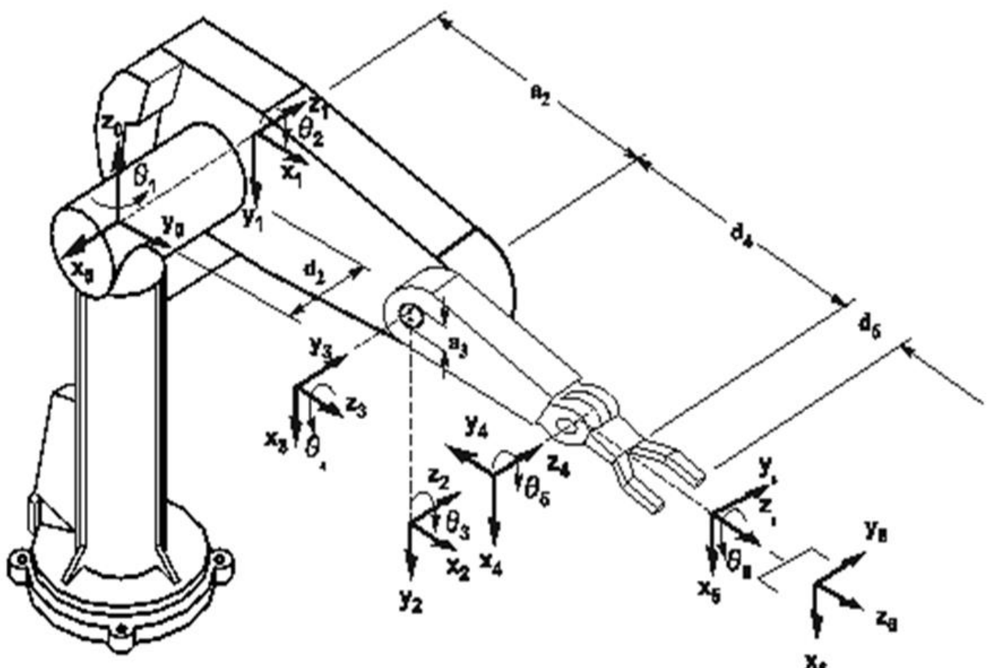
Forward Kinematics

- For a robotic arm, this would mean calculating the position and orientation of the end effector given all the joint variables.

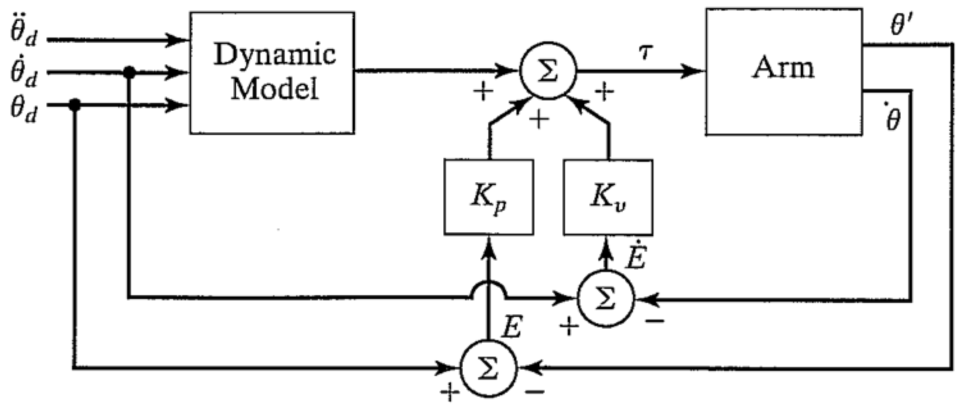


Inverse Kinematics

- Inverse Kinematics is the reverse of Forward Kinematics.
- It is the calculation of joint values given the positions, orientations, and geometries of mechanism's parts.
- It is useful for planning how to move a robot in a certain way.



Dynamics



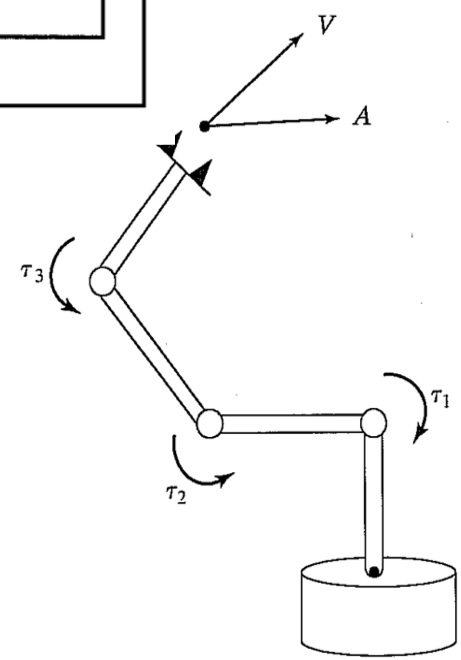
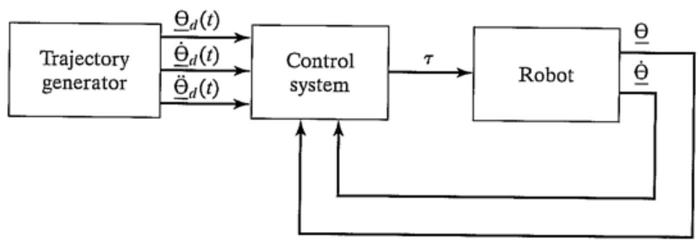


FIGURE 1.10:The relationship between the torques applied by the actuators and the resulting motion of the manipulator is embodied in the dynamic

Trajectory generating



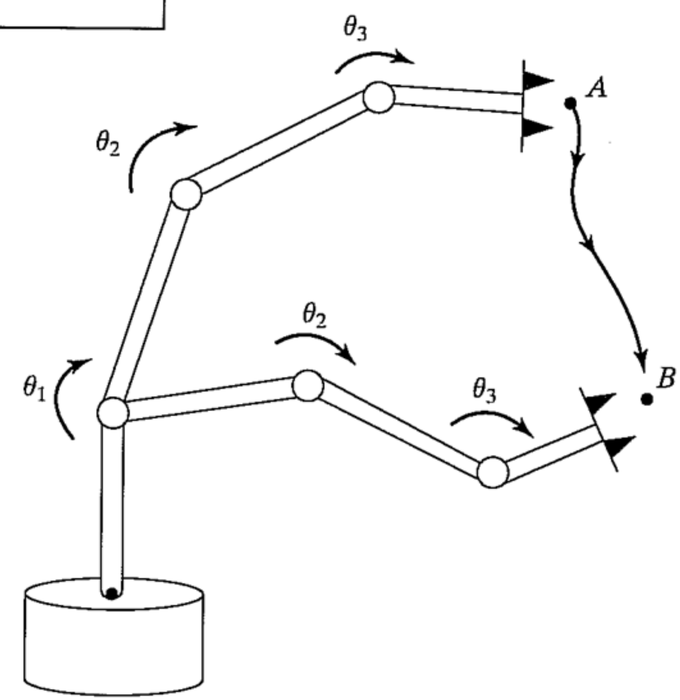


FIGURE 1.11:In order to move the end-effector through space from point A to point B , we must compute a trajectory for each joint to follow.

Position Control

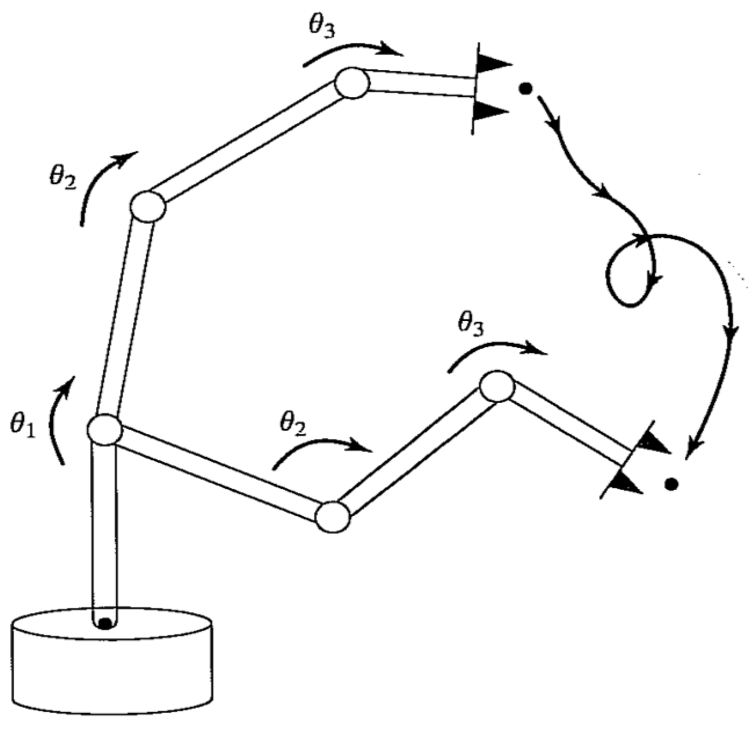


FIGURE 1.13: In order to cause the manipulator to follow the desired trajectory , a position-control system must be implemented. Such a system uses feedback from joint sensors to keep the manipulator on course.

Force Control

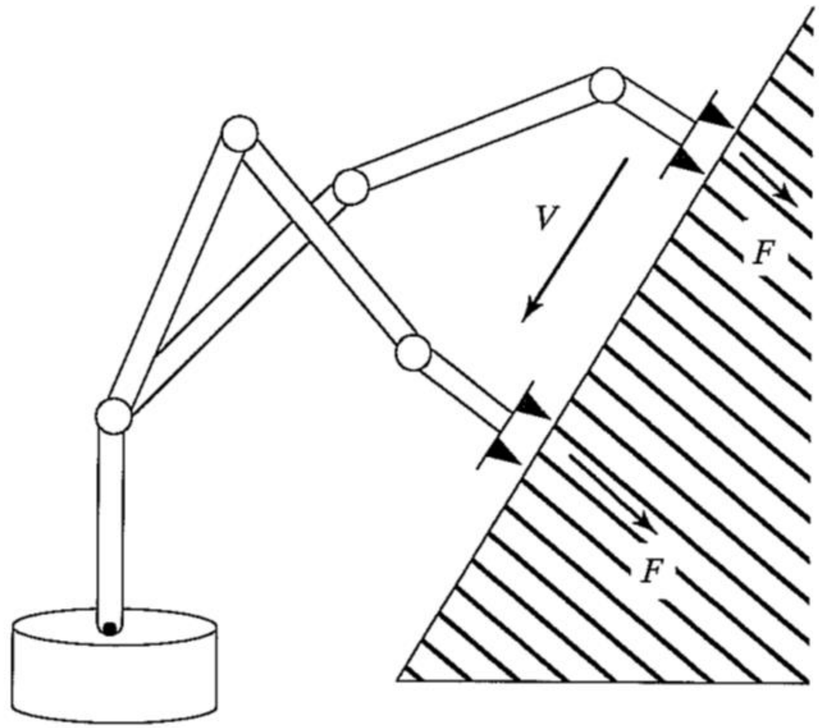
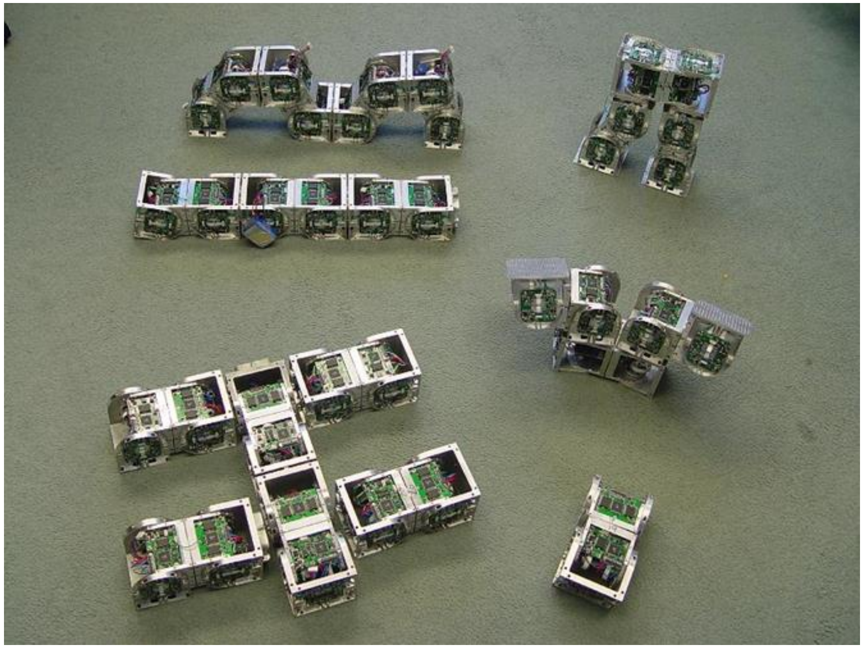


FIGURE 1.14: In order for a manipulator to slide across a surface while applying a constant force , a hybrid position-force control system must be used.

New direction

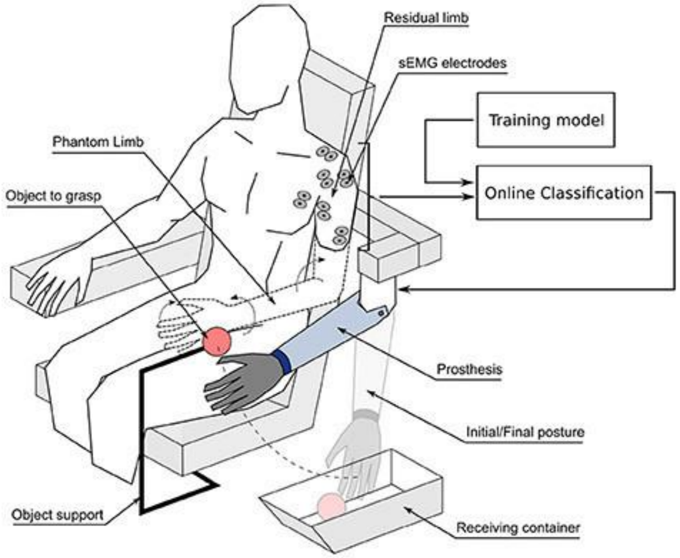
- Nanobots

Powered exoskeleton



- Reconfigurable Robot

Powered exoskeleton (Robotic suit) and robotics prosthetic limb



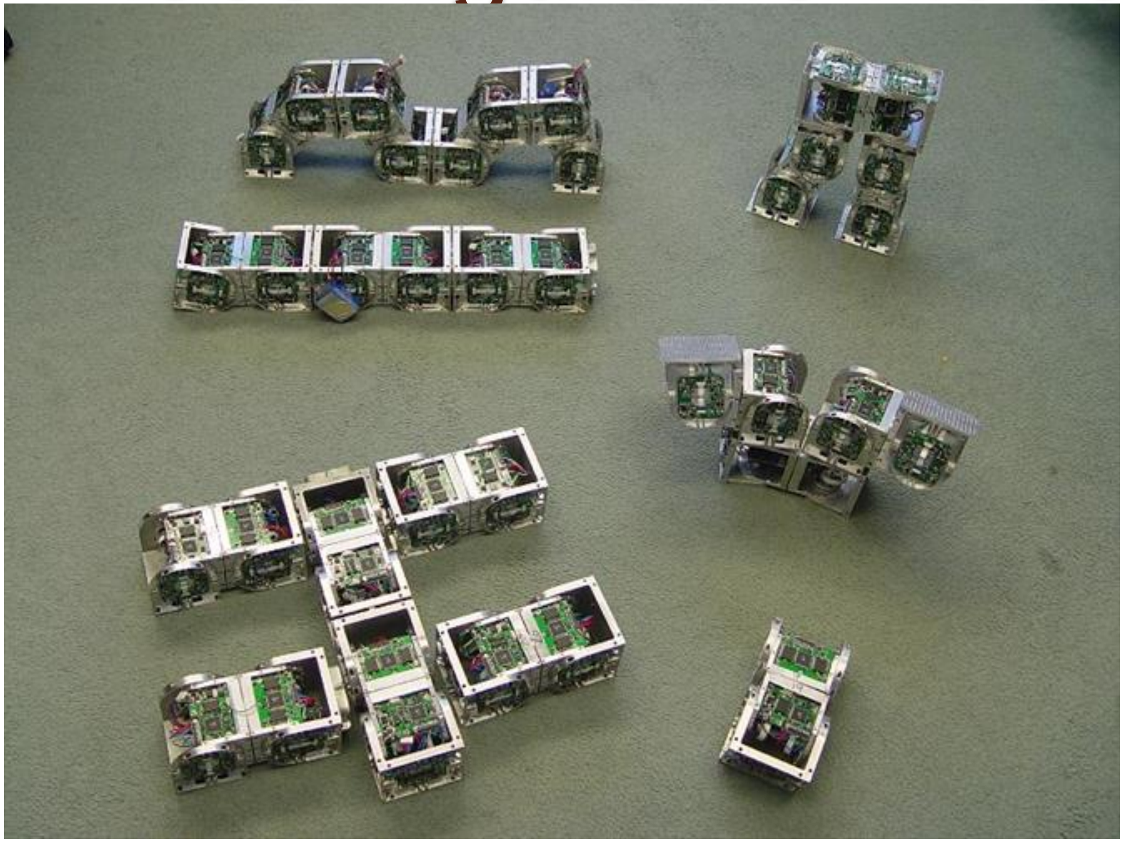
https://www.weforum.org/agenda/2018/12/a-new-prosthetic-arm-takes-the-place-of-a-phantom-limb

hybrid assistive limb
https://youtu.be/RCWw6LSuRCo?t=17

- # Nanobots

https://edition.cnn.com/videos/tv/2015/01/29/spc-make-create-innovate-nanobots.cnn

• Reconfigurable Robot



https://www.wevolver.com/wevolver.staff/superbot/

Chapter 2
Spatial descriptions

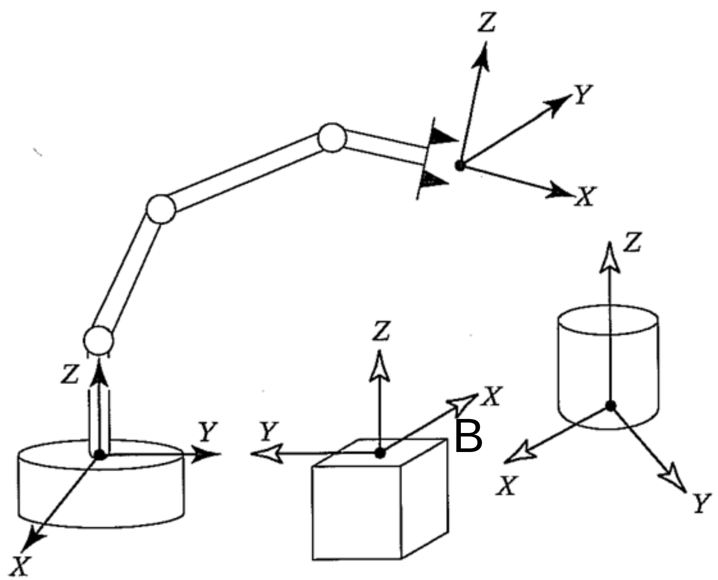
2.2 <u>DESCRIPTIONS</u> : POSITIONS,ORIENTATIONS,AND FRAMES

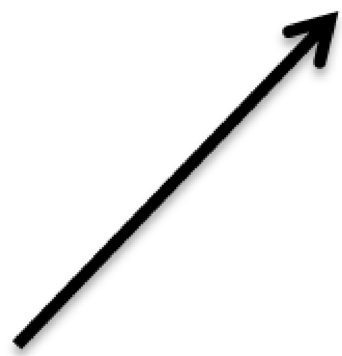
2.3 <u>MAPPINGS</u> :
CHANGING DESCRIPTION FROM FRAME TO FRAME

2.4 <u>OPERATORS</u> :
TRANSLATIONS ,ROTATIONS, AND TRANSFORMATIONS

Introduction:

We are constantly concerned with the location of objects in three-dimensional space. These objects are the links of the manipulator, the parts and tools with which it deals, and other objects in the manipulator's environment.





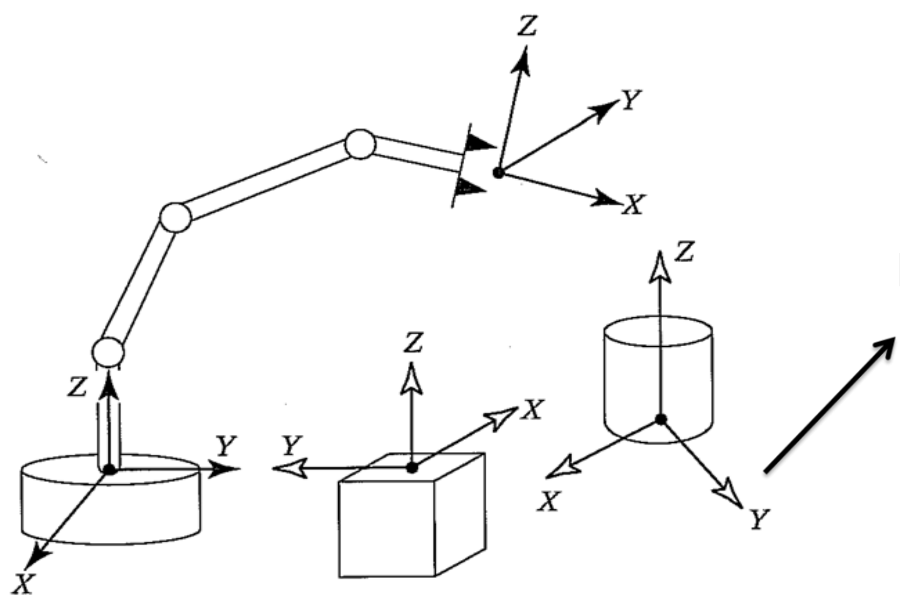
Position and orientation

FIGURE 1.5: Coordinate systems or “frames” are attached to the manipulator and to objects in the environment.

https://youtu.be/vxEU97BKHbk

Introduction: cont.

In order to describe the position and orientation of a body in space, we will always attach a coordinate system, or **frame**, rigidly to the object. We then proceed to describe the position and orientation of this frame with respect to some reference coordinate system. (See Fig. 1.5.)

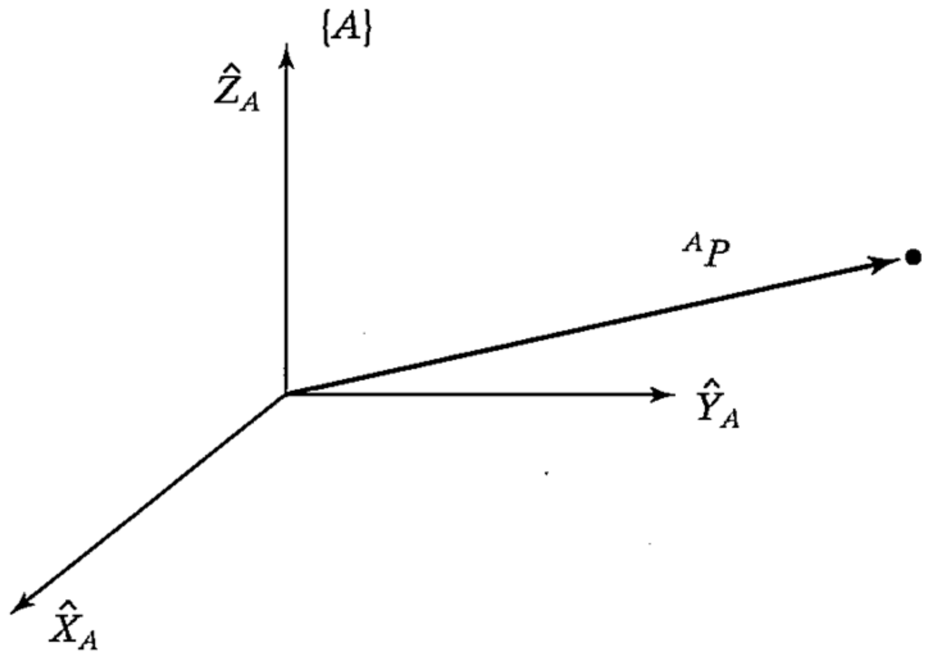


Position and orientation

FIGURE 1.5: Coordinate systems or "frames" are attached to the manipulator and to objects in the environment.

Description of a position

Once a coordinate system is established, we can locate any point in the universe with a 3 × 1 position vector. Because we will often define many coordinate systems, the vector will have the name of the coordinate



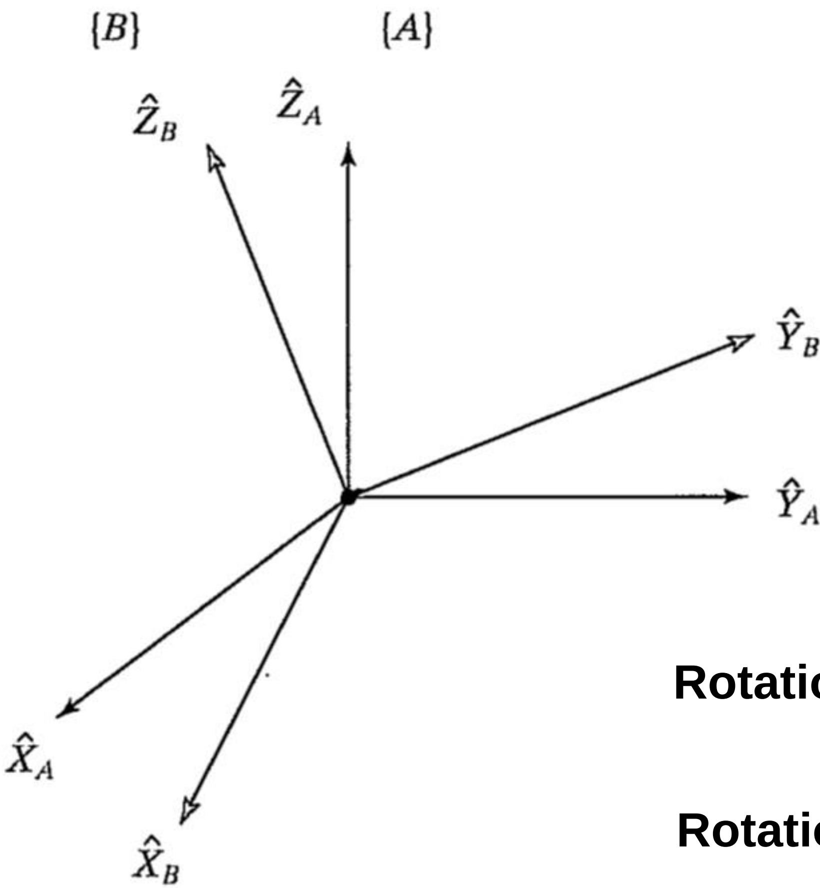
$$|\hat{x}| = 1,$$
$$|\hat{y}| = 1,$$
$$|\hat{z}| = 1,$$

$$A_P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

FIGURE 2.1: Vector relative to frame (example).

Description of an orientation

Often, we will find it necessary not only to represent a point in space but also to describe the **orientation** of a body in space .



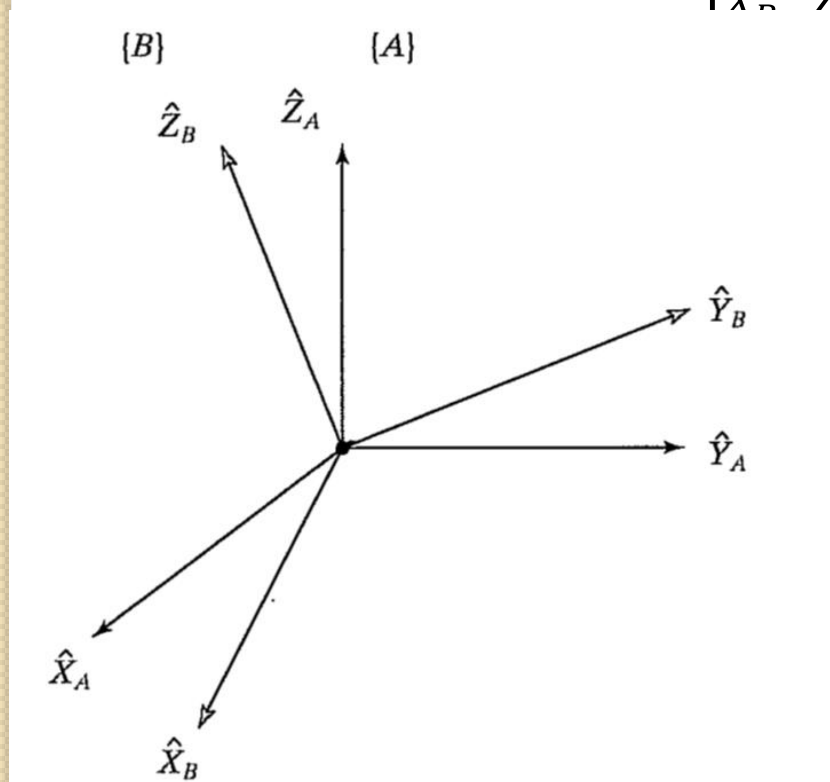
$$ {}^{A}_{B}R = \left[\widehat{{}^{A}X_B} \; \widehat{{}^{A}Y_B} \; \widehat{{}^{A}Z_B} \right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

Rotation matrix : {B} relative to { A}

Rotation matrix : {B} with respect to (w.r.t) A}

We can give expressions for the scalars r_{ij} in (2.2) by nothing that the components of any vector are simply the projections of that vector on to the unit directions of its reference frame. Hence , each component of ${}^{A}_{B}R$ in (2.2) can be written as the dot product of a pair of unit vectors :

$$ {}^{A}_{B}R = \left[\widehat{{}^{A}X}_B \quad \widehat{{}^{A}Y}_B \quad \widehat{{}^{A}Z}_B \right] = \begin{bmatrix} \hat{X}_B . \hat{X}_A & \hat{Y}_B . \hat{X}_A & \hat{Z}_B . \hat{X}_A \\ \hat{X}_B . \hat{Y}_A & \hat{Y}_B . \hat{Y}_A & \hat{Z}_B . \hat{Y}_A \\ \hat{X}_B . \hat{Z}_A & \hat{Y}_B . \hat{Z}_A & \hat{Z}_B . \hat{Z}_A \end{bmatrix} \quad (2.3) $$



Note: dot product for vector

Find ${}^{B}_{A}R$ given ${}^{A}_{B}R$?

$$ {}^{B}_{A}R = \left[\widehat{{}^{B}X}_A \;\; \widehat{{}^{B}Y}_A \; \widehat{{}^{B}Z}_A \right] $$

Or

Hence, ${}^{B}_{A}R$, the description of frame {A} relative to {B}, is given by the transpose of ${}^{A}_{B}R$; that is,

$$ {}^{B}_{A}R = {}^{A}_{B}R^{T}. $$

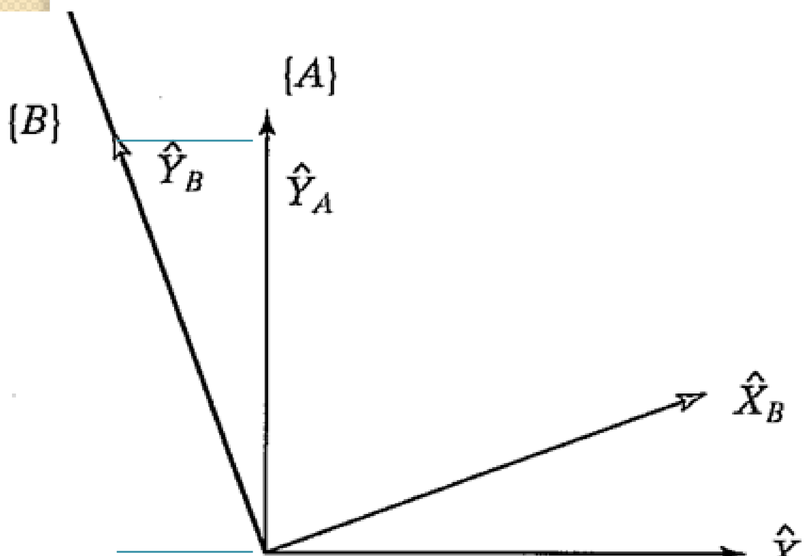
This suggests that the inverse of a rotation matrix is equal to its transpose:

$$ {}^{B}_{A}R = {}^{A}_{B}R^{T} = {}^{A}_{B}R^{-1} $$

$$ {}^{A}_{B}R^{T}\,{}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B^T \\ {}^{A}\hat{Y}_B^T \\ {}^{A}\hat{Z}_B^T \end{bmatrix} \left[{}^{A}\hat{X}_B \;\; {}^{A}\hat{Y}_B \;\; {}^{A}\hat{Z}_B \right] = I_3, $$

EXAMPLE 2.1

Figure 2.6 shows a frame {B} that is rotated relative to frame {A} about $\hat{z}$ by θ =30 degrees. Here , $\hat{z}$ is pointing out of the page . Find the rotation matrix {B} w.r.t {A}



$$ {}^{A}_{B}R = R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

$$ {}^{B}_{A}R = {}^{A}_{B}R^{T} = \begin{bmatrix} 0.866 & 0.500 & 0.000 \\ -0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

FIGURE 2.6: {B} rotated 30 degrees about $\hat{z}$.

APPENDIX A

Formulas for rotation about the principle axes by :

$$R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},$$

$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Mappings involving general frames

Very often, we know the description of a vector with respect to some frame {B},and we would like to know its description with respect to another frame,{A}.

We now consider the general case of mapping. Here, the origin of frame {B} is not coincident with that of frame {A} but has a general vector offset. The vector that locates {B}'s origin is called ${}^{A}P_{BORG}$. Also {B} is rotated with respect to {A}, as described by ${}^{A}_{B}R$.

Given ${}^{B}P$, we wish to compute ${}^{A}P$, as in Fig. 2.7.

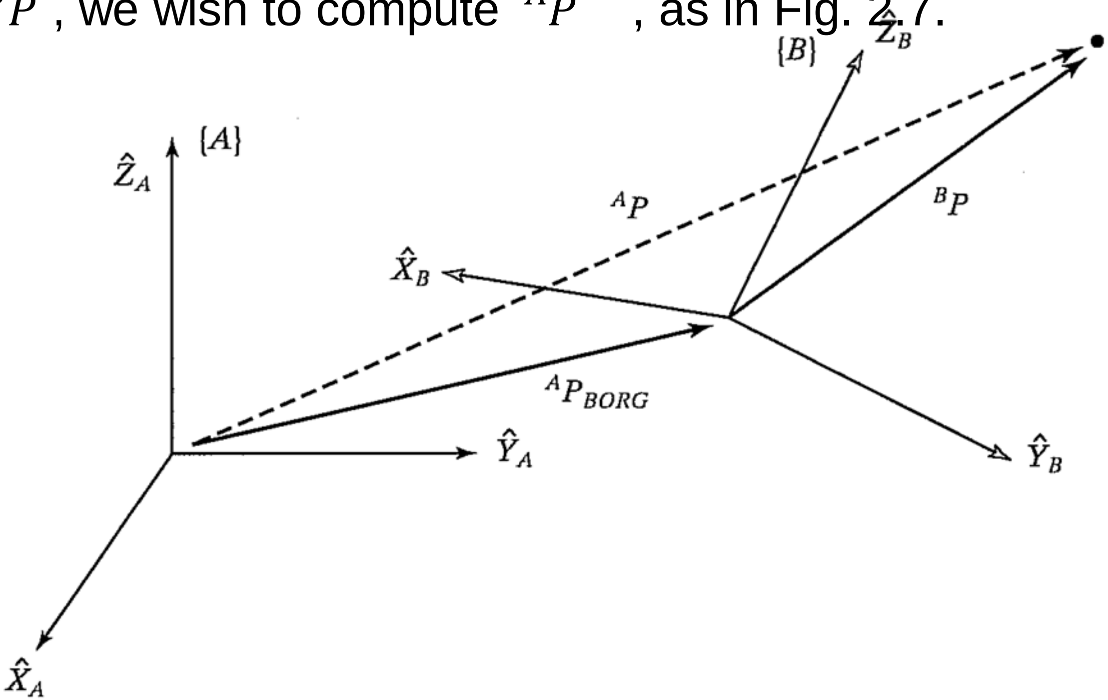
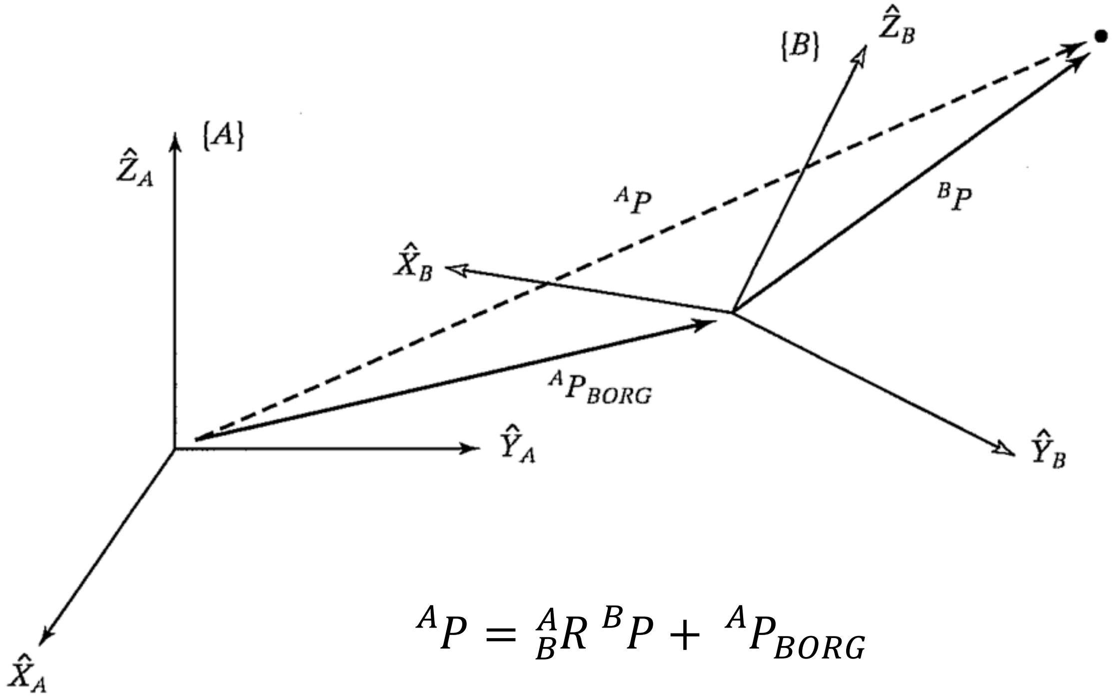


FIGURE 2.7 : General transform of a vector .



$$^AP = {}^A_BR\, {}^BP + {}^AP_{BORG}$$

Homogeneous transform

$$^AP = {}^A_BT\, {}^BP$$

$$\begin{bmatrix} {}^AP \\ 1 \end{bmatrix} = \begin{bmatrix} {}^A_BR & {}^AP_{BORG} \\ 0\ 0\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^BP \\ 1 \end{bmatrix}$$

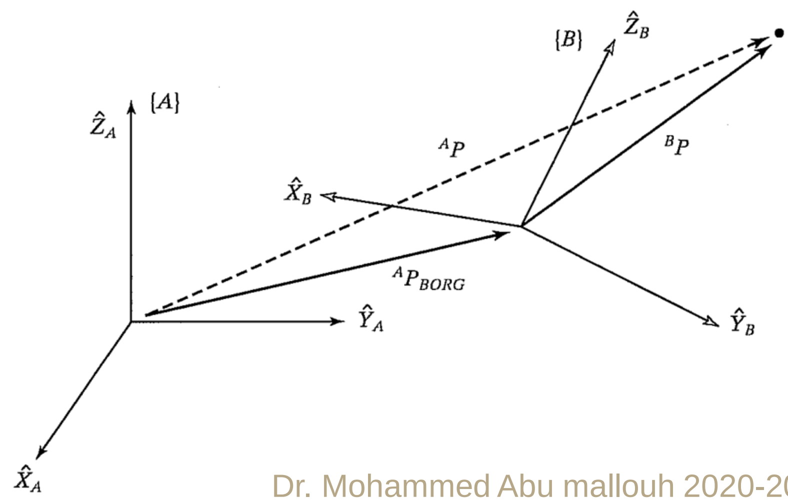
EXAMPLE 2.2

Figure shows a frame {B}, which is rotated relative to frame {A} about $\hat{Z}$ by 30 degree, and translated 10 units in $\hat{X}_A$, and translated 5 units in $\hat{Y}_A$. Find ${}^{A}P$, where ${}^{B}P = [3.0\ 7.0\ 0.0]^T$.

$$ {}^{A}_{B}R = R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

$$ {}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{BORG} \\ 0\ 0\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}P = {}^{A}_{B}T\, {}^{B}P = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix} $$



EXAMPLE 2.5

Figure 2.13 shows a frame {B} that is rotated relative to frame {A} about Z by 30 degrees and translated four units in $\hat{X}_A$ and three units in $\hat{Y}_A$. Thus, we have a description of ${}^{A}_{B}T$.Find ${}^{B}_{A}T$?

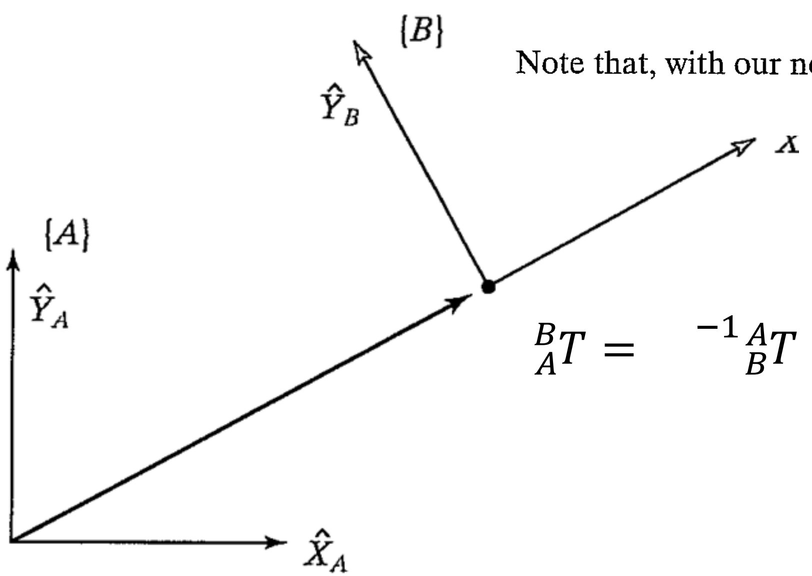
$$ {}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 4 \\ 0.5 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.68) $$

Using (2.42) and (2.44), we can write the form of ${}^{B}_{A}T$ as

$$ {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T}\,{}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right]. $$

Note that, with our notation,

$$ {}^{B}_{A}T = {}^{A}_{B}T^{-1}. $$

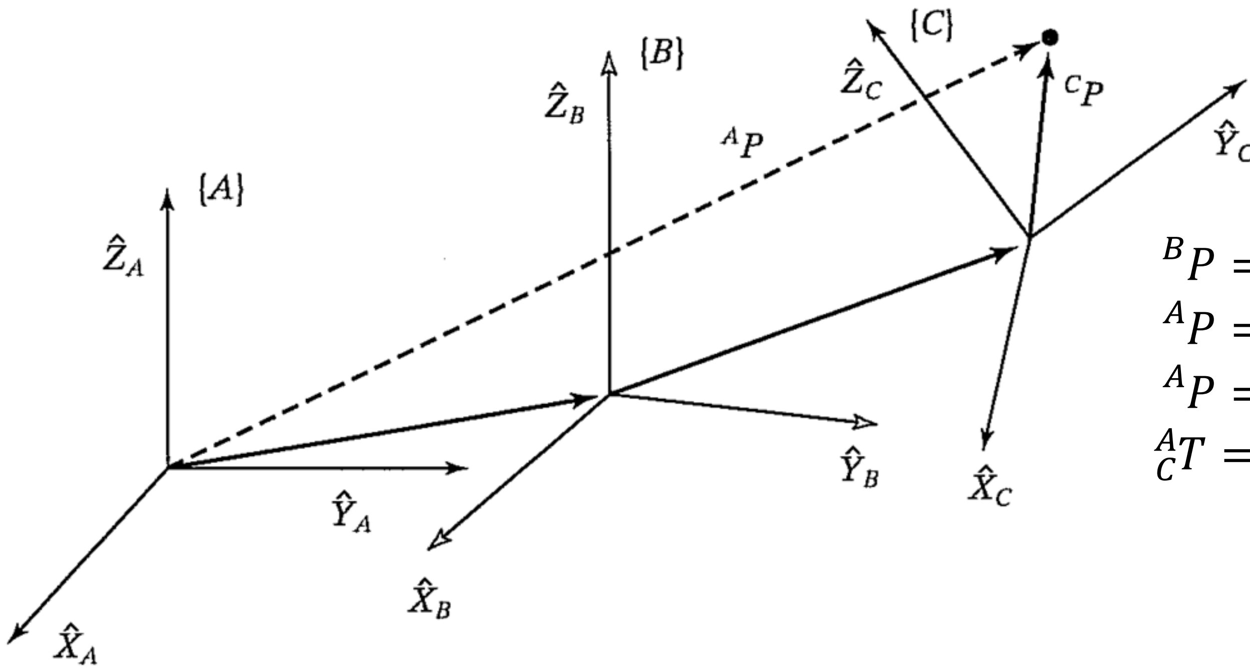


$$ {}^{B}_{A}T = {}^{-1}\,{}^{A}_{B}T = \begin{bmatrix} 0.866 & 0.500 & 0 & -4.964 \\ -0.5 & 0.866 & 0 & -0.598 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

FIGURE 2.13 : {B} relative to {A} .

2.6 Compound transformations

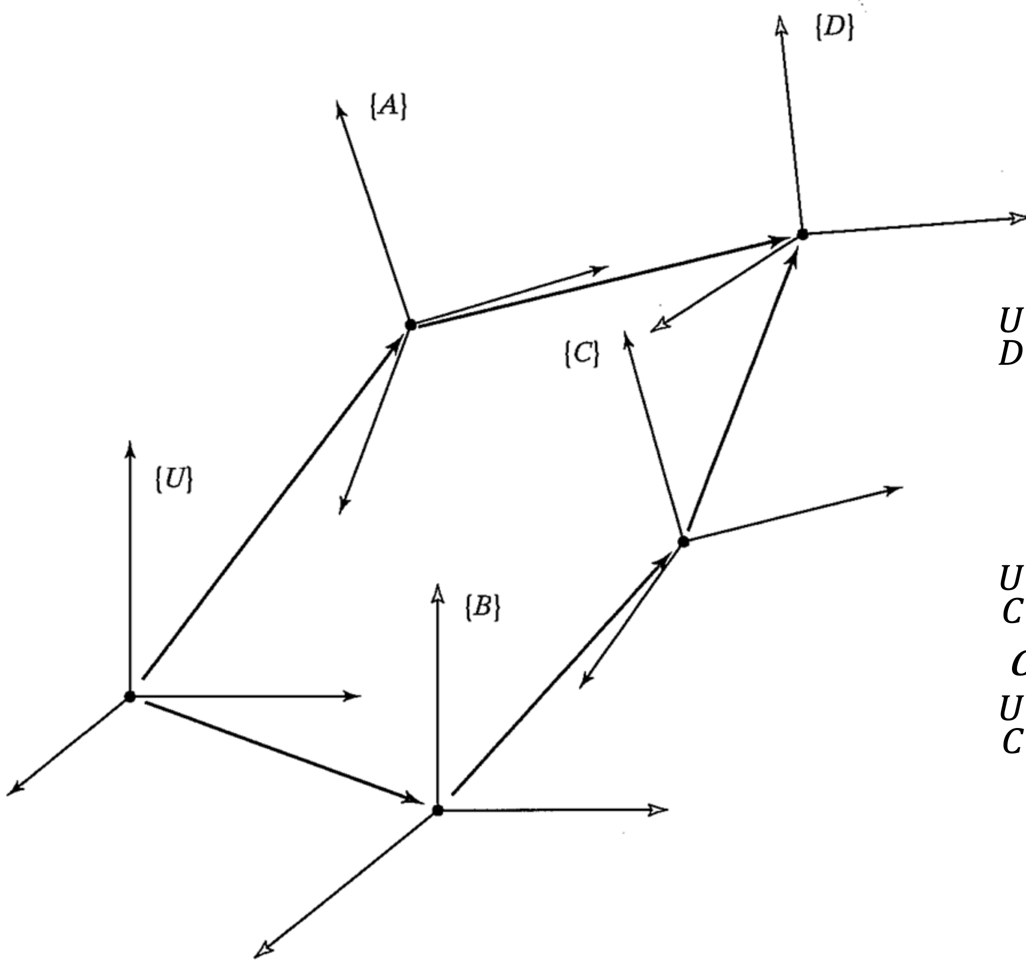
we have ${}^{C}P$ and wish to find ${}^{A}P$.



$$
\begin{aligned}
{}^{B}P &= {}^{B}_{C}T\,{}^{C}P \\
{}^{A}P &= {}^{A}_{B}T\,{}^{B}P \\
{}^{A}P &= {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P \\
{}^{A}_{C}T &= {}^{A}_{B}T\,{}^{B}_{C}T.
\end{aligned}
$$

FIGURE 2.12 : Compound frames: Each is known relative to previous one .

2.7 TRANSFORM EQUATIONS



$$^{U}_{D}T = {}^{U}_{A}T\, {}^{A}_{D}T, or = {}^{U}_{B}T\, {}^{B}_{C}T\, {}^{C}_{D}T.$$

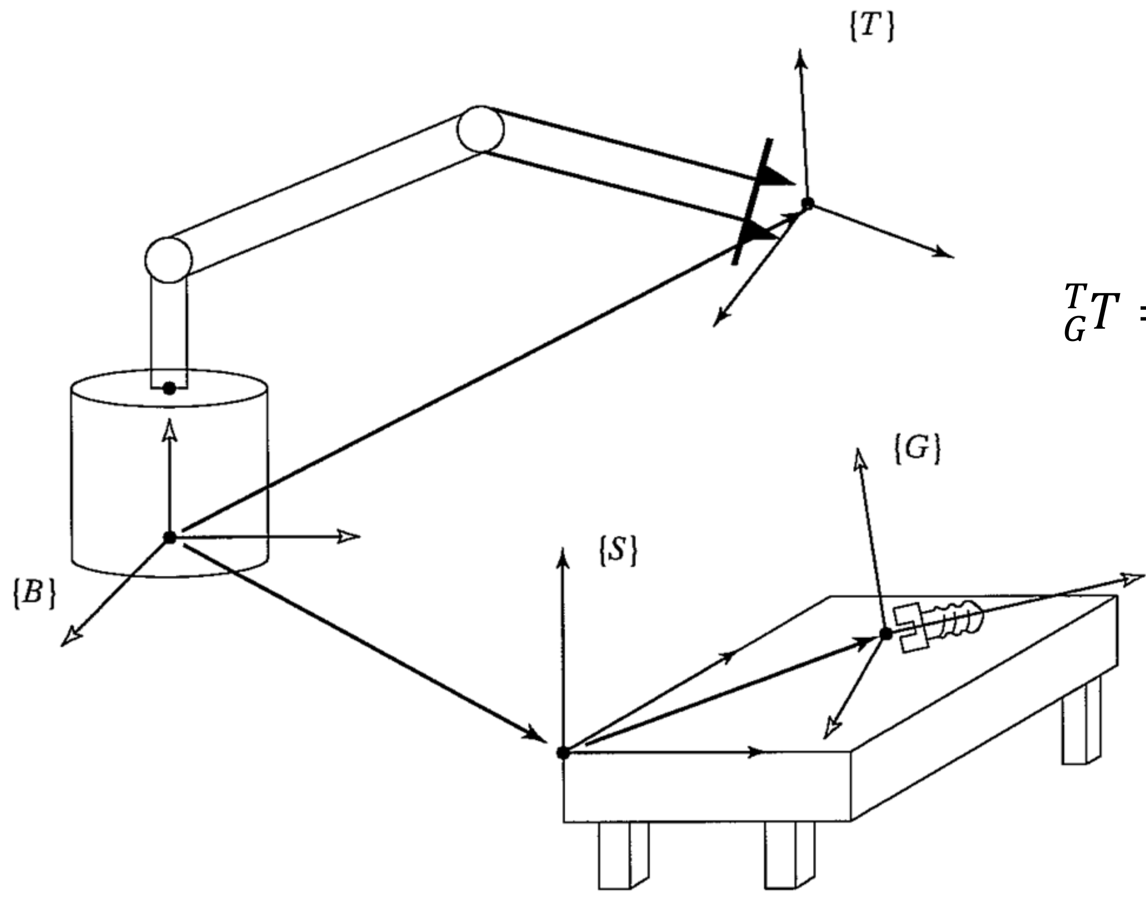
$$^{U}_{C}T = {}^{U}_{A}T\, {}^{A}_{D}T\, {}^{D}_{C}T = {}^{U}_{A}T\, {}^{A}_{D}T\, {}^{C}_{D}T^{-1}$$

or

$$^{U}_{C}T = {}^{U}_{B}T\, {}^{B}_{C}T.$$

FIGURE 2.14: Set of transforms forming a loop.

2.7 TRANSFORM EQUATIONS



$$ {}^{T}_{G}T = {}^{T}_{B}T\,{}^{B}_{S}T\,{}^{S}_{G}T = {}^{B}_{T}T^{-1}\,{}^{B}_{S}T\,{}^{S}_{G}T. $$

FIGURE 2.16 : Manipulator reaching for a bolt .

2.8 MORE ON REPRESENTATION OF ORIANTATION

Rotation matrix determinant is +1

$$R = [\hat{X}\ \hat{Y}\ \hat{Z}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$|\hat{X}| = 1,$$
$$|\hat{Y}| = 1,$$
$$|\hat{Z}| = 1,$$
$$\hat{X} . \hat{Y} = 0,$$
$$\hat{X} . \hat{Z} = 0,$$
$$\hat{Y} . \hat{Z} = 0.$$

- Clearly, the nine elements of a rotation matrix are not all independent .
- In fact, given a rotation matrix, R , it is easy to write down the six dependencies between the elements.
- Therefore, rotation matrix can be specified by just three parameters.

2.8 MORE ON REPRESENTATION OF ORIANTATION

- Rotation matrices are useful as operators (computer). Their matrix from is such that, when multiplied by a vector , they perform the rotation operation.

- Human operator at a computer terminal who wishes to type in the specification of the desired orientation of a robot’s hand would have a hard time inputting a nine-element matrix with orthonormal columns. A representation that requires only three numbers would be simpler .

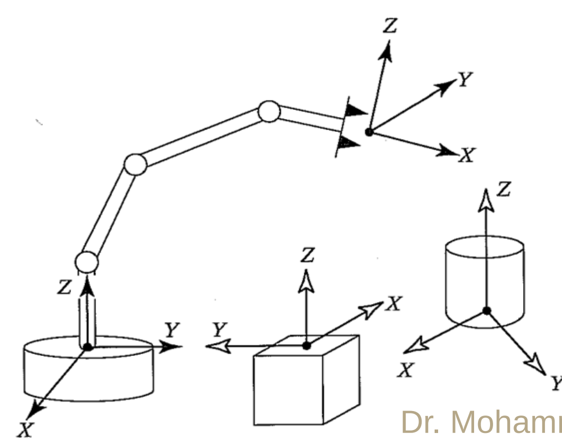
Fixed angles Orientation representation

X–Y–Z fixed angles

One method of describing the orientation of a frame $\{B\}$ is as follows:

> Start with the frame coincident with a known reference frame $\{A\}$. Rotate $\{B\}$ first about $\hat{X}_A$ by an angle γ, then about $\hat{Y}_A$ by an angle β, and, finally, about $\hat{Z}_A$ by an angle α.

Each of the three rotations takes place about an axis in the fixed reference frame $\{A\}$. We will call this convention for specifying an orientation **X–Y–Z fixed angles**. The word "fixed" refers to the fact that the rotations are specified about the fixed (i.e., nonmoving) reference frame (Fig. 2.17). Sometimes this convention is referred to as **roll, pitch, yaw angles**, but care must be used, as this name is often given to other related but different conventions.



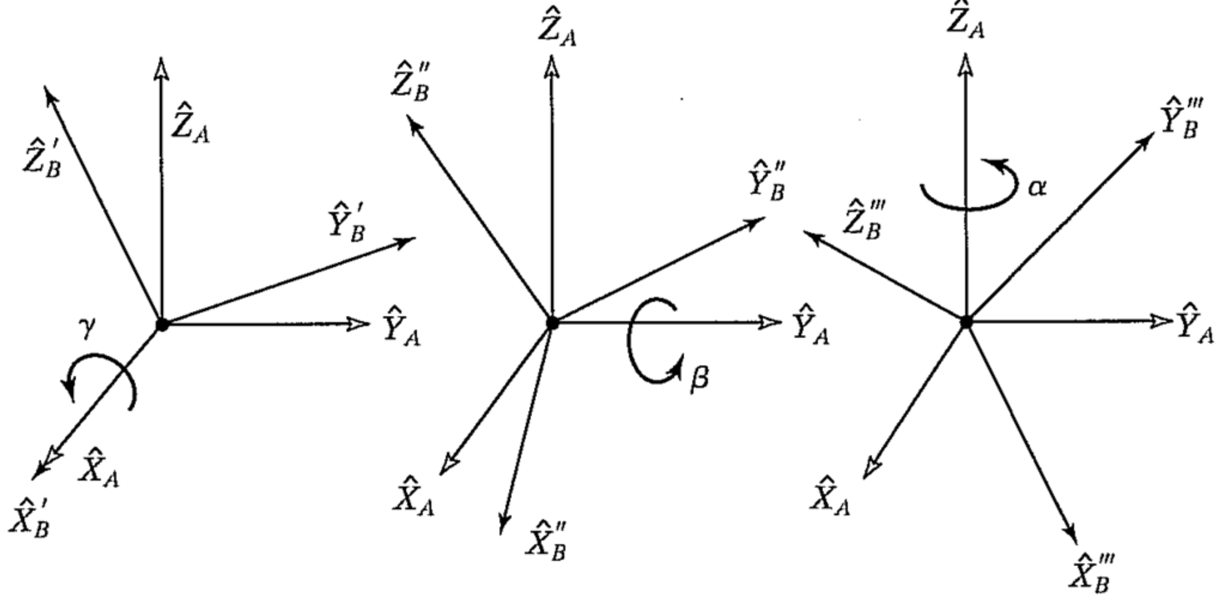


FIGURE 2.17:X-Y-Z fixed angles. Rotations are performed in the order

$$ {}^{A}_{B}\boldsymbol{R}_{\boldsymbol{XYZ}}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}. $$

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. $$

$$ \beta = \mathrm{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}), $$

$$ \alpha = \mathrm{Atan2}(r_{21}/c\beta, r_{11}/c\beta), $$

$$ \gamma = \mathrm{Atan2}(r_{32}/c\beta, r_{33}/c\beta), $$

Note: use handout

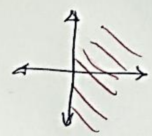
$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} C\alpha C\beta & C\alpha S\beta S\gamma - S\alpha C\gamma & C\alpha S\beta C\gamma + S\alpha S\gamma \\ S\alpha C\beta & S\alpha S\beta S\gamma + C\alpha C\gamma & S\alpha S\beta C\gamma - C\alpha S\gamma \\ -S\beta & C\beta S\gamma & C\beta C\gamma \end{bmatrix} $$

find γ, β, α given ${}^{A}_{B}R_{XYZ} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

$$ r_{11}^2 + r_{21}^2 = C_\alpha^2 C_\beta^2 + S_\alpha^2 C_\beta^2 $$
$$ = C_\beta^2 (C_\alpha^2 + S_\alpha^2) = C_\beta^2 \Rightarrow C\beta = \mp\sqrt{r_{11}^2 + r_{21}^2} $$

$C\beta = \sqrt{r_{11}^2 + r_{21}^2}$

$270 < \beta < 90$



$\Rightarrow \beta = \text{Atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$

Where: $\text{Atan2}(S, C) \approx \tan^{-1}\left(\frac{S}{C}\right)$
sing is important

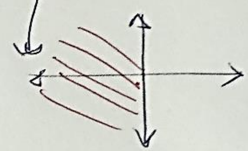
Ex: $\text{Atan2}(0.7, 0.7) = 45$
$\text{Atan2}(-0.7, -0.7) = 225$

$\Rightarrow \alpha = \text{Atan2}\left(\frac{r_{21}}{C\beta}, \frac{r_{11}}{C\beta}\right)$

$\Rightarrow \gamma = \text{Atan2}\left(\frac{r_{32}}{C\beta}, \frac{r_{33}}{C\beta}\right)$

$C\beta = -\sqrt{r_{11}^2 + r_{21}^2}$

$90 < \beta < 270$



$\Rightarrow \beta = \text{Atan2}\left(-r_{31}, -\sqrt{r_{11}^2 + r_{21}^2}\right)$

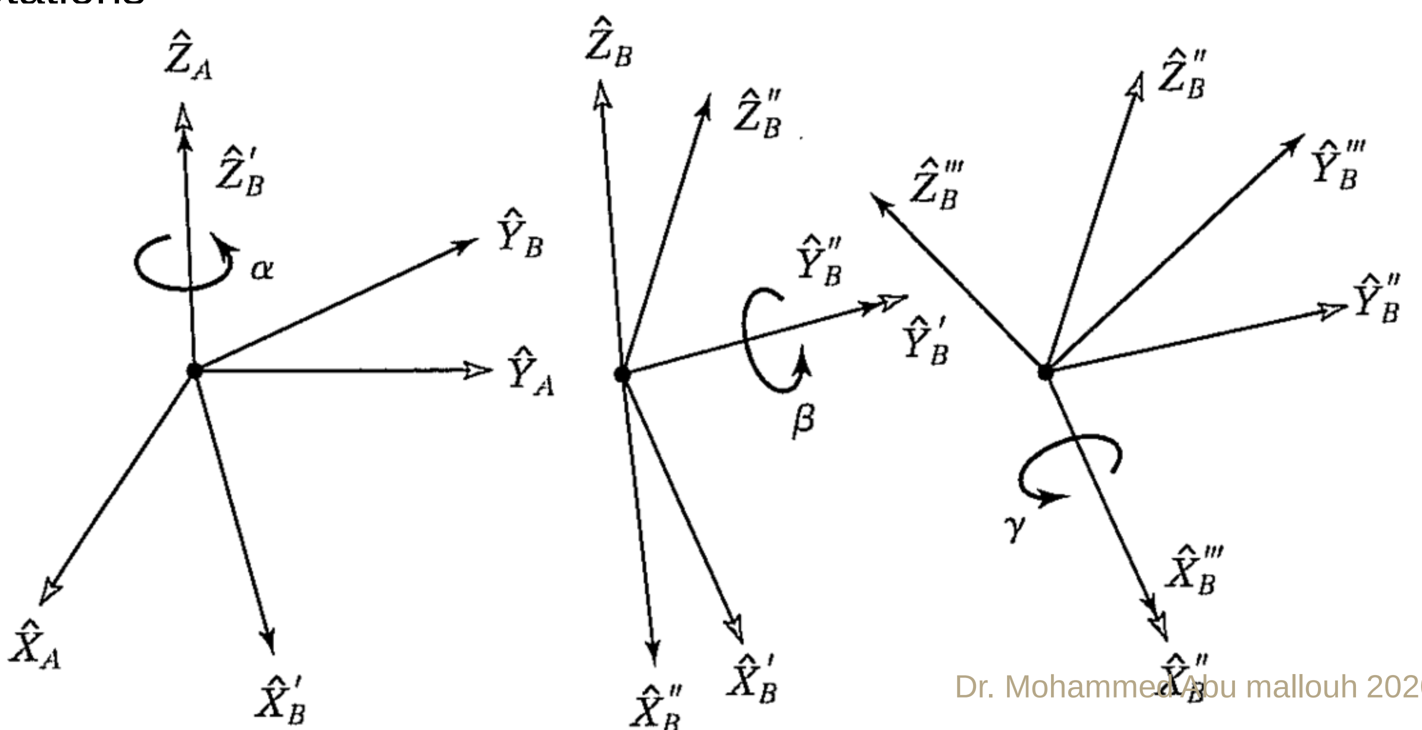
$\Rightarrow \alpha = \text{Atan2}\left(\frac{r_{21}}{C\beta}, \frac{r_{11}}{C\beta}\right)$

$\Rightarrow \gamma = \text{Atan2}\left(\frac{r_{32}}{C\beta}, \frac{r_{33}}{C\beta}\right)$

Z-Y-X Euler angles (current angles)

Another possible description of a frame {B} is as follows:
Start with the frame coincident with a known frame {A}. Rotate {B} first about $\hat{Z}_B$ by an angle α ,then about $\hat{Y}_B$ by an angle β, and , finally , about $\hat{X}_B$ by an angle γ.

In this representation ,each rotation is performed about an axis of the moving system {B} rather than one of the fixed reference {A}.Such sets of three rotations



$$ {}^{A}_{B}\boldsymbol{R}_{\acute{Z}\acute{Y}\acute{X}}(\alpha,\beta,\gamma\ ) = R_Z(\alpha), R_Y(\beta)R_X(\gamma) $$

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

Z-Y-X Euler angles

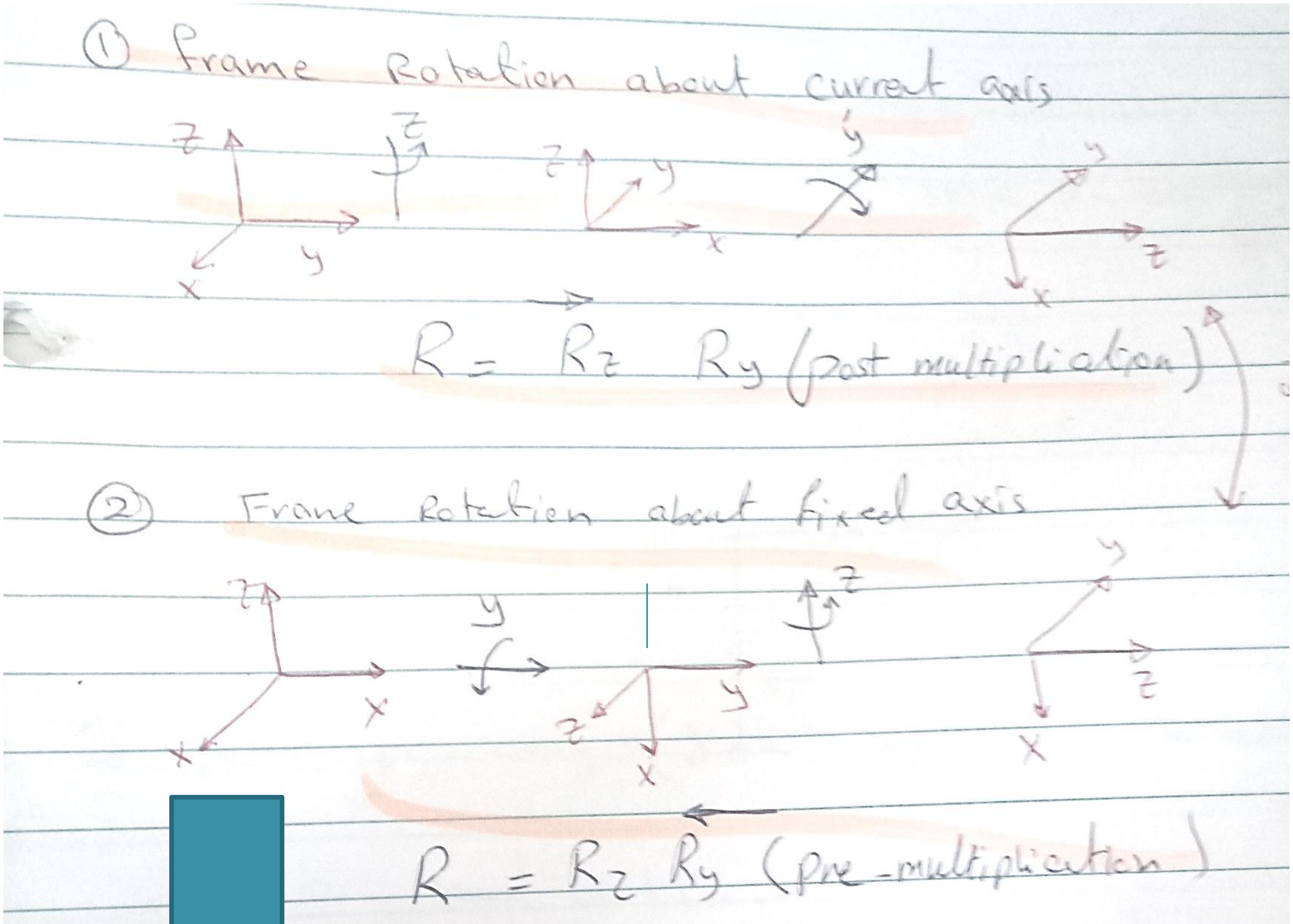
Equivalent ….so same solution for α,β,γ

$$ {}^{A}_{B}\boldsymbol{R}_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha), R_Y(\beta)R_X(\gamma) $$

X-Y-Z Fixed angles

$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

EXAMPLE


① frame Rotation about current axis
R = Rz Ry (post multiplication)
② Frame Rotation about fixed axis
R = Rz Ry (pre-multiplication)

EXAMPLE 2.7

Consider two rotations , one about $\hat{Z}$ by 30 degrees and one about $\hat{X}$ by 30 degrees

$$R_Z(30) = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \quad (2.60)$$

$$R_X(30) = \begin{bmatrix} 1.000 & 0.000 & 0.000 \\ 0.000 & 0.866 & -0.500 \\ 0.000 & 0.500 & 0.866 \end{bmatrix} \quad (2.61)$$

$$R_Z(30)R_X(30) = \begin{bmatrix} 0.87 & -0.43 & 0.25 \\ 0.50 & 0.75 & -0.43 \\ 0.00 & 0.50 & 0.87 \end{bmatrix}$$

APPENDIX B
The 24 angle-set conventions

The 12 Euler angle sets

The 12 fixed angle sets

2.27 [15] Referring to Fig. 2.25, give the value of ${}^{A}_{B}T$

2.28 [15] Referring to Fig. 2.25, give the value of ${}^{A}_{C}T$

2.29 [15] Referring to Fig. 2.25, give the value of ${}^{B}_{C}T$

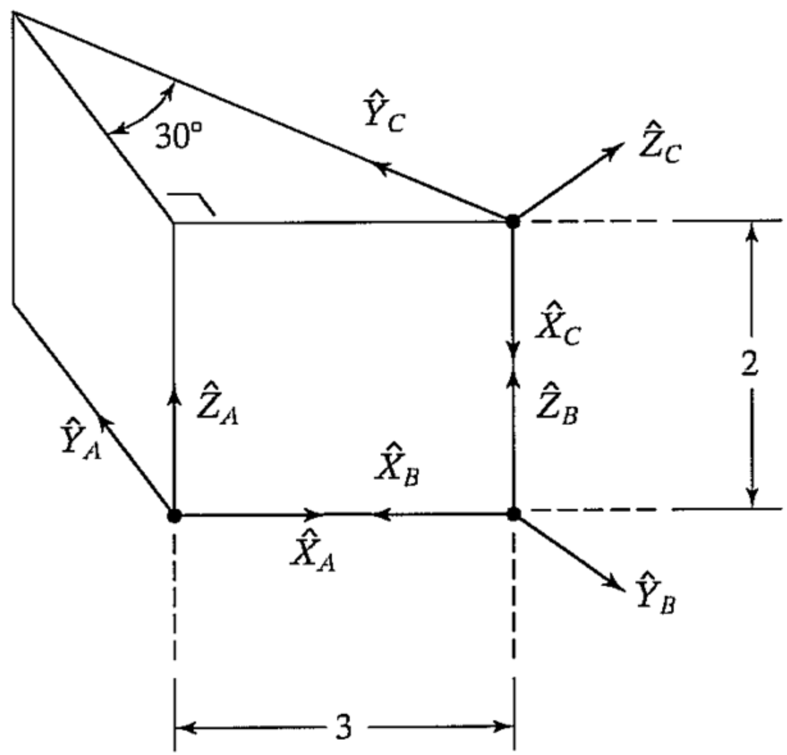
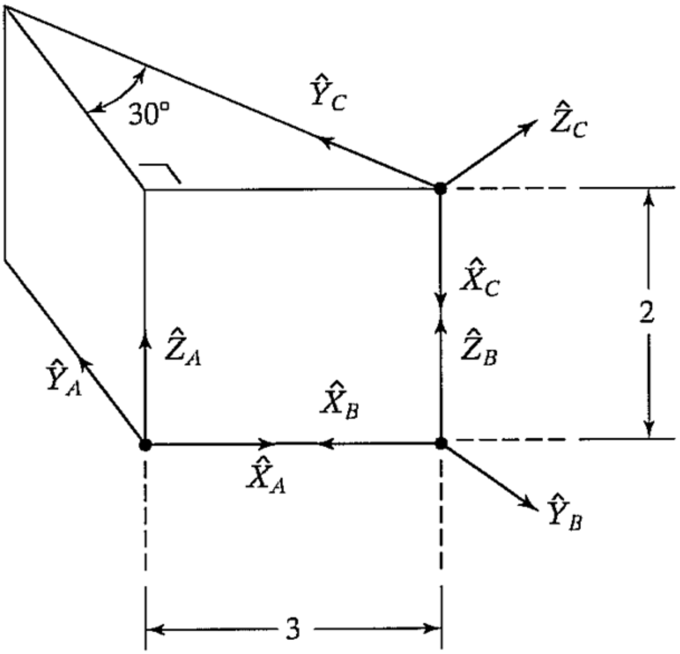
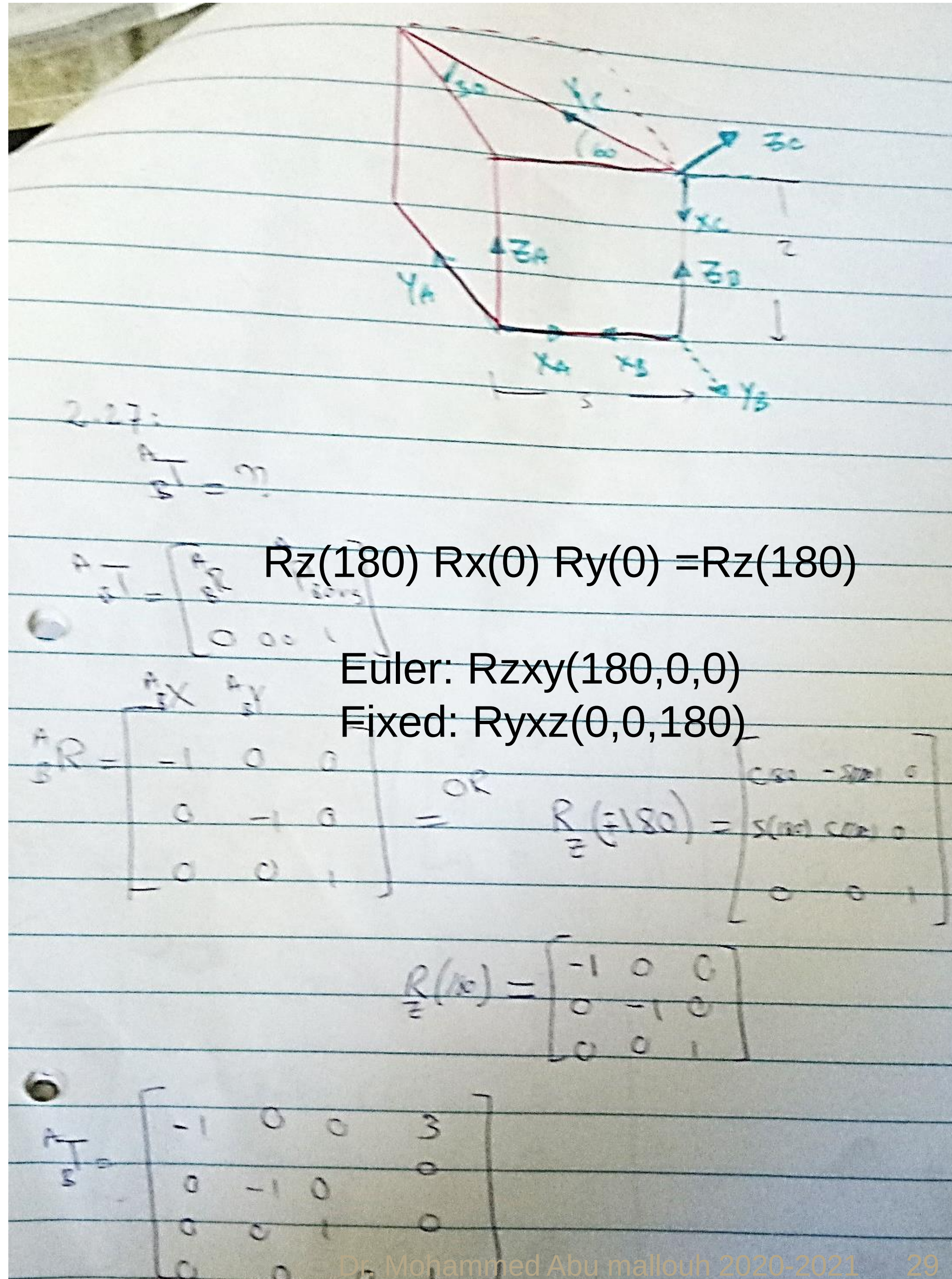


FIGURE 2.25: Frames at the corners of a wedge.

2.27 Fig. 2.25, give the value of ${}^{A}_{B}T$



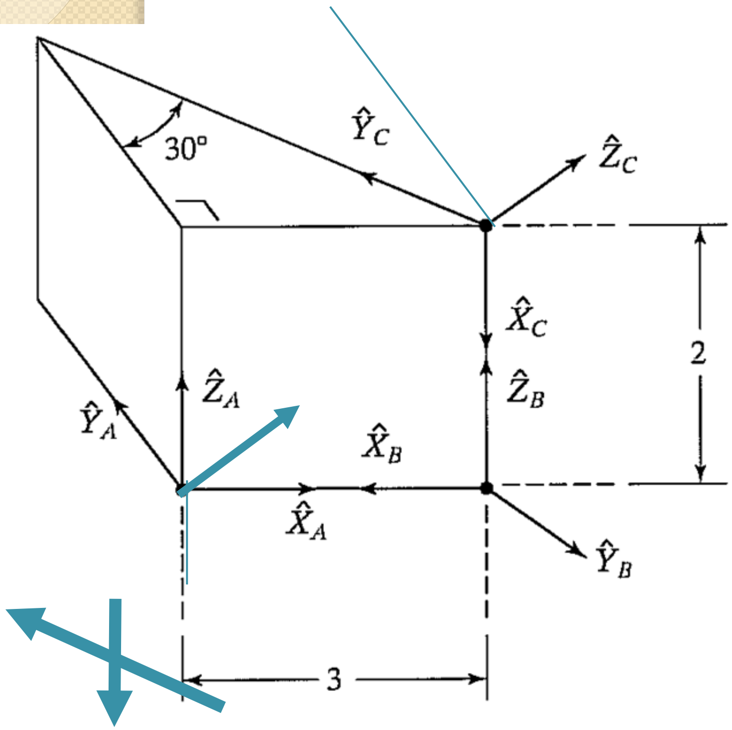


Rz(180) Rx(0) Ry(0) =Rz(180)

Euler: Rzxy(180,0,0)

Fixed: Ryxz(0,0,180)

2.28 Fig. 2.25,give the value of ${}^{A}_{C}T$

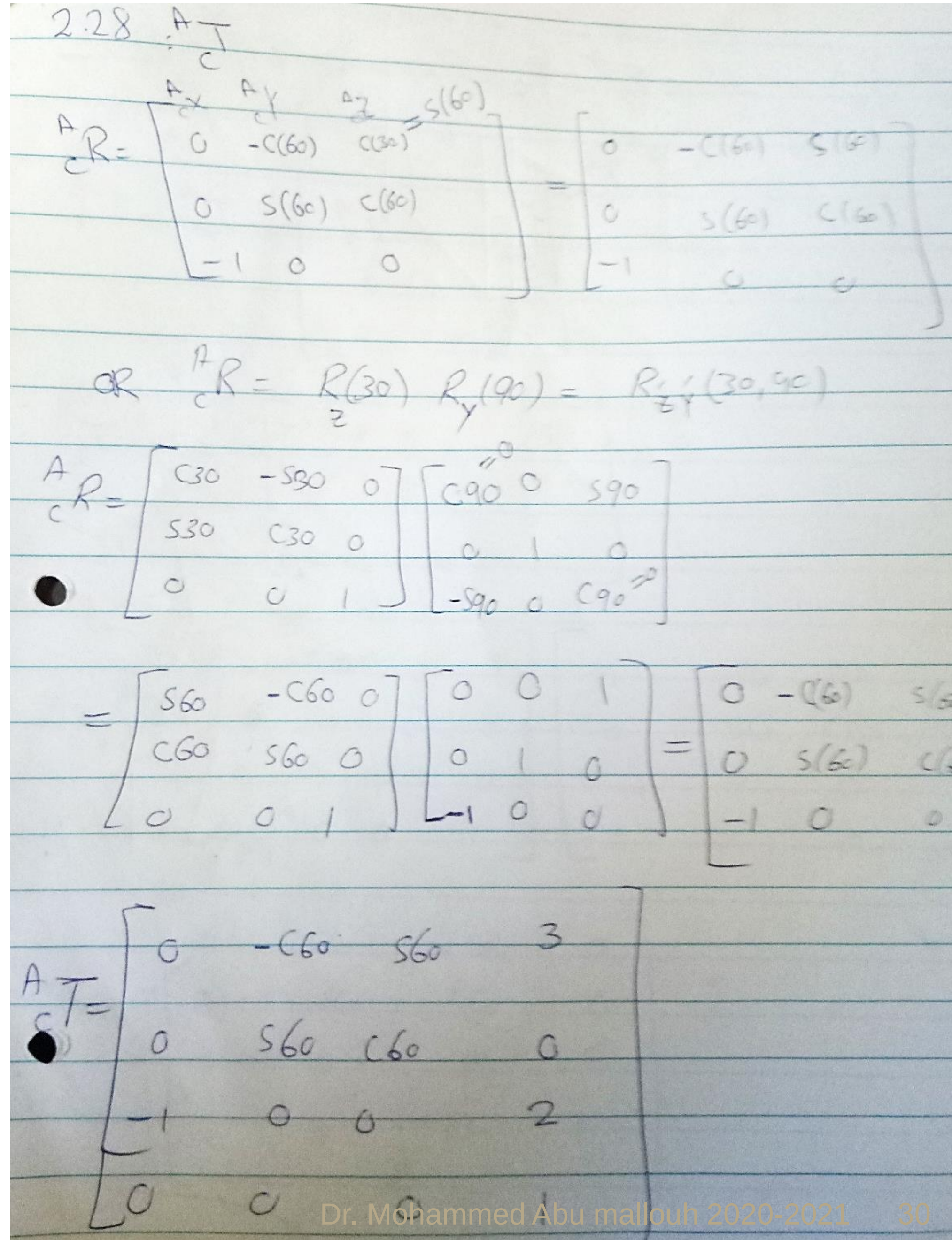


Ryzx (fixed)=Rx(0) Rz(30) Ry(90)

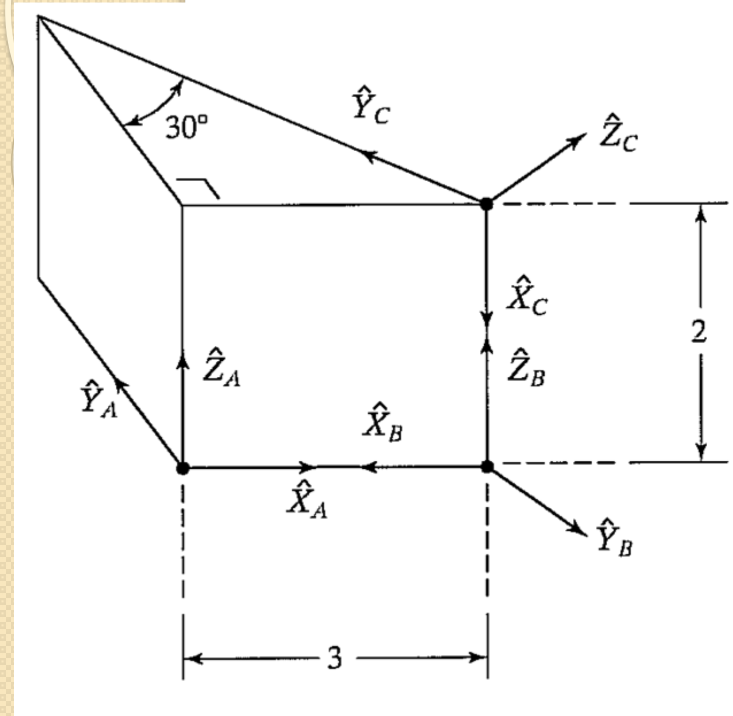
Rxzy (Euler)=Rx(0) Rz(30) Ry(90)

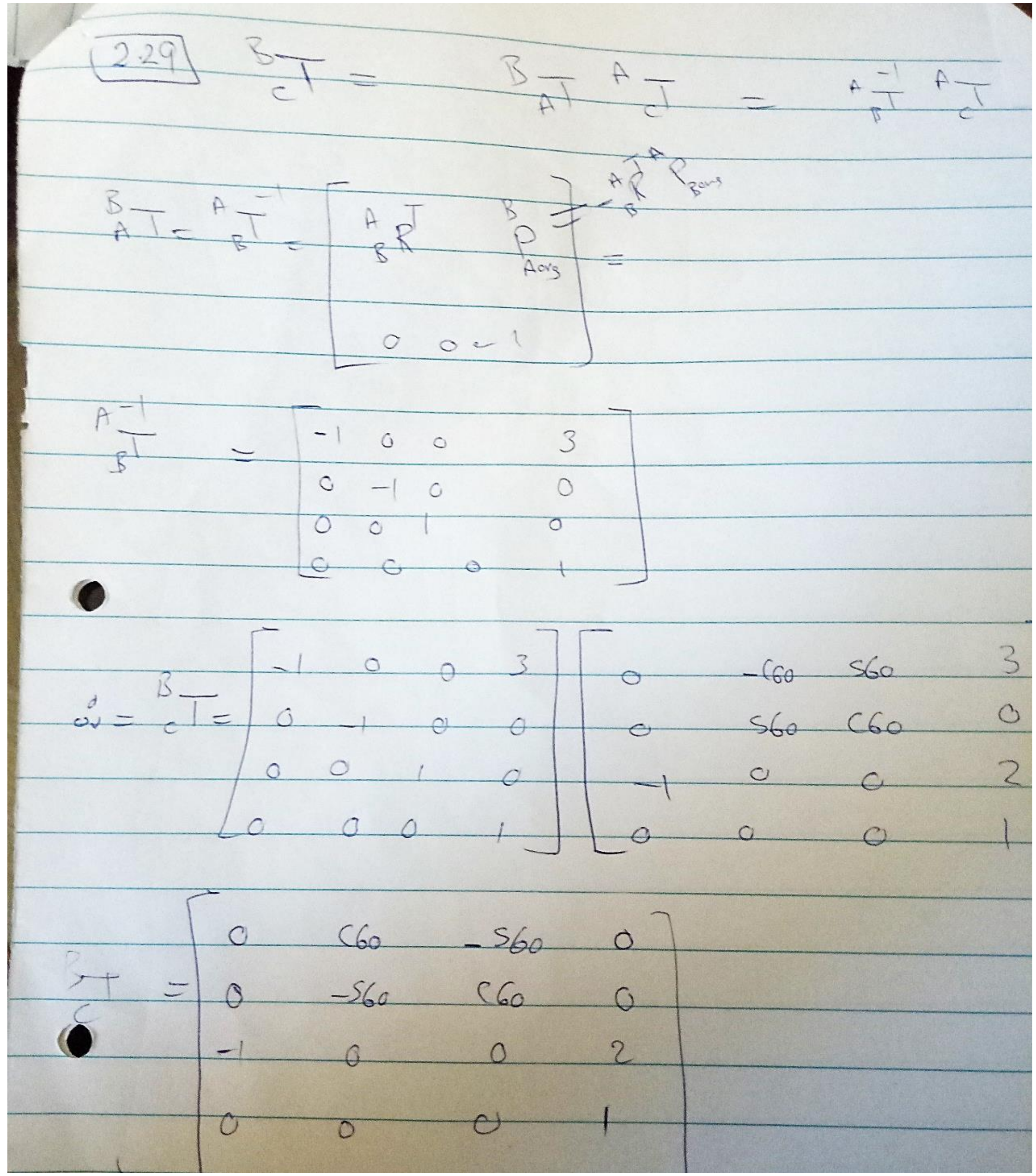
Other solutions??

Euler: Ry(90) Rx(-30)



2.29 give the value of ${}^{B}_{C}T$





Other solutions??

Chapter 3
Manipulator Kinematics

Kinematics

- Kinematics is the study of motion without regard for the forces that cause it.
- It refers to time-based and geometrical properties of motion.
- It ignores concepts such as torque, force, mass, energy, and inertia.

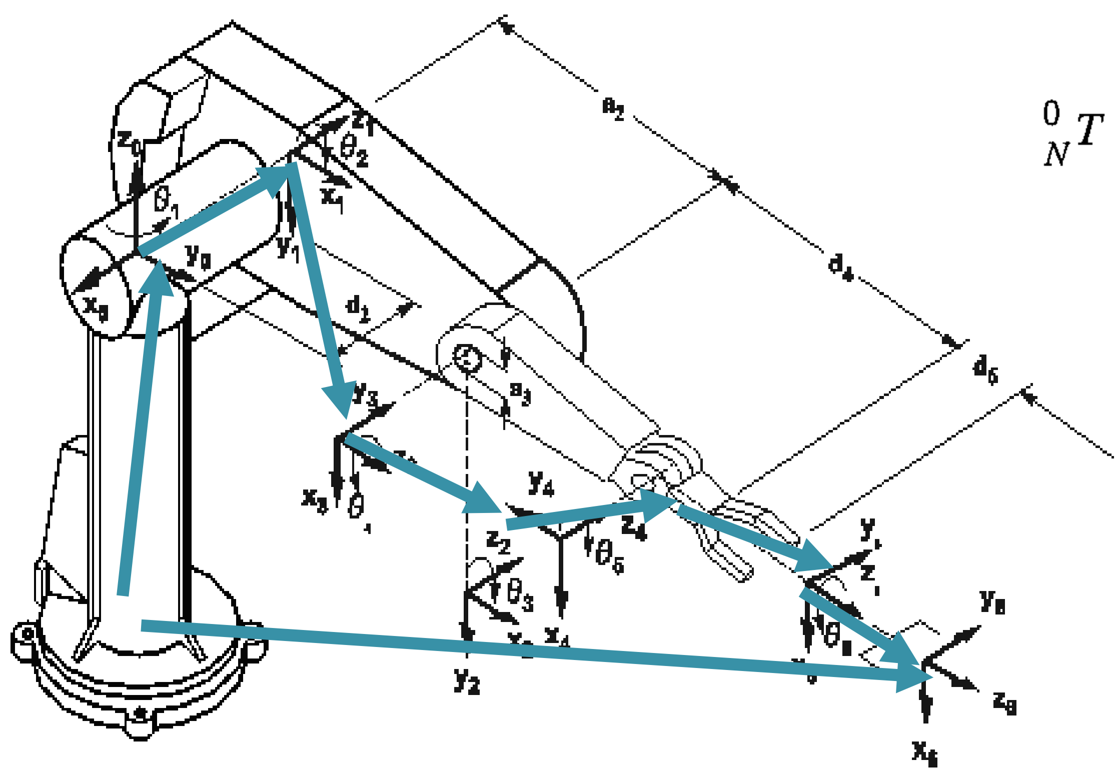
Teach pendent

Forward Kinematics

- For a robotic arm, this would mean calculating the position and orientation of the end effector given all the joint variables.

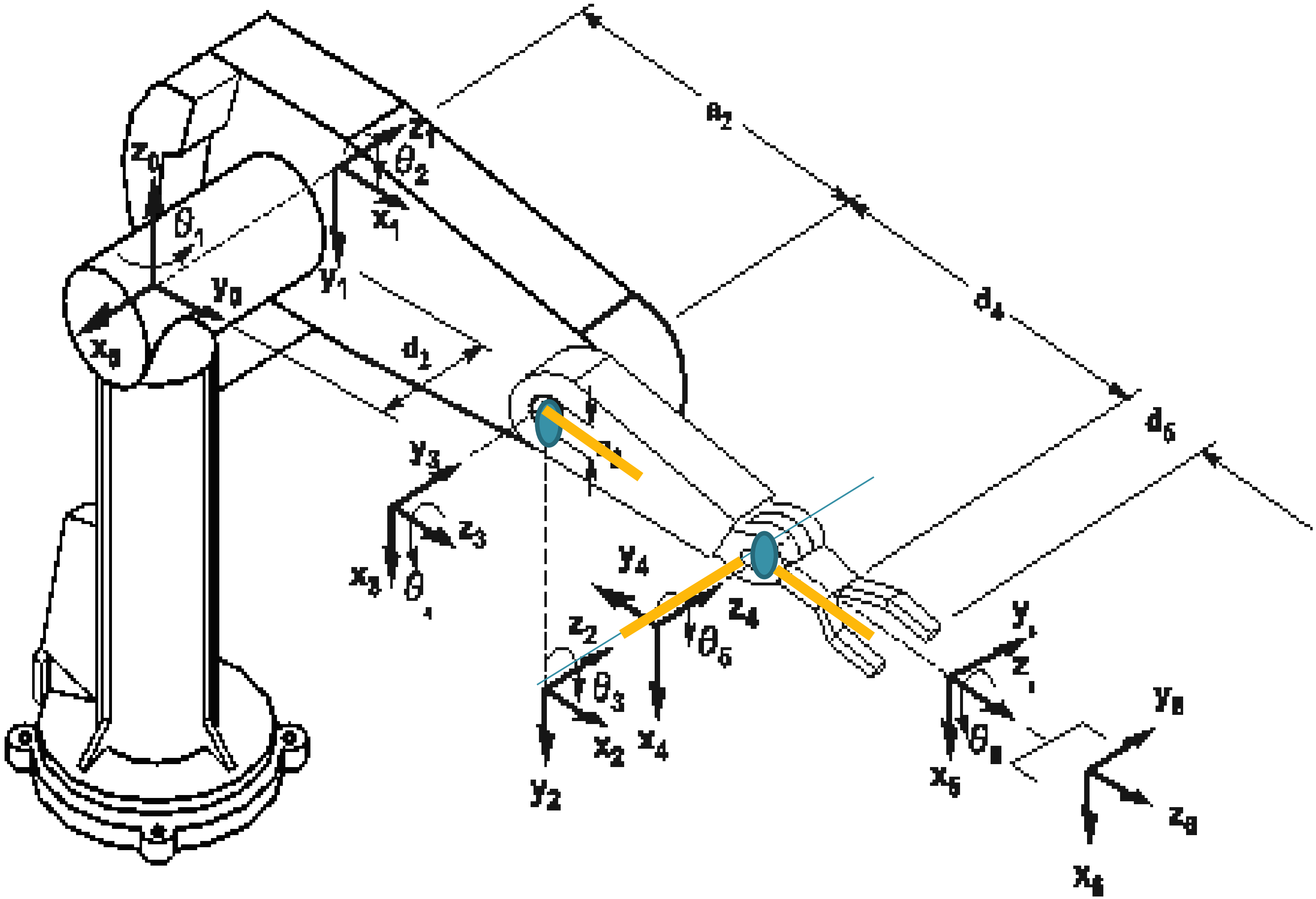
Position and orientation endeffector (x,y,z) w.r.t {base}=f(θ1, θ2, θ3…. Θn)

Note: assuming that all joints are revolute.



$$ {}^{0}_{N}T = {}^{0}_{1}T \, {}^{1}_{2}T \, {}^{2}_{3}T \ldots {}^{N-1}_{N}T. $$

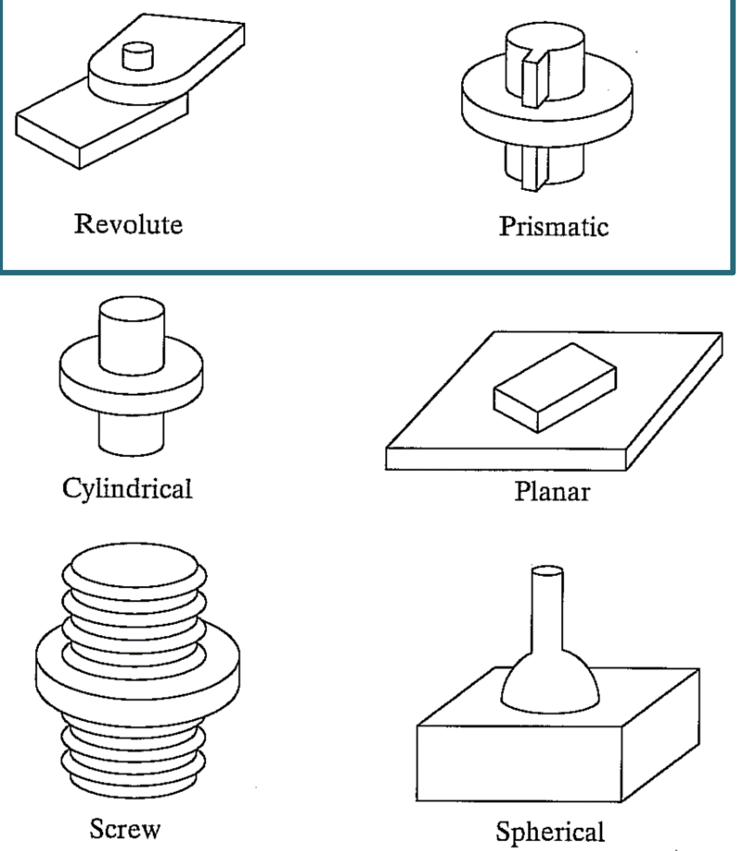
Inverse Kinematics

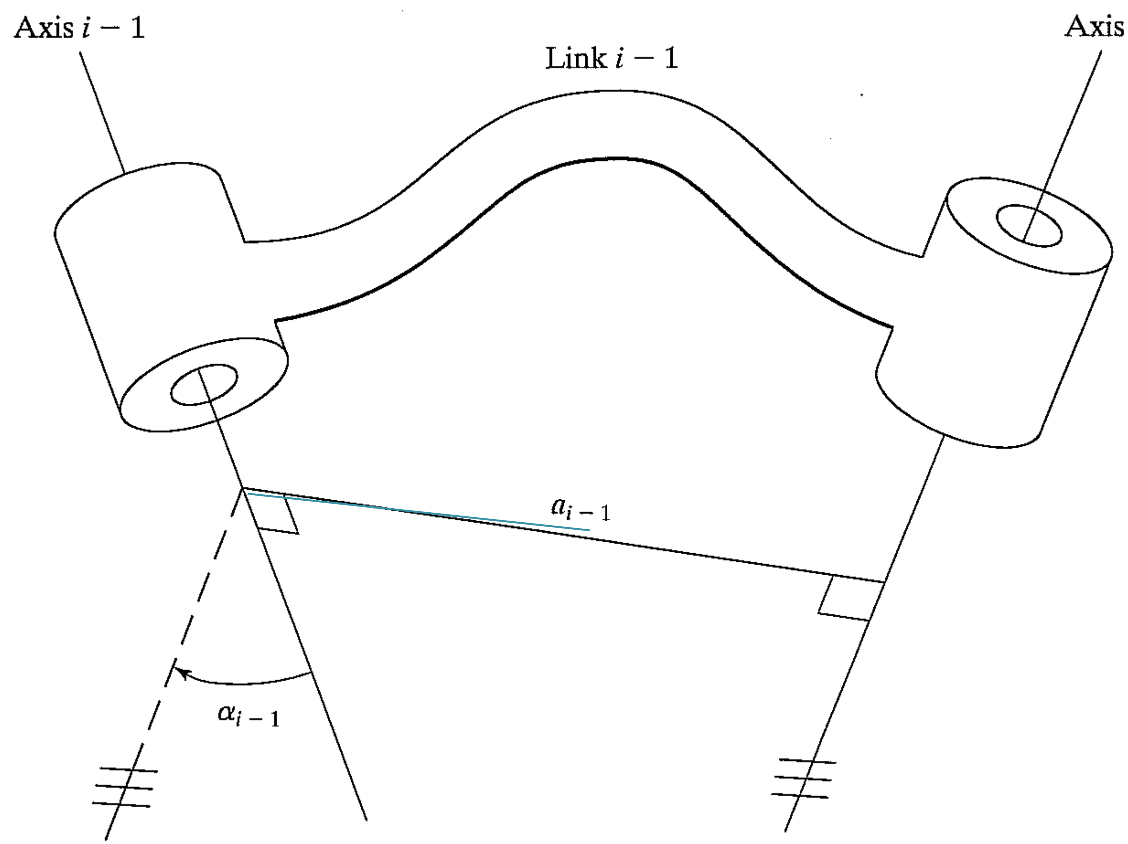


tion

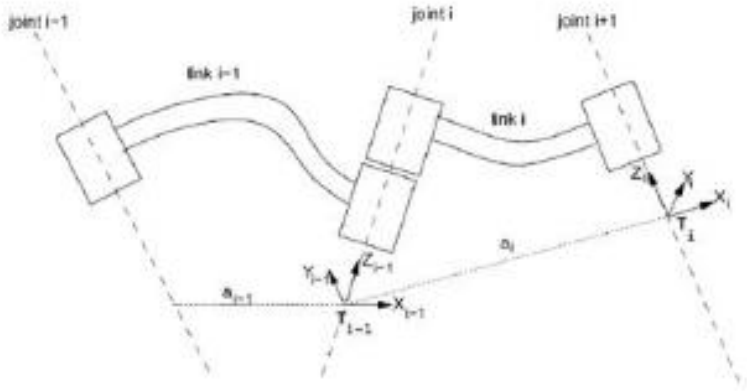
3.2 LINK DESCRIPTION

A manipulator may be thought of as a set of bodies connected in a chain by joints. These bodies are called links. Joints form a connection between a neighboring pair of links.

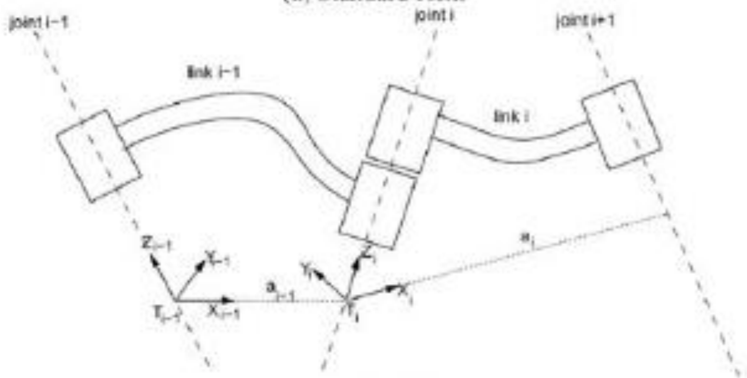




Standard Vs modified DH convention



(a) Standard form



(b) Modified form

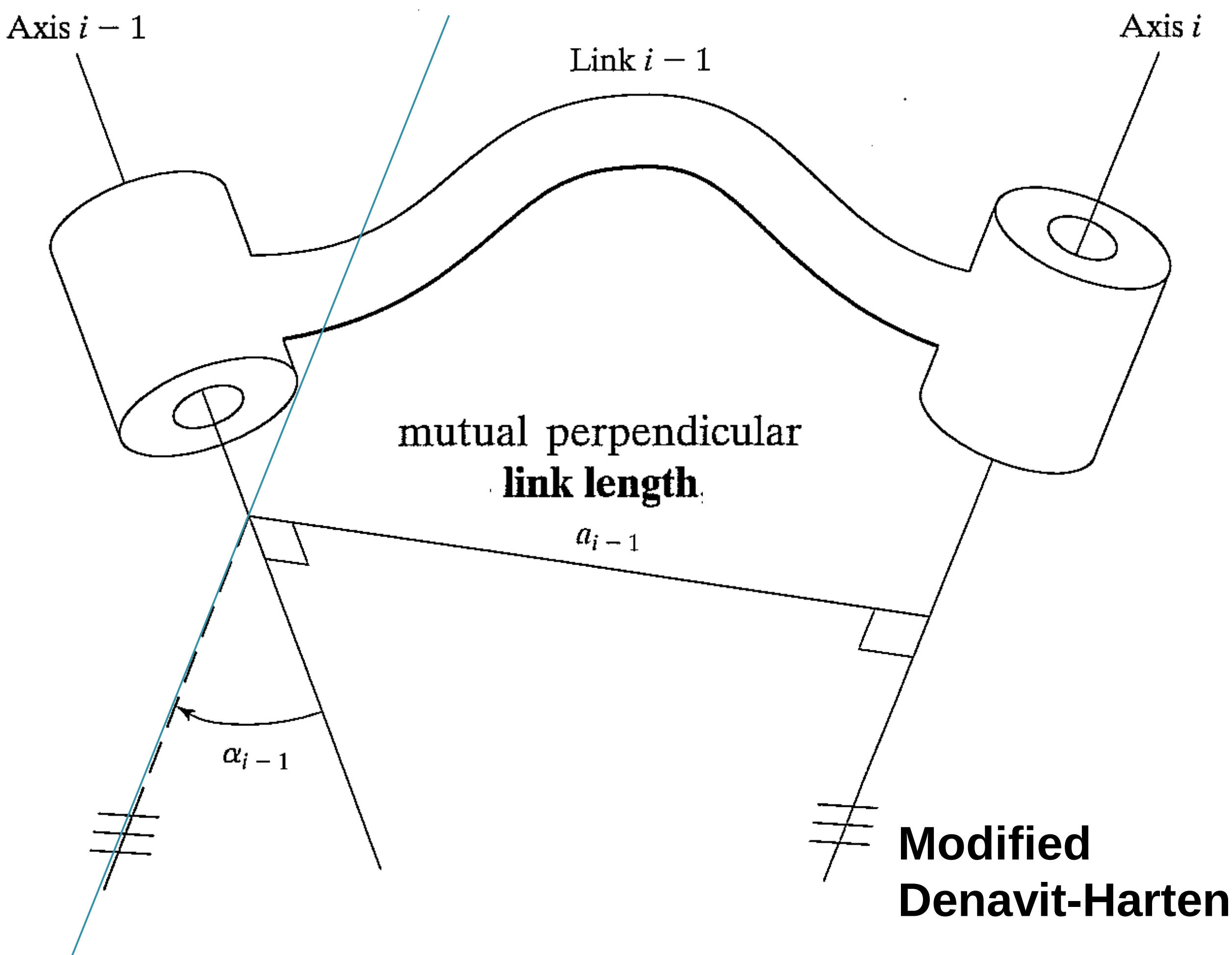
Standard Form

$$
{}^{i-1}_{\;\;i}T = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\alpha_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Modified Form

$$
{}^{i-1}_{\;\;i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

modified DH notation



Modified Denavit-Hartenberg Notation

link twist.

This angle is measured from axis $i-1$ to axis i in the right-hand sense about a_{i-1}.

3.3 LINK-CONNECTION DESCRIPTION

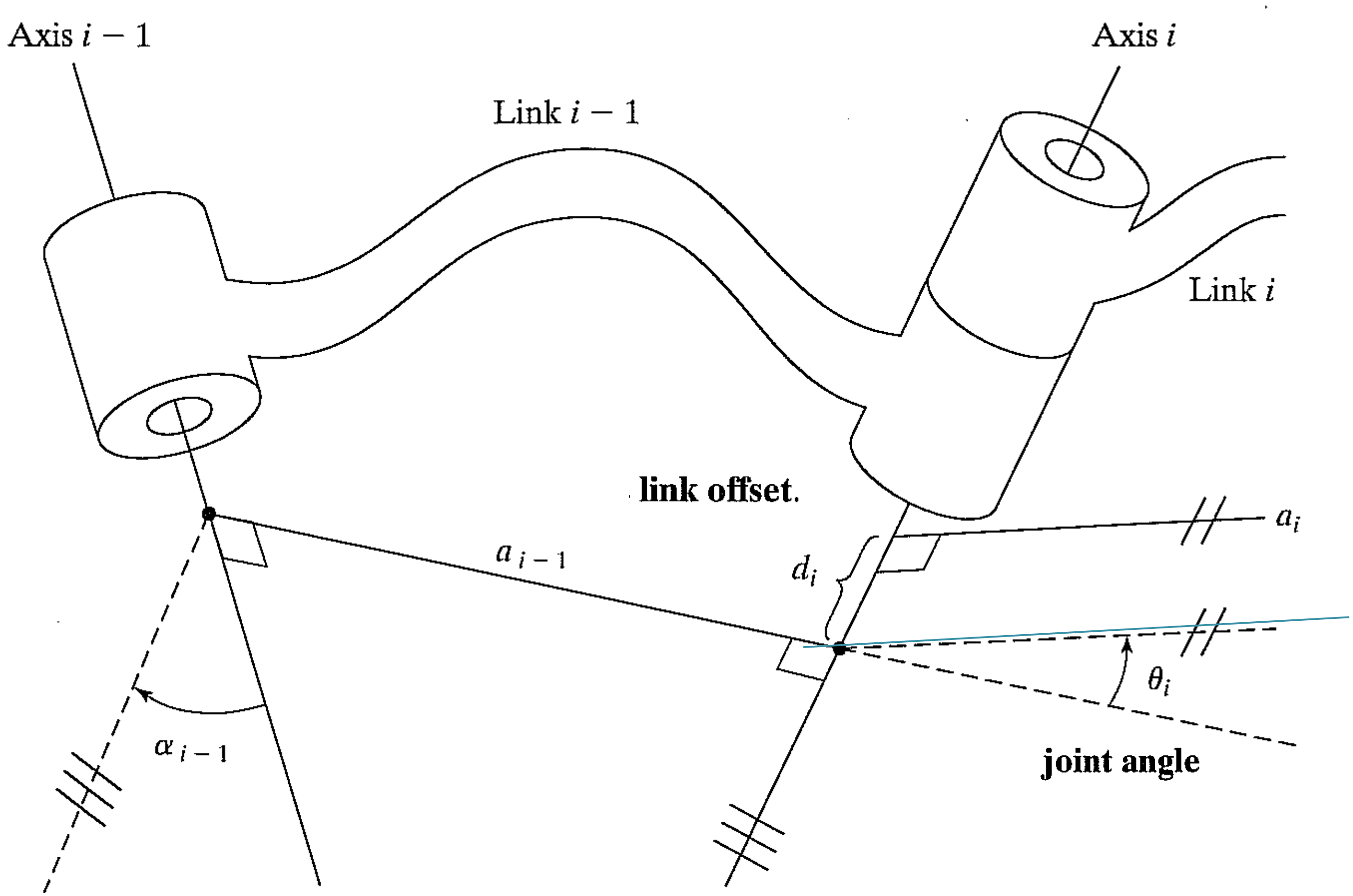
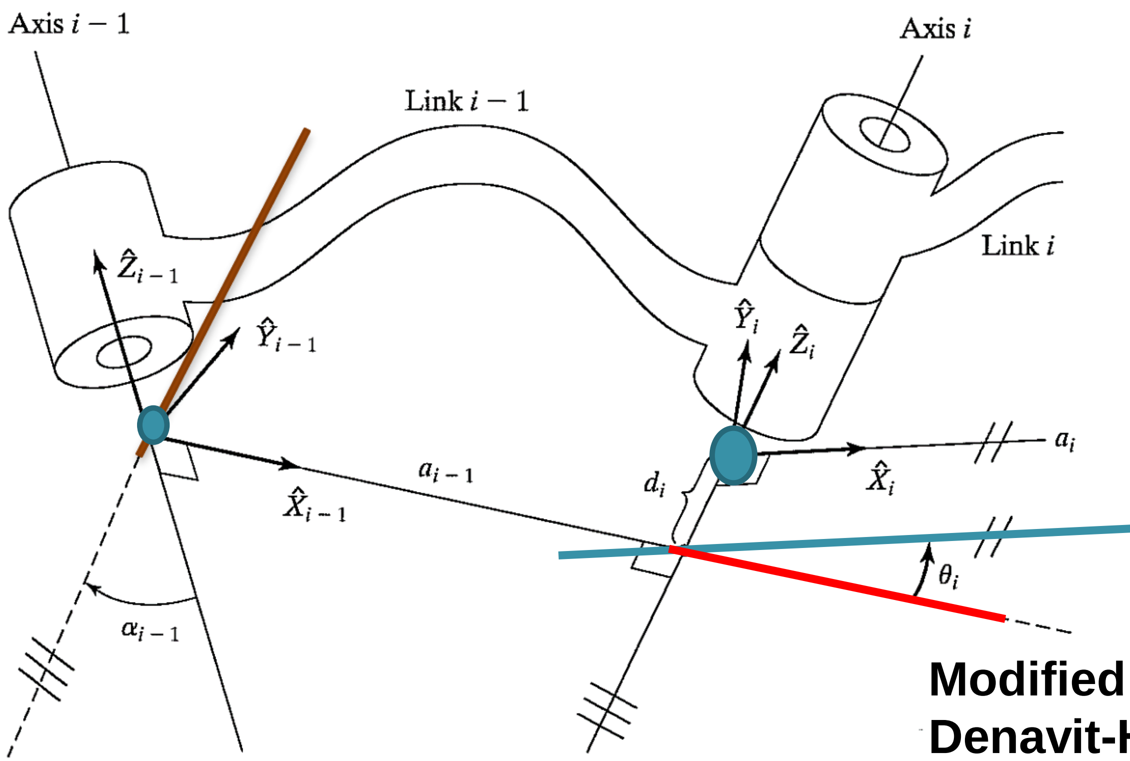


FIGURE 3.4: The link offset, d, and the joint angle, θ, are two parameters that may be used to describe the nature of the connection between neighboring links.



Modified Denavit-Hartenberg Notation

a_{i-1} = the distance from $\hat{Z}_{i-1}$ to $\hat{Z}_i$ measured along $\hat{X}_{i-1}$

α_{i-1} = the angle from $\hat{Z}_{i-1}$ to $\hat{Z}_i$ measured about $\hat{X}_{i-1}$

d_i = the distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$; and

θ_i = the angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$.

3.4 CONVENTION FOR AFFIXING FRAMES TO LINKS

Intermediate links in the chain

The convention we will use to locate frames on the links is as follows: The $\hat{Z}$-axis of frame $\{i\}$, called $\hat{Z}_i$, is coincident with the joint axis i. The origin of frame $\{i\}$ is located where the a_i perpendicular intersects the joint i axis. $\hat{X}_i$ points along a_i in the direction from joint i to joint $i+1$.

In the case of $a_i = 0$, $\hat{X}_i$ is normal to the plane of $\hat{Z}_i$ and $\hat{Z}_{i+1}$. We define α_i as being measured in the right-hand sense about $\hat{X}_i$, and so we see that the freedom of choosing the sign of α_i in this case corresponds to two choices for the direction of $\hat{X}_i$. $\hat{Y}_i$ is formed by the right-hand rule to complete the ith frame. Figure 3.5 shows the location of frames $\{i-1\}$ and $\{i\}$ for a general manipulator.

3.4 CONVENTION FOR AFFIXING FRAMES TO LINKS

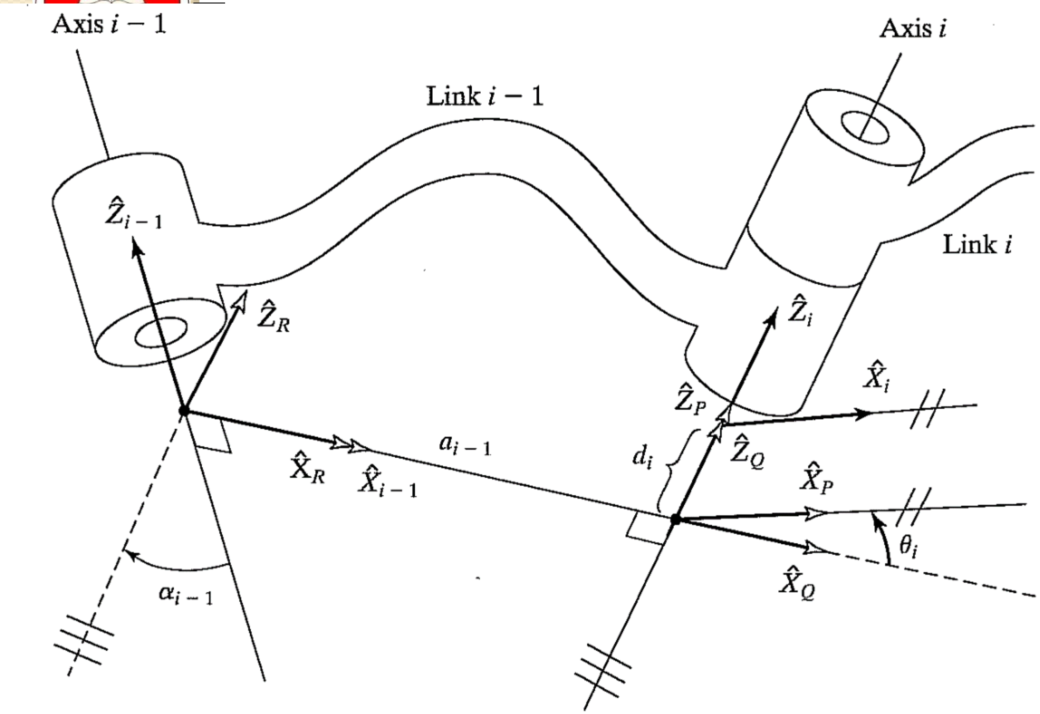
First and last links in the chain

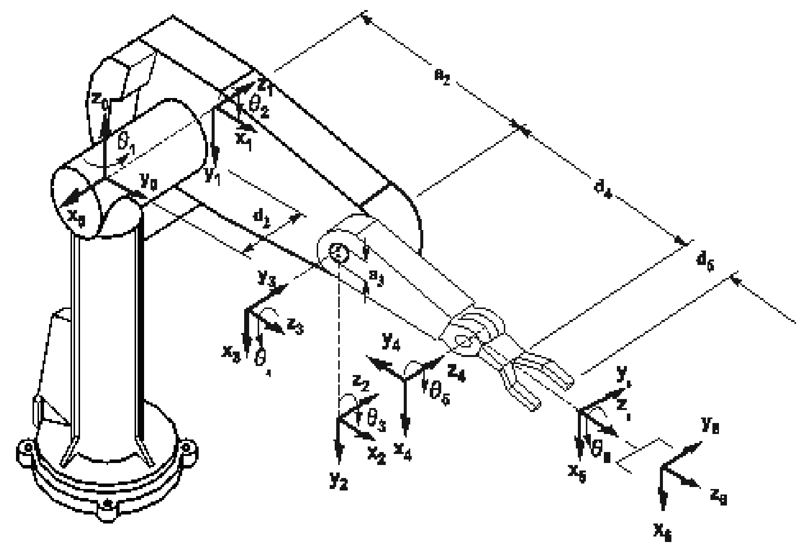
We attach a frame to the base of the robot, or link 0, called frame {0}. This frame does not move; for the problem of arm kinematics, it can be considered the reference frame. We may describe the position of all other link frames in terms of this frame.

Frame {0} is arbitrary, so it always simplifies matters to choose $\hat{Z}_0$ along axis 1 and to locate frame {0} so that it coincides with frame {1} when joint variable 1 is zero. Using this convention, we will always have $a_0 = 0.0$, $\alpha_0 = 0.0$. Additionally, this ensures that $d_1 = 0.0$ if joint 1 is revolute, or $\theta_1 = 0.0$ if joint 1 is prismatic.

For joint n revolute, the direction of $\hat{X}_N$ is chosen so that it aligns with $\hat{X}_{N-1}$ when $\theta_n = 0.0$, and the origin of frame $\{N\}$ is chosen so that $d_n = 0.0$. For joint n prismatic, the direction of $\hat{X}_N$ is chosen so that $\theta_n = 0.0$, and the origin of frame $\{N\}$ is chosen at the intersection of $\hat{X}_{N-1}$ and joint axis n when $d_n = 0.0$.

Derivation of link transformations





$$ {}^{i-1}_{i}T = {}^{i-1}_{R}T \; {}^{R}_{Q}T \; {}^{Q}_{P}T \; {}^{P}_{i}T. $$

$$ {}^{i-1}_{i}T = R_X(\alpha_{i-1}) D_X(a_{i-1}) R_Z(\theta_i) D_Z(d_i), $$

$$ {}^{0}_{N}T = {}^{0}_{1}T \; {}^{1}_{2}T \; {}^{2}_{3}T \ldots {}^{N-1}_{N}T. $$

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

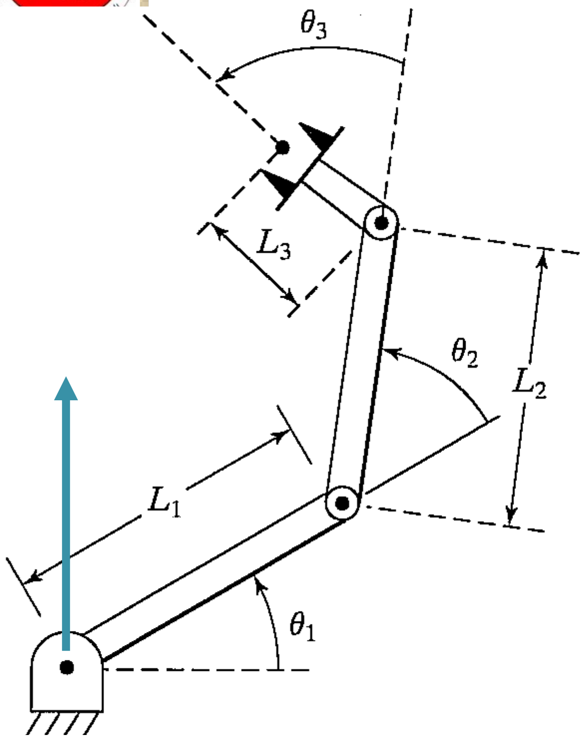
Summary of link-frame attachment procedure

The following is a summary of the procedure to follow when faced with a new mechanism, in order to properly attach the link frames:

1. Identify the joint axes and imagine (or draw) infinite lines along them. For steps 2 through 5 below, consider two of these neighboring lines (at axes i and $i+1$).
2. Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the ith axis, assign the link-frame origin.
3. Assign the $\hat{Z}_i$ axis pointing along the ith joint axis.
4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.
5. Assign the $\hat{Y}_i$ axis to complete a right-hand coordinate system.
6. Assign $\{0\}$ to match $\{1\}$ when the first joint variable is zero. For $\{N\}$, choose an origin location and $\hat{X}_N$ direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

RRR (or 3*R*) mechanism.

general configration all joints ≠ zero



1-Assign frames on robot based on Modified DH convention

2-fill in the Modified DH parameters table

3-find ${}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T$??

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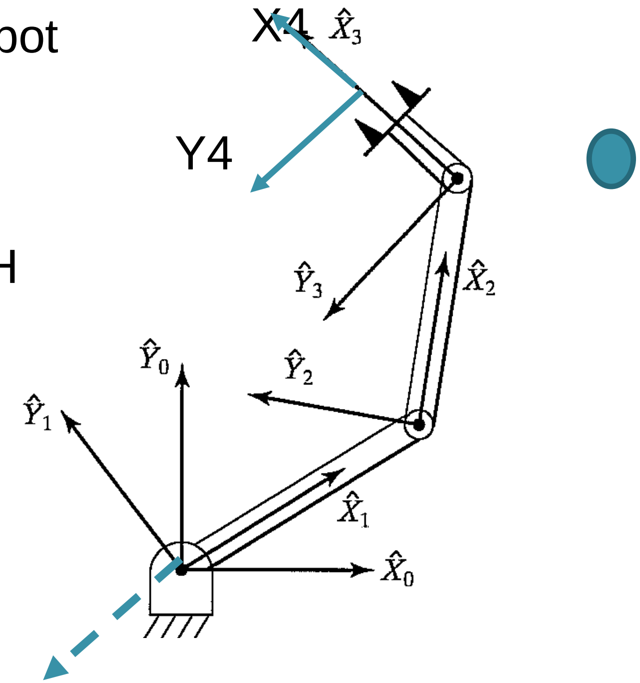


C12=cos(θ1+ θ2)≠ C1C2 ≠C1+C2

zero conf.>>>all joint =zero
if you assigned frams correctly>>all Xs in same direction.

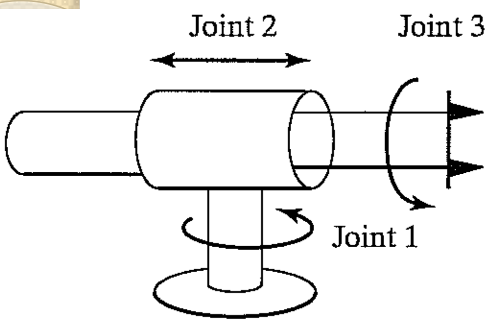
${}^{0}_{3}Rz$(θ1+ θ2+ θ3)

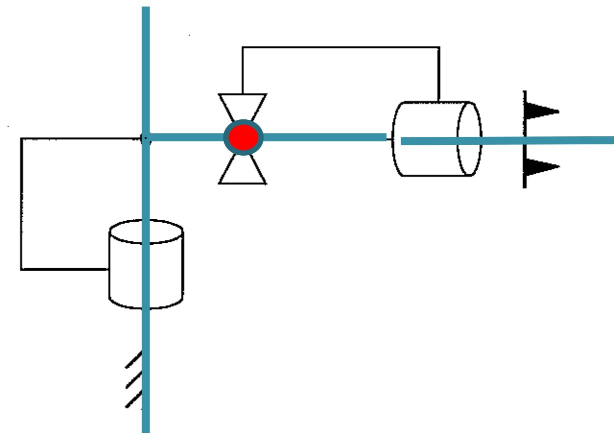
$$
{}^{0}_{3}T = \begin{bmatrix} C_{123} & -S_{123} & 0 & L_1C_1 + L_2C_{12} \\ S_{123} & C_{123} & 0 & L_1S_1 + L_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

i	α_{i-1}	a_{i-1}	d_i	θ_i	i-1
1	0	0	0	θ_1	0
2	0	L_1	0	θ_2	1
3	0	L_2	0	θ_3	2

RPR mechanism

general configration all joints ≠ zero





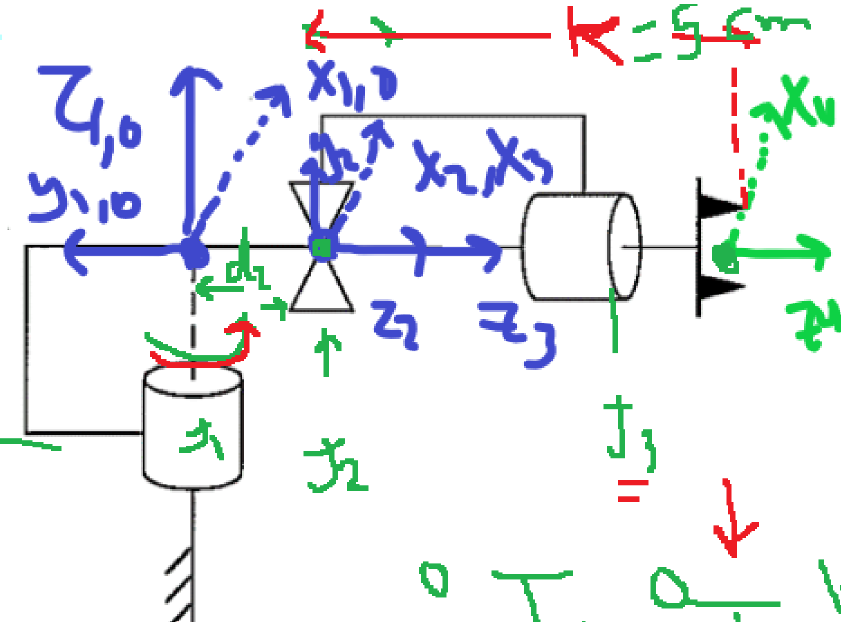
1-Assign frames on robot based on Modified DH convention

2-fill in the Modified DH parameters table

3-find ${}^0_3T$?

general configration all joints ≠ zero

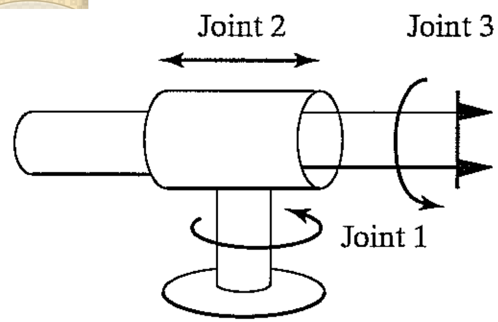
i+1	α	a	d	θ	i
1	0	0	0	θ1(t)	0
2	90	0	d2(t)	0	1
3	0	0	0	θ3(t)	2
4	0	0	K	Zero	3

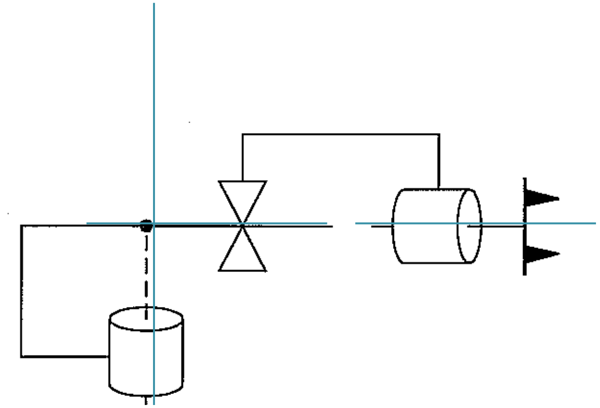


$${}^0_3T = {}^0_1T\,{}^1_2T\,{}^2_3T$$

zero conf.>>>all joint =zero
if you assigned frams correctly>>all Xs in same direction.

RPR mechanism



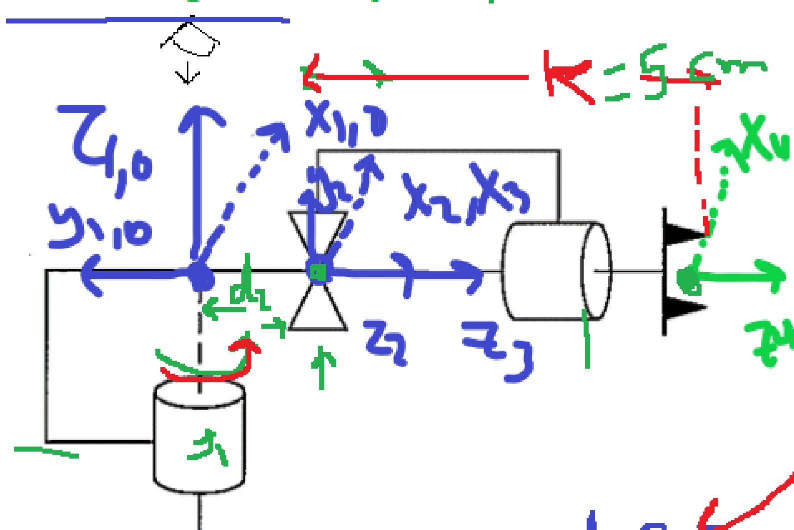


1-Assign frames on robot based on Modified DH convention

2-fill in the Modified DH parameters table

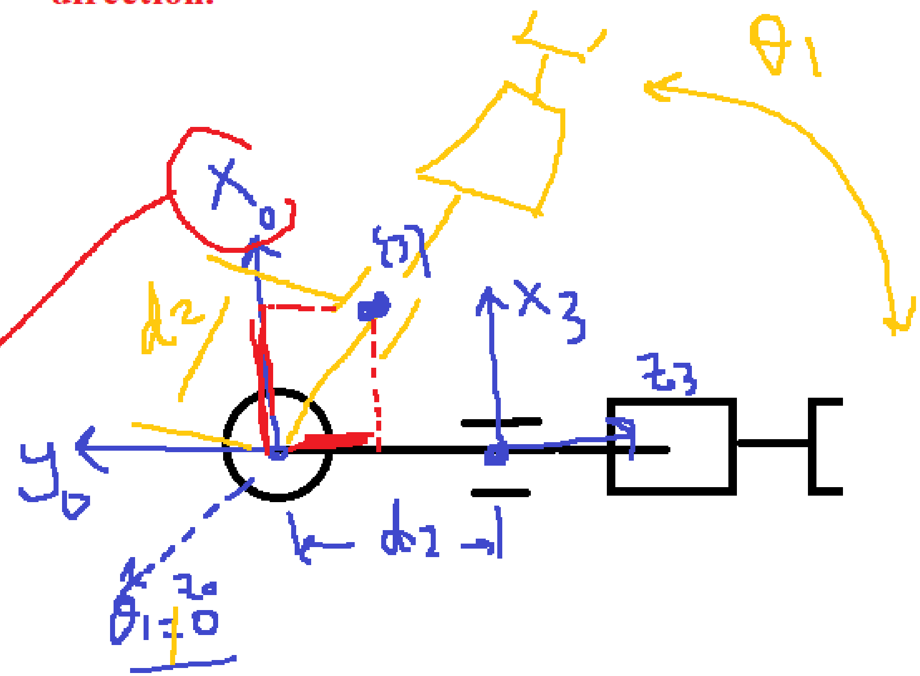
general configration all joints ≠ zero

zero conf.>>>all joint =zero
if you assigned frams correctly>>all Xs in same direction.

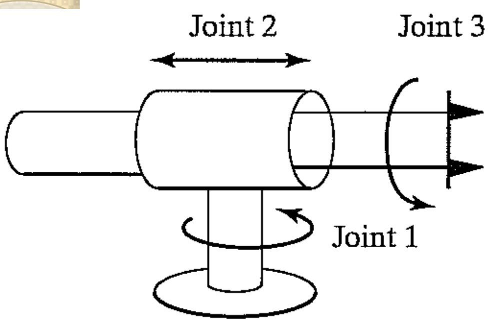


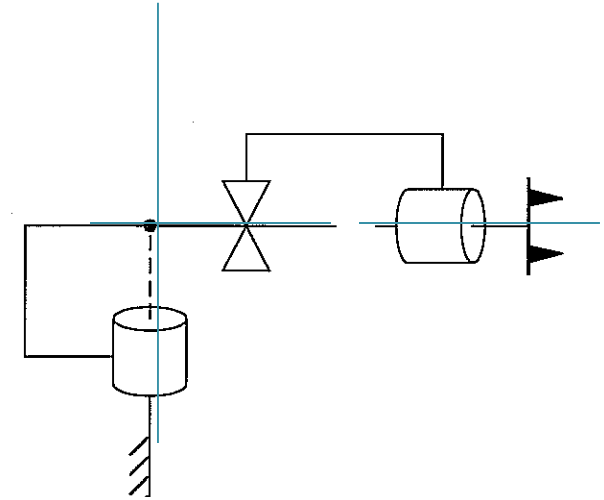
$${}^0_3T = \begin{bmatrix} \ \end{bmatrix}$$

$${}^0_3P = \begin{bmatrix} d_2 S_1 \\ -d_2 C_1 \\ 0 \end{bmatrix}$$



RPR mechanism



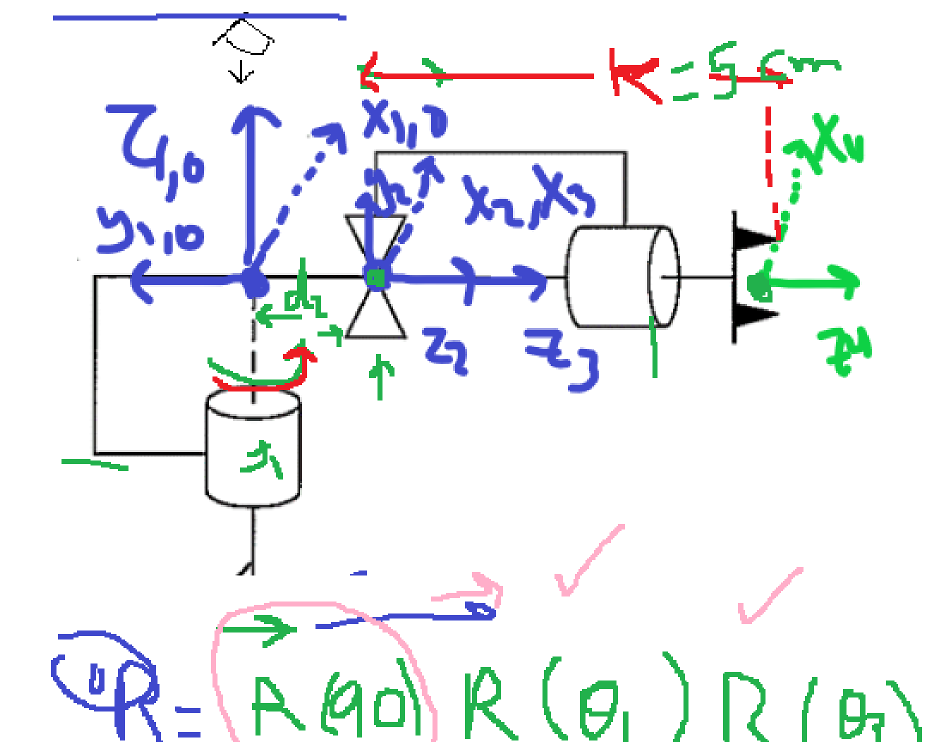


1-Assign frames on robot based on Modified DH convention

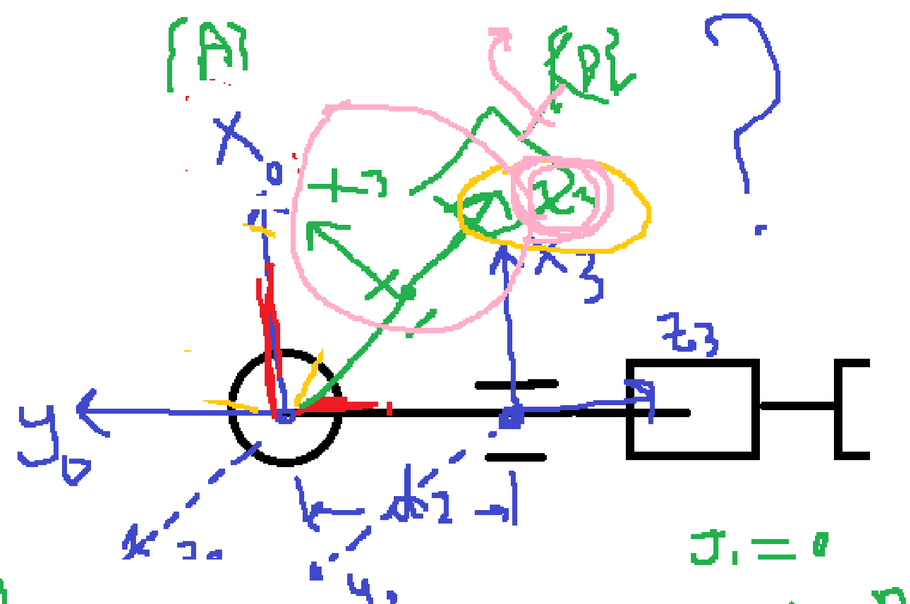
2-fill in the Modified DH parameters table

3-find ${}^{0}_{3}T$?

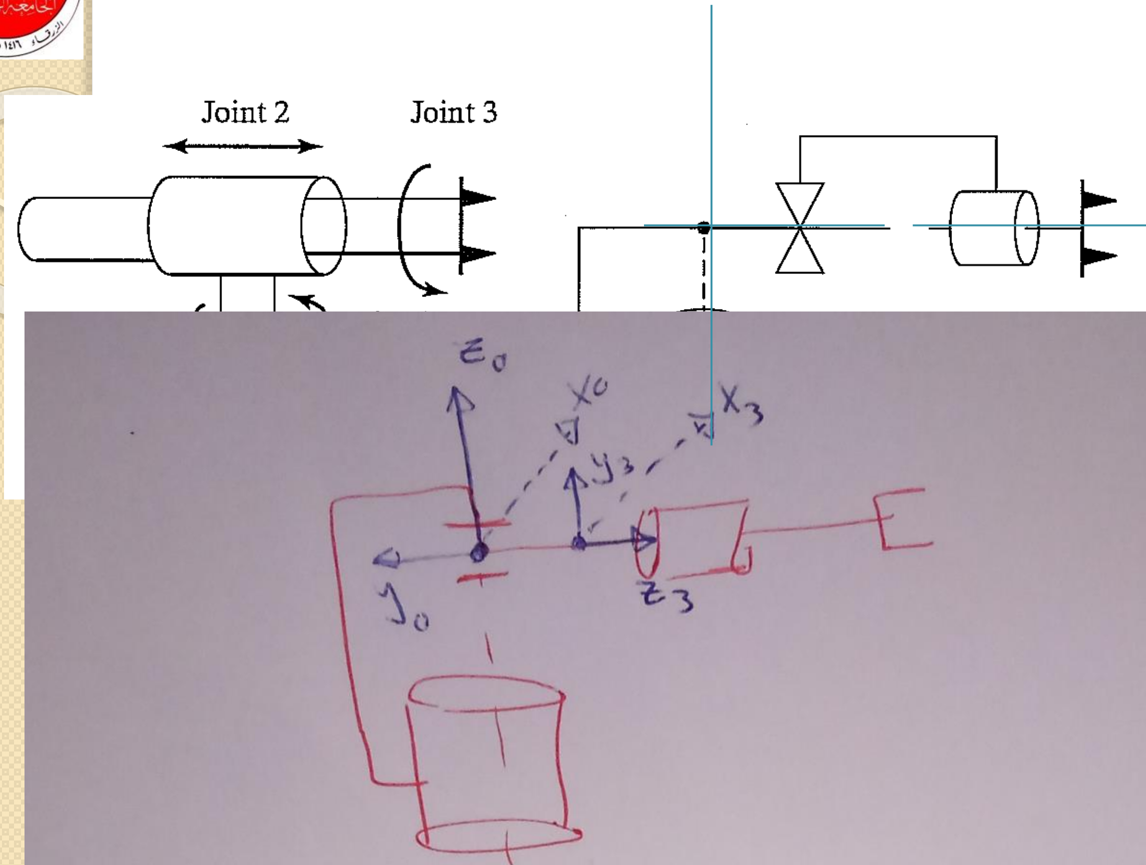
general configration all joints ≠ zero



zero conf>>>all joint =zero
if you assigned frams correctly>>all Xs in same direction.



RPR mechanism

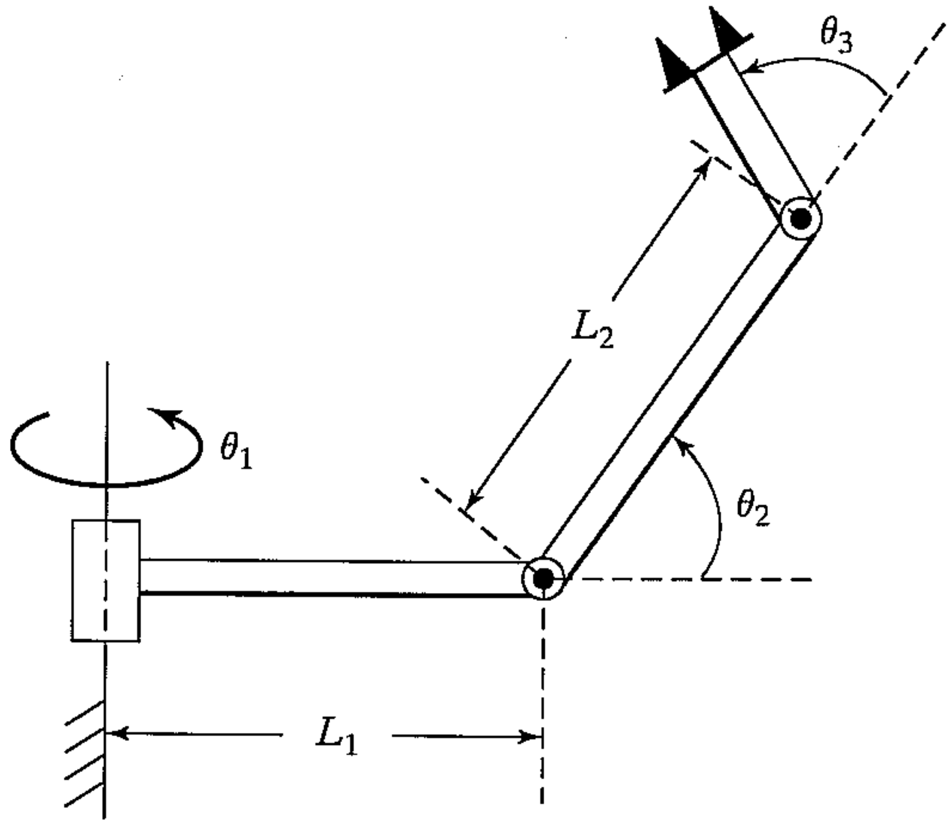


1-Assign frames on robot based on Modified DH convention

2-fill in the Modified DH parameters table

3-find ${}^{0}_{3}T$?

$$ {}^{0}_{3}T = \begin{bmatrix} C_1C_3 & -C_1S_3 & S_1 & S_1 d_2 \\ S_1C_3 & -S_1S_3 & -C_1 & -C_1 d_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



FIGURE

1-Assign frames on robot based on Modified DH convention

2-fill in the Modified DH parameters table

3-find ${}^{0}_{3}T$?

$$
{}^{0}_{3}T = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & (L_1+L_2C_2)C_1 \\ S_1S_{23} & -S_1S_{23} & +C_1 & (L_1+L_2C_2)S_1 \\ S_{23} & C_{23} & 0 & L_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

See note

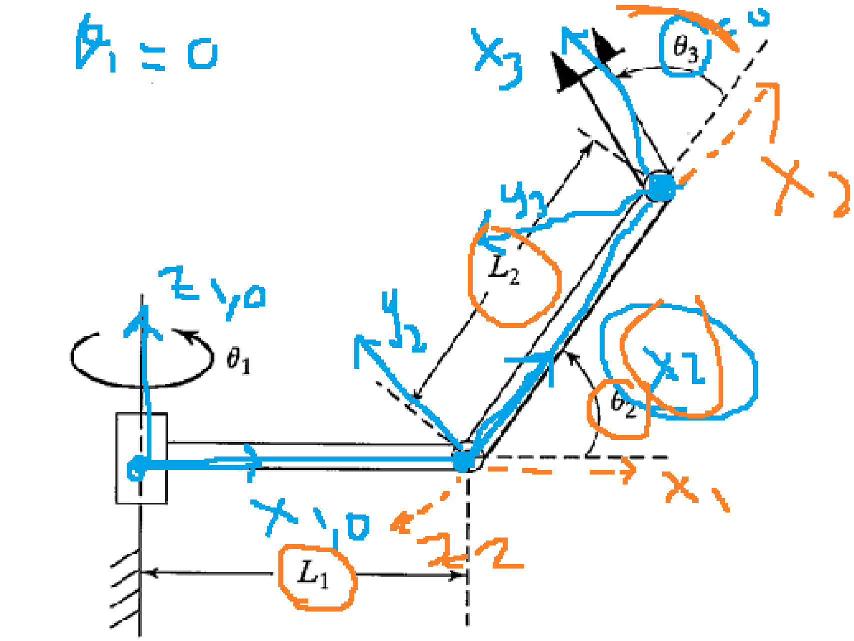


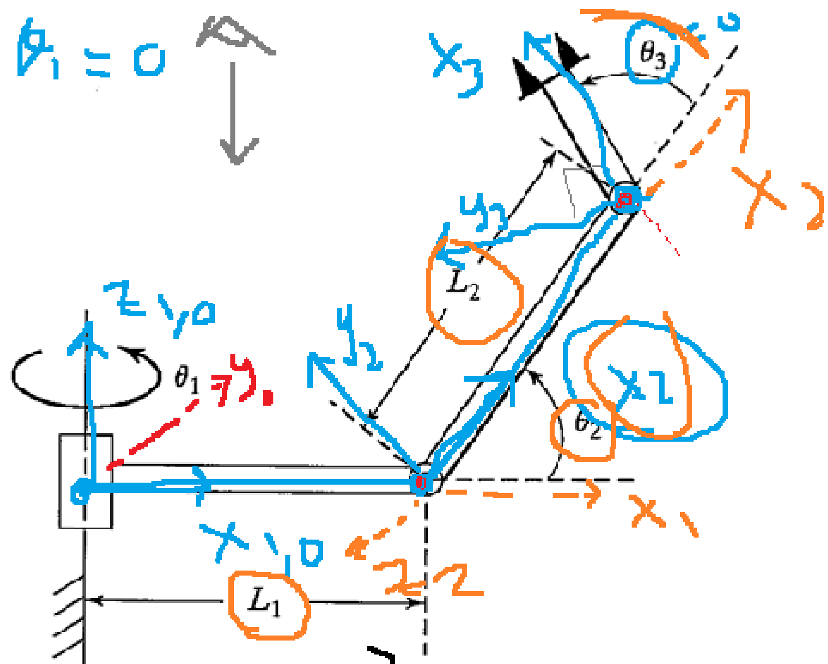
FIGURE 3.29: The 3R nonplanar arm (Exercise 3.3).

$^0_3T = DH = {}^0_1T \; {}^1_2T \; {}^2_3T$

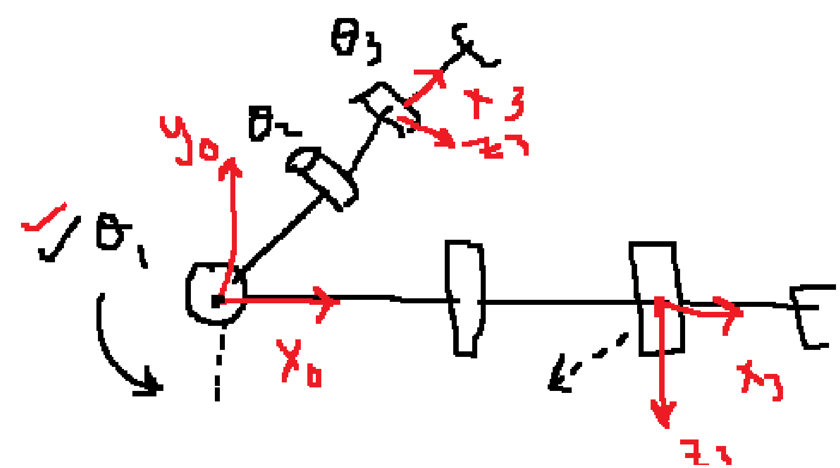
DH Table

i+1	α	a	d	θ	i
1	0	0	0	θ1 (t)	0
2	90	L1	0	θ2 (t)	1
3	0	L2	0	θ3 (t)	2

See notes



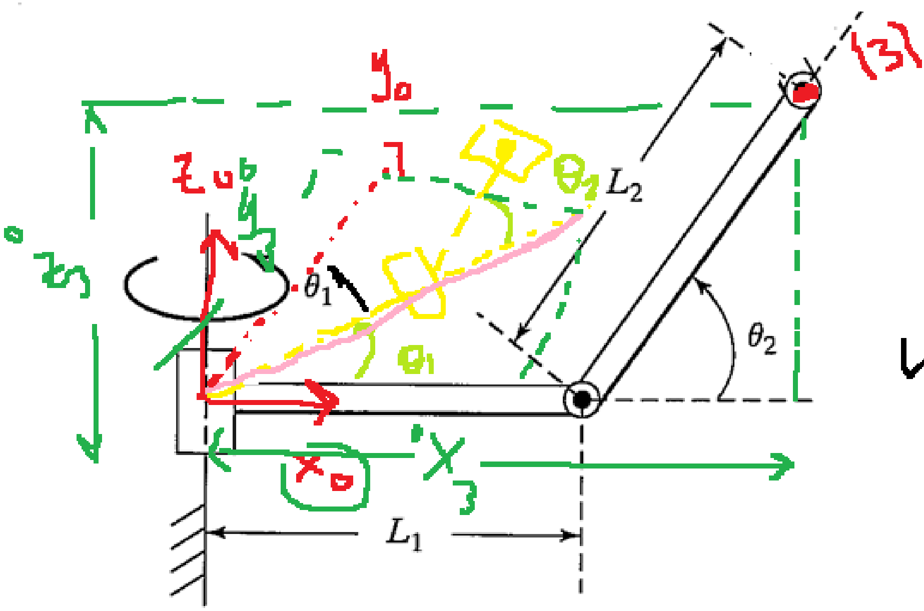
Zero Conf. →



$$^0_3T = \begin{bmatrix} ^0_3R & ^0_3P \\ 0\ 0\ 0 & 1 \end{bmatrix}$$

$$^0_3R = R_{xyz\ Eular} = R_x(90)\, R_y(\theta_1)\, R_z(\theta_2+\theta_3)$$

See notes



$$^0_3P = \begin{bmatrix} ^0x_3 = (L_1 + L_2C_2)C_1 \\ ^0y_3 = (L_1 + L_2C_2)S_1 \\ ^0z_3 = L_2 S_2 \end{bmatrix}$$

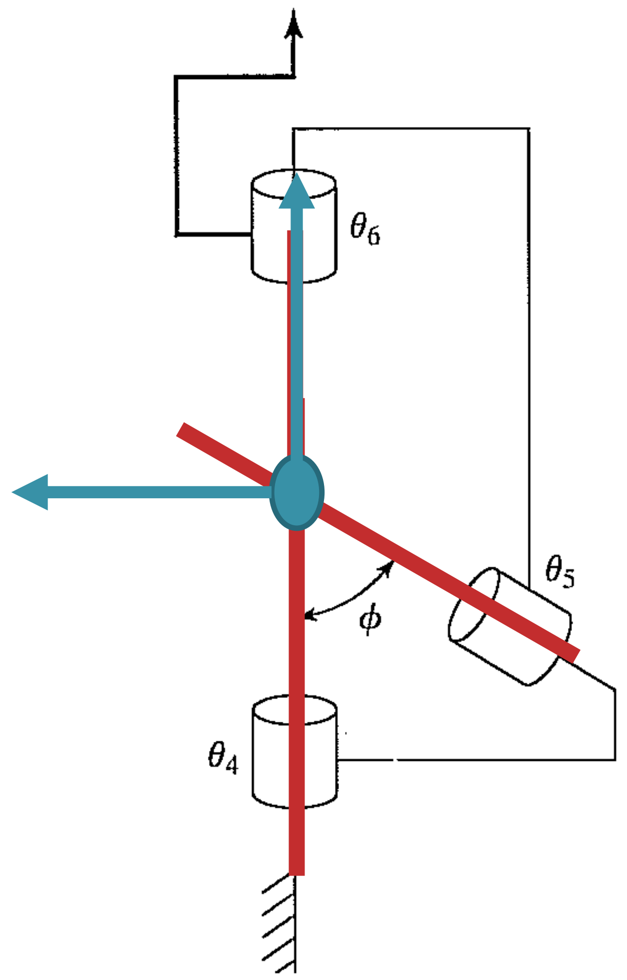
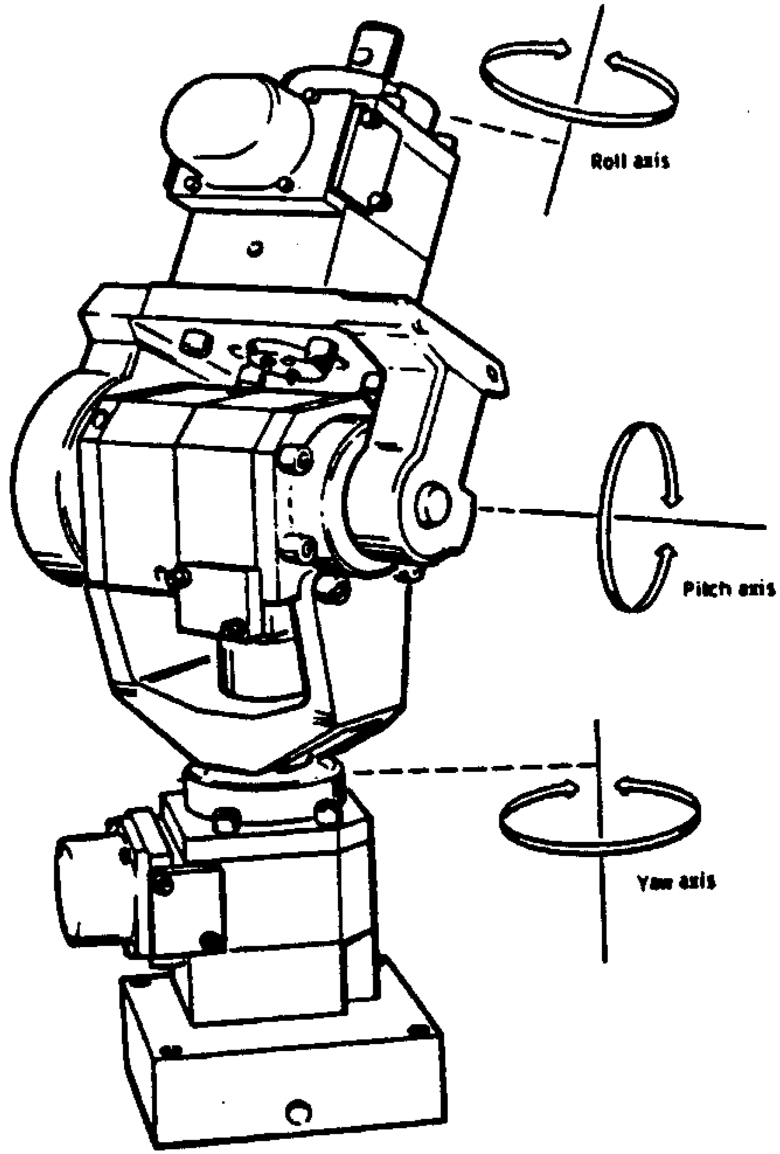


FIGURE 3.33: $3R$ nonorthogonal-axis robot (Exercise 3.11).

See notes


Roll axis
Pitch axis
Yaw axis

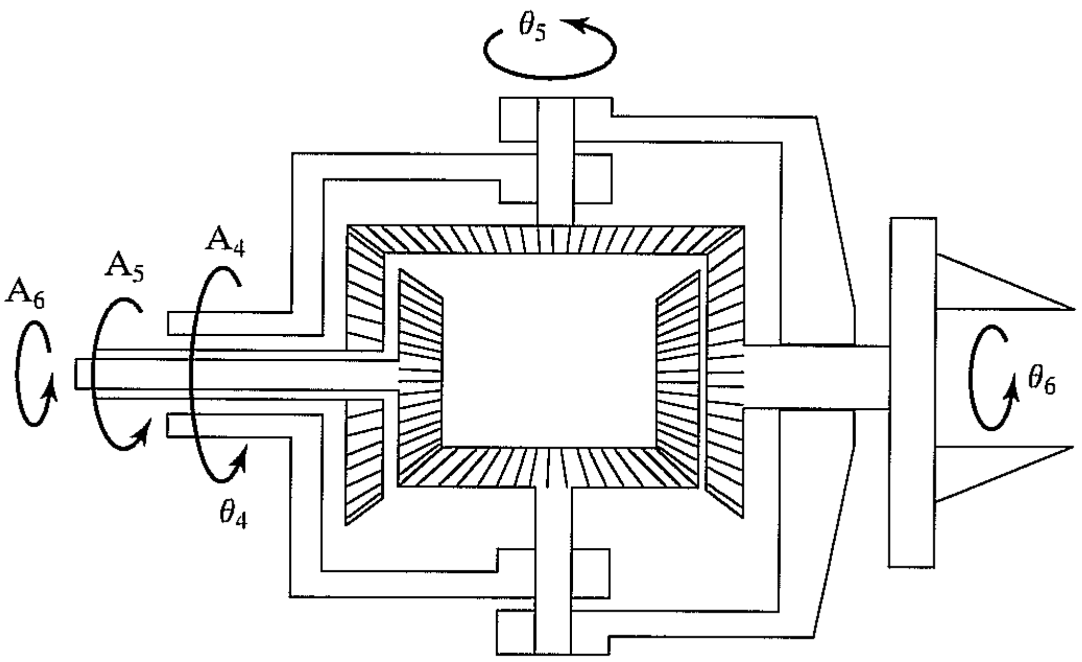
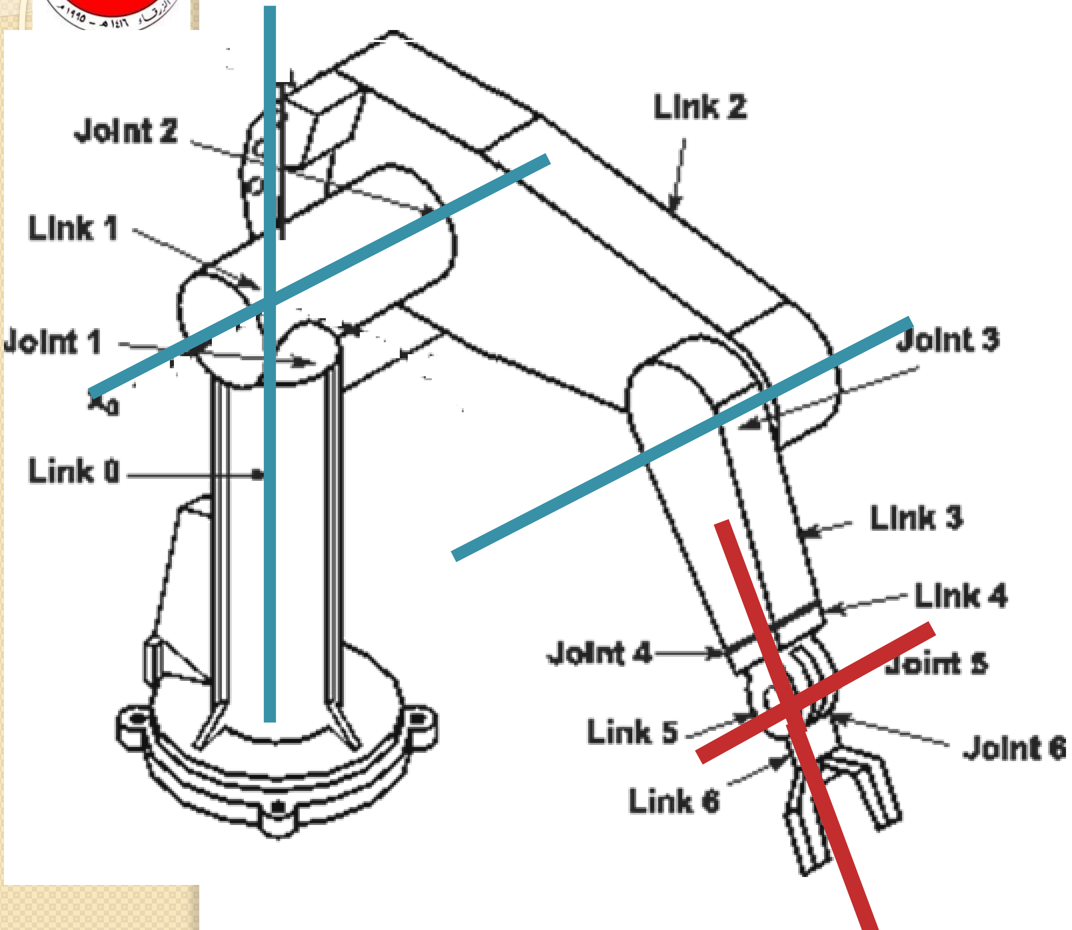
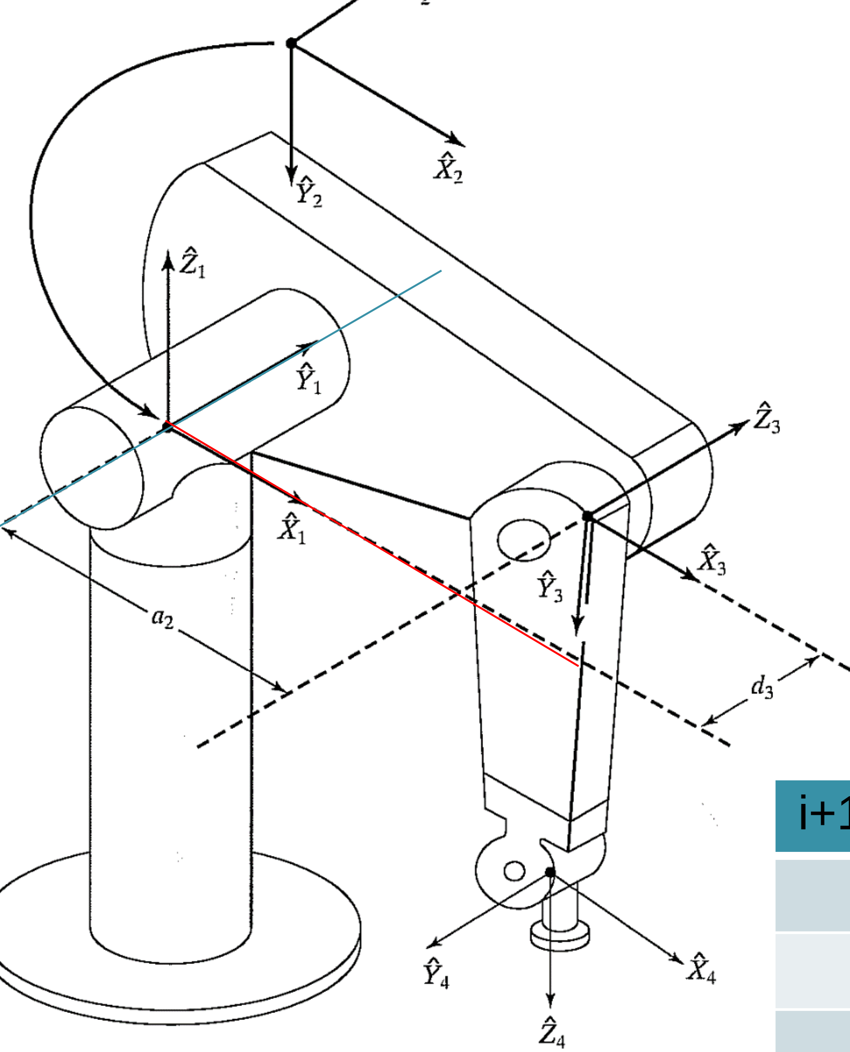
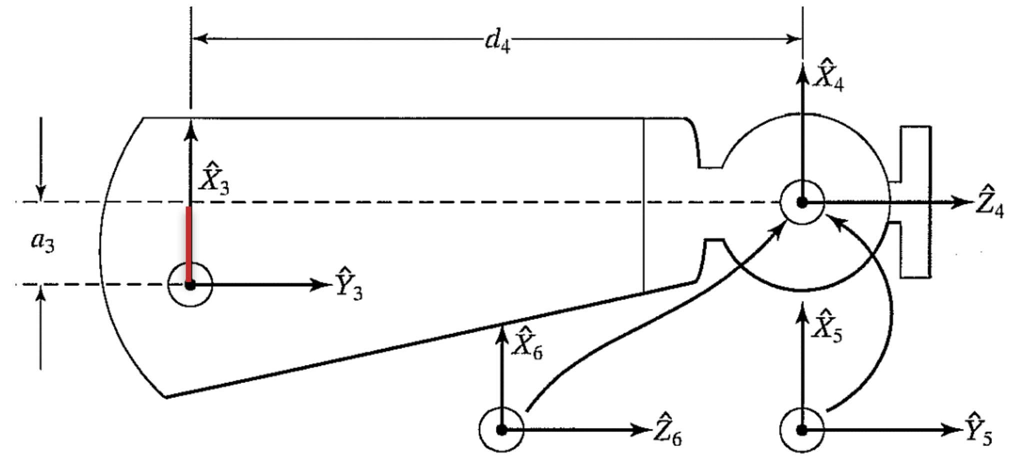


FIGURE 8.8: An orthogonal-axis wrist driven by remotely located actuators via three concentric shafts.



PUMA robot 6DOF





i+1	αi-1	ai-1	di	Theta	i
1	0	0	0	θ_1	0
2	-90	0	0	θ_2	1
3	0	a2	d3	θ_3	2
4	-90	a3	d4	θ_4	3
5	90	0	0	θ_5	4
6	-90	0	0	θ_6	5

See notes

$$r_{11} = c_1[c_{23}(c_4c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6),$$

$$r_{21} = s_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6 - c_1(s_4c_5c_6 + c_4s_6),$$

$$r_{31} = -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6,$$

$$r_{12} = c_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] + s_1(c_4c_6 - s_4c_5s_6),$$

$$r_{22} = s_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6),$$

$$r_{32} = -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6,$$

$$r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5,$$

$$r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5,$$

$$r_{33} = s_{23}c_4s_5 - c_{23}c_5,$$

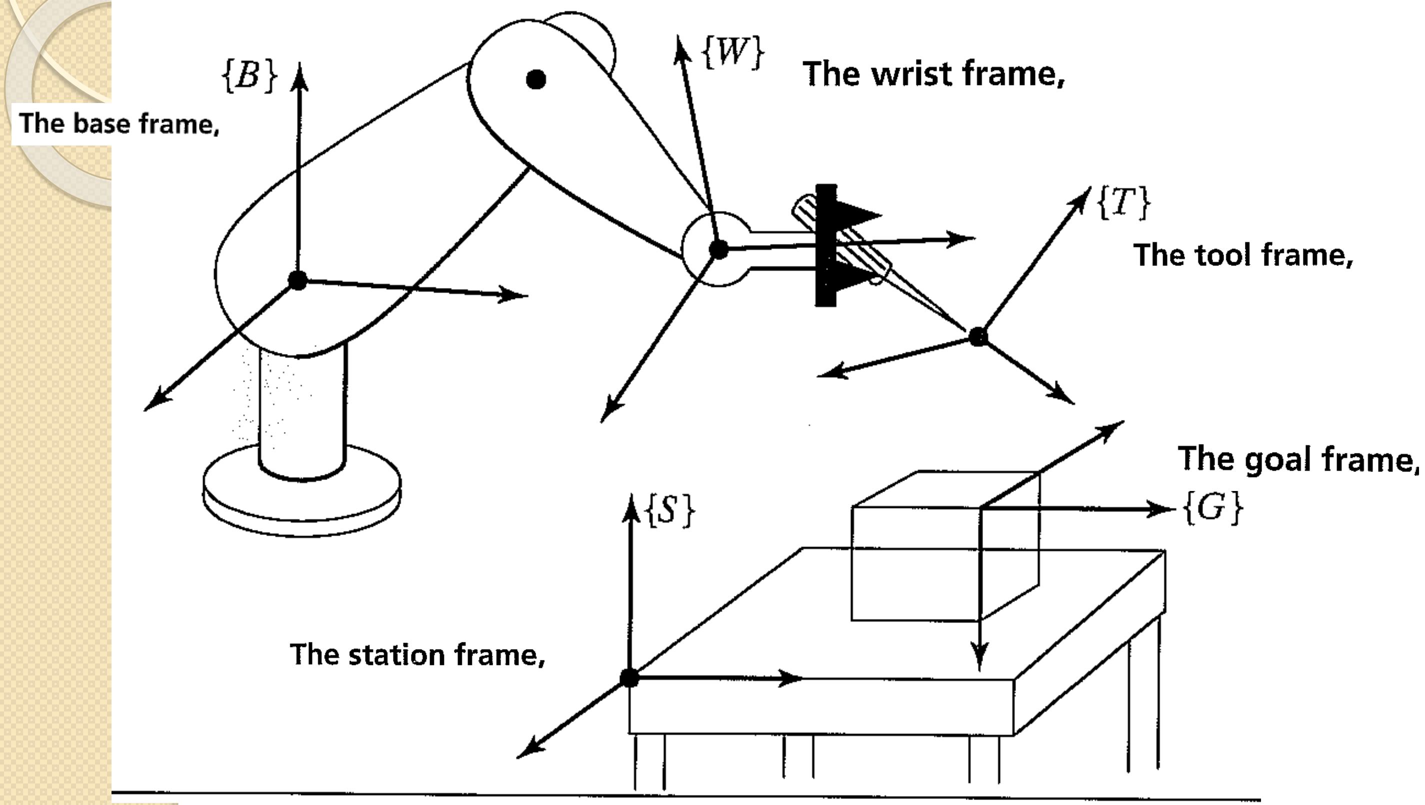
$$p_x = c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1,$$

$$p_y = s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] + d_3c_1,$$

$$p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}.$$

$$ {}^0_6T = {}^0_1T\, {}^1_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3.8 FRAMES WITH STANDARD NAMES



3.6 ACTUATOR SPACE, JOINT SPACE, AND CARTESIAN SPACE

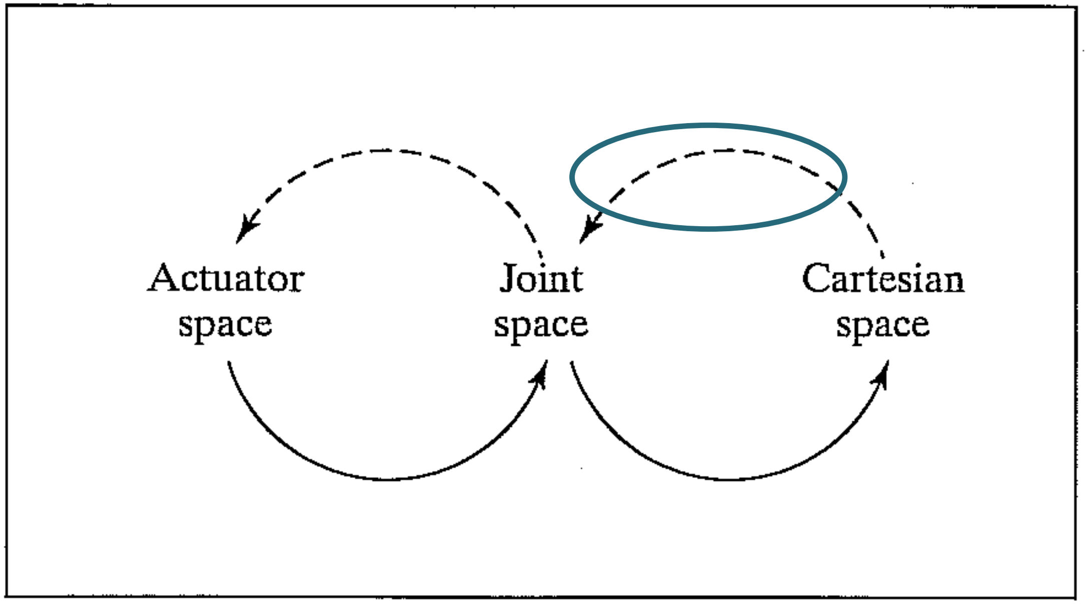
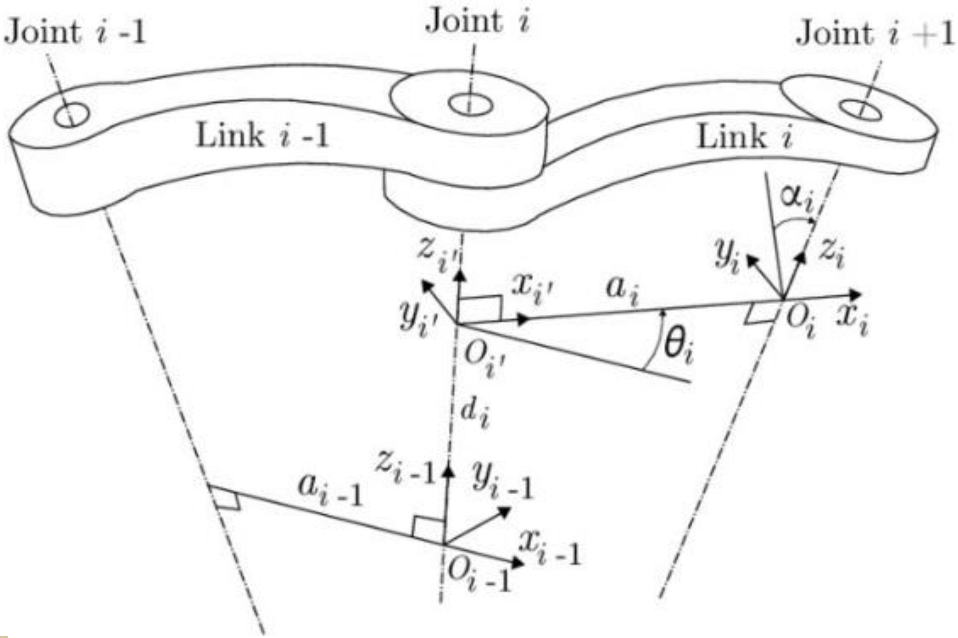


FIGURE 3.16: Mappings between kinematic description

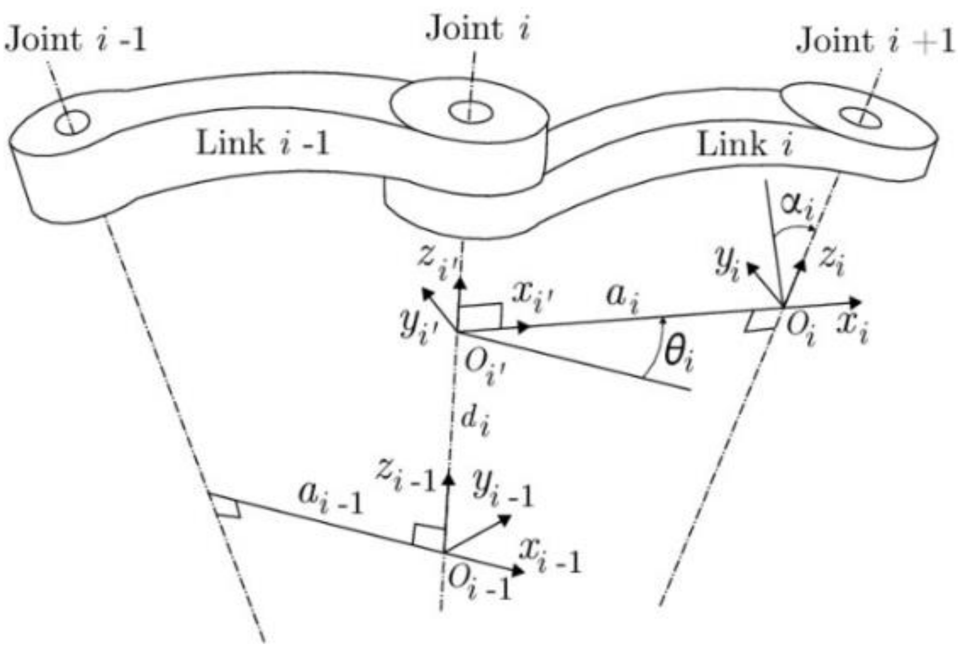
Below slides are self reading for your information only

Standard DH notation



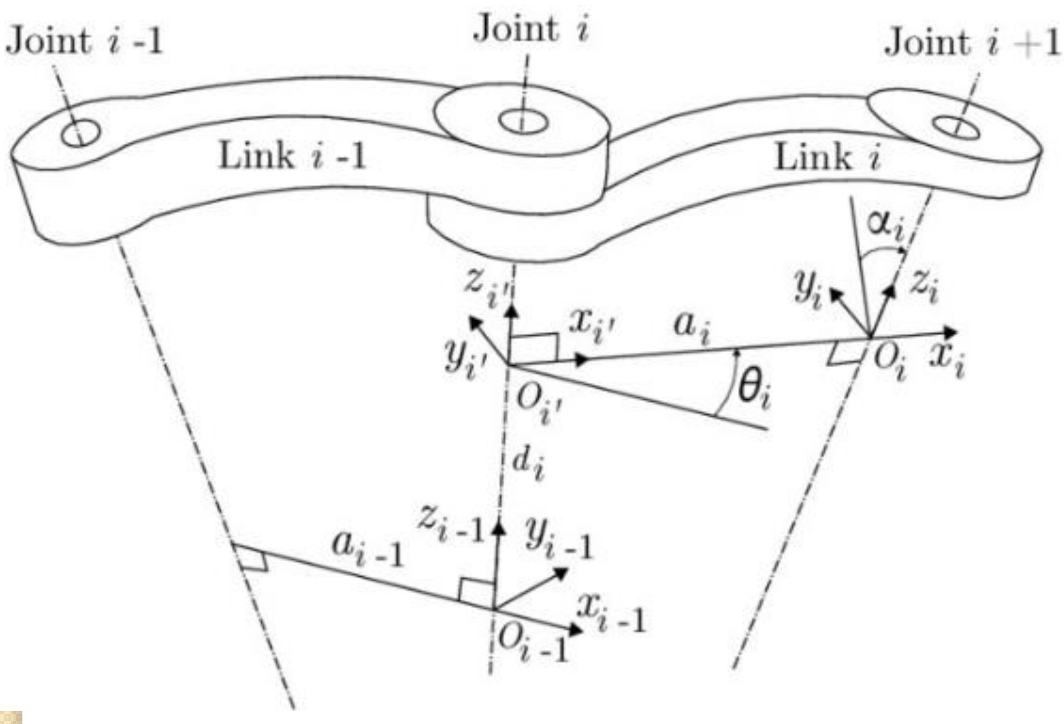
- Choose axis z_i along the axis of Joint $i + 1$.
- Locate the origin O_i at the intersection of axis z_i with the common normal[9] to axes z_{i-1} and z_i. Also, locate $O_{i'}$ at the intersection of the common normal with axis z_{i-1}.
- Choose axis x_i along the common normal to axes z_{i-1} and z_i with positive direction from Joint i to Joint $i + 1$.
- Choose axis y_i so as to complete a right-handed frame.

Standard DH notation



- For Frame 0, only the direction of axis z_0 is specified; then O_0 and x_0 can be arbitrarily chosen.
- For Frame n, since there is no Joint $n+1$, z_n is not uniquely defined while x_n has to be normal to axis z_{n-1}. Typically, Joint n is revolute, and thus z_n can be aligned with the direction of z_{n-1}.

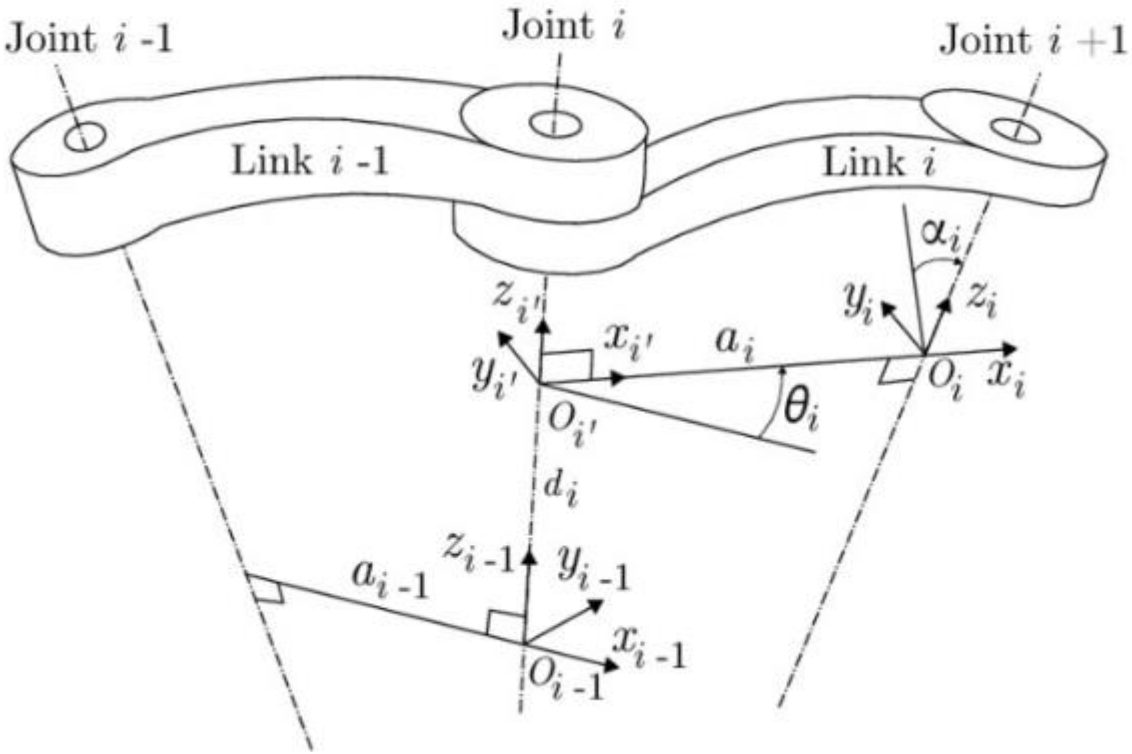
Standard DH notation



Frame i with respect to Frame $i-1$ are completely specified by the following *parameters*:

- a_i distance between O_i and $O_{i'}$,
- d_i coordinate of $O_{i'}$ along z_{i-1},
- α_i angle between axes z_{i-1} and z_i about axis x_i to be taken positive when rotation is made counter-clockwise,
- θ_i angle between axes x_{i-1} and x_i about axis z_{i-1} to be taken positive when rotation is made counter-clockwise.

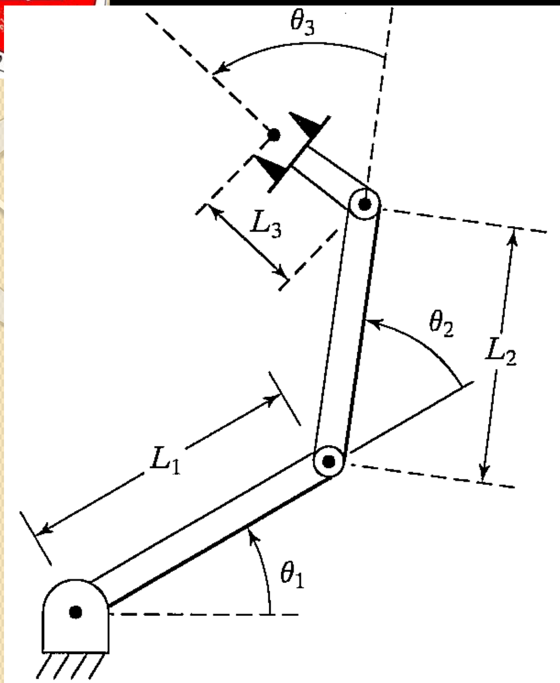
Standard DH notation



Standard Form

$$
{}^{i-1}_{\;\;i}T = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\alpha_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Standard DH notation



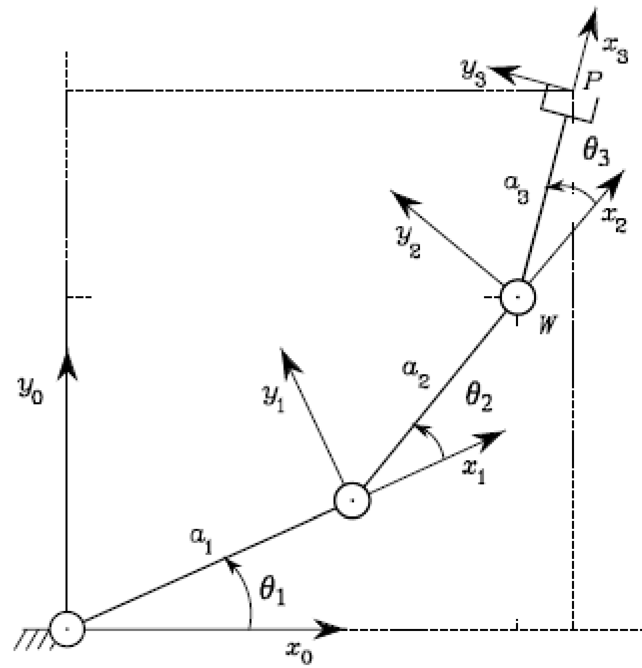


Table 2.1. DH parameters for the three-link planar arm

Link	a_i	α_i	d_i	ϑ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

$$ {}^{0}T_3 = \begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & r_{14} \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & r_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ r_{14} = l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) $$
$$ r_{24} = l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) $$

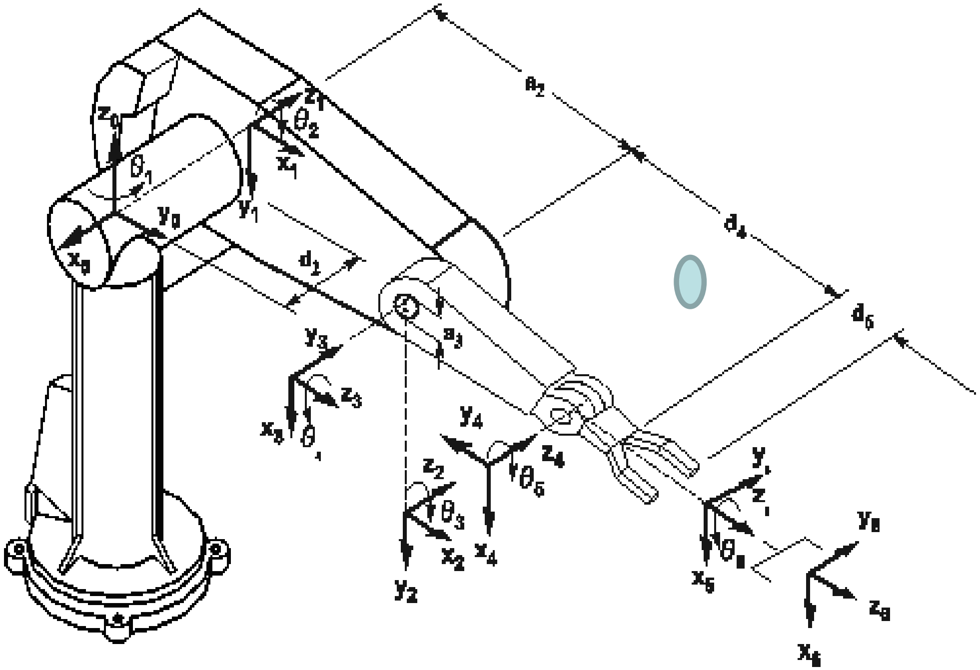
Robotics

Chapter 4
Inverse Manipulator Kinematics

4.1 INTRODUCTION

- Inverse Kinematics is the reverse of Forward Kinematics. (!)
- It is the calculation of joint values given the positions, orientations, and geometries of mechanism's parts

Given the numerical value of ${}^{0}_{N}T$, we attempt to find values of $\theta_1, \theta_2, \ldots, \theta_n$.



4.2 SOLVABILITY

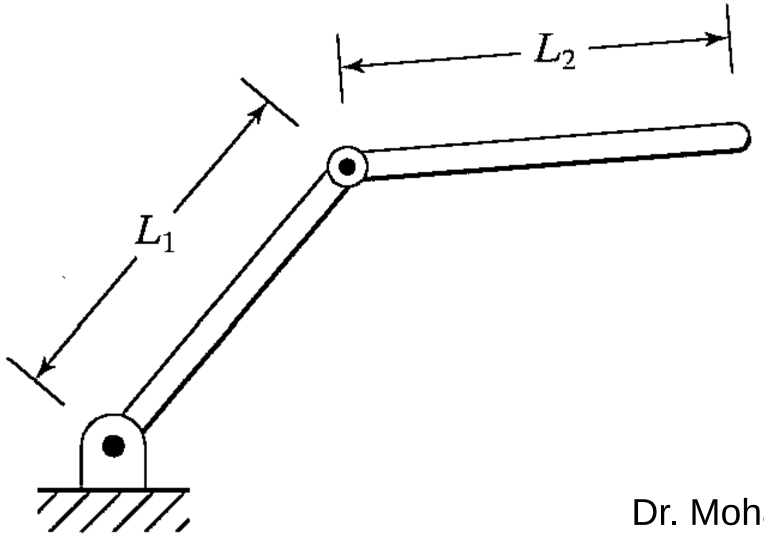
Existence of solutions

The question of whether any solution exists at all raises the question of the manipulator's **workspace**. Roughly speaking, workspace is that volume of space that the end-effector of the manipulator can reach. For a solution to exist, the specified goal point must lie within the workspace. Sometimes, it is useful to consider two definitions of workspace: **Dextrous workspace** is that volume of space that the robot end-effector can reach with all orientations. That is, at each point in the dextrous workspace, the end-effector can be arbitrarily oriented. The **reachable workspace** is that volume of space that the robot can reach in at least one orientation. Clearly, the dextrous workspace is a subset of the reachable workspace.

If $l_1 = l_2$: If $l_1 \neq l_2$,

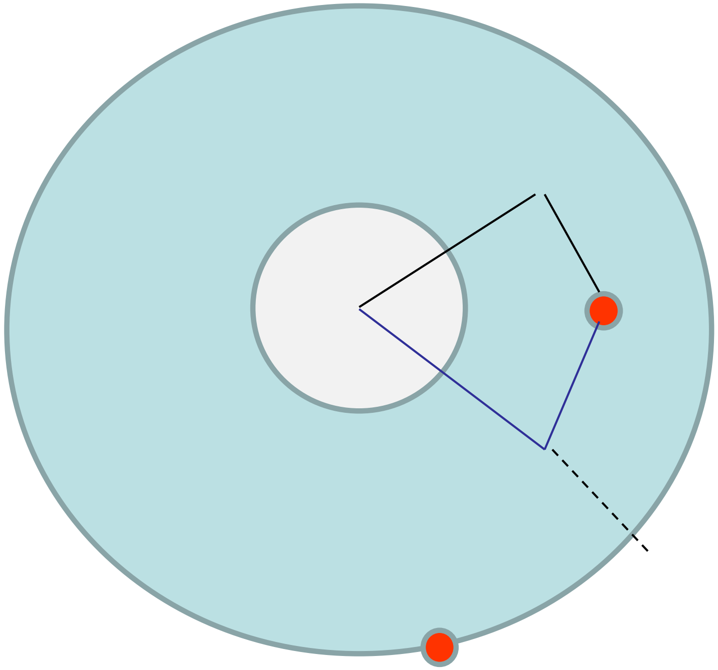
Consider the workspace of the two-link manipulator in Fig. 4.1. If $l_1 = l_2$, then the reachable workspace consists of a disc of radius $2l_1$. The dextrous workspace consists of only a single point, the origin. If $l_1 \neq l_2$, then there is no dextrous workspace, and the reachable workspace becomes a ring of outer radius $l_1 + l_2$ and inner radius $|l_1 - l_2|$. Inside the reachable workspace there are two possible orientations of the end-effector. On the boundaries of the workspace there is only one possible orientation.

$$0 \le \theta_1 \le 360, \quad 0 \le \theta_2 \le 360$$



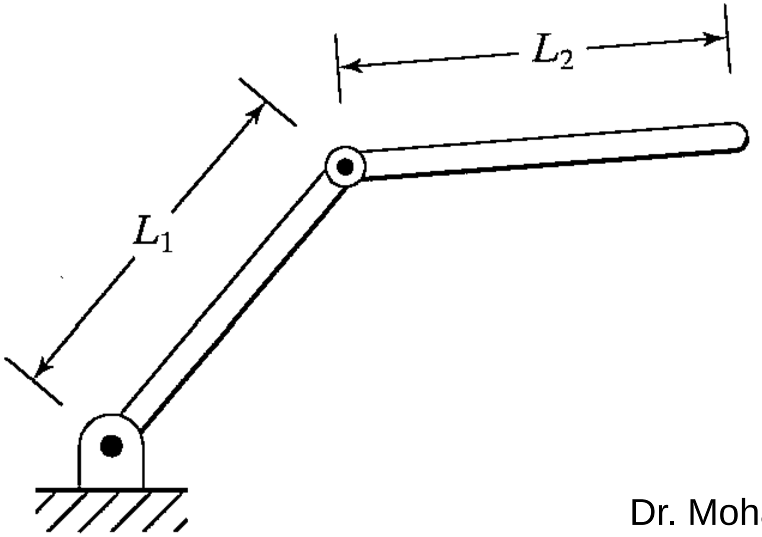
- See notes

If $L_1 > L_2$



- Reachable Workspace
- Dextrous workspace
- No of solutions (inner and boundary)

$$0 \le \theta_1 \le 360, \quad 0 \le \theta_2 \le 360$$

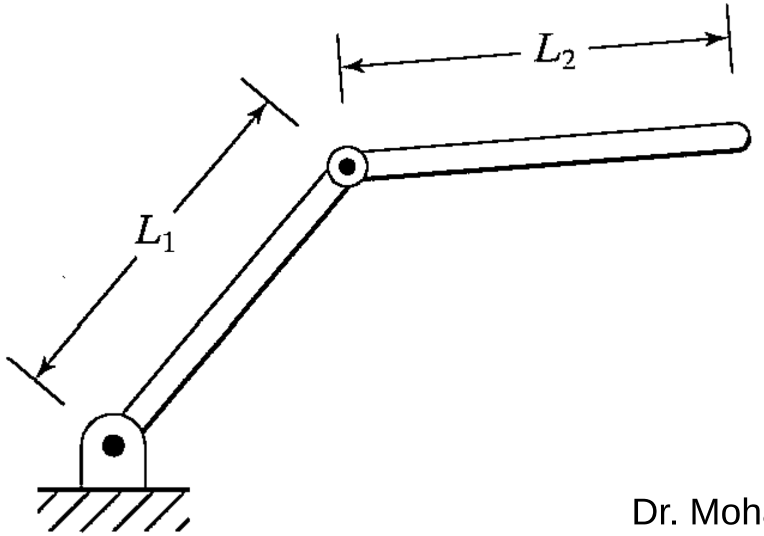


- See notes

If $L_1 > L_2$

- Reachable Workspace
- Dextrous workspace
- No of solutions (inner and boundary)

$$0 \le \theta_1 \le 360, \quad 0 \le \theta_2 \le 180$$

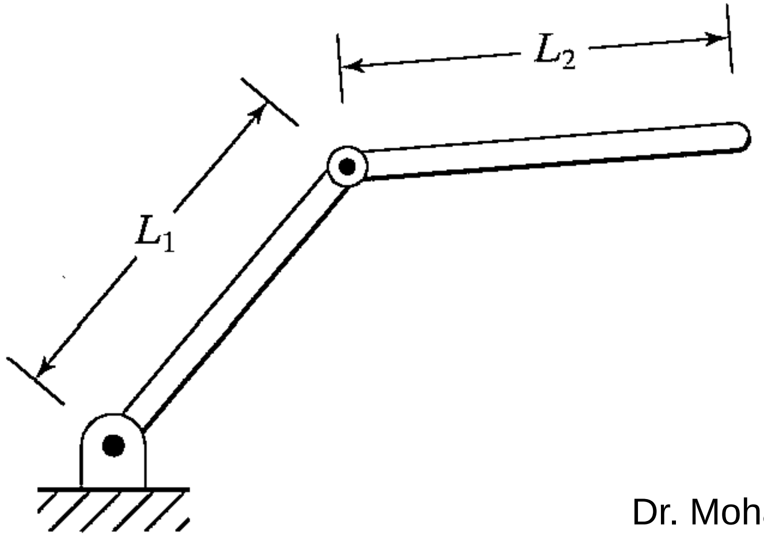


- See notes

If $L_2 > L_1$

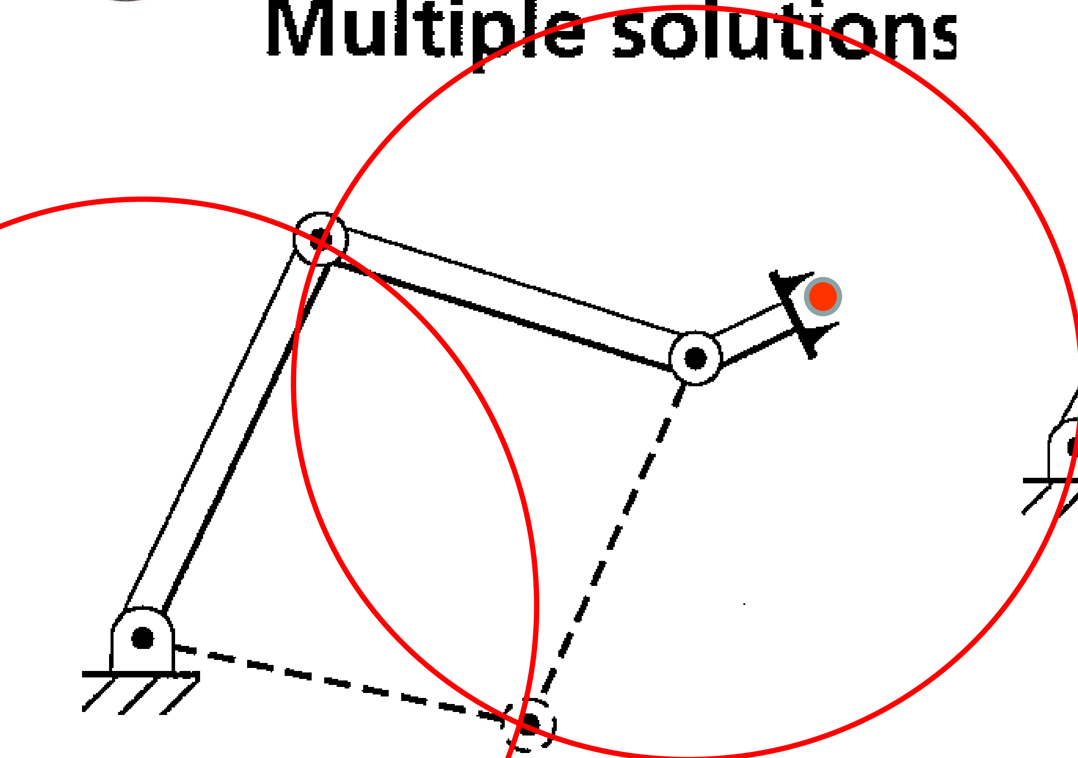
- Reachable Workspace
- Dextrous workspace
- No of solutions (inner and boundary)

$$0 \le \theta_1 \le 360, \quad 0 \le \theta_2 \le 360$$

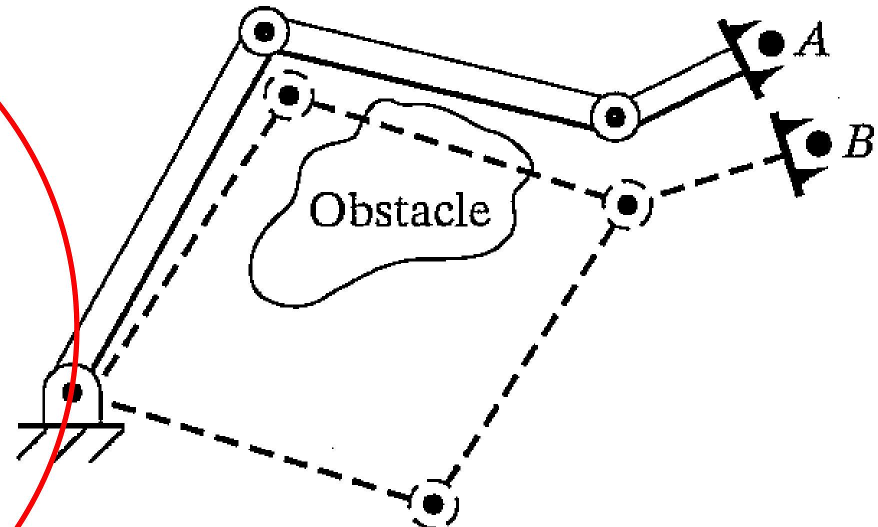


- See notes

Multiple solutions



Dashed lines indicate a second solution.



closest solution.

. The presence of obstacles

moving smaller joints

PUMA 560 can reach certain goals with eight different solutions.

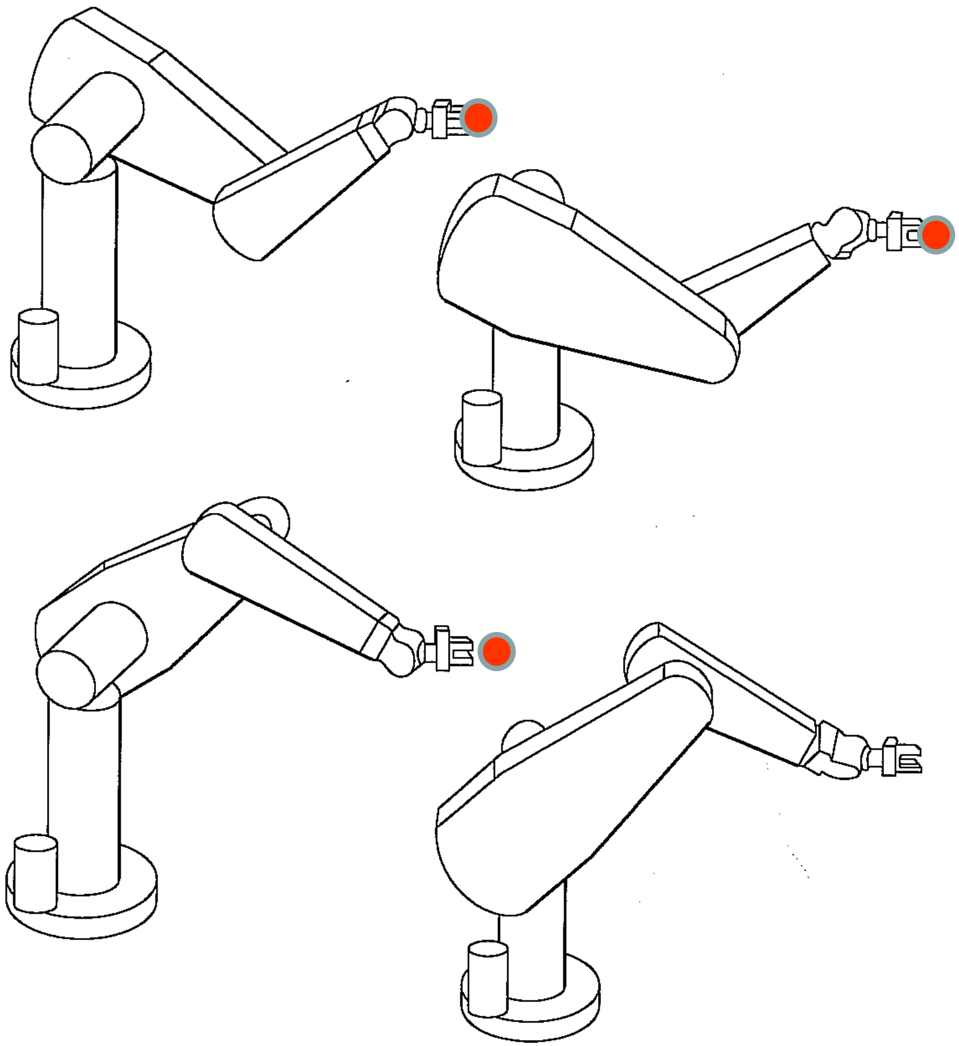


FIGURE 4.4: Four solutions of the PUMA 560.

$$\theta_4' = \theta_4 + 180°,$$

$$\theta_5' = -\theta_5,$$

$$\theta_6' = \theta_6 + 180°.$$

a_i	Number of solutions
$a_1 = a_3 = a_5 = 0$	$\leqslant 4$
$a_3 = a_5 = 0$	$\leqslant 8$
$a_3 = 0$	$\leqslant 16$
All $a_i \neq 0$	$\leqslant 16$

FIGURE 4.5: Number of solutions vs. nonzero a_i.

Method of solution

closed-form solutions and **numerical solutions**.

We will restrict our attention to closed-form solution methods.

"closed form" means a solution method based on analytic expressions

Within the class of closed-form solutions, we distinguish two methods of obtaining the solution: **algebraic** and **geometric**. These distinctions are somewhat hazy: Any geometric methods brought to bear are applied by means of algebraic expressions, so the two methods are similar. The methods differ perhaps in approach only.

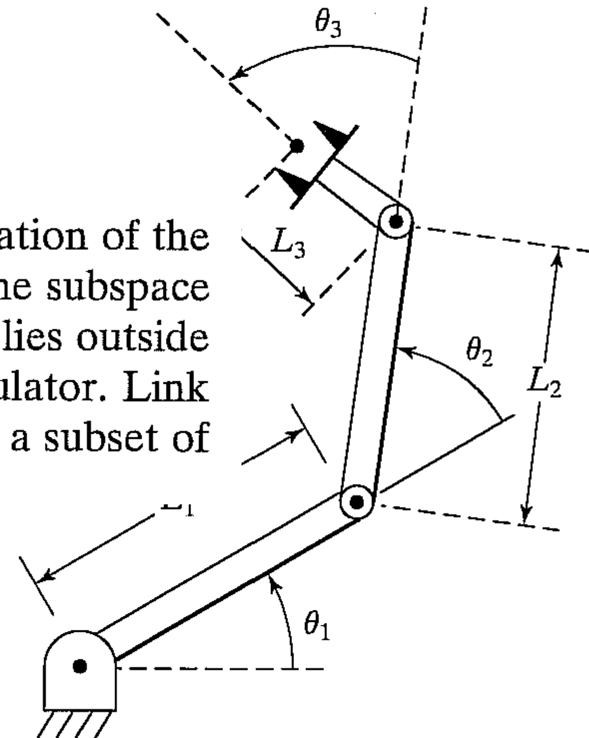
A major recent result in kinematics is that, according to our definition of solvability, *all systems with revolute and prismatic joints having a total of six degrees of freedom in a single series chain are solvable*. However, this general solution is a numerical one. Only in special cases can robots with six degrees of freedom be solved analytically. These robots for which an analytic (or closed-form) solution exists are characterized either by having several intersecting joint axes or by having many α_i equal to 0 or ± 90 degrees. Calculating numerical solutions is generally time consuming relative to evaluating analytic expressions; hence, it is considered very important to design a manipulator so that a closed-form solution exists. Manipulator designers discovered this very soon, and now virtually all industrial manipulators are designed sufficiently simply that a closed-form solution can be developed.

4.3 THE NOTION OF MANIPULATOR SUBSPACE WHEN $n < 6$

Give a description of the subspace of ${}^B_W T$ for the three-link manipulator

The subspace of ${}^B_W T$ is given by

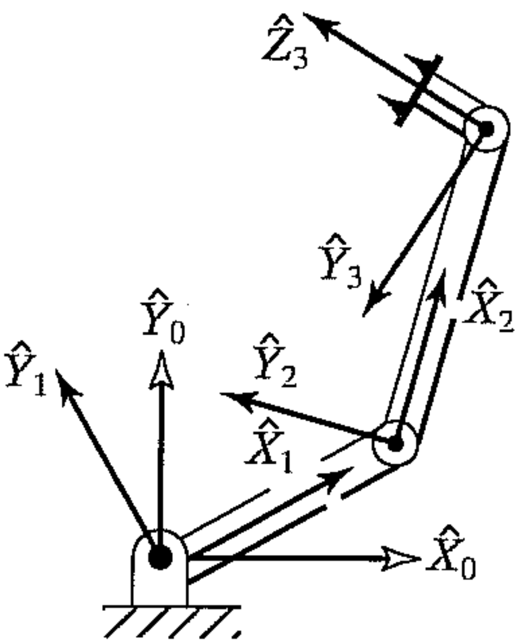
$$
{}^B_W T = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$



where x and y give the position of the wrist and ϕ describes the orientation of the terminal link. As x, y, and ϕ are allowed to take on arbitrary values, the subspace is generated. Any wrist frame that does not have the structure of (4.2) lies outside the subspace (and therefore lies outside the workspace) of this manipulator. Link lengths and joint limits restrict the workspace of the manipulator to be a subset of this subspace.

Algebraic solution

$$ {}^{0}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0.0 & l_1s_1 + l_2s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$



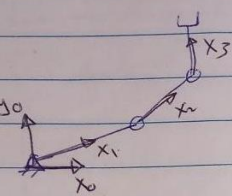
given

$$ {}^{0}_{3}T = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Algebraic solution

Inverse Kinematic (Algebric method).

$$^{0}_{3}T = \begin{bmatrix} C_{123} & -S_{123} & 0 & L_1C_1+L_2C_{12} \\ S_{123} & C_{123} & 0 & L_1S_1+L_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\phi & -S\phi & 0 & X \\ S\phi & C\phi & 0 & Y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$X = L_1C_1 + L_2C_{12}$$

$$Y = L_1S_1 + L_2S_{12}$$

But $C_1C_2 \mp S_1S_2 = C(1\pm 2)$

$$X^2+Y^2 = (L_1^2C_1^2 + L_2^2C_{12}^2 + 2L_1C_1L_2C_{12}) + (L_1^2S_1^2 + L_2^2S_{12}^2 + 2L_1S_1L_2S_{12})$$

$$X^2+Y^2 = L_1^2 + L_2^2 + 2L_1L_2(C_1C_{12} + S_1S_{12})$$

$$X^2+Y^2 = L_1^2 + L_2^2 + 2L_1L_2\, C(\theta_1 - (\theta_1+\theta_2)) = L_1^2 + L_2^2 + 2L_1L_2C_2$$

$$\therefore C_2 = \frac{X^2+Y^2-L_1^2-L_2^2}{2L_1L_2}$$

$-1 \leq C_2 \leq 1$ solution exist. otherwise point out of workspace

$$S_2 = \mp\sqrt{1-C_2^2}$$

$$\theta_2 = A\tan 2(S_2, C_2)$$

$$X = L_1C_1 + L_2(C_1C_2 - S_1S_2) = (L_1+L_2C_2)C_1 - (L_2S_2)S_1$$

$$y = L_1S_1 + L_2(C_1S_2 + S_1C_2) = (L_1+L_2C_2)S_1 + (L_2S_2)C_1$$

$K_1 = L_1 + L_2C_2 \qquad K_2 = L_2S_2$

$$\therefore X = K_1C_1 - K_2S_1$$

$$[y = K_2C_1 + K_1S_1]\ \frac{K_2}{K_1}$$

Algebraic solu

$$K_1^2 + K_2^2 = (l_1 + l_2 C_2)^2 + (l_2 S_2)^2 = l_1^2 + l_2^2 C_2^2 + 2 l_1 l_2 C_2 + l_2^2 S_2^2$$

$$K_1^2 + K_2^2 = l_1^2 + l_2^2 + 2 l_1 l_2 C_2$$

$$C_1 = \frac{x K_1 + K_2 y}{K_1^2 + K_2^2} = \frac{(l_1 + l_2 C_2) x + (l_2 S_2) y}{l_1^2 + l_2^2 + 2 l_1 l_2 C_2}$$

Same way

$$S_1 = \frac{(l_1 + l_2 C_2) y - (l_2 S_2) x}{l_1^2 + l_2^2 + 2 l_1 l_2 C_2}$$

$$\theta_1 = Atan2(S_1, C_1)$$

$$\phi = Atan2(S\phi, C\phi) = Atan2(r_{21}, r_{11})$$

$$\theta_3 = \phi - \theta_1 - \theta_2$$

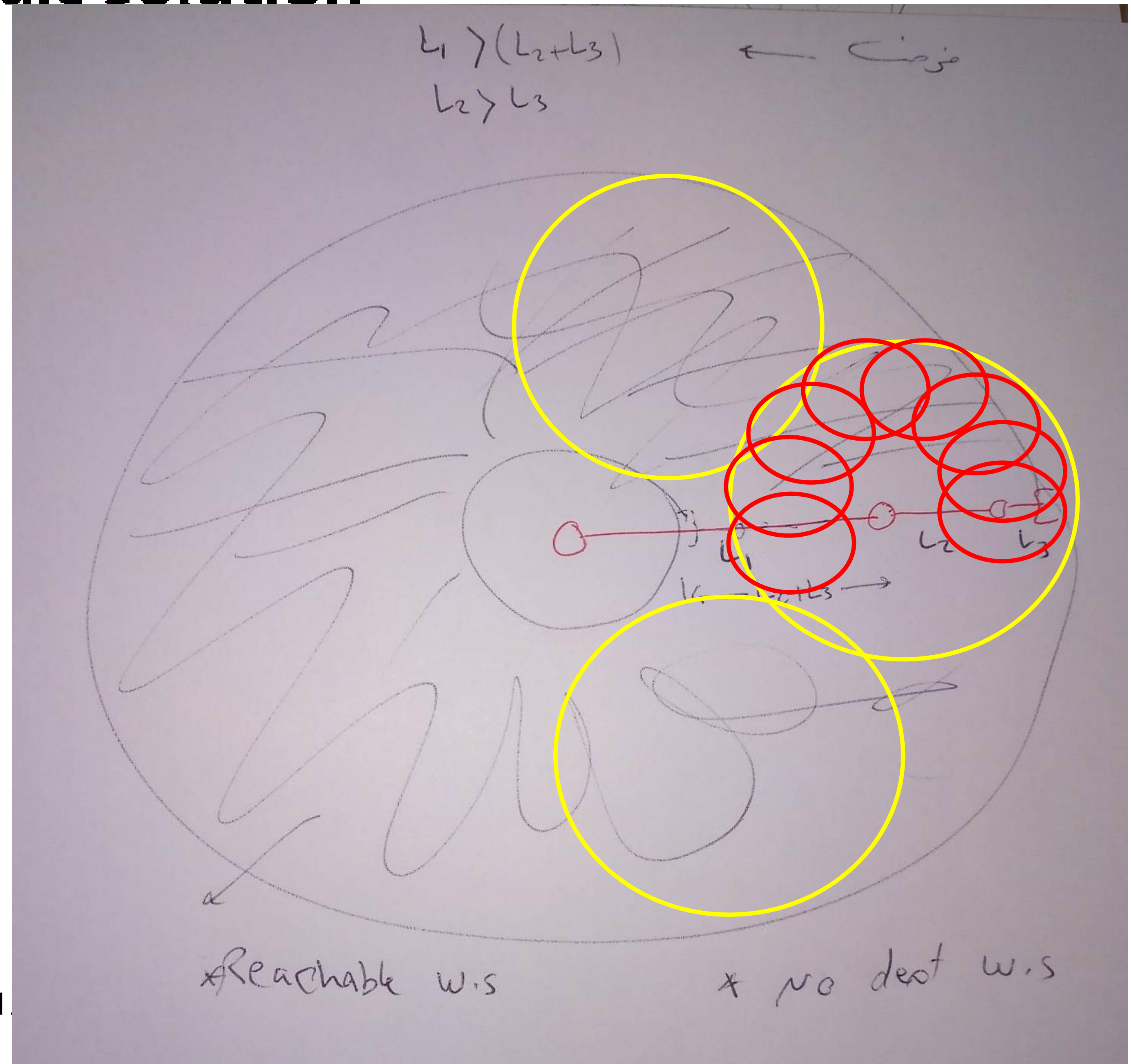
Appendix C-12

$$a \cos\theta - b \sin\theta = c$$

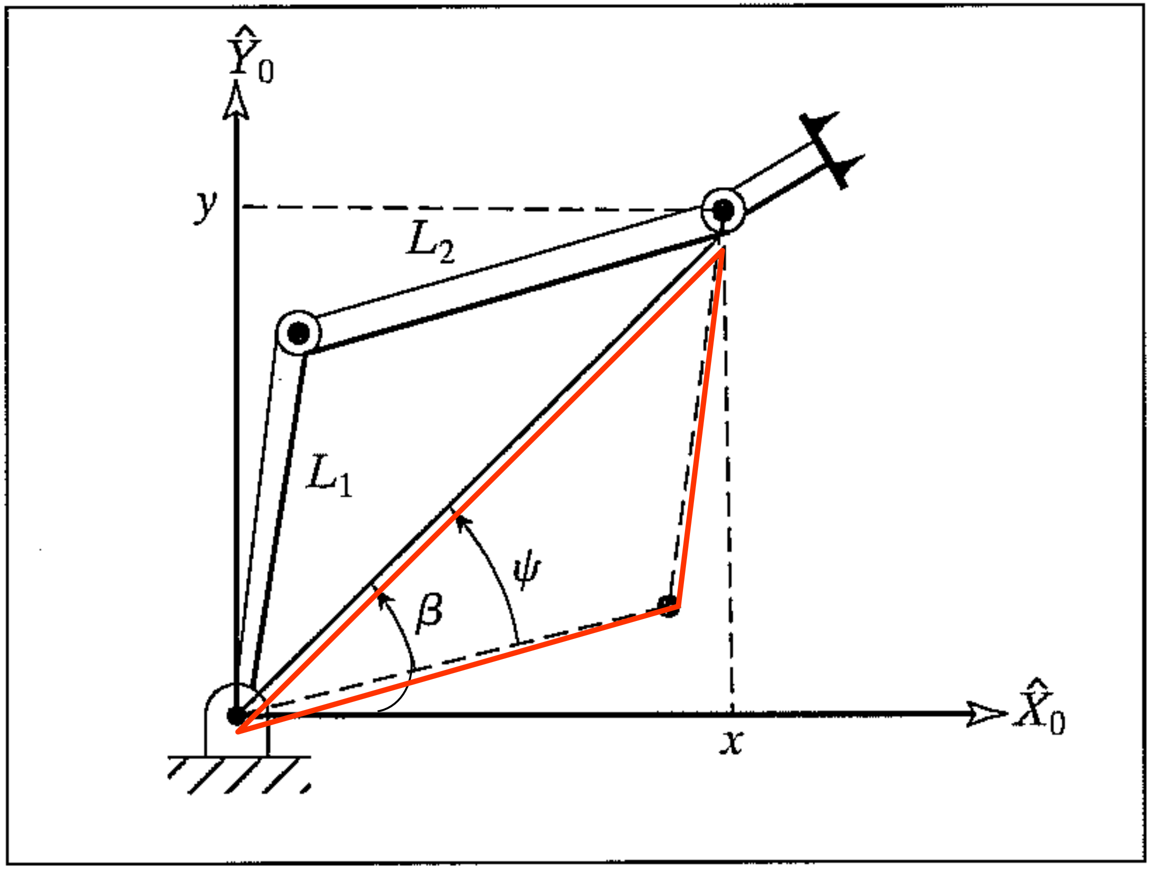
$$a \sin\theta + b \cos\theta = d$$

$$\theta = Atan2(ad - bc, ac + bd)$$

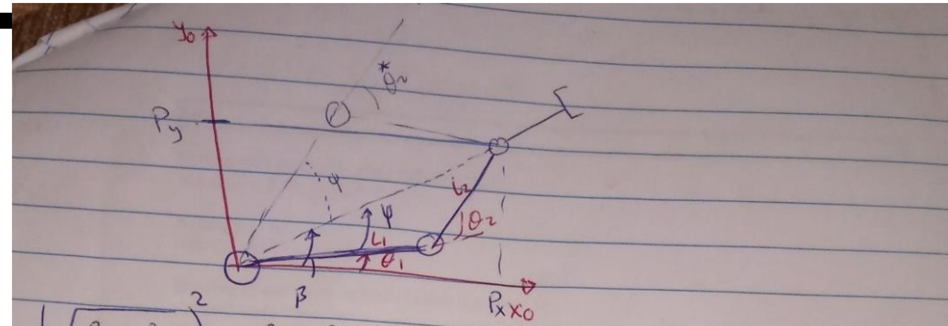
Algebraic solution



Geometric solution



Geometric solution



$$\left(\sqrt{P_x^2+P_y^2}\right)^2 = l_1^2 + l_2^2 - 2l_1l_2\, C(180-\theta_2)$$

$$P_x^2+P_y^2 = l_1^2+l_2^2 - 2l_1l_2\left[C180\, C_2 + S180\, S_2\right] = l_1^2+l_2^2-2l_1l_2(-C_2)$$

$$P_x^2+P_y^2 = l_1^2+l_2^2+2l_1l_2C_2$$

$$\Rightarrow C_2 = \frac{P_x^2+P_y^2-l_1^2-l_2^2}{2l_1l_2} \qquad 0<\theta_2<180 \text{ (elbow down)}$$

$$\theta_2 = \cos^{-1}\left(\frac{P_x^2+P_y^2-l_1^2-l_2^2}{2l_1l_2}\right) \qquad \theta_2^* = -\theta_2 \text{ (elbow up)}$$

$$-180<\theta_2^*<0$$

* $\beta = Atan2(P_y, P_x)$ $\qquad 0<\beta<360$

* $l_2^2 = P_x^2+P_y^2+l_1^2-2l_1\sqrt{P_x^2+P_y^2}\, C\psi$

$$C\psi = \frac{P_x^2+P_y^2+l_1^2-l_2^2}{2l_1\sqrt{x^2+y^2}} \qquad 0\le\psi\le180$$

$$\psi = \cos^{-1}\left(\frac{P_x^2+P_y^2+l_1^2-l_2^2}{2l_1\sqrt{x^2+y^2}}\right)$$

$\theta_1 = \beta-\psi$ $\qquad \theta_2>0$

$\theta_1 = \beta+\psi$ $\qquad \theta_2^*$: $\theta_2<0$

$\theta_3 = \phi-\theta_1-\theta_2$

ALGEBRAIC SOLUTION BY REDUCTION TO POLYNOMIAL

Transcendental equations are often difficult to solve because, even when there is only one variable (say, θ), it generally appears as $\sin\theta$ and $\cos\theta$. Making the following substitutions, however, yields an expression in terms of a single variable, u:

$$u = \tan\frac{\theta}{2},$$

$$\cos\theta = \frac{1-u^2}{1+u^2}, \qquad (4.35)$$

$$\sin\theta = \frac{2u}{1+u^2}.$$

See notes

EXAMPLE 4.3

Convert the transcendental equation

$$a\cos\theta + b\sin\theta = c$$

into a polynomial in the tangent of the half angle, and solve for θ.

Substituting from (4.35) and multiplying through by $1+u^2$, we have

$$a(1-u^2) + 2bu = c(1+u^2).$$

Collecting powers of u yields

$$(a+c)u^2 - 2bu + (c-a) = 0,$$

which is solved by the quadratic formula:

$$u = \frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a+c}.$$

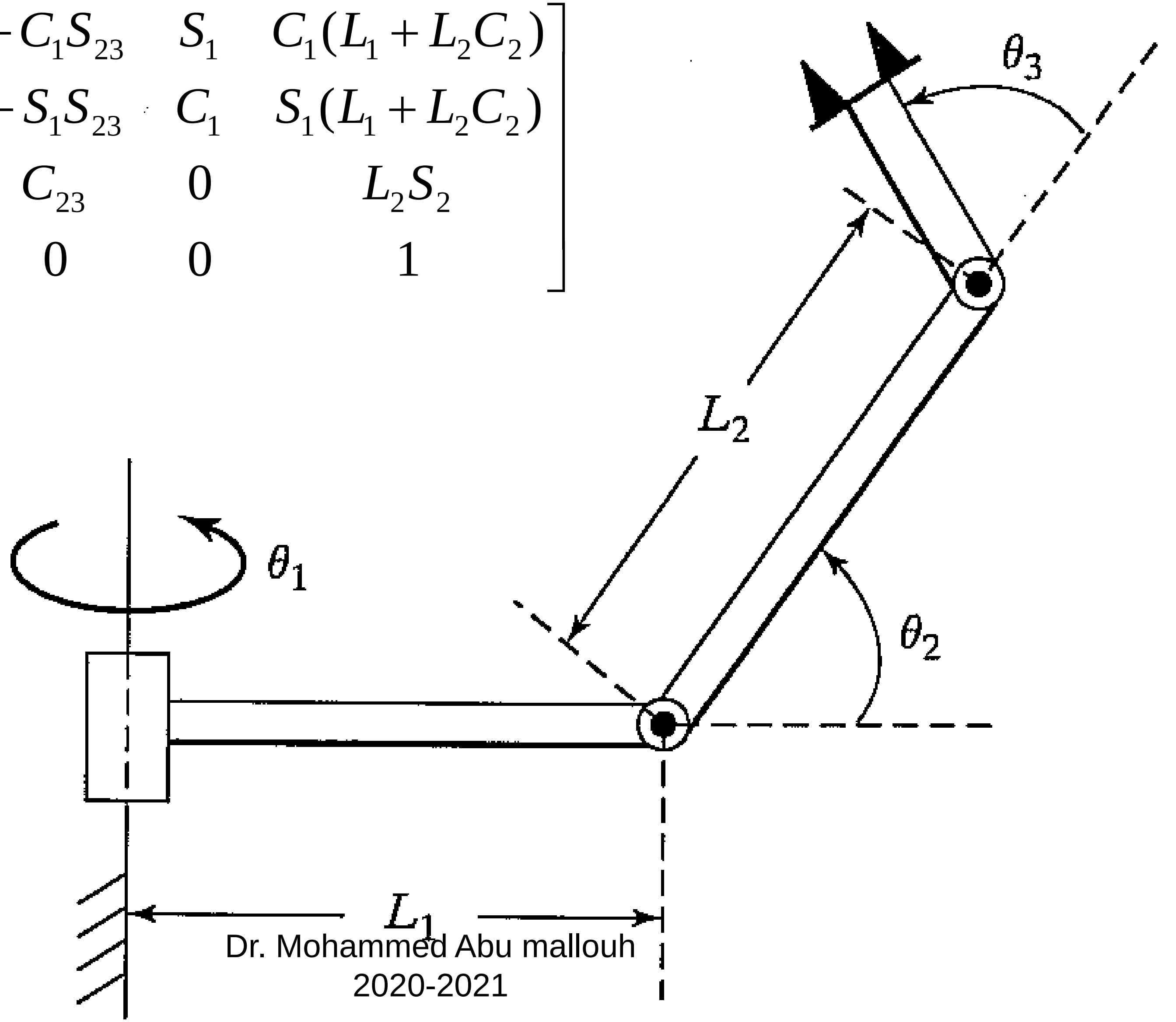
Hence,

$$\theta = 2\tan^{-1}\left(\frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a+c}\right).$$

Appendix C another solution

- Given the below transformation matrix solve inverse kinematic problem
- Sketch the workspace

$$
{}^{0}_{3}T = \begin{bmatrix} C_1 C_{23} & -C_1 S_{23} & S_1 & C_1(L_1 + L_2 C_2) \\ S_1 S_{23} & -S_1 S_{23} & C_1 & S_1(L_1 + L_2 C_2) \\ S_{23} & C_{23} & 0 & L_2 S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



3	0	L_2	0	θ_3	2

$$
{}^0_3T = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & (L_1+L_2C_2)C_1 \\ S_1S_{23} & -S_1S_{23} & +C_1 & (L_1+L_2C_2)S_1 \\ S_{23} & C_{23} & 0 & L_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$\theta_1 = \text{Atan2}(r_{13}, r_{23})$ or $\text{Atan2}(P_y, P_x)$

$$P_x^2 + P_y^2 = (L_1+L_2C_2)^2 C_1^2 + (L_1+L_2C_2)^2 S_1^2 = (L_1+L_2C_2)^2$$

$$C_2 = \frac{\pm\sqrt{P_x^2+P_y^2} - L_1}{L_2}$$

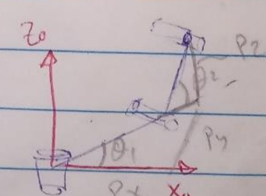
$$S_2 = \frac{P_z}{L_2}$$

$\Rightarrow \theta_2 = \text{Atan2}(S_2, C_2)$

$= \text{Atan2}\left(P_z, \sqrt{P_x^2+P_y^2} - L\right.$

$\theta_{23} = \text{Atan2}(r_{31}, r_{32})$

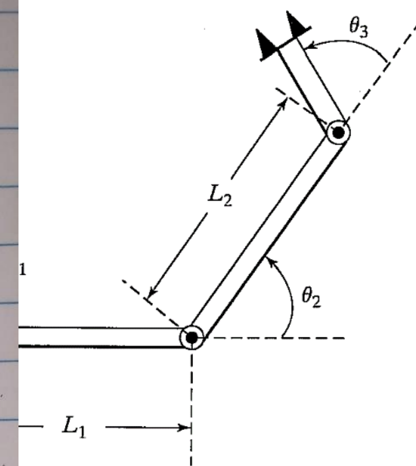
$\theta_3 = \theta_{23} - \theta_2$

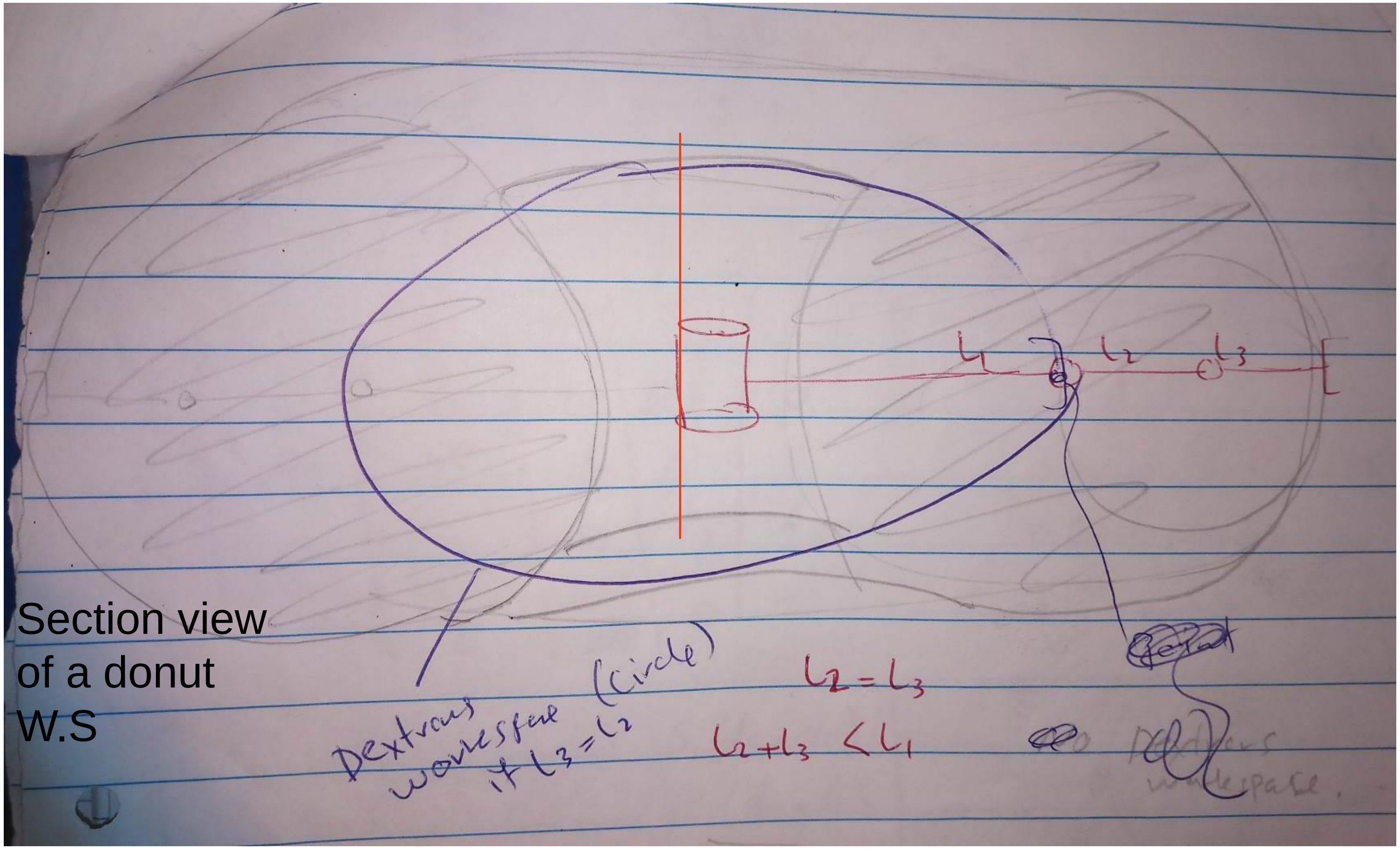


$\theta_2 = \text{Atan2}\left(P_z, \sqrt{P_x^2+P_y^2} - L_1\right)$

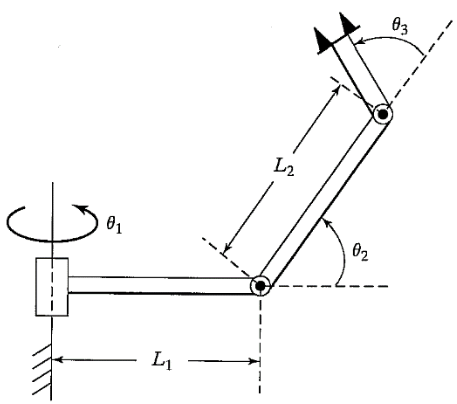
$\theta_2 = \sin^{-1}\left(\frac{r_{34}}{L_2}\right)$

$\theta_1 = \text{At}\ldots$

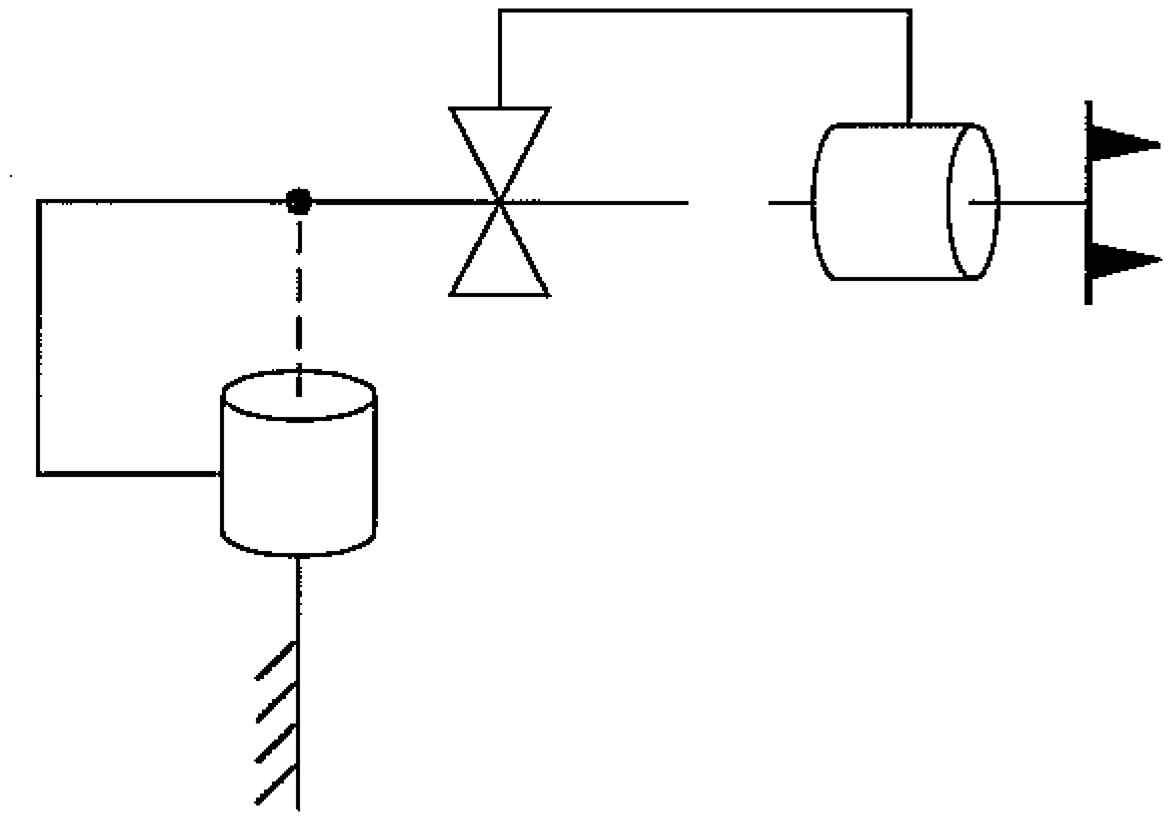




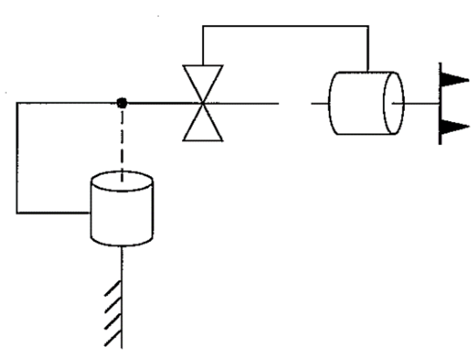
Section view of a donut W.S



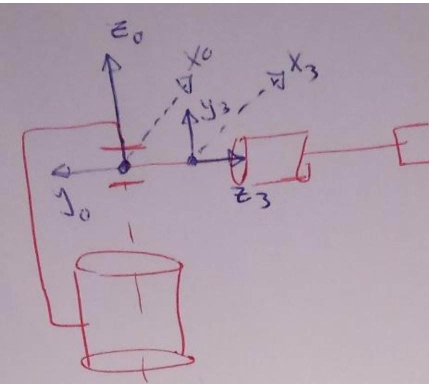
- solve inverse kinematic problem
- Sketch the workspace



(b)



(b)



$$
{}^{0}_{3}T = \begin{bmatrix} C_1C_3 & -C_1S_3 & S_1 & S_1 d_2 \\ S_1C_3 & -S_1S_3 & -C_1 & -C_1 d_2 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Algebric

$\theta_1 = Atan2(r_{13}, -r_{23})$ OR

OR

$\theta_1 = Atan2(r_{14}, -r_{24}) = Atan2(P_x, -P_y)$ (with $r_{14} = P_x$, $r_{24} = P_y$)

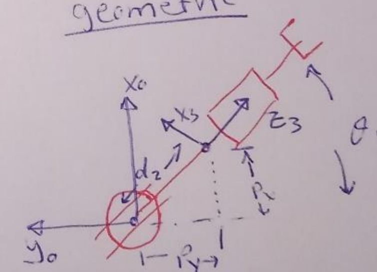
$S_1 d_2 = P_x \Rightarrow d_2 = \frac{P_x}{S_1}$

OR

$-C_1 d_2 = P_y \Rightarrow d_2 = \frac{P_y}{-C_1}$

$\theta_3 = Atan2(r_{31}, r_{32})$

geometric



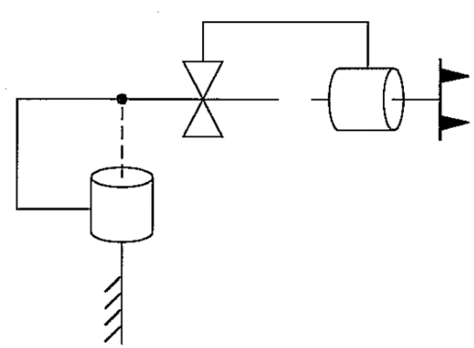
$\theta_1 = Atan2(P_x, -P_y)$

Note $S_1 = \frac{P_x}{d_2}$, $C_1 = \frac{-P_y}{d_2}$

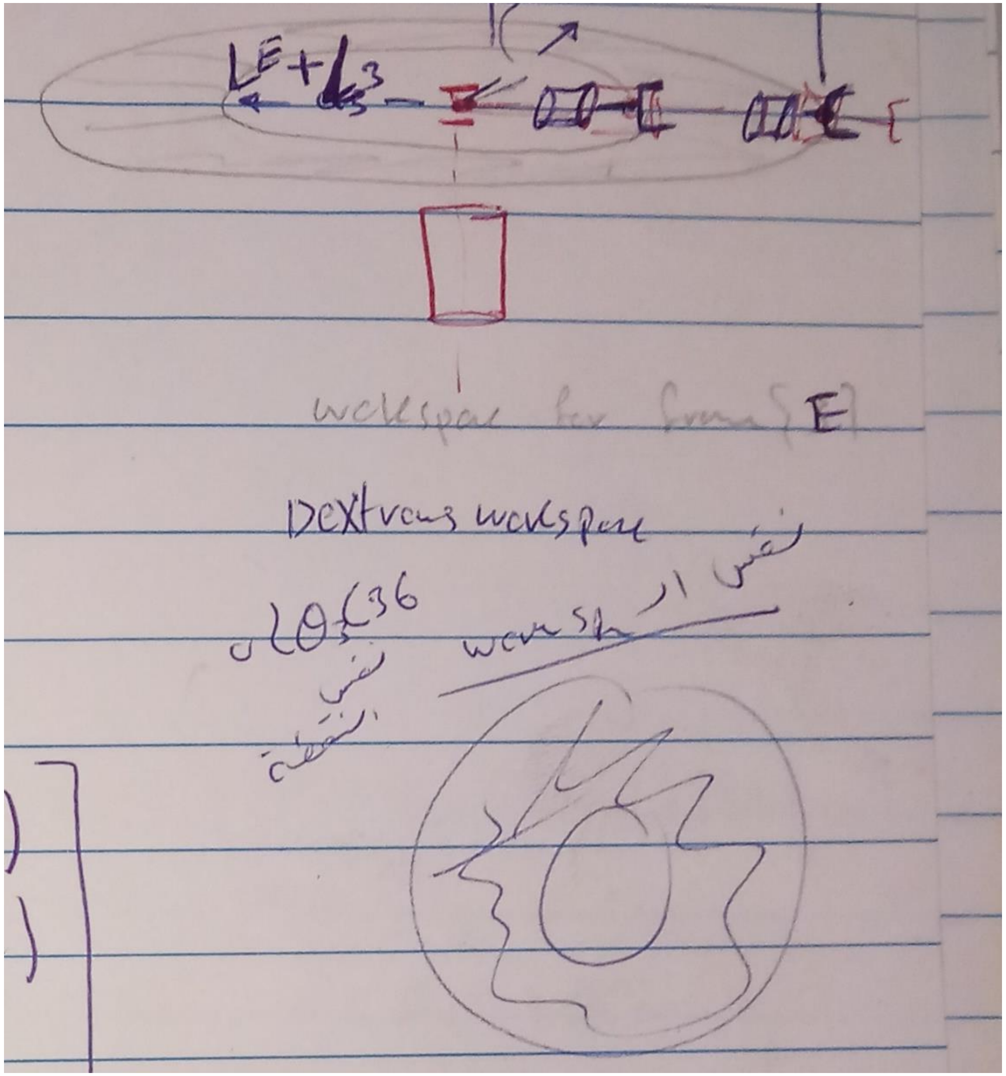
$d_2 = \frac{P_x}{S_1}$ or $d_2 = \frac{-P_y}{C_1}$

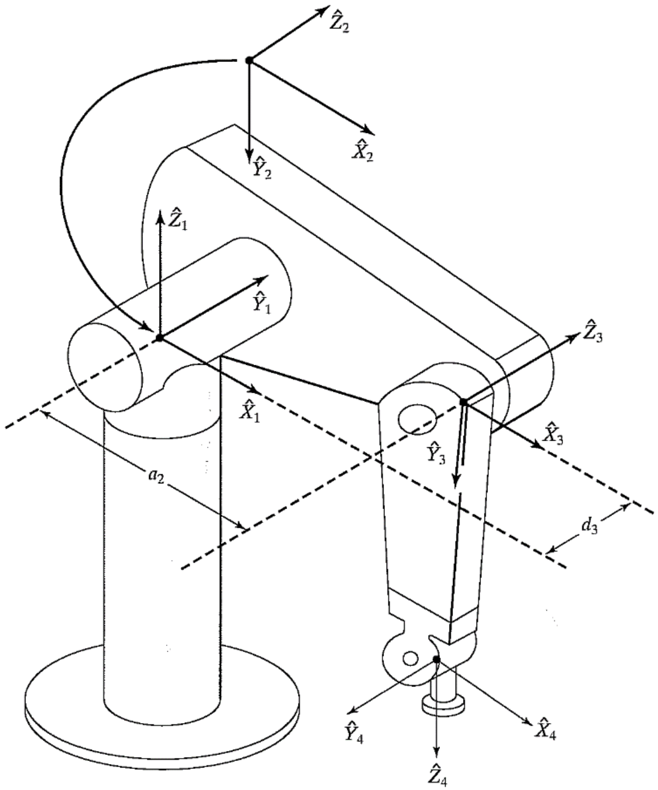
OR $d_2 = \sqrt{P_x^2 + P_y^2}$

25*



(b)





$$[{}^0_1T(\theta_1)]^{-1}\,{}^0_6T = {}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6). \tag{4.55}$$

Inverting ${}^0_1T$, we write (4.55) as

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T, \tag{4.56}$$

$$ {}^{1}_{6}T = {}^{1}_{3}T \, {}^{3}_{6}T = \begin{bmatrix} {}^{1}r_{11} & {}^{1}r_{12} & {}^{1}r_{13} & {}^{1}p_x \\ {}^{1}r_{21} & {}^{1}r_{22} & {}^{1}r_{23} & {}^{1}p_y \\ {}^{1}r_{31} & {}^{1}r_{32} & {}^{1}r_{33} & {}^{1}p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{1}_{6}T, $$

$$
\begin{aligned}
{}^{1}r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6, \\
{}^{1}r_{21} &= -s_4c_5c_6 - c_4s_6, \\
{}^{1}r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6, \\
{}^{1}r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6, \\
{}^{1}r_{22} &= s_4c_5s_6 - c_4c_6, \\
{}^{1}r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6, \\
{}^{1}r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\
{}^{1}r_{23} &= s_4s_5, \\
{}^{1}r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\
{}^{1}p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\
{}^{1}p_y &= d_3, \\
{}^{1}p_{\mathbf{z}} &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T,$$

$${}^1_6T = {}^1_3T\, {}^3_6T = \begin{bmatrix} {}^1r_{11} & {}^1r_{12} & {}^1r_{13} & {}^1p_x \\ {}^1r_{21} & {}^1r_{22} & {}^1r_{23} & {}^1p_y \\ {}^1r_{31} & {}^1r_{32} & {}^1r_{33} & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$-s_1 p_x + c_1 p_y = d_3.$$

$$\begin{aligned}
{}^1r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6, \\
{}^1r_{21} &= -s_4c_5c_6 - c_4s_6, \\
{}^1r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6, \\
{}^1r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6, \\
{}^1r_{22} &= s_4c_5s_6 - c_4c_6, \\
{}^1r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6, \\
{}^1r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\
{}^1r_{23} &= s_4s_5, \\
{}^1r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\
{}^1p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\
{}^1p_y &= d_3, \\
{}^1p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}$$

$$ {}^1_6T = {}^1_3T \, {}^3_6T = \begin{bmatrix} {}^1r_{11} & {}^1r_{12} & {}^1r_{13} & {}^1p_x \\ {}^1r_{21} & {}^1r_{22} & {}^1r_{23} & {}^1p_y \\ {}^1r_{31} & {}^1r_{32} & {}^1r_{33} & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T, $$

$$ -s_1 p_x + c_1 p_y = d_3. $$

$$
\begin{aligned}
{}^1r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6, \\
{}^1r_{21} &= -s_4c_5c_6 - c_4s_6, \\
{}^1r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6, \\
{}^1r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6, \\
{}^1r_{22} &= s_4c_5s_6 - c_4c_6, \\
{}^1r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6, \\
{}^1r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\
{}^1r_{23} &= s_4s_5, \\
{}^1r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\
{}^1p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\
{}^1p_y &= d_3, \\
{}^1p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

$$ \theta_1 = \text{Atan2}(p_y, p_x) - \text{Atan2}\left(d_3, \pm\sqrt{p_x^2 + p_y^2 - d_3^2}\right). $$

$$ {}^1_6T = {}^1_3T\,{}^3_6T = \begin{bmatrix} {}^1r_{11} & {}^1r_{12} & {}^1r_{13} & {}^1p_x \\ {}^1r_{21} & {}^1r_{22} & {}^1r_{23} & {}^1p_y \\ {}^1r_{31} & {}^1r_{32} & {}^1r_{33} & {}^1p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T, $$

$$ \begin{aligned} {}^1r_{11} &= c_{23}[c_4c_5c_6 - s_4s_6] - s_{23}s_5s_6, \\ {}^1r_{21} &= -s_4c_5c_6 - c_4s_6, \\ {}^1r_{31} &= -s_{23}[c_4c_5c_6 - s_4s_6] - c_{23}s_5c_6, \\ {}^1r_{12} &= -c_{23}[c_4c_5s_6 + s_4c_6] + s_{23}s_5s_6, \\ {}^1r_{22} &= s_4c_5s_6 - c_4c_6, \\ {}^1r_{32} &= s_{23}[c_4c_5s_6 + s_4c_6] + c_{23}s_5s_6, \\ {}^1r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\ {}^1r_{23} &= s_4s_5, \\ {}^1r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\ {}^1p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\ {}^1p_y &= d_3, \\ {}^1p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}. \end{aligned} $$

$$ \begin{aligned} c_1p_x + s_1p_y &= a_3c_{23} - d_4s_{23} + a_2c_2, \\ -Pz &= a_3s_{23} + d_4c_{23} + a_2s_2. \end{aligned} $$

Note that we have found two possible solutions for θ_1, corresponding to the plus-or-minus sign in (4.64). Now that θ_1 is known, the left-hand side of (4.56) is known. If we equate both the (1,4) elements and the (3,4) elements from the two sides of (4.56), we obtain

$$c_1 p_x + s_1 p_y = a_3 c_{23} - d_4 s_{23} + a_2 c_2,$$

$$-p_z = a_3 s_{23} + d_4 c_{23} + a_2 s_2. \tag{4.65}$$

If we square equations (4.65) and (4.57) and add the resulting equations, we obtain

$$a_3 c_3 - d_4 s_3 = K, \tag{4.66}$$

where

$$K = \frac{p_z^2 + p_y^2 + p_x^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2}{2a_2}. \tag{4.67}$$

Same method as before

$$\theta_3 = \text{Atan2}(a_3, d_4) - \text{Atan2}(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}).$$

$$[{}^0_3T(\theta_2)]^{-1}{}^0_6T = {}^3_4T(\theta_4){}^4_5T(\theta_5){}^5_6T(\theta_6), \tag{4.69}$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^3_6T, \tag{4.70}$$

$${}^3_6T = {}^3_4T\,{}^4_6T = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

where ${}^3_6T$ is given by equation (3.11) developed in Chapter 3. Equating both the (1,4) elements and the (2,4) elements from the two sides of (4.70), we get

$$\begin{aligned} c_1c_{23}p_x + s_1c_{23}p_y - s_{23}p_z - a_2c_3 &= a_3, \\ -c_1s_{23}p_x - s_1s_{23}p_y - c_{23}p_z + a_2s_3 &= d_4. \end{aligned} \tag{4.71}$$

These equations can be solved simultaneously for s_{23} and c_{23}, resulting in

$$s_{23} = \frac{(-a_3 - a_2 c_3)p_z + (c_1 p_x + s_1 p_y)(a_2 s_3 - d_4)}{p_z^2 + (c_1 p_x + s_1 p_y)^2},$$

$$c_{23} = \frac{(a_2 s_3 - d_4)p_z - (a_3 + a_2 c_3)(c_1 p_x + s_1 p_y)}{p_z^2 + (c_1 p_x + s_1 p_y)^2}. \tag{4.72}$$

The denominators are equal and positive, so we solve for the sum of θ_2 and θ_3 as

$$\theta_{23} = \text{Atan2}[(-a_3 - a_2 c_3)p_z - (c_1 p_x + s_1 p_y)(d_4 - a_2 s_3), \\ (a_2 s_3 - d_4)p_z - (a_3 + a_2 c_3)(c_1 p_x + s_1 p_y)]. \tag{4.73}$$

Equation (4.73) computes four values of θ_{23}, according to the four possible combinations of solutions for θ_1 and θ_3; then, four possible solutions for θ_2 are computed as

$$\theta_2 = \theta_{23} - \theta_3, \tag{4.74}$$

Now the entire left side of (4.70) is known. Equating both the (1,3) elements and the (3,3) elements from the two sides of (4.70), we get

$$r_{13}c_1c_{23} + r_{23}s_1c_{23} - r_{33}s_{23} = -c_4s_5,$$
$$-r_{13}s_1 + r_{23}c_1 = s_4s_5. \tag{4.75}$$

$$\theta_4 = \text{Atan2}(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + r_{33}s_{23}). \tag{4.76}$$

$$[{}^0_4T(\theta_4)]^{-1}\,{}^0_6T = {}^4_5T(\theta_5){}^5_6T(\theta_6), \tag{4.77}$$

where $[{}^0_4T(\theta 4)]^{-1}$ is given by

$$\begin{bmatrix} c_1c_{23}c_4 + s_1s_4 & s_1c_{23}c_4 - c_1s_4 & -s_{23}c_4 & -a_2c_3c_4 + d_3s_4 - a_3c_4 \\ -c_1c_{23}s_4 + s_1c_4 & -s_1c_{23}s_4 - c_1c_4 & s_{23}s_4 & a_2c_3s_4 + d_3c_4 + a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.78}$$

$${}^4_6T = {}^4_5T\,{}^5_6T = \begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.10}$$

$$[{}^0_4T(\theta_4)]^{-1}\,{}^0_6T = {}^4_5T(\theta_5)\,{}^5_6T(\theta_6), \tag{4.77}$$

where $[{}^0_4T(\theta 4)]^{-1}$ is given by

$$\begin{bmatrix} c_1c_{23}c_4+s_1s_4 & s_1c_{23}c_4-c_1s_4 & -s_{23}c_4 & -a_2c_3c_4+d_3s_4-a_3c_4 \\ -c_1c_{23}s_4+s_1c_4 & -s_1c_{23}s_4-c_1c_4 & s_{23}s_4 & a_2c_3s_4+d_3c_4+a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3-d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.78}$$

$${}^4_6T = {}^4_5T\,{}^5_6T = \begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.10}$$

$$\begin{aligned} r_{13}(c_1c_{23}c_4+s_1s_4)+r_{23}(s_1c_{23}c_4-c_1s_4)-r_{33}(s_{23}c_4) &= -s_5, \\ r_{13}(-c_1s_{23})+r_{23}(-s_1s_{23})+r_{33}(-c_{23}) &= c_5. \end{aligned} \tag{4.79}$$

Hence, we can solve for θ_5 as

$$\theta_5 = \text{Atan2}(s_5, c_5), \tag{4.80}$$

Applying the same method one more time, we compute $({}^0_5T)^{-1}$ and write (4.54) in the form

$$({}^0_5T)^{-1}\,{}^0_6T = {}^5_6T(\theta_6). \tag{4.81}$$

Equating both the (3,1) elements and the (1,1) elements from the two sides of (4.77) as we have done before, we get

$$\theta_6 = \text{Atan2}(s_6, c_6), \tag{4.82}$$

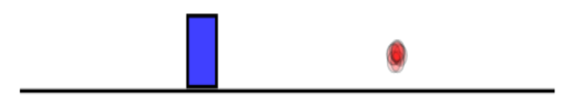
where

$$s_6 = -r_{11}(c_1c_{23}s_4 - s_1c_4) - r_{21}(s_1c_{23}s_4 + c_1c_4) + r_{31}(s_{23}s_4),$$
$$c_6 = r_{11}[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5] + r_{21}[(s_1c_{23}c_4 - c_1s_4)c_5 - s_1s_{23}s_5]$$
$$-r_{31}(s_{23}c_4c_5 + c_{23}s_5).$$

4.10 REPEATABILITY AND ACCURACY

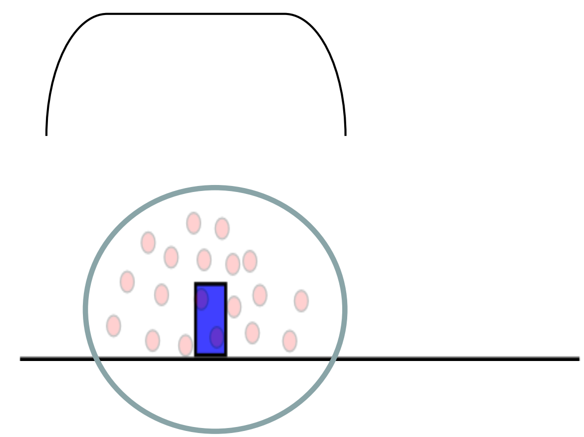
- Accuracy (ex:+-1 mm)
  - –The difference between the actual position of the robot and the programmed position

computed points.



- Repeatability (ex:+-0.5 mm)

  Will the robot always return to the same point under the same control conditions?



taught point

4.10 REPEATABILITY AND ACCURACY

Many industrial robots today move to goal points that have been taught. A **taught point** is one that the manipulator is moved to physically, and then the joint position sensors are read and the joint angles stored. When the robot is commanded to return to that point in space, each joint is moved to the stored value. In simple "teach and playback" manipulators such as these, the inverse kinematic problem never arises, because goal points are never specified in Cartesian coordinates. When a manufacturer specifies how precisely a manipulator can return to a taught point, he is specifying the **repeatability** of the manipulator.

4.10 REPEATABILITY AND ACCURACY

Any time a goal position and orientation are specified in Cartesian terms, the inverse kinematics of the device must be computed in order to solve for the required joint angles. Systems that allow goals to be described in Cartesian terms are capable of moving the manipulator to points that were never taught—points in its workspace to which it has perhaps never gone before. We will call such points **computed points**. Such a capability is necessary for many manipulation tasks. For example, if a computer vision system is used to locate a part that the robot must grasp, the robot must be able to move to the Cartesian coordinates supplied by the vision sensor. The precision with which a computed point can be attained is called the **accuracy** of the manipulator.

4.10 REPEATABILITY AND ACCURACY

The accuracy of a manipulator is bounded by the repeatability. Clearly, accuracy is affected by the precision of parameters appearing in the kinematic equations of the robot. Errors in knowledge of the Denavit–Hartenberg parameters will cause the inverse kinematic equations to calculate joint angle values that are in error. Hence, although the repeatability of most industrial manipulators is quite good, the accuracy is usually much worse and varies quite a bit from manipulator to manipulator. Calibration techniques can be devised that allow the accuracy of a manipulator to be improved through estimation of that particular manipulator's kinematic parameters [10].

Problem 1(80 points):

For the RRP 3 DOF manipulator shown in figure 1, the homogeneous transformation matrix between the end-effector frame (3) and the base frame (0) is given as:

$$ {}^{0}_{3}T = \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & C_1S_2d_3 - S_1 \\ S_1C_2 & C_1 & S_1S_2 & S_1S_2d_3 + C_1 \\ -S_2 & 0 & C_2 & C_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Find θ_1, d_3, θ_2 ??

$0 \leqslant \theta_1 \leqslant 360 \quad 0 \leqslant \theta_2 \leqslant 360$

$0 \leqslant d_3 \leqslant \sqrt{3}$

$$ {}^{0}_{3}T = \begin{bmatrix} & ?? & & 0 \\ & & & 2 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

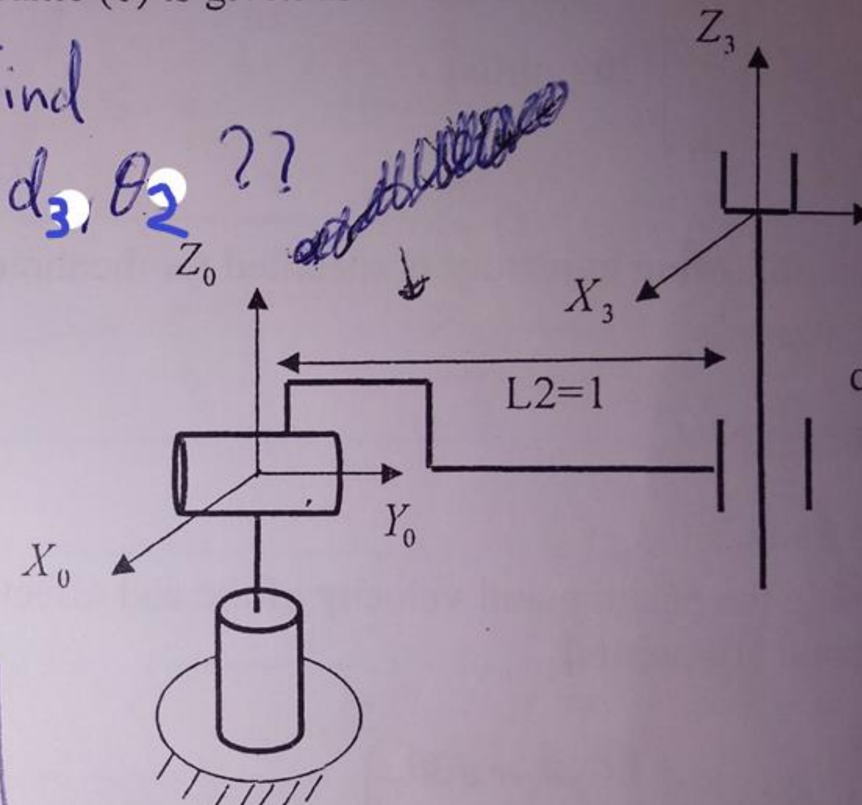


Fig 1: 3 DOF RRP manipulator

3 x 3 matrix

$P_x = C_1 S_2 d_3 - S_1$

$P_y = S_1 S_2 d_3 + C_1$

$P_z = C_2 d_3$

(5) $\Longleftrightarrow$

$0 = C_1 S_2 d_3 - S_1$ ... (1)

$2 = S_1 S_2 d_3 + C_1$ ... (2)

$0 = C_2 d_3$ ... (3)

(3)

From (3) $C_2 = 0$ or $d_3 = 0$

$\theta_2 = -\frac{\pi}{2}, \frac{\pi}{2}$ (5)

$\theta_2 = +\frac{\pi}{2}$, $S_2 = +1$, $C_2 = 0$

$\theta_2 = -\frac{\pi}{2}$, $S_2 = -1$, $C_2 = 0$

$\begin{pmatrix} 0 = C_1 d_3 - S_1 \\ 2 = S_1 d_3 + C_1 \end{pmatrix}^2$

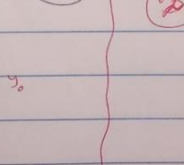
$0 = C_1^2 d_3^2 + S_1^2 - 2 S_1 C_1 d_3$

$4 = S_1^2 d_3^2 + C_1^2 + 2 S_1 C_1 d_3$ +

$4 = d_3^2 (C_1^2 + S_1^2) + (S_1^2 + C_1^2) + 0$

$4 - 1 = d_3^2 \Rightarrow d_3 = \pm\sqrt{3}$ (5)

(3)



$0 = C_1 d_3 - S_1$ (5)

$\frac{S_1}{C_1} = \frac{\sqrt{3}}{1} \Rightarrow \theta_1 = 60$

$\begin{pmatrix} 0 = -C_1 d_3 - S_1 \\ 2 = -S_1 d_3 + C_1 \end{pmatrix}^2$

$0 = C_1^2 d_3^2 + S_1^2 + 2 S_1 C_1 d_3$

$4 = C_1^2 + S_1^2 d_3^2 - 2 S_1 C_1 d_3$ +

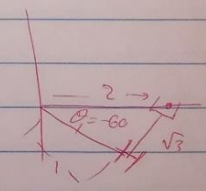
$4 = d_3^2 + 1$

$d_3 = \sqrt{3}$

$\frac{S_1}{C_1} = -d_3 = -\sqrt{3}$

$\theta_1 = -60$ (5)

(2)



If $d_3 = 0 \Rightarrow 0 = -S_1$, $2 = C_1$

$\hookrightarrow 2 \neq^{\text{never}} C_1$ not solution

$\therefore$ 2 solution (2) (5)

Robotics

Jacobians: Velocities and Static Forces

INTRODUCTION

- In this chapter ,we expand our consideration of robot manipulators beyond static-positioning Problems.

- We examine the notions of linear and angular velocity of a rigid body and use these Concepts to analyze the motion of a manipulator.

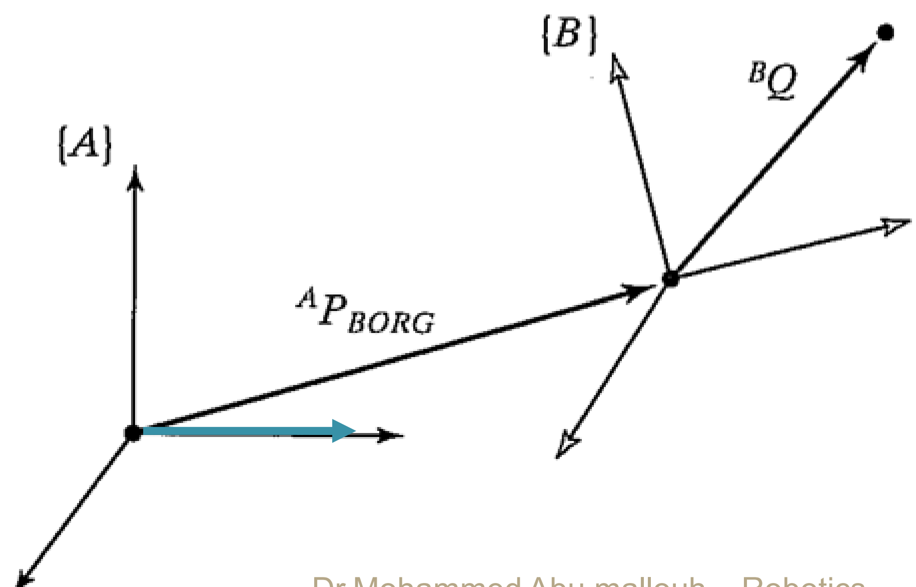
Differentiation of position vectors

$$^{B}V_{Q} = \frac{d}{dt}\,^{B}Q = \lim_{\Delta t \to 0} \frac{^{B}Q(t+\Delta t) - ^{B}Q(t)}{\Delta t}$$

$$^{A}\left(^{B}V_{Q}\right) = \frac{^{A}d}{dt}\,^{B}Q.$$

$$^{B}\left(^{B}V_{Q}\right) = ^{B}V_{Q}.$$

$$^{A}\left(^{B}V_{Q}\right) = ^{A}_{B}R\,^{B}V_{Q}.$$



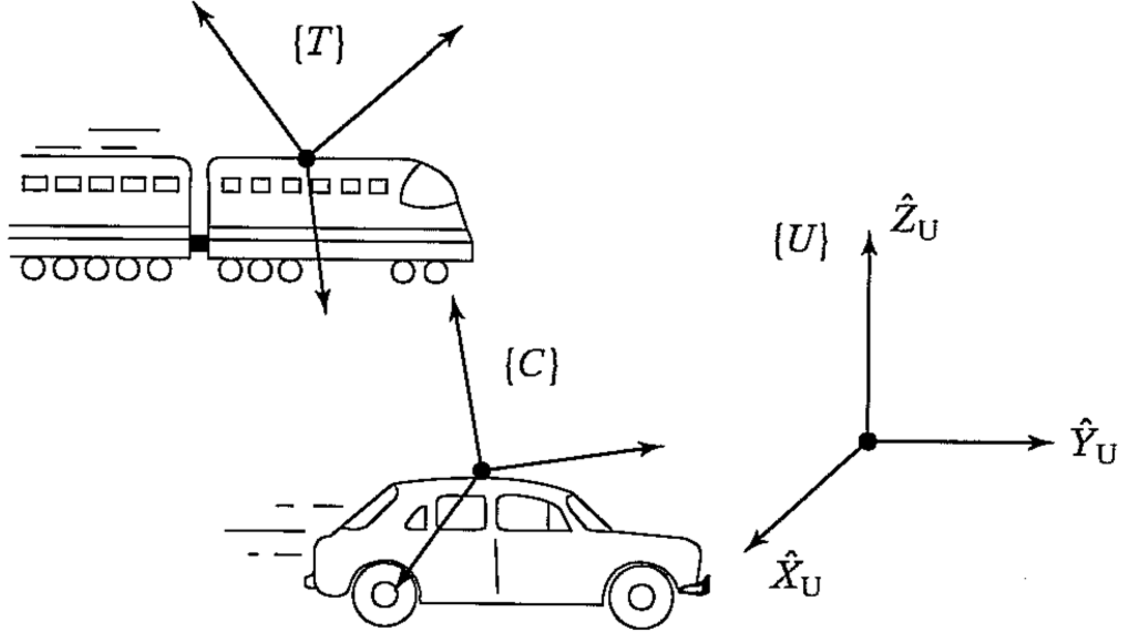


FIGURE 5.1: Example of some frames in linear motion.

FIGURE 5.1 shows a fixed universe frame,{U},a frame attached to a train traveling at 100 mph, {T},and a frame attached to a car traveling at 30 mph,{C}. Both vehicles are heading in the X direction of {U} .the rotation matrices, ${}^{U}_{T}R$ and ${}^{U}_{C}R$,are known and constant.

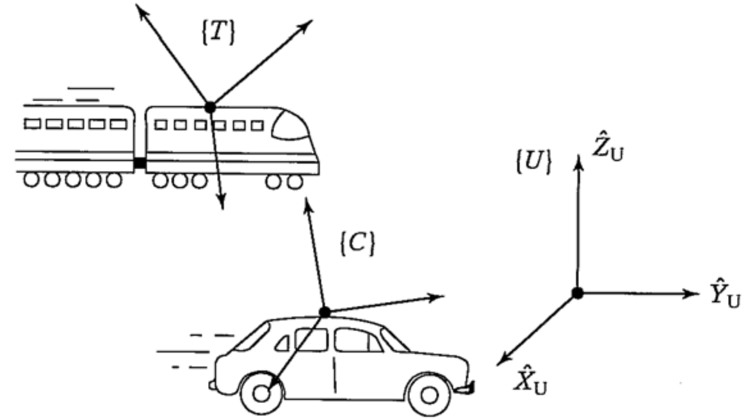


FIGURE 5.1: Example of some frames in linear motion

What is $\frac{{}^{U}d}{dt}\,{}^{U}P_{CORG}$?

$$\frac{{}^{U}d}{dt}\,{}^{U}P_{CORG} = {}^{U}V_{CORG} = \upsilon_C = 30\hat{X}.$$

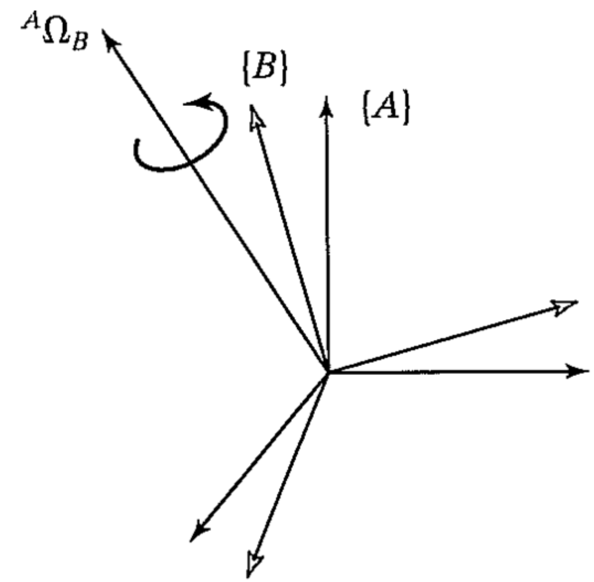
What is ${}^{C}({}^{U}V_{TORG})$?

$${}^{C}({}^{U}V_{TORG}) = {}^{C}\upsilon_T = {}^{C}_{U}R\upsilon_T = {}^{C}_{U}R(100\hat{X}) = {}^{U}_{C}R^{-1}100\hat{X}.$$

What is ${}^{C}({}^{T}V_{CORG})$?

$${}^{C}({}^{T}V_{CORG}) = {}^{C}_{T}R\,{}^{T}V_{CORG} = -{}^{C}_{U}R^{-1}\,{}^{U}_{T}R\;70\hat{X}.$$

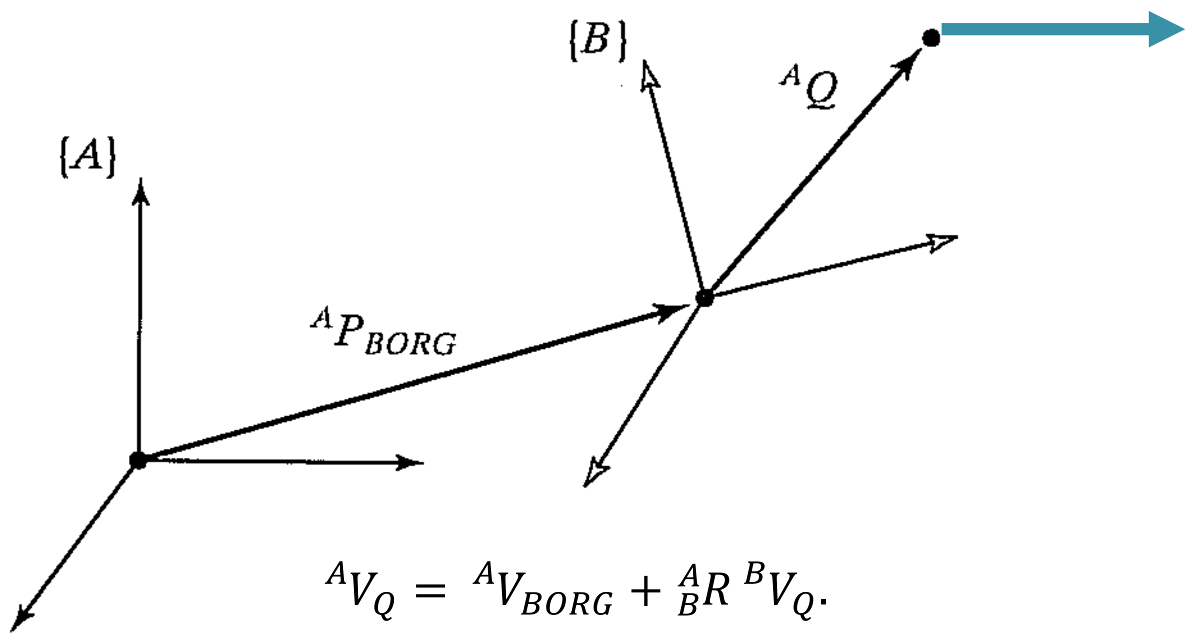
The angular velocity vector



In Fig. 5.2, $^{A}\Omega_{B}$ describes the rotation of frame {B} relative to {A}.Physically , at any instant, the direction of $^{A}\Omega_{B}$ indicates the instantaneous axis of rotation of {B} relative {A} , and the magnitude of $^{A}\Omega_{B}$ indicates the speed of rotation .Again ,like any vector , an angular velocity vector may be expressed in any coordinate system, and so anther leading superscript may be added; for example , $^{C}(^{A}\Omega_{B})$ is the angular velocity of frame {B} relative to {A} expressed In terms of frame {C}.

Linear and rotational velocity of rigid bodies

$^{A}\Omega_{B} = zero$



$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q}.$$

Equation (5.7) is for only that case in which relative orientation of {B} and {A} remains constant. **{B} dose not rotate w.r.t {A}**

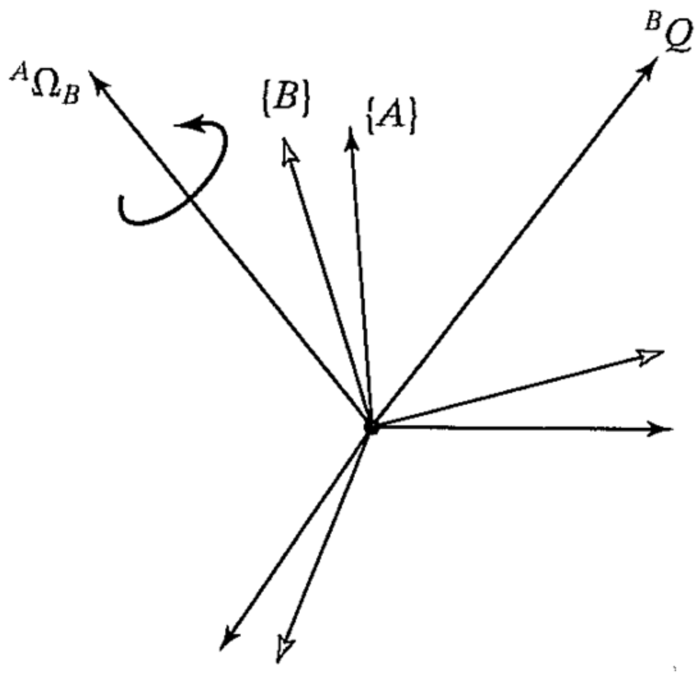
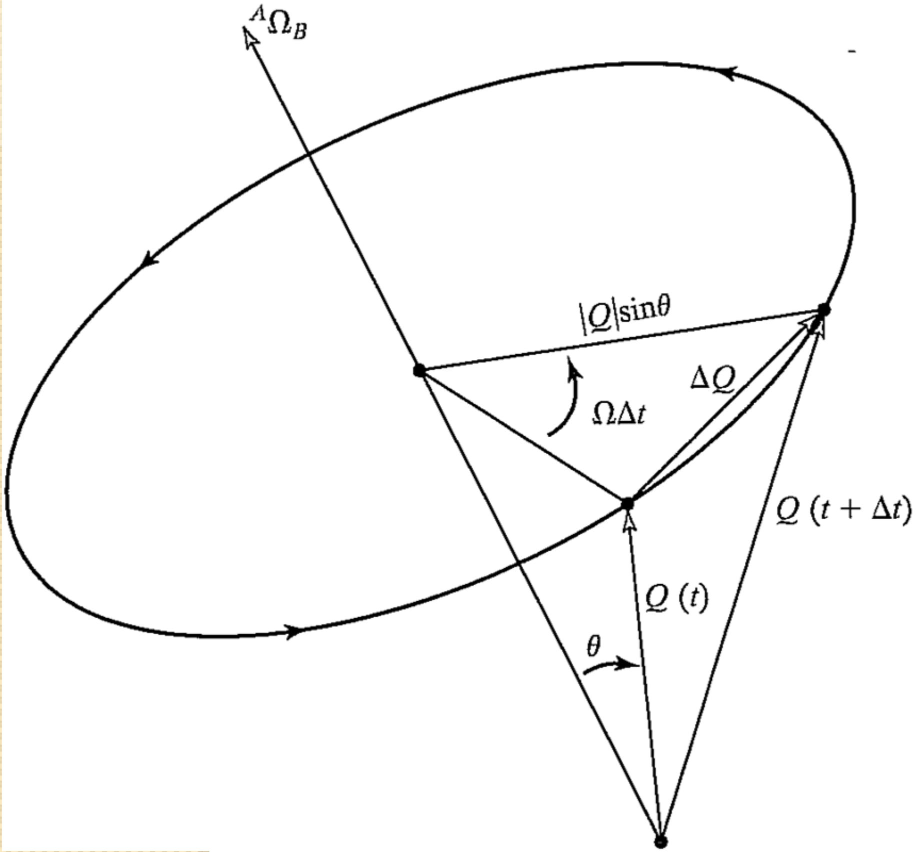


Figure 5.4 : Vector ^BQ , fixed in frame {B}, is rotating with respect to frame {A} with angular velocity $^A\Omega_B$.



{B rotating w.r.t {A} with angular velocity

$^BQ \quad fixed \quad w.r.t \quad \{B\}$

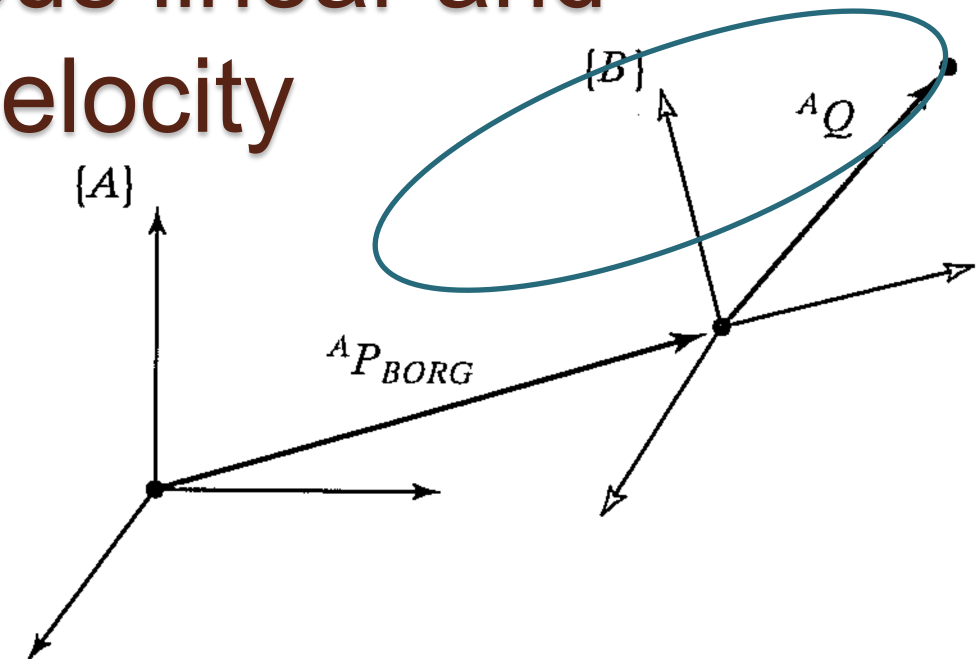
$$|\triangle Q| = (|\,^AQ|.\sin\theta)(|\,^A\Omega_B|.\triangle t).$$

$$^AV_Q = \frac{|\triangle Q|}{\triangle t} = \,^A\Omega_B \times \,^AQ.$$

The vector Q could also be changing with respect to frame {B}.

$$^AV_Q = A \quad (^BV_Q) + \,^A\Omega_B \times \,^AQ. \qquad ^AV_Q = \,^A_BR\,^BV_Q + \,^A\Omega_B \times \,^A_BR\,^BQ.$$

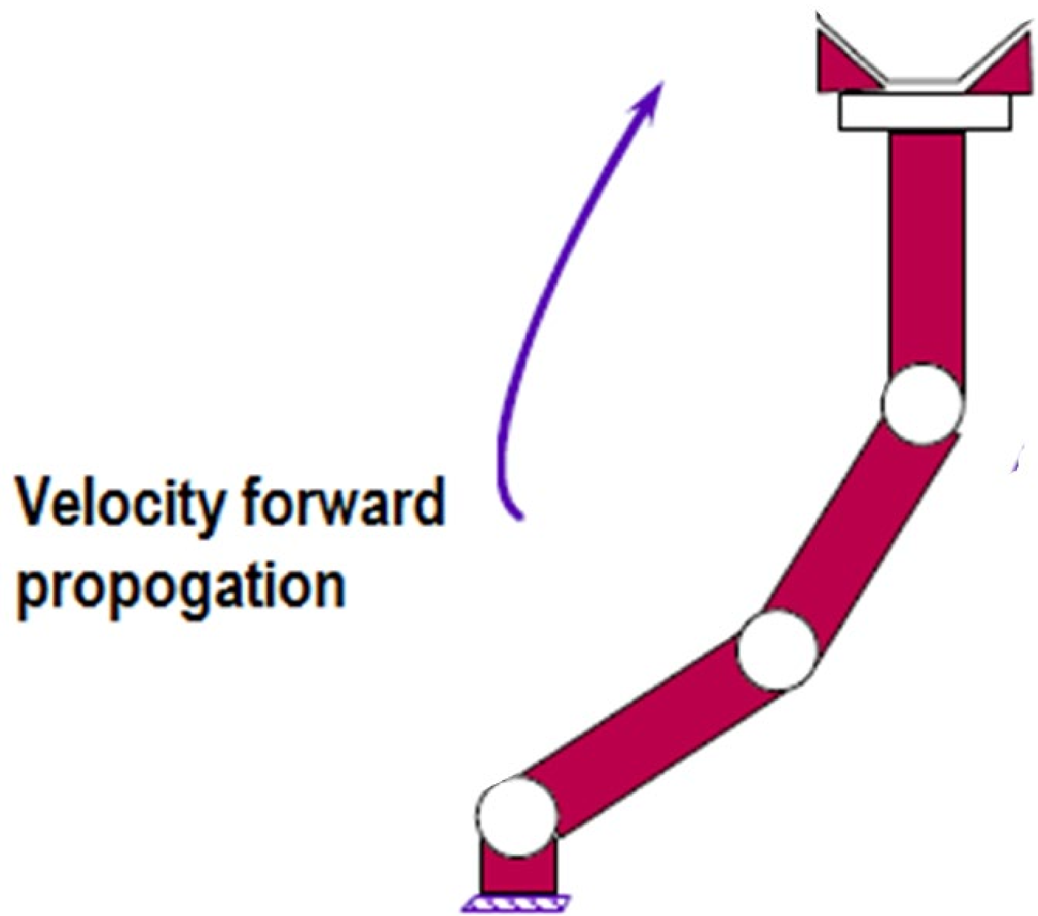
Simultaneous linear and rotational velocity



- {B} rotate w.r.t {A}
- Q changing w.r.t {B}.
- $^AP_{Borg}$ is changing
- the general formula for velocity of a vector in frame {B} as seen from frame {A} is :

$$^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A\Omega_B \times {}^A_BR\,{}^BQ. \qquad (5.13)$$

5.6 Velocity "propagation" from link to link



5.6 Velocity "propagation" from link to link

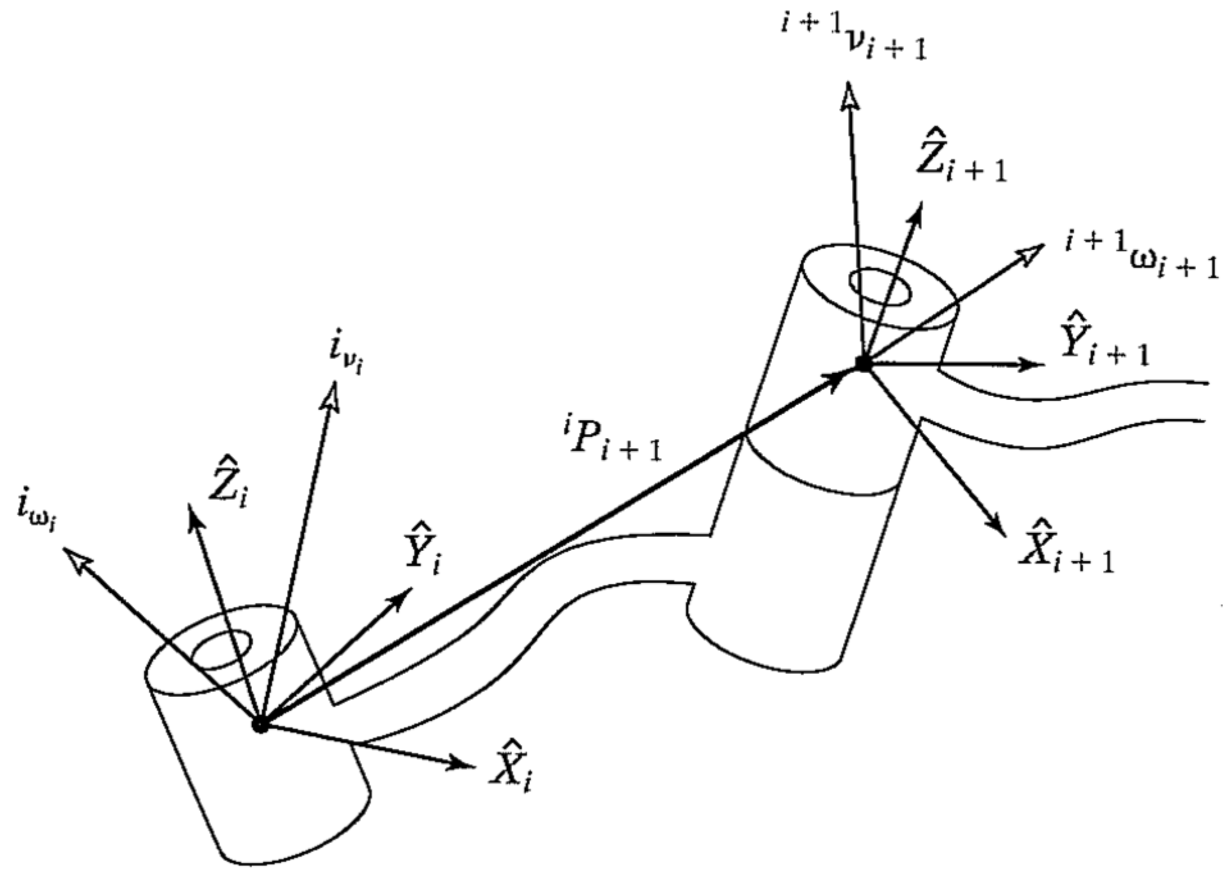


FIGURE 5.7 : Velocity vectors of neighboring links.

Velocity “propagation” for revolute joints

- the angular velocity of link i+1 is the same as that of link i plus a new component caused by rotational velocity at joint i+1 .
- Rotational velocities can be added when both vectors are written w.r.t the same frame .

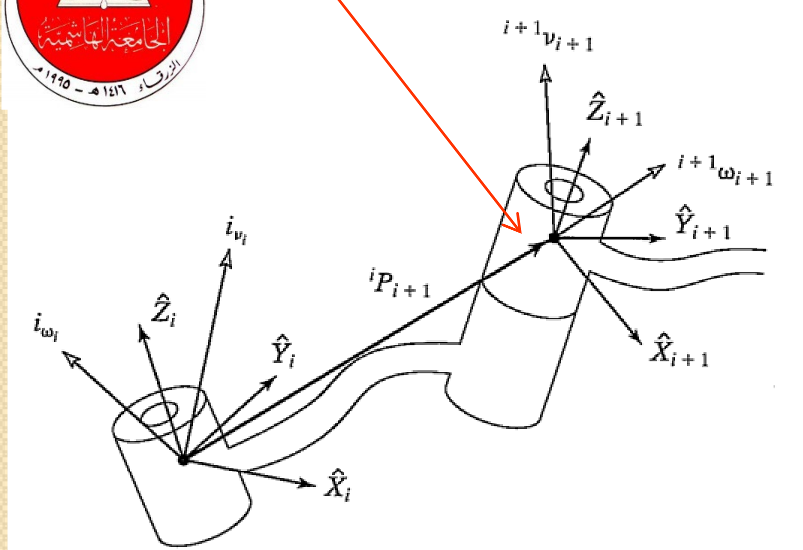
This can be written in terms of frame {i} as

For revolute

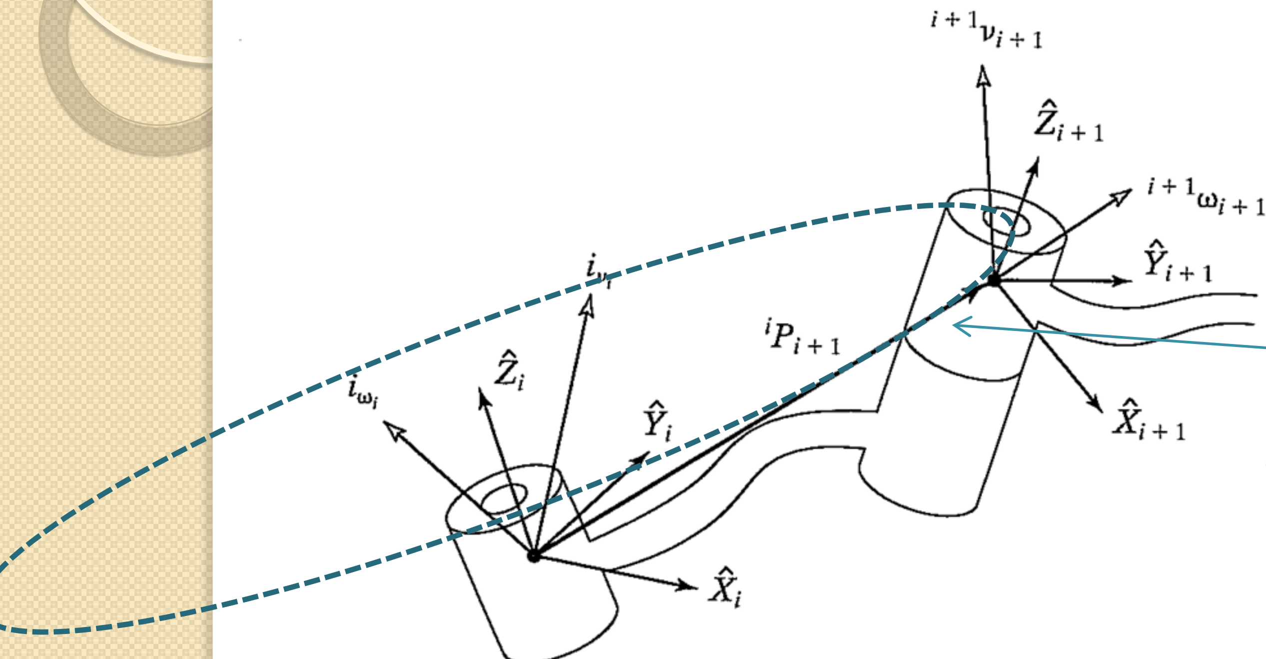
$$^{i}\omega_{i+1} = {}^{i}\omega_{i} + {}^{i}_{i+1}R\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}\ .$$

$$\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}.$$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}.$$



Velocity "propagation" for revolute joints



For revolute

$$ {}^{i}v_{i+1} = {}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}. $$
$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_{i} + {}^{i}\omega_{i} x\ {}^{i}P_{i+1}). $$

Velocity "propagation" for prismatic joints

The corresponding relationships for the case that i+1 is prismatic are

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_{i},$$
$$^{i+1}v_{i+1} = {}^{i+1}_{i}R(\ {}^{i}v_{i} +\ {}^{i}\omega_{i} \times\ {}^{i}P_{i+1}) + \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}. \quad (5.48)$$

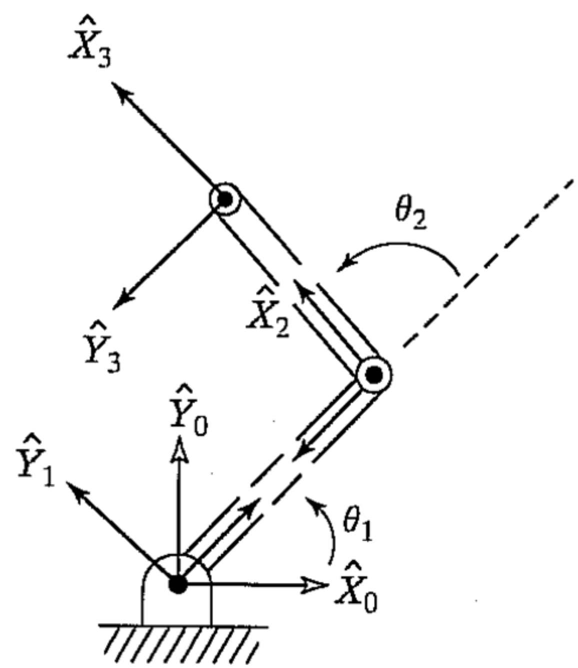
Applying these equations successively from link to link, we can compute $^{N}\omega_{N}$ and $^{N}v_{N}$,the rotational and linear velocities of the last link.

$$^{0}\omega_{N} = {}^{0}_{N}R\ {}^{N}\omega_{N}$$
$$^{0}V_{N} = {}^{0}_{N}R\ {}^{N}V_{N}$$

Note that the resulting velocities are expressed in terms of frame {N}.this turns out to be useful ,as we will see later. If the velocities are desired in terms of the base coordinate system, they can be rotated in to base coordinates by multiplication with $^{0}_{N}R$.

EXAMPLE 1

A two-link manipulator with rotational joints is shown below, Calculate the velocity of the tip of the arm (end effector) as a function of joint rates (joint velocity). Give the answer in two forms- in terms of frame {3} ${}^3_3V$ and also in terms of frame {0} ${}^0_3V$.



$$
{}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

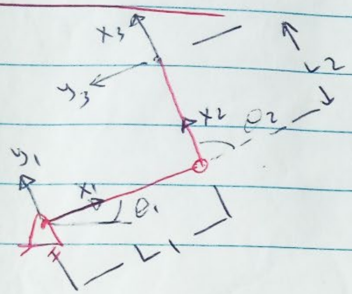
$$
{}^1_2T = \begin{bmatrix} c_1 & -s_1 & 0 & l_1 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

- See notes

EXAMPLE 1

Ex:



$^3v_3 = ??$

$^0v_3 = ??$

$$^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & L_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^3v_3 = {}^3_2R\left({}^2v_2 + {}^2\omega_2 \times {}^2P_3\right) \quad (3)$$

$$^2v_2 = {}^2_1R\left({}^1v_1 + {}^1\omega_1 \times {}^1P_2\right) \quad (2)$$

(1) $$^1v_1 = {}^1_0R\left({}^0v_0 + {}^0\omega_0 \times {}^0P_1\right) \quad (1)$$

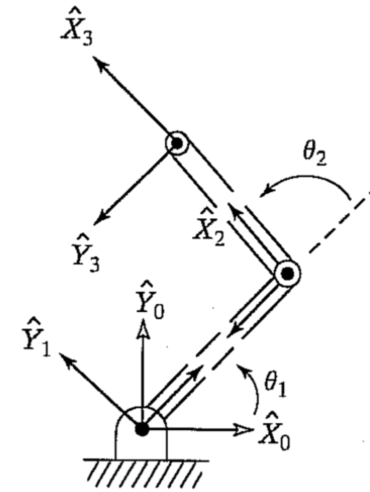
$$^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad {}^1_0R = {}^0_1R^{-1} = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad {}^0P_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$^1v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(2) $^1P_2$, $$^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix}, \quad {}^2_1R = {}^1_2R^{-1} = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad {}^1P_2 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix}$$

$$^2v_2 = {}^2_1R\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \dot\theta_1 {}^1\hat Z_1 \times L_1 {}^1\hat X_1\right) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot\theta_1 L_1 \sin 90 \\ 0 \end{bmatrix}\right)$$

$$^2v_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \dot\theta_1 L_1 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 s_2 \dot\theta_1 \\ L_1 c_2 \dot\theta_1 \\ 0 \end{bmatrix}$$



(3)

$${}^3v_3 = {}^3_2R\left({}^2v_2 + {}^2\omega_2 \times {}^2P_3\right) \qquad {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \dot\theta_2\,{}^2\hat Z_2$$

$${}^2\omega_2 = \begin{bmatrix} C_2 & S_2 & 0 \\ -S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix}$$

$${}^2P_3 = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}, \quad {}^3_2R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

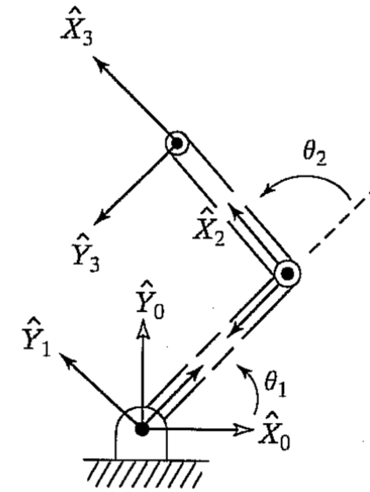
$${}^3v_3 = {}^3_2R\left(\begin{bmatrix} L_1S_2\dot\theta_1 \\ L_1C_2\dot\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1+\dot\theta_2 \end{bmatrix} \times \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}\right)$$

$${}^3v_3 = {}^3_2R\left(\begin{bmatrix} L_1S_2\dot\theta_1 \\ L_1C_2\dot\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ L_2(\dot\theta_1+\dot\theta_2) \\ 0 \end{bmatrix}\right) = I\left(\begin{bmatrix} L_1S_2\dot\theta_1 \\ L_1C_2\dot\theta_1 + L_2(\dot\theta_1+\dot\theta_2) \\ 0 \end{bmatrix}\right)$$

$${}^0v_3 = {}^0_3R\,{}^3v_3 = \begin{bmatrix} C_{12} & -S_{12} & 0 \\ S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} L_1S_2\dot\theta_1 \\ L_1C_2\dot\theta_1 + L_2(\dot\theta_1+\dot\theta_2) \\ 0 \end{bmatrix}$$

$${}^0_3R = {}^0_1R\,{}^1_2R\,{}^2_3R = \begin{bmatrix} C_{12} & -S_{12} & 0 \\ S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = {}^0v_3 = \begin{bmatrix} -L_1S_1\dot\theta_1 - L_2S_{12}(\dot\theta_1+\dot\theta_2) \\ L_1C_1\dot\theta_1 + L_2C_{12}(\dot\theta_1+\dot\theta_2) \\ 0 \end{bmatrix} = \begin{bmatrix} -L_1S_1 - L_2S_{12} & -L_2S_{12} \\ L_1C_1 + L_2C_{12} & L_2C_{12} \end{bmatrix}\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}$$



$${}^0v = {}^0J(\Theta)\dot\Theta,$$

5.7 Jacobians

$$y_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6),$$

$$y_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6),$$

$$\vdots$$

$$y_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6).$$

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6,$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6}\delta x_6,$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6,$$

$$\delta Y = \frac{\partial F}{\partial X}\delta X.$$

$$\delta Y = J(X)\delta X.$$

$$\dot{Y} = J(X)\dot{X}.$$

$${}^{0}\nu = {}^{0}J(\Theta)\dot{\Theta},$$

5.7 Jacobians

v : Cartesian velocity = End-effector velocity.

$$v = \begin{bmatrix} \text{linear velocity} \\ \text{angular velocity} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ w_x \\ w_y \\ w_z \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

$$P_x(\theta_1,\theta_2 \dots \theta_n) \Rightarrow \frac{\partial P_x}{\partial t} = v_x = \frac{\partial P_x}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial P_x}{\partial \theta_2}\dot{\theta}_2 + \dots \frac{\partial P_x}{\partial \theta_n}\dot{\theta}_n$$

$$P_y(\theta_1,\theta_2 \dots \theta_n) \Rightarrow \frac{\partial P_y}{\partial t} = v_y = \frac{\partial P_y}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial P_y}{\partial \theta_2}\dot{\theta}_2 + \dots \frac{\partial P_y}{\partial \theta_n}\dot{\theta}_n$$

$$P_z(\theta_1,\theta_2 \dots \theta_n) \Rightarrow \frac{\partial P_z}{\partial t} = v_z = \frac{\partial P_z}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial P_z}{\partial \theta_2}\dot{\theta}_2 + \dots \frac{\partial P_z}{\partial \theta_n}\dot{\theta}_n$$

$$\phi_x(\theta_1,\theta_2 \dots \theta_n) \Rightarrow \frac{\partial \phi_x}{\partial t} = w_x = \frac{\partial \phi_x}{\partial \theta_1}\dot{\theta}_1 + \dots \frac{\partial \phi_x}{\partial \theta_n}\dot{\theta}_n$$

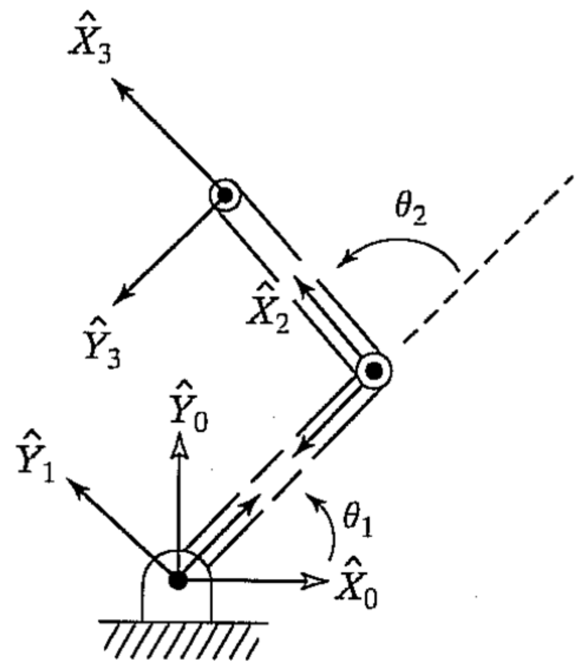
$$\phi_y(\theta_1,\theta_2 \dots \theta_n) \Rightarrow \frac{\partial \phi_y}{\partial t} = w_y = \frac{\partial \phi_y}{\partial \theta_1}\dot{\theta}_1 + \dots + \frac{\partial \phi_y}{\partial \theta_n}\dot{\theta}_n$$

$$\phi_z(\theta_1,\theta_2 \dots \theta_n) \Rightarrow \frac{\partial \phi_z}{\partial t} = w_z = \frac{\partial \phi_z}{\partial \theta_1}\dot{\theta}_1 + \dots + \frac{\partial \phi_z}{\partial \theta_n}\dot{\theta}_n$$

$$\begin{matrix} v \\ \\ \\ w \end{matrix}\begin{bmatrix} v_x \\ v_y \\ v_z \\ w_x \\ w_y \\ w_z \end{bmatrix} = \underbrace{\begin{bmatrix} \partial P_x/\partial\theta_1 & \partial P_x/\partial\theta_2 & \dots & \partial P_x/\partial\theta_n \\ \partial P_y/\partial\theta_1 & \partial P_y/\partial\theta_2 & \dots & \partial P_y/\partial\theta_n \\ \partial P_z/\partial\theta_1 & \partial P_z/\partial\theta_2 & \dots & \partial P_z/\partial\theta_n \\ \partial \phi_x/\partial\theta_1 & \partial \phi_x/\partial\theta_2 & \dots & \partial \phi_x/\partial\theta_n \\ \partial \phi_y/\partial\theta_1 & \partial \phi_y/\partial\theta_2 & \dots & \partial \phi_y/\partial\theta_n \\ \partial \phi_z/\partial\theta_1 & \partial \phi_z/\partial\theta_3 & \dots & \partial \phi_z/\partial\theta_n \end{bmatrix}}_{6 \times n} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

EXAMPLE 1 (another method)

A two-link manipulator with rotational joints is shown below, Calculate the velocity of the tip of the arm (end effector) as a function of joint rates (joint velocity). Give the answer in two forms- in terms of frame {3} ${}^{3}_{3}V$ and also in terms of frame {0} ${}^{0}_{3}V$.



$$ {}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

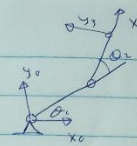
$$ {}^{1}_{2}T = \begin{bmatrix} c_1 & -s_1 & 0 & l_1 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

- See notes

- Example 1 (another method)

$$^0X_3 = L_1C_1 + L_2C_{12}$$
$$^0Y_3 = L_1S_1 + L_2S_{12}$$
$$^0Z_3 = 0$$



$$^0V_{x_3} = {}^0\dot{X}_3 = -L_1S_1\dot{\theta}_1 - L_2S_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$^0V_{y_3} = {}^0\dot{y}_3 = L_1C_1\dot{\theta}_1 + L_2C_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$
$$^0V_{z_3} = {}^0\dot{z}_3 = 0$$

$$^0V_3 = \underbrace{\begin{bmatrix} -L_1S_1 - L_2S_{12} & -L_2S_{12} \\ L_1C_1 + L_2C_{12} & L_2C_{12} \\ 0 & 0 \end{bmatrix}}_{3\times2,\ {}^0J} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}}_{2\times1} = \underbrace{\begin{bmatrix} ^0V_{3x} \\ ^0V_{3y} \\ ^0V_{3z} \end{bmatrix}}_{3\times1}$$

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \\ 0 & 0 \end{bmatrix}}_{^0J} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$^0J = {}^0_3R\ {}^3J = \underbrace{\begin{bmatrix} C_{12} & -S_{12} & 0 \\ S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3\times3} \underbrace{\begin{bmatrix} L_1S_2 & 0 \\ L_1C_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix}}_{3\times2} =$$

$$^3_3V = \begin{bmatrix} ^3_3V_x \\ ^3_3V_y \\ ^3_3V_z \end{bmatrix} = \begin{bmatrix} L_1S_2\dot{\theta}_1 \\ L_1C_2\dot{\theta}_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} L_1S_2 & 0 \\ L_1C_2 + L_2 & L_2 \\ 0 & 0 \end{bmatrix}}_{^3J,\ 3\times2} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}}_{2\times1}$$

Example 1 (another method)

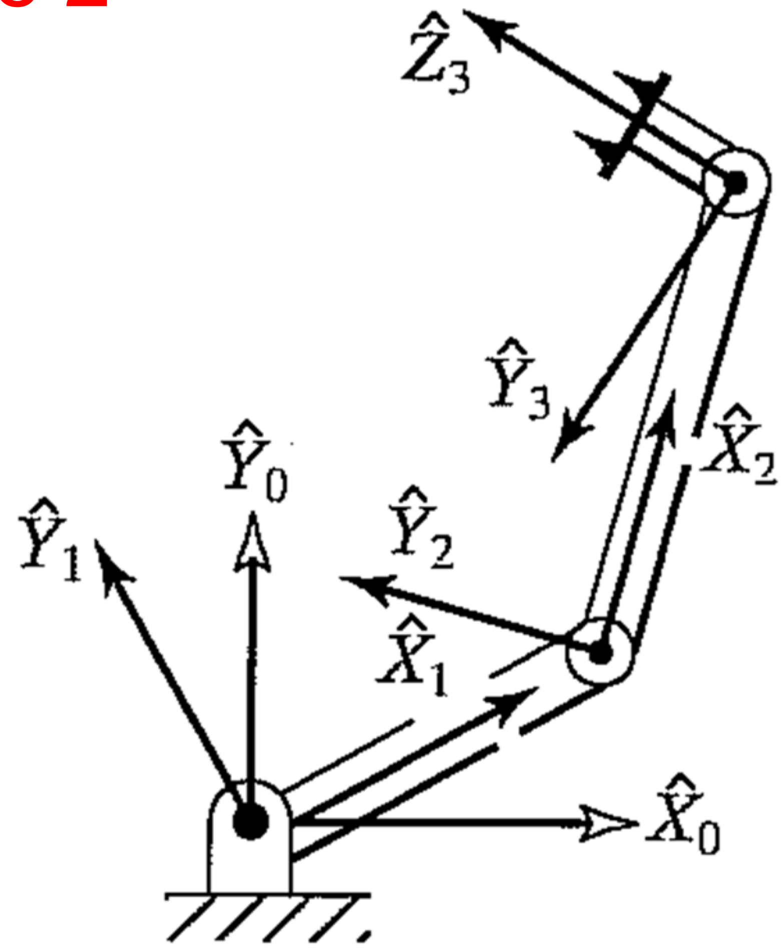
$$^{0}J = \begin{bmatrix} C_{12}L_1S_2 - S_{12}L_1C_2 - S_{12}L_2 & -S_{12}L_2 \\ S_{12}L_1S_2 + L_1C_2C_{12} + C_{12}L_2 & C_{12}L_2 \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

$$^{0}J = \begin{bmatrix} L_1\left[S(\theta_2 - (\theta_1 + \theta_2))\right] - S_{12}L_2 & -L_2S_{12} \\ L_1\left[C(\theta_2 - \theta_1 - \theta_2)\right] + C_{12}L_2 & L_2C_{12} \\ 0 & 0 \end{bmatrix}$$

$$^{0}J = \begin{bmatrix} L_1S(-\theta_1) - S_{12}L_2 & -L_2S_{12} \\ L_1C(-\theta_1) + C_{12}L_2 & L_2C_{12} \\ 0 & 0 \end{bmatrix}$$

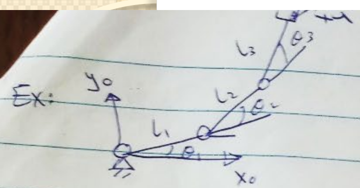
$$^{0}J = \begin{bmatrix} -L_1S_1 - S_{12}L_2 & -L_2S_{12} \\ L_1C_1 + C_{12}L_2 & L_2C_{12} \\ 0 & 0 \end{bmatrix}$$

- Example 2



- Example 2

Ex:



$${}^0P_{4x} = L_1C_1 + L_2C_{12} + L_3C_{123}$$

$${}^0P_{4y} = L_1S_1 + L_2S_{12} + L_3S_{123}$$

$${}^0P_{4z} = 0$$

$$\phi_x = 0$$

$$\phi_y = 0$$

$$\phi_z = \theta_1 + \theta_2 + \theta_3$$

$${}^0J = \begin{bmatrix} \frac{\partial P_x}{\partial \theta_1} & \frac{\partial P_x}{\partial \theta_2} & \frac{\partial P_x}{\partial \theta_3} \\ \vdots & & \\ \frac{\partial \phi_z}{\partial \theta_1} & \frac{\partial \phi_z}{\partial \theta_2} & \frac{\partial \phi_z}{\partial \theta_3} \end{bmatrix}$$

$$\Rightarrow$$

$${}^0J = \begin{bmatrix} -L_1S_1 - L_2S_{12} - L_3S_{123} & -L_2S_{12} - L_3S_{123} & -L_3S_{123} \\ L_1C_1 + L_2C_{12} + L_3C_{123} & L_2C_{12} + L_3C_{123} & L_3C_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

6×3

OR

$${}^0J = \begin{bmatrix} -L_1S_{12} - L_2S_{12} - L_3S_{123} & -L_2S_{12} - L_3S_{123} & -L_3S_{123} \\ L_1C_1 + L_2C_{12} + L_3C_{123} & L_2C_{12} + L_3C_{123} & L_3C_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

3×3

Manar (8)

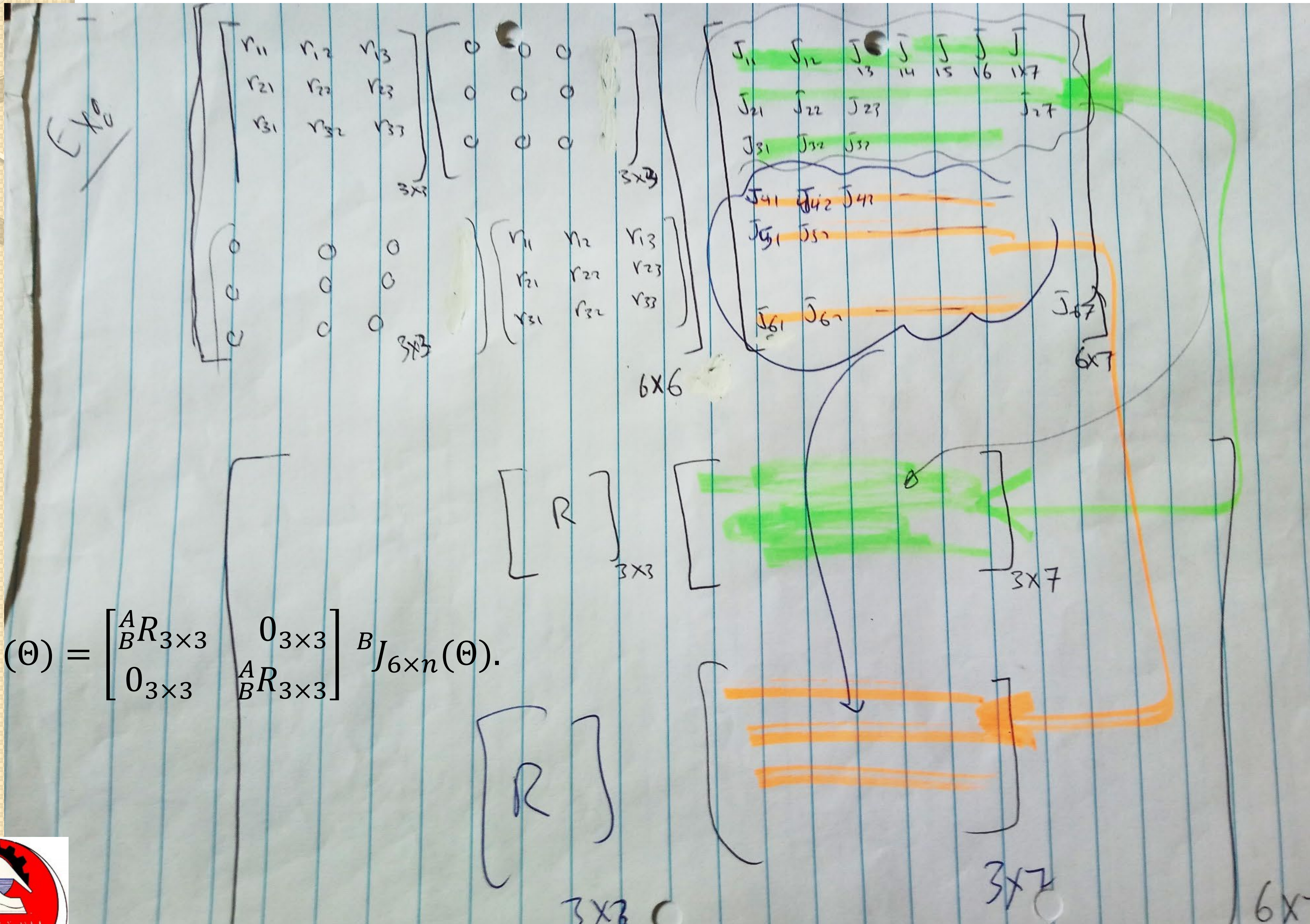
Changing a Jacobian's frame of reference

$$\begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix}_{6\times 1} = {}^{B}\nu = {}^{B}J(\Theta)_{6\times n}\dot{\Theta}_{n\times 1},$$

$$\begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & {}^{A}_{B}R_{3\times 3} \end{bmatrix} \begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix}.$$

$${}^{A}J_{6\times n}(\Theta) = \begin{bmatrix} {}^{A}_{B}R_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & {}^{A}_{B}R_{3\times 3} \end{bmatrix} {}^{B}J_{6\times n}(\Theta).$$

Changing a Jacobian's frame of reference



$$ {}^{A}J_{6\times n}(\Theta) = \begin{bmatrix} {}^{A}_{B}R_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & {}^{A}_{B}R_{3\times3} \end{bmatrix} {}^{B}J_{6\times n}(\Theta). $$

5.8 Singularities

- Most manipulators have values of joints where the Jacobian becomes singular (det(J)=0)

- Such locations are called **singularities of the mechanism** or **singularities** for short .

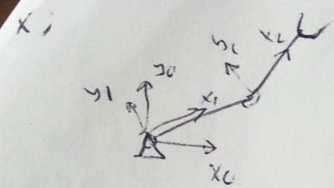
- All manipulators have singularities at the boundary of their workspace.

$$\mathrm{J} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ If } \mathrm{AD} - \mathrm{BC} \neq 0, \text{then J has an inverse,denoted } \mathrm{J}^{-1}$$

$$J^{-1} = \frac{1}{AD - BC} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}$$

$$\dot{\theta} = J^{-1} V \rightarrow \infty \text{ when det(J)=0 (singular configration)}$$

Singularities: Example 4



$S(\theta_1 - \theta_2) = S_1C_2 - C_1S_2$

$S(\theta_1 + \theta_2 - \theta_1) = S\,\theta_2 = S_2$

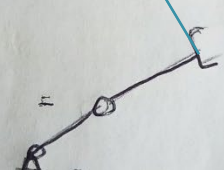
$$^0J = \begin{bmatrix} -L_1S_1 - L_2S_{12} & -L_2S_{12} \\ L_1C_1 + L_2C_{12} & L_2C_{12} \end{bmatrix}$$

$$|^0J| = (-L_1S_1 - L_2S_{12})(L_2C_{12}) - (-L_2S_{12})(L_1C_1 + L_2C_{12})$$

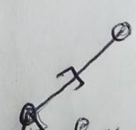
$$= -L_1L_2S_1C_{12} - L_2^2S_{12}C_{12} + L_1L_2S_{12}C_1 + L_2^2S_{12}C_{12}$$

$$= L_1L_2(C_1S_{12} - S_1C_{12}) = L_1L_2S_2 = 0 \Rightarrow$$

$$\theta_2 = 0,\ 180$$



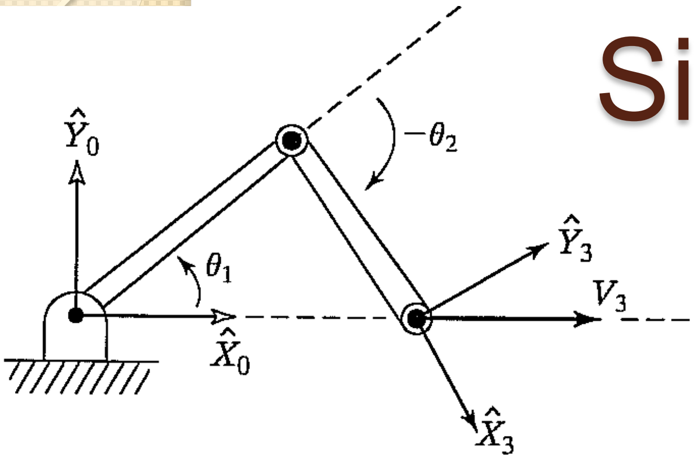
fully extended



fully folded back on itself

workspace boundary singularity.

Singularities: Example 4



$$ {}^{0}\nu = {}^{0}J(\Theta)\dot{\Theta}, $$

$$ \dot{\theta} = J^{-1}V $$

$$ \dot{\theta}_1 = \frac{c_{12}}{l_1 s_2}, $$

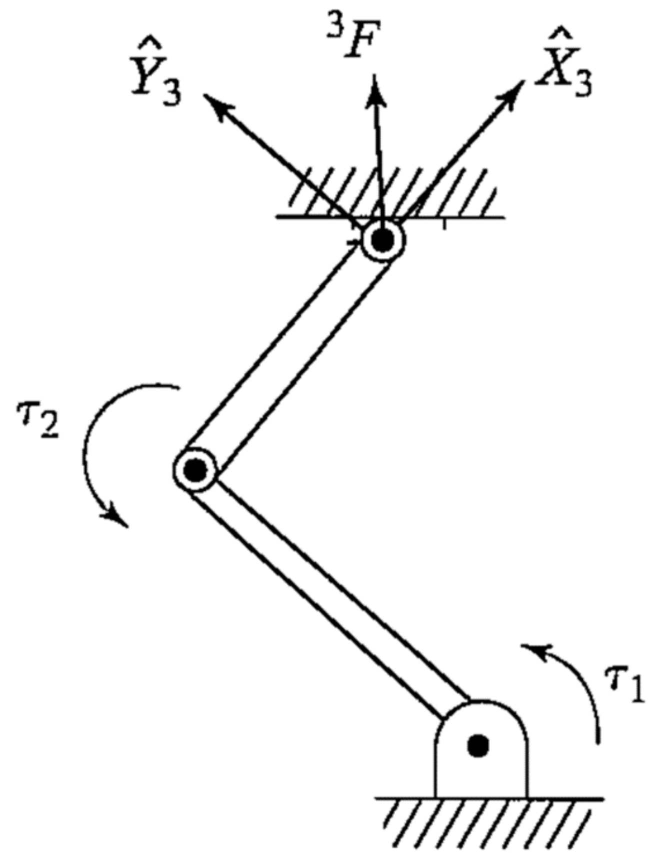
$$ \dot{\theta}_2 = -\frac{c_1}{l_2 s_2} - \frac{c_{12}}{l_1 s_2}. $$

At singular configuration:
1- $\dot{\theta} = \infty$ problem… det(J)=0
2- the robot may loose one or more DOF

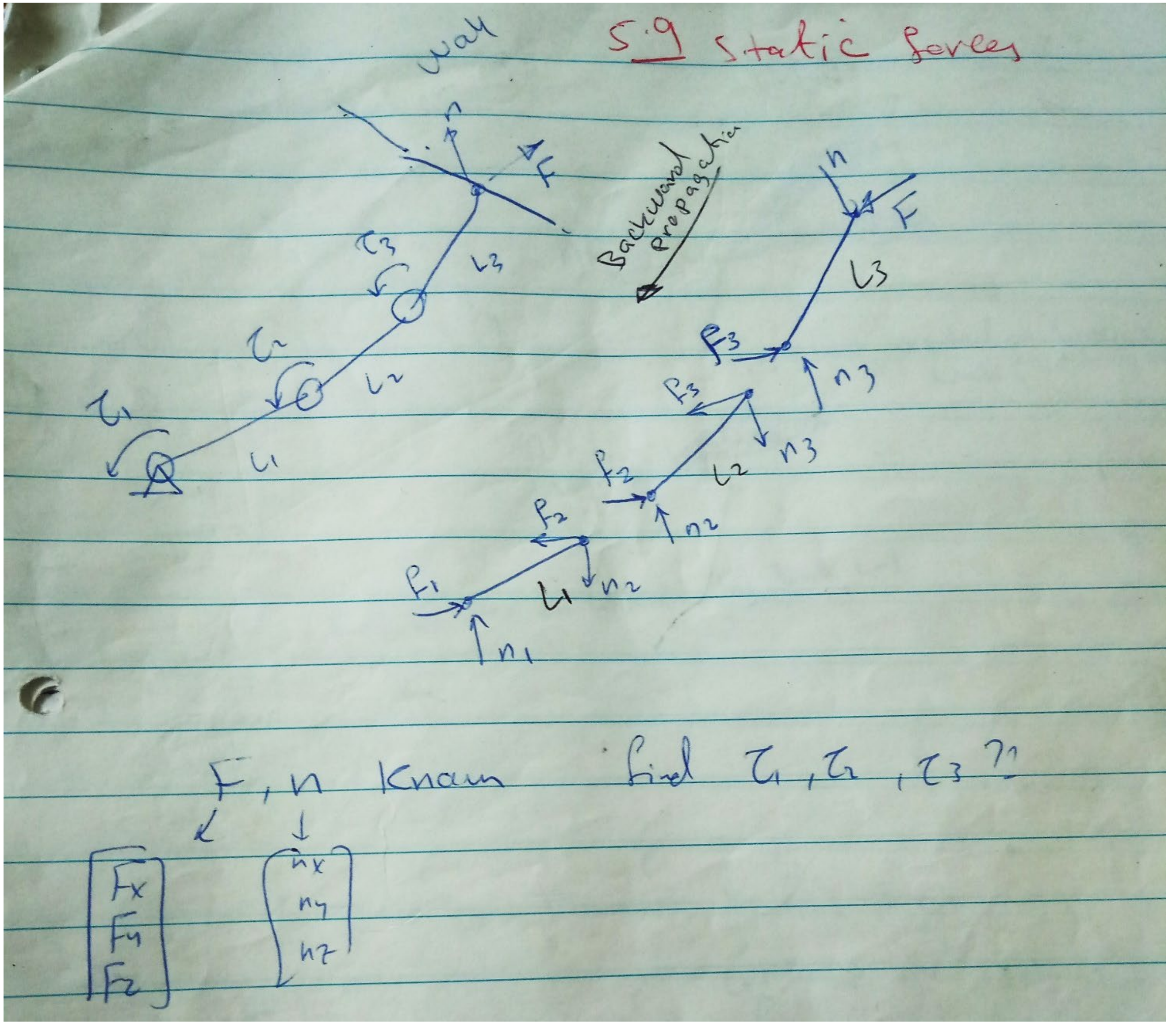
- When joint value ϴ is close to 0 or 180 , S2 is very close to zero, thus $\dot{\theta} = \infty$

5.9 Static forces in manipulators

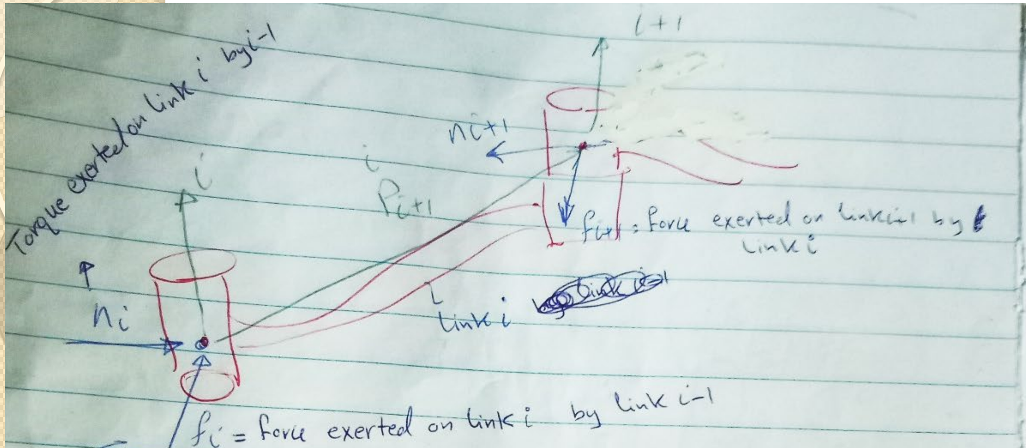
The robot is pushing on something in the environment with the end-effector or is perhaps supporting a load at the hand . We wish to solve for the joint torques that must be acting to keep the system in static equilibrium.



Static forces in manipulators



Static forces in manipulators



$\sum F_i = 0$

$${}^i f_i - {}^i f_{i+1} = 0$$

$\sum n_i = 0$

$${}^i n_i - {}^i n_{i+1} - {}^i P_{i+1} \times {}^i f_{i+1} = 0$$

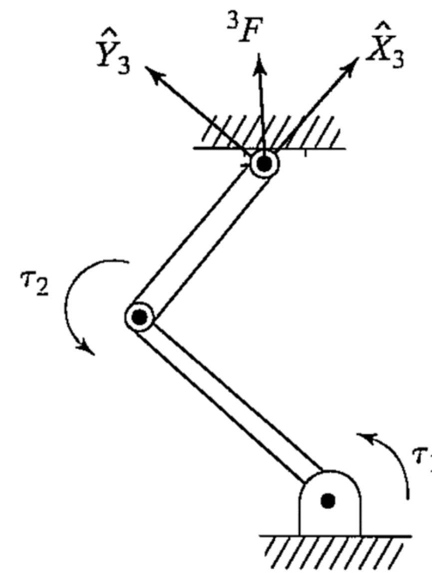
$${}^i f_i = {}^i f_{i+1} = {}^i_{i+1}R\; {}^{i+1} f_{i+1}$$

$${}^i n_i = {}^i n_{i+1} + {}^i P_{i+1} \times {}^i f_{i+1} = {}^i_{i+1}R\; {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i_{i+1}R\; {}^{i+1} f_{i+1}$$

$$= {}^i_{i+1}R\; {}^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_i$$

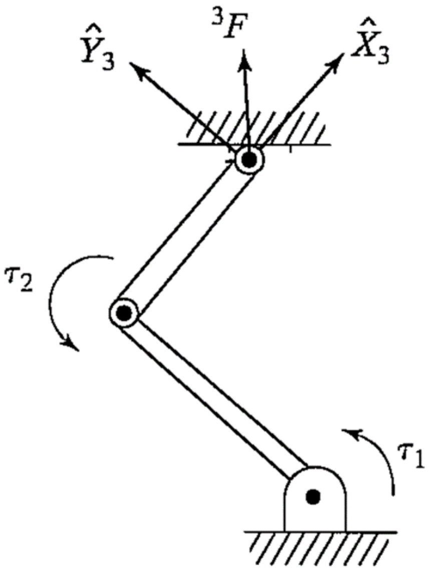
$$\tau_i = {}^i n_i^T\; {}^i \hat{Z}_i = [{}^i n_x \;\; {}^i n_y \;\; {}^i n_z] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = {}^i n_z$$ revolute

$$\tau_i = {}^i f_i^T\; {}^i Z_i = [{}^i f_x \;\; {}^i f_y \;\; {}^i f_z] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = {}^i f_z$$



Static forces

End effector dose not apply any moment on the object ${}^3_3N$=zero, ${}^3_3F$=[fx fy 0]^T



$$ {}^2f_2 = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}, $$

$$ {}^2n_2 = l_2\hat{X}_2 \times \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix}, $$

$$ {}^1f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix}, $$

$$ {}^1n_1 = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} + l_1\hat{X}_1 \times {}^1f_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix}. $$

$$ {}^if_i = {}^i_{i+1}R\,{}^{i+1}f_{i+1}, $$

$$ {}^in_i = {}^i_{i+1}R\,{}^{i+1}n_{i+1} + {}^iP_{i+1} \times {}^if_i. $$

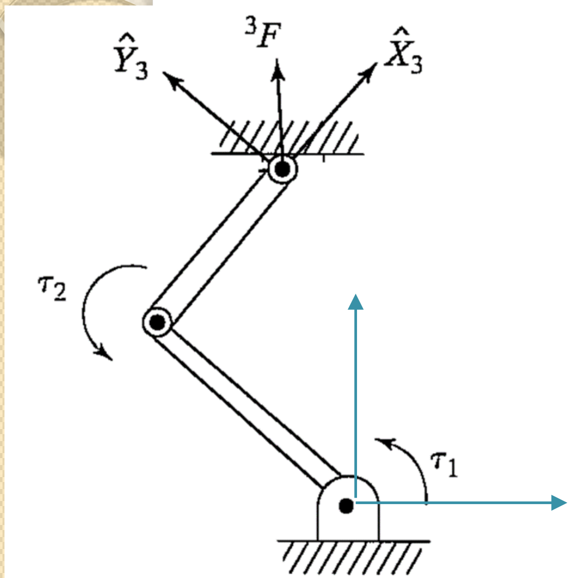
$$ \tau_1 = l_1 s_2 f_x + (l_2 + l_1 c_2) f_y, $$

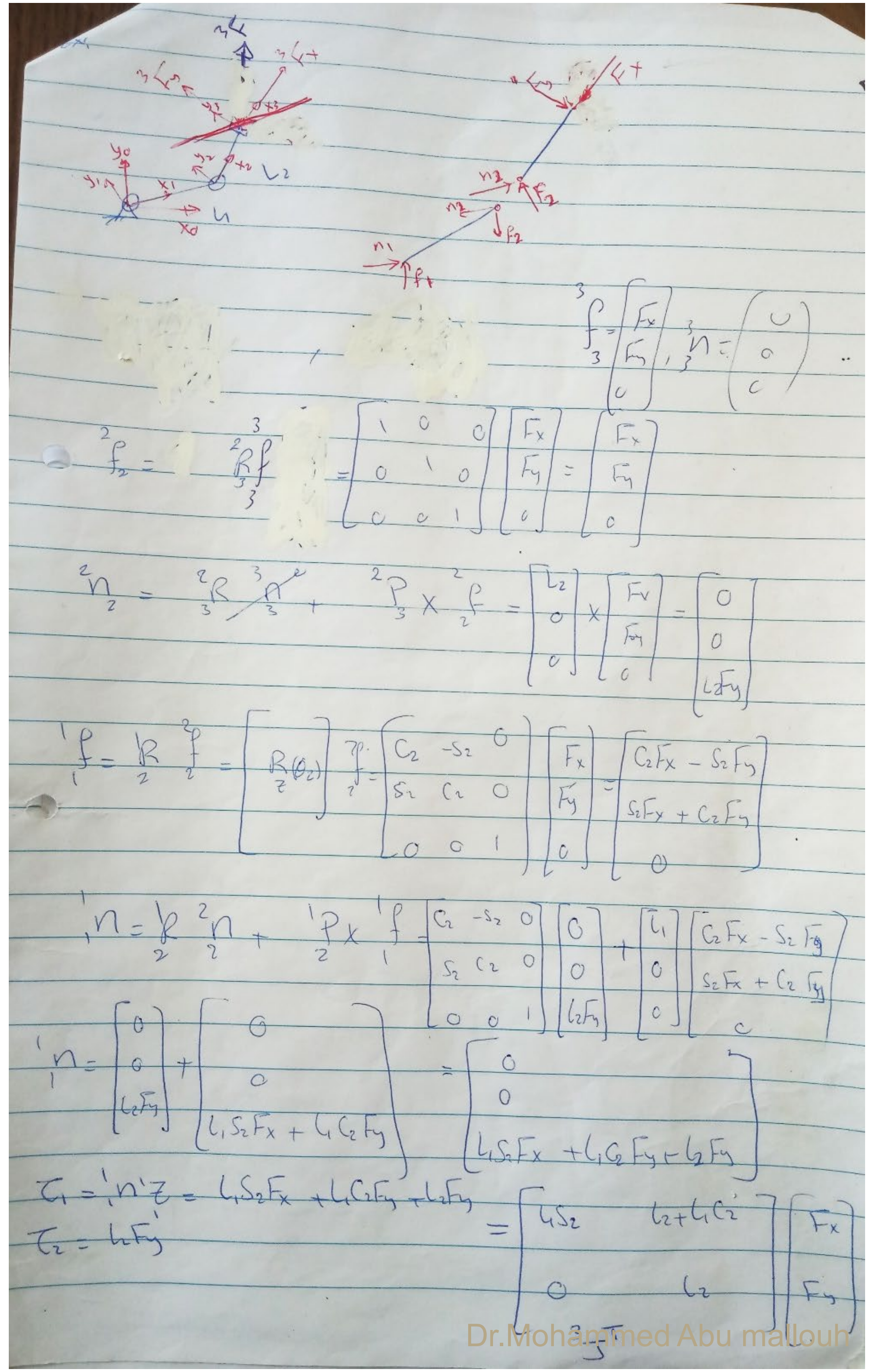
$$ \tau_2 = l_2 f_y. $$

$$ \tau_i = {}^in_i^T\,{}^i\hat{Z}_i. $$

$$ \tau_i = {}^if_i^T\,{}^i\hat{Z}_i. $$

Static forces





5.10 Jacobians in the force domain

$$\tau_1 = l_1 s_2 f_x + (l_2 + l_1 c_2) f_y,$$

$$\tau_2 = l_2 f_y.$$

$$\tau = \begin{bmatrix} l_1 s_2 & l_2 + l_1 c_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

$$\tau = {}^3J^T \, {}^3F$$

in general : $\tau = {}^0J^T \, {}^0F$

Robotics

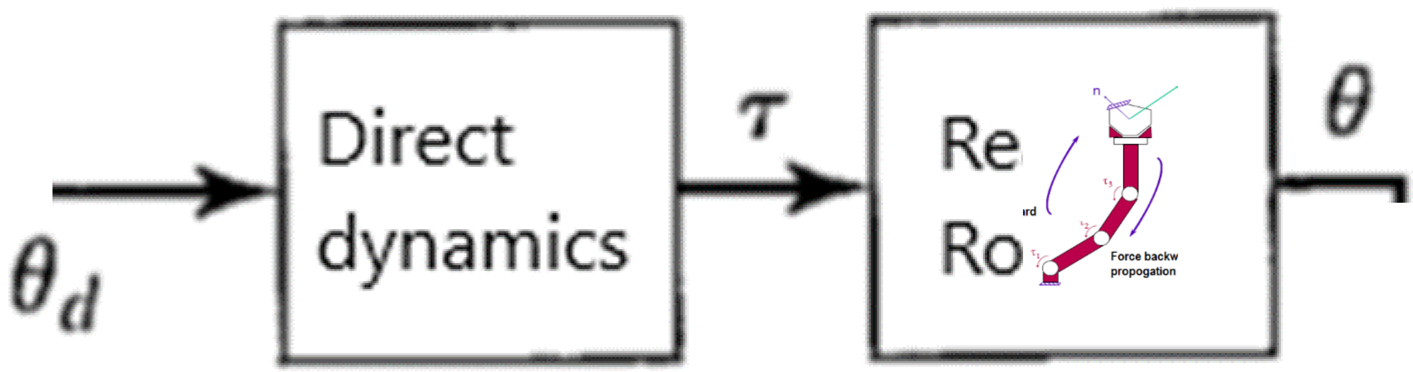
Chapter 6
Manipulator dynamics

6.1 INTRODUCTION

Our study of manipulators so far has focused on kinematic considerations only. We have studied static positions, static forces, and velocities; but we have never considered *the forces required to cause motion*. In this chapter, we consider the equations of motion for a manipulator—the way in which motion of the manipulator arises from torques applied by the actuators or from external forces applied to the manipulator.

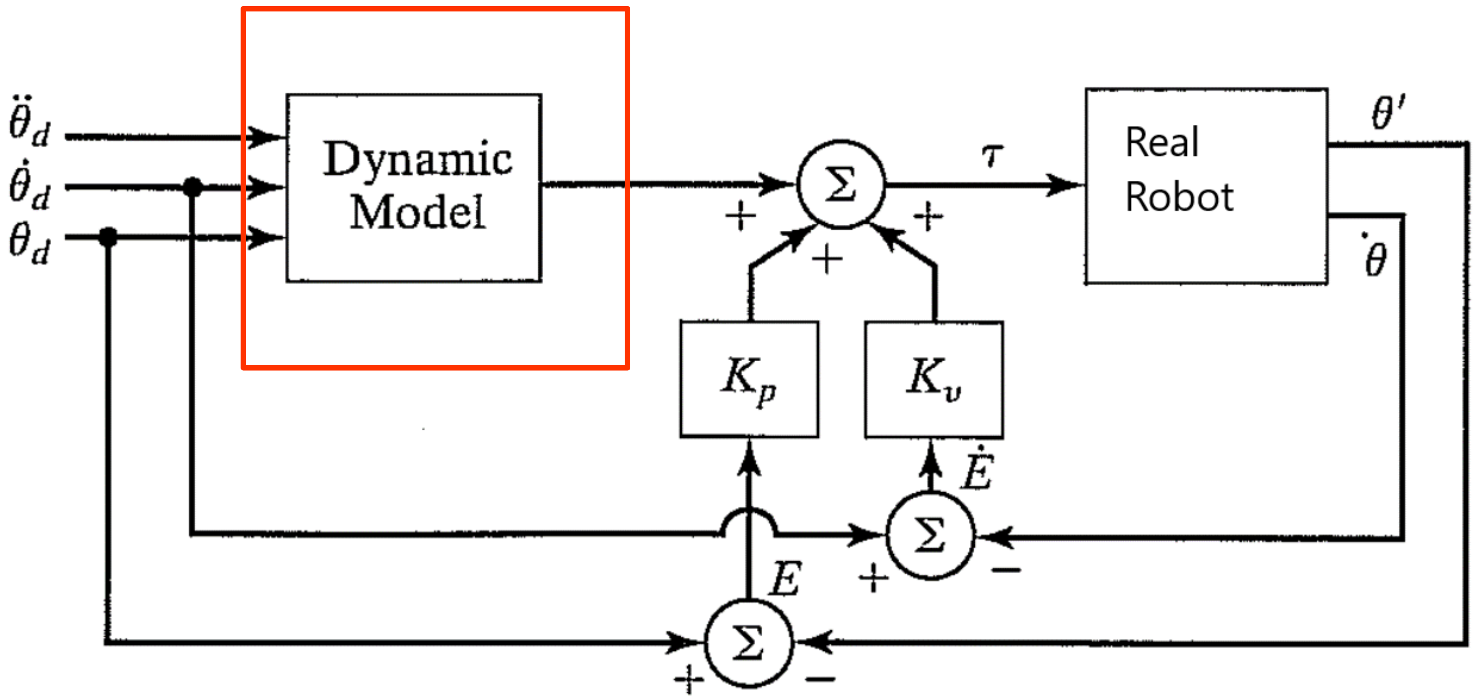
Direct dynamic model

There are two problems related to the dynamics of a manipulator that we wish to solve. In the first problem, we are given a trajectory point, Θ, $\dot{\Theta}$, and $\ddot{\Theta}$, and we wish to find the required vector of joint torques, τ. this is called Direct Dynamics



Direct dynamic model

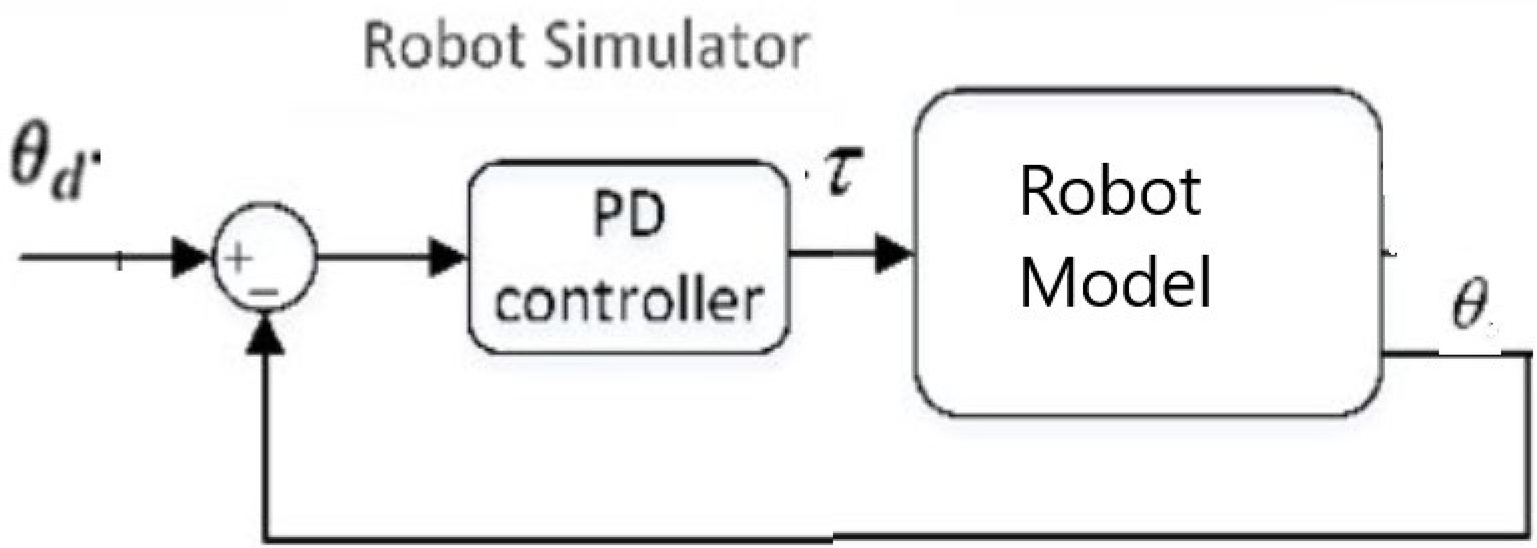
This formulation of dynamics is useful for the problem of controlling the manipulator



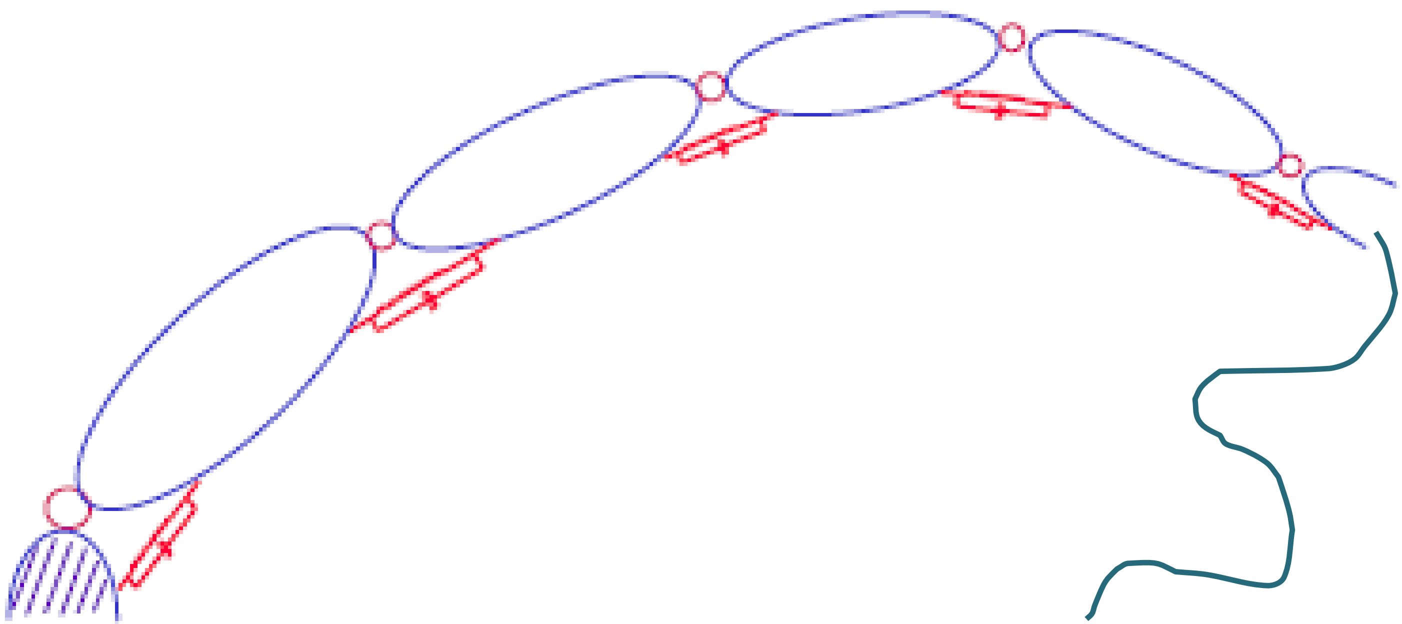
Inverse dynamic model

The second

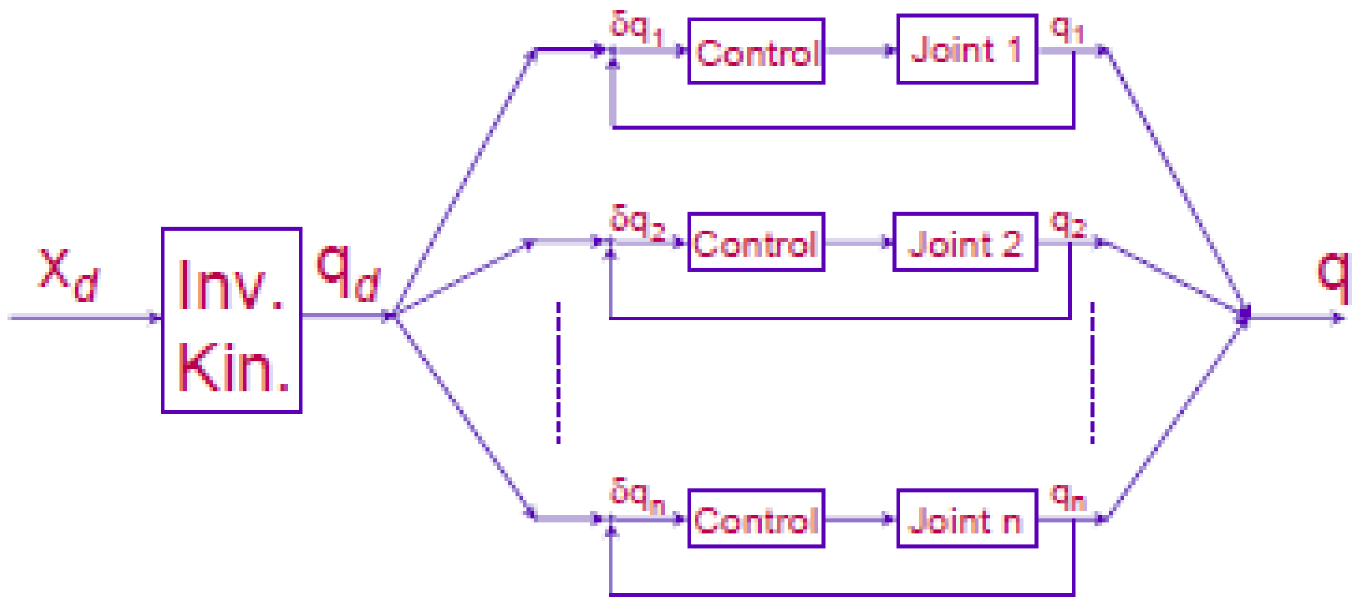
problem is to calculate how the mechanism will move under application of a set of joint torques. That is, given a torque vector, τ, calculate the resulting motion of the manipulator, Θ, $\dot{\Theta}$, and $\ddot{\Theta}$. This is useful for simulating the manipulator.

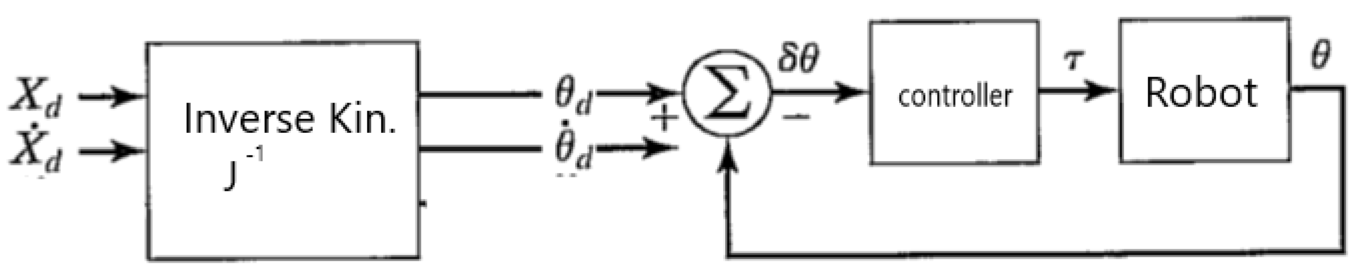


Joint-Space Control

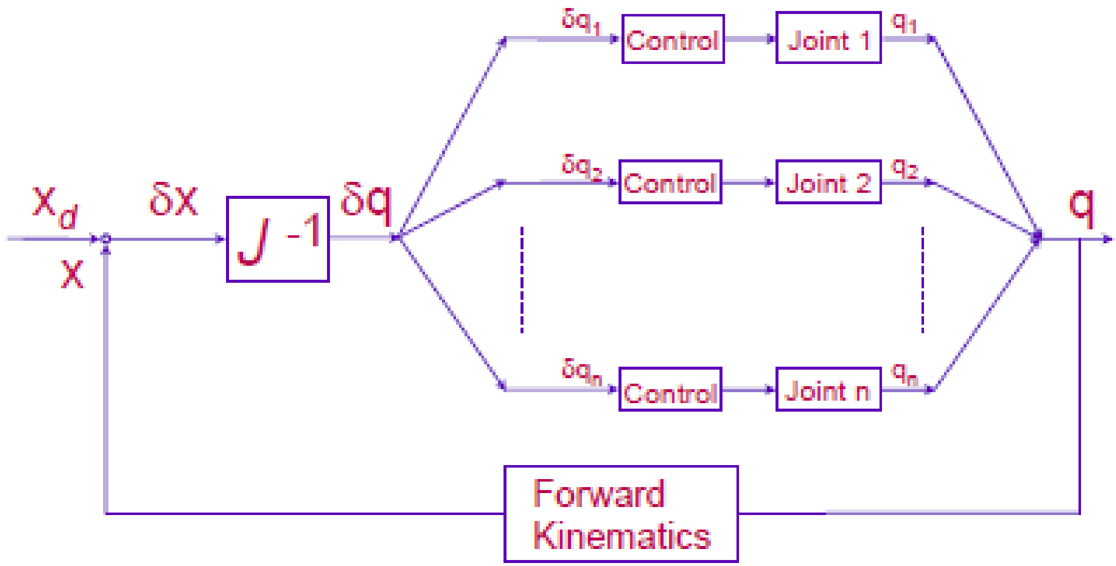


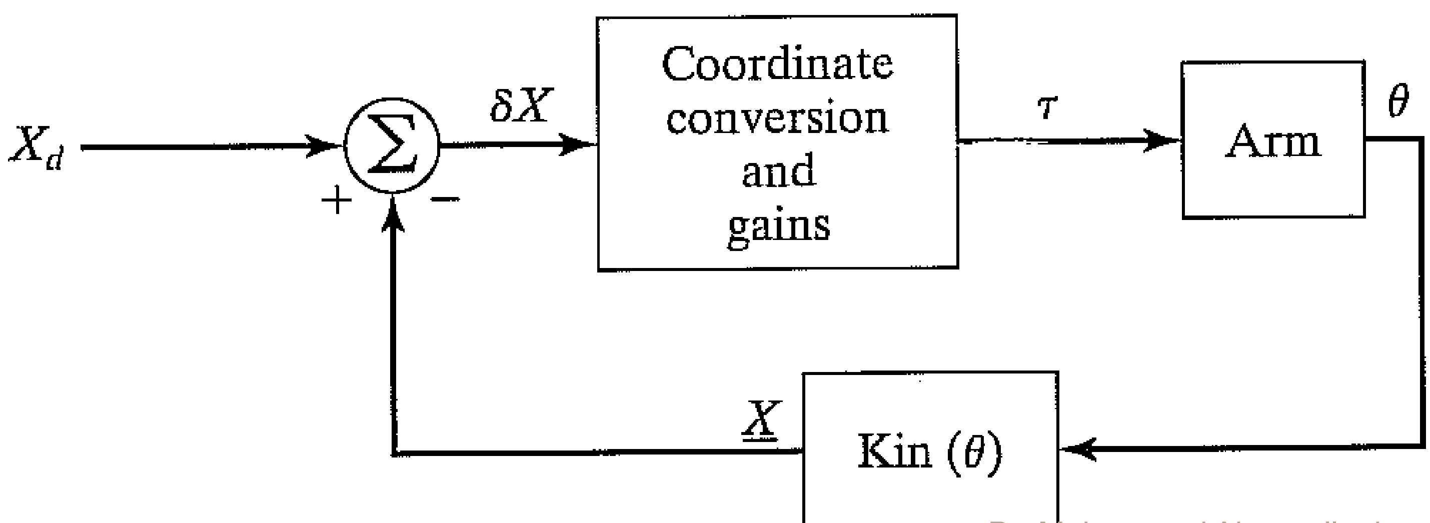
Joint Space Control



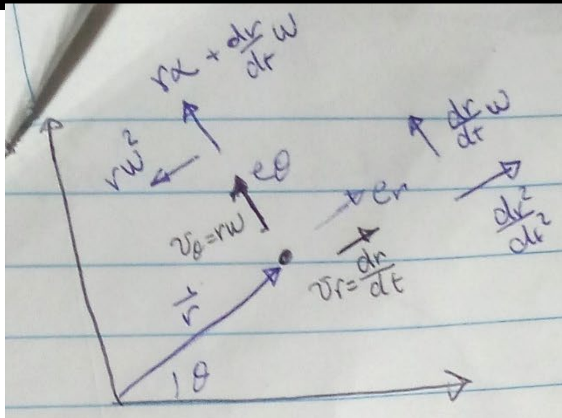


Cartesian space control





Dynamics review (2D)



$$\vec{r} = r\,\vec{e}_r$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt}$$

But $\frac{d\vec{e}_r}{dt} = \frac{d\theta}{dt}\vec{e}_\theta = \omega\vec{e}_\theta$

(due to change in magnitude) ↳ due to change in direction

$$\vec{v} = \frac{dr}{dt}\vec{e}_r + r\omega\vec{e}_\theta$$

$$\vec{a} = \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\omega\vec{e}_\theta + r\frac{d\omega}{dt}\vec{e}_\theta + r\omega\frac{d\vec{e}_\theta}{dt}$$

But $\frac{d\vec{e}_\theta}{dt} = -\frac{d\theta}{dt}\vec{e}_r$

$$\vec{a} = \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\omega\vec{e}_\theta + \frac{dr}{dt}\omega\vec{e}_\theta + r\alpha\vec{e}_\theta + r\omega\left(-\frac{d\theta}{dt}\right)\vec{e}_r$$

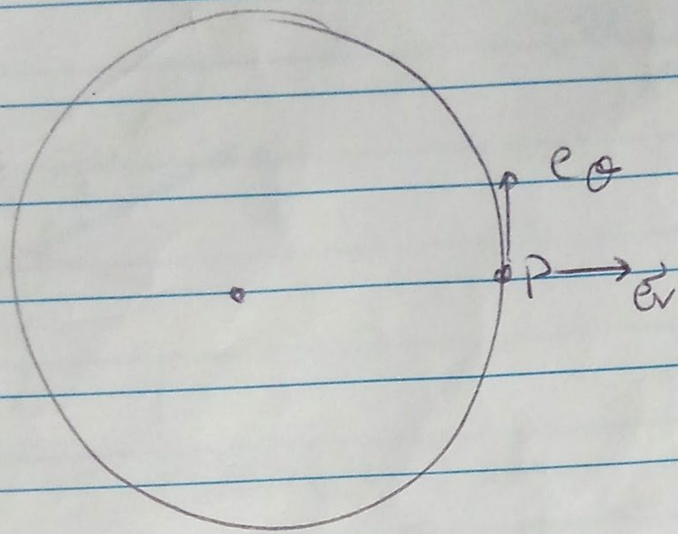
$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\omega^2\right]\vec{e}_r + \left[r\alpha + 2\frac{dr}{dt}\omega\right]\vec{e}_\theta$$

$-r\omega^2$ ↓ centripetal acceleration

$2\frac{dr}{dt}\omega$ ↳ Coriolis acceleration

Dynamics review (2D)

circular motion



$$\vec{v} = \frac{dr}{dt}\vec{e_r} + r\omega\vec{e_\theta} = r\omega\vec{e_\theta}$$

($\frac{dr}{dt}$ crossed out, $= 0$)

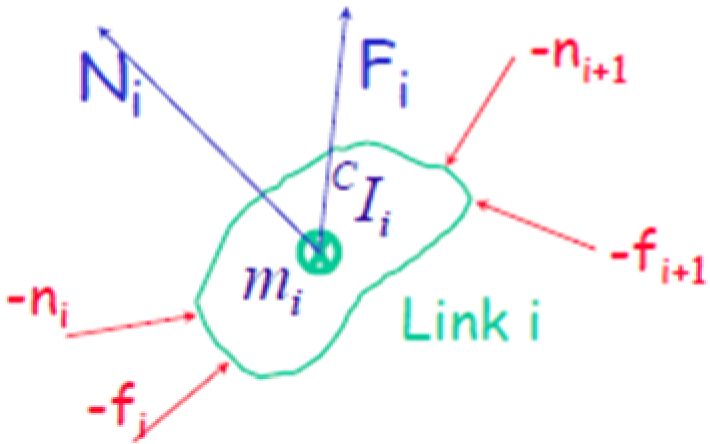
$$\vec{a} = \frac{dr}{dt}\omega\vec{e_\theta} + r\alpha\vec{e_\theta} + r\omega^2\vec{e_r} = (r\alpha)\vec{e_\theta} - (r\omega^2)\vec{e_r}$$

($\frac{dr}{dt}$ crossed out)

Two methods for Dynamics modeling

Formulations

Newton-Euler



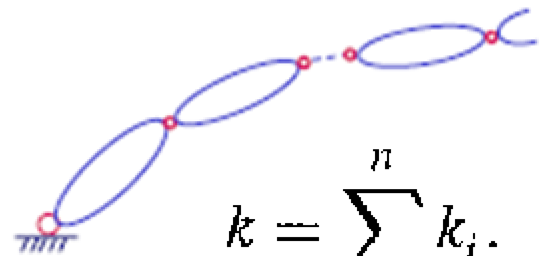
Newton: $F_i = m_i \dot{v}_{C_i}$

Euler: $N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i$

Eliminate Internal Forces and Moments

$$\Gamma_i = \begin{cases} n_i^T.Z_i & \text{revolute} \\ f_i^T.Z_i & \text{prismatic} \end{cases}$$

Lagrange



$$k = \sum_{i=1}^{n} k_i.$$

$$u = \sum_{i=1}^{n} u_i.$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\Theta}} - \frac{\partial \mathcal{L}}{\partial \Theta} = \tau,$$

$$\mathcal{L}(\Theta, \dot{\Theta}) = k(\Theta, \dot{\Theta}) - u(\Theta).$$

Two methods for Dynamics modeling

$$^{i}f_i = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} + {}^{i}F_i,$$

$$^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i$$

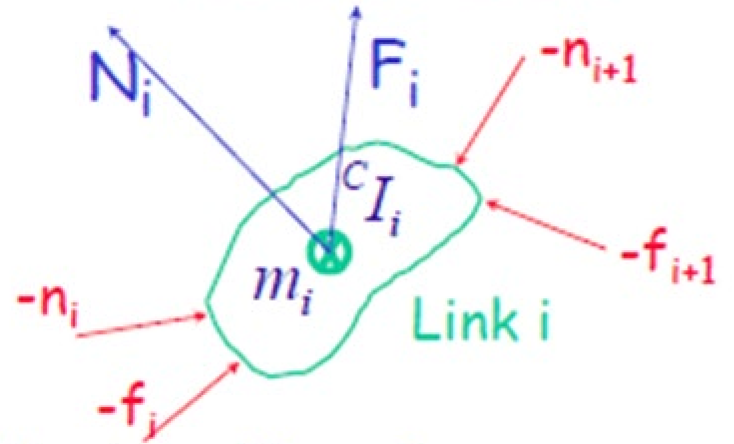
$$+ {}^{i}P_{i+1} \times {}^{i}_{i+1}R\, {}^{i+1}f_{i+1},$$

$$\tau_i = {}^{i}n_i^T\, {}^{i}\hat{Z}_i.$$

$$\Gamma_i = \begin{cases} n_i^T.Z_i & \text{revolute} \\ f_i^T.Z_i & \text{prismatic} \end{cases}$$

Formulations

Newton-Euler

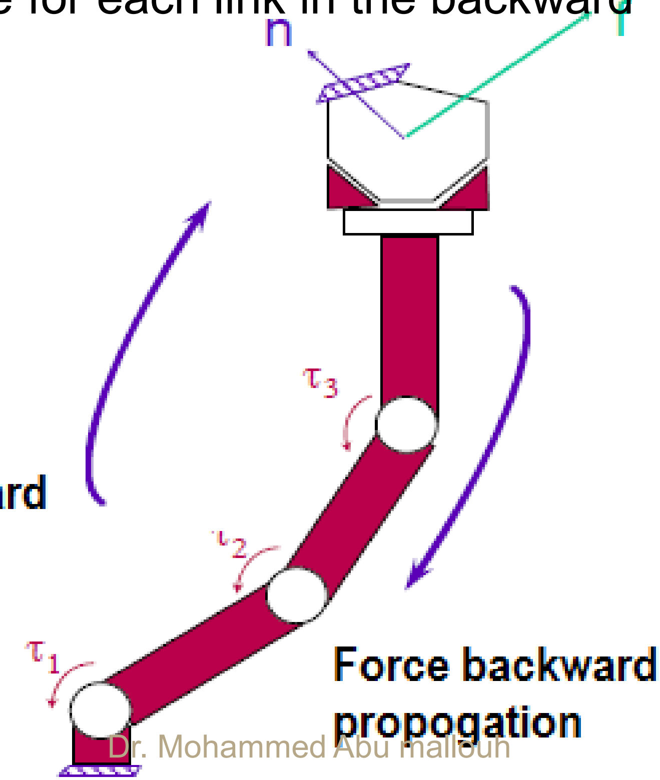


Newton: $F_i = m_i \dot{v}_{C_i}$

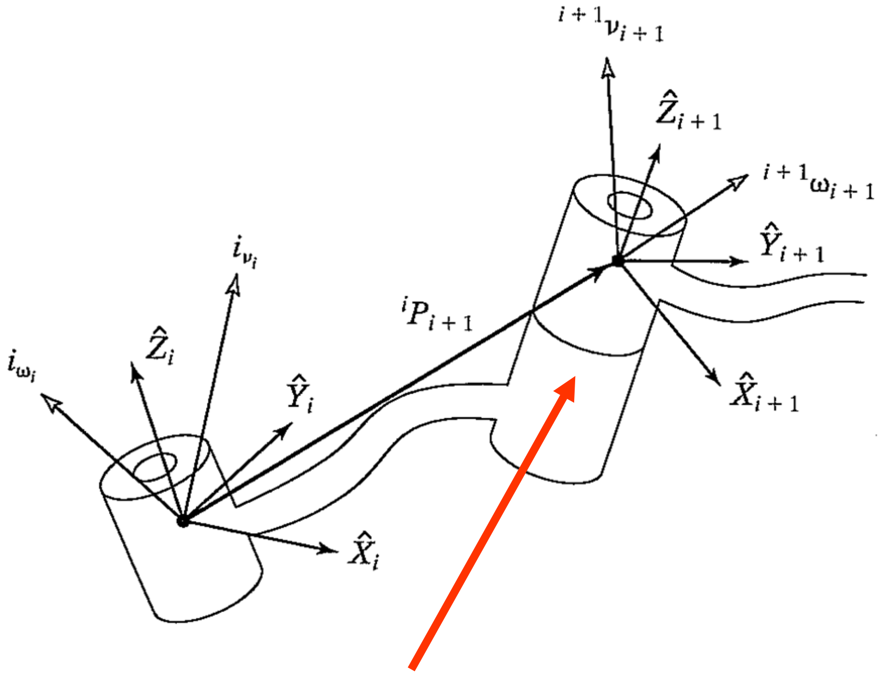
Euler: $N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i$

Newton Euler method

- Apply Newton's force equation and Euler moment equation for each link
- To do so , one needs linear and angular acceleration for each link
- this is done by forward propagation of acceleration
- Then apply force and moment balance for each link in the backward direction (backward).



Newton Euler method

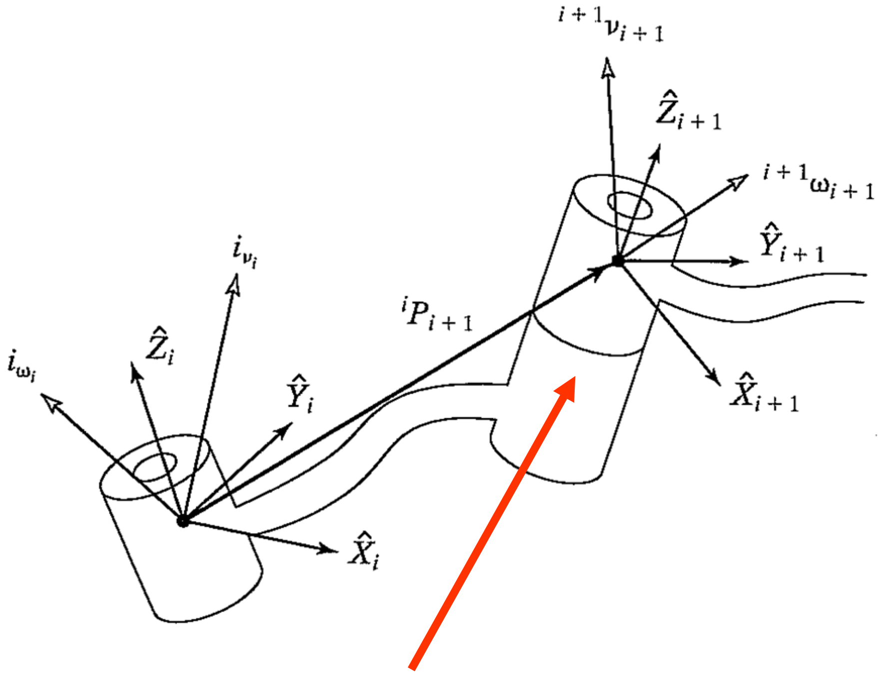


For revolute joint

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}. $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}. $$

Newton Euler method

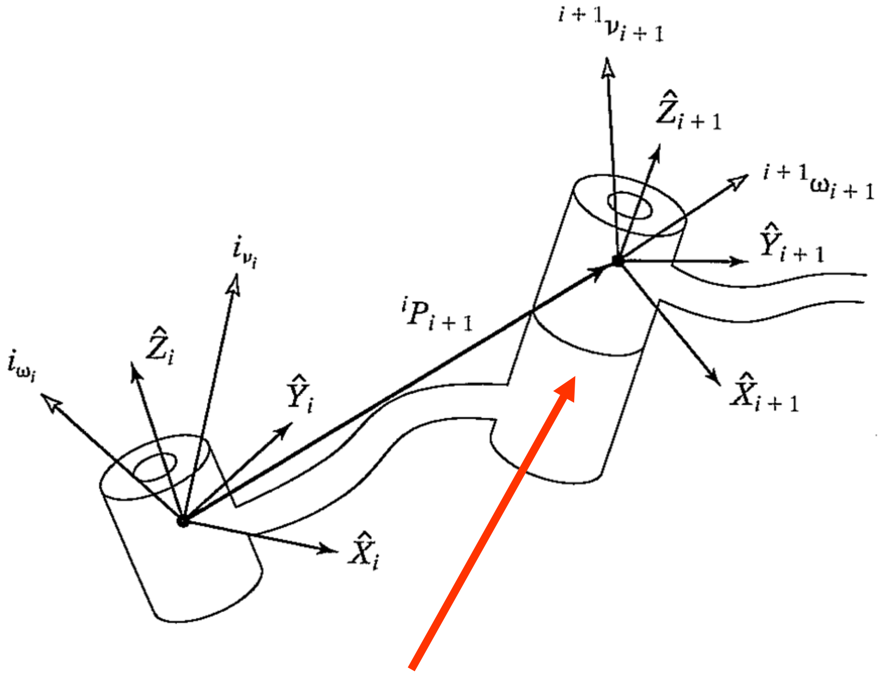


For revolute joint

$$ {}^{i+1}\upsilon_{i+1} = {}^{i+1}_{i}R({}^{i}\upsilon_{i} + {}^{i}\omega_{i} X\, {}^{i}P_{i+1}). $$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R[{}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times ({}^{i}\omega_{i} \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_{i}], $$

Newton Euler method

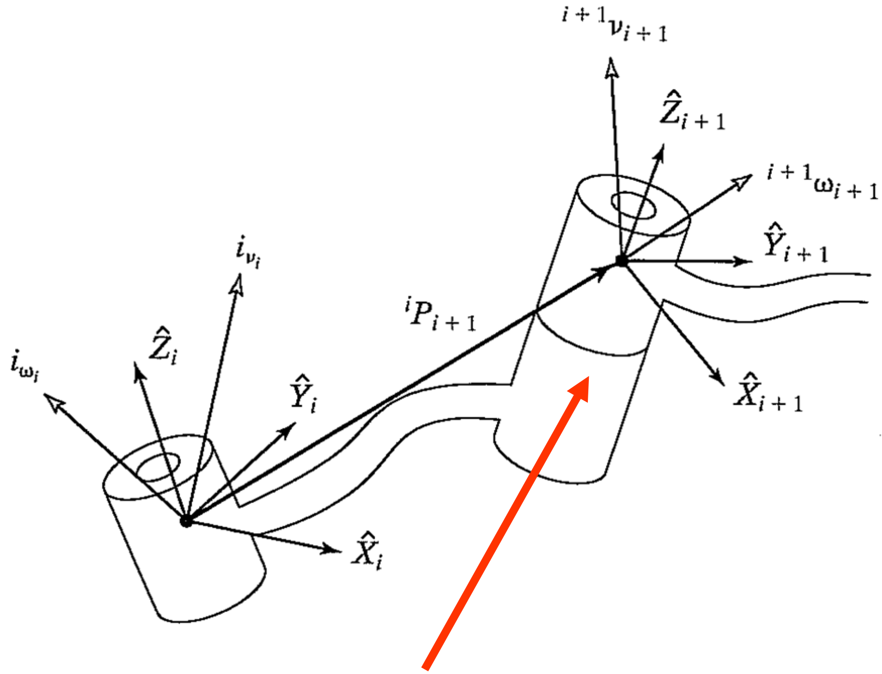


For prismatic joint

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i .$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i .$$

Newton Euler method

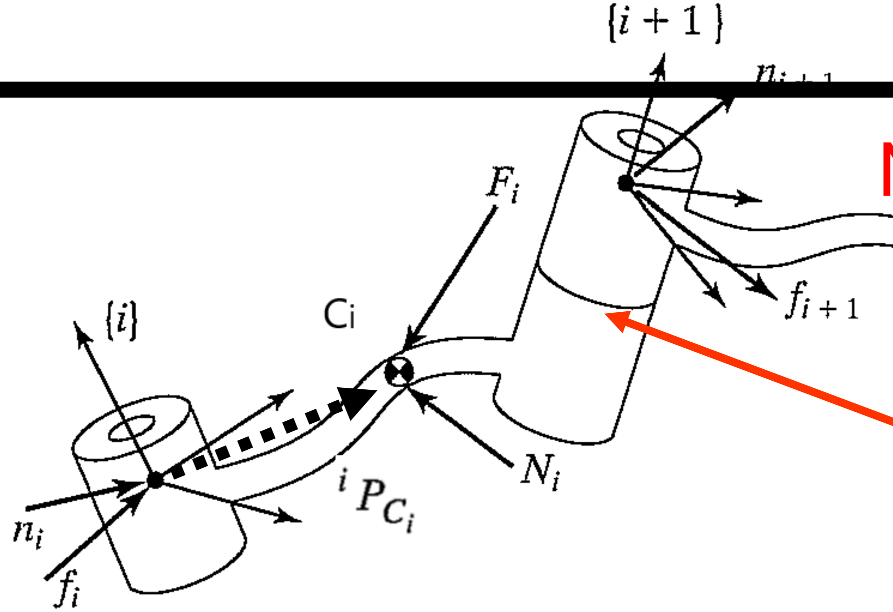


For prismatic joint

$$^{i+1}\upsilon_{i+1} = {}^{i+1}_{i}R({}^{i}\upsilon_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}) + \dot{d}_{i+1}\,{}^{i+1}\hat{z}_{i+1}$$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times ({}^{i}\omega_{i} \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_{i}) + 2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}.$$

Newton Euler method



For prismatic and revolute joint

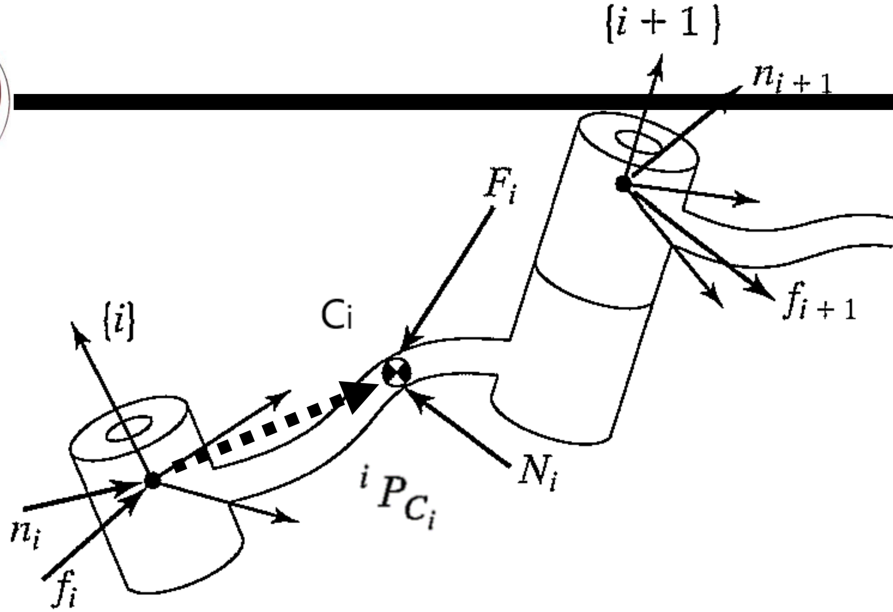
$$ {}^{i}\omega_{ci} = {}^{i}\omega_{i} \qquad {}^{i}\dot{\omega}_{ci} = {}^{i}\dot{\omega}_{i} $$

We also will need the linear acceleration of the center of mass of each link, which also can be found by applying (6.12):

$$ {}^{i}\dot{v}_{C_i} = {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times ({}^{i}\omega_i + {}^{i}P_{C_i}) + {}^{i}\dot{v}_i, \qquad (6.36) $$

Here, we imagine a frame, $\{C_i\}$, attached to each link, having its origin located at the center of mass of the link and having the same orientation as the link frame,

Newton Euler method



Inertia tensor for link i in {Ci}

$$F_i = m\dot{v}_{C_i},$$

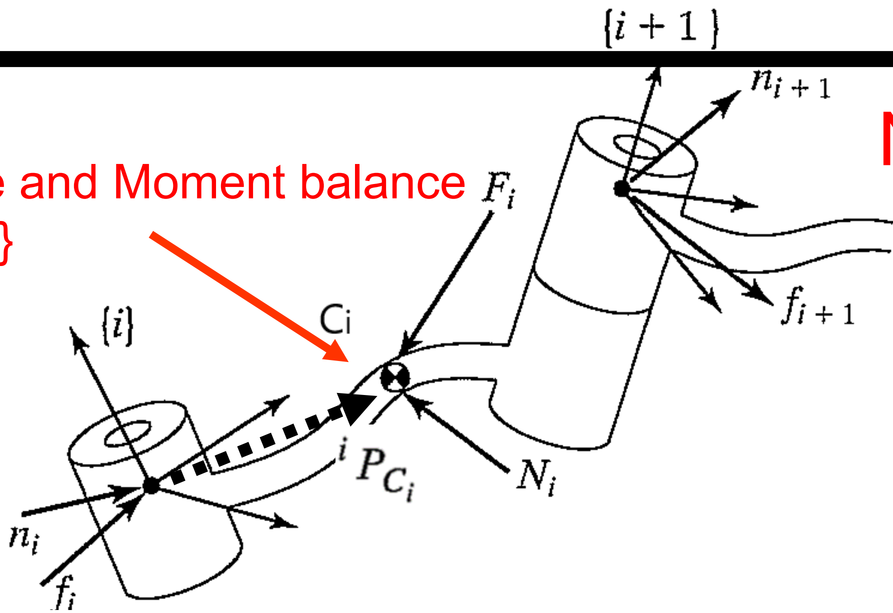
$$N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i, \quad (6.37)$$

where $\{C_i\}$ has its origin at the center of mass of the link and has the same orientation as the link frame, $\{i\}$.

$$f_i = \text{force exerted on link } i \text{ by link } i-1,$$
$$n_i = \text{torque exerted on link } i \text{ by link } i-1.$$

Newton Euler method

Force and Moment balance at {Ci}



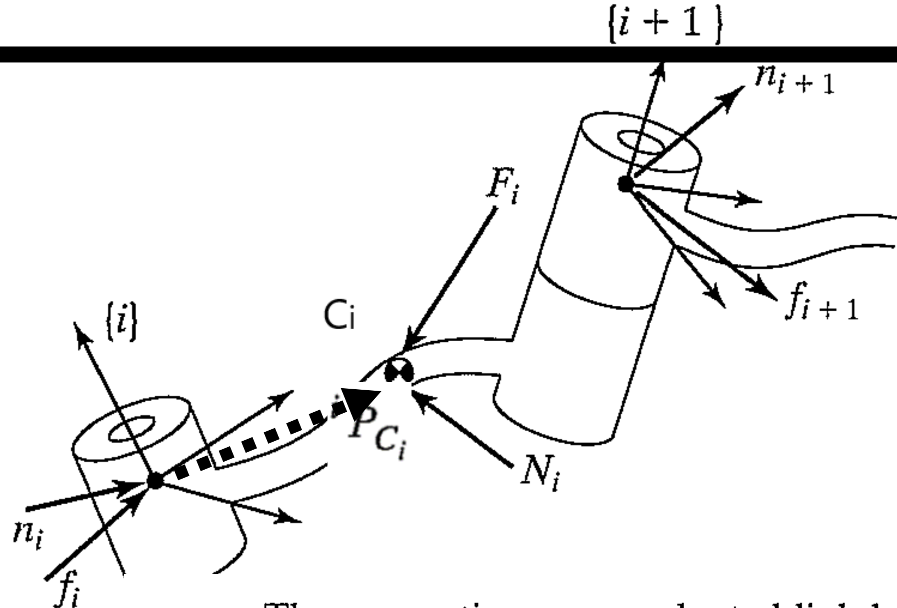
By summing torques about the center of mass and setting them equal to zero, we arrive at the torque-balance equation:

$$ {}^iN_i = {}^in_i - {}^in_{i+1} + (-{}^iP_{C_i}) \times {}^if_i - ({}^iP_{i+1} - {}^iP_{C_i}) \times {}^if_{i+1}. \tag{6.39}$$

$$ {}^iF_i = {}^if_i - {}^i_{i+1}R^{i+1}f_{i+1}.$$

$$ {}^if_i = {}^i_{i+1}R\,{}^{i+1}f_{i+1} + {}^iF_i,$$

$$ {}^in_i = {}^iN_i + {}^i_{i+1}R\,{}^{i+1}n_{i+1} + {}^iP_{C_i} \times {}^iF_i + {}^iP_{i+1} \times {}^i_{i+1}R\,{}^{i+1}f_{i+1}.$$



These equations are evaluated link by link, starting from link n and working inward toward the base of the robot. These *inward force iterations* are analogous to the static force iterations introduced in Chapter 5, except that inertial forces and torques are now considered at each link.

As in the static case, the required joint torques are found by taking the $\hat{Z}$ component of the torque applied by one link on its neighbor:

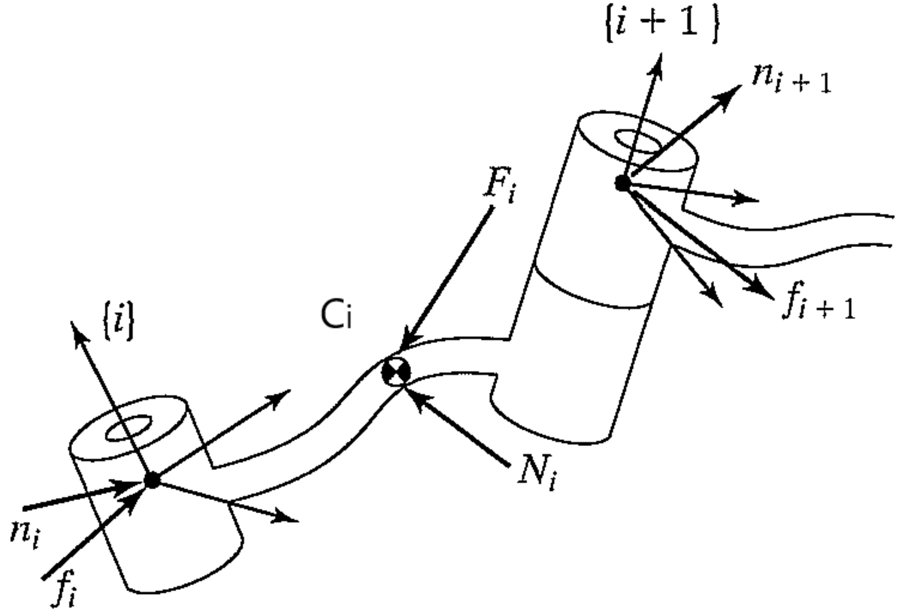
For revolute joint

$$\tau_i = {}^i n_i^T \, {}^i\hat{Z}_i. \tag{6.43}$$

For joint i prismatic, we use

$$\tau_i = {}^i f_i^T \, {}^i\hat{Z}_i, \tag{6.44}$$

For prismatic joint



where we have used the symbol τ for a linear actuator force.

Note that, for a robot moving in free space, ${}^{N+1}f_{N+1}$ and ${}^{N+1}n_{N+1}$ are set equal to zero, and so the first application of the equations for link n is very simple. If the robot is in contact with the environment, the forces and torques due to this contact can be included in the force balance by having nonzero ${}^{N+1}f_{N+1}$ and ${}^{N+1}n_{N+1}$.

Inward iterations to compute forces and torques

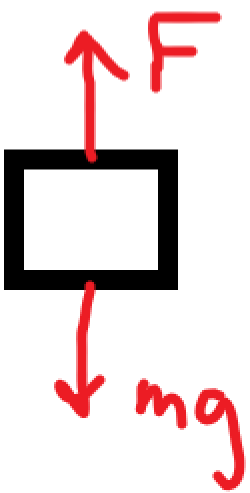
Having computed the forces and torques acting on each link, we now need to calculate the joint torques that will result in these net forces and torques being applied to each link.

We can do this by writing a force-balance and moment-balance equation based on a free-body diagram of a typical link. (See Fig. 6.5.) Each link has forces and torques exerted on it by its neighbors and in addition experiences an inertial force and torque. In Chapter 5, we defined special symbols for the force and torque exerted by a neighbor link, which we repeat here:

f_i = force exerted on link i by link $i-1$,
n_i = torque exerted on link i by link $i-1$.

Inclusion of gravity forces in the dynamics algorithm

The effect of gravity loading on the links can be included quite simply by setting ${}^{0}\dot{v}_0 = G$, where G has the magnitude of the gravity vector but points in the opposite direction. This is equivalent to saying that the base of the robot is accelerating upward with 1 g acceleration. This fictitious upward acceleration causes exactly the same effect on the links as gravity would. So, with no extra computational expense, the gravity effect is calculated.

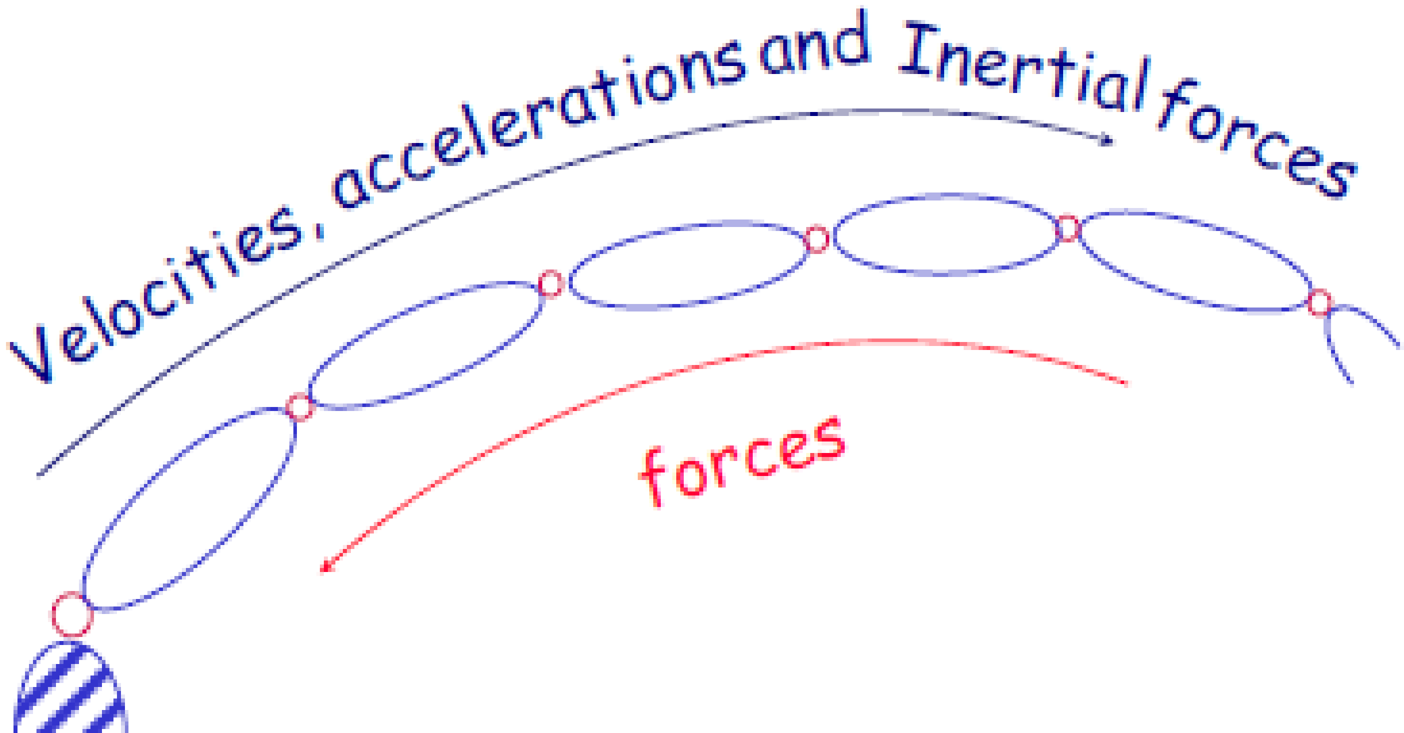


$$F - mg = ma$$

$$F = m(a+g)$$

The iterative Newton–Euler dynamics algorithm

The complete algorithm for computing joint torques from the motion of the joints is composed of two parts. First, link velocities and accelerations are iteratively computed from link 1 out to link n and the Newton–Euler equations are applied to each link. Second, forces and torques of interaction and joint actuator torques are computed recursively from link n back to link 1. The equations are summarized next for the case of all joints rotational:



Outward iterations: $i : 0 \to 5$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}, \tag{6.45}$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\, {}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\, {}^{i}\omega_i \times \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}, \tag{6.46}$$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i), \tag{6.47}$$

$$\begin{aligned} ^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} \\ &\quad + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1}, \end{aligned} \tag{6.48}$$

$$^{i+1}F_{i+1} = m_{i+1}\, {}^{i+1}\dot{v}_{C_{i+1}}, \tag{6.49}$$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\omega_{i+1}. \tag{6.50}$$

Inward iterations: $i : 6 \to 1$

$$^{i}f_i = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} + {}^{i}F_i, \tag{6.51}$$

$$\begin{aligned} ^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i \\ &\quad + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\, {}^{i+1}f_{i+1}, \end{aligned} \tag{6.52}$$

$$\tau_i = {}^{i}n_i^T\, {}^{i}\hat{Z}_i. \tag{6.53}$$

6.3 MASS DISTRIBUTION

In systems with a single degree of freedom, we often talk about the mass of a rigid body. In the case of rotational motion about a single axis, the notion of the *moment of inertia* is a familiar one. For a rigid body that is free to move in three dimensions, there are infinitely many possible rotation axes. In the case of rotation about an arbitrary axis, we need a complete way of characterizing the mass distribution of a rigid body. Here, we introduce the **inertia tensor**

We shall now define a set of quantities that give information about the distribution of mass of a rigid body relative to a reference frame. Figure 6.1 shows a rigid body with an attached frame. Inertia tensors can be defined relative to any frame, but we will always consider the case of an inertia tensor defined for a frame attached to the rigid body. Where it is important, we will indicate, with a leading superscript, the frame of reference of a given inertia tensor. The inertia tensor relative to frame $\{A\}$ is expressed in the matrix form as the 3×3 matrix

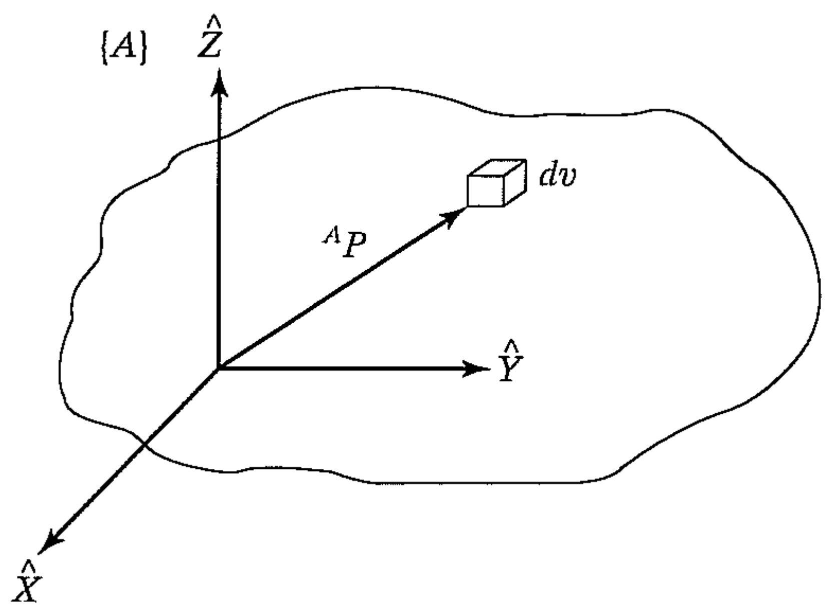


FIGURE 6.1: The inertia tensor of an object describes the object's mass distribution. Here, the vector ^{A}P locates the differential volume element, dv.

$$ {}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} $$

$$ I_{xx} = \iiint_V (y^2 + z^2)\rho dv, $$

$$ I_{yy} = \iiint_V (x^2 + z^2)\rho dv, $$

$$ I_{zz} = \iiint_V (x^2 + y^2)\rho dv, $$

$$ I_{xy} = \iiint_V xy\rho dv, $$

$$ I_{xz} = \iiint_V xz\rho dv, $$

$$ I_{yz} = \iiint_V yz\rho dv, $$

The elements I_{xx}, I_{yy}, and I_{zz} are called the **mass moments of inertia**.

The elements with mixed indices are called the **mass products of inertia**.

EXAMPLE 6.1

Find the inertia tensor for the rectangular body of uniform density ρ with respect to the coordinate system shown in Fig. 6.2.

First, we compute I_{xx}. Using volume element $dv = dx\,dy\,dz$, we get

$$\begin{aligned} I_{xx} &= \int_0^h \int_0^l \int_0^\omega (y^2+z^2)\rho\,dx\,dy\,dz \\ &= \int_0^h \int_0^l (y^2+z^2)\omega\rho\,dy\,dz \\ &= \int_0^h \left(\frac{l^3}{3}+z^2 l\right)\omega\rho\,dz \\ &= \left(\frac{hl^3\omega}{3}+\frac{h^3 l\omega}{3}\right)\rho \\ &= \frac{m}{3}(l^2+h^2), \end{aligned} \tag{6.18}$$

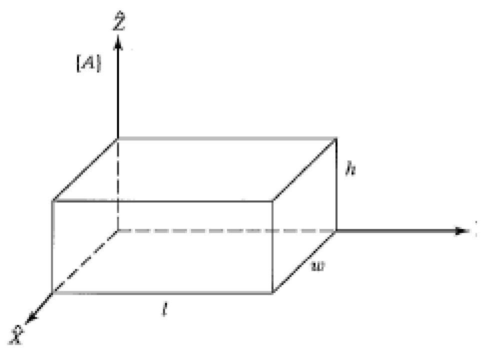


FIGURE 6.2: A body of uniform density.

where m is the total mass of the body. Permuting the terms, we can get I_{yy} and I_{zz} by inspection:

$$I_{yy} = \frac{m}{3}(\omega^2+h^2) \tag{6.19}$$

and

$$I_{zz} = \frac{m}{3}(l^2+\omega^2). \tag{6.20}$$

We next compute I_{xy}:

$$
\begin{aligned}
I_{xy} &= \int_0^h \int_0^l \int_0^\omega xy\rho \, dx \, dy \, dz \\
&= \int_0^h \int_0^l \frac{\omega^2}{2} y\rho \, dy \, dz \qquad (6.21) \\
&= \int_0^h \frac{\omega^2 l^2}{4} \rho dz \\
&= \frac{m}{4} \omega l.
\end{aligned}
$$

Permuting the terms, we get

$$I_{xz} = \frac{m}{4} h\omega \qquad (6.22)$$

and

$$I_{yz} = \frac{m}{4} hl. \qquad (6.23)$$

Hence, the inertia tensor for this object is

$$
{}^A I = \begin{bmatrix} \frac{m}{3}(l^2 + h^2) & -\frac{m}{4}\omega l & -\frac{m}{4}h\omega \\ -\frac{m}{4}\omega l & \frac{m}{3}(\omega^2 + h^2) & -\frac{m}{4}hl \\ -\frac{m}{4}h\omega & -\frac{m}{4}hl & \frac{m}{3}(l^2 + \omega^2) \end{bmatrix}. \qquad (6.24)
$$

As noted, the inertia tensor is a function of the location and orientation of the reference frame. A well-known result, the **parallel-axis theorem**, is one way of computing how the inertia tensor changes under *translations* of the reference coordinate system. The parallel-axis theorem relates the inertia tensor in a frame with origin at the center of mass to the inertia tensor with respect to another reference frame. Where $\{C\}$ is located at the center of mass of the body, and $\{A\}$ is an arbitrarily translated frame, the theorem can be stated [1] as

$$
\begin{aligned}
{}^{A}I_{zz} &= {}^{C}I_{zz} + m(x_c^2 + y_c^2), \\
{}^{A}I_{xy} &= {}^{C}I_{xy} - m x_c y_c,
\end{aligned} \tag{6.25}
$$

where $P_c = [x_c, y_c, z_c]^T$ locates the center of mass relative to $\{A\}$. The remaining moments and products of inertia are computed from permutations of x, y, and z in (6.25). The theorem may be stated in vector–matrix form as

$$
{}^{A}I = {}^{C}I + m[P_c^T P_c I_3 - P_c P_c^T], \tag{6.26}
$$

where I_3 is the 3×3 identity matrix.

EXAMPLE 6.2

Find the inertia tensor for the same solid body described for Example 6.1 when it is described in a coordinate system with origin at the body's center of mass.

We can apply the parallel-axis theorem, (6.25), where

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \omega \\ l \\ h \end{bmatrix}.$$

Next, we find

$$\begin{aligned} {}^{C}I_{zz} &= \frac{m}{12}(\omega^2 + l^2), \\ {}^{C}I_{xy} &= 0. \end{aligned} \tag{6.27}$$

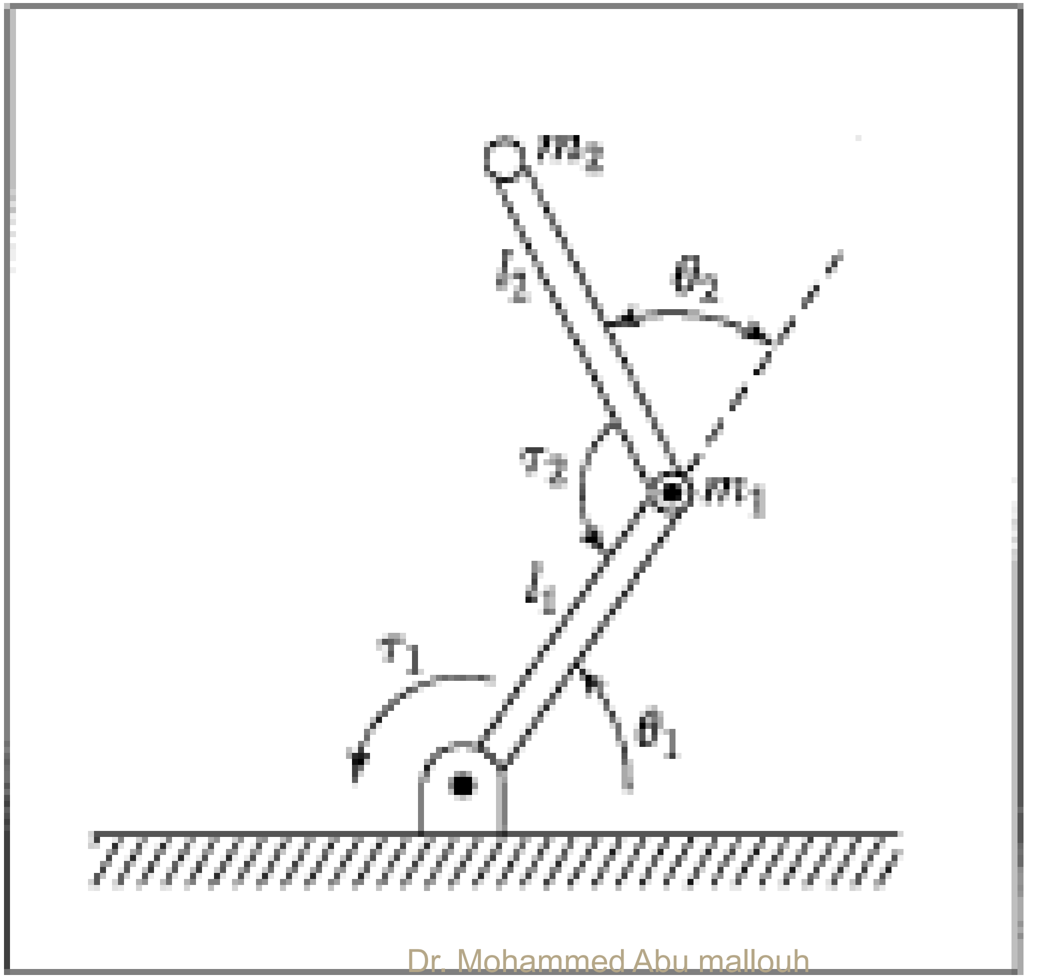
The other elements are found by symmetry. The resulting inertia tensor written in the frame at the center of mass is

$${}^{C}I = \begin{bmatrix} \frac{m}{12}(h^2 + l^2) & 0 & 0 \\ 0 & \frac{m}{12}(\omega^2 + h^2) & 0 \\ 0 & 0 & \frac{m}{12}(l^2 + \omega^2) \end{bmatrix}. \tag{6.28}$$

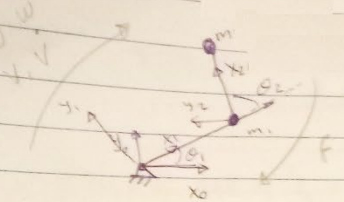
The result is diagonal, so frame $\{C\}$ must represent the principal axes of this body.

Example1

Here we compute the closed-form dynamic equations for the two-link planar manipulator shown in Fig. 6.6. For simplicity, we assume that the mass distribution is extremely simple: All mass exists as a point mass at the distal end of each link. These masses are m_1 and m_2.



Ex:



$$^0\omega_0=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad ^0\dot{\omega}_0=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad ^0\dot{v}_0=\begin{bmatrix}0\\g\\0\end{bmatrix},\quad ^0P_1=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad ^1_0R=\begin{bmatrix}c_1 & s_1 & 0\\-s_1 & c_1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$^1P_2={}^1P_{C_1}=\begin{bmatrix}L_1\\0\\0\end{bmatrix},\quad ^{C_2}I_2={}^{C_1}I_1=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad ^2P_{C_2}=\begin{bmatrix}L_2\\0\\0\end{bmatrix},\quad ^1P_2=\begin{bmatrix}L_1\\0\\0\end{bmatrix},\quad ^2_1R=\begin{bmatrix}c_2 & s_2 & 0\\-s_2 & c_2 & 0\\0 & 0 & 1\end{bmatrix}$$

$$^1\dot{v}_1={}^1_0R\left({}^0\dot{\omega}_0\times{}^0P_1+{}^0\omega_0\times({}^0\omega_0\times{}^0P_1)+{}^0\dot{v}_0\right)$$

$$^1\dot{v}_1=\begin{bmatrix}c_1 & s_1 & 0\\-s_1 & c_1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\g\\0\end{bmatrix}=\begin{bmatrix}s_1g\\c_1g\\0\end{bmatrix}$$

$$^1\omega_1={}^1_0R\,{}^0\omega_0+\dot{\theta}_1\,{}^1Z_1=\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}$$

$$^1\dot{\omega}_1={}^1_0R\,{}^0\dot{\omega}_0+{}^1_0R\,{}^0\omega_0\times\dot{\theta}_1\,{}^1Z_1+\ddot{\theta}_1\,{}^1Z_1=\begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix}$$

$$^1\dot{v}_{C_1}={}^1\dot{\omega}_1\times{}^1P_{C_1}+{}^1\omega_1\times({}^1\omega_1\times{}^1P_{C_1})+{}^1\dot{v}_1$$

$$^1\dot{v}_{C_1}=\begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix}\times\begin{bmatrix}L_1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}\times\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}\times\begin{bmatrix}L_1\\0\\0\end{bmatrix}+\begin{bmatrix}s_1g\\c_1g\\0\end{bmatrix}$$

$$^1\dot{v}_{C_1}=\begin{bmatrix}0\\\ddot{\theta}_1L_1\\0\end{bmatrix}+\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}\times\begin{bmatrix}0\\\dot{\theta}_1L_1\\0\end{bmatrix}+\begin{bmatrix}s_1g\\c_1g\\0\end{bmatrix}=\begin{bmatrix}0\\\ddot{\theta}_1L_1\\0\end{bmatrix}+\begin{bmatrix}-\dot{\theta}_1^2L_1\\0\\0\end{bmatrix}+\begin{bmatrix}s_1g\\c_1g\\0\end{bmatrix}=\begin{bmatrix}-\dot{\theta}_1^2L_1+s_1g\\\ddot{\theta}_1L_1+c_1g\\0\end{bmatrix}$$

$$^1F_1=m_1\,{}^1\dot{v}_{C_1}=\begin{bmatrix}-m_1L_1\dot{\theta}_1^2+m_1gs_1\\m_1L_1\ddot{\theta}_1+m_1gc_1\\0\end{bmatrix}$$

$$^1N_1={}^{C_1}I_1\,{}^1\dot{\omega}_1+{}^1\omega_1\times{}^{C_1}I_1\,{}^1\omega_1=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \dot\theta_2\,{}^2\hat Z_2 = \begin{bmatrix} C_2 & S_2 & 0\\ -S_2 & C_2 & 0 \\ 0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix} + \begin{bmatrix}0\\0\\\dot\theta_2\end{bmatrix} = \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix}$$

$$^2\dot\omega_2 = {}^2_1R\,{}^1\dot\omega_1 + {}^2_1R\,{}^1\omega_1 \times \dot\theta_2\,{}^2\hat Z_2 + \ddot\theta_2\,{}^2\hat Z_2$$

$$= \begin{bmatrix} C_2 & S_2 & 0\\ -S_2 & C_2 & 0 \\ 0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\\ddot\theta_1\end{bmatrix} + \begin{bmatrix} C_2 & S_2 & 0\\ -S_2 & C_2 & 0 \\ 0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix} \times \begin{bmatrix}0\\0\\\dot\theta_2\end{bmatrix} + \begin{bmatrix}0\\0\\\ddot\theta_2\end{bmatrix}$$

$$^2\dot\omega_2 = \begin{bmatrix}0\\0\\\ddot\theta_1\end{bmatrix} + \begin{bmatrix}0\\0\\\ddot\theta_2\end{bmatrix} = \begin{bmatrix}0\\0\\\ddot\theta_1+\ddot\theta_2\end{bmatrix}$$

$$^2\dot v_2 = {}^2_1R\left({}^1\dot\omega_1\times{}^1P_2 + {}^1\omega_1\times({}^1\omega_1\times{}^1P_2)+{}^1\dot v_1\right) = {}^2_1R\,{}^1\dot v_{C_1}$$

$$^2\dot v_2 = \begin{bmatrix} C_2 & S_2 & 0\\ -S_2 & C_2 & 0 \\ 0&0&1\end{bmatrix}\begin{bmatrix}-\dot\theta_1^2 L_1 + S_1 g\\ \ddot\theta_1 L_1 + C_1 g\\ 0\end{bmatrix} = \begin{bmatrix} C_2S_1g - C_2L_1\dot\theta_1^2 + S_2L_1\ddot\theta_1 + C_1S_2g\\ -S_2S_1g + S_2L_1\dot\theta_1^2 + C_2L_1\ddot\theta_1 - C_1C_2g\\ 0\end{bmatrix}$$

$$^2\dot v_2 = \begin{bmatrix} L_1\ddot\theta_1S_2 - L_1\dot\theta_1^2C_2 + gS_{12}\\ L_1\ddot\theta_1C_2 + L_1\dot\theta_1^2S_2 + gC_{12}\\ 0\end{bmatrix}$$

$$^2\dot v_{C_2} = {}^2\dot\omega_2\times{}^2P_{C_2} + {}^2\omega_2\times({}^2\omega_2\times{}^2P_{C_2}) + {}^2\dot v_2$$

(1)
$$\begin{bmatrix}0\\0\\\ddot\theta_1+\ddot\theta_2\end{bmatrix}\times\begin{bmatrix}L_2\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix}\times\left(\begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix}\times\begin{bmatrix}L_2\\0\\0\end{bmatrix}\right) + {}^2\dot v_2$$

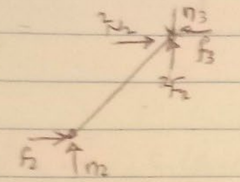
$$= \begin{bmatrix}0\\L_2(\ddot\theta_1+\ddot\theta_2)\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix}\times\begin{bmatrix}0\\L_2(\dot\theta_1+\dot\theta_2)\\0\end{bmatrix} + {}^2\dot v_2$$

$$= \begin{bmatrix}0\\L_2(\ddot\theta_1+\ddot\theta_2)\\0\end{bmatrix} + \begin{bmatrix}-L_2(\dot\theta_1+\dot\theta_2)^2\\0\\0\end{bmatrix} + \begin{bmatrix} L_1\ddot\theta_1S_2 - L_1\dot\theta_1^2C_2 + gS_{12}\\ L_1\ddot\theta_1C_2 + L_1\dot\theta_1^2S_2 + gC_{12}\\ 0\end{bmatrix}$$

$${}^2F_2 = m_2\,{}^2\dot{v}_{c_2} = \begin{bmatrix} m_2 l_1 \ddot\theta_1 S_2 - m_2 l_1 \dot\theta_1^2 C_2 + m_2 g S_{12} - m_2 l_2 (\dot\theta_1 + \dot\theta_2)^2 \\ m_2 l_1 \ddot\theta_1 C_2 + m_2 l_1 \dot\theta_1^2 S_2 + m_2 g C_{12} + m_2 l_2 (\ddot\theta_1 + \ddot\theta_2) \\ 0 \end{bmatrix}$$

$${}^2N_2 = {}^{C_2}I_2\,{}^2\dot\omega_2 + {}^2\omega_2 \times {}^{C_2}I_2\,{}^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

backwards force propagation



$$f_3 = n_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^2f_2 = {}^2_3R\,{}^3f_3 + {}^2F_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} m_2 l_1 \ddot\theta_1 S_2 - m_2 l_1 \dot\theta_1^2 C_2 + m_2 g S_{12} - m_2 l_2 (\dot\theta_1 + \dot\theta_2)^2 \\ m_2 l_1 \ddot\theta_1 C_2 - m_2 l_1 \dot\theta_1^2 S_2 + m_2 g C_{12} - m_2 l_2 (\ddot\theta_1 + \ddot\theta_2) \\ 0 \end{bmatrix}$$

$${}^2n_2 = {}^2N_2 + {}^2_3R\,{}^3n_3 + {}^2P_{C_2} \times {}^2F_2 + {}^2P_3 \times {}^2_3R\,{}^3f_3$$

$${}^2n_2 = \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} {}^2F_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 C_2 \ddot\theta_1 + m_2 l_1 l_2 S_2 \dot\theta_1^2 + m_2 l_2 g C_{12} + m_2 l_2^2 (\ddot\theta_1 + \ddot\theta_2) \end{bmatrix}$$

$${}^1f_1 = {}^1_2R\,{}^2f_2 + {}^1F_1 = \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^2f_2 \end{bmatrix} + \begin{bmatrix} -m_1 l_1 \dot\theta_1^2 + m_1 g S_1 \\ m_1 l_1 \ddot\theta_1 + m_1 g C_1 \\ 0 \end{bmatrix}$$

$${}^1n_1 = {}^1N_1 + {}^1_2R\,{}^2n_2 + {}^1P_{C_1} \times {}^1F_1 + {}^1P_2 \times {}^1_2R\,{}^2f_2$$

$$= \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 C_2 \ddot\theta_1 + m_2 l_1 l_2 S_2 \dot\theta_1^2 + m_2 l_2 g C_{12} + m_2 l_2^2 (\ddot\theta_1 + \ddot\theta_2) \end{bmatrix} +$$

$$\begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} -m_1 l_1 \dot\theta_1^2 + m_1 g S_1 \\ m_1 l_1 \ddot\theta_1 + m_1 g C_1 \\ 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^2f_2 \end{bmatrix}$$

$$ {}^1n_1 = \begin{bmatrix} 0 \\ 0 \\ m_2l_1l_2C_2\ddot\theta_1 + m_2l_1l_2S_2\dot\theta_1^2 + m_2l_2gC_{12} + m_2l_2^2(\ddot\theta_1+\ddot\theta_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ m_1l_1^2\ddot\theta_1 + m_1l_1gC_1 \end{bmatrix} + $$

$$ \begin{bmatrix} 0 \\ 0 \\ m_2l_1^2\ddot\theta_1 - m_2l_1l_2S_2(\dot\theta_1+\dot\theta_2)^2 + m_2l_1gS_2S_{12} + m_2l_1l_2C_2(\ddot\theta_1+\ddot\theta_2) + m_2l_1gC_2C_{12} \end{bmatrix} $$

$$ \tau_1 = {}^1n_1^T\,{}^1Z_1 = m_2l_2^2(\ddot\theta_1+\ddot\theta_2) + m_2l_1l_2C_2(2\ddot\theta_1+\ddot\theta_2) + (m_1+m_2)l_1^2\ddot\theta_1 - m_2l_1l_2S_2\dot\theta_2^2 - 2m_2l_1l_2S_2\dot\theta_1\dot\theta_2 + m_2l_2gC_{12} + (m_1+m_2)l_1gC_1 $$

$$ \tau_2 = {}^2n_2^T\,{}^2Z_2 = m_2l_1l_2C_2\ddot\theta_1 + m_2l_1l_2S_2\dot\theta_1^2 + m_2l_2gC_{12} + m_2l_2^2(\ddot\theta_1+\ddot\theta_2) $$

$$ \tau = M(\theta)\ddot\theta + V(\theta,\dot\theta) + G(\theta) $$

↳ mass matrix (symmetric and positive definite) ⇒ invertable ($Z^TMZ > 0$)

$$ \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix} l_2^2m_2 + 2l_1l_2m_2C_2 + l_1^2(m_1+m_2) & l_2^2m_2 + l_1l_2m_2C_2 \\ l_2^2m_2 + l_1l_2m_2C_2 & l_2^2m_2 \end{bmatrix}\begin{bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{bmatrix} + $$

$$ \underbrace{\begin{bmatrix} -m_2l_1l_2S_2\dot\theta_1^2 - 2m_2l_1l_2S_2\dot\theta_1\dot\theta_2 \\ m_2l_1l_2S_2\dot\theta_1^2 \end{bmatrix}}_{V(\theta,\dot\theta)} + \underbrace{\begin{bmatrix} m_2l_2gC_{12} + (m_2+m_1)l_1gC_1 \\ m_2l_2gC_{12} \end{bmatrix}}_{G(\theta)} $$

(Coriolis: $-2m_2l_1l_2S_2\dot\theta_1\dot\theta_2$; centrifugal: $m_2l_1l_2S_2\dot\theta_1^2$)

$V(\theta,\dot\theta)$ → centrifugal + Coriolis forces

$G(\theta)$ → gravity

First, we determine the values of the various quantities that will appear in the recursive Newton–Euler equations. The vectors that locate the center of mass for each link are

$$^{1}P_{C_1} = l_1 \hat{X}_1,$$
$$^{2}P_{C_2} = l_2 \hat{X}_2.$$

Because of the point-mass assumption, the inertia tensor written at the center of mass for each link is the zero matrix:

$$^{C_1}I_1 = 0,$$
$$^{C_2}I_2 = 0.$$

There are no forces acting on the end-effector, so we have

$$f_3 = 0,$$
$$n_3 = 0.$$

The base of the robot is not rotating; hence, we have

$$\omega_0 = 0,$$
$$\dot{\omega}_0 = 0.$$

To include gravity forces, we will use

$$ {}^{0}\dot{v}_0 = g\hat{Y}_0. $$

The rotation between successive link frames is given by

$$ {}^{i}_{i+1}R = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0.0 \\ s_{i+1} & c_{i+1} & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}, $$

$$ {}^{i+1}_{i}R = \begin{bmatrix} c_{i+1} & s_{i+1} & 0.0 \\ -s_{i+1} & c_{i+1} & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}. $$

We now apply equations (6.46) through (6.53).

The outward iterations for link 1 are as follows:

$$ {}^{1}\omega_1 = \dot{\theta}_1 \, {}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, $$

$$ {}^{1}\dot{\omega}_1 = \ddot{\theta}_1 \, {}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}, $$

$$ {}^{1}\dot{v}_1 = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix}, $$

$$ {}^{1}\dot{v}_{C_1} = \begin{bmatrix} 0 \\ l_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -l_1\dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 \\ l_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix}, $$

$$ {}^{1}F_1 = \begin{bmatrix} -m_1 l_1\dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1\ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}, $$

$$ {}^{1}N_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (6.54) $$

The outward iterations for link 2 are as follows:

$$^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix},$$

$$^{2}\dot{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix},$$

$$^{2}\dot{v}_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 \\ l_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\ddot{\theta}_1 s_2 - l_1\dot{\theta}_1^2 c_2 + gs_{12} \\ l_1\ddot{\theta}_1 c_2 + l_1\dot{\theta}_1^2 s_2 + gc_{12} \\ 0 \end{bmatrix},$$

$$^{2}\dot{v}_{C_2} = \begin{bmatrix} 0 \\ l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} l_1\ddot{\theta}_1 s_2 - l_1\dot{\theta}_1^2 c_2 + gs_{12} \\ l_1\ddot{\theta}_1 c_2 + l_1\dot{\theta}_1^2 s_2 + gc_{12} \\ 0 \end{bmatrix}, \qquad (6.55)$$

$$^{2}F_2 = \begin{bmatrix} m_2 l_1\ddot{\theta}_1 s_2 - m_2 l_1\dot{\theta}_1^2 c_2 + m_2 g s_{12} - m_2 l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1\ddot{\theta}_1 c_2 + m_2 l_1\dot{\theta}_1^2 s_2 + m_2 g c_{12} + m_2 l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix},$$

$$^{2}N_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The inward iterations for link 2 are as follows:

$$ {}^2f_2 = {}^2F_2, $$

$$ {}^2n_2 = \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}. \qquad (6.56) $$

The inward iterations for link 1 are as follows:

$$ {}^1f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 l_1 s_2 \ddot{\theta}_1 - m_2 l_1 c_2 \dot{\theta}_1^2 + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 c_2 \ddot{\theta}_1 + m_2 l_1 s_2 \dot{\theta}_1^2 + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} $$

$$ + \begin{bmatrix} -m_1 l_1 \dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}, $$

$$ {}^1n_1 = \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix} $$

$$ + \begin{bmatrix} 0 \\ 0 \\ m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 g c_1 \end{bmatrix} $$

$$ + \begin{bmatrix} 0 \\ 0 \\ m_2 l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 l_1 g s_2 s_{12} \\ + m_2 l_1 l_2 c_2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 g c_2 c_{12} \end{bmatrix}. \qquad (6.57) $$

$$ {}^if_i = {}^i_{i+1}R\,{}^{i+1} $$

$$ {}^in_i = {}^iN_i + {}^i_{i+1} $$

$$ + {}^iP_{i+1} $$

$$ \tau_i = {}^in_i^T\,{}^i\hat{Z}_i. $$

Extracting the $\hat{Z}$ components of the ${}^{i}n_i$, we find the joint torques:

$$\tau_1 = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2(2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 \dot{\theta}_2^2$$
$$-2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1,$$
$$\tau_2 = m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2). \quad (6.58)$$

Equations (6.58) give expressions for the torque at the actuators as a function of joint position, velocity, and acceleration. Note that these rather complex functions arose from one of the simplest manipulators imaginable. Obviously, the closed-form equations for a manipulator with six degrees of freedom will be quite complex.

THE STRUCTURE OF A MANIPULATOR'S DYNAMIC EQUATIONS

It is often convenient to express the dynamic equations of a manipulator in a single equation that hides some of the details, but shows some of the structure of the equations.

The state-space equation

When the Newton–Euler equations are evaluated symbolically for any manipulator, they yield a dynamic equation that can be written in the form

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta), \tag{6.59}$$

where $M(\Theta)$ is the $n \times n$ **mass matrix** of the manipulator, $V(\Theta, \dot{\Theta})$ is an $n \times 1$ vector of centrifugal and Coriolis terms, and $G(\Theta)$ is an $n \times 1$ vector of gravity terms. We use the term **state-space equation** because the term $V(\Theta, \dot{\Theta})$, appearing in (6.59), has both position and velocity dependence [3].

Each element of $M(\Theta)$ and $G(\Theta)$ is a complex function that depends on Θ, the position of all the joints of the manipulator. Each element of $V(\Theta, \dot{\Theta})$ is a complex function of both Θ and $\dot{\Theta}$.

We may separate the various types of terms appearing in the dynamic equations and form the mass matrix of the manipulator, the centrifugal and Coriolis vector, and the gravity vector.

$$M(\Theta) = \begin{bmatrix} l_2^2 m_2 + 2l_1 l_2 m_2 c_2 + l_1^2 (m_1 + m_2) & l_2^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2^2 m_2 + l_1 l_2 m_2 c_2 & l_2^2 m_2 \end{bmatrix}. \quad (6.60)$$

Any manipulator mass matrix is symmetric and positive definite, and is, therefore, always invertible.

The velocity term, $V(\Theta, \dot{\Theta})$, contains all those terms that have any dependence on joint velocity. Thus, we obtain

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}. \quad (6.61)$$

A term like $-m_2 l_1 l_2 s_2 \dot{\theta}_2^2$ is caused by a **centrifugal force**, and is recognized as such because it depends on the square of a joint velocity. A term such as $-2m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2$ is caused by a **Coriolis force** and will always contain the product of two different joint velocities.

The gravity term, $G(\Theta)$, contains all those terms in which the gravitational constant, g, appears. Therefore, we have

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}. \quad (6.62)$$

Note that the gravity term depends only on Θ and not on its derivatives.

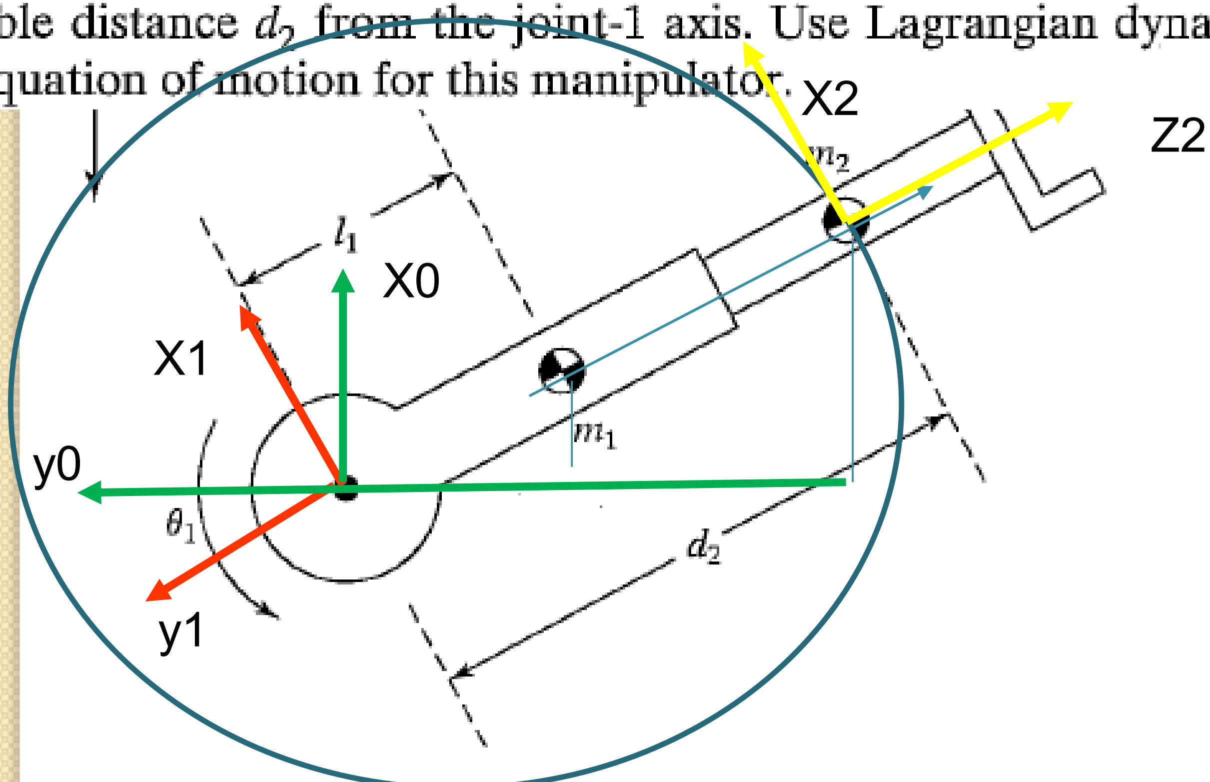
EXAMPLE 6.5

The links of an RP manipulator, shown in Fig. 6.7, have inertia tensors

$$^{c_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix},$$

$$^{c_2}I_2 = \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix},$$

and total mass m_1 and m_2. As shown in Fig. 6.7, the center of mass of link 1 is located at a distance l_1 from the joint-1 axis, and the center of mass of link 2 is at the variable distance d_2 from the joint-1 axis. Use Lagrangian dynamics to determine the equation of motion for this manipulator.



$$\tau_1 = (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2)\ddot{\theta}_1 + 2m_2 d_2 \dot{\theta}_1 \dot{d}_2$$
$$+(m_1 l_1 + m_2 d_2) g \cos(\theta_1),$$
$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g \sin(\theta_1).$$

From (6.89), we can see that

$$M(\Theta) = \begin{bmatrix} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2) & 0 \\ 0 & m_2 \end{bmatrix},$$

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix},$$

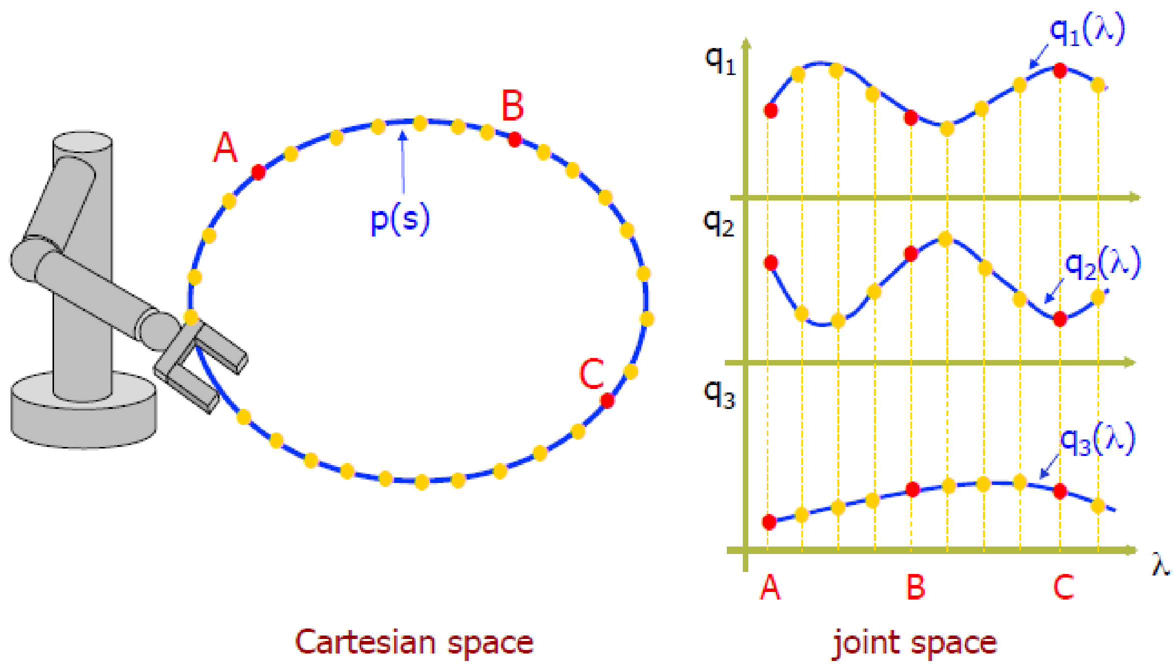
$$G(\Theta) = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g \cos(\theta_1) \\ m_2 g \sin(\theta_1) \end{bmatrix}.$$

Robotics

Chapter 7
Trajectory Generation

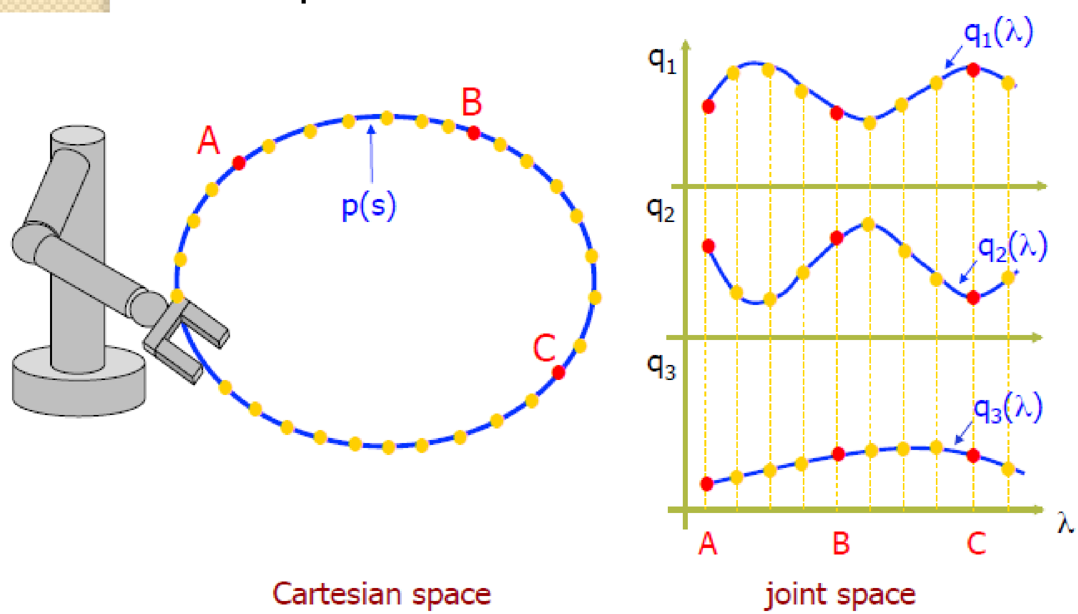
JOINT-SPACE SCHEMES

- the user specify the desired goal position and orientation of the end-effector, and leave it to the system to decide on the exact shape of the path to get there, the duration, the velocity profile, and other details.
- Each path point is usually specified in terms of a desired position and orientation of the tool frame, {*T* }, relative to the station frame, {*S*}.
- Each of these via points is “converted” into a set of desired joint angles by application of the inverse kinematics.
- Then, a smooth function is found for each of the *n* joints.



JOINT-SPACE SCHEMES

- The time required for each segment is the same for each joint so that all joints will reach the via point at the same time, thus resulting in the desired Cartesian position of $\{T\}$ at each via point.
- Joint-space schemes are usually the easiest to compute, and, because we make no continuous correspondence between joint space and Cartesian space, there is essentially no problem with singularities of the mechanism.
- smooth function is continuous and has continuous first derivative, sometimes a continuous second derivative is also desirable.
- Rough, jerky motions tend to cause increased wear on the mechanism, and cause vibrations by exciting resonances in the manipulator.



- Given the initial and goal position of the end effector and the travel time.
- Using Inverse kinematics to find set of joint angles that correspond to the goal and initial position of the manipulator.
- What is required is a function for each joint whose value at t_0 is the initial position of the joint θ_0, and whose value at t_f is the desired goal position θ_f, of that joint. As shown in Fig. 7.2.
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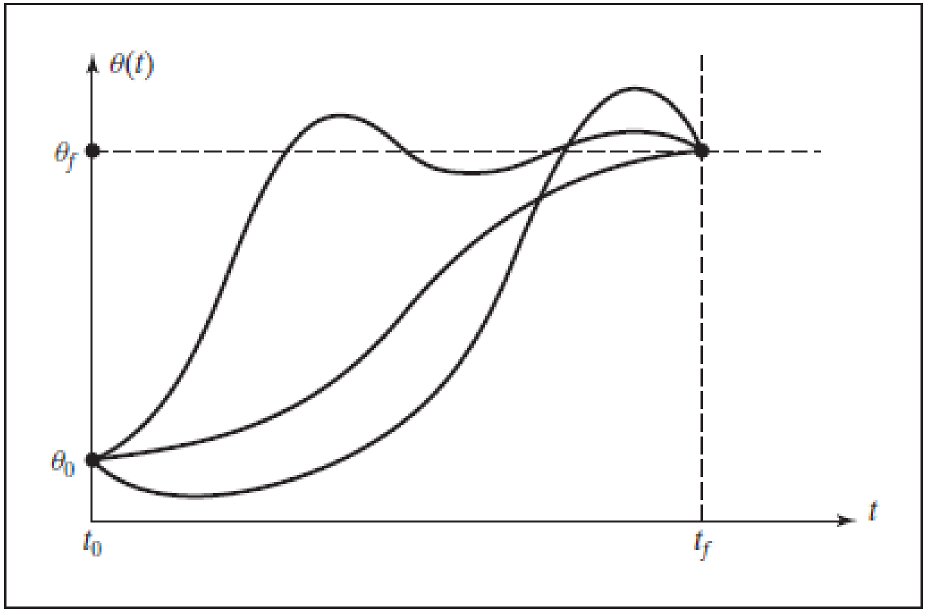


FIGURE 7.2: Several possible path shapes for a single joint.

- there are many smooth functions, *θ(t)*, that might be used to interpolate the joint value.
- In making a single smooth motion, at least four constraints on *θ(t)* are evident.
- Two constraints on the function's value come from the selection of initial and
- final values:

$$\theta(0) = \theta_0,$$

$$\theta(t_f) = \theta_f.$$

- An additional two constraints are that the function be continuous in velocity, which in this case means that the initial and final velocity are zero:

$$\dot{\theta}(0) = 0,$$

$$\dot{\theta}(t_f) = 0.$$

- These four constraints can be satisfied by a polynomial of at least third degree. (A cubic polynomial has four coefficients, so it can be made to satisfy the four constraints

cubic. A cubic has the form

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \qquad (7.3)$$

so the joint velocity and acceleration along this path are clearly

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2,$$
$$\ddot{\theta}(t) = 2a_2 + 6a_3 t. \qquad (7.4)$$

Combining (7.3) and (7.4) with the four desired constraints yields four equations in four unknowns:

$$\theta_0 = a_0,$$
$$\theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3,$$
$$0 = a_1, \qquad (7.5)$$
$$0 = a_1 + 2a_2 t_f + 3a_3 t_f^2.$$

Solving these equations for the a_i, we obtain

$$a_0 = \theta_0,$$

$$a_1 = 0,$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0),$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0).$$

EXAMPLE 7.1

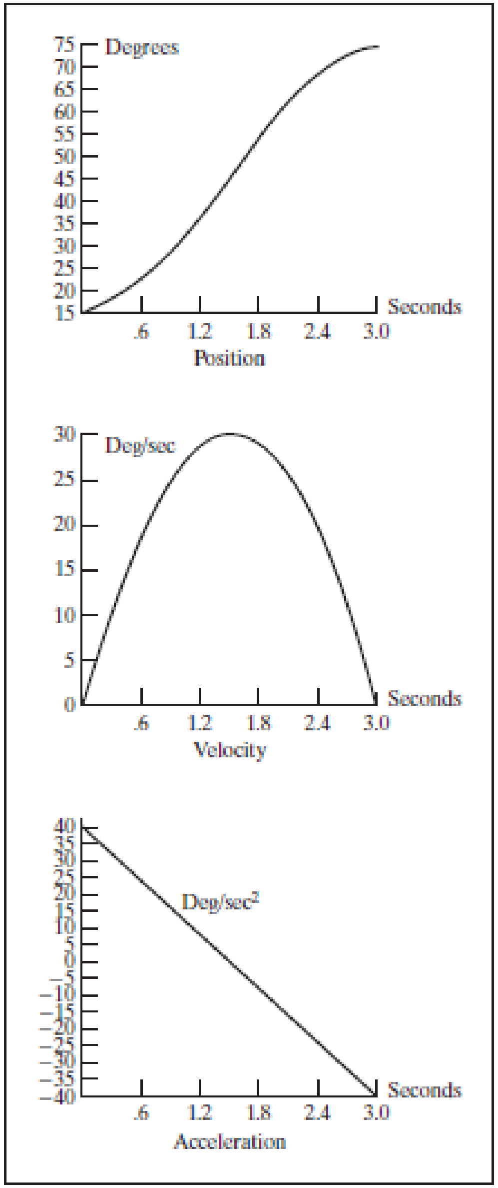
A single-link robot with a rotary joint is motionless at $\theta = 15$ degrees. It is desired to move the joint in a smooth manner to $\theta = 75$ degrees in 3 seconds. Find the coefficients of a cubic that accomplishes this motion and brings the manipulator to rest at the goal. Plot the position, velocity, and acceleration of the joint as a function of time.

Substituting into (7.6), we find that

$$\begin{aligned} a_0 &= 15.0, \\ a_1 &= 0.0, \\ a_2 &= 20.0, \\ a_3 &= -4.44. \end{aligned} \tag{7.7}$$

Using (7.3) and (7.4), we obtain

$$\begin{aligned} \theta(t) &= 15.0 + 20.0t^2 - 4.44t^3, \\ \dot{\theta}(t) &= 40.0t - 13.33t^2, \\ \ddot{\theta}(t) &= 40.0 - 26.66t. \end{aligned} \tag{7.8}$$



$$\theta(t) = 15.0 + 20.0t^2 - 4.44t^3,$$
$$\dot{\theta}(t) = 40.0t - 13.33t^2,$$
$$\ddot{\theta}(t) = 40.0 - 26.66t.$$

FIGURE 7.3: Position, velocity, and acceleration profiles for a single cubic segment that starts and ends at rest.