ElCoM
ELECTRICAL COMPUTER MECHATRONICS

تلخيص

معادلات تفاضلية 1

للطالـب المبدع

محمود مجدلاني

إرادة - ثقة - تغيير

CHAPTER (1)

Classification

Differentiable equations are classified by :

TYPE		
	ordinary DE (ODE)	1) y'' + y' = 3y 2) $\frac{dy}{dx}$ + 3y = Lnx
	partial DE (PDE)	1) $\frac{\partial y}{\partial x} + \frac{\partial y}{\partial z}$ = 3xz 2) $u_{xx} - u_{yy} = 0$

يعني أكم مرة نشتق
هون نشتق مرتين

ORDER

The order of DE is the order of the highest derivative in the equation

(باختصار هي رتبة اعلى مشتقة)

CLASSIFICATION AS LINEAR OR NON-LINEAR

The linear (ODE) of n^th order has the forms :

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_1(x)\frac{dy}{dx} + a(x)\,y = g(x)$$

- Linear

ال y ومشتقاتها الأس لها (1) وليست مضروبة ببعضها ومش موجودة داخل (sin , Ln , الجذر , المقام)

✓ $a(x)$ ⟶ x او ثوابت

- The general solution of (ODE) with n^th order is :

$$\frac{d^n y}{dx^n} = f(x, y, y', y'', \ldots, y^{n-1})$$

الشكل العام للمعدلة التفاضلية

Note :

المشتقة في البسط

$\frac{dy}{dx} \rightarrow y'$

$\frac{dx}{dt} \rightarrow x'$

$\frac{dz}{dx} \rightarrow z'$

SOLUTION

- Explicit soulution — solution is a function — Ex: $y = f(x)$
- Implicit solution — Relation between x and y — Ex: $x^2 + y^2 = 25$

CHAPTER (2)

INITIAL VALUE PROBLEM (I.V.P)

- Solve the following (I.V.P)

$$\frac{d^n y}{dx^n} = f(x , y , y' , \ldots , y^{n-1})$$

$y(x_0) = y_o$

$y(x_0) = y_o$

$y^{n-1}(x_0) = y_{n-1}$

FIRST ORDER DE

- Separable equation
- Linear equation
- Bernoulli equation
- Exact equation
- Homogeneous equation

- The DE has the form → $y' = f(x, y)$

Separable equation

Separable equation	
Standard form	Solution
$y' = f(x) \,.\, g(x)$ يجب ان يكون ضرب بينهم ✓ معادلة تفاضلية قابلة للفصل تفصل x عن y	$y' = f(x) \,.\, g(x)$ $\frac{dy}{dx} = f(x) \,.\, g(x)$ $\int \frac{1}{g(x)} \,.\, dy = \int f(x) \,.\, dx$

Linear equation

Linear equation		
Standard form	Integrating factor	solution
$y' + p(x)\, y = f(x)$	$\mu_{(x)} = e^{\int p(x).dx}$	$y = \frac{1}{\mu_{(x)}} (\int \mu_{(x)} . f(x). dx + C)$

Note : $p(x) = \frac{\mu'_{(x)}}{\mu_{(x)}}$

Bernoulli's equation

- Standard form : $y' + q(x)\,y = g(x)\,y^n$, for all (n ∈ R)

- If ($n = 0$) then the above equation becomes :

$$y' + q(x)\,y = g(x)$$ linear

- If ($n = 1$) then the above equation becomes :

$$y' + (\,q(x) - g(x)\,)\,y = 0$$ linear

- For ($n \neq 0$) and ($n \neq 1$) , The substution ($v = y^{1-n}$) transforms the Bernoulli equation in to Linear as :

$$y' + q(x)\,y = g(x)\,y^n$$

$(1-n)$

$v = y^{1-n}$

$$v' + (1-n)\,q(x)\,v = (1-n)\,g(x)$$

Exact equation

- Is exact if $\left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right]$ ⟶ ($M_y = N_x$)

Exact equation	
Standard form	Solution
M(x,y) .dx + N(x,y) .dy = 0	➢ $\int M(x,y).dx + \int$ الحدود التي لا تحتوي على N في x $.dy = C$ ➢ $\int$ الحدود التي لا تحتوي على M في y $.dy + \int M(x,y).dx = C$

- Is exact , the equation then :
  - $M(x,y) = \int N_x\,.dy + h(x)$ — يعتمد على x أو ثابت، Ex : x , 1 , Lnx
  - $N(x,y) = \int M_y\,.dx + g(y)$ — يعتمد على y أو ثابت، Ex : y , 1 , Lny

- If [$f(x,y) = C$] is a solution for exact equation :
- $ax\, M(x,y)\ .dx + by\, N(x,y).dy = 0$

$ax\, M(x,y) = \frac{\partial f}{\partial x}$, $by\, N(x,y) = \frac{\partial f}{\partial y}$

Special Integrating factor

✓ For the non-exact DE :

$$M(x,y)\ .dx + N(x,y).dy = 0 \qquad M_y \neq N_x$$

WE HAVE

1) If $\frac{M_y - N_x}{N(x,y)}$ depends on x only then the special integrating factor is :

$$\mu_{(x)} = e^{\int \frac{M_y - N_x}{N(x,y)}.dx}$$

Then the DE becomes exact if multiply the DE by $\mu_{(x)}$

2) If $\frac{N_x - M_y}{M(x,y)}$ depends on y only then the special integrating factor is :

$$\mu_{(y)} = e^{\int \frac{N_x - M_y}{M(x,y)}.dy}$$

Then the DE becomes exact if multiply the DE by $\mu_{(y)}$

- $M(x,y).dx + N(x,y).dy = 0$

Internating factor

$$\mu_{(x,y)} = x^p \cdot y^q$$

Then to find p and q we used the condition

$$M_y - N_x = p\,\frac{N(x,y)}{x} - q\,\frac{M(x,y)}{y}$$

Homogeneous equation

- A function G(x,y) is said to be homogeneous of order n if :

$$G(t_x - t_y) = t^n \, G(x,y)$$

- The homogeneous DE has the form :

$$\frac{dy}{dx} = h\left(\frac{y}{x}\right)$$

To solve homogeneous equation

we use the substution ($v = \frac{y}{x}$) then will be DE separable

✓ بالمختصر انا لازم أحاول اطلع صورة $\frac{y}{x}$ وافرضها v وبشتقها وبعوض بالمعادلة وبتطلع معي معادلة separable

---

Equation of the form

التعويض الخطي

$$\frac{dy}{dx} = f(ax + by + c)$$

$$V = ax + by + c$$

❖ دائما ال (x,y) قوتهم 1 ⟸ (x^1, y^1)

حتى نستعمل الفرض الخطي وبنحل علها عادي لو كانت

داخل : اس , Ln , ...

❖ تعويض الخطي ⟸ separable

Unique colution

- $y' + p(x)\,y = f(x)$ Where $y(x_0) = y_0$

❖ If $p(x)$, $f(x)$ are can't function on an open Interval (a,b) , $x_0 \in (a,b)$ contains x_0 , then there is a unique colution of I.V.P on that Interval

1) بنطلع مجال كل من p(x) , f(x) ⟸ نأخذ الفترة التي فيها x_0

2) بنقاطع المجالين

SECOND ORDER DE

- This DE has the form :

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = g(x)$$

$g(x) = 0$ → homogeneous , Linear

$g(x) \neq 0$ → non – homogeneous

Superposition principle (homogeneous)

- If y_1 , y_2 are solution for DE :

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$$

$$y = C_1\, y_1 + C_2\, y_2$$

Is the another solution

Linear second order DE

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = g(x)$$

- Can be written as :

$L[y](x) = g(x)$

where

$L[y](x) = a(x)\, y'' + b(x)\, y' + c(x)\, y$ ⟷ is a Linear operator

➢ Theorem :

If (y_1 , y_2 , …….. , y_n) are solution for DE :

$$a_n(x)\, y^n + a_{n-1}(x)\, y^{n-1} + \ldots.. + a_1(x)\, y' + a_0(x)\, y = 0$$

$$y = C_1\, y_1 + C_2\, y_2 + \ldots.. + C_n\, y_n$$

Is another solution

Wronskian

➢ $W\,[y_1\,, y_2\,, \ldots \ldots, y_n](x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$

Note :

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a.d - b.c$$

Linearly dependent

➢ We say that the set of function ($y_1\,, y_2\,, \ldots \ldots, y_n$) is Linearly Independent on Interval

If $W\,[y_1\,, y_2\,, \ldots \ldots, y_n](x) \neq 0$

يعني بقدر اختصر كل هاض الحكي انه اذا كان W

$\neq$ → Linearly Independent

$=$ → Linearly dependent

Fundamental solution

❖ من شروطها :

1) يجب ان تكون ال solutions ⟸ تحقق المعادلة

2) يجب ان تكون Linearly Independent أي (W $\neq$ 0)

Abel's Theorem

- If y_1 , y_2 are solution of the DE :

$$y'' + p(x)\, y' + q(x)\, y = 0$$

- $W\,[y_1\,, y_2](x) = C\, e^{-\int p(x).dx}$

Notes :

$p(x) = \frac{-W'}{W}$

ال $p(x)$ بتكون معامل $(\,n-1)$ من اعلى مشتقة

Ex: $y'' + p(x)\, y' + y = 0$

$y^{(5)} + p(x)\, y^{(4)} + y = 0$

Reduction of order

$$y'' + p(x)\, y' + q(x)\, y = 0$$

- $y_2 = y_1 \int \frac{e^{-\int p(x).dx}}{(y_1)^2}\, .dx = y_1 \int \frac{W}{(y_1)^2}\, .dx$

Linear , homogeneous , second order DE with constant coefficients

$$a(x)\,y'' + b(x)\,y' + c(x)\,y = 0$$

$$a(x)\,r^2 + b(x)\,r + c(x) = 0$$

Is called the auxiliary or chapteristic equation

$$\Delta = b^2 - 4ac$$

$\Delta > 0$

Then there are two different real roots say (r_1 , r_2)

Solution:

$y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$

$\Delta = 0$

Then there are two different real roots say (r_1 , r_2)

Solution :

$y_1 = e^{r_1 x}$, $y_2 = x e^{r_2 x}$

❖ كل ما يتكرر بنضرب بـ x

$\Delta < 0$

Then we have complex roots say ($r = \alpha \pm \beta i$)

where ($i = -\sqrt{-1}$)

Solution :

$y_1 = e^{\alpha x}\cos(\beta x)$

$y_2 = e^{\alpha x}\sin(\beta x)$

g.s : $y = C_1\, y_1 + C_2\, y_2$

Euler equation (homog , Linear , second order)

$$ax^2\, y'' + bx\, y' + c\, y = 0 \quad , \quad x > 0$$

- The substution :

$x = e^t \Longleftrightarrow \text{Ln}x = t$

$$a\,\frac{d^2y}{dt^2} + (\,b - a\,)\,\frac{dy}{dt} + c\,y = 0$$

$$y = C_1\, y_1 + C_2\, y_2$$

CHAPTER (5)

Non-homog , Linear DE of second order

Undermined coefficient

Variation of parameters

Undermined coefficient

$$a\,y'' + b\,y' + c\,y = g(x)$$

❖ Homogeneous

$$a\,y'' + b\,y' + c\,y = 0$$

$$y_h = C_1\,y_1 + C_2\,y_2$$

❖ Non-homogeneous

Polynomial	
g(x)	y_p
2	$(\,A\,)\,x^k$
$x + 2$	$(\,A\,x + B\,)\,x^k$
$x^2 + 2x$	$(\,A\,x^2 + B\,x + C\,)\,x^k$
k : عدد مرات ظهور (الصفر) في الجذور المعادلة التربيعية	

Exponential	
g(x)	y_p
$2e^{\alpha x}$	$(\,A\,e^{\alpha x}\,)\,x^m$
$x\,e^{\alpha x}$	$([\,A\,x + B\,]\,e^{\alpha x})\,x^m$
$(x^2 + x\,)\,e^{\alpha x}$	$(\,[\,A\,x^2 + B\,x + C\,]\,e^{\alpha x})\,x^m$
m : عدد مرات ظهور (α) في الجذور المعادلة التربيعية	

Trigonometric [sin (βx) or cos (βx)]	
g(x)	y_p
sin (βx)	(A cos (βx) + B sin (βx)) x^n
x sin (βx)	([A x + B] cos (βx) + [C x + D] sin (βx)) x^n
2 $e^{\alpha x}$ sin (βx)	([A cos (βx) + D sin (βx)] $e^{\alpha x}$) x^n
$x\, e^{\alpha x}$ sin (βx)	([(A x + B) cos (βx) + (C x + D) sin (βx)] $e^{\alpha x}$) x^n
n : عدد مرات ظهور ($\alpha + \beta i$) في الجذور المعادلة التربيعية للمرافقة	

- g.s: $y = y_h + y_p$

Variation of parameters

$$a\, y'' + b\, y' + c\, y = f(x) \quad , \quad g(x) = \frac{f(x)}{a}$$

➢ Find homogenous solutions { y_1 , y_2 }

$$y_h = C_1\, y_1 + C_2\, y_2$$

1) W $[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

2) $V_1 = -\int \frac{y_2 \cdot g(x)}{W} .dx$

3) $V_2 = \int \frac{y_1 \cdot g(x)}{W} .dx$

4) The particular solution is :

$$y_p = V_1\, y_1 + V_2\, y_2$$

5) g.s : $y = y_h + y_p$

CHAPTER (6)

Laplace Transform

- Let $f(t)$ be a function on [0 , ∞] then Laplace Transform of $f(t)$ written as $\mathcal{L}\{ f(t) \}$ or F(s) is defined by :

❖ $\mathcal{L}\{ f(t) \} = \int_0^{\infty} f(t).e^{-st}.dt = \lim_{h\to\infty} \int_0^{h} f(t).e^{-st}.dt = F(s)$

f(t)	$\mathcal{L}\{ f(t) \} = F(s)$
a	$\frac{a}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
sin (at)	$\frac{a}{s^2+a^2}$
cos (at)	$\frac{s}{s^2+a^2}$
sinh (at)	$\frac{a}{s^2-a^2}$
cosh (at)	$\frac{s}{s^2-a^2}$

$F(s) = \mathcal{L}\{ f(t) \}$	$\mathcal{L}^{-1}\{ F(s) \} = f(t)$
$\frac{a}{s}$	a
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2+a^2}$	$\frac{\sin(at)}{a}$
$\frac{s}{s^2+a^2}$	cos (at)
$\frac{1}{s^2-a^2}$	$\frac{\sinh(at)}{a}$
$\frac{1}{s^2-a^2}$	cosh (at)

✓ $\mathcal{L}\{ \alpha f(t) \pm \beta g(t) \} = \alpha \mathcal{L}\{ f(t) \} \pm \beta \mathcal{L}\{ g(t) \}$

✓ $\mathcal{L}\{ e^{at}. f(t) \} = \mathcal{L}\{ f(t) \}\Big|_{s=s-a} = F(s-a)$

✓ $\mathcal{L}\{ t^n. f(t) \} = (-1)^n. \frac{d^n}{d^n s}(\mathcal{L}\{ f(t) \}) \rightarrow F^{(n)}(s)$

✓ $\mathcal{L}^{-1}\{ f(s-a) \} = e^{at} \mathcal{L}^{-1}\{ F(s) \}$

✓ $\mathcal{L}^{-1}\{ F(s) \} = \frac{-1}{t} \mathcal{L}^{-1}\{ F'(s) \}$

✓ $\mathcal{L}\{ y^{(n)}(t) \} = s^n \mathcal{L}\{ y \} - s^{n-1}.y(0) - \ldots\ldots - y^{(n-1)}(0)$

Ex:

1) $\mathcal{L}\{ y'(t) \} = s \mathcal{L}\{ y \} - y(0)$ $n = 1$

2) $\mathcal{L}\{ y''(t) \} = s^2 \mathcal{L}\{ y \} - s.y(0) - \ldots\ldots - y'(0)$ $n = 2$

Unit-step function

Note :

$F(s) \xrightarrow[\text{Laplace}]{\text{لحتى تكون}} \lim_{s\to\infty} F(s) = 0$

➢ $u(t-a) = \begin{cases} 0\,, & t < a \\ 1\,, & t \geq a \end{cases}$

✓ $\mathscr{L}\{\, u(t-a)\,\} = \dfrac{e^{-as}}{s}$

✓ $\mathscr{L}\{\, u(t-a)\,.\, f(t)\} = e^{-as}\,.\,\mathscr{L}\{\, f(t+a)\,\}$

✓ $\mathscr{L}^{-1}\{e^{-as}\,.\,F(s)\,\} = u(t-a)\,.\,\mathscr{L}^{-1}\{\,F(s)\,\}$

حالة الاقتران المتشعب

$h(t) = \begin{cases} f(t) & , \quad 0 < t < a \\ g(t) & , \quad a \leq t < b \\ p(t) & , \quad t \geq b \end{cases}$

$h(t) = f(t) + [\,(g(t) - f(t))\,u(t-a)\,] + [\,(p(t) - g(t))\,u(t-b)\,]$

Theorem : (superposition principle) (Non-homog)

➢ Let y_1 be a solution to the DE :

$$a\,y'' + b\,y' + c\,y = f_1(x)$$

➢ Let y_2 be a solution to the DE :

$$a\,y'' + b\,y' + c\,y = f_2(x)$$

متساوي للمعادلتين

لازم يكون الحد الي على يسار المساواة

Then for any constant C_1 and C_2 the function

$$y = C_1\,y_1 + C_2\,y_2$$

❖ Is a solution to DE :

$$a\,y'' + b\,y' + c\,y = C_1\,f_1(x) + C_2\,f_2(x)$$