معادلات تفاضلية عادية 1

تقي الدين الشطناوي

للطالـب المبدع

عبد الرحمان فتيحة

إرادة - ثقـة - تغيير

Ch1: Introduction

Ex of D.E

1) $y' + 2xy = \sin x$
2) $y'' + 2xy' = \sin x - 3y$

note: IF Indep. vairble more than 1 use partial derivation $u(x,t)$

depended indep

* $\frac{\partial u}{\partial x} = u_x$
* $\frac{\partial^2 u}{\partial x^2} = u_{xx}$

*Classification of D.E

1 types

Ordinary D.E	Partial D.E
one indepen. vairble	At least two indep. vairble Ex: $u_{xx} + u_{yy} = 0$

2 order of D.E

(IS the highest derivatives)

Ex: $(\sin x)\, y = x^2 y''' - y^2 \longrightarrow$ order $= 3$

Ex₂: $u_{xx} + u_{yy} = \sin t \longrightarrow$ dep. vairble $= u$

independed. vairble $= x, y, t$

order $= 2$ $\quad u(x,y,t)$

3 linear of D.E

$$a_n(x)\, y^{(n)} + a_{n+1}(x)\, y^{(n+1)} + \cdots\cdots + a_1(x)\, y' + a_2 y = F(x)$$

Ex: 1) $(y')^2 + 5y = 3\tan x \longrightarrow$ Non linear

2) $y' - 5\cos x = 0 \longrightarrow$ linear

3) $x \dfrac{d^4y}{dx^4} - (y')^4 = 0 \longrightarrow$ Non linear

Ex: classify the following D.E interms of :-

1) type 2) order 3) linear

1) $x x' - t x''' = t^3$

t is the only indep. varible

order = 3

non linear

2) $y'''' = axy - e^y$

ODE

4

non linear

4 Solution D.E

Types of sol

1) Explicit : $y = P(x)$

2) Implicit : $y - \ln y = x$

Ex: $y = 2x + e^x$

$y' = 2 + e^x$

$y'' = e^x$

Ex: show that $g(x) = x^2 - x^{-1}$ is sol for $y'' - \frac{2}{x^2} y = 0, \; x \neq 0$

$g'(x) = 2x + x^{-2}, \quad g''(x) = 2 - 2x^{-3}$

$\rightarrow 2 - 2x^{-3} - \frac{2}{x^2}(x^2 - x^{-1})$

$2 - 2x^{-3} - 2 + 2x^{-3} = 0$

Ex: show that $y - \ln y = x^2 + 1$ is sol for $(y-1)y' = 2xy$

by implicit der: $y' - \frac{1}{y}y' = 2x$

$y'(1 - \frac{1}{y} = 2x)\, xy$

$y'(y-1) = 2xy$ #

Ex: If $y = e^{-3x}$ is a sol for $y'' + xy' + f(x)y = 0$

find $p(x)$?

$y' = -3e^{-3x}, \quad y'' = 9e^{-3x}$

in D.E $\longrightarrow 9e^{-3x} + x(-3e^{-3x}) + p(x) \cdot e^{-3x} \stackrel{!}{=} 0$

$e^{-3x}(9 - 3x + p(x)) = 0$

$0 \neq$

$\therefore 9 - 3x + p(x) = 0 \longrightarrow 3x - 9 = p(x)$ #

Ex: If $y = x^n$ is sol for D.E $x^2y'' + 6xy' + 4y = 0, \; x \neq 0$

find n?

$y' = nx^{n-1}, \quad y'' = n(n-1)x^{n-2}$

in D.E $\rightarrow x^2(n(n-1)x^{n-2}) + 6x(nx^{n-1}) + 4x^n = 0$

$n(n-1)x^n + 6nx^n + 4x^n = 0$

$x^n(n(n-1) + 6n + 4) = 0$

$0 \neq \; \therefore n(n-1) + 6n + 4 = 0$

$n^2 - n + 6n + 4 = 0$

$n^2 + 5n + 4 = 0$

$(n+1)(n+4) = 0$

$n = -1 \longrightarrow y = x^{-1}$

$n = -4 \longrightarrow y = x^{-4}$

Initial Value Problem = D.E + inital condition (IC

Def: let D.E of order, n $y^{(n)} = F(x, y, y', y'', \dots, y^{(n)})$

with

$y(x_0) = y_0$ عدد مص , $y'(x_0) = y_1$ عدد مص , ..., $y^{(n-1)}(x_0) = y_{n-1}$

where x_0 and y_0 are Const.

Ex: 1) $y' + 2xy = e^x$, $y(0) = 1$

is IVP

D.E order = 1

$x_0 = 0$

$y_0 = 1$

Note: The number of ICs is same order of D.E and (x_0) is fixed by all ICs.

ثابت

Ex:- 2) $x^2 y'' - xy - 2y = 0$, $y(\underline{0}) = 1$, $y(\underline{2}) = 3$

D.E order = 2

Not IVP

لأنه غيّر ال (x_0)

وما حط y'

* Use IC to find constant in general sol.

Ex: If $y = C_1 e^{-x} + C_2 e^{2x}$ is sol for IVP

$y'' - y' - 2y = 0$, $y(0) = 2$, $y'(0) = -3$

Find C_1 and C_2?

* $y'(x) = -C_1 e^{-x} + 2C_2 e^{2x}$

use IC $y(0) = 2$

$y(0) = 2 = C_1 + C_2$ —— ①

use IC $y'(0) = -3$

$y'(0) = -3 = -C_1 + 2C_2$ —— ②

① + ② $\longrightarrow C_2 = \frac{-1}{3} \longrightarrow C_1 = \frac{7}{3}$

حل واحد ووحيد

Exist and Unique

Consider, IVP $y' = F(X,y)$, $y(X_0) = y_0$

IF $f(x,y)$ and $\frac{\partial f}{\partial y}$ are cont. fn on $a<x<b$ and $c<y<d$

Contains (x_0, y_0) .

Thm: IVP has a Unique sol. $y(x)$ in some interval Contains X_0

standard form $y' = f(x,y)$

Ex: $y' + 2xy = \sin X \longrightarrow y' = \sin X - 2Xy$ (standard form)

where $\frac{\partial f}{\partial y}$:- partial derivative off (x,y) with respect to y .

Ex: $f(x,y) = x^2 + y^3 + e^x \sin y$

$$\frac{\partial f}{\partial y} = 0 + 3y^2 + e^x \cos y \longleftarrow$$ (ثابت (X) صون)

Ex:- Does IVP $3y' - xy^3 = x^2$, $y(1) = 3$ has Unique sol ?

$$y' = \frac{1}{3}x^2 + \frac{1}{3}xy^3$$

For all

$f(x,y) = \frac{1}{3}x^2 + \frac{1}{3}xy^3$ is cont $\forall (x,y) \in \mathbb{R}$

$\frac{\partial f}{\partial y} = 0 + \frac{1}{3}x(3y^2) = xy^2$ Cont $\forall \mathbb{R}$

$\rightarrow$ then has Unique sol #

فترة الاتصال التي تحتوي X_0

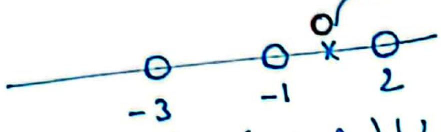
Ex: find the largest interval where IVP

$$y' = \frac{y}{(X+1)(X-2)(X+3)} \quad , \quad y(0) = 3$$ has Unique sol ?

sol $f(x,y) = \frac{1}{(X+1)(X-2)(X+3)} \, y$

Cont at $\mathbb{R} - \{X = -1, 2, -3\}$

The interval cont for $f(x,y)$ $X_0 = 0$



$(-\infty, -3) \cup (-3, -1) \cup (-1, 2) \cup (2, \infty)$

$$\frac{\partial f}{\partial y} = \frac{1}{(X+1)(X-2)(X+3)}$$ cont on $\mathbb{R} - \{X = -1, 2, -3\}$

as $X_0 = 0$ in $(-1, 2)$

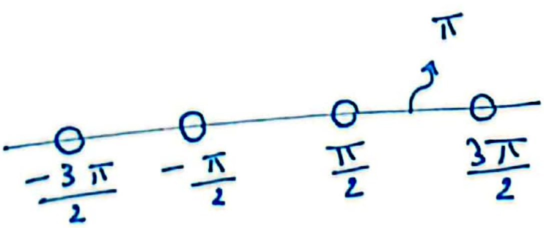
$\therefore$ largest interval for Unique sol is $(-1, 2)$ ※

Ex: $y' + (\tan x)y = \sin X$, $y(\pi) = 0$?

sol $y' = \sin X - (\tan x)y$

$f(x,y)$ is Cont on $\mathbb{R} - \{X = \pm(2n+1)\frac{\pi}{2}, \; n = 1, 2, 3, \dots\}$

$$\tan X = \frac{\sin X}{\cos X}$$



$$\frac{\partial f}{\partial y} = -\tan X$$ is Cont for all $\mathbb{R} - \{X = \pm(2n+1)\frac{\pi}{2}\}$

$\therefore$ The largest interval is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ ※

1 Separable Equations

$f(x)\,dx = g(x)\,dy$

$y' = \frac{dy}{dx} \Rightarrow$ Then D.E is Called Sep. eqn.

Ex: $y' = \frac{x+y}{3}$

$\frac{dy}{dx} = \frac{1}{3}x + \frac{1}{3}y$ ∴ not Sep. eqn

Method :- Integrate both sides

$$\int f(x)\,dx + c = \int g(x)\,dy$$

Ex: Solve or find the general sol ?

① $y' = \frac{x^2+1}{y^2-1}$

sep. D.E

$\frac{dy}{dx} = \frac{x^2+1}{y^2-1} \longrightarrow (y^2-1)\,dy = (x^2+1)\,dx$

$\int y^2-1\,dy = \int x^2+1\,dx$

$\frac{y^3}{3} - y = \frac{x^3}{3} + x + C$ #

② $y' = y^2-4$

$\frac{dy}{dx} = y^2-4$

$\frac{1}{y^2-4}\,dy = dx$ sep

$\int \frac{1}{y^2-4}\,dy = \int dx$

* using the partial fraction

$\frac{1}{y^2-4} = \frac{A}{y-2} + \frac{B}{y+2}$

$1 = A(y+2) + B(y-2)$

$y = 2 \longrightarrow 1 = 4A \longrightarrow A = \frac{1}{4}$

$y = -2 \longrightarrow 1 = -4B \longrightarrow B = -\frac{1}{4}$

$$\int \frac{\frac{1}{4}}{y-2} + \frac{-\frac{1}{4}}{y+2}\,dy = \int dx$$

$$\frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = x + C$$

$$* \quad \frac{y-2}{y+2} = e^{4x+4C}$$

$\boxed{2}$ linear Equations

$y' + p(x)y = Q(x) \longrightarrow$ linear in y "dep. var $= y$"

$x' + p(y)x = Q(y) \longrightarrow$ linear in x

(1) مركز

Ex₁ :- $y' + \sin y \; y = e^x \longrightarrow$ not linear in y

Ex₂ : $y' + 2xy = \sin x \longrightarrow$ linear in y with $p(x) = 2x$, $Q(x) = \sin x$

IVP : $y' + p(x)y = Q(x)$, $y(x_0) = y_0$

$$y' = Q(x) - p(x)y \rightsquigarrow f(x,y)$$

$$\frac{\partial f}{\partial y} = -p(x)$$

∴ IVP has Unique sol If $Q(x)$ and $p(x)$ are cont ✱

Method :- 1) find integrating factor

$$\mu_0(x) = e^{\int p(x)dx} \longrightarrow \text{y معامل}$$

$$\mu_0'(x) = e^{\int p(x)dx} \cdot p(x)$$

$$\mu' = \mu \cdot p(x) \rightarrow p(x) = \frac{\mu'(x)}{\mu(x)}$$

2) Multiply $\mu(x)$ by D.E

$$y' \cdot \mu(x) + \underbrace{\mu(x)\,p(x)}_{\mu'(x)}y = Q(x)\,\mu(x)$$

$$\underbrace{y' \cdot \mu(x) + \mu'(x)y}_{(y \cdot \mu(x))'} = Q(x)\,\mu(x)$$

$$(y \cdot \mu(x))' = Q(x)\,\mu(x)$$

3 solve for Soln $y(x)$

$$\int (y \cdot \mu(x))'\,dx = \int Q(x)\,\mu(x)\,dx$$

$$y \cdot \mu(x) = \int Q(x)\,\mu(x)\,dx + C$$

$$y = \frac{1}{\mu(x)}\left[\int Q(x)\,\mu(x)\,dx + C\right]$$ # (القانون)

Ex:- IF $\mu(x) = 3x^2$ is an I.F for

$2y' + x\,f(x)\,y = 3x^2 - 4$, find $f(x)$??

sol

$$y' + \underbrace{\frac{x\,f(x)}{2}}_{P(x)}y = \underbrace{\frac{3}{2}x^2 - 2}_{Q(x)}$$

$$P(x) = \frac{\mu'(x)}{\mu(x)}$$

Ex$_2$:- Solve $xy' + y = \sin x$

$$y' + \frac{1}{x} y = \frac{\sin x}{x} \quad , \quad x \neq 0$$

$$x\frac{dy}{dx} = \sin x - y$$

$$\frac{x}{dx} = \frac{\sin x - y}{dy}$$

linear in y , $P(x) = \frac{1}{x}$, $Q(x) = \frac{\sin x}{x}$

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$y = \frac{1}{\mu}\left[\int \mu . Q \, dx + c\right]$$

$$= \frac{1}{x}\left[\int x \frac{\sin x}{x} dx + c\right]$$

$$y = \frac{1}{x}\left[-\cos x + c\right]$$

Ex$_3$:- solve $ty' - 2y = 5t^2$, $y(1) = 2$, $t \neq 0$, $t > 0$

$$y' - \frac{2}{t} y = 5t$$

$$\mu(t) = e^{\int \frac{-2}{t} dt} = e^{-2\ln t} = t^{-2}$$

$$y = t^2\left[\int t^{\cancel{-2}} 5\cancel{t} \, dt + c\right]$$

$$= t^2\left[\int \frac{5}{t} dt + c\right]$$

$$y(t) = t^2\left[5\ln t + c\right]$$

Use the $y(1) = 2$ to find Constant C

$$2 = 1[5\cancel{\ln 1} + c] \Rightarrow C = 2$$

The general Solu

$$\boxed{y(t) = 5t^2 \ln t + 2t^2}$$

2.6 :- Bernoulli Equation

Standard form : $y' + P(x)y = Q(x)\, y^n$, $n \neq 0$, $n \neq 1$

$n \neq 0 \downarrow$ linear eqn

$n \neq 1 \downarrow$ *

* $n=1$

$$y' + P(x)y = Q(x)\, y$$

$$\frac{dy}{dx} + Py = Q(x)y$$

$$\frac{dy}{dx} = Q(x)y - P(x)y$$

$$\frac{dy}{dx} = y(Q(x) - P(x))$$

$$\frac{1}{y}dy = dx\,(Q(x) - P(x))$$

Method :-

① use $V = y^{1-n}$

* Transform Bernoulli eqn to linear in V

② $V' + (1-n)P(x)V = (1-n)Q(x)$ (linear in V)

③ find soln $V(x)$ by I.F

④ use $V = y^{1-n}$ to find solv interm of y

Ex, :- solve $y' - 5y = \frac{-5}{2}x y^3$,

Bernoulli eqn : $P(x) = -5$, $Q(x) = \frac{-5}{2}x$, $n = 3$

① $v = y^{1-n} \Rightarrow v = y^{-2}$

② linear in V : $V' + (1-n)P(x)V = (1-n)Q(x)$

$$v' + (-2)(-5)v = (-2)\frac{-5}{2}x$$

$$v' + 10v = 5x$$

③ $\mu(x) = e^{\int 10\,dx} = e^{10x}$

$v(x) = \frac{1}{e^{10x}}\left[\int e^{10x} \cdot 5x\,dx + c\right] \longrightarrow$

v'		$\frac{dv}{dx}$
$5x$	$+$	e^{10x}
5	$-$	$\frac{e^{10x}}{10}$
0		$\frac{e^{10x}}{100}$

$= e^{-10x}\left[\frac{1}{2} x e^{10x} - \frac{1}{20} e^{10x} + C\right]$

$v(x) = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}$

④ soln for Bernoulli eqn use $v = y^{-2}$

$y^{-2} = \frac{1}{2}x - \frac{1}{20} + Ce^{-10x}$

Ex$_2$:- $(x-2)y' - 5(x-2)^2 y^{\frac{1}{2}} = y$, $y(3) = 4$, $x \neq 2$

$y' - 5(x-2)\,y^{\frac{1}{2}} = \frac{y}{(x-2)}$

$y' - \frac{y}{(x-2)} = 5(x-2)\,y^{\frac{1}{2}}$

Bernoulli in y : with $p(x) = \frac{1}{(x-2)}$, $Q(x) = 5(x-2)$, $n = \frac{1}{2}$

① $v = y^{1-n} = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$

② $v' + \frac{1}{2} - \frac{1}{x-2}v = \frac{1}{2}5\,(x-2)$

$v' - \frac{1}{2}\frac{1}{x-2}v = \frac{5}{2}(x-2)$

③ $\mu(x) = e^{-\frac{1}{2}\int \frac{1}{x-2}dx} = \cancel{e}^{-\frac{1}{2}\cancel{\ln}(x-2)} = (x-2)^{-\frac{1}{2}}$

$v(x) = (x-2)^{-\frac{1}{2}}\left[\int (x-2)^{-\frac{1}{2}} \cdot \frac{5}{2}(x-2)\,dx + c\right]$

$= (x-2)^{\frac{1}{2}}\left[\frac{5}{2}\int (x-2)^{\frac{1}{2}}\,dx + c\right]$

$= (x-2)^{\frac{1}{2}}\left[\frac{5}{\cancel{2}} * \frac{\cancel{2}}{3}(x-2)^{\frac{3}{2}} + c\right]$

$v(x) = \frac{5}{3}(x-2)^{2} + C(x-2)^{\frac{1}{2}}$

④ $y^{\frac{1}{2}} = \frac{5}{3}(x-2)^2 + c(x-2)^{\frac{1}{2}}$

⑤ $y(3) = 4$, $x = 3$

$2 = \frac{5}{3} + C \Rightarrow C = 2 - \frac{5}{3} = \frac{1}{3}$

the general form

$y^{\frac{1}{2}} = \frac{5}{3}(x-2)^2 + \frac{1}{3}(x-2)^{\frac{1}{2}}$

2.4 : Exact Equation

① $M(x,y) + N(x,y)y' = 0$

Replace : $\frac{dy}{dx}$ and mutiply By dx

② $M(x,y)\,dx + N(x,y)\,dy = 0$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then D.E is Exact

Ex : let $f(x,y) = \ln x^2+y^3 + x^3 + 3y^2x$

find $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial y} = \frac{3y^2}{x^2+y^3} + 0 + 6xy$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^3} + 3x^2 + 3y^2$$

Ex : which of the following D.E is exact

① $(\cos x \cos y + 2x)\,dx - (\sin x \sin y + 2y)\,dy = 0$

$$\frac{\partial M}{\partial y} = -\cos x \sin y \stackrel{?}{=} \frac{\partial N}{\partial x} = -(\sin y \cos x)$$

As $M_y = N_x \Rightarrow$ D.E is exact

② $(2xy + \pi) = (x^2-1)y'$

$2xy + \pi - (x^2-1)y' = 0$

$M_y = 2x \neq N_x = -2x \Rightarrow$ D.E is not exact

Ex₁: Find $N(x,y)$ such that the following D.E is exact

1) $\left(x^{-\frac{1}{2}} y^{\frac{1}{2}} + \frac{x}{x^2+y}\right) dx + N(x,y)\, dy = 0$

As D.E is exact $M_y = N_x$

$$M_y = x^{-\frac{1}{2}} \frac{1}{2} y^{-\frac{1}{2}} + \frac{-x}{(x^2+y)^2}$$

$$= \frac{1}{2} x^{-\frac{1}{2}} y^{-\frac{1}{2}} - \frac{x}{(x^2+y)^2} = N_x$$

$$N(x,y) = \int N_x\, dx$$

$$= \int \frac{1}{2} x^{-\frac{1}{2}} y^{-\frac{1}{2}} - \frac{x}{(x^2+y)^2}$$

$$= \frac{1}{2} y^{-\frac{1}{2}} \int x^{-\frac{1}{2}} dx - \int \frac{x}{(x+y)^2}\, dx$$

$$= \frac{1}{2} y^{-\frac{1}{2}} \cdot 2 \cdot x^{\frac{1}{2}} + \frac{1}{2}(x^2+y)^{-1} + g(y)$$

$$= y^{-\frac{1}{2}} x^{\frac{1}{2}} + \frac{1}{2}(x^2+y)^{-1} + g(y)$$

$$\int \frac{x}{(x+y)^2}\, dx$$

let $u = x^2+y$

$du = 2x\, dx$

$dx = \frac{du}{2x}$

$$\int \frac{-\cancel{x}}{u^2} * \frac{du}{2\cancel{x}} =$$

$$-\frac{1}{2} u^{-1} =$$

$$-\frac{1}{2}(x^2+y)^{-1}$$

② Find $M(x,y)$ such that the D.E

$M(x,y)\, dx + \left(x e^{xy} + \frac{1}{x}\right) dy = 0$ is exact (h.w)

Ex₂: Find a and b such that the D.E

$(ayx^2 - by^2)\, dx - (x^3 + 2xy)\, dy = 0$ is exact

As D.E is exact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$ax^2 - 2by = -(3x^2 + 2y)$$

$ax^2 = -3x^2$

$a = -3$

$-2by = -2y$

$b = 1$

* Method to Solve Exact eqn $M(x,y)\,dx + N(x,y)\,dy = 0$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then D.E is exact

1. let the soln is $F(x,y) = C$
2. to find $F(x,y)$ we use

$$\frac{\partial F}{\partial x} = M \quad \text{or} \quad \frac{\partial F}{\partial y} = N$$

3. $F(x,y) = \int M(x,y)\,dx + g(y)$
4. to Find $g(y)$ use $\frac{\partial F}{\partial y} = N$
5. Find $g'(y) = \ldots\ldots$
6. Then Integrate to Find $g(y)$

Ex1: Solve $(\cos x \sin x - xy^2)\,dx + y(1-x^2)\,dy = 0$

$M_y = -2xy$, $N_x = y(0-2x) = -2xy$

As $M_y = N_x \Rightarrow$ D.E is exact

1. let soln $F(x,y) = C$
2. $\frac{\partial F}{\partial x} = M = \cos x \sin x - xy^2$
3. $F(x,y) = \int \cos x \sin x - xy^2\,dx$

$F(x,y) = \int \frac{1}{2}\sin 2x - xy^2\,dx$

$= -\frac{1}{2}\frac{\cos 2x}{2} - \frac{x^2}{2}y^2 + g(y)$

4. $\frac{\partial F}{\partial y} = N$

$0 - \frac{1}{2}x^2(2y) + g'(y) = y - x^2y$

$-\frac{1}{\cancel{2}} * \cancel{2} * \cancel{x^2y} + g'(y) = y - \cancel{x^2y} \Rightarrow g'(y) = y \Rightarrow g(y) = \frac{y^2}{2}$

general soln $F(x,y) = C$

$$-\frac{1}{4}\cos(2x) - \frac{1}{2}x^2y^2 + \frac{y^2}{2} = C$$

* Method 2 (لنفس السؤال)

$$F(x,y) = C$$

$$\frac{\partial F}{\partial y} = N = y - yx^2$$

$$F(x,y) = \int (y - yx^2)\,dy = \frac{y^2}{2} - \frac{y^2}{2}x^2 + h(x)$$

$$\frac{\partial F}{\partial x} = M$$

$$\cancel{0 - \frac{y^2}{2}2x} + h'(x) = \cos x \sin x - \cancel{xy^2}$$

$$h'(x) = \cos x \sin x$$

$$= \frac{1}{2}\sin 2x$$

$$h(x) = \frac{-1}{4}\cos 2x$$

$$\frac{y^2}{2} - \frac{y^2}{2}x^2 - \frac{1}{4}\cos 2x = C$$

Ex$_2$: solve $(1 + e^x y + xe^x y)\,dx + (xe^x + 2)\,dy = 0$

$$y' + P(x)y = Q(x)$$

$$(xe^x + 2)y' + 1 + y(e^x + xe^x) = 0$$

$$(xe^x + 2)y' + (e^x + xe^x)y = 1$$

$$y' + \frac{e^x + xe^x}{xe^x + 2}y = \frac{-1}{xe^x + 2}$$

$M_y = e^x + xe^x$

$N_x = xe^x + e^x$

As $M_y = N_x$ D.E is exact

soln is $F(x,y) = C$

$F(x,y) = \int (xe^x + 2)\,dy$

$F(x,y) = (xe^x + 2) + h(x)$

use $\frac{\partial f}{\partial x} = M$

$(xe^x + e^x)y + h'(x) = 1 + e^x y + xe^x y$

$\cancel{xe^x y} + \cancel{e^x y} + h'(x) = 1 + \cancel{e^x y} + \cancel{xe^x y}$

$h'(x) = 1 \Rightarrow h(x) = x$

$(xe^x + 2)y + x = C$

2.5 :- Special Integrating factor

Defn : let D.E $M(x,y)\,dx + N(x,y)\,dy = 0$
where $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ not exact

If $\mu(x,y)$: is fn such that

$$\Rightarrow \underbrace{\mu(x,y)\,M}_{M^*}\,dx + \underbrace{\mu(x,y)\,N}_{N^*}\,dy = 0$$

and satisfy condition

$$\frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

Then, $\mu(x,y)$ is called special I.F

$\mu(x,y)$: not exact $\xrightarrow[\text{Multiply}]{}$ exact

* Ex : If $(12 + 5xy)\,dx + (6xy^{-1} + 3x^2)\,dy = 0$
has I.F of form $\mu(x,y) = x^n y^2$ Find n

$$5x = \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} = 6y^{-1} + 6x$$

* Multiply D.E by $\mu = x^n y^2$

$$(12\,x^n y^2 + 5x^{n+1} y^3)\,dx + (6x^{n+1} y + 3x^{n+2} y^2)\,dy = 0$$

$$\Rightarrow \frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

$$24\,x^n y + 15\,x^{n+1} y^2 = 6(n+1)\,x^n y + 3(n+2)\,x^{n+1} y^2$$

$$24 = 6n + 6 \Rightarrow n = 3$$

Method to find Special I.F

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$M_y \neq N_x$ "not exact"

Case 1 : If $\dfrac{M_y - N_x}{N}$ is continuous and depends only on x,

then : $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} = \ldots$

Case 2 : If $\dfrac{N_x - M_y}{M}$ is fn depends only on y,

then : $\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$

Ex : If $\mu(y) = y$ is I.F that transform D.E

$M\,dx + N\,dy = 0$ to exact find $N_x - M_y$?

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

$$y = e^{\int \frac{N_x - M_y}{M} dy} \xrightarrow{\ln} \ln y = \int \frac{N_x + M_y}{M} dy \xrightarrow{'}$$

$$\frac{1}{y} = \frac{N_x - M_y}{M} \longrightarrow N_x - M_y = \frac{M}{y}$$

Ex : Find I.F that transform D.E into exact

1) $(2x^2y + xe^x)dx + (x^3 + xy^3)\,dy = 0$

$M_y = 2x^2$, $N_x = 3x^2 + y^3$

$$\frac{M_y - N_x}{N} = \frac{2x^2 - 3x^2 - y^3}{x^3 + xy^3} = \frac{-x^2 - y^3}{x^3 + xy^3}$$

$$\frac{-\cancel{(x^2+y^3)}}{x\cancel{(x^2+y^3)}} = \frac{-1}{x} \rightarrow \text{fn depends only on } x$$

$$\text{I.F } \mu(x) = e^{\int \frac{-1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

2) $(y^2+3xy^3)dx + (1-xy)dy = 0$

$$M_y = 2y + 9xy^2 \quad , \quad N_x = -y$$

(A) $$\frac{M_y - N_x}{N} = \frac{2y+9xy^2+y}{1-xy} = \frac{3y+9xy^2}{1-xy} = \frac{3y(1+3xy)}{1-xy}$$

depends on both x and y
(فشلت)

(B) $$\frac{N_x - M_y}{M} = \frac{-y-2y-9xy^2}{y^2+3xy^3} = \frac{-3y-9xy^2}{y^2(1+3xy)}$$

$$= \frac{-3\cancel{y}}{y^{\cancel{2}}}\left(\frac{\cancel{1+3xy}}{\cancel{(1+3xy)}}\right) = \frac{-3}{y}$$

$$\text{I.F } \mu(y) = e^{\int \frac{-3}{y}dy} = e^{-3\ln y} = y^{-3} = \frac{1}{y^3}$$

Ex : Solve $\quad y\,dx + (2x - ye^y)dy = 0$

$$M_y = 1 \quad \neq \quad N_x = 2 \qquad \text{"not exact"}$$

(A) $$\frac{M_y - N_x}{N} = \frac{1-2}{2x-ye^y} = \frac{-1}{2x-ye^y}$$ (فشلت)

(B) $$\frac{N_x - M_y}{M} = \frac{2-1}{y} = \frac{1}{y}$$

$$\text{I.f } \mu(y) = e^{\int \frac{1}{y}dy} = e^{\ln y} = y$$

Multiply $\frac{\text{D.E}}{\text{not exact}}$ by I.F $M(y) = y$

$$\underbrace{y^2}_{M^*} dx + \underbrace{(2xy - y^2e^y)}_{N^*} dy = 0$$

$$M_y = 2y = N_x = 2y$$

1) soln $F(x,y) = C$

2) $F(x,y) = \int y^2 dx = y^2 x + g(y)$

3) $\frac{\partial F}{\partial y} = 2xy - y^2 e^y$

$$\cancel{2xy} + g'(y) = \cancel{2xy} - y^2 e^y$$

$$g'(y) = -ye^y$$

u	dv
y^2	e^y
$2y$	e^y
2	e^y
0	e^y

$$g(y) = -y^2 e^y + 2ye^y - 2e^y$$

general Soln

$$y^2 x - e^y(y^2 - 2y + 2) = C$$

Q3 If $\frac{M(x,y) - N(x,y)}{N(x,y)} = x$ Find the general solu

to D.E $\frac{M}{x} dx - \frac{N}{y} dy = 0$, $x > y > 0$

$$\frac{M(x,y)}{N(x,y)} - \cancel{\frac{N(x,y)}{N(x,y)}} = x$$

$$\frac{M(x-y)}{N(x-y)} = x + 1$$

$$\left(\frac{M}{x} dx - \frac{N}{y} dy = 0\right) \frac{1}{N} \Rightarrow \frac{M}{N} * \frac{1}{x} dx - \frac{1}{y} dy = 0$$

$$\frac{(x+1)}{x} dx = \frac{1}{y} dy \quad (\text{sep}) \quad \vdots$$

2.6 : Homogeneous Equation

If any first ODE can be written

$y' = F\left(\frac{y}{x}\right)$ or $y' = F\left(\frac{x}{y}\right)$

Ex: IS the follwing D.E is Homog.

① $y' = \dfrac{y - 4x}{x - y} = \dfrac{\cancel{y}\left(1 - 4\frac{x}{y}\right)}{\cancel{y}\left(\frac{x}{y} - 1\right)}$ is Homog

$$= \frac{1 - 4\frac{x}{y}}{\frac{x}{y} - 1} = F\left(\frac{x}{y}\right)$$

② $y' = \dfrac{x}{y} + \ln x$ $\times$ not Homog *

~~③ $y = x$~~

③ $(x - y)\cancel{\dfrac{dx}{dx}} + x\dfrac{dy}{dx} = 0$

$$y' = \frac{(y - x)}{x}$$

$$y' = \frac{y}{x} - 1 = F\left(\frac{y}{x}\right) \quad \text{Homog}$$

④ $y' = \dfrac{y}{x} e^{\frac{x}{y}} = \left(\dfrac{x}{y}\right)^{-1} e^{\frac{x}{y}} = F\left(\dfrac{x}{y}\right)$

* Method to solve Homog

To solve Homog use $u = \frac{y}{x}$ if form $y' = F(\frac{y}{x})$

If Form $y' = F(\frac{x}{y})$ use $u = \frac{x}{y}$

This u transform Homog $\Rightarrow$ sep .

$y' = F(\frac{y}{x}) \longrightarrow$ use $u = \frac{y}{x}$

$$\left.\begin{aligned} x u' + u &= F(u) \\ x u' &= F(u) - u \end{aligned}\right\} \begin{aligned} y &= x \cdot u \\ y' &= x u' + u \cdot 1 \end{aligned}$$

$$x \frac{du}{dx} = F(u) - u$$

$$\frac{x}{dx} = \frac{F(u) - u}{du}$$

$$\frac{A}{B} = \frac{C}{D} \Longleftrightarrow \frac{B}{A} = \frac{D}{C}$$

$$\frac{1}{x} dx = \frac{1}{F(u) - u} du$$

soln $u = \ldots$

To find soln y : use $u = \frac{y}{x}$

Ex : solve $2xyy' = 4x^2 + 3y^2$.

$$y' = \frac{2x}{y} + \frac{3}{2}\left(\frac{y}{x}\right)$$

$$y' = 2\left(\frac{y}{x}\right)^{-1} + \frac{3}{2}\left(\frac{y}{x}\right)$$

$$u = \frac{y}{x} \,,\; y' = xu' + u$$

$$xu' + u = 2(u)^{-1} + \frac{3}{2}(u)$$

$$\frac{x\,du}{dx} + u = 2(u)^{-1} + \frac{3}{2}u$$

$$\frac{x}{dx} = \frac{2(u)^{-1} + \frac{3}{2}u - u}{du}$$

$$\left(\frac{2}{u} + \frac{u}{2}\right) = \frac{4+u^2}{2u}$$

$$\frac{1}{x}dx = \frac{2u}{4+u^2}\,du$$

$$\int \frac{1}{x}dx = \int \frac{2u}{4+u^2}\,du$$

$$\ln x + c = \ln|4+u^2| \qquad u = \frac{y}{x}$$

$$\ln x + c = \ln\left|4+\left(\frac{y}{x}\right)^2\right|$$

Ex: $x^2y' = x^2 + xy + y^2$

$$y' = 1 + \frac{y}{x} + \frac{y^2}{x^2}$$

$$y' = 1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 = F\left(\frac{y}{x}\right)$$

let $u = \frac{y}{x} \Rightarrow y' = xu' + u$

$$xu' + \cancel{u} = 1 + \cancel{u} + u^2$$

$$x\frac{du}{dx} = 1 + u^2 \Rightarrow \frac{x}{dx} = \frac{1+u^2}{du}$$

$$\int \frac{1}{x}dx = \int \frac{1}{1+u^2}du$$

$$\ln x + c = \tan^{-1}(u) \Rightarrow u = \tan(\ln x + c)$$

use $u = \frac{y}{x}$

$$y = x\tan(\ln x + c)$$

* Equation of the form $y' = G(ax + by)$ $a \neq 0, b \neq 0$

Ex.: ① $y' = \sin(9x - y)$

$a = 9$, $b = -1$

$G(x) = \sin(\ )$

$u = 9x - y$

② $y' = e^{3x+y+5}$ $\qquad$ $u = 3x + y + 5$

$G = e^{u+5}$, $u = 3x + y$

Method to solve $y' = G(ax+by)$

use $u = ax + by$

Transform D.E to sep.

$u' = a + by' \Rightarrow y' = \frac{u' - a}{b}$

$\frac{u'-a}{b} = G(u) \Rightarrow \frac{1}{b}u' - \frac{a}{b} = G(u)$

$\frac{1}{b} * \frac{du}{dx} = G(u) + \frac{a}{b}$

$\frac{1}{b} \cdot \frac{1}{dx} = \frac{G(u) + \frac{a}{b}}{du}$

Ex: Solve $y' = \sin(x-y)$

let $u = x - y$

$u' = 1 - y \Rightarrow y' = 1 - u'$

$1 - u' = \sin(u)$

$1 - \sin(u) = \frac{du}{dx}$

$\frac{1-\sin(u)}{du} = \frac{1}{dx}$

$\int dx = \int \frac{1}{1-\sin u} du$

$x + c = \int \frac{1+\sin u}{(1+\sin u)(1-\sin u)} du$

$= \int \frac{1+\sin u}{\cos^2 u} du$

$x + c = \int (\sec^2 a + \tan u \cdot \sec u) du$

$x + c = \tan u + \sec u$

$x + c = \tan(x-y) + \sec(x-y)$

Ch4: Second Order Liner D.E

1 Form Second Order liner

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

where $a(x), b(x), c(x), g(x)$ are fns.

If $g(x) = 0 \Rightarrow$ D.E is called Homog.

If $g(x) \neq 0 \Rightarrow$ D.E in not called Homog. (Non - Homog)

2 IVP: second order

2 $y'' + p(x)y' + q(x)y = f(x)$, with $y = y_0$, $y'(x_0) = y_1$

Note: For IVP in *2* has unique solve if The fns $p(x), q(x)$ and $f(x)$ are cont on interval containg x_0.

Ex: Find the largest Inteval such that the D.E has Unique soln:

$$(1+x^2)y'' + xy - y = \tan x \quad , y(1) = 0 \ , y'(1) = \pi$$

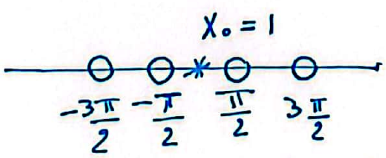
$$y'' + \underbrace{\frac{x}{1+x^2}}_{p(x)} y' - \underbrace{\frac{1}{1+x^2}}_{q(x)} y = \underbrace{\frac{\tan x}{1+x^2}}_{F(x)}$$

$p(x)$ and $q(x)$ are cont. $\forall x \in \mathbb{R}$

But $f(x) = \dfrac{\tan x}{1+x^2}$ has discont. pts at

$$x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots\ldots$$

* Interval of Cont



$x_0 = 1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow$ is largest Interval for Unique soln.

3 linear Independent / dependent

Defn: Two functions $f(x)$ and $g(x)$ are linear dep. on I, If there exist constant C such that $g(x) = C f(x)$

Test

1) If $\frac{g(x)}{f(x)} = C$ (constant) $\Rightarrow$ $f(x)$ and $g(x)$ are lin. dep.

2) If $\frac{g(x)}{f(x)} \neq C$ $\Rightarrow$ $f(x)$ and $g(x)$ are lin. indep.

Ex. Is the following fns lin. dep / indep:

1) $f(x) = e^{2x}$, $g(x) = e^{5x}$

$$\frac{g}{f} = \frac{e^{5x}}{e^{2x}} = e^{3x} \neq C \Rightarrow \text{lin. indep}$$

2) $f(x) = \sin 2x$ $\qquad$ $g(x) = \cos x \sin x$ $\left(\frac{1}{2}\sin 2x\right)$

$$\frac{g}{f} = \frac{\frac{1}{2}\cancel{\sin 2x}}{\cancel{\sin 2x}} = \frac{1}{2} = C \Rightarrow \text{lin. dep.}$$

* Matrix

$A_{n \times m}$ where n is the number of rows. and m is the number of columns.

$$A_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

* $\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Defn : The wronskian for any two fns f(x) and g(x) is

$$W[f(x), g(x)] = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = fg' - gf'$$

Note₁ :

1) If $W[g,f] \neq 0 \rightarrow f(x)$ and $g(x)$ are lin. indep.

2) If $W[g,f] = 0 \Rightarrow$ " " " " lin. dep.

Note₂ : The general soln for any D.E second Order is

$$y(x) = C_1 y_1 + C_2 y_2$$

Defn : Fundemental Set of Soln (F.S.S)

For D.E $ay'' + by' + cy = 0$

where a, b and c are constant

we called the set $\{y_1, y_2\}$ is F.S.S iff

① y_1 and y_2 is Soln.

② y_1 and y_2 are lin. indep $W \neq 0$

Note₃ :

IF $F.S.S = \{y_1, y_2, \dots, y_n\} \Rightarrow$ Then, general soln is

$$y(x) = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$$

Constant Coeff $y^{(n)} \longrightarrow r^n$

$ay'' + by' + cy = 0$ (Homog)

where a, b, c are constant

soln has the form $y = e^{rx}$

D.E $\xrightarrow[\text{char method}]{}$ eqn quadratic

As $y = e^{rx}$ is soln

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

I : Second Order Homog with constant Coeff.

$ay'' + by + cy = 0$

where $a, b, c \in \mathbb{R}$ "Constant"

let soln $y(x) = e^{rx}$ (r is root for Char. eqn)

$y' = re^{rx}$, $y'' = r^2 e^{rx}$

$a(r^2 e^{rx}) + b(re^{rx}) + c e^{rx} = 0$

$\underset{\neq 0}{e^x}(ar^2 + br + c) = 0$

$ar^2 + br + c = 0$ (is characteristic eqn)

~~in general~~

ingeneral $y^{(n)} \longrightarrow r^n$

derivative $\longrightarrow$ Power

Case 1 :

If $\Delta = b^2 - 4ac$

$\Delta > 0$

we have two different roots

$r_1 \neq r_2$

$F.S.S = | y_1 = e^{r_1 x}, y_2 = e^{r_2 x} |$

general soln

$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

Case$_2$:

If $\Delta = 0$

Then we have repeated roots

$r_1 = r_2 = r_3 = r$

$F.S.S = | y_1 = e^{r_1 x}, y_2 = x e^{r_1 x} |$

case 3 :-

If $\Delta < 0$

Then the roots are complex Conjugate

$r = \alpha \mp i\beta$

* To find r use

$r = \dfrac{-b \mp \sqrt{\Delta}}{2a}$

Note : complex number $z = \alpha + i\beta$

where $i = \sqrt{-1}$

$i^2 = -1$

$F.S.S = | e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x |$

Ex : Solve the following D.E

1) $y'' - 5y' + 6y = 0$

$a = 1 \quad , b = -5 \quad , c = 6$

Char. eqn $\quad ar^2 + br + c = 0$

$r^2 - 5r + 6 = 0$

$\Delta = 25 - 4(1)(6) = 25 - 24 = 1 > 0$

$(-2 + r)(r - 3) = 0$

$r_1 = 2 \quad , r_2 = 3$

F.S.S $= \{e^{2x}, e^{3x}\} \Rightarrow y(x) = C_1 e^{2x} + C_2 e^{3x}$

2) $y'' - 4y' + 4y = 0$

$r^2 + (-4y') + 4 = 0$

$\Delta = 16 - 4(1)(4) = 0$

$(r-2)^2 = 0$

$r_1 = r_2 = 2$

F.S.S $= \{e^{2x}, x e^{2x}\}$

general soln $\quad y(x) = C_1 e^{2x} + C_1 x e^{2x}$

3) $y'' + 16y = 0 \quad , y(0) = 2 \quad , y'(0) = -2$

$a = 1 \quad , b = 0 \quad , c = 16$

Char. eqn : $r^2 + 16 = 0$

$\Delta = 0 - 4(1)(16) = -64 < 0$

Root is complex

Find Root by $\quad r = \dfrac{-b \mp \sqrt{\Delta}}{2a} = \dfrac{0 \mp \sqrt{-64}}{2 \cdot 1} = 0 \mp 4i$

$\alpha = 0, \beta = 4$

$F.S.S = \{e^{0x}\cos(4x),\ e^{0x}\sin(4x)\}$

$\{\cos(4x),\ \sin(4x)\}$

general soln

$y(x) = C_1\cos(4x) + C_2\sin(4x)$

Use IC To find C_1 and C_2

$y(0) = 2 \Rightarrow 2 = C_1 \cdot 1 + C_2 \cdot 0$

$C_1 = 2$

$y'(x) = -4C_1\sin 4x + 4C_2\cos 4x$

use $y'(0) = 2$

$-2 = 0 + 4C_2 \Rightarrow C_2 = -\frac{1}{2}$

$y(x) = 2\cos 4x - \frac{1}{2}\sin 4x$

Q: If $y = xe^{4x}$ is soln for D.E

$y'' + by' + cy = 0$

Find b, c?

$r = 4$

$r_1 = r_2 = 4 \Rightarrow (r-4)(r-4) = 0$

$r^2 - 8r + 16 = 0$

$y'' - 8y' + 16y = 0$

$b = -8$

$c = 16$

2 Reduction of Order

$\text{I } y'' + p(x) y' + q(x) y = 0$

where one soln is given y_1

$F.S.S = \{ y_1 , y_2 \}$

let $y_2(x) = y_1(x) U(x)$ is soln (lin . indep)

$\Rightarrow y_2'' + p(x) y_2' + q(x) y_2 = 0$

$y_2' = y_1 U' + U y_1'$

$y_2'' = y_1 U'' + U' y_1' + U y_1'' + y_1' U'$

$= y_1 U'' + 2U' y_1' + U y_1''$ $\xrightarrow{\text{in D.E}}$

$y_1 U'' + 2U' y_1' + U y_1'' + p(x) (y_1 U' + U y_1') + q(x) y_1 U = 0$

$U(\underbrace{y_1'' + p(x) y_1' + q(x) y_1}_{0}) + y_1 U'' + 2U' y_1' + p(x) y_1 U' = 0$

As y_1 is soln $\Rightarrow$ $y_1'' + p(x) y_1' + q(x) y_1 = 0$

$y_1 U'' + U' (2y_1' + p(x) y_1) = 0$

solve for $U(x)$

$\div y_1$ and $U' \Rightarrow$

$\frac{U''}{U'} = - \left(2 \frac{y_1'}{y_1} + p(x) \right)$

integerate $\Rightarrow$

$\ln U' = -2 \ln y_1 - \int p(x)\, dx$

exp ⇒

$$V' = \frac{1}{y^2} e^{-\int p(x)dx}$$

$$* \quad V(x) = \int \frac{1}{y^2} e^{-\int p(x)dx} dx$$ (القانون)

$$* \quad y_2 = y_1 \cdot V$$

Ex : Constant Coeff $ay'' + by' + cy = 0$

when $\Delta = 0 \Rightarrow y_2 = xe^{rx}$

Show that by Reduction Method $V(x) = x$?

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$$

$$\Delta = 0 , r = \frac{-b}{2a} \Rightarrow 2r = \frac{-b}{a}$$

$$y_2 = V_{(x)} \, y_1$$

$$V(x) = \int \frac{1}{y_1^2} e^{\int p(x)dx} dx$$

$$= \int \frac{1}{e^{2rx}} e^{-\int \frac{b}{a} dx} dx$$

$$= \int 1 \, dx = x$$

Abel's Formula

* $y'' + p(x)y' + q(x)y = 0$

If y_1, y_2 are soln, then

$$W[y_1, y_2](x) = C e^{-\int p(x)dx}$$

where C is any constant

$$W' = C e^{-\int p(x)dx} \cdot -p(x)$$

$$W' = W(-p(x))$$

$$p(x) = \frac{-W'(x)}{W(x)}$$

$\boxed{3}$ Cauchy Euler Method $x^n : y^{(n)}$

Def: ① $(ax^2y'' + bxy' + cy = 0)$, $x > 0$, $a \neq 0$

Second Order Homog. Cauchy Euler eqn

② $(ax^3y''' + bx^2y'' + cxy' + dy = 0)$

3rd Order Homog Cauchy Euler eqn

let soln $y = x^r$

where r is the root for char. eqn

D.E "cauchy" $\longrightarrow$ quadrat. eqn

$y' = r x^{r-1}$, $y'' = r(r-1) x^{r-2}$

in D.E $\Rightarrow$ $ax^2\left(r(r-1)x^{r-2}\right)+bx\,r\,x^{r-1}+c\,x^r=0$

$$ar(r-1)x^r+br\,x^r+Cx^r=0$$

$$\underset{0\neq}{x^r}\left[ar(r-1)+br+C\right]=0$$

$$\therefore\ ar(r-1)+br+C=0$$

$$ar^2+(b-a)r+C=0$$

$$\Delta=(b-a)^2-4ac$$

Case 1 :

$\Delta>0$

There are Two different roots

$r_1\neq r_2$

$F.S.S=\{x^{r_1},x^{r_2}\}$

$y(x)=C_1x^{r_1}+C_2x^{r_2}$

Case 2 :

$\Delta=0$

$r_1=r_2=r$ Repeated root

$F.S.S=\{x^r,\ln x\ x^r\}$

$y_2=x^r\underbrace{\ln x}_{v(x)}$

Case 3 :

$\Delta < 0$

There are a complex roots

$r_{1,2} = \alpha \mp i\beta$, $r_1 = \alpha + i\beta$

soln $\rightarrow y = x^r = x^{\alpha+i\beta} = x^{\alpha} \cdot x^{i\beta}$

$= x^{\alpha} \cdot e^{\ln x^{i\beta}} = x^{\alpha} \cdot e^{i\beta \ln x}$

$= x^{\alpha}(\cos(\beta \ln x) + i\sin(\beta \ln x))$

$$F.S.S = \{ \underbrace{x^{\alpha}\cos(\beta \ln x)}_{y_1} , \underbrace{x^{\alpha}\sin(\beta \ln x)}_{y_2} \}$$

Ex : Solve the folling D.E

1) $3x^2y'' + 11xy' - 3y = 0$, $x > 0$

char - eqn

$3r^2 + 8r - 3 = 0$

$\Delta = 64 - 4(3)(-3) = 100 > 0$

$r = \frac{-8 \mp 10}{6}$, $r_1 = \frac{-8+10}{6} = \frac{2}{6} = \frac{1}{3}$

$r_2 = \frac{-8-10}{6} = \frac{-18}{6} = -3$

$\therefore y_1 = x^{\frac{1}{3}}$

$y_2 = x^{-3}$

general soln $y(x) = C_1 x^{\frac{1}{3}} + C_2 x^{-3}$

* ملاحظة : إذا أعطاني أن $x<0$ بفرض $t=-x$
ثم تعوض لتصبح $t>0$ وفي نهاية المعادلة
أعيد الفرض كما كان .

2) $x^2y'' + xy' = 0$

Char - eqn

$r^2 + 0r + 0 = 0$

$r^2 = 0$

$r_{1,2} = 0$

$F.S.S = \{ x^0, x^0 \ln x \}$

$y = c_1 + c_2 \ln x$

3) $x^2y'' + 5xy' + 5y = 0$

Char - eqn

$r^2 + 4r + 5 = 0$

$\Delta = 16 - 4(1)(5) = -4 < 0$

$r = \dfrac{-4 \mp \sqrt{-4}}{2} = \dfrac{-4 \mp 2i}{2} = -2 \mp i$

$\alpha = -2 \qquad \beta = +1$

$F.S.S = \{ x^{-2}\cos(\ln x), x^{-2}\sin(\ln x) \}$

4) $t^2y'' - 4ty' + 4y = 0$, $y(1) = -2$, $y'(1) = -11$, $t > 0$

Char - eqn

$r^2 - 5r + 4 = 0$

$\Delta = 25 - 4(1)(4) = 9 > 0$

$(r-1)(r-4) = 0$

$y(t) = C_1 t^1 + C_2 t^4$

$y(1) = -2 \longrightarrow -2 = C_1 + C_2$ — ①

$y'(1) = -11 \longrightarrow -11 = C_1 + 4C_2$ — ②

① − ② $\Rightarrow 9 = -3C_2 \Rightarrow C_2 = -3$

$\therefore C_1 = 1$

Sec 4.4 Non - Homog D.E

The method of undetermined coeff

$$ay'' + by' + Cy = f(x)$$

general soln

$y(x) = y_h + y_p$

$y_h \rightarrow F.S.S = \{y_1, y_2\}$

$y_h = C_1 y_1 + C_2 y_2$

where y_p is called the particular soln

Note : we use the Undetermined method only if f(x) one of the folowing case :

1) f(x) is exp $f(x) = e^{ax+b}$
2) $\sin(\theta)$ or $\cos(\theta)$
3) polynomial of any degree

$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$

$P_1(x) = a_1 x^1 + a_0$

or any combination between these three cases.

Note : To find the form of $y_{p(x)}$

	F(x)	$y_{p(x)}$
1)	e^{ax+b}	Ae^{ax+b}
2)	$\sin\theta$ or $\cos\theta$	$A\cos\theta + B\sin\theta$
3)	$P_n(x)$	$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0$

Q: How Use the method of Undetermined Coeff To Solve Non-Homog

1) Find the Homog. Soln $\Leftrightarrow$ F.S.S $= \{y_1, y_2\}$

$(f(x) = 0)$ $\quad y_h = C_1 y_1 + C_2 y_2$

2) Find the form of y_p (use the Table)

3) Find the general soln $y(x) = C_1 y_1 + C_2 y_2 + y_p$

Note : $ay'' + by' + Cy = f(x)$

1) To Find the constant in $y_p(x)$ by using $y_p(x)$ as soln for Non-Homog D.E

2) The general soln has only the constant C_1 & C_2

Ex : solve $y'' - 5y' + 4y = 8e^x$

1) Find y_h : $y'' - 5y' + 4y = 0$

Char-eqn : $r^2 - 5r + 4 = 0$

$\Delta = 25 - 4(1)(4) = 25 - 16 = 9 > 0$

$(r-1)(r-4) = 0$

$r_1 = 1, \; r_2 = 4$

F.S.S $= \{e^x, e^{4x}\} \Rightarrow y_{h(x)} = C_1 e^x + C_2 e^{4x}$

2) Find y_p

As $f(x) = 8e^x \longrightarrow y_{p(x)} \overset{\text{case}}{=} Ae^x$

$y_{p(x)} = Axe^x$

$y_p'' - 5y_p' + 4y_p = 8e^x$

$y_p' = Axe^x + Ae^x$

$y_p'' = Axe^x + Ae^x + Ae^x = 2Ae^x + Axe^x$

$2Ae^x + Axe^x - 5Axe^x - 5Ae^x + 4Axex = 8e^x$

$2A - 5A = 8 \rightarrow -3A = 8 \rightarrow A = -\frac{8}{3}$

$y_{p(x)} = \frac{-8}{3} xe^x$

general soln : $y_{(x)} = y_h + y_p$

$= C_1 e^x + C_2 e^{4x} + \frac{-8}{3} xe^x$

Ex : Find the Form of $y_{p(x)}$

1) $y'' + y = \sin x \cos x$

Homog : $y'' + y = 0$

$r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$\alpha = 0$

$\beta = 1$

$F.S.S = \{\cos x, \sin x\}$

$y_h = c_2 \sin x + c_1 \cos x$

$* y_p$: $f(x) = \sin x \cos x = \frac{1}{2} \sin 2x$

$y_{p(x)} = A \sin x(2x) + B \cos(2x)$

2) $y'' - y = x^2 e^{2x}$

Homog : $y'' - y = 0 \Rightarrow$ Char - eqn : $r^2 - 1 = 0$

$(r-1)(r+1) = 0$

$r_1 = 1, \; r_2 = -1$

$F.S.S = \{e^x, e^{-x}\}$

$f(x) = x^2 e^{2x} = P_{2(x)} e^{2x}$

$y_{p(x)} = (a_2 x^2 + a_1 x + a_0) e^{2x}$

Ex: Solve $y'' + 2y' + 2y = 5\cos x + 5\sin x$
with $y(0) = 1$, $y'(0) = 2$

Homog:

$y'' + 2y' + 2y = 0$

char-eqn

$r^2 + 2r + 2 = 0$

$\Delta = 4 - 4(1)(2) = -4 < 0$

$r = \frac{-2 \mp 2i}{2} = -1 \mp i$

$\alpha = -1$, $\beta = 1$

$F.S.S = \{ e^{-x}\cos(x), e^{-x}\sin x \}$

$y_h = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x$

$y_{p(x)} = A\cos\overset{(x)}{\theta} + B\sin\overset{(x)}{\theta}$

$F(x) = 5\cos x + 5\sin x$

* To Find constant A & B in y_p

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

in D.E $\Rightarrow -A\cos x - B\sin x - 2A\sin x + 2B\cos x + 2A\cos x + 2B\sin x =$
$5\cos x + 5\sin x$

$[\sin x]: -B - 2A + 2B = 5 \Rightarrow \boxed{B = 5 + 2A}$ $\mathbb{G}$

$[\cos x]: -A + 2B + 2A = 5 \Rightarrow \boxed{A + 2B = 5}$ $\mathbb{G}_2$

$A + 10 + 4A = 5$

$5A = 5 - 10$

$A = -1$

$B = 3$

Undetermined Coeff Methods

The form of $y_{p(x)}$ where $f(x)$ is product

Rule1 : for the D.E $ay'' + by' + Cy = K x^m e^{rx}$

where $m \geq 0$ positive integer $m = 0, 1, 2, \ldots$

OR x^m is poly of degree m

Then the form of $y_{p(x)}$ is

$$y_{p(x)} = x^s \left(A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0\right) e^{rx}$$

*where s is the number of repeated root r in char-eqn.

$s = 0$, If r is not root in char-eqn

$s = 2$, If r is repeated root.

Ex: Find the form of $y_{p(x)}$ for D.E

$y'' + 2y' - 3y = f(x)$ where

$f(x) = 3xe^x$

Homog: $y'' + 2y' - 3y = 0$

char eqn: $r^2 + 2r - 3 = 0$

$(r-1)(r+3) = 0$

$r_1 = 1, \; r_2 = -3$

But $f(x) = 3xe^x \rightarrow m = 1$

$r = 1 \rightsquigarrow s = 1$

$$y_{p(x)} = x(a_1 x + a_0)e^x$$

Variation of parameter

* II $y'' + p(x)y' + q(x)y = g(x)$

Homog - soln : $(g(x) \equiv 0) \Rightarrow F.S.S = (y_1, y_2)$

$$y_h = C_1 y_1 + C_2 y_2$$

gole : Find the particular soln $y_{p(x)}$

* No constant in $y_{p(x)}$.
* No special Case in $g(x)$

$$y_{p(x)} = y_1 u_1 + y_2 u_2$$

<u>steps</u>

① Find $F.S.S = \{y_1, y_2\}$

② Find $W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

③ $y_p = y_1 u_1 + y_2 u_2$ where

$$u_1 = -\int \frac{y_2 \, g(x)}{w(x)} dx$$

$$u_2 = \int \frac{y_1 \, g(x)}{w(x)} dx$$

Ex : Solve $x^2 y'' - 2xy' + 2y = x^3 e^{-x}$, $x > 0$

By Variation of parameter.

Homog : $x^2 y'' - 2xy' + 2y = 0$

Charegn : $ar^2 + (b-a)r + C = 0$

$r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$r_1 = 1 \ , r_2 = 2$

$F.S.S = \{x^1, x^2\} \Rightarrow y_h = C_1 x + C_2 x^2$

$w[x, x^2] = 2x^2 - x^2 = x^2$

$u_1 = -\int \frac{y_2 \, g}{w} dx = -\int \frac{\cancel{x} \cdot e^{-x} \cdot x}{\cancel{x}}$

by variation

$g(x) = \frac{x^3 e^{-x}}{x^2} = x e^{-x}$

$u_1 = -\int e^{-x} \cdot x = -(-xe^{-x} - e^{-x}) = xe^{-x} + e^{-x}$

$u_2 = \int \frac{x \cdot xe^{-x}}{x^2} = \int e^{-x} dx = -e^{-x}$

$y_p = xe^{-x}$ ※

The Superposition principle

Theorem:

Let $L[y] = ay'' + by' + cy = 0$ Homo
$\neq 0$ non-Homo

if y_1 be a solution to $L[y] = f(x)$ and
y_2 be a solution to $L[y] = g(x)$

* الفرق بينهم هو فقط بالطرف الأيمن اذا كان Homo او لا.

Then,
The solution for $L[y] = C_1 f(x) + C_2 g(x)$ is $C_1 y_1 + C_2 y_2$

Ex: Let y_1 be solution for $y'' + x^2 y' + y = x^2 - 2x$
y_2 " " for $y'' + x^2 y' + y = 5x + 3$

Find the solution for $y'' + x^2 y' + y = 2x^2 + 11x + 9$?

$L[y] = y'' + x^2 y' + y$

$f(x) = x^2 - 2x$ $\qquad$ $g(x) = 5x + 3$

By Superposition thrm: soln for $L[y] = 2x^2 + 11x + 9$ is $C_1 y_1 + C_2 y_2$

But $C_1 f(x) + C_2 g(x) = 2x^2 + 11x + 9$

To find C_1 and C_2

$C_1 (x^2 - 2x) + C_2 (5x + 3) = 2x^2 + 11x + 9$

$C_1 x^2 - 2C_1 x + 5C_2 x + 3C_2 = 2x^2 + 11x + 9$

$3C_2 = 9 \Rightarrow C_2 = 3$

$C_1 = 2$

so the Soln $C_1 y_1 + C_2 y_2$ is
$2y_1 + 3y_2$

Higher Order Differential Equation

The Constant Coefficient D.E of Order n is given by

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)$$

where $a_n, \dots, a_0$ are constant

* if $g(x) = 0$, then the D.E is Homog.
* if $g(x) \neq 0$, the D.E is called Non-Homog
* The solution For Homog D.E is $y = e^{rx}$ were r is the root For the Char. eqn (Auxiliary equation):

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$$

* there are three Cases in the second order D.E:

Case 1: There are n different roots $r_1 \neq r_2 \neq \dots \neq r_n$

$$F.S.S = \{e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}\}$$

Case 2: There are one r (repeated K times), then

$$F.S.S = \{e^{rx}, xe^{rx}, \dots, x^{K-1} e^{rx}\}$$

Case 3: Complex root $r = a \pm i\beta$

$$F.S.S = \{e^{ax}\cos(\beta x), e^{ax}\sin(\beta x)\}$$

Ex: Solve the Following D.E

① $y^5 + 2y^4 + y^3 = 0$

The Char. Eqn $\longrightarrow r^5 + 2r^4 + r^3 = 0 \longrightarrow r^3(r^2 + 2r + 1) = 0 \longrightarrow r^3(r+1)(r+1) = 0$

The root For Char. Eqn are: $r = 0, 0, 0, -1, -1$

$F.S.S = \{e^{0x}, xe^{0x}, x^2 e^{0x}, e^{-1x}, xe^{-x}\}$

general slon $y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^{-x} + C_5 x e^{-x}$

② $y^{(3)}+3y''+4y=0$

The char. eqn $\rightarrow r^3+3r^2-4=0$

* نجد عدد يحقق المعادلة
مثلاً ال $\boxed{1}$.

$r=1 \rightarrow 1+3-4=0$ ✓

$(r-1)(r^2+4r+4)=0$

$(r-1)(r+2)^2=0$

$r=1,-2,-2$

$F.S.S=\{e^{x}, e^{-2x}, xe^{-2x}\}$

$$\begin{array}{r|l} & r^2+4r+4 \\ \hline r-1 & r^3+3r^2-4 \\ - & \cancel{r^3}-r^2 \\ \hline & \quad 4r^2-4 \\ - & \quad \cancel{4r^2}-4r \\ \hline & \qquad 4r-4 \\ - & \qquad \cancel{4r-4} \\ \hline & \qquad 0 \end{array}$$

③ $y^{(4)}+2y''+y=0$

The Char. eqn $\rightarrow r^4+2r^2+r^0=0$

$t=r^2 \longrightarrow t^2+2t+1 \longrightarrow (t+1)^2 \longrightarrow (r^2+1)^2 \longrightarrow r^2+1=0 \longrightarrow r_{1,2}=\pm i \longrightarrow \alpha=0, \beta=1$

$r_{3,4}=\pm i$

$F.S.S=\{\underset{y_1}{\cos x}, \underset{y_2}{\sin x}, \underset{y_3}{x\cos x}, \underset{y_4}{x\sin x}\}$

Ex:- if the roots for the auxiliry equation are $r_1=4, r_2=r_3=-5$

Find the D.E ?

From the roots of char. eqn

$(r-4)(r+5)^2=0$

$r^3+6r^2-15r-100=0$

But $r^n \longrightarrow y^{(n)}$

$y^{(3)}+6y''-15y'-100y=0$

Ex: Find the form of $y_p(x)$ for the D.E

① $y^{4} + 2y''' + 5y'' = 4x^2$

The char eqn: $r^4 + 2r^3 + 5r^2 = 0$

$\Rightarrow r^2(r^2 + 2r + 5) = 0$

The roots for Char. eqn are:

$r = 0, 0, -1 \pm 2i$

As $F(x) = 4x^2$ By Rule 1, $m = 2, r = 0, s = 2$

$\Rightarrow y_p(x) = (a_2x^2 + a_1x + a_0)\,x^2$

② $y''' + y'' - 2y = xe^x$

The char. eqn: $r^3 + r^2 - 2 = 0$

$\Rightarrow (r-1)(r^2 + 2r + 2) = 0$

The roots for Char. eqn are:

$r = 1, -1 \pm i$

$F.S.S = \{e^x, e^{-x}\cos x, e^{-x}\sin(x)\}$

$y_p(x) = (a_1x + a_0)e^x$ but e^x is in F.S.S

$\Rightarrow y_p(x) = (a_1x + a_0)xe^x$

③ $y^{(3)} + 9y' = x\sin(3x) + x^2e^{2x}$

The char. eqn: $r^3 + 9r = 0 \rightarrow r(r^2 + 9) = 0$

The roots are:

$r = 0 \pm 3i$

$F.S.S = \{1, \cos(3x), \sin(3x)\}$

$F_1(x) = x\sin 3x$

$\Rightarrow y_{p_1}(x) = (ax + b)\sin 3x + (cx + d)\cancel{x}\cos 3x$

But $\sin 3x$ and $\cos 3x$ are soln for homog, then multiply by x

$\Rightarrow y_{p_1}(x) = (ax^2 + bx)\sin 3x + (cx^2 + dx)\cos 3x$

$F_2(x) = x^2e^{2x} \Rightarrow y_{p_2} = (a_2x^2 + a_1x + a_0)e^{2x}$

so, $y_p(x) = y_{p_1} + y_{p_2}$

Ch7: Laplace Transform

$$F(t) \longrightarrow F(s)$$
$$F(t) \longleftarrow \mathcal{L}^{-1}$$

Defn: If $f(t)$ is cont on $(0,\infty)$ then Laplace Transform of $f(t)$ is:

$$F(s) = \mathcal{L}[f(t)]$$
$$= \int_0^\infty e^{-st} f(t)\,dt \,, \quad s>0 \qquad \text{(Improper Integral)}$$
$$= \lim_{K\to\infty} \int_0^K e^{-st} f(t)\,dt$$

If limit exist $\Rightarrow$ Integral conv.

Ex: ① $\mathcal{L}[C] = \int_0^\infty e^{-st} . C\,dt$

where C any constant

$$= \lim_{K\to\infty} \int_0^K e^{-st} C\,dt = \lim_{K\to\infty} C \frac{e^{-st}}{-s}\Big|_0^K$$
$$= \lim_{K\to\infty} C \frac{1}{-s}\left(\overset{t=K}{e^{-sK}} - \overset{t=0}{1}\right) = C\frac{1}{-s}(0-1)$$
$$= \frac{C}{s}$$

$$\therefore \mathcal{L}[C] = \frac{C}{s}$$

② $\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} . e^{at}\,dt$

$$= \lim_{K\to\infty} \int_0^K e^{-st+at}\,dt = \lim_{K\to\infty} \frac{e^{(a-s)t}}{a-s}\Big|_0^K$$
$$= \frac{1}{a-s} \lim_{K\to\infty} \left(e^{(a-s)K} - 1\right)$$

$\Rightarrow$ IF $a<s \Rightarrow \frac{1}{a-s}(-1) = \frac{1}{s-a}$

* $a-s<0$
  $a<s$

* $a-s>0$
  Fail

* $\lim_{s\to\infty} F(s) = 0$

Note: Table of Laplace Transform

	$F(t)$	$F(s) = \mathcal{L}[F(t)]$
1	C	$\frac{C}{s}$
2	e^{at}	$\frac{1}{s-a}$
3	$\sin(at)$	$\frac{a}{s^2+a^2}$
4	$\sinh(at)$	$\frac{a}{s^2+b^2}$
5	$\cos(bt)$	$\frac{s}{s^2+b^2}$
6	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
7	t^n	$\frac{n!}{s^{n+1}}$

* properties of $\mathcal{L}$aplace

1. $\mathcal{L}[B.f(t)] = B\,\mathcal{L}[f(t)]$
2. $\mathcal{L}[F(t) \pm g(t)] = \mathcal{L}[f(t)] \pm \mathcal{L}[g(t)]$

Ex: Find the value of K such that

$$F(s) = \frac{2Ks}{s+1} - \frac{s^3-1}{s^3+1}$$

$$\lim_{s\to\infty} F(s) = 0$$

* اعلى اس معامله

$$\lim_{s\to\infty} \frac{2K\cancel{s}}{\cancel{s}} - \frac{\cancel{s^3}}{\cancel{s^3}} = 0$$

$$2K - 1 = 0$$

$$K = \frac{1}{2}$$

$$\mathcal{L}[F(t).g(t)] \neq \mathcal{L}[F(t)].\mathcal{L}[g(t)]$$

Rule 1 : shifting

$$\mathcal{L}[e^{at}F(t)] = \mathcal{L}[F(t)] = F(s)\Big|_{s \to s-a} = F(s-a)$$

Rule 2 : Derivatives

$$\mathcal{L}[t^n F(t)] = (-1)^n \cdot \frac{d^n}{ds^n}(\mathcal{L}[F(t)])$$
$$= (-1)^n \frac{d^n}{ds^n}(F(s))$$
$$= (-1)^n F^{(n)}(s)$$

Ex : Evaluate

1) $\mathcal{L}[\sin t \, t] = (-1)\frac{d}{ds}(\mathcal{L}[\sin t])$
$$= -\frac{d}{ds}\left(\frac{1}{(s^2+1)}\right) = \frac{+2s}{(s^2+1)^2}$$

2) $\mathcal{L}[t^2 e^t] = (-1)^2 \frac{d^2}{ds^2}[\mathcal{L}[e^t]]$
$$= 1 \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right) = \left(\frac{-1}{(s-1)^2}\right)'$$

* $\mathcal{L}[t^n e^{at} F(t)] = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}(e^{at}F(t)))$
$$= (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}[F(t)]\Big|_{s \to s-a}\right)$$

* Laplace Transform For derivatives :

1) $\mathcal{L}[F^{(n)}(t)] = s^n F(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - F^{(n-1)}(0)$

Laplace Inverse

If $F(s) = \mathcal{L}[F(t)]$

$\Rightarrow F(t) = \mathcal{L}^{-1}[F(s)]$

$\mathcal{L}^{-1}[F(s-a)] = e^{at}F(t)$

Ex: Evaluate

1) $\mathcal{L}^{-1}\left[\frac{2}{s^5} - \frac{3}{5}\right] = \mathcal{L}^{-1}\left[\frac{2}{s^5}\right] - \mathcal{L}^{-1}\left[\frac{3}{5}\right]$

$= \frac{2}{4!}\mathcal{L}\left[\frac{1*4}{s^5}\right] - 3 = \frac{1}{12}t^4 - 3$

2) $\mathcal{L}^{-1}\left[\frac{3}{(s-5)^4}\right] = \frac{1}{2}e^{5t}t^3$

$= \frac{1}{2}\mathcal{L}\left[\frac{3.2.1}{s^4}\right] = \frac{1}{2}t^3 \longrightarrow \boxed{F(t)} \times e^{5t}$

3) $\mathcal{L}^{-1}\left[\frac{7s-1}{(s+1)(s+2)(s-3)}\right]$

partial Fraction

$F(s) = \frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}$

$* \; 7s-1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$

$s=-1 \Rightarrow -8 = -4A \Rightarrow A=2$

$s=-2 \Rightarrow -15 = 5B \Rightarrow B=-3$

$s=3 \Rightarrow 20 = 20C \Rightarrow C=1$

$= \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{-3}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s-3}\right]$

$= 2e^{-t} - 3e^{-2t} + e^{3t}$

4) $\mathcal{L}^{-1}\left[\frac{3s+2}{s^2+2s+10}\right] \longrightarrow$ إكمال مربع $\longrightarrow s^2+2s+1-1+10 \rightarrow (s+1)^2+9$

$\mathcal{L}^{-1}\left(\frac{3s+2}{(s+1)^2+9}\right) = e^{-t}.F(x)$

$\mathcal{L}^{-1}\left(\frac{3s+2}{s^2+9}\right) \longrightarrow \mathcal{L}^{-1}\left(\frac{3s}{s^2+9}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) \longrightarrow \underbrace{3\cos 3t + \frac{2}{3}\sin 3t}_{F(x)}$

Ex: $\mathcal{L}^{-1}\left[\ln\left(\frac{s^2+1}{s-1}\right)\right] = F(t) = \frac{-2\cos t + e^t}{t}$

Derivative Rule $\xrightarrow[n=1]{}$ $\mathcal{L}[tF(t)] = -\frac{d}{ds}\left(\mathcal{L}[F(t)]\right)$

let $F(t) = \mathcal{L}^{-1}\left[\ln\left(\frac{s^2+1}{s-1}\right)\right]$

$\xrightarrow{\mathcal{L}}$ $\mathcal{L}F(t) = \ln\left(\frac{s^2+1}{s-1}\right) = \ln(s^2+1) - \ln(s-1)$

$\xrightarrow{\frac{-d}{ds}}$ $\frac{-d}{ds}\left(\mathcal{L}F(t)\right) = -\left(\frac{2s}{s^2+1} - \frac{1}{s-1}\right)$

$= -2\frac{s}{s^2+1} + \frac{1}{s-1}$

$= -2\,\mathcal{L}[\cos t] + \mathcal{L}[e^t]$

$\mathcal{L}[tF(t)] = \mathcal{L}[-2\cos t + e^t]$

Take $\mathcal{L}^{-1}$

$\Rightarrow \frac{\cancel{t}\cdot F(t)}{\cancel{t}} = \frac{-2\cos t + e^t}{t}$

Ex: $\mathcal{L}^{-1}[\tan^{-1}(3s)] = F(t)$

$\mathcal{L}F(t) = \tan^{-1}(3s)$

$\frac{-d}{ds}\mathcal{L}F(t) = \frac{-d}{ds}\left(\tan^{-1}(3s)\right) = -\left(\frac{1}{1+9s^2}\right)\cdot 3$

$= \frac{-3}{1+9s^2} = \frac{-3}{9\left(\frac{1}{9}+s^2\right)}$

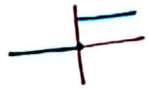
$\frac{-d}{ds}\left(\mathcal{L}(F(t))\right) = -\frac{\frac{1}{3}}{s^2+1/9} = -\mathcal{L}\left[\sin\left(\frac{1}{3}t\right)\right]$

$\mathcal{L}[tF(t)] = -\mathcal{L}\left[\sin\left(\frac{t}{3}\right)\right]$

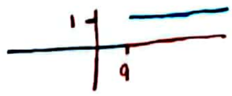
$F(t) = \frac{\sin\frac{t}{3}}{t}$

Unit Step Function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$*$ $$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$



Note : Let $0 < a_1 < a_2 < \cdots < a_n$

$$F(t) = \begin{cases} F_1(t), & t < a_1 \\ F_2(t), & a_1 < t < a_2 \\ \vdots & \\ F_n(t), & a_{n-1} < t < a_n \end{cases}$$

Then, $F(t)$ can be written in terms of unit step as

$$F(t) = F_1 + (F_2 - F_1)\,u(t-a_1) + (F_3 - F_2)\,u(t-a_2) + \cdots + (F_n - F_{n-1})\,u(t-a_{n-1})$$

$*$ Laplace For Unit Step fn.

$$F(t) = u(t-a)$$

$$\mathcal{L}[u(t-a)] = \int_0^\infty e^{-st}\,u(t-a)\,dt = \lim_{b\to\infty} \int_0^b e^{-st}\,u(t-a)\,dt$$

$$= \lim_{b\to\infty} \int_a^b e^{-st} \cdot 1 \cdot dt = \lim_{b\to\infty} \left. \frac{e^{-st}}{-s} \right|_a^b$$

$$= \lim_{b\to\infty} \frac{1}{-s}\left(\cancel{e^{-sb}}_{0} - e^{-sa}\right) = \frac{1}{-s}\left(-e^{-as}\right) = \frac{e^{-as}}{s}$$

Rule 1 : $\mathcal{L}[u(t-a)] = \dfrac{e^{-as}}{s}$

Rule 2 : $\mathcal{L}[F(t)\,u(t-a)] = e^{-as}\,\mathcal{L}[F(t+a)]$

Ex : Find $\mathcal{L}[F(t)]$ where,

$$F(t) = \begin{cases} 2, & t < 1 \\ 0, & 1 < t < 2 \\ e^t, & t \geq 2 \end{cases}$$

$$F(t) = 2 + (0+2)\,u(t-1) + (e^t - 0)\,u(t-2)$$

$$\mathcal{L}[F(t)] = \mathcal{L}\,2 - 2\,\mathcal{L}[u(t-1)] + \mathcal{L}[e^t u(t-2)] \longrightarrow e^{-2s}\,\mathcal{L}[F(t+2)]$$

$$= \frac{2}{s} - 2\,\frac{e^{-s}}{s} + e^{-2s+2} * \frac{1}{s-1}$$

$$e^{-2s}\,\mathcal{L}[e^{t+2}]$$

$$e^{-2s+2}\,\mathcal{L}[e^t]$$

$$\frac{e^{-2s+2}}{s-1}$$

Inverse For Unit Step

1) $\mathcal{L}^{-1}\left[\frac{e^{-as}}{s}\right] = u(t-a)$

2) $\mathcal{L}^{-1}\left[e^{-as}F(s)\right] = u(t-a)f(t-a)$

where $f(t) = \mathcal{L}^{-1}[F(s)]$

Ex: ① $\mathcal{L}^{-1}\left[\frac{e^{-\frac{\pi}{2}s}\,s}{s^2+9}\right]$

$a = \frac{\pi}{2}$, $F(s) = \frac{s}{s^2+9} \Rightarrow f(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right]$

$= \cos(3t)$

$= u(t-\frac{\pi}{2})\, f(t-\frac{\pi}{2})$

$= u(t-\frac{\pi}{2}) \cos(3t - \frac{\pi}{2})$

$= -u(t-\frac{\pi}{2}) \sin(3t)$

② $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s-4}\right]$ $\quad a = 2,\ F(s) = \frac{1}{s-4}$

$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s-4}\right] = e^{4t}$

↓

$= u(t-2)f(t-2) = u(t-2)e^{4(t-2)}$

$= u(t-2)\, e^{4t-8} = \begin{cases} 0, & t<2 \\ e^{4t-8}, & t>2 \end{cases}$

③ $\mathcal{L}\left[t\cos(2t-2)\, u(t-1)\right]$

$= e^{-s}\mathcal{L}\left[(t+1)\cos(2(t+1)-2)\right]$

$= e^{-s}\left(\mathcal{L}[t\cos 2t] + \mathcal{L}[\cos(2t)]\right)$

$= e^{-s}\left(\left(\frac{-d}{ds}\left(\frac{s}{s^2+4}\right)\right) + \frac{s}{s^2+4}\right)$

Solving IVP by Using Laplace Transformation

$y_p'' + y_p = t$

$0 + at + b = t \longrightarrow \begin{matrix} a=1 \\ b=0 \end{matrix} \Rightarrow y_p = t$

general soln $y = C_1 \cos t + C_2 \sin t + t$

$y(0) = 0 \Rightarrow 0 = C_1 + 0 + 0 \Rightarrow C_1 = 0$

$y'(t) = -C_1 \sin t + C_2 \cos t + 1$

$y'(0) = 1 \Rightarrow 1 = -C_1 0 + C_2 1 + 1 \Rightarrow C_2 = 0$

$\boxed{y(t) = t}$

Steps:

1) Take Laplace for both side of D.E e use IC
2) solve for $Y(s) = \ldots\ldots$
3) Take $\mathcal{L}^{-1} \Rightarrow$ general soln

$y(t) = \mathcal{L}^{-1}[Y(s)]$

Ex: Using laplace Transform to solve

$y' - 2y = g(t), \; y(0) = 0$

where $g(t) = \begin{cases} 0, & t<1 \\ 1, & t>1 \end{cases} = u(t-1)$

$g(t) = 0 + (1-0)u(t-1)$

D.E: $y' - 2y = u(t-1)$

$\mathcal{L}[y'] - 2\mathcal{L}[y] = \mathcal{L}[u(t-1)]$

$sY - y(0) - 2Y = \frac{e^{-s}}{s}$

$* \; Y(s-2) = \frac{e^{-s}}{s}$

$Y(s) = \frac{e^{-s}}{s(s-2)}$

general soln $y(t) = \mathcal{L}^{-1}\left(\frac{e^{-s}}{s(s-2)}\right)$

$\mathcal{L}^{-1}[e^{-as} F(s)] = f(t-a)\,u(t-a)$

$= f(t-1)\,u(t-1)$

$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s-2)}\right]$

$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$ $\quad A = \frac{-1}{2}$, $B = \frac{1}{2}$

$f(t) = \mathcal{L}^{-1}\left[\frac{-\frac{1}{2}}{s}\right] + \mathcal{L}^{-1}\left[\frac{\frac{1}{2}}{(s-2)}\right]$

$= -\frac{1}{2} + \frac{1}{2} e^{2t}$

Ch8: Series Solution of D.E

Defn: Taylor series of $F(x)$

$$F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(x_0)}{n!} (x - x_0)^n$$

about $x = x_0$

* If $P_1(x), \dots, P_n(x)$ and $g(x)$ are cont at a point $x = x_0$ and y is a fn that have derivatives of all order at x_0 and y is soln of IVP

$$1\, y^{(n)} + P_1(x)\, y^{(n-1)} + \dots + P_{n-1}(x)\, y' + P_n(x_0)\, y = g(x)$$

$$y(x_0) = y_0 \,/\, y'(x_0) = y_1 / \dots / y^{(n-1)}(x_0) = y_{n-1}$$

Then y has a series form

$$* \quad y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x - x_0)^n$$

Ex: Find the first four non-zero terms in a power series expansion of soln to IVP

1) $y'' + y' - x y_{(x)} = 3$, $y(0) = 1$, $y'(0) = -2$

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x - x_0)^n$$

$$= y_{(0)} + \frac{y'(0)}{1!} x + \frac{y''(0)}{2} x^2 + \frac{y'''(0)}{6} x^3$$

$$= 1 - 2x + \frac{5}{2} x^2 + \frac{-4}{6} x^3$$

in D.E $\Rightarrow$ $y''(0) + y'(0) - 0 = 3$ $\leftarrow$ (للتوضيح)

$$y''(0) = 3 + 2 = 5$$

Diff $\Rightarrow$ $y''' + y'' - (x y' + y) = 0$

$$y'''(0) + 5 - (0\, y' + 1) = 0$$

$$y'''(0) = -4$$

2) $y'' - y = 0$ about $x = 1$

IC is $y(1) = a \neq 0$ / $y'(1) = b \neq 0$

$$y = \sum \frac{y^{(n)}(1)}{n!} (x-1)^n$$

$$y = a + \frac{b}{1}(x-1)^1 + \frac{a}{2!}(x-1)^2 + \frac{b}{3!}(x-1)^3 + \dots$$

$$y'' = y$$

At $x = 1 \Rightarrow y''(1) = y(1) = a$

$$y'''(1) = y'(1) = b$$

$$y^{(4)}(1) = y''(1) = a$$

$$y = a\left(1 + \frac{1}{2}(x-1)^2 + \frac{1}{4!}(x-1)^4 + \dots\right) + b\left((x-1) + \frac{1}{3!}(x-1)^3 + \frac{1}{5!}(x-1)^5 + \dots\right)$$

$$y = a.\left(\sum_{k=0} \frac{1}{2k!}(x-1)^{2k}\right) + b.\left(\sum_{k=0} \frac{1}{(2k+1)}(x-1)^{2k+1}\right)$$